2021 Midterm Solution

Bongjung Seung

November 1, 2021

Problem 1. Let $\ell(\theta)$ be the log-likelihood function. Then,

$$\begin{aligned}\ell(1) &= \log\left[\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x_0^2\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(x_1-2)^2\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(x_2-2)^2\right)\right] \\ &= -\frac{1}{2}x_0^2 - \frac{1}{2}(x_1-2)^2 - \frac{1}{2}(x_2-2)^2 - \frac{3}{2}\log(2\pi),\end{aligned} \tag{1}$$

$$\begin{aligned}\ell(2) &= \log\left[\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(x_0-1)^2\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(x_1-1)^2\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(x_2-3)^2\right)\right] \\ &= -\frac{1}{2}(x_0-1)^2 - \frac{1}{2}(x_1-1)^2 - \frac{1}{2}(x_2-3)^2 - \frac{3}{2}\log(2\pi).\end{aligned} \tag{2}$$

Hence, by (1) and (2),

$$\begin{aligned}\ell(2)-\ell(1) &= -\frac{1}{2}\left[(x_0-1)^2 - x_0^2\right] - \frac{1}{2}\left[(x_1-1)^2-(x_1-2)^2\right] - \frac{1}{2}\left[(x_2-3)^2-(x_2-2)^2\right] \\ &= -\frac{1}{2}\left(-2x_0+1\right) - \frac{1}{2}\left(2x_1-3\right) - \frac{1}{2}\left(-2x_2+5\right) \\ &= x_0 - x_1 + x_2 - \frac{3}{2}.\end{aligned}$$

This implies that

$$\ell(2) > \ell(1) \Leftrightarrow x_0 - x_1 + x_2 > \frac{3}{2}.$$

Likewise,

$$\ell(2) < \ell(1) \Leftrightarrow x_0 - x_1 + x_2 < \frac{3}{2}.$$

Therefore, MLE of θ, denoted by $\hat{\theta}$, is given as following:

$$\hat{\theta} = \begin{cases} 2, & \text{if } x_0 - x_1 + x_2 > \frac{3}{2} \\ 1, & \text{if } x_0 - x_1 + x_2 < \frac{3}{2} \end{cases}$$

with probability 1, since the probability of $X_0 - X_1 + X_2 = \frac{3}{2}$ is 0.

Problem 2. Suppose $(X, Y) \sim f$. Let R, Θ be independent random variables such that the p.d.f. $g(\cdot)$ of R is given by

$$g(t) = \frac{2t}{r^2} I(0 \le t \le r)$$

and $\Theta \sim U(0, 2\pi)$. Observe that $X \overset{d}{\equiv} R\cos\Theta$ and $Y \overset{d}{\equiv} R\sin\Theta$. Note that $R \overset{d}{\equiv} \sqrt{X^2+Y^2}$. Let $R_i \overset{d}{\equiv} \sqrt{X_i^2+Y_i^2}$. Then, the likelihood of $\theta = r^2$ is given by

$$\begin{aligned}L(\theta) &= \prod_{i=1}^{n} \frac{2r_i}{\theta} I(0 \le r_i \le \sqrt{\theta}) \\ &= \frac{\prod_{i=1}^{n} 2r_i I(0 \le r_i \le \sqrt{\theta})}{\theta^n}.\end{aligned}$$

Let $R_{(1)} \leq R_{(2)} \leq \cdots \leq R_{(n)}$ be order statistics of $R_1, R_2, \ldots, R_n$. Then, one may easily see that the MLE of θ, denoted by $\hat{\theta}$, is given by $\hat{\theta} = R_{(n)}^2$. We claim that the asymptotic distribution of $-n/\theta(\hat{\theta} - \theta)$ is equivalent to Exp(1), whose mean is 1.

Problem 3. See the textbook.

Problem 4 (a). Let $\mu = (\mu_1, \mu_2)$ and denote MLE of μ by $\hat{\mu} = (\hat{\mu}_1, \hat{\mu}_2)$. The log-likelihood ℓ of μ is given by

$$\ell(\mu) = -\frac{1}{2}(x - \mu_1)^2 - \frac{1}{2}(y - \mu_2)^2 + \text{const.}$$

Clearly, for each fixed μ_1, the term $-\frac{1}{2}(y - \mu_2)^2$ is uniquely maximized at $\mu_2 = y$. Hence, $\hat{\mu}_2 = y$. If $x \geq 0$, $-\frac{1}{2}(x - \mu_1)^2$ is uniquely maximized at $\mu_1 = x$, since $\mu_1 \geq 0$. Otherwise, $-\frac{1}{2}(x - \mu_1)^2$ is uniquely maximized at $\mu_1 = 0$. Thus,

$$\hat{\mu} = \begin{cases} (x, y) & \text{if } x \geq 0 \\ (0, y) & \text{otherwise} \end{cases}.$$

Problem 4 (b). Let $\hat{\mu}^0 = (0, 0)$. Then, LRT statistics is given by

$$2(\ell(\hat{\mu}) - \ell(\hat{\mu}^0)) = \begin{cases} X^2 + Y^2, & \text{if } X \geq 0 \\ Y^2, & \text{otherwise} \end{cases}.$$

One may wish to find $c \geq 0$ such that

$$\mathbb{P}_{\hat{\mu}^0}\left(2(\ell(\hat{\mu}) - \ell(\hat{\mu}^0)) \geq c\right) = \alpha.$$

Under H_0, as $X, Y \overset{i.i.d.}{\sim} N(0, 1)$, $X^2, Y^2 \overset{i.i.d.}{\sim} \chi^2(1)$ and $X^2 + Y^2 \sim \chi^2(2)$, since X, Y are independent so that X^2, Y^2 are independent. Since $\mathbb{P}_{\hat{\mu}^0}(X \geq 0) = \mathbb{P}_{\hat{\mu}^0}(X < 0) = 1/2$, this implies that

$$2(\ell(\hat{\mu}) - \ell(\hat{\mu}^0)) \sim \frac{1}{2}\chi^2(2) + \frac{1}{2}\chi^2(1).$$

Thus, the desired c can be estimated by generating sample by randomly obtaining sample from $\chi^2(2)$ or $\chi^2(1)$ and then finding $100(1 - \alpha)$th percentile in the given sample.