2021 Final Solution

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Problem 1. (a) The likelihood of θ is given as following:

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} \frac{1}{2\theta^2} I(|X_i| + |Y_i| \le \theta) \\ &= \left(\frac{1}{2\theta^2}\right)^n I(\max_{1\le i\le n} \{|X_i| + |Y_i|\} \le \theta). \end{aligned}$$

Thus, by factorization theorem, one can see that $Z_n = \max_{1\le i\le n}\{|X_i| + |Y_i|\}$ is a sufficient statistic for θ. We claim that Z_n is complete. Denote the C.D.F of Z_n by $F(\cdot)$. Then,

$$\begin{aligned} F(x) &= P(\max_{1\le i\le n} \{|X_i| + |Y_i|\} \le x) \\ &= [P(|X_1| + |Y_1| \le x)]^n \\ &= \begin{cases} 0, & x < 0 \\ \left(\frac{x^2}{\theta^2}\right)^n, & 0 \le x \le \theta \\ 1, & \theta < x \end{cases}. \end{aligned}$$

Suppose $E_\theta \varphi(Z_n) = 0$ for all $\theta > 0$. Then,

$$\begin{aligned} E_\theta \varphi(Z_n) &= \int \varphi(x) dF(x) \\ &= \int_0^\theta \varphi(x) F'(x) dx \\ &= \int_0^\theta \varphi(x) n \left(\frac{x^2}{\theta^2}\right)^{n-1} \frac{2x}{\theta^2} dx \\ &= \int_0^\theta \frac{2n}{\theta^{2n}} x^{2n-1} \varphi(x) dx \\ &= 0 \\ &\Leftrightarrow \int_0^\theta x^{2n-1} \varphi(x) dx = 0 \end{aligned}$$

for all $\theta > 0$. Thus, by Fundamental Theorem of Calculus, one can see that

$$\begin{aligned} \theta^{2n-1}\varphi(\theta) \overset{\theta}{=} 0 \Rightarrow \varphi(\theta) &\overset{\theta\ge 0}{=} 0 \\ \Rightarrow \varphi(x) &\overset{x\ge 0}{=} 0 \end{aligned}$$

and this implies Z_n is a complete sufficient statistic for θ. Also,

$$\begin{aligned} E_\theta Z_n &= \int_0^\theta \frac{2n}{\theta^{2n}} x^{2n-1} \cdot x dx \\ &= \int_0^\theta \frac{2n}{\theta^{2n}} x^{2n} dx \\ &= \frac{2n}{2n+1}\theta, \end{aligned}$$

which implies $E_\theta \frac{2n+1}{2n} Z_n = \theta$. Hence, by Lehmann-Scheffe theorem, $\frac{2n+1}{2n} Z_n$ is UMVUE of θ, as $\frac{2n+1}{2n} Z_n$ is a function of Z_n.

Problem 1. **(b)** The likelihood of θ is given by

$$
\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} \frac{2}{3\theta^2} I(\frac{\theta}{2} \leq |X_i| + |Y_i| \leq \theta) \\
&= \left(\frac{2}{3\theta^2}\right)^n I(\max_{1 \leq i \leq n}\{|X_i| + |Y_i|\} \leq \theta \leq 2 \min_{1 \leq i \leq n}\{|X_i| + |Y_i|\}).
\end{aligned}
$$

Thus, $(\max_{1\leq i\leq n}\{|X_i|+|Y_i|\}, 2\min_{1\leq i\leq n}\{|X_i|+|Y_i|\})$ is sufficient statistic for θ by factorization theorem. Denote the C.D.F of $\max_{1\leq i\leq n}\{|X_i| + |Y_i|\}$ and $\min_{1\leq i\leq n}\{|X_i| + |Y_i|\}$ by $G(\cdot)$ and $H(\cdot)$, respectively. Then,

$$
\begin{aligned}
G(x) &= P(\max_{1 \leq i \leq n}\{|X_i| + |Y_i|\} \leq x) \\
&= [P(|X_1| + |Y_1| \leq x)]^n \\
&= \begin{cases} 0, & x < \frac{\theta}{2} \\ \left(\frac{4x^2-\theta^2}{3\theta^2}\right)^n, & \frac{\theta}{2} \leq x \leq \theta \\ 1, & \theta < x \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
H(x) &= P(\min_{1 \leq i \leq n}\{|X_i| + |Y_i|\} \leq x) \\
&= 1 - P(\min_{1 \leq i \leq n}\{|X_i| + |Y_i|\} > x) \\
&= 1 - [P(|X_1| + |Y_1| > x)]^n \\
&= \begin{cases} 0, & x < \frac{\theta}{2} \\ 1 - \left(\frac{4\theta^2-4x^2}{3\theta^2}\right)^n, & \frac{\theta}{2} \leq x \leq \theta \ . \\ 1, & \theta < x \end{cases}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
E(\max_{1 \leq i \leq n}\{|X_i| + |Y_i|\}) &= \int_{\frac{\theta}{2}}^{\theta} x dG(x) \\
&= \int_{\frac{\theta}{2}}^{\theta} x G'(x) dx \\
&= [xG(x)]_{\frac{\theta}{2}}^{\theta} - \int_{\frac{\theta}{2}}^{\theta} G(x) dx \\
&= \theta - \int_{\frac{\theta}{2}}^{\theta} \left(\frac{4x^2 - \theta^2}{3\theta^2}\right)^n dx \\
&= \theta - \frac{\theta}{2} \int_0^1 \left(\frac{t(t+2)}{3}\right)^n dt \\
&= \theta \left[1 - \frac{1}{2} \int_0^1 \left(\frac{t(t+2)}{3}\right)^n dt\right]
\end{aligned} \tag{1}
$$

and

$$
\begin{aligned}
E(\min_{1\le i\le n}\{|X_i|+|Y_i|\}) &= \int_{\frac{\theta}{2}}^{\theta} x dH(x) \\
&= \int_{\frac{\theta}{2}}^{\theta} xH'(x)dx \\
&= [xH(x)]_{\frac{\theta}{2}}^{\theta} - \int_{\frac{\theta}{2}}^{\theta} H(x)dx \\
&= \theta - \int_{\frac{\theta}{2}}^{\theta}\left[1-\left(\frac{4\theta^2-4x^2}{3\theta^2}\right)^n\right]dx \\
&= \frac{\theta}{2} + \theta\int_{\frac{1}{2}}^{1}\left(\frac{4}{3}\right)^n (1-t^2)^n dt \\
&= \theta\left[\frac{1}{2}+\int_{\frac{1}{2}}^{1}\left(\frac{4}{3}\right)^n (1-t^2)^n dt\right].
\end{aligned} \tag{2}
$$

Here we used integration by substitution in the fifth equality in (1) and in (2) considering $x = \theta/2(t+1)$ and $x = \theta t$, respectively. Let $a_n = 1 - \frac{1}{2}\int_0^1 \left(\frac{t(t+2)}{3}\right)^n dt$ and $b_n = \frac{1}{2} + \int_{\frac{1}{2}}^1 \left(\frac{4}{3}\right)^n (1-t^2)^n dt$. One can easily deduce that the mean of $\max_{1\le i\le n}\{|X_i|+|Y_i|\} - a_n/b_n \min_{1\le i\le n}\{|X_i|+|Y_i|\}$ is 0 from (1) and (2). Also, $\max_{1\le i\le n}\{|X_i|+|Y_i|\} - a_n/b_n \min_{1\le i\le n}\{|X_i|+|Y_i|\}$ is a function of $(\max_{1\le i\le n}\{|X_i|+|Y_i|\}, 2\min_{1\le i\le n}\{|X_i|+|Y_i|\})$. Thus, this implies that $(\max_{1\le i\le n}\{|X_i|+|Y_i|\}, 2\min_{1\le i\le n}\{|X_i|+|Y_i|\})$ is not complete.

Problem 2. **(a)** Observe that $Z_i = \Delta_i X_i + (1-\Delta_i)Y_i$. Clearly, if $\Delta_i = 1$, $Z_i \overset{d}{=} X_i$ and $Z_i \overset{d}{=} Y_i$, otherwise. Thus, the p.d.f of (Z_i, Δ_i) can be expressed as

$$
\left(\frac{1}{\theta_x}\right)^{\Delta_i}\left(\frac{1}{\theta_y}\right)^{1-\Delta_i}\exp\left(-\frac{1}{\theta_x}\Delta_i X_i\right)\exp\left(-\frac{1}{\theta_y}(1-\Delta_i)Y_i\right).
$$

Hence, the joint p.d.f of (Z_i, Δ_i) $(i = 1, 2, \ldots, n)$ is

$$
\begin{aligned}
&\prod_{i=1}^n \left(\frac{1}{\theta_x}\right)^{\Delta_i}\left(\frac{1}{\theta_y}\right)^{1-\Delta_i}\exp\left(-\frac{1}{\theta_x}\Delta_i X_i\right)\exp\left(-\frac{1}{\theta_y}(1-\Delta_i)Y_i\right) \\
&= \left(\frac{1}{\theta_x}\right)^{\sum_{i=1}^n \Delta_i}\left(\frac{1}{\theta_y}\right)^{\sum_{i=1}^n (1-\Delta_i)}\exp\left(-\frac{1}{\theta_x}\sum_{i=1}^n \Delta_i X_i\right)\exp\left(-\frac{1}{\theta_y}\sum_{i=1}^n (1-\Delta_i)Y_i\right) \\
&= \left(\frac{1}{\theta_x}\right)^{\sum_{i=1}^n \Delta_i}\left(\frac{1}{\theta_y}\right)^{n-\sum_{i=1}^n \Delta_i}\exp\left(-\frac{1}{\theta_x}\sum_{i=1}^n \Delta_i X_i\right)\exp\left(-\frac{1}{\theta_y}\sum_{i=1}^n (1-\Delta_i)Y_i\right).
\end{aligned}
$$

This implies that $(\sum_{i=1}^n \Delta_i, \sum_{i=1}^n \Delta_i X_i, \sum_{i=1}^n (1-\Delta_i)Y_i)$ is the sufficient statistic for (θ_x, θ_y).

Problem 2. (b)

$$
\begin{aligned}
E(\Delta_1 X_1) &= \int_0^\infty \int_y^\infty x \frac{1}{\theta_x} \frac{1}{\theta_y} \exp\left(-\frac{x}{\theta_x}\right) \cdot \exp\left(-\frac{y}{\theta_y}\right) dx dy \\
&= \int_0^\infty \frac{1}{\theta_y} \exp\left(-\frac{y}{\theta_y}\right) \int_y^\infty \frac{x}{\theta_x} \exp\left(-\frac{x}{\theta_x}\right) dx dy \\
&= \int_0^\infty \frac{1}{\theta_y} \exp\left(-\frac{y}{\theta_y}\right) \left[-x \exp\left(-\frac{x}{\theta_x}\right) \Big|_y^\infty + \int_y^\infty \exp\left(-\frac{x}{\theta_x}\right) dx \right] dy \\
&= \int_0^\infty \frac{1}{\theta_y} \exp\left(-\frac{y}{\theta_y}\right) \left[y \exp\left(-\frac{y}{\theta_x}\right) + \theta_x \exp\left(-\frac{y}{\theta_x}\right) \right] dy \\
&= \int_0^\infty \left[\frac{y}{\theta_y} \exp\left(-\left(\frac{1}{\theta_y} + \frac{1}{\theta_x}\right) y\right) + \frac{\theta_x}{\theta_y} \exp\left(-\left(\frac{1}{\theta_y} + \frac{1}{\theta_x}\right) y\right) \right] dy \\
&= \frac{1}{\theta_y} / \left(\frac{1}{\theta_y} + \frac{1}{\theta_x}\right)^2 + \frac{\theta_x}{\theta_y} / \left(\frac{1}{\theta_y} + \frac{1}{\theta_x}\right) \\
&= \frac{\theta_x^2 \theta_y}{(\theta_x + \theta_y)^2} + \frac{\theta_x^2}{\theta_x + \theta_y} \\
&= \frac{\theta_x^3 + 2\theta_x^2 \theta_y}{(\theta_x + \theta_y)^2}.
\end{aligned} \tag{3}
$$

Similarly, changing the role of x and y in (3),

$$
E((1 - \Delta_1) Y_1) = \frac{\theta_y^3 + 2\theta_x \theta_y^2}{(\theta_x + \theta_y)^2}. \tag{4}
$$

By (3) and (4),

$$
\begin{aligned}
E(\Delta_1 X_1 - (1 - \Delta_1) Y_1) &= \frac{\theta_x^3 + 2\theta_x^2 \theta_y}{(\theta_x + \theta_y)^2} - \frac{\theta_y^3 + 2\theta_x \theta_y^2}{(\theta_x + \theta_y)^2} \\
&= \frac{(\theta_x - \theta_y)(\theta_x^2 + 3\theta_x \theta_y + \theta_y^2)}{(\theta_x + \theta_y)^2}.
\end{aligned}
$$

Therefore,

$$
E\left(\sum_{i=1}^n (\Delta_i X_i - (1 - \Delta_i) Y_i)\right) = n \frac{(\theta_x - \theta_y)(\theta_x^2 + 3\theta_x \theta_y + \theta_y^2)}{(\theta_x + \theta_y)^2}.
$$

Problem 3. The likelihood of μ is given by

$$
\begin{aligned}
L(\mu) &= \prod_{i=1}^n \frac{1}{x_i \sqrt{2\pi}} \exp\left(-\frac{1}{2} (\log x_i - \mu)^2\right) \\
&= \exp\left(\mu \sum_{i=1}^n \log x_i - \frac{n}{2} \mu^2\right) \cdot \exp\left(-\frac{1}{2} \sum_{i=1}^n \log^2 x_i\right) \prod_{i=1}^n \frac{1}{x_i \sqrt{2\pi}}.
\end{aligned}
$$

Thus, $\sum_{i=1}^n \log X_i$ is a sufficient statistic for μ. Also, one may easily see that $\sum_{i=1}^n \log X_i$ is a complete sufficient statistic (CSS) for μ by the fact that $\log X_i \overset{\text{i.i.d.}}{\sim} N(\mu, 1)$ and the properties of exponential family. Note that $Y = 1/n \sum_{i=1}^n \log X_i$ is also a CSS for μ as Y is one-to-one function of $\sum_{i=1}^n \log X_i$.

Observe that $Y \sim N(\mu, 1/n)$. Suppose $Z \sim N(\theta, \sigma^2)$ for $(\theta, \sigma^2) \in \mathbb{R} \times \mathbb{R}_+$. Then,

$$\begin{aligned}
E\exp(tZ) &= \int \exp(tz) \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(z-\theta)^2\right) dz \\
&= \int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(z-\theta)^2 + tz\right) dz \\
&= \int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(z^2 - 2(\theta+\sigma^2 t)z + \theta^2)\right) dz \\
&= \int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(z-(\theta+\sigma^2 t))^2\right) \cdot \exp\left(-\frac{1}{2\sigma^2}(\theta^2 - (\theta+\sigma^2 t)^2)\right) dz \\
&= \exp\left(-\frac{1}{2\sigma^2}(-2\sigma^2\theta t - \sigma^4 t^2)\right) \\
&= \exp(t\theta) \cdot \exp\left(\frac{t^2\sigma^2}{2}\right).
\end{aligned} \tag{5}$$

From (5), one can deduce that $E\exp(tY) = \exp(t\mu) \cdot \exp\left(\frac{t^2}{2n}\right)$ or equivalently, $E\exp\left(-\frac{t^2}{2n}\right) \cdot \exp(tY) = \exp(t\mu)$. Therefore, by Lehmann-Scheffe theorem, $\varphi(Y) = \exp\left(-\frac{t^2}{2n}\right) \cdot \exp(tY)$ is an UMVUE of $\eta = \exp(t\mu)$.

Now, we find the variance of $\varphi(Y)$ and see whether this attains the Cramer-Rao bound. Note that $E\varphi(Y)^2 = \exp\left(-\frac{t^2}{n}\right) E\exp(2tY)$. Again, from (5), one can deduce that $E\exp(2tY) = \exp(2t\mu) \cdot \exp\left(\frac{2t^2}{n}\right)$, which gives $E\varphi(Y)^2 = \exp\left(\frac{t^2}{n}\right) \cdot \exp(2t\mu)$. This implies

$$\begin{aligned}
\mathrm{Var}\varphi(Y) &= E\varphi(Y)^2 - [E\varphi(Y)]^2 \\
&= \exp\left(\frac{t^2}{n}\right) \cdot \exp(2t\mu) - \exp(2t\mu) \\
&= \left[\exp\left(\frac{t^2}{n}\right) - 1\right] \exp(2t\mu) \\
&= \left[\exp\left(\frac{t^2}{n}\right) - 1\right] \eta^2.
\end{aligned}$$

Since $\varphi(Y)$ is an UMVUE of η, the desired Cramer-Rao bound is $I(\eta)^{-1}$, where $I(\eta)$ is Fisher information of η. Note that $\mathrm{Var}\varphi(Y)$ attains the Cramer-Rao bound if $t = 0$ as $\eta = 1$. So we consider only when $t \neq 0$. Observe that $\mu = 1/t \log \eta$ so that

$$L(\eta) = \prod_{i=1}^{n} \frac{1}{x_i\sqrt{2\pi}} \exp\left(-\frac{1}{2}(\log x_i - 1/t \log \eta)^2\right).$$

Thus, the log-likelihood of η is

$$\ell(\eta) = -\frac{1}{2}\sum_{i=1}^{n}(\log x_i - 1/t \log \eta)^2 - \sum_{i=1}^{n} \log x_i + \text{const}.$$

Let $n = 1$. Since ℓ is twice continuously differentiable with respect to η,

$$\begin{aligned}
\frac{\partial \ell}{\partial \eta} &= -\frac{1}{t\eta}(1/t \log \eta - \log x_1) \\
\frac{\partial^2 \ell}{\partial \eta^2} &= \frac{1}{t\eta^2}(1/t \log \eta - \log x_1) - \frac{1}{t^2\eta^2} \\
&= \frac{\log \eta - 1}{t^2\eta^2} - \frac{\log x_1}{t\eta^2}.
\end{aligned}$$

By Bartlett identity,

$$\begin{aligned}
I(\eta) &= -nE\frac{\partial^2 \ell}{\partial \eta^2} \\
&= -n\left[\frac{\log \eta - 1}{t^2\eta^2} - \frac{1}{t\eta^2}E\log X_1\right] \\
&= -n\left[\frac{\log \eta - 1}{t^2\eta^2} - \frac{1}{t\eta^2}\cdot\frac{\log \eta}{t}\right] \\
&= \frac{n}{t^2\eta^2},
\end{aligned}$$

where the third equality holds by the fact that $\log X_1 \sim N(\mu, 1) = N(1/t\log\eta, 1)$. Thus,

$$\mathrm{Var}\varphi(Y) = \left[\exp\left(\frac{t^2}{n}\right) - 1\right]\eta^2 \geq I(\eta)^{-1} = \frac{t^2}{n}\cdot\eta^2,$$

which follows from the fact that $\exp(x) \geq x + 1$ for all $x \in \mathbb{R}$. Since the equality holds only when $t = 0$, $\varphi(Y)$ attains the Cramer-Rao bound only when $t = 0$.

Problem 4. Let $\Theta_0 = \{\theta \in \mathbb{R} : \theta \leq 0\}$ and take any $\theta_1 > 0$. Suppose there exists the MP level α test for testing

$$H_0 : \theta \in \Theta_0 \text{ versus } H_1 : \theta = \theta_1. \tag{6}$$

If such test does not depend on the choice of θ_1, then the MP level α test for (6) is the UMP level α test by the definition of UMP test. Also, if the critical region of such test is in form of $X > C$ for some constant C, we are done. Hence, we find the MP level α test for (6) whose critical region is in form of $X > C$. Choose any $\theta_0 \in \Theta_0$. Consider the following hypothesis problem:

$$H_0 : \theta = \theta_0 \text{ versus } H_1 : \theta = \theta_1. \tag{7}$$

For given $k > 0$, with simple calculations,

$$\begin{aligned}
\frac{f(x;\theta_1)}{f(x;\theta_0)} &= \frac{e^{x-\theta_1}}{(1+e^{x-\theta_1})^2}\cdot\frac{(1+e^{x-\theta_0})^2}{e^{x-\theta_0}} \\
&= e^{\theta_0-\theta_1}\cdot\frac{(1+e^{x-\theta_0})^2}{(1+e^{x-\theta_1})^2} > k \\
&\Leftrightarrow \frac{1+e^{x-\theta_0}}{1+e^{x-\theta_1}} > k'
\end{aligned}$$

for some $k' > 0$. Let $S(x) := (1+e^{x-\theta_0})/(1+e^{x-\theta_1})$. Since

$$\begin{aligned}
\frac{dS}{dx} &= \frac{e^{x-\theta_0}(1+e^{x-\theta_1}) - e^{x-\theta_1}(1+e^{x-\theta_0})}{(1+e^{x-\theta_1})^2} \\
&= e^x\frac{e^{-\theta_0} - e^{-\theta_1}}{(1+e^{x-\theta_1})^2} > 0
\end{aligned}$$

as $\theta_0 \leq 0 < \theta_1$, S is strictly increasing function. Thus, $S(x) > k'$ if and only if $x > k''$ for some constant k''. Now consider the following test for (7)

$$\varphi(x) = \begin{cases} 1, & x > K \\ 0, & x \leq K \end{cases}.$$

Note that

$$\begin{aligned}
E_\theta\varphi(X) &= \int_K^\infty \frac{e^{x-\theta}}{(1+e^{x-\theta})^2}dx \\
&= \left[-(1+e^{x-\theta})^{-1}\right]_K^\infty \\
&= \frac{1}{1+e^{K-\theta}}.
\end{aligned}$$

Thus, $E_\theta\varphi(X)$ is a strictly increasing function with respect to θ with K given. Also, there exists an unique K such that $E_\theta\varphi(X) = \alpha$ for each $0 < \alpha < 1$. Note that such K is $\theta + \log(\frac{1-\alpha}{\alpha})$. From these facts, letting $\theta = 0$, we claim that the test

$$\psi(x) = \begin{cases} 1, & x > \log\left(\frac{1-\alpha}{\alpha}\right) \\ 0, & x \leq \log\left(\frac{1-\alpha}{\alpha}\right) \end{cases}.$$

is the MP level α test for (6). If this claim holds, as the test ψ does not depend on the choice of θ_1, the test ψ is the UMP level α test for the given hypothesis testing problem.

By the above arguments, one can see that the test ψ is the MP test for (7) but at level $E_{\theta_0}\psi(X)$ by Neyman-Pearson Lemma (NP Lemma). Because $E_\theta\psi(X)$ is a strictly increasing function with respect to θ, one can deduce that $\max_{\theta\in\Theta_0} E_\theta\psi(X) = E_0\psi(X) = \alpha$. Let $A_0 = \{\phi : E_0\phi(X) \leq \alpha\}$ and $A = \{\phi : \max_{\theta\in\Theta_0} E_\theta\phi(X) \leq \alpha\}$. Then, $A \subset A_0$. By NP Lemma, ψ is MP among A_0 and thus among A.

Problem 5. See Exercise 10.7 in the textbook.