Lab Report 8

By Seunghyun Park 1003105855 & Juann Jeon 1005210166
Question 1 done by Juann Jeon
Question 2 done by Seunghyun Park
Lab Report done by Seunghyun Park & Juann Jeon

Q1 a)

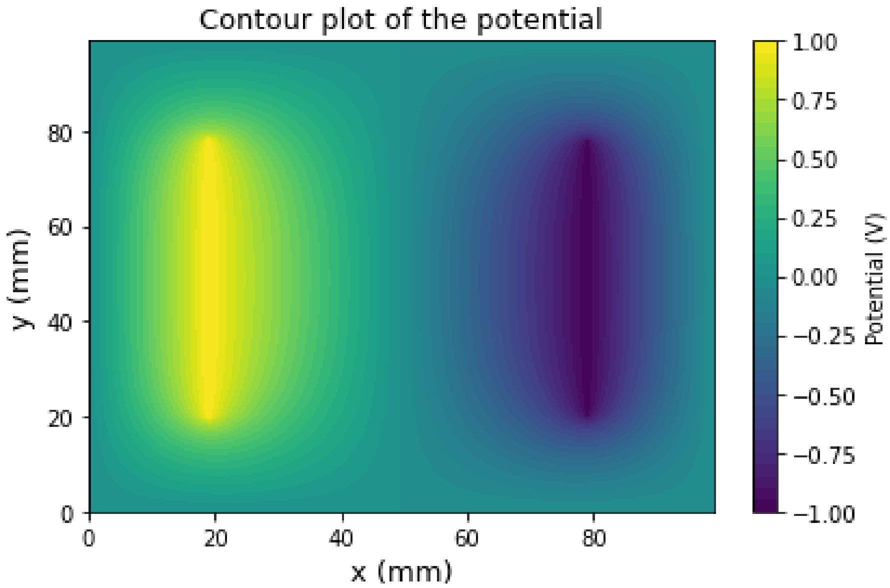

Figure 1: Contour plot of the potential of 2D electronic capacitor using Gauss-Seidal method

Q1 b)

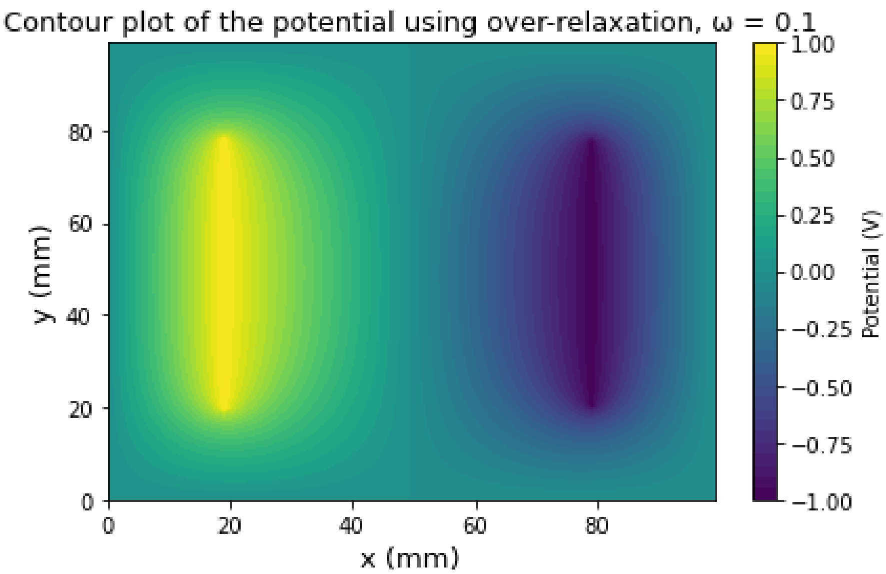

Figure 2: Contour plot of the potential of 2D electronic capacitor with over-relaxation $\omega = 0.1$

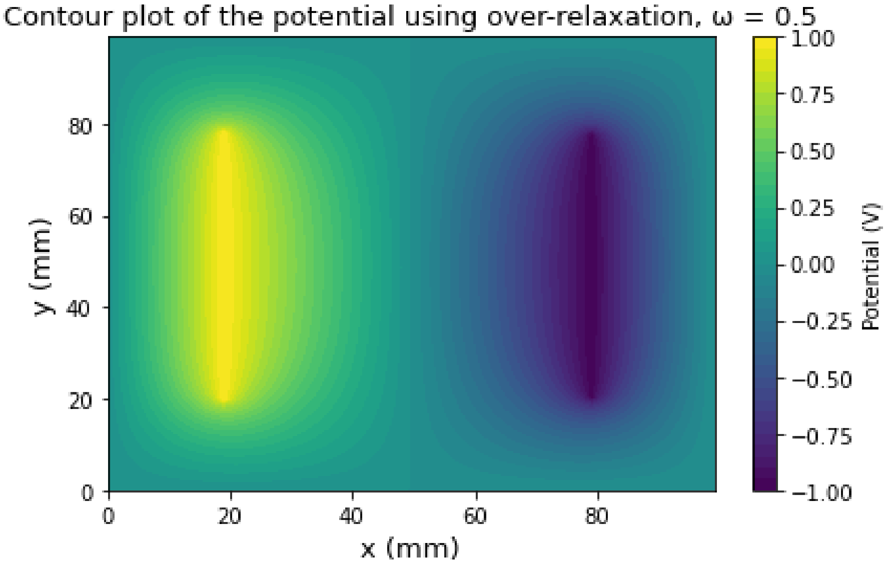

Figure 3: Contour plot of the potential of 2D electronic capacitor with over-relaxation $\omega = 0.5$

```
Total time it took for Q 1a) (in s): 15.919002532958984
Total time it took for omega = 0.1 (in s): 15.561467170715332
Total time it took for omega = 0.5 (in s): 6.342304944992065
```

Figure 4: Calculation time it took for Figure 1, 2 and 3 in respective order

Figure 2 and 3 shows the contour map of the potential energy with overrelaxation $\omega = 0.1$ (for Figure 2) and $\omega = 0.5$ (for Figure 3). Notice that Figure 2 and 3 have the same contour map as Figure 1, which is without over relaxation. The difference is in the time it takes for calculation; over relaxation is used to shorten the time it takes for calculation. Figure 4 shows the total time it took for Figure 1, 2 and 3 in respective order. Notice that Figure 1 took $\approx 15.92s$ to finish the calculation. Figure 2 took $\approx 15.56s$ which is slightly less compared to the time it took for Figure 1, but by not much. Figure 3 took $\approx 6.34s$ for calculation, which shows that overrelaxation indeed shortens the time it takes for calculation without deviating from the final result compared to without overrelaxation.

Q2 a)
Equation (6):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial (uh)}{\partial x} = 0$$

where

$$h = \eta - \eta_b \ , \ \ \vec{F}\ (u, \eta) = [\frac{1}{2}u^2 + g\eta, (\eta - \eta_b)\, u)]$$

Take partial derivative respect to x,

$$\frac{\partial \vec{F}\ (u,\eta)}{\partial x} = \left[\frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + g\eta\right), \frac{\partial}{\partial x}((\eta - \eta_b)\, u)\right]$$

$$= \left[u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x}, \frac{\partial\,(u\,(\eta - \eta_b))}{\partial x}\right]$$

From equation (6),

$$\frac{\partial u}{\partial t} = -\left(u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x}\right)$$

$$\frac{\partial \eta}{\partial t} = -\frac{\partial\,(uh)}{\partial x}$$

$$= -\frac{\partial\,(u\,(\eta - \eta_b))}{\partial x}$$

Since $\vec{u} = (u, \eta)$,

$$\frac{\partial \vec{u}}{\partial t} = [\frac{\partial u}{\partial t}, \frac{\partial \eta}{\partial t}]$$

$$\frac{\partial \vec{u}}{\partial t} = [-\left(u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x}\right), -\frac{\partial\,(u\,(\eta - \eta_b))}{\partial x}]$$

$$\therefore\ \frac{\partial \vec{u}}{\partial t} = -\frac{\partial \vec{F}\ (u,\eta)}{\partial x}$$

The forward-time centred-space scheme is

$$\frac{\partial u}{dt}|_j^n \approx \frac{1}{\Delta t}\left(u_j^{n+1} - u j^n\right)$$

$$\frac{\partial F}{\partial x}|_j^n \approx \frac{1}{2\Delta x}\left(F_{j+1}^n - F_{j-1}^n\right)$$

.

Substituting these approximations into the equation from previous part

$$\frac{\partial \overrightarrow{u}}{\partial t} = -\frac{\partial \overrightarrow{F}\ (u,\eta)}{\partial x}\ ,$$

then

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left(F(u)_{j+1}^n - F(u)_{j-1}^n\right)$$

$$\eta_j^{n+1} = \eta_j^n - \frac{\Delta t}{2\Delta x}\left(F(\eta)_{j+1}^n - F(\eta)_{j-1}^n\right)$$

$$\therefore \quad u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}(\frac{1}{2}\left(u_{j+1}^n\right)^2 + g\eta_{j+1}^n - \frac{1}{2}\left(u_{j-1}^n\right)^2 - g\eta_{j-1}^n)$$

$$\eta_j^{n+1} = \eta_j^n - \frac{\Delta t}{2\Delta x}(u_{j+1}^n(\eta_{j+1}^n - \eta_{b,j+1}) - (u_{j-1}^n(\eta_{j-1}^n - \eta_{b,j-1}))$$

Q2 (b)

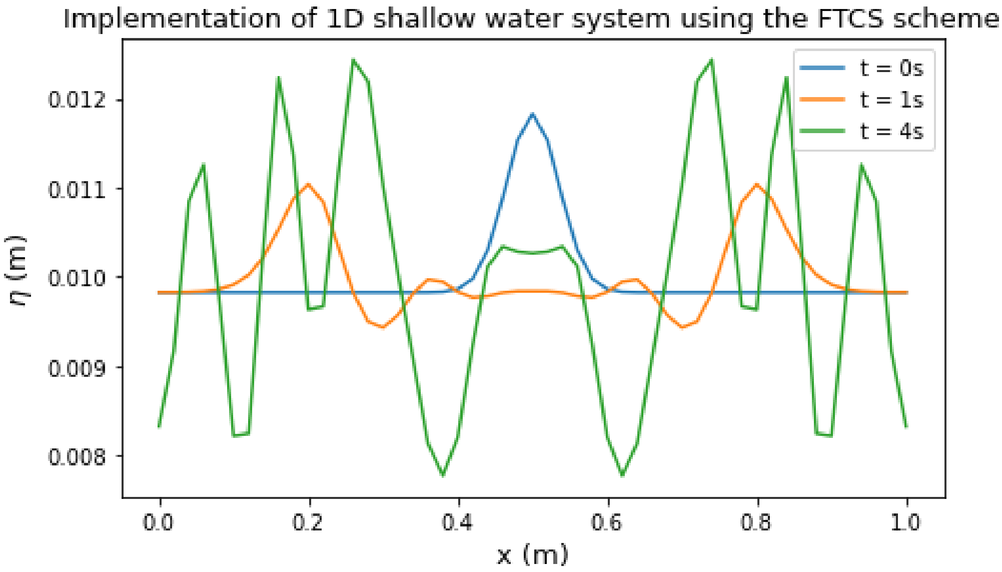

Figure 5: Implementation of the 1D shallow water system at given times using the FTCS scheme

Q2 (c)

Equation (6):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial (uh)}{\partial x} = 0$$

The linearized equation (6) about $(u, \eta) = (0, H)$ with $\eta_b = 0$ is

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial (u(\eta - \eta_b))}{\partial x} = 0$$

$$\frac{\partial \eta}{\partial t} + (\eta - \eta_b)\frac{\partial u}{\partial x} + u\frac{\partial (\eta - \eta_b)}{\partial x} = 0$$

$$\frac{\partial \eta}{\partial t} = -H\frac{\partial u}{\partial x}$$

$$\therefore \quad \frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial \eta}{\partial t} = -H\frac{\partial u}{\partial x}$$

The spatial varaion can be expressed in the form of a Fourier series.

$u(x,t) = \sum_{k=1}^{\infty} c_u(t)e^{ikx}$

$\eta(x,t) = \sum_{k=1}^{\infty} c_\eta(t))e^{ikx}$

Since the equation is linear, we can only study what happens to each term in the series.

$$
\begin{aligned}
u(x,t) &= \textstyle\sum_{k=1}^{\infty} c_u(t)e^{ikx} \\
u(x,t+h) &= c_u(t+h)e^{ikx} \\
&= c_u(t)e^{ikx} - \frac{\Delta t}{2\Delta x}(gc_\eta(t)e^{ik(x+\Delta x)} - gc_\eta(t)e^{ik(x-\Delta x)}) \\
&= c_u(t)e^{ikx} - \frac{g\Delta t}{2\Delta x}(c_\eta(t)e^{ik(x+\Delta x)} - c_\eta(t)e^{ik(x-\Delta x)}) \\
&= [c_u(t) - \frac{g\Delta t}{2\Delta x}c_\eta(t)(e^{ik\Delta x} - e^{-ik\Delta x})]e^{ikx} \\
\therefore c_u(t+h) &= c_u(t) - \frac{g\Delta t}{2\Delta x}c_\eta(t)(e^{ik\Delta x} - e^{-ik\Delta x}) \\
\eta(x,t) &= \textstyle\sum_{k=1}^{\infty} c_\eta(t)e^{ikx} \\
\eta(x,t+h) &= c_\eta(t+h)e^{ikx} \\
&= c_\eta(t)e^{ikx} - \frac{\Delta t}{2\Delta x}(Hc_u(t)e^{ik(x+\Delta x)} - Hc_u(t)e^{ik(x-\Delta x)}) \\
&= c_\eta(t)e^{ikx} - \frac{H\Delta t}{2\Delta x}(c_u(t)e^{ik(x+\Delta x)} - c_u(t)e^{ik(x-\Delta x)}) \\
&= [c_\eta(t) - \frac{H\Delta t}{2\Delta x}c_u(t)(e^{ik\Delta x} - e^{-ik\Delta x})]e^{ikx} \\
\therefore c_\eta(t+h) &= c_\eta(t) - \frac{H\Delta t}{2\Delta x}c_u(t)(e^{ik\Delta x} - e^{-ik\Delta x})
\end{aligned}
$$

$\mathbf{c}(t+h) = \mathbf{A}\mathbf{c}(t)$

Where $\mathbf{c}(t)$ is the vector (c_u, c_η) and $\mathbf{A}$ is the matrix

$$
\mathbf{A} = \begin{pmatrix} 1 & -\frac{g\Delta t}{2\Delta x}(e^{ik\Delta x} - e^{-ik\Delta x}) \\ -\frac{H\Delta t}{2\Delta x}(e^{ik\Delta x} - e^{-ik\Delta x}) & 1 \end{pmatrix}
$$

$$
\mathbf{A} = \begin{pmatrix} 1 & -i\frac{g\Delta t}{\Delta x}sin(k\Delta x) \\ -i\frac{H\Delta t}{\Delta x}sin(k\Delta x) & 1 \end{pmatrix}
$$

The determinant of the matrix $\mathbf{A}$ is,

$$det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -i\frac{g\Delta t}{\Delta x}sin(k\Delta x) \\ -i\frac{H\Delta t}{\Delta x}sin(k\Delta x) & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 + gH(\frac{\Delta t}{\Delta x})^2 sin^2(k\Delta x) = 0$$

$$\lambda = 1 \pm \sqrt{gH(\frac{\Delta t}{\Delta x})^2 sin^2(k\Delta x)}$$

$$\therefore |\lambda| = \sqrt{1 + gH\frac{\Delta t^2}{\Delta x^2}sin^2(k\Delta x)}$$

The magnitude of the eigenvalues λ is never less than one. This implies the Fourier component grows exponentially as it gets repeatedly multiplied by the same factor on each time-step. Therefore, the FTCS method is unstable for the 1D shallow water system.