$b_1 =$
$b_2 = b_2 - 3b_1$
$b_3 =$

Exercise 1A

1. Find $\underline{U}$ using forward elimination

$$\begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ \frac{3}{2} & \frac{13}{2} & 5 \\ 1 & 5 & 7 \end{bmatrix} \quad \underline{R_2 = R_2 - 3R_1} \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 2 & 2 \\ 1 & 5 & 7 \end{bmatrix} \quad \underline{R_3 = R_3 - 2R_1} \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix} \quad \underline{R_3 = R_3 - R_2}$$

$$\underline{\underline{U}} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

3 eleminations are done to find $\underline{U}$

$\underline{C} = T(\underline{b}) \leftarrow$ transformations on <b>

Write out $\underline{b}$ in terms of Elimination steps to construct $\underline{C}$, to satisfy the equation $\underline{U} \cdot \underline{x} = \underline{b}$

$$b = \begin{bmatrix} 6 \\ 24 \\ 21 \end{bmatrix} \quad \underline{C} = \begin{bmatrix} b_1 \\ b_2 - 3b_1 \\ b_3 - 2b_1 - b_2 \end{bmatrix} \quad \begin{matrix} 6 \\ (24 - 18) = 6 \\ 21 - 12 - 6 \rightarrow \quad 21 - 18 = 3 \end{matrix} \qquad C = \begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix}$$

$$\underline{\underline{L}} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

$\underline{L}$, though not Required here to Solve for $\underline{x}$.

Our objective is to solve for $\underline{x}$. We need to find c.

$$\underline{\underline{U}} \cdot \underline{x} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underline{C}$$

Plug $\underline{c}$ into $\underline{U} \cdot \underline{x} = \underline{C}$ to Solve $\underline{x}$.

$$\underline{U} \cdot \underline{x} = \underline{C} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix} =$$

using back Substitution

$\frac{1}{2}x_1 + \frac{3}{2}x_2 + x_3 = 6 \Rightarrow \cancel{\frac{1}{2}}(x_1) + \frac{3}{\cancel{2}}(\cancel{2}) + (\cancel{1}) = 6 - 1 - 3 \times 2 = 4 = x_1$

$2x_2 + 2x_3 = 6 \Rightarrow \cancel{2}x_2 + \cancel{2} = 6 - 2 / 2 = 2 = x_2$

$3x_3 = 3 \quad x_3 = 1$

$$\underline{x} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

Plug $\underline{x}$ into $\underline{A} \cdot \underline{x} = \underline{b}$ to verify if $\underline{x}$ is correct.

$$\underline{A} \cdot \underline{x} = \underline{b} \quad \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 1 \\ \frac{3}{2} & \frac{13}{2} & 5 \\ 1 & 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{2} + \frac{6}{2} + 1 \\ \frac{12}{2} + \frac{26}{2} + 5 \\ 4 + 10 + 7 \end{bmatrix} = \begin{bmatrix} 2 + 3 + 1 \\ 6 + 13 + 5 \\ 4 + 10 + 7 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \\ 21 \end{bmatrix} = \underline{b}$$

it is.

$$\underline{x} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

Write $f = x_1^2 + 10x_1x_2 + x_2^2$ as a difference of squares, $f = x_1^2 + 10x_1x2 + 30x_2^2$ as a sum of squares. What symmetric matriees correspond to these quadratic forms by $f = \underline{x}^T \cdot \underline{\underline{A}} \cdot \underline{x}$? Produce plots of the level curves of f for these two cases.

$f = a x_1^2 + 2b x_1 x_2 + c x_2^2$ $\qquad \underline{\underline{A}} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

1. Write $f = x_1^2 + 10x_1x_2 + x_2^2$ as a difference of squares.

$a=1 \qquad 10 = 2b \qquad c=1$

$b=5$

Using $f = a\left(x_1 + \frac{b}{a} x_2\right)^2 + \left(c - \frac{b^2}{a}\right) x_2^2$ to complete the square

$\frac{b}{a} = 5 \qquad c - \frac{b^2}{a} = 1 - \frac{25}{1} = -24$

$f = (x_1 + 5x_2)^2 - 24x_2^2$ is the difference of squares.

$\underline{\underline{A}} = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$

for $f = x_1^2 + 10x_1x_2 + 30x_2^2$

$c - \frac{b^2}{a} = 30 - \frac{25}{1} = 5$

$a=1 \qquad b=5 \qquad c=30$

$b/a = 5$

$f = (x_1 + 5x_2)^2 + 5x_2^2$ the sum of squares.

$\underline{\underline{A}} = \begin{bmatrix} 1 & 5 \\ 5 & 30 \end{bmatrix}$

Decide on the positive definiteness of the following matrices:

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \quad \underline{\underline{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Bulvan:

Write each of these matrices as $\sum \underline{l}_k d_k \underline{l}_k^T$ (LDL decomposition) and explain how you obtained the result.

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \quad \underline{\underline{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Both of these matrices have positive diagonals so they both <u>could</u> be positive definite because they meet that minimum requirement but positive definiteness requires positive pivots.

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\lambda = 1, 1, 1$ $\quad$ $\underline{\underline{A}}$ is positive definite, all pivots are positive.

$$\underline{\underline{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\lambda = 1, 0, 0$

$\underline{\underline{B}}$ is <u>Not</u> positive definite because it does not have only positive pivots.

$$\sum \underline{l}_k d_k \underline{l}_k^T$$

LDL Decomposition results from Python Script:

$\underline{\underline{A}}$: $\underline{\underline{L}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad \underline{\underline{D}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underline{\underline{L}}^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\sum \underline{l}_k d_k \underline{l}_k^T = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot 1 \cdot \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{1} \underset{+}{+} \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot 1 \cdot \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}}_{1} \underset{+}{+} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot 1 \cdot \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}_{1} = 3$$

$\underline{\underline{B}}$: $\underline{\underline{L}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \underline{\underline{D}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{\underline{L}}^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\sum \underline{l}_k d_k \underline{l}_k^T = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot 1 \cdot \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{1} \underset{+}{+} \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot 1 \cdot \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}}_{1} \underset{+}{+} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot 1 \cdot \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}}_{1} = 3$$

Using the output LDL decomposition of $\underline{\underline{A}}$ and $\underline{\underline{B}}$ I constructed the $\sum \underline{l}_k d_k \underline{l}_k^T$ for each, using the k column from $\underline{\underline{L}}$ dot the k pivot of d and the k column from $\underline{\underline{L}}$ transpose where $k = 1, 2, 3$ (or $0, 1, 2$).

The result is then found by computing the dot product for each term and computing the sum.

A function $F(x,y)$ has a local minimum at any point where its first derivatives vanish, and its Hessian matrix

$$\underline{\underline{\mathbf{H}}} = \begin{bmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{bmatrix}$$

is positive definite. Is this true for $F_1 = x^2 - x^2y^2 + y^2 + y^3$ and $F_2 = \cos x \cos y$ at $x = 0$ and $y = 0$? Does F_1 have a global minimum or can it approach $-\infty$?

$$F_1 = x^2 - x^2y^2 + y^2 + y^3$$

$$\frac{\partial F}{\partial x} = 2x - 2xy^2 \qquad \frac{\partial F}{\partial y} = -2yx^2 + 2y + 3y^2$$

$$\frac{\partial^2 F}{\partial x^2} = 2 - 2y^2 \qquad \frac{\partial^2 F}{\partial y^2} = -2x^2 + 2 + 6y$$

$$\frac{\partial^2 F}{\partial x \partial y} = -4xy \qquad \frac{\partial^2 F}{\partial y \partial x} = -4yx$$

$$H = \begin{bmatrix} 2-2y^2 & -4xy \\ -4yx & -2x^2 + 6y + 2 \end{bmatrix}$$

for $x = 0$, $y = 0$ $\quad H = \begin{bmatrix} 2-(0) & (0) \\ (0) & (0)+(0)+2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\frac{\partial F}{\partial x} = 2x - 2xy = 0$ where $x=0$ $y=0$

$\frac{\partial F}{\partial y} = -2yx^2 + 2y + 3y^2 = 0$ where $x=0$ $y=0$

The first derivitave of F_1 approaches zero at $x=0, y=0$ and the resulting Hessian matrix at that point is Positive Definite so F_1 has a local minimum at $x=0$ $y=0$.

F_1 does not have a global minimum. because of the y^3 term, F_1 can / does approach $\pm\infty$ even though f_1 @ x,y is a local minimum, this point likely is a saddle point.

$$F_2 = \cos x \cos y$$

$$\frac{\partial F_2}{\partial x} = -\sin(x)\cos(y) \qquad \frac{\partial F_2}{\partial y} = -\cos(x)\sin(y)$$

$$\frac{\partial^2 F_2}{\partial x^2} = -\cos(x)\cos(y) \qquad \frac{\partial^2 F_2}{\partial y^2} = -\cos(x)\cos(y)$$

$$\frac{\partial^2 F_2}{\partial x \partial y} = \sin(x)\sin(y) \qquad \frac{\partial^2 F_2}{\partial y \partial x} = \sin(x)\sin(y)$$

$$H = \begin{bmatrix} -\cos(x)\cos(y) & \sin(x)\sin(y) \\ \sin(x)\sin(y) & -\cos(x)\cos(y) \end{bmatrix}$$

for $x=0$ $y=0$

$$H = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Because the pivots of the Hessian matrix of F_2 are Negative, $(0,0)$ is Not a local minimum of F_2.

Question #7

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This paper uses Cholesky decomposition as a method to reduce the data footprint and computational resources required to process the outputs from various types of prediction algorithms. Specifically they use Cholesky decomposition in the context of 3D climate and numerical weather prediction applications.

They use this decomposition on positive definite covariance matrices, which by definition are symmetric. The large-scale nature of the solution to these models produces a dense memory footprint and a heavy computational demand. In an attempt to optimize this process, the authors using TLR Cholesky factorization to reduce the memory footprint from the dense covariance matrices by using “the data sparsity structure” of the off-diagonal tiles of the matrix using low-rank approximations. The “why” is structured around optimizing the utilization of computational resources.

The specifics of how they utilized the Cholesky factorization indicates that the “flavor” is more developed and utilizes other numerical and linear algebra techniques in order to conduct their study. They specifically use singular value decomposition to decay the singular values in the resulting covariance matrix (which I believe was an output component for a step involved in the TLR Cholesky factorization) into two sets of spatial locations in order to apply an admissibility condition which considers the distance between the two sets to their maximum diameters.

This process ultimately produces a directed acyclic graph (DAG) which maps the to the utilization of computation resources that are used to solve the model. They say that “L is the number of tasks along the critical path, D is all of the other operations except L”, and that the number of computational resources is represented by C. The resulting execution time is the maximum between L and D/C. Other specifics about the number of nodes that are utilized, and other useful computational resources are also identified using their methods.