Numerical Simulation Analysis

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We provide the summary of the numerical simulations under general factor type core covariance model (see Section 7.1 for details).

```
################ Libraries

library(tidyverse)
```

```
## -- Attaching core tidyverse packages ------------------------ tidyverse 2.0.0 --
## v dplyr     1.1.4     v readr     2.1.6
## v forcats   1.0.1     v stringr   1.6.0
## v ggplot2   4.0.1     v tibble    3.3.0
## v lubridate 1.9.4     v tidyr     1.3.1
## v purrr     1.2.0
## -- Conflicts ------------------------------------------ tidyverse_conflicts() --
## x dplyr::filter() masks stats::filter()
## x dplyr::lag()    masks stats::lag()
## i Use the conflicted package (<http://conflicted.r-lib.org/>) to force all conflicts to become errors
```

```
library(latex2exp)
library(ggpubr)

################ Loading the simulation results
file.list=list.files(path="./numerical.simul_gfcore",pattern=".rds")
file.list=file.list[-c(1:4)]

for(i in file.list){
  load(paste("./numerical.simul_gfcore/",i,sep=""))
}

################ Utensils

stat.analy=function(fit){

  kro.mle=fit$kro.mle
  cse=fit$cse
  init.ai=fit$pi.core.init.ai
  init.chol=fit$pi.core.init.chol
  fit.ai=fit$pi.core.ai
  fit.chol=fit$pi.core.chol

  kro.mle=cbind(init.ai[,1],kro.mle)
  cse=cbind(init.ai[,1],cse[,1:2])
  init.chol=cbind(init.ai[,1],init.chol)
```

```
  colnames(kro.mle)=colnames(cse)=colnames(init.chol)=c("K","C","Sigma")

  fit.ai.analy=NULL; fit.chol.analy=NULL
  for(i in 1:100){
    fit.ai.analy=rbind(fit.ai.analy,fit.ai[[i]]$eval)
    fit.chol.analy=rbind(fit.chol.analy,fit.chol[[i]]$eval)
  }
  return(list(kro.mle=kro.mle,cse=cse,init.ai=init.ai,init.chol=init.chol,fit.ai.analy=fit.ai.analy,fit
}

conv.analy=function(fit1,fit2,fit3,fit4,fit5,name){

  fit.ai1=fit1$pi.core.ai
  fit.chol1=fit1$pi.core.chol
  fit.ai2=fit2$pi.core.ai
  fit.chol2=fit2$pi.core.chol
  fit.ai3=fit3$pi.core.ai
  fit.chol3=fit3$pi.core.chol
  fit.ai4=fit4$pi.core.ai
  fit.chol4=fit4$pi.core.chol
  fit.ai5=fit5$pi.core.ai
  fit.chol5=fit5$pi.core.chol

  rel.conv.ai=NULL
  conv.ai=NULL
  rel.conv.chol=NULL
  conv.chol=NULL

  for(i in 1:100){
    temp1=c(fit.ai1[[i]]$rel.conv,fit.ai2[[i]]$rel.conv,fit.ai3[[i]]$rel.conv,fit.ai4[[i]]$rel.conv,fit
    temp2=c(fit.ai1[[i]]$convergence,fit.ai2[[i]]$convergence,fit.ai3[[i]]$convergence,fit.ai4[[i]]$con
    rel.conv.ai=cbind(rel.conv.ai,temp1)
    conv.ai=cbind(conv.ai,temp2)

    temp1=c(fit.chol1[[i]]$rel.conv,fit.chol2[[i]]$rel.conv,fit.chol3[[i]]$rel.conv,fit.chol4[[i]]$rel.
    temp2=c(fit.chol1[[i]]$convergence,fit.chol2[[i]]$convergence,fit.chol3[[i]]$convergence,fit.chol4[
    rel.conv.chol=cbind(rel.conv.chol,temp1)
    conv.chol=cbind(conv.chol,temp2)
  }
  rownames(rel.conv.ai)=rownames(conv.ai)=rownames(rel.conv.chol)=rownames(conv.chol)=name

  return(list(rel.conv.ai=rel.conv.ai,conv.ai=conv.ai,rel.conv.chol=rel.conv.chol,conv.chol=conv.chol))
}

plot.K.analy=function(fit1,fit2,fit3,fit4,fit5,samp.size,lambda){

  method=rep(c("Base-AI","PI-AI","PI-Chol"),each=100)

  dat1=tibble(stat=c(fit1$init.ai[,1],fit1$fit.ai.analy[,1],fit1$fit.chol.analy[,1]),Method=method,samp
  dat2=tibble(stat=c(fit2$init.ai[,1],fit2$fit.ai.analy[,1],fit2$fit.chol.analy[,1]),Method=method,samp
  dat3=tibble(stat=c(fit3$init.ai[,1],fit3$fit.ai.analy[,1],fit3$fit.chol.analy[,1]),Method=method,samp
  dat4=tibble(stat=c(fit4$init.ai[,1],fit4$fit.ai.analy[,1],fit4$fit.chol.analy[,1]),Method=method,samp
  dat5=tibble(stat=c(fit5$init.ai[,1],fit5$fit.ai.analy[,1],fit5$fit.chol.analy[,1]),Method=method,samp
```

```
  dat=rbind(dat1,dat2,dat3,dat4,dat5)
  dat$samp.size=factor(dat$samp.size,levels=samp.size)
  dat$Method=factor(dat$Method,levels=c("Base-AI","PI-AI","PI-Chol"))
  return(dat)
}

plot.C.analy=function(fit1,fit2,fit3,fit4,fit5,samp.size,lambda){

  method=rep(c("CSE","Base-AI","Base-Chol","PI-AI","PI-Chol"),each=100)

  dat1=tibble(stat=c(fit1$cse[,2],fit1$init.ai[,2],fit1$init.chol[,2],
                     fit1$fit.ai.analy[,2],fit1$fit.chol.analy[,2]),Method=method,samp.size=rep(samp.si
              lambda=rep(lambda,500))
  dat2=tibble(stat=c(fit2$cse[,2],fit2$init.ai[,2],fit2$init.chol[,2],
                     fit2$fit.ai.analy[,2],fit2$fit.chol.analy[,2]),Method=method,samp.size=rep(samp.si
              lambda=rep(lambda,500))
  dat3=tibble(stat=c(fit3$cse[,2],fit3$init.ai[,2],fit3$init.chol[,2],
                     fit3$fit.ai.analy[,2],fit3$fit.chol.analy[,2]),Method=method,samp.size=rep(samp.si
              lambda=rep(lambda,500))
  dat4=tibble(stat=c(fit4$cse[,2],fit4$init.ai[,2],fit4$init.chol[,2],
                     fit4$fit.ai.analy[,2],fit4$fit.chol.analy[,2]),Method=method,samp.size=rep(samp.si
              lambda=rep(lambda,500))
  dat5=tibble(stat=c(fit5$cse[,2],fit5$init.ai[,2],fit5$init.chol[,2],
                     fit5$fit.ai.analy[,2],fit5$fit.chol.analy[,2]),Method=method,samp.size=rep(samp.si
              lambda=rep(lambda,500))

  dat=rbind(dat1,dat2,dat3,dat4,dat5)
  dat$samp.size=factor(dat$samp.size,levels=samp.size)
  dat$Method=factor(dat$Method,levels=c("KMLE","CSE","Base-AI","Base-Chol","PI-AI","PI-Chol"))
  return(dat)

}

plot.Sigma.analy=function(fit1,fit2,fit3,fit4,fit5,samp.size,lambda){

  method=rep(c("KMLE","CSE","Base-AI","PI-AI","PI-Chol"),each=100)

  dat1=tibble(stat=c(fit1$kro.mle[,3],fit1$cse[,3],fit1$init.ai[,3],
                     fit1$fit.ai.analy[,3],fit1$fit.chol.analy[,3]),Method=method,samp.size=rep(samp.si
              lambda=rep(lambda,500)
  )
  dat2=tibble(stat=c(fit2$kro.mle[,3],fit2$cse[,3],fit2$init.ai[,3],
                     fit2$fit.ai.analy[,3],fit2$fit.chol.analy[,3]),Method=method,samp.size=rep(samp.si
              lambda=rep(lambda,500))
  dat3=tibble(stat=c(fit3$kro.mle[,3],fit3$cse[,3],fit3$init.ai[,3],
                     fit3$fit.ai.analy[,3],fit3$fit.chol.analy[,3]),Method=method,samp.size=rep(samp.si
              lambda=rep(lambda,500))
  dat4=tibble(stat=c(fit4$kro.mle[,3],fit4$cse[,3],fit4$init.ai[,3],
                     fit4$fit.ai.analy[,3],fit4$fit.chol.analy[,3]),Method=method,samp.size=rep(samp.si
              lambda=rep(lambda,500))
  dat5=tibble(stat=c(fit5$kro.mle[,3],fit5$cse[,3],fit5$init.ai[,3],
                     fit5$fit.ai.analy[,3],fit5$fit.chol.analy[,3]),Method=method,samp.size=rep(samp.si
              lambda=rep(lambda,500))
```

```
  dat=rbind(dat1,dat2,dat3,dat4,dat5)
  dat$samp.size=factor(dat$samp.size,levels=samp.size)
  dat$Method=factor(dat$Method,levels=c("KMLE","CSE","Base-AI","PI-AI","PI-Chol"))
  return(dat)

}
```

(p1,p2,r)=(16,12,3)

```
##### mean of the relative spectral norms for $K$, $C$, and $\Sigma$ (in order) across 100 iterations
##### under different choices of the sample size $n$ and the value of non-spiked eigenvalue $\lambda$.
###lambda=0.2
### n=p/8
fit.16.12.3.24.lambda1=stat.analy(pi.core.fit.16.12.3.24.lambda1)
colMeans(fit.16.12.3.24.lambda1$kro.mle)
```

```
##         K         C     Sigma
## 0.3265857 0.9838769 0.9641926
```

```
colMeans(fit.16.12.3.24.lambda1$cse)
```

```
##         K         C     Sigma
## 0.3265857 0.3677388 0.5053063
```

```
colMeans(fit.16.12.3.24.lambda1$init.ai)
```

```
##         K         C     Sigma
## 0.3265857 0.3052293 0.4945858
```

```
colMeans(fit.16.12.3.24.lambda1$init.chol)
```

```
##         K         C     Sigma
## 0.3265857 0.3380701 0.4945858
```

```
colMeans(fit.16.12.3.24.lambda1$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.2463290 0.1862703 0.2912795
```

```
colMeans(fit.16.12.3.24.lambda1$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.2479897 0.2126027 0.2961559
```

```
### n=p/4
fit.16.12.3.48.lambda1=stat.analy(pi.core.fit.16.12.3.48.lambda1)
colMeans(fit.16.12.3.48.lambda1$kro.mle)
```

```
##         K         C     Sigma
## 0.2254017 0.9838769 0.9631214
```

```
colMeans(fit.16.12.3.48.lambda1$cse)
```

```
##         K         C     Sigma
## 0.2254017 0.2922604 0.3771772
```

```
colMeans(fit.16.12.3.48.lambda1$init.ai)
```

```
##         K         C     Sigma
```

```
## 0.2254017 0.2123285 0.3583381
```

```
colMeans(fit.16.12.3.48.lambda1$init.chol)
```

```
##         K         C     Sigma
## 0.2254017 0.2356957 0.3583381
```

```
colMeans(fit.16.12.3.48.lambda1$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1721506 0.1235124 0.2073536
```

```
colMeans(fit.16.12.3.48.lambda1$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1754525 0.1476541 0.2193698
```

```
### n=p/2
fit.16.12.3.96.lambda1=stat.analy(pi.core.fit.16.12.3.96.lambda1)
colMeans(fit.16.12.3.96.lambda1$kro.mle)
```

```
##         K         C     Sigma
## 0.1560104 0.9838769 0.9638397
```

```
colMeans(fit.16.12.3.96.lambda1$cse)
```

```
##         K         C     Sigma
## 0.1560104 0.2344615 0.2971959
```

```
colMeans(fit.16.12.3.96.lambda1$init.ai)
```

```
##         K         C     Sigma
## 0.1560104 0.1488375 0.2458674
```

```
colMeans(fit.16.12.3.96.lambda1$init.chol)
```

```
##         K         C     Sigma
## 0.1560104 0.1642483 0.2458674
```

```
colMeans(fit.16.12.3.96.lambda1$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.12360824 0.08780297 0.15035869
```

```
colMeans(fit.16.12.3.96.lambda1$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1279440 0.1064034 0.1614918
```

```
### n=p
fit.16.12.3.192.lambda1=stat.analy(pi.core.fit.16.12.3.192.lambda1)
colMeans(fit.16.12.3.192.lambda1$kro.mle)
```

```
##         K         C     Sigma
## 0.1043306 0.9838769 0.9639923
```

```
colMeans(fit.16.12.3.192.lambda1$cse)
```

```
##         K         C     Sigma
## 0.1043306 0.1665930 0.2117369
```

```
colMeans(fit.16.12.3.192.lambda1$init.ai)
```

```
##         K         C     Sigma
```

```
## 0.1043306 0.1029376 0.1697907
```

```
colMeans(fit.16.12.3.192.lambda1$init.chol)
```

```
##         K         C     Sigma
## 0.1043306 0.1132618 0.1697907
```

```
colMeans(fit.16.12.3.192.lambda1$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.08638520 0.06316136 0.10815034
```

```
colMeans(fit.16.12.3.192.lambda1$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.09093156 0.07932828 0.12169739
```

```
### n=2p
fit.16.12.3.384.lambda1=stat.analy(pi.core.fit.16.12.3.384.lambda1)
colMeans(fit.16.12.3.384.lambda1$kro.mle)
```

```
##          K          C      Sigma
## 0.07168778 0.98387694 0.96373742
```

```
colMeans(fit.16.12.3.384.lambda1$cse)
```

```
##          K          C      Sigma
## 0.07168778 0.11128662 0.14422089
```

```
colMeans(fit.16.12.3.384.lambda1$init.ai)
```

```
##          K          C      Sigma
## 0.07168778 0.07140406 0.11793333
```

```
colMeans(fit.16.12.3.384.lambda1$init.chol)
```

```
##          K          C      Sigma
## 0.07168778 0.07871915 0.11793333
```

```
colMeans(fit.16.12.3.384.lambda1$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.06132030 0.04809253 0.08290504
```

```
colMeans(fit.16.12.3.384.lambda1$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.06456753 0.06037039 0.09274870
```

```
###lambda=0.4
### n=p/8
fit.16.12.3.24.lambda2=stat.analy(pi.core.fit.16.12.3.24.lambda2)
colMeans(fit.16.12.3.24.lambda2$kro.mle)
```

```
##         K         C     Sigma
## 0.3136047 0.9786279 0.9526608
```

```
colMeans(fit.16.12.3.24.lambda2$cse)
```

```
##         K         C     Sigma
## 0.3136047 0.4929949 0.5573743
```

```r
colMeans(fit.16.12.3.24.lambda2$init.ai)
```

```
##         K         C     Sigma
## 0.3136047 0.3873812 0.5300031
```

```r
colMeans(fit.16.12.3.24.lambda2$init.chol)
```

```
##         K         C     Sigma
## 0.3136047 0.4149365 0.5300031
```

```r
colMeans(fit.16.12.3.24.lambda2$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.2641896 0.3121133 0.3792918
```

```r
colMeans(fit.16.12.3.24.lambda2$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.2645486 0.3295145 0.3768277
```

```r
### n=p/4
fit.16.12.3.48.lambda2=stat.analy(pi.core.fit.16.12.3.48.lambda2)
colMeans(fit.16.12.3.48.lambda2$kro.mle)
```

```
##         K         C     Sigma
## 0.2161752 0.9786279 0.9514918
```

```r
colMeans(fit.16.12.3.48.lambda2$cse)
```

```
##         K         C     Sigma
## 0.2161752 0.4314219 0.4659944
```

```r
colMeans(fit.16.12.3.48.lambda2$init.ai)
```

```
##         K         C     Sigma
## 0.2161752 0.2684603 0.3804179
```

```r
colMeans(fit.16.12.3.48.lambda2$init.chol)
```

```
##         K         C     Sigma
## 0.2161752 0.2880863 0.3804179
```

```r
colMeans(fit.16.12.3.48.lambda2$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1841089 0.1989654 0.2629906
```

```r
colMeans(fit.16.12.3.48.lambda2$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1847223 0.2150795 0.2645488
```

```r
### n=p/2
fit.16.12.3.96.lambda2=stat.analy(pi.core.fit.16.12.3.96.lambda2)
colMeans(fit.16.12.3.96.lambda2$kro.mle)
```

```
##         K         C     Sigma
## 0.1495447 0.9786279 0.9520875
```

```r
colMeans(fit.16.12.3.96.lambda2$cse)
```

```
##         K         C     Sigma
## 0.1495447 0.3650456 0.3964961
```

```r
colMeans(fit.16.12.3.96.lambda2$init.ai)
```

```
##         K         C     Sigma
## 0.1495447 0.1847706 0.2591688
```

```r
colMeans(fit.16.12.3.96.lambda2$init.chol)
```

```
##         K         C     Sigma
## 0.1495447 0.1975206 0.2591688
```

```r
colMeans(fit.16.12.3.96.lambda2$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1297676 0.1353599 0.1836283
```

```r
colMeans(fit.16.12.3.96.lambda2$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1302317 0.1467162 0.1840015
```

```r
### n=p
fit.16.12.3.192.lambda2=stat.analy(pi.core.fit.16.12.3.192.lambda2)
colMeans(fit.16.12.3.192.lambda2$kro.mle)
```

```
##         K         C     Sigma
## 0.1023154 0.9786279 0.9522382
```

```r
colMeans(fit.16.12.3.192.lambda2$cse)
```

```
##         K         C     Sigma
## 0.1023154 0.2725151 0.2983929
```

```r
colMeans(fit.16.12.3.192.lambda2$init.ai)
```

```
##         K         C     Sigma
## 0.1023154 0.1281013 0.1800143
```

```r
colMeans(fit.16.12.3.192.lambda2$init.chol)
```

```
##         K         C     Sigma
## 0.1023154 0.1367102 0.1800143
```

```r
colMeans(fit.16.12.3.192.lambda2$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.08991233 0.09476181 0.12881254
```

```r
colMeans(fit.16.12.3.192.lambda2$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.09055613 0.10327352 0.13004171
```

```r
### n=2p
fit.16.12.3.384.lambda2=stat.analy(pi.core.fit.16.12.3.384.lambda2)
colMeans(fit.16.12.3.384.lambda2$kro.mle)
```

```
##         K         C     Sigma
## 0.0707069 0.9786279 0.9519791
```

```r
colMeans(fit.16.12.3.384.lambda2$cse)
```

```
##         K         C     Sigma
## 0.0707069 0.1828351 0.2052671
```

```
colMeans(fit.16.12.3.384.lambda2$init.ai)
```

```
##          K          C      Sigma
## 0.07070690 0.08857798 0.12510831
```

```
colMeans(fit.16.12.3.384.lambda2$init.chol)
```

```
##          K          C      Sigma
## 0.07070690 0.09469716 0.12510831
```

```
colMeans(fit.16.12.3.384.lambda2$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.06252754 0.06631239 0.08970775
```

```
colMeans(fit.16.12.3.384.lambda2$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.06280986 0.07280027 0.09175251
```

```
###lambda=0.6
### n=p/8
fit.16.12.3.24.lambda3=stat.analy(pi.core.fit.16.12.3.24.lambda3)
colMeans(fit.16.12.3.24.lambda3$kro.mle)
```

```
##         K         C     Sigma
## 0.3058981 0.9683142 0.9300139
```

```
colMeans(fit.16.12.3.24.lambda3$cse)
```

```
##         K         C     Sigma
## 0.3058981 0.6160712 0.6348471
```

```
colMeans(fit.16.12.3.24.lambda3$init.ai)
```

```
##         K         C     Sigma
## 0.3058981 0.5345289 0.6243393
```

```
colMeans(fit.16.12.3.24.lambda3$init.chol)
```

```
##         K         C     Sigma
## 0.3058981 0.5563877 0.6243393
```

```
colMeans(fit.16.12.3.24.lambda3$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.2777582 0.5240068 0.5549240
```

```
colMeans(fit.16.12.3.24.lambda3$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.2770631 0.5350337 0.5493208
```

```
### n=p/4
fit.16.12.3.48.lambda3=stat.analy(pi.core.fit.16.12.3.48.lambda3)
colMeans(fit.16.12.3.48.lambda3$kro.mle)
```

```
##         K         C     Sigma
## 0.2092660 0.9683142 0.9287170
```

```
colMeans(fit.16.12.3.48.lambda3$cse)
```

```
##         K         C     Sigma
```

```
## 0.2092660 0.5731654 0.5761007
```

```
colMeans(fit.16.12.3.48.lambda3$init.ai)
```

```
##         K         C     Sigma
## 0.2092660 0.3583258 0.4350614
```

```
colMeans(fit.16.12.3.48.lambda3$init.chol)
```

```
##         K         C     Sigma
## 0.2092660 0.3738246 0.4350614
```

```
colMeans(fit.16.12.3.48.lambda3$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1905991 0.3139881 0.3566524
```

```
colMeans(fit.16.12.3.48.lambda3$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1904458 0.3251092 0.3556601
```

```
### n=p/2
fit.16.12.3.96.lambda3=stat.analy(pi.core.fit.16.12.3.96.lambda3)
colMeans(fit.16.12.3.96.lambda3$kro.mle)
```

```
##         K         C     Sigma
## 0.1454748 0.9683142 0.9290889
```

```
colMeans(fit.16.12.3.96.lambda3$cse)
```

```
##         K         C     Sigma
## 0.1454748 0.5023790 0.5061985
```

```
colMeans(fit.16.12.3.96.lambda3$init.ai)
```

```
##         K         C     Sigma
## 0.1454748 0.2390687 0.2901929
```

```
colMeans(fit.16.12.3.96.lambda3$init.chol)
```

```
##         K         C     Sigma
## 0.1454748 0.2493207 0.2901929
```

```
colMeans(fit.16.12.3.96.lambda3$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1339808 0.2027266 0.2387039
```

```
colMeans(fit.16.12.3.96.lambda3$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1338445 0.2111148 0.2383885
```

```
### n=p
fit.16.12.3.192.lambda3=stat.analy(pi.core.fit.16.12.3.192.lambda3)
colMeans(fit.16.12.3.192.lambda3$kro.mle)
```

```
##         K         C     Sigma
## 0.1010763 0.9683142 0.9292238
```

```
colMeans(fit.16.12.3.192.lambda3$cse)
```

```
##         K         C     Sigma
```

```
## 0.1010763 0.3946557 0.3997634
```

```
colMeans(fit.16.12.3.192.lambda3$init.ai)
```

```
##         K         C     Sigma
## 0.1010763 0.1642501 0.2019827
```

```
colMeans(fit.16.12.3.192.lambda3$init.chol)
```

```
##         K         C     Sigma
## 0.1010763 0.1713734 0.2019827
```

```
colMeans(fit.16.12.3.192.lambda3$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.09310959 0.14060300 0.16780429
```

```
colMeans(fit.16.12.3.192.lambda3$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.09314804 0.14705106 0.16774213
```

```
### n=2p
fit.16.12.3.384.lambda3=stat.analy(pi.core.fit.16.12.3.384.lambda3)
colMeans(fit.16.12.3.384.lambda3$kro.mle)
```

```
##          K          C      Sigma
## 0.06970459 0.96831425 0.92895706
```

```
colMeans(fit.16.12.3.384.lambda3$cse)
```

```
##          K          C      Sigma
## 0.06970459 0.27284543 0.28250466
```

```
colMeans(fit.16.12.3.384.lambda3$init.ai)
```

```
##          K          C      Sigma
## 0.06970459 0.11368383 0.14152126
```

```
colMeans(fit.16.12.3.384.lambda3$init.chol)
```

```
##          K          C      Sigma
## 0.06970459 0.11869856 0.14152126
```

```
colMeans(fit.16.12.3.384.lambda3$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.06497537 0.09826459 0.11876989
```

```
colMeans(fit.16.12.3.384.lambda3$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.0649370 0.1030292 0.1187266
```

```
###lambda=0.8
### n=p/8
fit.16.12.3.24.lambda4=stat.analy(pi.core.fit.16.12.3.24.lambda4)
colMeans(fit.16.12.3.24.lambda4$kro.mle)
```

```
##         K         C     Sigma
## 0.3051467 0.9388075 0.8659407
```

```r
colMeans(fit.16.12.3.24.lambda4$cse)
```

```
##         K         C     Sigma
## 0.3051467 0.7338528 0.7024756
```

```r
colMeans(fit.16.12.3.24.lambda4$init.ai)
```

```
##         K         C     Sigma
## 0.3051467 0.9460708 0.9704359
```

```r
colMeans(fit.16.12.3.24.lambda4$init.chol)
```

```
##         K         C     Sigma
## 0.3051467 0.9593618 0.9704359
```

```r
colMeans(fit.16.12.3.24.lambda4$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.2936206 0.9895081 0.9582223
```

```r
colMeans(fit.16.12.3.24.lambda4$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.2928567 0.9937253 0.9537380
```

```r
### n=p/4
fit.16.12.3.48.lambda4=stat.analy(pi.core.fit.16.12.3.48.lambda4)
colMeans(fit.16.12.3.48.lambda4$kro.mle)
```

```
##         K         C     Sigma
## 0.2063058 0.9388075 0.8642007
```

```r
colMeans(fit.16.12.3.48.lambda4$cse)
```

```
##         K         C     Sigma
## 0.2063058 0.7049302 0.6622426
```

```r
colMeans(fit.16.12.3.48.lambda4$init.ai)
```

```
##         K         C     Sigma
## 0.2063058 0.5943627 0.6234684
```

```r
colMeans(fit.16.12.3.48.lambda4$init.chol)
```

```
##         K         C     Sigma
## 0.2063058 0.6036444 0.6234684
```

```r
colMeans(fit.16.12.3.48.lambda4$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1976472 0.5987239 0.6016877
```

```r
colMeans(fit.16.12.3.48.lambda4$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1972240 0.6041615 0.5999553
```

```r
### n=p/2
fit.16.12.3.96.lambda4=stat.analy(pi.core.fit.16.12.3.96.lambda4)
colMeans(fit.16.12.3.96.lambda4$kro.mle)
```

```
##         K         C     Sigma
## 0.1430761 0.9388075 0.8638360
```

```r
colMeans(fit.16.12.3.96.lambda4$cse)
```

```
##         K         C     Sigma
## 0.1430761 0.6338553 0.5969361
```

```r
colMeans(fit.16.12.3.96.lambda4$init.ai)
```

```
##         K         C     Sigma
## 0.1430761 0.3650225 0.3877695
```

```r
colMeans(fit.16.12.3.96.lambda4$init.chol)
```

```
##         K         C     Sigma
## 0.1430761 0.3716336 0.3877695
```

```r
colMeans(fit.16.12.3.96.lambda4$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1376453 0.3450316 0.3600371
```

```r
colMeans(fit.16.12.3.96.lambda4$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1375037 0.3495769 0.3592209
```

```r
### n=p
fit.16.12.3.192.lambda4=stat.analy(pi.core.fit.16.12.3.192.lambda4)
colMeans(fit.16.12.3.192.lambda4$kro.mle)
```

```
##         K         C     Sigma
## 0.1015027 0.9388075 0.8638388
```

```r
colMeans(fit.16.12.3.192.lambda4$cse)
```

```
##         K         C     Sigma
## 0.1015027 0.5220679 0.4935599
```

```r
colMeans(fit.16.12.3.192.lambda4$init.ai)
```

```
##         K         C     Sigma
## 0.1015027 0.2423196 0.2660160
```

```r
colMeans(fit.16.12.3.192.lambda4$init.chol)
```

```
##         K         C     Sigma
## 0.1015027 0.2474535 0.2660160
```

```r
colMeans(fit.16.12.3.192.lambda4$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.09697991 0.22841660 0.25053383
```

```r
colMeans(fit.16.12.3.192.lambda4$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.09700115 0.23271000 0.25039783
```

```r
### n=2p
fit.16.12.3.384.lambda4=stat.analy(pi.core.fit.16.12.3.384.lambda4)
colMeans(fit.16.12.3.384.lambda4$kro.mle)
```

```
##          K          C      Sigma
## 0.06905348 0.93880749 0.86351950
```

```r
colMeans(fit.16.12.3.384.lambda4$cse)
```

```
##          K          C      Sigma
## 0.06905348 0.37887159 0.36456528
```

```r
colMeans(fit.16.12.3.384.lambda4$init.ai)
```

```
##          K          C      Sigma
## 0.06905348 0.16697357 0.19369262
```

```r
colMeans(fit.16.12.3.384.lambda4$init.chol)
```

```
##          K          C      Sigma
## 0.06905348 0.17033767 0.19369262
```

```r
colMeans(fit.16.12.3.384.lambda4$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.06705794 0.15936713 0.18546182
```

```r
colMeans(fit.16.12.3.384.lambda4$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.06703104 0.16245143 0.18544336
```

```r
##### boxplots of the relative spectral norms for K, C, and Sigma across 100 iterations
##### under different choices of the sample size $n$ and the value of non-spiked eigenvalue $\lambda$.
### K

K.analy.16.12.3.lambda1=plot.K.analy(fit.16.12.3.24.lambda1,fit.16.12.3.48.lambda1,fit.16.12.3.96.lambda
K.analy.16.12.3.lambda2=plot.K.analy(fit.16.12.3.24.lambda2,fit.16.12.3.48.lambda2,fit.16.12.3.96.lambda
K.analy.16.12.3.lambda3=plot.K.analy(fit.16.12.3.24.lambda3,fit.16.12.3.48.lambda3,fit.16.12.3.96.lambda
K.analy.16.12.3.lambda4=plot.K.analy(fit.16.12.3.24.lambda4,fit.16.12.3.48.lambda4,fit.16.12.3.96.lambda

K.analy.16.12.3=rbind(K.analy.16.12.3.lambda1,K.analy.16.12.3.lambda2,K.analy.16.12.3.lambda3,K.analy.1
K.analy.16.12.3.lambda=factor(K.analy.16.12.3$lambda,levels=c("lambda1","lambda2","lambda3","lambda4"))
levels(K.analy.16.12.3.lambda)=c(TeX("$\\lambda=0.2$"),TeX("$\\lambda=0.4$"),TeX("$\\lambda=0.6$"),TeX(
K.analy.16.12.3$lambda=K.analy.16.12.3.lambda

png("./numerical.simul_gfcore/K.16.12.3.png",width=900,height=600)
ggplot(K.analy.16.12.3,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambda,la
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("Separable Component Consistency : $(p_1,p_2,r)=(16,12,3)$"))+theme(plot.title=element_tex
dev.off()
```

```
## pdf
##   2
```

```r
### C

C.analy.16.12.3.lambda1=plot.C.analy(fit.16.12.3.24.lambda1,fit.16.12.3.48.lambda1,fit.16.12.3.96.lambda
C.analy.16.12.3.lambda2=plot.C.analy(fit.16.12.3.24.lambda2,fit.16.12.3.48.lambda2,fit.16.12.3.96.lambda
C.analy.16.12.3.lambda3=plot.C.analy(fit.16.12.3.24.lambda3,fit.16.12.3.48.lambda3,fit.16.12.3.96.lambda
C.analy.16.12.3.lambda4=plot.C.analy(fit.16.12.3.24.lambda4,fit.16.12.3.48.lambda4,fit.16.12.3.96.lambda

C.analy.16.12.3=rbind(C.analy.16.12.3.lambda1,C.analy.16.12.3.lambda2,C.analy.16.12.3.lambda3,C.analy.1
C.analy.16.12.3.lambda=factor(C.analy.16.12.3$lambda,levels=c("lambda1","lambda2","lambda3","lambda4"))
levels(C.analy.16.12.3.lambda)=c(TeX("$\\lambda=0.2$"),TeX("$\\lambda=0.4$"),TeX("$\\lambda=0.6$"),TeX(
```

```
C.analy.16.12.3$lambda=C.analy.16.12.3.lambda

png("./numerical.simul_gfcore/C.16.12.3.png",width=900,height=600)
ggplot(C.analy.16.12.3,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambda,la
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("Core Component Consistency : $(p_1,p_2,r)=(16,12,3)$"))+theme(plot.title=element_text(hju
dev.off()
```

```
## pdf
##   2
```

```
### Sigma

Sigma.analy.16.12.3.lambda1=plot.Sigma.analy(fit.16.12.3.24.lambda1,fit.16.12.3.48.lambda1,fit.16.12.3.9
Sigma.analy.16.12.3.lambda2=plot.Sigma.analy(fit.16.12.3.24.lambda2,fit.16.12.3.48.lambda2,fit.16.12.3.9
Sigma.analy.16.12.3.lambda3=plot.Sigma.analy(fit.16.12.3.24.lambda3,fit.16.12.3.48.lambda3,fit.16.12.3.9
Sigma.analy.16.12.3.lambda4=plot.Sigma.analy(fit.16.12.3.24.lambda4,fit.16.12.3.48.lambda4,fit.16.12.3.9

Sigma.analy.16.12.3=rbind(Sigma.analy.16.12.3.lambda1,Sigma.analy.16.12.3.lambda2,Sigma.analy.16.12.3.la
Sigma.analy.16.12.3.lambda=factor(Sigma.analy.16.12.3$lambda,levels=c("lambda1","lambda2","lambda3","lam
levels(Sigma.analy.16.12.3.lambda)=c(TeX("$\\lambda=0.2$"),TeX("$\\lambda=0.4$"),TeX("$\\lambda=0.6$"),T
Sigma.analy.16.12.3$lambda=Sigma.analy.16.12.3.lambda

png("./numerical.simul_gfcore/Sigma.16.12.3.png",width=900,height=600)
ggplot(Sigma.analy.16.12.3,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambd
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("Sigma Consistency : $(p_1,p_2,r)=(16,12,3)$"))+theme(plot.title=element_text(hjust=0.5))
dev.off()
```

```
## pdf
##   2
```

(p1,p2,r)=(18,8,3)

```
##### mean of the relative spectral norms for $K$, $C$, and $\Sigma$ (in order) across 100 iterations
##### under different choices of the sample size $n$ and the value of non-spiked eigenvalue $\lambda$.
###lambda=0.2
### n=p/8
fit.18.8.3.18.lambda1=stat.analy(pi.core.fit.18.8.3.18.lambda1)
colMeans(fit.18.8.3.18.lambda1$kro.mle)
```

```
##         K         C     Sigma
## 0.4498574 0.9772074 0.9456202
```

```
colMeans(fit.18.8.3.18.lambda1$cse)
```

```
##         K         C     Sigma
## 0.4498574 0.3563862 0.5250413
```

```
colMeans(fit.18.8.3.18.lambda1$init.ai)
```

```
##         K         C     Sigma
## 0.4498574 0.2315587 0.4821864
```

```
colMeans(fit.18.8.3.18.lambda1$init.chol)
```

```
##         K         C     Sigma
```

```
## 0.4498574 0.3087617 0.4821864
```

```
colMeans(fit.18.8.3.18.lambda1$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.2974022 0.1956993 0.3077567
```

```
colMeans(fit.18.8.3.18.lambda1$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.2984345 0.2345839 0.3083591
```

```
### n=p/4
fit.18.8.3.36.lambda1=stat.analy(pi.core.fit.18.8.3.36.lambda1)
colMeans(fit.18.8.3.36.lambda1$kro.mle)
```

```
##         K         C     Sigma
## 0.2953311 0.9772074 0.9449694
```

```
colMeans(fit.18.8.3.36.lambda1$cse)
```

```
##         K         C     Sigma
## 0.2953311 0.2966762 0.4224137
```

```
colMeans(fit.18.8.3.36.lambda1$init.ai)
```

```
##         K         C     Sigma
## 0.2953311 0.1596066 0.3695388
```

```
colMeans(fit.18.8.3.36.lambda1$init.chol)
```

```
##         K         C     Sigma
## 0.2953311 0.2128795 0.3695388
```

```
colMeans(fit.18.8.3.36.lambda1$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.2150499 0.1286145 0.2215223
```

```
colMeans(fit.18.8.3.36.lambda1$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.2164757 0.1575058 0.2252247
```

```
### n=p/2
fit.18.8.3.72.lambda1=stat.analy(pi.core.fit.18.8.3.72.lambda1)
colMeans(fit.18.8.3.72.lambda1$kro.mle)
```

```
##         K         C     Sigma
## 0.1985010 0.9772074 0.9459904
```

```
colMeans(fit.18.8.3.72.lambda1$cse)
```

```
##         K         C     Sigma
## 0.1985010 0.2261084 0.3184601
```

```
colMeans(fit.18.8.3.72.lambda1$init.ai)
```

```
##         K         C     Sigma
## 0.1985010 0.1093975 0.2514982
```

```
colMeans(fit.18.8.3.72.lambda1$init.chol)
```

```
##         K         C     Sigma
```

```
## 0.1985010 0.1468992 0.2514982
```

```
colMeans(fit.18.8.3.72.lambda1$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.15221467 0.08781594 0.15696918
```

```
colMeans(fit.18.8.3.72.lambda1$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1544559 0.1086482 0.1620174
```

```
### n=p
fit.18.8.3.144.lambda1=stat.analy(pi.core.fit.18.8.3.144.lambda1)
colMeans(fit.18.8.3.144.lambda1$kro.mle)
```

```
##         K         C     Sigma
## 0.1326098 0.9772074 0.9456880
```

```
colMeans(fit.18.8.3.144.lambda1$cse)
```

```
##         K         C     Sigma
## 0.1326098 0.1586276 0.2218311
```

```
colMeans(fit.18.8.3.144.lambda1$init.ai)
```

```
##          K          C      Sigma
## 0.13260978 0.07476232 0.16653618
```

```
colMeans(fit.18.8.3.144.lambda1$init.chol)
```

```
##         K         C     Sigma
## 0.1326098 0.0997750 0.1665362
```

```
colMeans(fit.18.8.3.144.lambda1$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.10074511 0.05960863 0.10286507
```

```
colMeans(fit.18.8.3.144.lambda1$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.10364725 0.07497804 0.10907388
```

```
### n=2p
fit.18.8.3.288.lambda1=stat.analy(pi.core.fit.18.8.3.288.lambda1)
colMeans(fit.18.8.3.288.lambda1$kro.mle)
```

```
##          K          C      Sigma
## 0.09825791 0.97720739 0.94557122
```

```
colMeans(fit.18.8.3.288.lambda1$cse)
```

```
##          K          C      Sigma
## 0.09825791 0.10684172 0.16182729
```

```
colMeans(fit.18.8.3.288.lambda1$init.ai)
```

```
##          K          C      Sigma
## 0.09825791 0.05289287 0.12233814
```

```
colMeans(fit.18.8.3.288.lambda1$init.chol)
```

```
##          K          C      Sigma
```

```
## 0.09825791 0.07146648 0.12233814
```

```
colMeans(fit.18.8.3.288.lambda1$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.07340397 0.04224106 0.07599137
```

```
colMeans(fit.18.8.3.288.lambda1$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.07736585 0.05447332 0.08420297
```

```
###lambda=0.4
### n=p/8
fit.18.8.3.18.lambda2=stat.analy(pi.core.fit.18.8.3.18.lambda2)
colMeans(fit.18.8.3.18.lambda2$kro.mle)
```

```
##         K         C     Sigma
## 0.4211118 0.9698372 0.9283628
```

```
colMeans(fit.18.8.3.18.lambda2$cse)
```

```
##         K         C     Sigma
## 0.4211118 0.5049964 0.5678640
```

```
colMeans(fit.18.8.3.18.lambda2$init.ai)
```

```
##         K         C     Sigma
## 0.4211118 0.3425508 0.5112613
```

```
colMeans(fit.18.8.3.18.lambda2$init.chol)
```

```
##         K         C     Sigma
## 0.4211118 0.3941785 0.5112613
```

```
colMeans(fit.18.8.3.18.lambda2$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.3213489 0.3341068 0.3999743
```

```
colMeans(fit.18.8.3.18.lambda2$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.3226641 0.3660907 0.3989263
```

```
### n=p/4
fit.18.8.3.36.lambda2=stat.analy(pi.core.fit.18.8.3.36.lambda2)
colMeans(fit.18.8.3.36.lambda2$kro.mle)
```

```
##         K         C     Sigma
## 0.2738012 0.9698372 0.9279139
```

```
colMeans(fit.18.8.3.36.lambda2$cse)
```

```
##         K         C     Sigma
## 0.2738012 0.4602136 0.4957487
```

```
colMeans(fit.18.8.3.36.lambda2$init.ai)
```

```
##         K         C     Sigma
## 0.2738012 0.2316718 0.3796743
```

```r
colMeans(fit.18.8.3.36.lambda2$init.chol)
```

```
##         K         C     Sigma
## 0.2738012 0.2669882 0.3796743
```

```r
colMeans(fit.18.8.3.36.lambda2$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.2307429 0.2145546 0.2837751
```

```r
colMeans(fit.18.8.3.36.lambda2$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.2306523 0.2393813 0.2841754
```

```r
### n=p/2
fit.18.8.3.72.lambda2=stat.analy(pi.core.fit.18.8.3.72.lambda2)
colMeans(fit.18.8.3.72.lambda2$kro.mle)
```

```
##         K         C     Sigma
## 0.1866046 0.9698372 0.9287377
```

```r
colMeans(fit.18.8.3.72.lambda2$cse)
```

```
##         K         C     Sigma
## 0.1866046 0.3744440 0.4076197
```

```r
colMeans(fit.18.8.3.72.lambda2$init.ai)
```

```
##         K         C     Sigma
## 0.1866046 0.1568689 0.2571139
```

```r
colMeans(fit.18.8.3.72.lambda2$init.chol)
```

```
##         K         C     Sigma
## 0.1866046 0.1821242 0.2571139
```

```r
colMeans(fit.18.8.3.72.lambda2$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1605010 0.1441183 0.1980337
```

```r
colMeans(fit.18.8.3.72.lambda2$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1605900 0.1618087 0.1980828
```

```r
### n=p
fit.18.8.3.144.lambda2=stat.analy(pi.core.fit.18.8.3.144.lambda2)
colMeans(fit.18.8.3.144.lambda2$kro.mle)
```

```
##         K         C     Sigma
## 0.1245042 0.9698372 0.9282489
```

```r
colMeans(fit.18.8.3.144.lambda2$cse)
```

```
##         K         C     Sigma
## 0.1245042 0.2759209 0.3035303
```

```r
colMeans(fit.18.8.3.144.lambda2$init.ai)
```

```
##         K         C     Sigma
## 0.1245042 0.1074134 0.1711953
```

```r
colMeans(fit.18.8.3.144.lambda2$init.chol)
```

```
##         K         C     Sigma
## 0.1245042 0.1244917 0.1711953
```

```r
colMeans(fit.18.8.3.144.lambda2$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.10860704 0.09784872 0.13240060
```

```r
colMeans(fit.18.8.3.144.lambda2$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1088086 0.1101353 0.1326317
```

```r
### n=2p
fit.18.8.3.288.lambda2=stat.analy(pi.core.fit.18.8.3.288.lambda2)
colMeans(fit.18.8.3.288.lambda2$kro.mle)
```

```
##          K          C      Sigma
## 0.09094657 0.96983724 0.92815287
```

```r
colMeans(fit.18.8.3.288.lambda2$cse)
```

```
##          K          C      Sigma
## 0.09094657 0.19003553 0.21643343
```

```r
colMeans(fit.18.8.3.288.lambda2$init.ai)
```

```
##          K          C      Sigma
## 0.09094657 0.07616145 0.12542944
```

```r
colMeans(fit.18.8.3.288.lambda2$init.chol)
```

```
##          K          C      Sigma
## 0.09094657 0.08862987 0.12542944
```

```r
colMeans(fit.18.8.3.288.lambda2$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.07874344 0.06936801 0.09765789
```

```r
colMeans(fit.18.8.3.288.lambda2$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.07894997 0.07799759 0.09834846
```

```r
###lambda=0.6
### n=p/8
fit.18.8.3.18.lambda3=stat.analy(pi.core.fit.18.8.3.18.lambda3)
colMeans(fit.18.8.3.18.lambda3$kro.mle)
```

```
##         K         C     Sigma
## 0.4104129 0.9554282 0.8950440
```

```r
colMeans(fit.18.8.3.18.lambda3$cse)
```

```
##         K         C     Sigma
## 0.4104129 0.6307429 0.6280749
```

```r
colMeans(fit.18.8.3.18.lambda3$init.ai)
```

```
##         K         C     Sigma
```

```
## 0.4104129 0.5411863 0.6235305
```

```
colMeans(fit.18.8.3.18.lambda3$init.chol)
```

```
##         K         C     Sigma
## 0.4104129 0.5730460 0.6235305
```

```
colMeans(fit.18.8.3.18.lambda3$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.3550601 0.5622683 0.5806514
```

```
colMeans(fit.18.8.3.18.lambda3$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.3519299 0.5872006 0.5753340
```

```
### n=p/4
fit.18.8.3.36.lambda3=stat.analy(pi.core.fit.18.8.3.36.lambda3)
colMeans(fit.18.8.3.36.lambda3$kro.mle)
```

```
##         K         C     Sigma
## 0.2700667 0.9554282 0.8948009
```

```
colMeans(fit.18.8.3.36.lambda3$cse)
```

```
##         K         C     Sigma
## 0.2700667 0.6028948 0.5848852
```

```
colMeans(fit.18.8.3.36.lambda3$init.ai)
```

```
##         K         C     Sigma
## 0.2700667 0.3434887 0.4306280
```

```
colMeans(fit.18.8.3.36.lambda3$init.chol)
```

```
##         K         C     Sigma
## 0.2700667 0.3673228 0.4306280
```

```
colMeans(fit.18.8.3.36.lambda3$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.2443865 0.3384581 0.3797775
```

```
colMeans(fit.18.8.3.36.lambda3$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.2443401 0.3580142 0.3798059
```

```
### n=p/2
fit.18.8.3.72.lambda3=stat.analy(pi.core.fit.18.8.3.72.lambda3)
colMeans(fit.18.8.3.72.lambda3$kro.mle)
```

```
##         K         C     Sigma
## 0.1815981 0.9554282 0.8953597
```

```
colMeans(fit.18.8.3.72.lambda3$cse)
```

```
##         K         C     Sigma
## 0.1815981 0.5163395 0.5074762
```

```
colMeans(fit.18.8.3.72.lambda3$init.ai)
```

```
##         K         C     Sigma
```

```
## 0.1815981 0.2263502 0.2863259
```

```
colMeans(fit.18.8.3.72.lambda3$init.chol)
```

```
##         K         C     Sigma
## 0.1815981 0.2439588 0.2863259
```

```
colMeans(fit.18.8.3.72.lambda3$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1654531 0.2190650 0.2534906
```

```
colMeans(fit.18.8.3.72.lambda3$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1655234 0.2333235 0.2534680
```

```
### n=p
fit.18.8.3.144.lambda3=stat.analy(pi.core.fit.18.8.3.144.lambda3)
colMeans(fit.18.8.3.144.lambda3$kro.mle)
```

```
##         K         C     Sigma
## 0.1225344 0.9554282 0.8945109
```

```
colMeans(fit.18.8.3.144.lambda3$cse)
```

```
##         K         C     Sigma
## 0.1225344 0.4016269 0.3967448
```

```
colMeans(fit.18.8.3.144.lambda3$init.ai)
```

```
##         K         C     Sigma
## 0.1225344 0.1532243 0.1926403
```

```
colMeans(fit.18.8.3.144.lambda3$init.chol)
```

```
##         K         C     Sigma
## 0.1225344 0.1652233 0.1926403
```

```
colMeans(fit.18.8.3.144.lambda3$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1137948 0.1469625 0.1734070
```

```
colMeans(fit.18.8.3.144.lambda3$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1137909 0.1569719 0.1733690
```

```
### n=2p
fit.18.8.3.288.lambda3=stat.analy(pi.core.fit.18.8.3.288.lambda3)
colMeans(fit.18.8.3.288.lambda3$kro.mle)
```

```
##          K          C      Sigma
## 0.08894988 0.95542823 0.89447983
```

```
colMeans(fit.18.8.3.288.lambda3$cse)
```

```
##          K          C      Sigma
## 0.08894988 0.28576214 0.28785822
```

```
colMeans(fit.18.8.3.288.lambda3$init.ai)
```

```
##          K          C      Sigma
```

```
## 0.08894988 0.10870413 0.14371344
```

```
colMeans(fit.18.8.3.288.lambda3$init.chol)
```

```
##          K          C      Sigma
## 0.08894988 0.11703857 0.14371344
```

```
colMeans(fit.18.8.3.288.lambda3$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.08177674 0.10449885 0.13199318
```

```
colMeans(fit.18.8.3.288.lambda3$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.08177845 0.11114489 0.13196083
```

```
###lambda=0.8
### n=p/8
fit.18.8.3.18.lambda4=stat.analy(pi.core.fit.18.8.3.18.lambda4)
colMeans(fit.18.8.3.18.lambda4$kro.mle)
```

```
##         K         C     Sigma
## 0.4067328 0.9147403 0.8045424
```

```
colMeans(fit.18.8.3.18.lambda4$cse)
```

```
##         K         C     Sigma
## 0.4067328 0.7305303 0.6673306
```

```
colMeans(fit.18.8.3.18.lambda4$init.ai)
```

```
##         K         C     Sigma
## 0.4067328 1.1306747 1.0779444
```

```
colMeans(fit.18.8.3.18.lambda4$init.chol)
```

```
##         K         C     Sigma
## 0.4067328 1.1429158 1.0779444
```

```
colMeans(fit.18.8.3.18.lambda4$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.3794338 1.1591675 1.0618951
```

```
colMeans(fit.18.8.3.18.lambda4$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.3795983 1.1631404 1.0575969
```

```
### n=p/4
fit.18.8.3.36.lambda4=stat.analy(pi.core.fit.18.8.3.36.lambda4)
colMeans(fit.18.8.3.36.lambda4$kro.mle)
```

```
##         K         C     Sigma
## 0.2708083 0.9147403 0.8038019
```

```
colMeans(fit.18.8.3.36.lambda4$cse)
```

```
##         K         C     Sigma
## 0.2708083 0.7169463 0.6396534
```

```r
colMeans(fit.18.8.3.36.lambda4$init.ai)
```

```
##         K         C     Sigma
## 0.2708083 0.6687164 0.6489945
```

```r
colMeans(fit.18.8.3.36.lambda4$init.chol)
```

```
##         K         C     Sigma
## 0.2708083 0.6814926 0.6489945
```

```r
colMeans(fit.18.8.3.36.lambda4$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.2592537 0.6695038 0.6185169
```

```r
colMeans(fit.18.8.3.36.lambda4$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.2592278 0.6805267 0.6223999
```

```r
### n=p/2
fit.18.8.3.72.lambda4=stat.analy(pi.core.fit.18.8.3.72.lambda4)
colMeans(fit.18.8.3.72.lambda4$kro.mle)
```

```
##         K         C     Sigma
## 0.1799346 0.9147403 0.8033260
```

```r
colMeans(fit.18.8.3.72.lambda4$cse)
```

```
##         K         C     Sigma
## 0.1799346 0.6400597 0.5773397
```

```r
colMeans(fit.18.8.3.72.lambda4$init.ai)
```

```
##         K         C     Sigma
## 0.1799346 0.4046526 0.3971683
```

```r
colMeans(fit.18.8.3.72.lambda4$init.chol)
```

```
##         K         C     Sigma
## 0.1799346 0.4148062 0.3971683
```

```r
colMeans(fit.18.8.3.72.lambda4$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1721204 0.3887763 0.3801580
```

```r
colMeans(fit.18.8.3.72.lambda4$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1722787 0.3967741 0.3783453
```

```r
### n=p
fit.18.8.3.144.lambda4=stat.analy(pi.core.fit.18.8.3.144.lambda4)
colMeans(fit.18.8.3.144.lambda4$kro.mle)
```

```
##         K         C     Sigma
## 0.1218588 0.9147403 0.8013305
```

```r
colMeans(fit.18.8.3.144.lambda4$cse)
```

```
##         K         C     Sigma
## 0.1218588 0.5254340 0.4735317
```

```
colMeans(fit.18.8.3.144.lambda4$init.ai)
```

```
##         K         C     Sigma
## 0.1218588 0.2549076 0.2730879
```

```
colMeans(fit.18.8.3.144.lambda4$init.chol)
```

```
##         K         C     Sigma
## 0.1218588 0.2620751 0.2730879
```

```
colMeans(fit.18.8.3.144.lambda4$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1183634 0.2451186 0.2667792
```

```
colMeans(fit.18.8.3.144.lambda4$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1183298 0.2511977 0.2666631
```

```
### n=2p
fit.18.8.3.288.lambda4=stat.analy(pi.core.fit.18.8.3.288.lambda4)
colMeans(fit.18.8.3.288.lambda4$kro.mle)
```

```
##          K          C      Sigma
## 0.08849394 0.91474029 0.80138339
```

```
colMeans(fit.18.8.3.288.lambda4$cse)
```

```
##          K          C      Sigma
## 0.08849394 0.39006185 0.35754660
```

```
colMeans(fit.18.8.3.288.lambda4$init.ai)
```

```
##          K          C      Sigma
## 0.08849394 0.18025084 0.23552895
```

```
colMeans(fit.18.8.3.288.lambda4$init.chol)
```

```
##          K          C      Sigma
## 0.08849394 0.18381900 0.23552895
```

```
colMeans(fit.18.8.3.288.lambda4$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.08499674 0.17618698 0.23460397
```

```
colMeans(fit.18.8.3.288.lambda4$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.08497281 0.17941423 0.23459973
```

```
##### boxplots of the relative spectral norms for K, C, and Sigma across 100 iterations
##### under different choices of the sample size $n$ and the value of non-spiked eigenvalue $\lambda$.
### K

K.analy.18.8.3.lambda1=plot.K.analy(fit.18.8.3.18.lambda1,fit.18.8.3.36.lambda1,fit.18.8.3.72.lambda1,fi
K.analy.18.8.3.lambda2=plot.K.analy(fit.18.8.3.18.lambda2,fit.18.8.3.36.lambda2,fit.18.8.3.72.lambda2,fi
K.analy.18.8.3.lambda3=plot.K.analy(fit.18.8.3.18.lambda3,fit.18.8.3.36.lambda3,fit.18.8.3.72.lambda3,fi
K.analy.18.8.3.lambda4=plot.K.analy(fit.18.8.3.18.lambda4,fit.18.8.3.36.lambda4,fit.18.8.3.72.lambda4,fi

K.analy.18.8.3=rbind(K.analy.18.8.3.lambda1,K.analy.18.8.3.lambda2,K.analy.18.8.3.lambda3,K.analy.18.8.3
```

```
K.analy.18.8.3.lambda=factor(K.analy.18.8.3$lambda,levels=c("lambda1","lambda2","lambda3","lambda4"))
levels(K.analy.18.8.3.lambda)=c(TeX("$\\lambda=0.2$"),TeX("$\\lambda=0.4$"),TeX("$\\lambda=0.6$"),TeX("$
K.analy.18.8.3$lambda=K.analy.18.8.3.lambda

png("./numerical.simul_gfcore/K.18.8.3.png",width=900,height=600)
ggplot(K.analy.18.8.3,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambda,lab
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("Separable Component Consistency : $(p_1,p_2,r)=(18,8,3)$"))+theme(plot.title=element_text
dev.off()
```

```
## pdf
##   2
```

```
### C

C.analy.18.8.3.lambda1=plot.C.analy(fit.18.8.3.18.lambda1,fit.18.8.3.36.lambda1,fit.18.8.3.72.lambda1,fi
C.analy.18.8.3.lambda2=plot.C.analy(fit.18.8.3.18.lambda2,fit.18.8.3.36.lambda2,fit.18.8.3.72.lambda2,fi
C.analy.18.8.3.lambda3=plot.C.analy(fit.18.8.3.18.lambda3,fit.18.8.3.36.lambda3,fit.18.8.3.72.lambda3,fi
C.analy.18.8.3.lambda4=plot.C.analy(fit.18.8.3.18.lambda4,fit.18.8.3.36.lambda4,fit.18.8.3.72.lambda4,fi

C.analy.18.8.3=rbind(C.analy.18.8.3.lambda1,C.analy.18.8.3.lambda2,C.analy.18.8.3.lambda3,C.analy.18.8.3
C.analy.18.8.3.lambda=factor(C.analy.18.8.3$lambda,levels=c("lambda1","lambda2","lambda3","lambda4"))
levels(C.analy.18.8.3.lambda)=c(TeX("$\\lambda=0.2$"),TeX("$\\lambda=0.4$"),TeX("$\\lambda=0.6$"),TeX("$
C.analy.18.8.3$lambda=C.analy.18.8.3.lambda

png("./numerical.simul_gfcore/C.18.8.3.png",width=900,height=600)
ggplot(C.analy.18.8.3,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambda,lab
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("Core Component Consistency : $(p_1,p_2,r)=(18,8,3)$"))+theme(plot.title=element_text(hjus
dev.off()
```

```
## pdf
##   2
```

```
### Sigma

Sigma.analy.18.8.3.lambda1=plot.Sigma.analy(fit.18.8.3.18.lambda1,fit.18.8.3.36.lambda1,fit.18.8.3.72.la
Sigma.analy.18.8.3.lambda2=plot.Sigma.analy(fit.18.8.3.18.lambda2,fit.18.8.3.36.lambda2,fit.18.8.3.72.la
Sigma.analy.18.8.3.lambda3=plot.Sigma.analy(fit.18.8.3.18.lambda3,fit.18.8.3.36.lambda3,fit.18.8.3.72.la
Sigma.analy.18.8.3.lambda4=plot.Sigma.analy(fit.18.8.3.18.lambda4,fit.18.8.3.36.lambda4,fit.18.8.3.72.la

Sigma.analy.18.8.3=rbind(Sigma.analy.18.8.3.lambda1,Sigma.analy.18.8.3.lambda2,Sigma.analy.18.8.3.lambda
Sigma.analy.18.8.3.lambda=factor(Sigma.analy.18.8.3$lambda,levels=c("lambda1","lambda2","lambda3","lambd
levels(Sigma.analy.18.8.3.lambda)=c(TeX("$\\lambda=0.2$"),TeX("$\\lambda=0.4$"),TeX("$\\lambda=0.6$"),Te
Sigma.analy.18.8.3$lambda=Sigma.analy.18.8.3.lambda

png("./numerical.simul_gfcore/Sigma.18.8.3.png",width=900,height=600)
ggplot(Sigma.analy.18.8.3,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambda
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("Sigma Consistency : $(p_1,p_2,r)=(18,8,3)$"))+theme(plot.title=element_text(hjust=0.5))
dev.off()
```

```
## pdf
##   2
```

(p1,p2,r)=(16,12,5)

```
##### mean of the relative spectral norms for $K$, $C$, and $\Sigma$ (in order) across 100 iterations
##### under different choices of the sample size $n$ and the value of non-spiked eigenvalue $\lambda$.
###lambda=0.2
### n=p/8
fit.16.12.5.24.lambda1=stat.analy(pi.core.fit.16.12.5.24.lambda1)
colMeans(fit.16.12.5.24.lambda1$kro.mle)
```

```
##         K         C     Sigma
## 0.3104290 0.9734961 0.9362013
```

```
colMeans(fit.16.12.5.24.lambda1$cse)
```

```
##         K         C     Sigma
## 0.3104290 0.4938516 0.5927379
```

```
colMeans(fit.16.12.5.24.lambda1$init.ai)
```

```
##         K         C     Sigma
## 0.3104290 0.5158918 0.6527251
```

```
colMeans(fit.16.12.5.24.lambda1$init.chol)
```

```
##         K         C     Sigma
## 0.3104290 0.5423546 0.6527251
```

```
colMeans(fit.16.12.5.24.lambda1$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.2351154 0.3325902 0.3872793
```

```
colMeans(fit.16.12.5.24.lambda1$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.2368727 0.3467676 0.3907216
```

```
### n=p/4
fit.16.12.5.48.lambda1=stat.analy(pi.core.fit.16.12.5.48.lambda1)
colMeans(fit.16.12.5.48.lambda1$kro.mle)
```

```
##         K         C     Sigma
## 0.2260572 0.9734961 0.9342592
```

```
colMeans(fit.16.12.5.48.lambda1$cse)
```

```
##         K         C     Sigma
## 0.2260572 0.3997424 0.4535451
```

```
colMeans(fit.16.12.5.48.lambda1$init.ai)
```

```
##         K         C     Sigma
## 0.2260572 0.3612767 0.4842754
```

```
colMeans(fit.16.12.5.48.lambda1$init.chol)
```

```
##         K         C     Sigma
## 0.2260572 0.3777756 0.4842754
```

```
colMeans(fit.16.12.5.48.lambda1$fit.ai.analy)
```

```
##         K         C     Sigma
```

```
## 0.1644800 0.2136253 0.2703601
```

```
colMeans(fit.16.12.5.48.lambda1$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1676738 0.2297647 0.2812347
```

```
### n=p/2
fit.16.12.5.96.lambda1=stat.analy(pi.core.fit.16.12.5.96.lambda1)
colMeans(fit.16.12.5.96.lambda1$kro.mle)
```

```
##         K         C     Sigma
## 0.1501236 0.9734961 0.9355565
```

```
colMeans(fit.16.12.5.96.lambda1$cse)
```

```
##         K         C     Sigma
## 0.1501236 0.3125959 0.3516430
```

```
colMeans(fit.16.12.5.96.lambda1$init.ai)
```

```
##         K         C     Sigma
## 0.1501236 0.2532748 0.3220968
```

```
colMeans(fit.16.12.5.96.lambda1$init.chol)
```

```
##         K         C     Sigma
## 0.1501236 0.2638246 0.3220968
```

```
colMeans(fit.16.12.5.96.lambda1$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1138481 0.1486114 0.1943679
```

```
colMeans(fit.16.12.5.96.lambda1$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1195041 0.1671750 0.2089434
```

```
### n=p
fit.16.12.5.192.lambda1=stat.analy(pi.core.fit.16.12.5.192.lambda1)
colMeans(fit.16.12.5.192.lambda1$kro.mle)
```

```
##         K         C     Sigma
## 0.1022652 0.9734961 0.9353922
```

```
colMeans(fit.16.12.5.192.lambda1$cse)
```

```
##         K         C     Sigma
## 0.1022652 0.2270632 0.2556009
```

```
colMeans(fit.16.12.5.192.lambda1$init.ai)
```

```
##         K         C     Sigma
## 0.1022652 0.1735713 0.2227232
```

```
colMeans(fit.16.12.5.192.lambda1$init.chol)
```

```
##         K         C     Sigma
## 0.1022652 0.1805594 0.2227232
```

```
colMeans(fit.16.12.5.192.lambda1$fit.ai.analy)
```

```
##          K          C      Sigma
```

```
## 0.08185204 0.10625991 0.13962643
```

```
colMeans(fit.16.12.5.192.lambda1$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.08731071 0.12314650 0.15497967
```

```
### n=2p
fit.16.12.5.384.lambda1=stat.analy(pi.core.fit.16.12.5.384.lambda1)
colMeans(fit.16.12.5.384.lambda1$kro.mle)
```

```
##          K          C      Sigma
## 0.07149767 0.97349609 0.93543888
```

```
colMeans(fit.16.12.5.384.lambda1$cse)
```

```
##          K          C      Sigma
## 0.07149767 0.15690850 0.17751254
```

```
colMeans(fit.16.12.5.384.lambda1$init.ai)
```

```
##          K          C      Sigma
## 0.07149767 0.12121670 0.15784446
```

```
colMeans(fit.16.12.5.384.lambda1$init.chol)
```

```
##          K          C      Sigma
## 0.07149767 0.12637367 0.15784446
```

```
colMeans(fit.16.12.5.384.lambda1$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.05951156 0.08290782 0.11109733
```

```
colMeans(fit.16.12.5.384.lambda1$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.06519095 0.09938809 0.12771482
```

```
###lambda=0.4
### n=p/8
fit.16.12.5.24.lambda2=stat.analy(pi.core.fit.16.12.5.24.lambda2)
colMeans(fit.16.12.5.24.lambda2$kro.mle)
```

```
##         K         C     Sigma
## 0.3044358 0.9649725 0.9157926
```

```
colMeans(fit.16.12.5.24.lambda2$cse)
```

```
##         K         C     Sigma
## 0.3044358 0.5955302 0.6100796
```

```
colMeans(fit.16.12.5.24.lambda2$init.ai)
```

```
##         K         C     Sigma
## 0.3044358 0.6395278 0.7234977
```

```
colMeans(fit.16.12.5.24.lambda2$init.chol)
```

```
##         K         C     Sigma
## 0.3044358 0.6623373 0.7234977
```

```
colMeans(fit.16.12.5.24.lambda2$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.2568204 0.5821725 0.5715966
```

```
colMeans(fit.16.12.5.24.lambda2$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.2592574 0.5843323 0.5631781
```

```
### n=p/4
fit.16.12.5.48.lambda2=stat.analy(pi.core.fit.16.12.5.48.lambda2)
colMeans(fit.16.12.5.48.lambda2$kro.mle)
```

```
##         K         C     Sigma
## 0.2150196 0.9649725 0.9137175
```

```
colMeans(fit.16.12.5.48.lambda2$cse)
```

```
##         K         C     Sigma
## 0.2150196 0.5169921 0.5156825
```

```
colMeans(fit.16.12.5.48.lambda2$init.ai)
```

```
##         K         C     Sigma
## 0.2150196 0.4345164 0.5156927
```

```
colMeans(fit.16.12.5.48.lambda2$init.chol)
```

```
##         K         C     Sigma
## 0.2150196 0.4489509 0.5156927
```

```
colMeans(fit.16.12.5.48.lambda2$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1805681 0.3461878 0.3720314
```

```
colMeans(fit.16.12.5.48.lambda2$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1798905 0.3515555 0.3671819
```

```
### n=p/2
fit.16.12.5.96.lambda2=stat.analy(pi.core.fit.16.12.5.96.lambda2)
colMeans(fit.16.12.5.96.lambda2$kro.mle)
```

```
##         K         C     Sigma
## 0.1466835 0.9649725 0.9150164
```

```
colMeans(fit.16.12.5.96.lambda2$cse)
```

```
##         K         C     Sigma
## 0.1466835 0.4282396 0.4344362
```

```
colMeans(fit.16.12.5.96.lambda2$init.ai)
```

```
##         K         C     Sigma
## 0.1466835 0.2948627 0.3405850
```

```
colMeans(fit.16.12.5.96.lambda2$init.chol)
```

```
##         K         C     Sigma
## 0.1466835 0.3035842 0.3405850
```

```r
colMeans(fit.16.12.5.96.lambda2$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1243395 0.2206655 0.2500256
```

```r
colMeans(fit.16.12.5.96.lambda2$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1244847 0.2247570 0.2472811
```

```r
### n=p
fit.16.12.5.192.lambda2=stat.analy(pi.core.fit.16.12.5.192.lambda2)
colMeans(fit.16.12.5.192.lambda2$kro.mle)
```

```
##         K         C     Sigma
## 0.1012752 0.9649725 0.9147030
```

```r
colMeans(fit.16.12.5.192.lambda2$cse)
```

```
##         K         C     Sigma
## 0.1012752 0.3261250 0.3305807
```

```r
colMeans(fit.16.12.5.192.lambda2$init.ai)
```

```
##         K         C     Sigma
## 0.1012752 0.2015946 0.2367018
```

```r
colMeans(fit.16.12.5.192.lambda2$init.chol)
```

```
##         K         C     Sigma
## 0.1012752 0.2074128 0.2367018
```

```r
colMeans(fit.16.12.5.192.lambda2$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.08672925 0.14981423 0.17184528
```

```r
colMeans(fit.16.12.5.192.lambda2$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.08763299 0.15453220 0.17205159
```

```r
### n=2p
fit.16.12.5.384.lambda2=stat.analy(pi.core.fit.16.12.5.384.lambda2)
colMeans(fit.16.12.5.384.lambda2$kro.mle)
```

```
##          K          C      Sigma
## 0.06985333 0.96497254 0.91483943
```

```r
colMeans(fit.16.12.5.384.lambda2$cse)
```

```
##          K          C      Sigma
## 0.06985333 0.22863624 0.23495745
```

```r
colMeans(fit.16.12.5.384.lambda2$init.ai)
```

```
##          K          C      Sigma
## 0.06985333 0.14183750 0.16752617
```

```r
colMeans(fit.16.12.5.384.lambda2$init.chol)
```

```
##          K          C      Sigma
## 0.06985333 0.14611897 0.16752617
```

```
colMeans(fit.16.12.5.384.lambda2$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.05965283 0.10417473 0.12033358
```

```
colMeans(fit.16.12.5.384.lambda2$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.06028816 0.10872521 0.12214493
```

```
###lambda=0.6
### n=p/8
fit.16.12.5.24.lambda3=stat.analy(pi.core.fit.16.12.5.24.lambda3)
colMeans(fit.16.12.5.24.lambda3$kro.mle)
```

```
##         K         C     Sigma
## 0.3028983 0.9483784 0.8765024
```

```
colMeans(fit.16.12.5.24.lambda3$cse)
```

```
##         K         C     Sigma
## 0.3028983 0.6868453 0.6554850
```

```
colMeans(fit.16.12.5.24.lambda3$init.ai)
```

```
##         K         C     Sigma
## 0.3028983 0.8699324 0.8881842
```

```
colMeans(fit.16.12.5.24.lambda3$init.chol)
```

```
##         K         C     Sigma
## 0.3028983 0.8876704 0.8881842
```

```
colMeans(fit.16.12.5.24.lambda3$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.2877193 0.9922696 0.9515841
```

```
colMeans(fit.16.12.5.24.lambda3$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.2836092 1.0051523 0.9407846
```

```
### n=p/4
fit.16.12.5.48.lambda3=stat.analy(pi.core.fit.16.12.5.48.lambda3)
colMeans(fit.16.12.5.48.lambda3$kro.mle)
```

```
##         K         C     Sigma
## 0.2078564 0.9483784 0.8741131
```

```
colMeans(fit.16.12.5.48.lambda3$cse)
```

```
##         K         C     Sigma
## 0.2078564 0.6273561 0.5891855
```

```
colMeans(fit.16.12.5.48.lambda3$init.ai)
```

```
##         K         C     Sigma
## 0.2078564 0.5638015 0.5925996
```

```
colMeans(fit.16.12.5.48.lambda3$init.chol)
```

```
##         K         C     Sigma
```

```
## 0.2078564 0.5754675 0.5925996
```

```
colMeans(fit.16.12.5.48.lambda3$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1906219 0.5606743 0.5482304
```

```
colMeans(fit.16.12.5.48.lambda3$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1901923 0.5648567 0.5438706
```

```
### n=p/2
fit.16.12.5.96.lambda3=stat.analy(pi.core.fit.16.12.5.96.lambda3)
colMeans(fit.16.12.5.96.lambda3$kro.mle)
```

```
##         K         C     Sigma
## 0.1452901 0.9483784 0.8753477
```

```
colMeans(fit.16.12.5.96.lambda3$cse)
```

```
##         K         C     Sigma
## 0.1452901 0.5441668 0.5186403
```

```
colMeans(fit.16.12.5.96.lambda3$init.ai)
```

```
##         K         C     Sigma
## 0.1452901 0.3666798 0.3842529
```

```
colMeans(fit.16.12.5.96.lambda3$init.chol)
```

```
##         K         C     Sigma
## 0.1452901 0.3735314 0.3842529
```

```
colMeans(fit.16.12.5.96.lambda3$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1314784 0.3302199 0.3338574
```

```
colMeans(fit.16.12.5.96.lambda3$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1318877 0.3351211 0.3337000
```

```
### n=p
fit.16.12.5.192.lambda3=stat.analy(pi.core.fit.16.12.5.192.lambda3)
colMeans(fit.16.12.5.192.lambda3$kro.mle)
```

```
##         K         C     Sigma
## 0.1008365 0.9483784 0.8746988
```

```
colMeans(fit.16.12.5.192.lambda3$cse)
```

```
##         K         C     Sigma
## 0.1008365 0.4358806 0.4127736
```

```
colMeans(fit.16.12.5.192.lambda3$init.ai)
```

```
##         K         C     Sigma
## 0.1008365 0.2481951 0.2669292
```

```
colMeans(fit.16.12.5.192.lambda3$init.chol)
```

```
##         K         C     Sigma
```

```
## 0.1008365 0.2526943 0.2669292
```

```
colMeans(fit.16.12.5.192.lambda3$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.09119008 0.21975786 0.23046286
```

```
colMeans(fit.16.12.5.192.lambda3$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.0911462 0.2234025 0.2299500
```

```
### n=2p
fit.16.12.5.384.lambda3=stat.analy(pi.core.fit.16.12.5.384.lambda3)
colMeans(fit.16.12.5.384.lambda3$kro.mle)
```

```
##          K          C      Sigma
## 0.06882021 0.94837839 0.87495462
```

```
colMeans(fit.16.12.5.384.lambda3$cse)
```

```
##          K          C      Sigma
## 0.06882021 0.31608363 0.30333192
```

```
colMeans(fit.16.12.5.384.lambda3$init.ai)
```

```
##          K          C      Sigma
## 0.06882021 0.17425233 0.18921746
```

```
colMeans(fit.16.12.5.384.lambda3$init.chol)
```

```
##          K          C      Sigma
## 0.06882021 0.17762619 0.18921746
```

```
colMeans(fit.16.12.5.384.lambda3$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.06271681 0.15136129 0.16210931
```

```
colMeans(fit.16.12.5.384.lambda3$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.06271812 0.15404777 0.16201927
```

```
###lambda=0.8
### n=p/8
fit.16.12.5.24.lambda4=stat.analy(pi.core.fit.16.12.5.24.lambda4)
colMeans(fit.16.12.5.24.lambda4$kro.mle)
```

```
##         K         C     Sigma
## 0.3041913 0.9020679 0.7678933
```

```
colMeans(fit.16.12.5.24.lambda4$cse)
```

```
##         K         C     Sigma
## 0.3041913 0.7566865 0.6591615
```

```
colMeans(fit.16.12.5.24.lambda4$init.ai)
```

```
##         K         C     Sigma
## 0.3041913 1.4754942 1.3498422
```

```
colMeans(fit.16.12.5.24.lambda4$init.chol)
```

```
##         K         C     Sigma
## 0.3041913 1.4856620 1.3498422
```

```
colMeans(fit.16.12.5.24.lambda4$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.2900649 1.5162239 1.3136048
```

```
colMeans(fit.16.12.5.24.lambda4$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.2898473 1.5354981 1.3175943
```

```
### n=p/4
fit.16.12.5.48.lambda4=stat.analy(pi.core.fit.16.12.5.48.lambda4)
colMeans(fit.16.12.5.48.lambda4$kro.mle)
```

```
##         K         C     Sigma
## 0.2036019 0.9020679 0.7639503
```

```
colMeans(fit.16.12.5.48.lambda4$cse)
```

```
##         K         C     Sigma
## 0.2036019 0.7225537 0.6165195
```

```
colMeans(fit.16.12.5.48.lambda4$init.ai)
```

```
##         K         C     Sigma
## 0.2036019 0.9120027 0.8330766
```

```
colMeans(fit.16.12.5.48.lambda4$init.chol)
```

```
##         K         C     Sigma
## 0.2036019 0.9191349 0.8330766
```

```
colMeans(fit.16.12.5.48.lambda4$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1999062 1.0354435 0.9047564
```

```
colMeans(fit.16.12.5.48.lambda4$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.2000332 1.0541739 0.9149034
```

```
### n=p/2
fit.16.12.5.96.lambda4=stat.analy(pi.core.fit.16.12.5.96.lambda4)
colMeans(fit.16.12.5.96.lambda4$kro.mle)
```

```
##         K         C     Sigma
## 0.1443709 0.9020679 0.7646570
```

```
colMeans(fit.16.12.5.96.lambda4$cse)
```

```
##         K         C     Sigma
## 0.1443709 0.6496015 0.5613273
```

```
colMeans(fit.16.12.5.96.lambda4$init.ai)
```

```
##         K         C     Sigma
## 0.1443709 0.5595856 0.5123391
```

```r
colMeans(fit.16.12.5.96.lambda4$init.chol)
```

```
##         K         C     Sigma
## 0.1443709 0.5642255 0.5123391
```

```r
colMeans(fit.16.12.5.96.lambda4$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1388005 0.5964871 0.5281645
```

```r
colMeans(fit.16.12.5.96.lambda4$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1389816 0.5980790 0.5253968
```

```r
### n=p
fit.16.12.5.192.lambda4=stat.analy(pi.core.fit.16.12.5.192.lambda4)
colMeans(fit.16.12.5.192.lambda4$kro.mle)
```

```
##         K         C     Sigma
## 0.1002124 0.9020679 0.7630015
```

```r
colMeans(fit.16.12.5.192.lambda4$cse)
```

```
##         K         C     Sigma
## 0.1002124 0.5460538 0.4676843
```

```r
colMeans(fit.16.12.5.192.lambda4$init.ai)
```

```
##         K         C     Sigma
## 0.1002124 0.3641955 0.3509813
```

```r
colMeans(fit.16.12.5.192.lambda4$init.chol)
```

```
##         K         C     Sigma
## 0.1002124 0.3674308 0.3509813
```

```r
colMeans(fit.16.12.5.192.lambda4$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.09485962 0.36689647 0.34472112
```

```r
colMeans(fit.16.12.5.192.lambda4$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.09494741 0.36948894 0.34390920
```

```r
### n=2p
fit.16.12.5.384.lambda4=stat.analy(pi.core.fit.16.12.5.384.lambda4)
colMeans(fit.16.12.5.384.lambda4$kro.mle)
```

```
##          K          C      Sigma
## 0.06818306 0.90206794 0.76334113
```

```r
colMeans(fit.16.12.5.384.lambda4$cse)
```

```
##          K          C      Sigma
## 0.06818306 0.41542549 0.36177254
```

```r
colMeans(fit.16.12.5.384.lambda4$init.ai)
```

```
##          K          C      Sigma
## 0.06818306 0.24884379 0.25686814
```

```r
colMeans(fit.16.12.5.384.lambda4$init.chol)
```

```
##          K          C      Sigma
## 0.06818306 0.25120927 0.25686814
```

```r
colMeans(fit.16.12.5.384.lambda4$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.06570237 0.24455042 0.25152637
```

```r
colMeans(fit.16.12.5.384.lambda4$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.06568136 0.24647748 0.25142503
```

```r
##### boxplots of the relative spectral norms for K, C, and Sigma across 100 iterations
##### under different choices of the sample size $n$ and the value of non-spiked eigenvalue $\lambda$.
### K

K.analy.16.12.5.lambda1=plot.K.analy(fit.16.12.5.24.lambda1,fit.16.12.5.48.lambda1,fit.16.12.5.96.lambda
K.analy.16.12.5.lambda2=plot.K.analy(fit.16.12.5.24.lambda2,fit.16.12.5.48.lambda2,fit.16.12.5.96.lambda
K.analy.16.12.5.lambda3=plot.K.analy(fit.16.12.5.24.lambda3,fit.16.12.5.48.lambda3,fit.16.12.5.96.lambda
K.analy.16.12.5.lambda4=plot.K.analy(fit.16.12.5.24.lambda4,fit.16.12.5.48.lambda4,fit.16.12.5.96.lambda

K.analy.16.12.5=rbind(K.analy.16.12.5.lambda1,K.analy.16.12.5.lambda2,K.analy.16.12.5.lambda3,K.analy.16
K.analy.16.12.5.lambda=factor(K.analy.16.12.5$lambda,levels=c("lambda1","lambda2","lambda3","lambda4"))
levels(K.analy.16.12.5.lambda)=c(TeX("$\\lambda=0.2$"),TeX("$\\lambda=0.4$"),TeX("$\\lambda=0.6$"),TeX(
K.analy.16.12.5$lambda=K.analy.16.12.5.lambda

png("./numerical.simul_gfcore/K.16.12.5.png",width=900,height=600)
ggplot(K.analy.16.12.5,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambda,la
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("Separable Component Consistency : $(p_1,p_2,r)=(16,12,5)$"))+theme(plot.title=element_tex
dev.off()
```

```
## pdf
##   2
```

```r
### C

C.analy.16.12.5.lambda1=plot.C.analy(fit.16.12.5.24.lambda1,fit.16.12.5.48.lambda1,fit.16.12.5.96.lambda
C.analy.16.12.5.lambda2=plot.C.analy(fit.16.12.5.24.lambda2,fit.16.12.5.48.lambda2,fit.16.12.5.96.lambda
C.analy.16.12.5.lambda3=plot.C.analy(fit.16.12.5.24.lambda3,fit.16.12.5.48.lambda3,fit.16.12.5.96.lambda
C.analy.16.12.5.lambda4=plot.C.analy(fit.16.12.5.24.lambda4,fit.16.12.5.48.lambda4,fit.16.12.5.96.lambda

C.analy.16.12.5=rbind(C.analy.16.12.5.lambda1,C.analy.16.12.5.lambda2,C.analy.16.12.5.lambda3,C.analy.16
C.analy.16.12.5.lambda=factor(C.analy.16.12.5$lambda,levels=c("lambda1","lambda2","lambda3","lambda4"))
levels(C.analy.16.12.5.lambda)=c(TeX("$\\lambda=0.2$"),TeX("$\\lambda=0.4$"),TeX("$\\lambda=0.6$"),TeX(
C.analy.16.12.5$lambda=C.analy.16.12.5.lambda

png("./numerical.simul_gfcore/C.16.12.5.png",width=900,height=600)
ggplot(C.analy.16.12.5,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambda,la
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("Core Component Consistency : $(p_1,p_2,r)=(16,12,5)$"))+theme(plot.title=element_text(hju
dev.off()
```

```
## pdf
```

```
##   2
```

```
### Sigma

Sigma.analy.16.12.5.lambda1=plot.Sigma.analy(fit.16.12.5.24.lambda1,fit.16.12.5.48.lambda1,fit.16.12.5.9
Sigma.analy.16.12.5.lambda2=plot.Sigma.analy(fit.16.12.5.24.lambda2,fit.16.12.5.48.lambda2,fit.16.12.5.9
Sigma.analy.16.12.5.lambda3=plot.Sigma.analy(fit.16.12.5.24.lambda3,fit.16.12.5.48.lambda3,fit.16.12.5.9
Sigma.analy.16.12.5.lambda4=plot.Sigma.analy(fit.16.12.5.24.lambda4,fit.16.12.5.48.lambda4,fit.16.12.5.9

Sigma.analy.16.12.5=rbind(Sigma.analy.16.12.5.lambda1,Sigma.analy.16.12.5.lambda2,Sigma.analy.16.12.5.la
Sigma.analy.16.12.5.lambda=factor(Sigma.analy.16.12.5$lambda,levels=c("lambda1","lambda2","lambda3","lam
levels(Sigma.analy.16.12.5.lambda)=c(TeX("$\\lambda=0.2$"),TeX("$\\lambda=0.4$"),TeX("$\\lambda=0.6$"),T
Sigma.analy.16.12.5$lambda=Sigma.analy.16.12.5.lambda

png("./numerical.simul_gfcore/Sigma.16.12.5.png",width=900,height=600)
ggplot(Sigma.analy.16.12.5,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambd
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("Sigma Consistency : $(p_1,p_2,r)=(16,12,5)$"))+theme(plot.title=element_text(hjust=0.5))
dev.off()
```

```
## pdf
##   2
```

(p1,p2,r)=(18,8,5)

```
##### mean of the relative spectral norms for $K$, $C$, and $\Sigma$ (in order) across 100 iterations
##### under different choices of the sample size $n$ and the value of non-spiked eigenvalue $\lambda$.
###lambda=0.2
### n=p/8
fit.18.8.5.18.lambda1=stat.analy(pi.core.fit.18.8.5.18.lambda1)
colMeans(fit.18.8.5.18.lambda1$kro.mle)
```

```
##         K         C     Sigma
## 0.4144783 0.9638268 0.9099092
```

```
colMeans(fit.18.8.5.18.lambda1$cse)
```

```
##         K         C     Sigma
## 0.4144783 0.5369506 0.6936303
```

```
colMeans(fit.18.8.5.18.lambda1$init.ai)
```

```
##         K         C     Sigma
## 0.4144783 0.5110386 0.7482357
```

```
colMeans(fit.18.8.5.18.lambda1$init.chol)
```

```
##         K         C     Sigma
## 0.4144783 0.5545233 0.7482357
```

```
colMeans(fit.18.8.5.18.lambda1$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.2936365 0.3924721 0.4602142
```

```
colMeans(fit.18.8.5.18.lambda1$fit.chol.analy)
```

```
##         K         C     Sigma
```

```
## 0.2942171 0.4073266 0.4563046
```

```
### n=p/4
fit.18.8.5.36.lambda1=stat.analy(pi.core.fit.18.8.5.36.lambda1)
colMeans(fit.18.8.5.36.lambda1$kro.mle)
```

```
##         K         C     Sigma
## 0.2760968 0.9638268 0.9076500
```

```
colMeans(fit.18.8.5.36.lambda1$cse)
```

```
##         K         C     Sigma
## 0.2760968 0.4232770 0.5076847
```

```
colMeans(fit.18.8.5.36.lambda1$init.ai)
```

```
##         K         C     Sigma
## 0.2760968 0.3531599 0.5232493
```

```
colMeans(fit.18.8.5.36.lambda1$init.chol)
```

```
##         K         C     Sigma
## 0.2760968 0.3818112 0.5232493
```

```
colMeans(fit.18.8.5.36.lambda1$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.2078077 0.2322876 0.3115257
```

```
colMeans(fit.18.8.5.36.lambda1$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.2090947 0.2498853 0.3158213
```

```
### n=p/2
fit.18.8.5.72.lambda1=stat.analy(pi.core.fit.18.8.5.72.lambda1)
colMeans(fit.18.8.5.72.lambda1$kro.mle)
```

```
##         K         C     Sigma
## 0.1818612 0.9638268 0.9088839
```

```
colMeans(fit.18.8.5.72.lambda1$cse)
```

```
##         K         C     Sigma
## 0.1818612 0.3243982 0.3822353
```

```
colMeans(fit.18.8.5.72.lambda1$init.ai)
```

```
##         K         C     Sigma
## 0.1818612 0.2511521 0.3626540
```

```
colMeans(fit.18.8.5.72.lambda1$init.chol)
```

```
##         K         C     Sigma
## 0.1818612 0.2710691 0.3626540
```

```
colMeans(fit.18.8.5.72.lambda1$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1349511 0.1548443 0.2114134
```

```
colMeans(fit.18.8.5.72.lambda1$fit.chol.analy)
```

```
##         K         C     Sigma
```

```
## 0.1377418 0.1718151 0.2201081
```

```
### n=p
fit.18.8.5.144.lambda1=stat.analy(pi.core.fit.18.8.5.144.lambda1)
colMeans(fit.18.8.5.144.lambda1$kro.mle)
```

```
##         K         C     Sigma
## 0.1292909 0.9638268 0.9070238
```

```
colMeans(fit.18.8.5.144.lambda1$cse)
```

```
##         K         C     Sigma
## 0.1292909 0.2261573 0.2731532
```

```
colMeans(fit.18.8.5.144.lambda1$init.ai)
```

```
##         K         C     Sigma
## 0.1292909 0.1716378 0.2531209
```

```
colMeans(fit.18.8.5.144.lambda1$init.chol)
```

```
##         K         C     Sigma
## 0.1292909 0.1848695 0.2531209
```

```
colMeans(fit.18.8.5.144.lambda1$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.09809259 0.10594491 0.14970630
```

```
colMeans(fit.18.8.5.144.lambda1$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1016628 0.1208852 0.1611105
```

```
### n=2p
fit.18.8.5.288.lambda1=stat.analy(pi.core.fit.18.8.5.288.lambda1)
colMeans(fit.18.8.5.288.lambda1$kro.mle)
```

```
##          K          C      Sigma
## 0.09276499 0.96382683 0.90741171
```

```
colMeans(fit.18.8.5.288.lambda1$cse)
```

```
##          K          C      Sigma
## 0.09276499 0.15927818 0.19766765
```

```
colMeans(fit.18.8.5.288.lambda1$init.ai)
```

```
##          K          C      Sigma
## 0.09276499 0.12248783 0.17721627
```

```
colMeans(fit.18.8.5.288.lambda1$init.chol)
```

```
##          K          C      Sigma
## 0.09276499 0.13231778 0.17721627
```

```
colMeans(fit.18.8.5.288.lambda1$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.07223125 0.07843427 0.11286105
```

```
colMeans(fit.18.8.5.288.lambda1$fit.chol.analy)
```

```
##          K          C      Sigma
```

```
## 0.07712413 0.09271966 0.12463097
```

```
###lambda=0.4
### n=p/8
fit.18.8.5.18.lambda2=stat.analy(pi.core.fit.18.8.5.18.lambda2)
colMeans(fit.18.8.5.18.lambda2$kro.mle)
```

```
##         K         C     Sigma
## 0.4057991 0.9523191 0.8819741
```

```
colMeans(fit.18.8.5.18.lambda2$cse)
```

```
##         K         C     Sigma
## 0.4057991 0.6365949 0.6861166
```

```
colMeans(fit.18.8.5.18.lambda2$init.ai)
```

```
##         K         C     Sigma
## 0.4057991 0.6648972 0.8144429
```

```
colMeans(fit.18.8.5.18.lambda2$init.chol)
```

```
##         K         C     Sigma
## 0.4057991 0.7008474 0.8144429
```

```
colMeans(fit.18.8.5.18.lambda2$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.3323418 0.7127249 0.7176785
```

```
colMeans(fit.18.8.5.18.lambda2$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.3268141 0.7153985 0.6932218
```

```
### n=p/4
fit.18.8.5.36.lambda2=stat.analy(pi.core.fit.18.8.5.36.lambda2)
colMeans(fit.18.8.5.36.lambda2$kro.mle)
```

```
##         K         C     Sigma
## 0.2700714 0.9523191 0.8791812
```

```
colMeans(fit.18.8.5.36.lambda2$cse)
```

```
##         K         C     Sigma
## 0.2700714 0.5473665 0.5570896
```

```
colMeans(fit.18.8.5.36.lambda2$init.ai)
```

```
##         K         C     Sigma
## 0.2700714 0.4407324 0.5521882
```

```
colMeans(fit.18.8.5.36.lambda2$init.chol)
```

```
##         K         C     Sigma
## 0.2700714 0.4640202 0.5521882
```

```
colMeans(fit.18.8.5.36.lambda2$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.2263432 0.3871662 0.4265793
```

```r
colMeans(fit.18.8.5.36.lambda2$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.2278224 0.3993820 0.4247423
```

```r
### n=p/2
fit.18.8.5.72.lambda2=stat.analy(pi.core.fit.18.8.5.72.lambda2)
colMeans(fit.18.8.5.72.lambda2$kro.mle)
```

```
##         K         C     Sigma
## 0.1791041 0.9523191 0.8802166
```

```r
colMeans(fit.18.8.5.72.lambda2$cse)
```

```
##         K         C     Sigma
## 0.1791041 0.4466527 0.4565753
```

```r
colMeans(fit.18.8.5.72.lambda2$init.ai)
```

```
##         K         C     Sigma
## 0.1791041 0.3064962 0.3773873
```

```r
colMeans(fit.18.8.5.72.lambda2$init.chol)
```

```
##         K         C     Sigma
## 0.1791041 0.3220549 0.3773873
```

```r
colMeans(fit.18.8.5.72.lambda2$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1513856 0.2450460 0.2827887
```

```r
colMeans(fit.18.8.5.72.lambda2$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1506006 0.2535995 0.2783266
```

```r
### n=p
fit.18.8.5.144.lambda2=stat.analy(pi.core.fit.18.8.5.144.lambda2)
colMeans(fit.18.8.5.144.lambda2$kro.mle)
```

```
##         K         C     Sigma
## 0.1263310 0.9523191 0.8779679
```

```r
colMeans(fit.18.8.5.144.lambda2$cse)
```

```
##         K         C     Sigma
## 0.1263310 0.3263830 0.3383861
```

```r
colMeans(fit.18.8.5.144.lambda2$init.ai)
```

```
##         K         C     Sigma
## 0.1263310 0.2096952 0.2629223
```

```r
colMeans(fit.18.8.5.144.lambda2$init.chol)
```

```
##         K         C     Sigma
## 0.1263310 0.2195196 0.2629223
```

```r
colMeans(fit.18.8.5.144.lambda2$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1083000 0.1641962 0.1956011
```

```r
colMeans(fit.18.8.5.144.lambda2$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1083325 0.1714500 0.1949438
```

```r
### n=2p
fit.18.8.5.288.lambda2=stat.analy(pi.core.fit.18.8.5.288.lambda2)
colMeans(fit.18.8.5.288.lambda2$kro.mle)
```

```
##          K          C      Sigma
## 0.09209303 0.95231913 0.87836726
```

```r
colMeans(fit.18.8.5.288.lambda2$cse)
```

```
##          K          C      Sigma
## 0.09209303 0.23310270 0.24608279
```

```r
colMeans(fit.18.8.5.288.lambda2$init.ai)
```

```
##          K          C      Sigma
## 0.09209303 0.14831331 0.18505801
```

```r
colMeans(fit.18.8.5.288.lambda2$init.chol)
```

```
##          K          C      Sigma
## 0.09209303 0.15577123 0.18505801
```

```r
colMeans(fit.18.8.5.288.lambda2$fit.ai.analy)
```

```
##          K          C      Sigma
## 0.07909111 0.11615252 0.14261143
```

```r
colMeans(fit.18.8.5.288.lambda2$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.07931654 0.12171446 0.14271111
```

```r
###lambda=0.6
### n=p/8
fit.18.8.5.18.lambda3=stat.analy(pi.core.fit.18.8.5.18.lambda3)
colMeans(fit.18.8.5.18.lambda3$kro.mle)
```

```
##         K         C     Sigma
## 0.4021752 0.9301092 0.8298895
```

```r
colMeans(fit.18.8.5.18.lambda3$cse)
```

```
##         K         C     Sigma
## 0.4021752 0.7147875 0.6952151
```

```r
colMeans(fit.18.8.5.18.lambda3$init.ai)
```

```
##         K         C     Sigma
## 0.4021752 0.9693051 1.0104817
```

```r
colMeans(fit.18.8.5.18.lambda3$init.chol)
```

```
##         K         C     Sigma
## 0.4021752 0.9949916 1.0104817
```

```r
colMeans(fit.18.8.5.18.lambda3$fit.ai.analy)
```

```
##         K         C     Sigma
```

```
## 0.3736036 1.1031034 1.0567785
```

```
colMeans(fit.18.8.5.18.lambda3$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.3718204 1.1481597 1.0754789
```

```
### n=p/4
fit.18.8.5.36.lambda3=stat.analy(pi.core.fit.18.8.5.36.lambda3)
colMeans(fit.18.8.5.36.lambda3$kro.mle)
```

```
##         K         C     Sigma
## 0.2678544 0.9301092 0.8258900
```

```
colMeans(fit.18.8.5.36.lambda3$cse)
```

```
##         K         C     Sigma
## 0.2678544 0.6536238 0.6146057
```

```
colMeans(fit.18.8.5.36.lambda3$init.ai)
```

```
##         K         C     Sigma
## 0.2678544 0.6074925 0.6437303
```

```
colMeans(fit.18.8.5.36.lambda3$init.chol)
```

```
##         K         C     Sigma
## 0.2678544 0.6246319 0.6437303
```

```
colMeans(fit.18.8.5.36.lambda3$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.2444111 0.6550876 0.6436697
```

```
colMeans(fit.18.8.5.36.lambda3$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.2441419 0.6622649 0.6414292
```

```
### n=p/2
fit.18.8.5.72.lambda3=stat.analy(pi.core.fit.18.8.5.72.lambda3)
colMeans(fit.18.8.5.72.lambda3$kro.mle)
```

```
##         K         C     Sigma
## 0.1786960 0.9301092 0.8259245
```

```
colMeans(fit.18.8.5.72.lambda3$cse)
```

```
##         K         C     Sigma
## 0.1786960 0.5613413 0.5271455
```

```
colMeans(fit.18.8.5.72.lambda3$init.ai)
```

```
##         K         C     Sigma
## 0.1786960 0.4014418 0.4247961
```

```
colMeans(fit.18.8.5.72.lambda3$init.chol)
```

```
##         K         C     Sigma
## 0.1786960 0.4129383 0.4247961
```

```
colMeans(fit.18.8.5.72.lambda3$fit.ai.analy)
```

```
##         K         C     Sigma
```

```
## 0.1614605 0.3826035 0.3845216
```

```
colMeans(fit.18.8.5.72.lambda3$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1615469 0.3907311 0.3839501
```

```
### n=p
fit.18.8.5.144.lambda3=stat.analy(pi.core.fit.18.8.5.144.lambda3)
colMeans(fit.18.8.5.144.lambda3$kro.mle)
```

```
##         K         C     Sigma
## 0.1259822 0.9301092 0.8229050
```

```
colMeans(fit.18.8.5.144.lambda3$cse)
```

```
##         K         C     Sigma
## 0.1259822 0.4310073 0.4092298
```

```
colMeans(fit.18.8.5.144.lambda3$init.ai)
```

```
##         K         C     Sigma
## 0.1259822 0.2711719 0.2938958
```

```
colMeans(fit.18.8.5.144.lambda3$init.chol)
```

```
##         K         C     Sigma
## 0.1259822 0.2782223 0.2938958
```

```
colMeans(fit.18.8.5.144.lambda3$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1163362 0.2490371 0.2619241
```

```
colMeans(fit.18.8.5.144.lambda3$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1163361 0.2547225 0.2619640
```

```
### n=2p
fit.18.8.5.288.lambda3=stat.analy(pi.core.fit.18.8.5.288.lambda3)
colMeans(fit.18.8.5.288.lambda3$kro.mle)
```

```
##          K          C      Sigma
## 0.09152124 0.93010925 0.82320524
```

```
colMeans(fit.18.8.5.288.lambda3$cse)
```

```
##          K          C      Sigma
## 0.09152124 0.31660668 0.30349057
```

```
colMeans(fit.18.8.5.288.lambda3$init.ai)
```

```
##          K          C      Sigma
## 0.09152124 0.19085376 0.21021350
```

```
colMeans(fit.18.8.5.288.lambda3$init.chol)
```

```
##          K          C      Sigma
## 0.09152124 0.19647315 0.21021350
```

```
colMeans(fit.18.8.5.288.lambda3$fit.ai.analy)
```

```
##          K          C      Sigma
```

```
## 0.08397158 0.17354527 0.19325987
```

```
colMeans(fit.18.8.5.288.lambda3$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.08393267 0.17755959 0.19317341
```

```
###lambda=0.8
### n=p/8
fit.18.8.5.18.lambda4=stat.analy(pi.core.fit.18.8.5.18.lambda4)
colMeans(fit.18.8.5.18.lambda4$kro.mle)
```

```
##         K         C     Sigma
## 0.4020173 0.8697154 0.7045681
```

```
colMeans(fit.18.8.5.18.lambda4$cse)
```

```
##         K         C     Sigma
## 0.4020173 0.7560302 0.6553992
```

```
colMeans(fit.18.8.5.18.lambda4$init.ai)
```

```
##         K         C     Sigma
## 0.4020173 1.7584930 1.5358708
```

```
colMeans(fit.18.8.5.18.lambda4$init.chol)
```

```
##         K         C     Sigma
## 0.4020173 1.7712599 1.5358708
```

```
colMeans(fit.18.8.5.18.lambda4$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.3726658 1.7827162 1.4502935
```

```
colMeans(fit.18.8.5.18.lambda4$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.3779525 1.8247102 1.4893652
```

```
### n=p/4
fit.18.8.5.36.lambda4=stat.analy(pi.core.fit.18.8.5.36.lambda4)
colMeans(fit.18.8.5.36.lambda4$kro.mle)
```

```
##         K         C     Sigma
## 0.2681783 0.8697154 0.6956824
```

```
colMeans(fit.18.8.5.36.lambda4$cse)
```

```
##         K         C     Sigma
## 0.2681783 0.7238837 0.6062452
```

```
colMeans(fit.18.8.5.36.lambda4$init.ai)
```

```
##         K         C     Sigma
## 0.2681783 1.0801473 0.9188810
```

```
colMeans(fit.18.8.5.36.lambda4$init.chol)
```

```
##         K         C     Sigma
## 0.2681783 1.0893361 0.9188810
```

```r
colMeans(fit.18.8.5.36.lambda4$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.2601817 1.1308325 0.9241194
```

```r
colMeans(fit.18.8.5.36.lambda4$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.2609636 1.1471661 0.9355235
```

```r
### n=p/2
fit.18.8.5.72.lambda4=stat.analy(pi.core.fit.18.8.5.72.lambda4)
colMeans(fit.18.8.5.72.lambda4$kro.mle)
```

```
##         K         C     Sigma
## 0.1790717 0.8697154 0.6902683
```

```r
colMeans(fit.18.8.5.72.lambda4$cse)
```

```
##         K         C     Sigma
## 0.1790717 0.6515751 0.5398188
```

```r
colMeans(fit.18.8.5.72.lambda4$init.ai)
```

```
##         K         C     Sigma
## 0.1790717 0.6783766 0.5910795
```

```r
colMeans(fit.18.8.5.72.lambda4$init.chol)
```

```
##         K         C     Sigma
## 0.1790717 0.6850662 0.5910795
```

```r
colMeans(fit.18.8.5.72.lambda4$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1722436 0.7420626 0.6265671
```

```r
colMeans(fit.18.8.5.72.lambda4$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1736625 0.7471213 0.6258211
```

```r
### n=p
fit.18.8.5.144.lambda4=stat.analy(pi.core.fit.18.8.5.144.lambda4)
colMeans(fit.18.8.5.144.lambda4$kro.mle)
```

```
##         K         C     Sigma
## 0.1272286 0.8697154 0.6840101
```

```r
colMeans(fit.18.8.5.144.lambda4$cse)
```

```
##         K         C     Sigma
## 0.1272286 0.5293522 0.4432124
```

```r
colMeans(fit.18.8.5.144.lambda4$init.ai)
```

```
##         K         C     Sigma
## 0.1272286 0.4378673 0.3901200
```

```r
colMeans(fit.18.8.5.144.lambda4$init.chol)
```

```
##         K         C     Sigma
## 0.1272286 0.4415391 0.3901200
```

```r
colMeans(fit.18.8.5.144.lambda4$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.1241308 0.4475903 0.3922296
```

```r
colMeans(fit.18.8.5.144.lambda4$fit.chol.analy)
```

```
##         K         C     Sigma
## 0.1240305 0.4494564 0.3905622
```

```r
### n=2p
fit.18.8.5.288.lambda4=stat.analy(pi.core.fit.18.8.5.288.lambda4)
colMeans(fit.18.8.5.288.lambda4$kro.mle)
```

```
##          K          C      Sigma
## 0.09088225 0.86971541 0.68370502
```

```r
colMeans(fit.18.8.5.288.lambda4$cse)
```

```
##          K          C      Sigma
## 0.09088225 0.40560740 0.34441138
```

```r
colMeans(fit.18.8.5.288.lambda4$init.ai)
```

```
##          K          C      Sigma
## 0.09088225 0.32840957 0.29652039
```

```r
colMeans(fit.18.8.5.288.lambda4$init.chol)
```

```
##          K          C      Sigma
## 0.09088225 0.32938751 0.29652039
```

```r
colMeans(fit.18.8.5.288.lambda4$fit.ai.analy)
```

```
##         K         C     Sigma
## 0.0881248 0.3256475 0.2956644
```

```r
colMeans(fit.18.8.5.288.lambda4$fit.chol.analy)
```

```
##          K          C      Sigma
## 0.08809078 0.32644097 0.29551683
```

```r
##### boxplots of the relative spectral norms for K, C, and Sigma across 100 iterations
##### under different choices of the sample size $n$ and the value of non-spiked eigenvalue $\lambda$.
### K

K.analy.18.8.5.lambda1=plot.K.analy(fit.18.8.5.18.lambda1,fit.18.8.5.36.lambda1,fit.18.8.5.72.lambda1,fi
K.analy.18.8.5.lambda2=plot.K.analy(fit.18.8.5.18.lambda2,fit.18.8.5.36.lambda2,fit.18.8.5.72.lambda2,fi
K.analy.18.8.5.lambda3=plot.K.analy(fit.18.8.5.18.lambda3,fit.18.8.5.36.lambda3,fit.18.8.5.72.lambda3,fi
K.analy.18.8.5.lambda4=plot.K.analy(fit.18.8.5.18.lambda4,fit.18.8.5.36.lambda4,fit.18.8.5.72.lambda4,fi

K.analy.18.8.5=rbind(K.analy.18.8.5.lambda1,K.analy.18.8.5.lambda2,K.analy.18.8.5.lambda3,K.analy.18.8.5
K.analy.18.8.5.lambda=factor(K.analy.18.8.5$lambda,levels=c("lambda1","lambda2","lambda3","lambda4"))
levels(K.analy.18.8.5.lambda)=c(TeX("$\\lambda=0.2$"),TeX("$\\lambda=0.4$"),TeX("$\\lambda=0.6$"),TeX("$
K.analy.18.8.5$lambda=K.analy.18.8.5.lambda

png("./numerical.simul_gfcore/K.18.8.5.png",width=900,height=600)
ggplot(K.analy.18.8.5,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambda,lab
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
```

```
  ggtitle(TeX("Separable Component Consistency : $(p_1,p_2,r)=(18,8,5)$"))+theme(plot.title=element_tex
dev.off()
```

```
## pdf
##   2
```

```
### C

C.analy.18.8.5.lambda1=plot.C.analy(fit.18.8.5.18.lambda1,fit.18.8.5.36.lambda1,fit.18.8.5.72.lambda1,fi
C.analy.18.8.5.lambda2=plot.C.analy(fit.18.8.5.18.lambda2,fit.18.8.5.36.lambda2,fit.18.8.5.72.lambda2,fi
C.analy.18.8.5.lambda3=plot.C.analy(fit.18.8.5.18.lambda3,fit.18.8.5.36.lambda3,fit.18.8.5.72.lambda3,fi
C.analy.18.8.5.lambda4=plot.C.analy(fit.18.8.5.18.lambda4,fit.18.8.5.36.lambda4,fit.18.8.5.72.lambda4,fi

C.analy.18.8.5=rbind(C.analy.18.8.5.lambda1,C.analy.18.8.5.lambda2,C.analy.18.8.5.lambda3,C.analy.18.8.5
C.analy.18.8.5.lambda=factor(C.analy.18.8.5$lambda,levels=c("lambda1","lambda2","lambda3","lambda4"))
levels(C.analy.18.8.5.lambda)=c(TeX("$\\lambda=0.2$"),TeX("$\\lambda=0.4$"),TeX("$\\lambda=0.6$"),TeX("$
C.analy.18.8.5$lambda=C.analy.18.8.5.lambda

png("./numerical.simul_gfcore/C.18.8.5.png",width=900,height=600)
ggplot(C.analy.18.8.5,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambda,lab
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("Core Component Consistency : $(p_1,p_2,r)=(18,8,5)$"))+theme(plot.title=element_text(hjus
dev.off()
```

```
## pdf
##   2
```

```
### Sigma

Sigma.analy.18.8.5.lambda1=plot.Sigma.analy(fit.18.8.5.18.lambda1,fit.18.8.5.36.lambda1,fit.18.8.5.72.la
Sigma.analy.18.8.5.lambda2=plot.Sigma.analy(fit.18.8.5.18.lambda2,fit.18.8.5.36.lambda2,fit.18.8.5.72.la
Sigma.analy.18.8.5.lambda3=plot.Sigma.analy(fit.18.8.5.18.lambda3,fit.18.8.5.36.lambda3,fit.18.8.5.72.la
Sigma.analy.18.8.5.lambda4=plot.Sigma.analy(fit.18.8.5.18.lambda4,fit.18.8.5.36.lambda4,fit.18.8.5.72.la

Sigma.analy.18.8.5=rbind(Sigma.analy.18.8.5.lambda1,Sigma.analy.18.8.5.lambda2,Sigma.analy.18.8.5.lambda
Sigma.analy.18.8.5.lambda=factor(Sigma.analy.18.8.5$lambda,levels=c("lambda1","lambda2","lambda3","lambd
levels(Sigma.analy.18.8.5.lambda)=c(TeX("$\\lambda=0.2$"),TeX("$\\lambda=0.4$"),TeX("$\\lambda=0.6$"),Te
Sigma.analy.18.8.5$lambda=Sigma.analy.18.8.5.lambda

png("./numerical.simul_gfcore/Sigma.18.8.5.png",width=900,height=600)
ggplot(Sigma.analy.18.8.5,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambda
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("Sigma Consistency : $(p_1,p_2,r)=(18,8,5)$"))+theme(plot.title=element_text(hjust=0.5))
dev.off()
```

```
## pdf
##   2
```

Combining figures

```
### K

K1=ggplot(K.analy.16.12.3,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambda
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("$K$ Consistency : $(p_1,p_2,r)=(16,12,3)$"))+theme(plot.title=element_text(hjust=0.5))
```

```
K2=ggplot(K.analy.16.12.5,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambda
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("$K$ Consistency : $(p_1,p_2,r)=(16,12,5)$"))+theme(plot.title=element_text(hjust=0.5))
K3=ggplot(K.analy.18.8.3,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambda
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("$K$ Consistency : $(p_1,p_2,r)=(18,8,3)$"))+theme(plot.title=element_text(hjust=0.5))
K4=ggplot(K.analy.18.8.5,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambda
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("$K$ Consistency : $(p_1,p_2,r)=(18,8,5)$"))+theme(plot.title=element_text(hjust=0.5))

png("./numerical.simul_gfcore/K.rank3.png",width=900,height=600)
ggarrange(K1,K3,ncol=2,common.legend=TRUE)
dev.off()
```

```
## pdf
##   2
```

```
png("./numerical.simul_gfcore/K.rank5.png",width=900,height=600)
ggarrange(K2,K4,ncol=2,common.legend=TRUE)
dev.off()
```

```
## pdf
##   2
```

```
### C

C1=ggplot(C.analy.16.12.3,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambda
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("$C$ Consistency : $(p_1,p_2,r)=(16,12,3)$"))+theme(plot.title=element_text(hjust=0.5))
C2=ggplot(C.analy.16.12.5,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambda
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("$C$ Consistency : $(p_1,p_2,r)=(16,12,5)$"))+theme(plot.title=element_text(hjust=0.5))
C3=ggplot(C.analy.18.8.3,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambda
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("$C$ Consistency : $(p_1,p_2,r)=(18,8,3)$"))+theme(plot.title=element_text(hjust=0.5))
C4=ggplot(C.analy.18.8.5,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(.~lambda
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("$C$ Consistency : $(p_1,p_2,r)=(18,8,5)$"))+theme(plot.title=element_text(hjust=0.5))

png("./numerical.simul_gfcore/C.rank3.png",width=900,height=600)
ggarrange(C1,C3,ncol=2,common.legend=TRUE)
dev.off()
```

```
## pdf
##   2
```

```
png("./numerical.simul_gfcore/C.rank5.png",width=900,height=600)
ggarrange(C2,C4,ncol=2,common.legend=TRUE)
dev.off()
```

```
## pdf
##   2
```

```
### Sigma

sigma1=ggplot(Sigma.analy.16.12.3,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
```

```
  ggtitle(TeX("$\\Sigma$ Consistency : $(p_1,p_2,r)=(16,12,3)$"))+theme(plot.title=element_text(hjust=0
sigma2=ggplot(Sigma.analy.16.12.5,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("$\\Sigma$ Consistency : $(p_1,p_2,r)=(16,12,5)$"))+theme(plot.title=element_text(hjust=0
sigma3=ggplot(Sigma.analy.18.8.3,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("$\\Sigma$ Consistency : $(p_1,p_2,r)=(18,8,3)$"))+theme(plot.title=element_text(hjust=0.5
sigma4=ggplot(Sigma.analy.18.8.5,mapping=aes(x=samp.size,y=stat,col=Method))+geom_boxplot()+facet_wrap(
  theme_bw()+xlab("Sample Size")+ylab("Relative Norm")+
  ggtitle(TeX("$\\Sigma$ Consistency : $(p_1,p_2,r)=(18,8,5)$"))+theme(plot.title=element_text(hjust=0.5

png("./numerical.simul_gfcore/Sigma.rank3.png",width=900,height=600)
ggarrange(sigma1,sigma3,ncol=2,common.legend=TRUE)
dev.off()
```

```
## pdf
##   2
```

```
png("./numerical.simul_gfcore/Sigma.rank5.png",width=900,height=600)
ggarrange(sigma2,sigma4,ncol=2,common.legend=TRUE)
dev.off()
```

```
## pdf
##   2
```

C Consistency for KMLE

```
### Consistency of KMLE with respect to C
### In fact, this is given by ||C-I||/||C|| for true C

library(RSpectra)
load("./numerical.simul_gfcore/para.16.12.3.rds")
load("./numerical.simul_gfcore/para.18.8.3.rds")
load("./numerical.simul_gfcore/para.16.12.5.rds")
load("./numerical.simul_gfcore/para.18.8.5.rds")

## (p1,p2,r)=(16,12,3)
A=para.16.12.3$A; D=para.16.12.3$D

svds(0.8*A%*%t(A)+0.2*D-diag(192),1)$d/svds(0.8*A%*%t(A)+0.2*D,1)$d
```

```
## [1] 0.9838769
```

```
svds(0.6*A%*%t(A)+0.4*D-diag(192),1)$d/svds(0.6*A%*%t(A)+0.4*D,1)$d
```

```
## [1] 0.9786279
```

```
svds(0.4*A%*%t(A)+0.6*D-diag(192),1)$d/svds(0.4*A%*%t(A)+0.6*D,1)$d
```

```
## [1] 0.9683142
```

```
svds(0.2*A%*%t(A)+0.8*D-diag(192),1)$d/svds(0.2*A%*%t(A)+0.8*D,1)$d
```

```
## [1] 0.9388075
```

```
## (p1,p2,r)=(18,8,3)
A=para.18.8.3$A; D=para.18.8.3$D
```

```
svds(0.8*A%*%t(A)+0.2*D-diag(144),1)$d/svds(0.8*A%*%t(A)+0.2*D,1)$d
```

```
## [1] 0.9772074
```

```
svds(0.6*A%*%t(A)+0.4*D-diag(144),1)$d/svds(0.6*A%*%t(A)+0.4*D,1)$d
```

```
## [1] 0.9698372
```

```
svds(0.4*A%*%t(A)+0.6*D-diag(144),1)$d/svds(0.4*A%*%t(A)+0.6*D,1)$d
```

```
## [1] 0.9554282
```

```
svds(0.2*A%*%t(A)+0.8*D-diag(144),1)$d/svds(0.2*A%*%t(A)+0.8*D,1)$d
```

```
## [1] 0.9147403
```

```
## (p1,p2,r)=(16,12,5)
A=para.16.12.5$A; D=para.16.12.5$D

svds(0.8*A%*%t(A)+0.2*D-diag(192),1)$d/svds(0.8*A%*%t(A)+0.2*D,1)$d
```

```
## [1] 0.9734961
```

```
svds(0.6*A%*%t(A)+0.4*D-diag(192),1)$d/svds(0.6*A%*%t(A)+0.4*D,1)$d
```

```
## [1] 0.9649725
```

```
svds(0.4*A%*%t(A)+0.6*D-diag(192),1)$d/svds(0.4*A%*%t(A)+0.6*D,1)$d
```

```
## [1] 0.9483784
```

```
svds(0.2*A%*%t(A)+0.8*D-diag(192),1)$d/svds(0.2*A%*%t(A)+0.8*D,1)$d
```

```
## [1] 0.9020679
```

```
## (p1,p2,r)=(18,8,5)
A=para.18.8.5$A; D=para.18.8.5$D

svds(0.8*A%*%t(A)+0.2*D-diag(144),1)$d/svds(0.8*A%*%t(A)+0.2*D,1)$d
```

```
## [1] 0.9638268
```

```
svds(0.6*A%*%t(A)+0.4*D-diag(144),1)$d/svds(0.6*A%*%t(A)+0.4*D,1)$d
```

```
## [1] 0.9523191
```

```
svds(0.4*A%*%t(A)+0.6*D-diag(144),1)$d/svds(0.4*A%*%t(A)+0.6*D,1)$d
```

```
## [1] 0.9301092
```

```
svds(0.2*A%*%t(A)+0.8*D-diag(144),1)$d/svds(0.2*A%*%t(A)+0.8*D,1)$d
```

```
## [1] 0.8697154
```