1 The noise spectral density

The output will be a combination of a true GW signal and of noise as follows :

$$s(t) = h(t) + n(t),$$
$$n(t) \sim N(0, \sigma^2).$$

- Definition of auto correlation function of the noise and the noise spectral density

  $R(\tau) \equiv cov(n(t+\tau), n(t)) = \mathbb{E}[n(t+\tau), n(t)].$

  $\frac{1}{2}S_n(f) \equiv \int_{-\infty}^{\infty} R(\tau)e^{2\pi i f\tau}d\tau.$

- The ensemble average of the Fourier components of the noise

$$\tilde{n}(f) := \int_{-\infty}^{\infty} n(t)e^{-2\pi i f t}dt,\ \tilde{n}^*(f) := \int_{-\infty}^{\infty} n(t)e^{2\pi i f t}dt.$$

$$\begin{aligned}
\langle \tilde{n}^*(f), \tilde{n}(f') \rangle &= \left\langle \int_{-\infty}^{\infty} n(t)e^{2\pi i f t}dt \int_{-\infty}^{\infty} n(t') e^{-2\pi i f' t'}dt' \right\rangle \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle n(t)n(t') \rangle e^{2\pi i f t - 2\pi i f' t'}dtdt' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(t-t') e^{2\pi i f t - 2\pi i f' t'}dtdt',\ (\tau = t - t' \Rightarrow t = \tau + t') \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(\tau)e^{2\pi i f(\tau + t') - 2\pi i f' t'}d\tau dt' \\
&= \int_{-\infty}^{\infty} R(\tau)e^{2\pi i f\tau}d\tau \int_{-\infty}^{\infty} e^{2\pi i (f - f')t'}dt \\
&= \delta(f - f')\frac{1}{2}S_n(f),
\end{aligned}$$

where $\langle \cdot \rangle$ denotes the expectation symbol, and Dirac delta function is defined by

$$\delta(f) = \int_{-\infty}^{\infty} e^{2\pi i f t}dt.$$

2 Matched filtering

- What we want : Digging out the GW signal from a much larger noise.

$$\int_0^T S(t)h(t)dt = \int_0^T (n(t)+h(t))h(t)dt = \int_0^T h(t)^2 dt + \int_0^T n(t)h(t)dt. \tag{1}$$

  The equation (1) means that when $s(t)$ is multiplied by $h(t)$ and integrated with respect to t, the true signal $h(t)$ becomes stronger than the cross-correlation between the noise and the signal. Therefore, if we can appropriately find $h(t)$, we can enhance the signal relative to the noise.

- Definitions

  $\hat{s} = \int_{-\infty}^{\infty} s(t)k(t)dt$, where $k(t)$ is called filter function.

  S is the expectation value of $\hat{s}$ when the true signal is present.

  N is the root mean square of $\hat{s}^2$ when the true signal is absent.

$$\begin{aligned}
S &= \mathbb{E}\left[\int_{-\infty}^{\infty} s(t)k(t)dt\right] \\
&= \mathbb{E}\left[\int_{-\infty}^{\infty} n(t)k(t) + n(t)k(t)dt\right] \\
&= \int_{-\infty}^{\infty} \mathbb{E}(n(t))k(t)dt + \int_{-\infty}^{\infty} h(t)k(t)dt \\
&= \int_{-\infty}^{\infty} h(t)k(t)dt, \quad (\because n(t)=0) \\
&= \int \left[\int \tilde{h}(f)e^{2\pi i f t}df \int \tilde{k}\left(f'\right)e^{2\pi i f' t}df'\right] dt, \\
&\quad \left(\because h(t) = \int \tilde{h}(f)e^{2\pi i f t}df, k(t) = \int \tilde{k}\left(f'\right)e^{2\pi \pi f' t}df'\right) \\
&= \iiint \tilde{h}(f)\tilde{k}\left(f'\right)e^{2\pi i\left(f+f'\right)t}df df' dt \\
&= \iint \tilde{h}(f)\tilde{k}\left(f'\right)df df' \int e^{2\pi i\left(f+f'\right)t}dt \\
&= \iint \tilde{h}(f)\tilde{k}\left(f'\right)\delta\left(f+f'\right)df df' \\
&= \int \tilde{h}(f) \int \tilde{k}\left(f'\right)\delta\left(f+f'\right)df' \\
&= \int \tilde{h}(f)\tilde{k}(-f)df \\
&= \int \tilde{h}(f)\tilde{k}^*(f)df.
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}\left(\hat{s}^2\right) &= \mathbb{E}\left[\int s(t)k(t)dt \int s\left(t'\right)k\left(t'\right)dt'\right] \\
&= \mathbb{E}\left[\int n(t)k(t)dt \int n\left(t'\right)k\left(t'\right)dt'\right] \\
&= \iint \mathbb{E}\left(n(t)n\left(t'\right)\right)k(t)k\left(t'\right)dtdt' \\
&\quad \left(\because \mathbb{E}\left[n(t)n\left(t'\right)\right] = \mathbb{E}\left[\int \tilde{n}^*(f)e^{-2\pi ift}df \int \tilde{n}\left(f'\right)e^{2\pi f't'}df'\right]\right) \\
&= \iint \mathbb{E}\left[\tilde{n}^*(f)\tilde{n}\left(f'\right)\right]\int k(t)e^{-2\pi ift}dt \int k\left(t'\right)e^{2\pi if't}dt'dfdf' \\
&= \iint \delta\left(f-f'\right)\frac{1}{2}S_n(f)\tilde{k}(f)\tilde{k}^*\left(f'\right)dfdf' \\
&= \int \frac{1}{2}S_n(f)\tilde{k}(f)\int \tilde{k}^*\left(f'\right)\delta\left(f-f'\right)df'df \\
&= \int \frac{1}{2}S_n(f)\tilde{k}(f)\tilde{k}^*(f)df \\
&= \int \frac{1}{2}S_n(f)|\tilde{k}(f)|^2df,
\end{aligned}
$$

where $|\tilde{x}(f)|^2 := \tilde{x}(f)\cdot\tilde{x}^*(f)$.

$$
\therefore \quad \frac{S}{N} = \frac{\int \tilde{h}(f)\tilde{k}^*(f)df}{\left(\int \frac{1}{2}S_n(f)|\tilde{k}(f)|^2df\right)^{1/2}}.
$$

3 Probability and statistics

- The distribution of the noise $n(t)$

$$
\begin{gathered}
s(t) = h(t) + n(t), \\
n(t) \sim N(0,\sigma^2), \quad \text{where } \sigma^2 := \left\langle n(t)^2\right\rangle = \frac{1}{2}\int S_n(f)df.
\end{gathered}
$$

- The distribution of $\tilde{n}(f)$

  WTS : $\tilde{n}(f)$ follows a normal distribution.

$$
\tilde{n}(f) = \int n(t)e^{-2\pi ift}dt = \lim_{\Delta t\to 0}\sum_k n\left(t_k\right)e^{-2\pi ift_k}\Delta t.
$$

  $\tilde{n}(f)$ is a linear combination of $n(t_k)$ following the normal distribution, so $\tilde{n}(f)$ follows a normal distribution.

- Mean and Variance of $\tilde{n}(f)$

$$
\begin{aligned}
\mathbb{E}\left[\tilde{n}(f)\right] &= \mathbb{E}\left[\int n(t) e^{-2\pi i f t} dt\right] = \int \mathbb{E}[n(t)] e^{-2\pi i f t} dt = 0. \\
\mathrm{var}\left[\tilde{n}(f)\right] &= \mathbb{E}\left[\tilde{n}(f)\tilde{n}^*(f)\right] - \mathbb{E}\left[\tilde{n}(f)\right]\mathbb{E}\left[\tilde{n}^*(f)\right] = \langle \tilde{n}(f), \tilde{n}^*(f)\rangle = S_n(f)/2. \\
\therefore \tilde{n}(f) &\sim \mathcal{N}\left(0, \frac{S_n(f)}{2}\right).
\end{aligned}
$$

- Derive the log likelihood function

$$
\begin{aligned}
L &= \prod_f \mathrm{Prob}(\tilde{n}(f)) = \prod_f \left[\frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{S_n(f)}} \cdot \exp\left\{-\frac{1}{2}\frac{|\tilde{n}(f)|^2}{\frac{2\cdot S_n(f)}{2}}\right\}\right] \\
&= \prod_f \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{S_n(f)}} \cdot \exp\left\{\int_f -\frac{1}{2}\frac{|\tilde{n}(f)|^2}{S_n(f)} df\right\} \\
&= N \cdot \exp\left\{\int_f -\frac{1}{2}\frac{|\tilde{n}(f)|^2}{S_n(f)/2} df\right\} \\
&= N \cdot \exp\left\{-\frac{1}{2}\left(\tilde{n}(f)|\tilde{n}(f)\right)\right\} \\
&= N \cdot \exp\left\{-\frac{1}{2}\left(s - h(\theta_t)|s - h(\theta_t)\right)\right\} \\
&= N \cdot \exp\left\{-\frac{1}{2}\cdot 4\,\mathrm{Re}\left(\int_0^\infty \frac{\left(s - h(\theta_t)\right)^* \left(s - h(\theta_t)\right)}{S_n(f)/2} df\right)\right\} \\
&= N \cdot \exp\left\{-\frac{1}{2}\cdot 4\,\mathrm{Re}\left(\int_0^\infty \frac{s^* s - h(\theta_t)^* s - s^* h(\theta_t) + h(\theta_t)^* h(\theta_t)}{S_n(f)/2} df\right)\right\} \\
&= N \cdot \exp\left\{-\frac{1}{2}(s|s) + (h(\theta_t)|s) - \frac{(h(\theta_t)|h(\theta_t))}{2}\right\} \\
&\propto N \cdot \exp\left\{(h(\theta_t)|s) - \frac{1}{2}\left(h(\theta_t)|h(\theta_t)\right)\right\}.
\end{aligned}
$$

$$
\therefore \theta_t = \arg\max\left(h(\theta_t \mid s) - \frac{1}{2}\left(h(\theta_t) \mid h(\theta_t)\right)\right).
$$

Here, N is a normalization constant, and $(\cdot|\cdot)$ is defined by

$$
(A|B) = \mathrm{Re}\int_{-\infty}^{\infty} df \frac{\tilde{A}^*(f)\tilde{B}(f)}{(1/2)S_n(f)} = 4\,\mathrm{Re}\int_0^\infty df \frac{\tilde{A}^*(f)\tilde{B}(f)}{S_n(f)}.
$$

*** To ensure that the point where the objective function's derivative equals zero is indeed a maximum, it is essential to check the convexity of the objective function.***

- Maximum likelihood estimator (MLE) of θ_t is equivalent to the value that provides the highest signal-to-noise ratio in matched filtering.

- Maximum posterior probability

  MLE : the estimator that maximizes the **log likelihood function**.

  Here, we use the estimator maximizing the **posterior probability** $p(\theta_t|s)$.

  (CAUTION : Likelihood is not equal to probability.)

Proposition 1 *If there is $\theta = (\theta_1, \theta_2)$, and the joint probability of θ_1 and θ_2 is flat, then*

$$\arg\max_{\theta_1} p(\theta_1, \theta_2 \mid s) = \arg\max_{\theta_1} \int p(\theta_1, \theta_2 \mid s)\, d\theta_2.$$

- Bayes estimator

  $\hat{\theta}^i_B$ is the Bayes estimator of the i-th parameter.

$$\hat{\theta}^i_B \equiv \mathbb{E}\left[\theta^i \mid s\right] = \int \theta^i\, p(\theta^i \mid s)\, d\theta^i = \int_\theta \theta^i\, p(\theta \mid s)\, d\theta.$$

4 Matched filtering statistics

- What we want

  Investigating what is the statistical significance of the fact that we found events at a given level of signal-to-noise ratio.

- Two kinds of noise

  Gaussian noise : The probability of extremely large values occurring is low, and most of the values are distributed near 0. Therefore, we can eliminate them through a hard thresholding.

  Non-Gaussian noise : generated from a heavy-tailed distribution and is sometimes mistaken for a meaningful event.

- The definition of the signal-to-noise

$$(\text{signal-to-noise}) = \rho := \frac{\hat{s}}{N}, \quad \text{where} \quad \hat{s} = \int_{-\infty}^{\infty} s(t)k(t)\, dt = \int_{-\infty}^{\infty} \left(h(t) + n(t)\right) k(t)\, dt.$$

- The probability density function (PDF) of ρ_0 in the absence of a GW signal

$$
\begin{gathered}
\rho_0 = \frac{\hat{s}}{N} = \frac{\int n(t)\, k(t)\, dt}{N} = \frac{\int n(t)\, k(t)\, dt}{\langle \hat{s}^2(t) \rangle}, \\
\rho_0 \sim N(0, 1^2), \\
P(\rho_0 \mid h = 0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\rho_0^2}{2}}.
\end{gathered}
$$

- The PDF of ρ with a true signal-to-noise ratio $\bar{\rho}$

$$
\begin{gathered}
\rho = \frac{\int (h(t) + n(t)) k(t)\, dt}{N} = \frac{\int h(t) k(t)\, dt}{N} + \frac{\int n(t) k(t)\, dt}{N} = \overline{\rho} + \rho_0, \\
\rho - \overline{\rho} \sim N(0, 1^2) \iff \rho \sim N\left(\overline{\rho}, 1^2\right), \\
P(\rho \mid \overline{\rho}) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(\rho - \overline{\rho})^2}{2}}.
\end{gathered}
$$

- The PDF of the signal-to-noise ratio in energy ($R \equiv \rho^2$)

$$
\begin{aligned}
P(R \mid \overline{R}) &= P_\rho\left(\sqrt{R} \mid \overline{\rho}\right) \left| \frac{d\sqrt{R}}{dR} \right| + P_\rho\left(-\sqrt{R} \mid \overline{\rho}\right) \left| \frac{d(-\sqrt{R})}{dR} \right| \\
&= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(\sqrt{R} - \overline{\rho})^2}{2}} \cdot \frac{1}{2\sqrt{R}} + \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(\sqrt{R} + \overline{\rho})^2}{2}} \cdot \frac{1}{2\sqrt{R}} \\
&= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2\sqrt{R}} \cdot \left(e^{-\frac{(\sqrt{R} - \overline{\rho})^2}{2}} + e^{-\frac{(\sqrt{R} + \overline{\rho})^2}{2}} \right).
\end{aligned}
$$

- The Mean of R : $\langle R \rangle = \langle \rho^2 \rangle = 1 + \bar{\rho}^2 = 1 + \bar{R}$.

* Recall

If a random variable X follows $N(\mu, \sigma^2)$, then $\mathbb{E}[X^2] = \mu^2 + \sigma^2$.

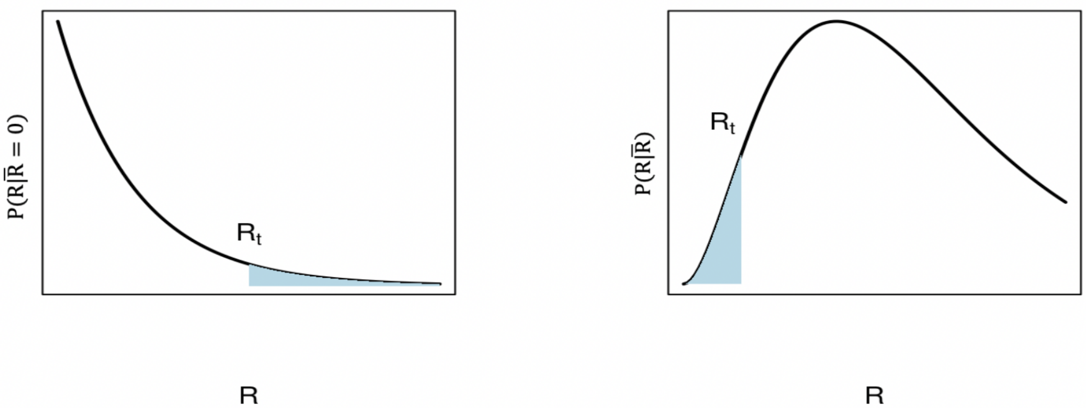

Figure 1: The left panel is the probability density function of R in the absence of a GW signal. In this panel, the blue area means false alarm probability. The right panel is the probability density function of R when a GW signal is present. In this panel, the blue area means false dismissal probability.

- False alarm probability

  The probability of observing an event that is considered a true signal, but actually consists of only noise.

$$P_{FA} = \int_{R_t}^{\infty} P(R \mid \bar{R} = 0) dR,$$

  where R_t is the threshold that is fixed deciding what is the maximum false alarm level that we are willing to tolerate.

- False dismissal probability

$$P_{FD} = \int_{0}^{R_t} P(R \mid \bar{R}) dR.$$

- The signal-to-noise ratio that consists of two components

  Assume that we have the signal-to-noise ratio which is a combination of x and y in quadrature, each one with its Gaussian noise.

$$\rho^2 = x^2 + y^2$$

- The PDF of $\rho^2 = x^2 + y^2$ in absence of a GW signal

$$p(x, y \mid h = 0) = p(x \mid h = 0) \cdot p(y \mid h = 0) = \frac{1}{2\pi} \cdot e^{-(x^2+y^2)/2}$$

$$x = \rho \cos\theta, \quad y = \rho \sin\theta, \quad |J| = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = \rho\cos^2\theta + \rho\sin^2\theta = \rho$$

$$p(\rho, \theta \mid h = 0) = \frac{1}{2\pi} \cdot e^{-(x^2+y^2)/2} = \frac{1}{2\pi} \cdot e^{-\rho^2/2} \cdot \rho$$

$$p(\rho \mid h = 0) = \int_0^{2\pi} p(\rho, \theta \mid h = 0)\, d\theta = \frac{1}{2\pi} \cdot e^{-\rho^2/2} \cdot \rho \int_0^{2\pi} 1\, d\theta = \rho \cdot e^{-\rho^2/2}$$

- The PDF of $\rho^2 = x^2 + y^2$ when a GW signal is present.

Derive it yourself by referring to the book!