Mathematical Statistics2 Tutoring7

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Optimality in Testing Problem

Suppose we observe a random sample $X_1, X_2, \ldots, X_n$ from a population with pdf $f(\cdot, \theta), \theta \in \Theta \subset \mathbb{R}^d$, and want to test

$$H_0 : \theta \in \Theta_0 \text{ versus } H_1 : \theta \in \Theta_1$$

at a given level α $(0 < \alpha < 1)$, where Θ_j are subsets of Θ such that $\Theta = \Theta_0 \cup \Theta_1$. Let $X = (X_1, \ldots, X_n)$.

- Critical function: A (randomized) test may be represented by a function $\varphi(X)$ of X whose value is the probability of rejecting H_0.
- Size of test: $\max_{\theta \in \Theta_0} E_\theta \phi(X)$ is called the size of the test φ.
- A test $\varphi(X)$ has level α if its size is less than or equal to α.
- We seek a test ϕ that masimizes the power $E_\theta \phi(X)$ for $\theta \in \Theta_1$ among all level α tests.

(Uniformly) Most Powerful Test

- Most powerful test: Suppose $\Theta = \{\theta_1\}$, i.e., H_1 is simple. A test ϕ is most powerful (MP) at level α if its level is α and its power is the largest among all level α tests, i.e.,
  (1) $\max_{\theta \in \Theta_0} E_\theta \phi(X) \leq \alpha$;
  (2) $E_{\theta_1} \phi(X) \geq E_{\theta_1} \varphi(X)$ for any test φ with $\max_{\theta \in \Theta_0} E_\theta \varphi(X) \leq \alpha$.
- Uniformly most powerful test: A test ϕ is uniformly most powerful (MP) at level α if its level is α and its power is the largest among all level α tests, i.e.,
  (1) $\max_{\theta \in \Theta_0} E_\theta \phi(X) \leq \alpha$;
  (2) for all $\theta_1 \in \Theta_1$, $E_{\theta_1} \phi(X) \geq E_{\theta_1} \varphi(X)$ for any test φ with $\max_{\theta \in \Theta_0} E_\theta \varphi(X) \leq \alpha$.
- If the MP level α test for testing $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta = \theta_1$ does not depend on $\theta_1 \in \Theta_1$, then it is UMP level α.

Neyman-Pearson Lemma

Suppose we want to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, where $\theta_1 \neq \theta_0$ are fixed. Let $L(\theta; x) = \prod_{i=1}^{n} f(x_i; \theta)$. If there exists a test ϕ such that

$$(1)\ \phi(x) = \begin{cases} 1 & \text{if } L(\theta_1; x)/L(\theta_0; x) > k \\ \gamma & \text{if } L(\theta_1; x)/L(\theta_0; x) = k \\ 0 & \text{if } L(\theta_1; x)/L(\theta_0; x) < k \, (k \geq 0) \end{cases}$$

$$(2)\ E_{\theta_0}\phi(X) = \alpha,$$

then ϕ is MP level α for testing H_0 versus H_1.

Neyman-Pearson Lemma

- There exists a test that satisfies (1) and (2). Also, any MP level α test satisfies the condition (1). It also satisfies (2) unless there exsits a test with size$< \alpha$ and power 1.
- Determination of γ and k: If

$$P_{\theta_0}\left(\frac{L(\theta_1;X)}{L(\theta_0;X)} > k\right) = \alpha_0 < \alpha < \alpha_1 = P_{\theta_0}\left(\frac{L(\theta_1;x)}{L(\theta_0;X)} \geq k\right),$$

  then $\gamma = (\alpha - \alpha_0)/(\alpha_1 - \alpha_0)$.
- Any MP level α test has power$\geq \alpha$.

UMP Test: Normal Model

Let $X_1, \ldots, X_n$ be a random sample from $N(\theta, \sigma^2)$ with $\sigma^2 > 0$ being fixed (known). Let $X \equiv (X_1, \ldots, X_n)$ and x denote its realization.

- Suppose we test $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$. Then, the test ϕ defined by

$$\phi(x) = \begin{cases} 1 & \text{if } \frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}} > z_\alpha \\ 0 & \text{otherwise} \end{cases}$$

  is UMP level α.

- Similarly, if one wishes to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta < \theta_0$, the test φ defined by

$$\varphi(x) = \begin{cases} 1 & \text{if } \frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}} < -z_\alpha \\ 0 & \text{otherwise} \end{cases}$$

  is UMP level α.

UMP Test: Poisson Model

Let $X_1, \ldots, X_n$ be a random sample Poisson(θ). Assume $n = 10$.

- Testing $H_0 : \theta = 1/10$ versus $H_1 : \theta > 1/10$: The UMP level $\alpha = 0.05$ test is given by

$$\phi(x) = \begin{cases} 1 & \text{if } x_1 + \cdots + x_{10} \geq 4 \\ 31/61 & \text{if } x_1 + \cdots + x_{10} = 3 \\ 0 & \text{if } x_1 + \cdots + x_{10} \leq 2 \end{cases}$$

- The test that rejects H_0 when $x_1 + \cdots + x_{10} \geq 3$ is UMP level $\alpha = 0.08$ (the size of the test).

UMP Test: Bernoulli Model

Let $X_1, \ldots, X_n$ be a random sample Bernoulli(θ). Assume $n = 5$.

- Testing $H_0 : \theta = 1/2$ versus $H_1 : \theta < 1/2$: The UMP level $\alpha = 0.1$ test is given by

$$\phi(x) = \begin{cases} 1 & \text{if } x_1 + \cdots + x_5 \leq 0 \\ 11/25 & \text{if } x_1 + \cdots + x_5 = 1 \\ 0 & \text{if } x_1 + \cdots + x_5 \geq 2 \end{cases}$$

- The test that rejects H_0 when $x_1 + \cdots + x_5 \geq 1$ is UMP level $\alpha = 3/16$ (the size of the test).

UMP Test: Exponential Family

Theorem. Let $X_1, \ldots, X_n$ be a random sample from a population with pdf $f(x;\theta) = \exp(\eta(\theta)T(x) - B(\theta)) \cdot h(x)$, where $\theta \in \Theta \subset \mathbb{R}$. Assume that $\eta(\theta)$ is a strictly increasing function of θ. Then, the UMP level α $(0 < \alpha < 1)$ test for

$$H_0 : \theta \leq \theta_0 \text{ versus } H_1 : \theta > \theta_0$$

is given by ϕ that satisfies

$$(1)\ \phi(x) = \begin{cases} 1 & \text{if } \sum_{i=1}^n T(x_i) > c \\ \gamma & \text{if } \sum_{i=1}^n T(x_i) = c \\ 0 & \text{if } \sum_{i=1}^n T(x_i) < c \end{cases}$$

$$(2)\ E_{\theta_0}\phi(X) = \alpha.$$

In fact, it is UMP level α for testing $H_0' : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$.