2021 Final Solution

Bongjung Seung

December 2, 2021

Problem 1 (a). Following the argument in the solution of Exercise 8.11 (a) in the textbook, one can see that $(X_{(1)}, \sum_{i=1}^m (X_i - X_{(1)}))$ and $(Y_{(1)}, \sum_{j=1}^n (Y_j - Y_{(1)}))$ are complete sufficient statistics for (a_x, θ_x) and (a_y, θ_y), respectively, where $X_{(1)} = \min_{1\le i\le m} X_i$ and $Y_{(1)} = \min_{1\le j\le n} Y_j$. Note that by the independence of X_i's and Y_j's, $(X_{(1)}, \sum_{i=1}^m (X_i - X_{(1)}), Y_{(1)}, \sum_{j=1}^n (Y_j - Y_{(1)}))$ is a complete sufficient statistic for $\theta = (a_x, \theta_x, a_y, \theta_y)$. Also, from the solutions of Exercise 8.11 (b) and (c), one can easily deduce that $(X_{(1)} - 1/m \sum_{i=1}^m (X_i - X_{(1)})/(m-1), \sum_{i=1}^m (X_i - X_{(1)})/(m-1), Y_{(1)} - 1/n \sum_{j=1}^n (Y_j - Y_{(1)})/(n-1), \sum_{j=1}^n (Y_j - Y_{(1)})/(n-1))$ is an unbiased estimator of θ. Thus, $X_{(1)} - 1/m \sum_{i=1}^m (X_i - X_{(1)})/(m-1) - (Y_{(1)} - 1/n \sum_{j=1}^n (Y_j - Y_{(1)})/(n-1))$ is an UMVUE of $a_x - a_y$ by Theorem 8.3.1 (b) in the textbook.

Let $X_{(1)} \le \cdots \le X_{(m)}$ be order statistics of $X_1, \ldots, X_m$. Observe that

$$\begin{aligned}
\sum_{i=1}^m (X_i - X_{(1)}) &= \sum_{i=1}^m (X_{(i)} - X_{(1)}) \\
&= \sum_{i=1}^m \{(X_{(i)} - X_{(i-1)}) + (X_{(i-1)} - X_{(i-2)}) + \cdots + (X_{(2)} - X_{(1)})\} \\
&= \sum_{i=1}^m \sum_{r=2}^i (X_{(r)} - X_{(r-1)}) \\
&= \sum_{r=2}^m \sum_{i=r}^m (X_{(r)} - X_{(r-1)}) \\
&= \sum_{r=2}^m (m-r+1)(X_{(r)} - X_{(r-1)}).
\end{aligned}$$

Since $X_i \overset{d}{\equiv} \theta_x U_i + a_x$ for $U_1, \ldots, U_m \overset{i.i.d.}{\sim} E(0,1)$,

$$\begin{aligned}
\sum_{i=1}^m (X_i - X_{(1)})/\theta_x &= \sum_{r=2}^m (m-r+1)(X_{(r)} - X_{(r-1)})/\theta_x \\
&\overset{d}{\equiv} \sum_{r=2}^m (m-r+1)(\theta_x U_{(r)} + a_x - \theta_x U_{(r-1)} - a_x)/\theta_x \\
' &= \sum_{r=2}^m (m-r+1)(U_{(r)} - U_{(r-1)}).
\end{aligned}$$

By Example 4.3.3 in the textbook,

$$(m-r+1)(U_{(r)} - U_{(r-1)}) \overset{i.i.d.}{\sim} \text{Gamma}(1,1)$$

for $r = 2, \ldots, m$. Therefore,

$$\sum_{i=1}^m (X_i - X_{(1)})/\theta_x \sim \text{Gamma}(m-1, 1).$$

Similarly,

$$\sum_{j=1}^n (Y_j - Y_{(1)})/\theta_y \sim \text{Gamma}(n-1, 1).$$

Clearly,

$$E_\theta\left(\sum_{i=1}^{m}(X_i - X_{(1)})/\theta_x\right) = m-1$$

so that $\sum_{i=1}^{m}(X_i - X_{(1)})/(m-1)$ is an unbiased estimator of θ_x. Also, for sufficiently large n,

$$\begin{aligned}
E_\theta\left(\theta_y/\sum_{j=1}^{n}(Y_j - Y_{(1)})\right) &= EV^{-1} \\
&= \int v^{-1}\frac{1}{\Gamma(n-1)}v^{n-2}\exp(-v)dv \\
&= \int \frac{1}{\Gamma(n-1)}v^{n-3}\exp(-v)dv \\
&= \frac{1}{\Gamma(n-1)}\cdot\Gamma(n-2) \\
&= \frac{(n-3)!}{(n-2)!} \\
&= \frac{1}{n-2},
\end{aligned}$$

where $V \sim \text{Gamma}(n-1,1)$. This implies

$$E_\theta\left((n-2)/\sum_{j=1}^{n}(Y_j - Y_{(1)})\right) = 1/\theta_y.$$

Because X_i's and Y_j's are independent so that $\sum_{i=1}^{m}(X_i - X_{(1)})/(m-1)$ and $(n-2)/\sum_{j=1}^{n}(Y_j - Y_{(1)})$ are independent, $\frac{\sum_{i=1}^{m}(X_i - X_{(1)})/(m-1)}{(n-2)/\sum_{j=1}^{n}(Y_j - Y_{(1)})}$ is an unbiased estimator of θ_x/θ_y and so the desired UMVUE of θ_x/θ_y by Theorem 8.3.1 (b).

Problem 1 (b). Let z_α be such that $P(Z > z_\alpha) = \alpha$, where $Z \sim N(0,1)$. Since

$$P(X_1 > \xi) = P((X_1-\mu)/\sigma > (\xi-\mu)/\sigma) = \alpha$$

and $(X_1-\mu)/\sigma \sim N(0,1)$, $(\xi-\mu)/\sigma = z_\alpha$ or $\xi = \sigma z_\alpha + \mu$. By Example 8.3.4 in the textbook, $(\bar{X}, S^2)$ are complete sufficient statistic for (μ,σ^2), where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n}X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$. Also, as $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1) \equiv \text{Gamma}((n-1)/2, 2)$. Thus, for sufficiently large n,

$$\begin{aligned}
E_{\mu,\sigma^2}\frac{\sqrt{n-1}S}{\sigma} &= EW^{1/2} \\
&= \int w^{1/2}\cdot\frac{1}{\Gamma((n-1)/2)2^{(n-1)/2}}w^{(n-1)/2-1}\exp\left(-\frac{w}{2}\right)dw \\
&= \int \frac{1}{\Gamma((n-1)/2)2^{(n-1)/2}}w^{n/2-1}\exp\left(-\frac{w}{2}\right)dw \\
&= \frac{1}{\Gamma((n-1)/2)2^{(n-1)/2}}\cdot\Gamma(n/2)2^{n/2} \\
&= \sqrt{2}\frac{\Gamma(n/2)}{\Gamma((n-1)/2)},
\end{aligned}$$

where $W \sim \chi^2(n-1)$. Consequently,

$$E_{\mu,\sigma^2}\sqrt{\frac{n-1}{2}}\frac{\Gamma((n-1)/2)}{\Gamma(n/2)}S = \sigma.$$

Since

$$E_{\mu,\sigma^2}(\sqrt{\frac{n-1}{2}}\frac{\Gamma((n-1)/2)}{\Gamma(n/2)}Sz_\alpha + \bar{X}) = \sigma z_\alpha + \mu$$

and ξ and $\sqrt{\frac{n-1}{2}}\frac{\Gamma((n-1)/2)}{\Gamma(n/2)}Sz_\alpha + \bar{X}$ are function of (μ, σ^2) and $(\bar{X}, S^2)$, which is a complete sufficient statistic for (μ, σ^2), respectively, $\sqrt{\frac{n-1}{2}}\frac{\Gamma((n-1)/2)}{\Gamma(n/2)}Sz_\alpha + \bar{X}$ is UMVUE for ξ, by Theorem 8.3.1 (b).

Problem 2. See Exercise 9.4 in the textbook.

Problem 3 (a). Let $X_{(1)} = \min_{1\le i\le n} X_i$. Note that since we assume that $X_1, \ldots, X_n$ are random samples from $f(\cdot; a_i, \theta)$ $(i = 1, 2)$, $X_{(1)} > a_1$ as $a_1 < a_0$. Also,

$$\begin{aligned}
\frac{\prod_{i=1}^n f(x_i; a_1, \theta)}{\prod_{i=1}^n f(x_i; a_0, \theta)} &= \frac{\prod_{i=1}^n \theta a_1^\theta x_i^{-(\theta+1)} I(x_i > a_1)}{\prod_{i=1}^n \theta a_0^\theta x_i^{-(\theta+1)} I(x_i > a_0)} \\
&= \left(\frac{a_1}{a_0}\right)^{n\theta} \frac{I(X_{(1)} > a_1)}{I(X_{(1)} > a_0)} \\
&= \begin{cases} \infty, & a_1 < X_{(1)} \le a_0 \\ \left(\frac{a_1}{a_0}\right)^{n\theta}, & X_{(1)} > a_0 \end{cases}.
\end{aligned} \tag{1}$$

Thus, by Theorem 9.1.1 (Neyman-Pearson Lemma or NP Lemma), the MP test for testing $H_0 : a = a_0$ vs $H_1 : a = a_1$ at level α should be in the form of one of followings:

$$T_1(X) = \begin{cases} 1, & a_1 < X_{(1)} \le a_0 \\ \gamma, & X_{(1)} > a_0 \end{cases},$$

$$T_2(X) = \begin{cases} \gamma, & a_1 < X_{(1)} \le a_0 \\ 0, & X_{(1)} > a_0 \end{cases}.$$

By NP Lemma, the test cannot be in the form of T_2 as

$$\begin{aligned}
E_{a_0} T_2(X) &= \gamma \cdot P_{a_0}(a_1 < X_{(1)} \le a_0) + 0 \cdot P_{a_0}(X_{(1)} > a_0) \\
&= \gamma \cdot 0 + 0 \cdot 1 = 0 \ne \alpha.
\end{aligned}$$

Hence, the test should be in form of T_1 and so

$$\begin{aligned}
E_{a_0} T_1(X) &= 1 \cdot P_{a_0}(a_1 < X_{(1)} \le a_0) + \gamma \cdot P_{a_0}(X_{(1)} > a_0) \\
&= 1 \cdot 0 + \gamma \cdot 1 = \gamma = \alpha.
\end{aligned}$$

Thus, the MP test for the given hypothesis testing problem at level α is given by

$$T(X) = \begin{cases} 1, & a_1 < X_{(1)} \le a_0 \\ \alpha, & X_{(1)} > a_0 \end{cases}. \tag{2}$$

Problem 3 (b). We claim that the test ϕ defined by

$$\phi(x) = \begin{cases} 1, & X_{(1)} \le a_0 \\ \alpha, & X_{(1)} > a_0 \end{cases}$$

is the UMP test for given hypothesis testing problem at level α. Clearly,

$$E_{a_0}\phi(X) = 1 \cdot P_{a_0}(X_{(1)} \le a_0) + \alpha \cdot P_{a_0}(X_{(1)} > a_0) = 1 \cdot 0 + \alpha \cdot 1 = \alpha.$$

Hence, it suffices to show that the test ϕ is the MP test for

$$H_0 : a = a_0 \text{ vs } H_1 : a = a_1$$

for any $a_1 < a_0$. But this can be established if $E_{a_1}\phi(X) = E_{a_1}T(X)$, where the test T is defined in (2). Let $p(\cdot;\cdot,\theta)$ be pdf of $X_{(1)}$ and $F(\cdot;\cdot,\theta)$ be the cdf corresponding to pdf $f(\cdot;\cdot,\theta)$. Then, under H_1, one can see that

$$\begin{aligned}
p(t;a_i,\theta) &= \frac{n!}{(1-1)!(n-1)!}[F(t;a_i,\theta)]^{1-1}\cdot f(t;a_i,\theta)\cdot[1-F(t;a_i,\theta)]^{n-1} \\
&= nf(t;a_i,\theta)\left[1-\int_{a_i}^{t} f(x;a_i,\theta)dx\right]^{n-1} I(a_i<t) \\
&= nf(t;a_i,\theta)\left\{1-\left[-a_i^{\theta}x^{-\theta}\right]_{a_i}^{t}\right\}^{n-1} I(a_i<t) \\
&= n\theta a_i^{\theta}t^{-(\theta+1)}[1-(-a_i^{\theta}t^{-\theta}+1)]^{n-1}I(a_i<t) \\
&= n\theta a_i^{\theta}t^{-(\theta+1)}(a_i^{\theta}t^{-\theta})^{n-1}I(a_i<t) \\
&= n\theta a_i^{n\theta}t^{-(n\theta+1)}I(a_i<t).
\end{aligned}$$

for $i = 1, 2$.

Thus,

$$\begin{aligned}
E_{a_1}T(X) &= 1\cdot\int_{a_1}^{a_0} p(t;a_1,\theta)dt + \alpha\cdot\int_{a_0}^{\infty} p(t;a_1,\theta)dt \\
&= 1\cdot[-a_1^{n\theta}t^{-n\theta}]_{a_1}^{a_0} + \alpha\cdot[-a_1^{n\theta}t^{-n\theta}]_{a_0}^{\infty} \\
&= 1\cdot[-a_1^{n\theta}a_0^{-n\theta} + a_1^{n\theta}a_1^{-n\theta}] + \alpha\cdot(a_1^{n\theta}a_0^{-n\theta}) \\
&= 1\cdot\left[-\left(\frac{a_1}{a_0}\right)^{n\theta}+1\right] + \alpha\cdot\left(\frac{a_1}{a_0}\right)^{n\theta} \\
&= 1+(\alpha-1)\cdot\left(\frac{a_1}{a_0}\right)^{n\theta}.
\end{aligned} \tag{3}$$

Also, as

$$\begin{aligned}
E_{a_1}\phi(X) &= 1\cdot P_{a_1}(a_1 < X_{(1)} \le a_0) + \alpha\cdot P_{a_1}(a_0 < X_{(1)}) \\
&= 1\cdot\int_{a_1}^{a_0} p(t;a_1,\theta)dt + \alpha\cdot\int_{a_0}^{\infty} p(t;a_1,\theta)dt,
\end{aligned}$$

$E_{a_1}T(X) = E_{a_1}\phi(X)$ for any $a_1 < a_0$ so that ϕ is the UMP test for the given hypothesis testing problem at level α by the definition of UMP test. Note that the result in (3) is to be used in **Problem 3 (c).**

Problem 3 (c). We first find the MP test at level α for following hypothesis testing problem:

$$H_0 : a = a_0 \text{ vs } H_1 : a = a_1 \tag{4}$$

for $a_1 > a_0$. Following the calculations in (1) but considering $a_1 > a_0$, one can easily see that

$$\frac{\prod_{i=1}^{n} f(x_i;a_1,\theta)}{\prod_{i=1}^{n} f(x_i;a_0,\theta)} = \begin{cases} 0, & a_0 < X_{(1)} \le a_1 \\ \left(\frac{a_1}{a_0}\right)^{n\theta}, & a_1 < X_{(1)} \end{cases}.$$

Thus, the MP test φ for (4) at level α should be in form of one of followings:

$$R_1(X) = \begin{cases} 1, & a_1 < X_{(1)} \\ \gamma, & a_0 < X_{(1)} \le a_1 \end{cases},$$

$$R_2(X) = \begin{cases} \gamma, & a_1 < X_{(1)} \\ 0, & a_0 < X_{(1)} \le a_1 \end{cases}.$$

If φ is in form of R_1,

$$\begin{aligned}E_{a_0}R_1(X) &= 1\cdot P_{a_0}(a_1 < X_{(1)}) + \gamma\cdot P_{a_0}(a_0 < X_{(1)} \le a_1)\\ &= \int_{a_1}^{\infty} p(t;a_0,\theta)dt + \gamma\cdot\int_{a_0}^{a_1} p(t;a_0,\theta)dt\\ &= [-a_0^{n\theta}t^{-n\theta}]_{a_1}^{\infty} + \gamma\cdot[-a_0^{n\theta}t^{-n\theta}]_{a_0}^{a_1}\\ &= \left(\frac{a_0}{a_1}\right)^{n\theta} + \gamma\cdot\left[1-\left(\frac{a_0}{a_1}\right)^{n\theta}\right].\end{aligned} \tag{5}$$

Similarly, if φ is in form of R_2,

$$\begin{aligned}E_{a_0}R_2(X) &= \gamma\cdot P_{a_0}(a_1 < X_{(1)}) + 0\cdot P_{a_0}(a_0 < X_{(1)} \le a_1)\\ &= \gamma\int_{a_1}^{\infty} p(t;a_0,\theta)dt\\ &= \gamma\cdot[-a_0^{n\theta}t^{-n\theta}]_{a_1}^{\infty}\\ &= \gamma\cdot\left(\frac{a_0}{a_1}\right)^{n\theta}.\end{aligned} \tag{6}$$

If $a_0\alpha^{-1/n\theta} < a_1$ so that $(a_0/a_1)^{n\theta} < \alpha$, equating $E_{a_0}R_1(X)$ in (5) to α gives

$$\gamma = \frac{\alpha-\left(\frac{a_0}{a_1}\right)^{n\theta}}{1-\left(\frac{a_0}{a_1}\right)^{n\theta}}.$$

Therefore, if $a_0\alpha^{-1/n\theta} < a_1$, by NP Lemma, the test φ_1 defined by (7) is the MP test for testing (4) at level α:

$$\varphi_1(X) = \begin{cases} 1, & a_1 < X_{(1)}\\ \dfrac{\alpha-\left(\frac{a_0}{a_1}\right)^{n\theta}}{1-\left(\frac{a_0}{a_1}\right)^{n\theta}}, & a_0 < X_{(1)} \le a_1\end{cases}. \tag{7}$$

Note that the test φ_1 exists only when $a_0\alpha^{-1/n\theta} < a_1$ as $0<\gamma<1$.

Otherwise $a_0\alpha^{-1/n\theta} \ge a_1$ so that $(a_0/a_1)^{n\theta} \ge \alpha$, equating $E_{a_0}R_2(X)$ in (6) to α gives

$$\gamma = \alpha/\left(\frac{a_0}{a_1}\right)^{n\theta}.$$

Thus, by NP Lemma, the test φ_2 defined by (8) is the MP test for testing (4) at level α:

$$\varphi_2(X) = \begin{cases} \alpha/\left(\frac{a_0}{a_1}\right)^{n\theta}, & a_1 < X_{(1)}\\ 0, & a_0 < X_{(1)} \le a_1\end{cases}. \tag{8}$$

Note that the test φ_2 exists only when $a_0\alpha^{-1/n\theta} \ge a_1$ as $0<\gamma\le 1$. Assuming $a_0\alpha^{-1/n\theta} < a_1$, the power of the test φ_1 for testing (4) is

$$\begin{aligned}E_{a_1}\varphi_1(X) &= 1\cdot P_{a_1}(a_1 < X_{(1)}) + \frac{\alpha-\left(\frac{a_0}{a_1}\right)^{n\theta}}{1-\left(\frac{a_0}{a_1}\right)^{n\theta}}\cdot P_{a_1}(a_0 < X_{(1)} \le a_1)\\ &= 1\cdot 1 + \frac{\alpha-\left(\frac{a_0}{a_1}\right)^{n\theta}}{1-\left(\frac{a_0}{a_1}\right)^{n\theta}}\cdot 0\\ &= 1.\end{aligned} \tag{9}$$

Furthermore, assuming $a_0\alpha^{-1/n\theta} \ge a_1$, the power of the test φ_2 for testing (4) is

$$\begin{aligned} E_{a_1}\varphi_2(X) &= \alpha/\left(\frac{a_0}{a_1}\right)^{n\theta} \cdot P_{a_1}(a_1 < X_{(1)}) + 0 \cdot P_{a_1}(a_0 < X_{(1)} \le a_1) \\ &= \alpha/\left(\frac{a_0}{a_1}\right)^{n\theta} \cdot 1 \\ &= \alpha/\left(\frac{a_0}{a_1}\right)^{n\theta}. \end{aligned} \tag{10}$$

Now we find the UMP test for the given hypothesis testing problem at level α. Recall that the rejection rule of ϕ for the hypothesis testing problem in **Problem 3 (b)** is $X_{(1)} \le a_0$. Also, the rejection rule of the MP test for testing (4) depends on $a_0\alpha^{-1/n\theta}$ and a_1 for $a_1 > a_0$. Under H_1 in (4),

$$a_0\alpha^{-1/n\theta} \le a_1 < X_{(1)}.$$

Hence, one may conjecture that the test ψ defined by

$$\psi(X) = \begin{cases} 1, & X_{(1)} \le a_0 \text{ or } a_0\alpha^{-1/n\theta} < X_{(1)} \\ 0, & \text{otherwise} \end{cases}$$

is the UMP test for the given hypothesis testing problem at level α. We claim that the test ψ is indeed the desired test. First, observe that

$$\begin{aligned} E_{a_0}\psi(X) &= P_{a_0}(X_{(1)} \le a_0 \text{ or } a_0\alpha^{-1/n\theta} < X_{(1)}) \\ &= \int_{a_0\alpha^{-1/n\theta}}^{\infty} p(t; a_0, \theta)dt \\ &= [-a_0^{n\theta}t^{-n\theta}]_{a_0\alpha^{-1/n\theta}}^{\infty} \\ &= a_0^{n\theta}(a_0\alpha^{-1/n\theta})^{-n\theta} \\ &= \alpha. \end{aligned}$$

Thus, it remains to show that the power of ψ for testing

$$H_0 : a = a_0 \text{ vs } H_1 : a = a_1 \tag{11}$$

for $a_1 \ne a_0$ attains the power in (3), (9), and (10) in case $a_1 < a_0$, $a_1 > a_0$ and $a_0\alpha^{-1/n\theta} < a_1$, and $a_1 > a_0$ and $a_0\alpha^{-1/n\theta} \ge a_1$, respectively. If $a_1 < a_0$,

$$\begin{aligned} E_{a_1}\psi(X) &= P_{a_1}(X_{(1)} \le a_0) + P_{a_1}(X_{(1)} > a_0\alpha^{-1/n\theta}) \\ &= \int_{a_1}^{a_0} p(t; a_1, \theta)dt + \int_{a_0\alpha^{-1/n\theta}}^{\infty} p(t; a_1, \theta)dt \\ &= [-a_1^{n\theta}t^{-n\theta}]_{a_1}^{a_0} + [-a_1^{n\theta}t^{-n\theta}]_{a_0\alpha^{-1/n\theta}}^{\infty} \\ &= 1 - \left(\frac{a_1}{a_0}\right)^{n\theta} + \alpha\left(\frac{a_1}{a_0}\right)^{n\theta} \\ &= 1 + (\alpha - 1)\left(\frac{a_1}{a_0}\right)^{n\theta}, \end{aligned}$$

which is equivalent to the power of T defined in (2) for testing (11) at level α when $a_1 < a_0$. Also, if $a_1 > a_0$ and $a_0\alpha^{-1/n\theta} < a_1$, the probability of $X_{(1)} \le a_0$ under H_1 is 0 and $a_0\alpha^{-1/n\theta} < X_{(1)}$ always holds because $a_1 < X_{(1)}$ so that

$$\begin{aligned} E_{a_1}\psi(X) &= P_{a_1}(X_{(1)} > a_0\alpha^{-1/n\theta}) \\ &= 1. \end{aligned}$$

which is equivalent to the power of φ_1 defined in (7) for testing (11) at level α when $a_1 > a_0$ and

$a_0\alpha^{-1/n\theta} < a_1$. Similarly, if $a_1 > a_0$ and $a_0\alpha^{-1/n\theta} \geq a_1$,

$$\begin{aligned} E_{a_1}\psi(X) &= P_{a_1}(X_{(1)} > a_0\alpha^{-1/n\theta}) \\ &= \int_{a_0\alpha^{-1/n\theta}}^{\infty} p(t; a_1, \theta)dt \\ &= [-a_1^{n\theta} t^{-n\theta}]_{a_0\alpha^{-1/n\theta}}^{\infty} \\ &= \alpha / \left(\frac{a_0}{a_1}\right)^{n\theta}, \end{aligned}$$

which is equivalent to the power of φ_2 defined in (8) for testing (11) at level α when $a_1 > a_0$ and $a_0\alpha^{-1/n\theta} \geq a_1$. Therefore, by the definition of UMP test, the test ψ is the desired test.

Problem 4. Note that $Z_i \overset{d}{\equiv} bX_i + a$ for $i = 1, 2, \ldots, n$, where $Z_i \overset{i.i.d.}{\sim} E(0, 1)$. Then,

$$\begin{aligned} \left(\frac{X_2 - X_1}{X_3 - X_2}, \ldots, \frac{X_{n-1} - X_{n-2}}{X_n - X_{n-1}}\right) &\overset{d}{\equiv} \left(\frac{X_2 - X_1}{X_3 - X_2}, \ldots, \frac{X_{n-1} - X_{n-2}}{X_n - X_{n-1}}\right) \\ &\overset{d}{\equiv} \left(\frac{bZ_2 + a - bZ_1 - a}{bZ_3 + a - bZ_2 - a}, \ldots, \frac{bZ_{n-1} + a - bZ_{n-2} - a}{bZ_n + a - bZ_{n-1} - a}\right) \\ &\overset{d}{\equiv} \left(\frac{Z_2 - Z_1}{Z_3 - Z_2}, \ldots, \frac{Z_{n-1} - Z_{n-2}}{Z_n - Z_{n-1}}\right). \end{aligned}$$

Since the distribution of $\left(\frac{Z_2-Z_1}{Z_3-Z_2}, \ldots, \frac{Z_{n-1}-Z_{n-2}}{Z_n-Z_{n-1}}\right)$ does not depend on a and b, $\left(\frac{X_2-X_1}{X_3-X_2}, \ldots, \frac{X_{n-1}-X_{n-2}}{X_n-X_{n-1}}\right)$ is an ancillary statistics for (a, b). Also, following the argument in the solution of Exercise 8.11 in the textbook, $(X_{(1)}X_i, \sum_{i=1}^n (X_i - X_{(1)}))$ is a complete sufficient statistic for (a,b), where $X_{(1)} = \min_{1\leq i\leq n} X_i$. Therefore, by Theorem 8.3.3 (Basu's Theorem), $\left(\frac{X_2-X_1}{X_3-X_2}, \ldots, \frac{X_{n-1}-X_{n-2}}{X_n-X_{n-1}}\right)$ and $(X_{(1)}X_i, \sum_{i=1}^n (X_i - X_{(1)}))$ are independent. Since $X_{(1)}$ is a function of $(X_{(1)}X_i, \sum_{i=1}^n (X_i - X_{(1)}))$ (consider the projection map $\pi(a, b) = a$), $\left(\frac{X_2-X_1}{X_3-X_2}, \ldots, \frac{X_{n-1}-X_{n-2}}{X_n-X_{n-1}}\right)$ and $X_{(1)}$ are also independent.