Mathematical Statistics2 Tutoring4

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October 7, 2021

Exercises

2019 Final Problem5

Consider the following model

$$Y_{ij} = \mu + \alpha_i + \beta x_{ij} + \epsilon_{ij}$$

where $\epsilon_{ij} \overset{\text{i.id.}}{\sim} N(0, \sigma^2)$, $(i = 1, \ldots, I, j = 1, \ldots, n_i)$. Here $\mu, \alpha_i, \beta, \sigma^2$ are unknown parameters ($\mu, \alpha_i, \beta \in \mathbb{R}$, $\sigma^2 \in \mathbb{R}_+$) and x_{ij} are known covariates. We further assume that

$$\sum_{i=1}^{I} n_i \alpha_i = 0 \text{ and } \sum_{i=1}^{I} \sum_{j=1}^{n_i} (x_{ij} - \bar{x})^2 \neq 0 \text{ where } \bar{x}_{i.} = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij}.$$

(a) Find the MLE of $(\mu, \alpha_i, \beta, \sigma^2)$ under the full model.
(b) For the following hypothesis $H_0 : \beta = 0$ versus $H_1 : \beta \neq 0$, find the LRT with significance level $0 < \alpha < 1$.

Exercises

2019 Final Problem4

Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be i.i.d. samples where

$$(X_i, Y_i) \sim N\left(\left(\begin{array}{c} \mu_1 \\ \mu_2 \end{array}\right), \left(\begin{array}{cc} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{array}\right)\right).$$

Here $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)^t \in \mathbb{R}^2 \times \mathbb{R}^2 \times (-1, 1)$. Assume $n \geq 5$ and denote the sample correlation by $\hat{\rho}$.

(a) Find the distribution of $\hat{\rho}/\sqrt{1 - \hat{\rho}^2}$.

(b) Find the LRT with significance level $0 < \alpha < 1$ for testing hypothesis $H_0 : \rho = 0$ versus $H_1 : \rho \neq 0$.

Exercises

2018 Final Problem1

Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be i.i.d. samples where

$$(X_i, Y_i) \sim N\left(\left(\begin{array}{c} \theta_1 \\ \theta_2 \end{array}\right), \left(\begin{array}{cc} 1 & \rho \\ \rho & 1 \end{array}\right)\right) \text{ for known } \rho \in (-1, 1).$$

We wish to test

$$H_0 : \theta_1 = \theta_2 = 0 \text{ versus } H_1 : \text{not } H_0.$$

(a) Suppose that the parameter space is given by $\Theta = \{(\theta_1, \theta_2) : \theta_2 \leq c\theta_1, \theta_1 \geq 0\}$ for $c \in \mathbb{R}$ and $\rho = 0$. Derive LRT of significance level $0 < \alpha < 1$.

(b)Suppose that the parameter space is given by $\Theta = \{(\theta_1, \theta_2) : \theta_1, \theta_2 \geq 0\}$. Derive LRT of significance level $0 < \alpha < 1$.

Approximation of LRT: Simple Null

Let $X_1, \ldots, X_n$ be a random sample from a distribution with pdf $f(\cdot;\theta)$, $\theta \in \Theta \subset \mathbb{R}^d$. Assume that the regularity conditions $(R0) - (R7)$ hold. Consider the problem of testing $H_0 : \theta = \theta_0$. Let $\hat{\theta}$ denote the MLE over Θ. For the LRT statistic $2(l(\hat{\theta}) - l(\hat{\theta_0}))$, it holds that, under H_0,

- $2(l(\hat{\theta}) - l(\theta_0)) \xrightarrow{\text{d}} \chi^2(d)$
- $2(l(\hat{\theta}) - l(\theta_0)) = W_n + o_p(1)$
- $2(l(\hat{\theta}) - l(\theta_0)) = R_n + o_p(1)$

where $W_n = (\hat{\theta} - \theta_0)^t (nI(\theta_0))(\hat{\theta} - \theta_0)$ (Wald test statistic) and $R_n = \dot{l}(\theta_0)^t (nI(\theta_0))^{-1} \dot{l}(\theta_0)$ (Rao/score test statistic). Note that (1) is called Wilk's phenomenon. Which test is better?

Approximation of LRT: Exponential Model

Suppose we observe a random sample $X_1, \ldots, X_n$ from Exp(θ) for $\theta > 0$.

- Hypothesis: $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.
- Asymptotic LRT: Reject H_0 if

$$2n(\bar{x}/\theta_0 - \log(\bar{x}/\theta_0) - 1) \geq \chi^2_\alpha(1).$$

- Wald and Rao tests: Reject H_0 if

$$n(\bar{x} - \theta_0)^2/\theta_0^2 \geq \chi^2_\alpha(1).$$

Approximation of LRT: Beta Model

Suppose we observe a random sample $X_1, \ldots, X_n$ from Beta$(\theta, 1)$ for $\theta > 0$.

- Hypothesis: $H_0 : \theta = \theta_0$ versus $H_0 : \theta \neq \theta_0$.
- MLE: $\hat{\theta} = -n/(\sum_{i=1}^n \log X_i)$.
- LRT statistic: $2(l(\hat{\theta}) - l(\theta_0)) = 2n(\log \hat{\theta} - 1 + 1/\hat{\theta} - \log \theta_0)$.
- Wald test statistic: $W_n(\theta_0) = n/\theta_0(\hat{\theta} - \theta_0)^2$.
- Rao test statistic: $R_n(\theta_0) = n\theta_0^2(1/\theta_0 + n^{-1} \sum_{i=1}^n \log X_i)^2$.

Approximation of LRT: Double Exponential Model

Suppose we observe a random sample $X_1, \ldots, X_n$ from DE$(\theta, 1)$ for $-\infty < \theta < \infty$.

- Hypothesis: $H_0 : \theta = \theta_0$ versus $H_0 : \theta \neq \theta_0$.
- MLE: $\hat{\theta} =$ med(X_i).
- LRT statistic: $2(l(\hat{\theta}) - l\theta_0)) = 2(\sum_{i=1}^n |x_i - \theta_0| - \sum_{i=1}^n |x_i - \hat{\theta}|)$.
- Wald test statistic: $W_n(\theta_0) = n(\hat{\theta} - \theta_0)^2$.
- Rao test statistic: $R_n(\theta_0) = (\sum_{i=1}^n \mathrm{sgn}(X_i - \theta_0))^2/n$.

Approximation of LRT: Composite Null

Here, we discuss the approximation of LRT when the null hypothesis is composite. For this, let $X_1, \dots, X_n$ be a random sample from a distribution with pdf $f(\cdot;\theta)$, $\theta \in \Theta \in \mathbb{R}^d$ that satisfies the regularity conditions (R0)-(R7). Let $\theta = (\xi^t, \eta^t)^t$ with $\eta \in \mathbb{R}^{d_0}$, and suppose we wish to test

$$H_0 : \xi = \xi_0 \text{ versus } H_1 : \xi \neq \xi_0.$$

The null parameter space is of dimension $\dim(\Theta_0) = d_0$ given by

$$\Theta_0 = \left\{ (\xi_0^t, \eta^t)^t : \eta \in \mathbb{R}^{d_0} \text{ and } (\xi^t, \eta^t)^t \in \Theta \right\}.$$

Let $\hat{\theta}^{\Theta}$ and $\hat{\theta}^{\Theta_0}$ denote the MLEs in Θ and Θ_0, respectively.

Approximation of LRT: Composite Null

Under the compositve null hypothesis $H_0 : \xi = \xi_0 \Leftrightarrow H_0 : \theta \in \Theta_0$, it holds that

- $2(l(\hat{\theta}^{\Theta}) - l(\hat{\theta}^{\Theta_0})) \xrightarrow{\text{d}} \chi^2(d)$
- $2(l(\hat{\theta}^{\Theta}) - l(\hat{\theta}^{\Theta_0})) = W_n + o_p(1)$
- $2(l(\hat{\theta}^{\Theta}) - l(\hat{\theta}^{\Theta_0})) = R_n + o_p(1)$

where $W_n = (\hat{\theta}^{\Theta} - \hat{\theta}^{\Theta_0})^t(nI(\hat{\theta}^{\Theta_0}))(\hat{\theta}^{\Theta} - \hat{\theta}^{\Theta_0})$ (Wald test statistic) and $R_n = \dot{l}(\hat{\theta}^{\Theta_0})^t(nI(\hat{\theta}^{\Theta_0}))^{-1}\dot{l}(\hat{\theta}^{\Theta_0})$ (Rao/score test statistic).