2020 Mathematical Statistics2 Midterm Solution

November 1, 2021

Problem 1 (a). Log-likelihood is given by

$$\begin{aligned}\ell(\theta) = \sum_{i=1}^{n} \log f(x_i;\theta) &= -\frac{1}{2\theta}\sum_{i=1}^{n}(\log x_i - \theta)^2 - \frac{n}{2}\log\theta + \text{const} \\ &= -\frac{n}{2}\theta - \frac{1}{2\theta}\sum_{i=1}^{n}\log^2 x_i - \frac{n}{2}\log\theta + \text{const.}\end{aligned}$$

Clearly, $\ell(\theta)$ is twice continuosuly differerentiable and

$$\frac{d\ell}{d\theta} = -\frac{n}{2} + \frac{1}{2\theta^2}\sum_{i=1}^{n}\log^2 x_i - \frac{n}{2\theta} = \frac{-n\theta^2 - n\theta + \sum_{i=1}^{n}\log^2 x_i}{2\theta^2} \tag{1}$$

$$\frac{d^2\ell}{d\theta^2} = -\frac{1}{\theta^3}\sum_{i=1}^{n}\log^2 x_i + \frac{n}{2\theta^2} = \frac{n\theta - 2\sum_{i=1}^{n}\log^2 x_i}{2\theta^3}. \tag{2}$$

Observe that $\ell(\theta)$ is neither convex nor concave as

$$\frac{d^2\ell}{d\theta^2}\begin{cases} < 0, & \text{if } 0 < \theta < \frac{2}{n}\sum_{i=1}^{n}\log^2 x_i \\ > 0, & \text{if } \frac{2}{n}\sum_{i=1}^{n}\log^2 x_i < \theta < \infty \end{cases}. \tag{3}$$

However, $\ell(\theta)$ has an unique maximizer and hence one can obtain unique MLE $\hat{\theta}$ of θ. Indeed, note that $\frac{d\ell}{d\theta}$ tends to ∞ as $\theta \downarrow 0$ and to $-\frac{n}{2}$ as $\theta \to \infty$. Also, (3) implies that $\frac{d\ell}{d\theta}$ is uniquely minimized at $\theta = \frac{2}{n}\sum_{i=1}^{n}\log^2 x_i$. Suppose $\tilde{\theta}$ is the solution of equation $\frac{d\ell}{d\theta} = 0$. Then,

$$\begin{aligned}\frac{d\ell}{d\theta} = 0 &\Leftrightarrow n\theta^2 + n\theta - \sum_{i=1}^{n}\log^2 x_i = 0 \\ &\Leftrightarrow \theta = \frac{-n \pm \sqrt{n^2 + 4n\sum_{i=1}^{n}\log^2 x_i}}{2n} \\ &\Rightarrow \tilde{\theta} = \frac{-n + \sqrt{n^2 + 4n\sum_{i=1}^{n}\log^2 x_i}}{2n} \quad (\because \tilde{\theta} > 0).\end{aligned}$$

Since $-n\theta^2 - n\theta + \sum_{i=1}^{n}\log^2 x_i$ is positive on $(\frac{-n-\sqrt{n^2+4n\sum_{i=1}^{n}\log^2 x_i}}{2n}, \frac{-n+\sqrt{n^2+4n\sum_{i=1}^{n}\log^2 x_i}}{2n})$ and negative on $(\frac{-n+\sqrt{n^2+4n\sum_{i=1}^{n}\log^2 x_i}}{2n}, \infty)$,

$$\frac{d\ell}{d\theta}\begin{cases} > 0 & \text{if } 0 < \theta < \frac{-n+\sqrt{n^2+4n\sum_{i=1}^{n}\log^2 x_i}}{2n} \\ < 0 & \text{if } \frac{-n+\sqrt{n^2+4n\sum_{i=1}^{n}\log^2 x_i}}{2n} < \theta < \infty \end{cases}.$$

Thus, this implies $\hat{\theta} = \frac{-n+\sqrt{n^2+4n\sum_{i=1}^{n}\log^2 x_i}}{2n}$.

Problem 1 (b). Set $n = 1$ in (2) and we have

$$\frac{d^2\ell}{d\theta^2} = -\frac{\log^2 X_1}{\theta^3} + \frac{1}{2\theta^2}.$$

Since ℓ is twice continuously differentiable, by Bartlett identity,

$$\begin{aligned} I(\theta) = \mathbb{E}_\theta(-\frac{d^2\ell}{d\theta^2}) &= -\frac{1}{2\theta^2} + \frac{1}{\theta^3}\mathbb{E}_\theta \log^2 X_1 \\ &= -\frac{1}{2\theta^2} + \frac{1}{\theta^3}\int_0^\infty \frac{\log^2 x}{x\sqrt{2\pi\theta}} \exp\left(-\frac{1}{2\theta}(\log x - \theta)^2\right) dx \\ &= -\frac{1}{2\theta^2} + \frac{1}{\theta^3}\int_{-\infty}^\infty \frac{t^2}{\sqrt{2\pi\theta}} \exp(-\frac{1}{2\theta}(t-\theta)^2)dt \\ &= -\frac{1}{2\theta^2} + \frac{1}{\theta^3}(\theta + \theta^2) = \frac{1}{\theta} + \frac{1}{2\theta^2}. \end{aligned} \tag{4}$$

Here the last equation in (4) holds because

$$\int_{-\infty}^\infty \frac{t^2}{\sqrt{2\pi\theta}} \exp(-\frac{1}{2\theta}(t-\theta)^2)dt = \mathbb{E}_\theta W,$$

where $W \sim N(\theta, \theta)$.

Problem 2 (a). Log-likelihood is given by

$$\begin{aligned} \ell(\mu) = \sum_{i=1}^n \log f(x_i;\mu) &= \sum_{i=1}^n \log \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}(x_i - \mu))^2 \\ &= -\frac{1}{2}\sum_{i=1}^n (x_i - \mu)^2 + \text{const} \\ &= -\frac{1}{2}\sum_{i=1}^n (x_i - \bar{x} + \bar{x} - \mu)^2 + \text{const} \\ &= -\frac{1}{2}\sum_{i=1}^n [(x_i - \bar{x})^2 + 2(x_i - \bar{x})(\bar{x} - \mu) + (\bar{x} - \mu)^2] + \text{const} \\ &= -\frac{1}{2}\sum_{i=1}^n (x_i - \bar{x})^2 - (\bar{x} - \mu)\sum_{i=1}^n (x_i - \bar{x}) - n(\bar{x} - \mu)^2 + \text{const} \\ &= -\frac{1}{2}\sum_{i=1}^n (x_i - \bar{x})^2 - (\bar{x} - \mu)(n\bar{x} - n\bar{x}) - n(\bar{x} - \mu)^2 + \text{const} \\ &= -n(\bar{x} - \mu)^2 + \text{const}. \end{aligned} \tag{5}$$

Here $\bar{x} = 1/n\sum_{i=1}^n x_i$. If $\bar{x} \geq 0$, clearly, $\ell(\mu)$ is uniquely maximized at $\mu = \bar{x}$ as $0 < \mu < \infty$. Otherwise, $\ell(\mu)$ is uniquely maximized at 0 due to the constraint $[0, \infty)$. Thus,

$$\hat{\mu} = \begin{cases} \bar{x}, & \text{if } \bar{x} \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

Problem 2 (b). Note that $\bar{X} \sim N(\mu, \frac{1}{n})$. Hence,

$$\mathbb{P}_\mu(\bar{X} \leq 0) = \mathbb{P}_\mu(\frac{\bar{X} - \mu}{1/\sqrt{n}} \leq -\sqrt{n}\mu) = \Phi(-\sqrt{n}\mu),$$

where $\Phi(\cdot)$ is c.d.f. of $N(0,1)$. Take any $\epsilon > 0$. Then,

$$\begin{aligned} \mathbb{P}_\mu(|\hat{\mu} - \mu| > \epsilon) &= \mathbb{P}_\mu(|\hat{\mu} - \mu| > \epsilon, \bar{X} < 0) + \mathbb{P}_\mu(|\hat{\mu} - \mu| > \epsilon, \bar{X} \geq 0) \\ &= \mathbb{P}_\mu(|\hat{\mu} - \mu| > \epsilon | \bar{X} < 0)\mathbb{P}_\mu(\bar{X} < 0) + \mathbb{P}_\mu(|\hat{\mu} - \mu| > \epsilon | \bar{X} \geq 0)\mathbb{P}_\mu(\bar{X} \geq 0). \end{aligned} \tag{6}$$

If $\mu > 0$,

$$
\begin{aligned}
\mathbb{P}_\mu(|\hat{\mu} - \mu| > \epsilon) &\le \mathbb{P}_\mu(\bar{X} < 0) + \mathbb{P}_\mu(|\hat{\mu} - \mu| > \epsilon | \bar{X} \ge 0)\mathbb{P}_\mu(\bar{X} \ge 0) \\
&\le \Phi(-\sqrt{n}\mu) + \mathbb{P}_\mu(|\hat{\mu} - \mu| > \epsilon | \bar{X} \ge 0) \\
&= \Phi(-\sqrt{n}\mu) + \mathbb{P}_\mu(|\bar{X} - \mu| > \epsilon | \bar{X} \ge 0) \\
&= \Phi(-\sqrt{n}\mu) + \frac{\mathbb{P}_\mu(|\bar{X} - \mu| > \epsilon, \bar{X} \ge 0)}{\mathbb{P}_\mu(\bar{X} \ge 0)} \\
&\le \Phi(-\sqrt{n}\mu) + \frac{\mathbb{P}_\mu(|\bar{X} - \mu| > \epsilon)}{1 - \Phi(-\sqrt{n}\mu)} \\
&\le \Phi(-\sqrt{n}\mu) + \frac{\mathrm{Var}_\mu(\bar{X})}{\epsilon^2} \cdot \frac{1}{1 - \Phi(-\sqrt{n}\mu)} \\
&= \Phi(-\sqrt{n}\mu) + \frac{1}{n\epsilon^2} \cdot \frac{1}{1 - \Phi(-\sqrt{n}\mu)}
\end{aligned} \tag{7}
$$

Here we used Chebyshev's inequality in the 4th inequality in (7). The last equation in (7) tends to 0 as $n \to \infty$. Otherwise, $\mu = 0$ and so $\mathbb{P}_\mu(\bar{X} \le 0) = \frac{1}{2}$. Thus, similar to (7), one can see that

$$
\begin{aligned}
\mathbb{P}_\mu(|\hat{\mu} - \mu| > \epsilon) &= \mathbb{P}_\mu(|\hat{\mu} - \mu| > \epsilon | \bar{X} < 0)\mathbb{P}_\mu(\bar{X} < 0) + \mathbb{P}_\mu(|\hat{\mu} - \mu| > \epsilon | \bar{X} \ge 0)\mathbb{P}_\mu(\bar{X} \ge 0) \\
&= \mathbb{P}_\mu(|0 - 0| > \epsilon | \bar{X} < 0)\mathbb{P}_\mu(\bar{X} < 0) + \mathbb{P}_\mu(|\hat{\mu} - \mu| > \epsilon | \bar{X} \ge 0)\mathbb{P}_\mu(\bar{X} \ge 0) \\
&= 0 \cdot \mathbb{P}_\mu(\bar{X} < 0) + \mathbb{P}_\mu(|\hat{\mu} - \mu| > \epsilon | \bar{X} \ge 0)\mathbb{P}_\mu(\bar{X} \ge 0) \\
&= \mathbb{P}_\mu(|\bar{X} - \mu| > \epsilon | \bar{X} \ge 0)\mathbb{P}_\mu(\bar{X} \ge 0) \\
&= \mathbb{P}_\mu(|\bar{X} - \mu| > \epsilon, \bar{X} \ge 0) \\
&\le \mathbb{P}_\mu(|\bar{X} - \mu| > \epsilon) \\
&\le \frac{\mathrm{Var}_\mu(\bar{X})}{\epsilon^2} \\
&\le \frac{1}{n\epsilon^2} \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{8}
$$

By (7) and (8), $\hat{\mu}$ is a consistent estimator of μ.

Problem 2 (c). Let $\mu = 0$. Then,

$$
\begin{aligned}
\mathbb{P}(\sqrt{n}\hat{\mu} \le x) &= \mathbb{P}(\sqrt{n}\hat{\mu} \le x | \bar{X} \ge 0)\mathbb{P}(\bar{X} \ge 0) + \mathbb{P}(\sqrt{n}\hat{\mu} \le x | \bar{X} < 0)\mathbb{P}(\bar{X} < 0) \\
&= \frac{1}{2}\mathbb{P}(\sqrt{n}\hat{\mu} \le x | \bar{X} \ge 0) + \frac{1}{2}\mathbb{P}(\sqrt{n}\hat{\mu} \le x | \bar{X} < 0) \\
&= \frac{1}{2}\mathbb{P}(\sqrt{n}\bar{X} \le x | \bar{X} \ge 0) + \frac{1}{2}\mathbb{P}(0 \le x | \bar{X} < 0).
\end{aligned} \tag{9}
$$

Here we used the fact that $\bar{X} \sim N(0, \frac{1}{n})$ so that $\mathbb{P}(\bar{X} \ge 0) = \mathbb{P}(\bar{X} < 0) = \frac{1}{2}$ in the second equation of (9). If $x < 0$, the last equation of (9) becomes 0 because $\sqrt{n}\bar{X} \le x$ does not hold if $\bar{X} \ge 0$ and $0 \le x$ also does not hold. Otherwise, $x \ge 0$ and thus

$$
\begin{aligned}
\mathbb{P}(\sqrt{n}\bar{X} \le x) &= \frac{1}{2}\mathbb{P}(\sqrt{n}\bar{X} \le x | \bar{X} \ge 0) + \frac{1}{2} \\
&= \frac{1}{2} \cdot \frac{\mathbb{P}(\sqrt{n}\bar{X} \le x, \bar{X} \ge 0)}{\mathbb{P}(\bar{X} \ge 0)} + \frac{1}{2} \\
&= \mathbb{P}(\sqrt{n}\bar{X} \le x, \bar{X} \ge 0) + \frac{1}{2} \\
&= \mathbb{P}(0 \le \sqrt{n}\bar{X} \le x) + \frac{1}{2} \\
&= \Phi(x) - \Phi(0) + \frac{1}{2} \\
&= \Phi(x),
\end{aligned}
$$

where $\Phi(\cdot)$ is c.d.f. of $N(0, 1)$. The 5th equation holds because $\sqrt{n}\bar{X} \sim N(0, 1)$. Therefore, $\sqrt{n}\bar{X}$ converges to random variable Y in distribution, whose c.d.f. F is

$$
F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \Phi(x), & \text{otherwise} \end{cases}.
$$

Problem 3. See the textbook p.587-588.

Problem 4. Let $\Theta = (0, \infty)$ and $\Theta_0 = \{1\}$. Let $\hat{\theta}^{\Theta}$ be MLE over Θ. and $\hat{\theta}^{\Theta_0}$ be that over Θ_0. Clearly, $\hat{\theta}^{\Theta_0} = 1$. Also, log-likelihood is given by

$$\begin{aligned}\ell(\theta) = \sum_{i=1}^{n} \log f(x_i; \theta) &= \sum_{i=1}^{n} \log \theta c^{\theta} x_i^{-\theta-1} \\ &= n \log \theta + \theta n \log c - \theta \sum_{i=1}^{n} \log x_i + \text{const.}\end{aligned}$$

This gives

$$\begin{aligned}\frac{d\ell}{d\theta} = \frac{n}{\theta} + n \log c - \sum_{i=1}^{n} \log x_i &= \frac{n}{\theta} - \sum_{i=1}^{n} \log(x_i/c) \\ &= \frac{n - \theta \sum_{i=1}^{n} \log(x_i/c)}{\theta}. \end{aligned} \tag{10}$$

Note that $\sum_{i=1}^{n} \log(x_i/c)$ is positive with probability tending to one as $c \le x_i$ for all $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \log(x_i/c) = 0$ if and only if $x_i = c$ for all i. So we may assume $\sum_{i=1}^{n} \log(x_i/c) > 0$. As $\theta > 0$, from (10),

$$\frac{d\ell}{d\theta} \begin{cases} > 0, & \text{if } 0 < \theta < 1/(\frac{1}{n}\sum_{i=1}^{n} \log(x_i/c)) \\ < 0, & \text{if } 1/(\frac{1}{n}\sum_{i=1}^{n} \log(x_i/c)) < \theta < \infty \end{cases}$$

Thus ℓ is strictly increasing on $(0, 1/(\frac{1}{n}\sum_{i=1}^{n} \log(x_i/c)))$ and strictly decreasing on $(1/(\frac{1}{n}\sum_{i=1}^{n} \log(x_i/c)), \infty)$. Hence, $\theta = \frac{1}{n}\sum_{i=1}^{n} \log(x_i/c)$ is an unique maximizer of ℓ over Θ. Thus, $\hat{\theta}^{\Theta} = 1/(\frac{1}{n}\sum_{i=1}^{n} \log(x_i/c))$. For a given hypothesis, LRT rejects H_0 if

$$\begin{aligned}2(\ell(\hat{\theta}^{\Theta}) - \ell(\hat{\theta}^{\Theta_0})) &= 2(n \log \hat{\theta}^{\Theta} + \hat{\theta}^{\Theta}(n \log c - \sum_{i=1}^{n} \log x_i) - (n \log c - \sum_{i=1}^{n} \log x_i)) \\ &= 2(n \log \hat{\theta}^{\Theta} - \hat{\theta}^{\Theta} \sum_{i=1}^{n} \log(x_i/c) + \sum_{i=1}^{n} \log(x_i/c)) \\ &= 2(n \log \hat{\theta}^{\Theta} - n + \sum_{i=1}^{n} \log(x_i/c)) \\ &= 2n(\log \hat{\theta}^{\Theta} - 1 + \frac{1}{n}\sum_{i=1}^{n} \log(x_i/c)) \\ &= 2n(\log \hat{\theta}^{\Theta} - 1 + 1/\hat{\theta}^{\Theta}) \\ &= 2n(-\log 1/\hat{\theta}^{\Theta} - 1 + 1/\hat{\theta}^{\Theta}) \ge k\end{aligned}$$

for some k. For a function function $f(t) = -\log t - 1 + t$ defined on $(0, \infty)$, f is strictly decreasing on $(0, 1)$ and strictly increasing on $(1, \infty)$. Also, $f(1) = 0$. Hence, there exist $0 < k_1 < k_2 < \infty$ such that $f(k_1) = f(k_2) = k$ for given $0 < k < \infty$ and so $f(t) \ge k$ if $0 < t \le k_1$ and $k_2 \le t < \infty$. This implies that the desired critical region is in the form of

$$\frac{1}{n}\sum_{i=1}^{n} \log(x_i/c) \ge c_2 \text{ or } \frac{1}{n}\sum_{i=1}^{n} \log(x_i/c) \le c_1$$

for some constants $0 < c_1 < c_2 < \infty$. Under H_0, if $X \sim f(\cdot; 1)$, $Y = \log(X/c) \sim \text{Exp}(1) \overset{\text{d}}{\equiv} \text{Gamma}(1, 1)$. This is because if we denote the p.d.f. of Y by $g(\cdot)$,

$$y = \log(x/c) \Leftrightarrow x = c \exp(y) \Rightarrow \frac{dx}{dy} = c \exp(y) > 0$$

an so

$$g(y) = f(c \exp(y); 1) c \exp(y) = \exp(-y) I_{[c,\infty)}(c \exp(y)) = \exp(-y) I_{[1,\infty)}(y),$$

which is p.d.f. of Exp(1). Since

$$2n(\frac{1}{n}\sum_{i=1}^{n}\log(X_i/c)) = 2\sum_{i=1}^{n}\log(X_i/c) \sim 2\text{Gamma}(n,1) \stackrel{\text{d}}{\equiv} \text{Gamma}(n,2) \stackrel{\text{d}}{\equiv} \chi^2(2n),$$

we reject H_0 if

$$2n(\frac{1}{n}\sum_{i=1}^{n}\log(X_i/c)) \geq \chi^2_{\alpha/2}(2n) \text{ or } 2n(\frac{1}{n}\sum_{i=1}^{n}\log(X_i/c)) \leq \chi^2_{1-\alpha/2}(2n)$$

, or equivalently,

$$\frac{1}{n}\sum_{i=1}^{n}\log(X_i/c) \geq \frac{1}{2n}\chi^2_{\alpha/2}(2n) \text{ or } \frac{1}{n}\sum_{i=1}^{n}\log(X_i/c)) \leq \frac{1}{2n}\chi^2_{1-\alpha/2}(2n).$$

Problem 5. See the textbook p.287-288.