Exercise 1 (based on Strang 1.6.9)

Describe the vectors (length, components) in the column space, nullspace, row space, and left nullspace of the matrix $\underline{\underline{\mathbf{A}}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

$\mathcal{C}(\underline{\underline{A}})$ $\underline{\underline{A}}\,\underline{x} = \underline{b}$

$\underline{\underline{A}}\underline{x} = b$ $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = b$ $x_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ x_1, x_4 = Free

$\mathcal{C}(\underline{\underline{A}}) = x_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Span$([\;],[\;])$, or $([\;]+[\;])$

Rank of $\underline{\underline{A}} = 2$ each $[3\times1]$ with one non-zero component in M-dimensional Subspace. each vector with length $\sqrt{1}$

$\mathcal{N}(\underline{\underline{A}}) = \underline{\underline{A}}\,\underline{x} = \underline{0}$ $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_2 = 0$

$x_3 = 0$

$x_1 = x_1$, $x_4 = x_4$

$\mathcal{N}(\underline{\underline{A}}) = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

rank of $N(\underline{\underline{A}}) = 2$. each vector is $[4\times1]$, one non-zero vector in N-Dimensional space. each vector has length $1/\sqrt{1}$

$\mathcal{C}(\underline{\underline{B}})$ $(\underline{\underline{B}} = \underline{\underline{A}}^T)$ $= \underline{\underline{B}}\,\underline{y} = \underline{f}$ 4×1 in terms of y

$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b$ $x_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $x_3 = x_3$ a free variable.

the Row space of $\underline{\underline{A}}$ is the Column Space of $\underline{\underline{B}}$ $\mathcal{C}(\underline{\underline{B}}) = x_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

each vector has 4 N-dimensional components, each with one non-zero component. each vector has length $\sqrt{1}$. Rank $(\underline{\underline{B}}) = 2$.

$\mathcal{N}(\underline{\underline{B}})$ $\underline{\underline{B}}\,\underline{y} = 0$ $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

3×1

$x_1 = 0$

$x_2 = 0$

$x_3 = x_3$

$x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\mathcal{N}(\underline{\underline{B}}) = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

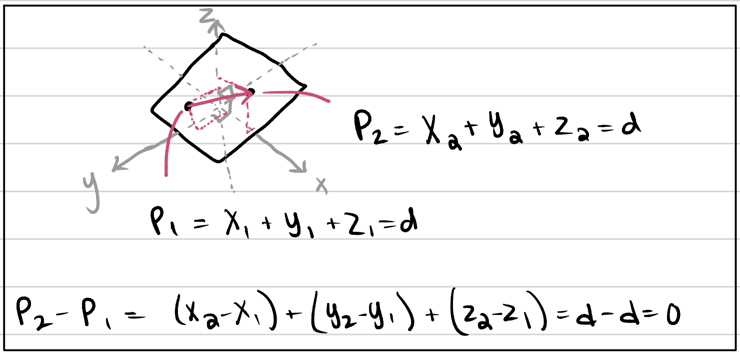
The rank of the left nullspace of $\underline{\underline{A}}$ is one. The vector is $[3\times1]$, with 3 Components in N dimensional space. one nonzero component and the length of the vector is $\sqrt{1}$.

Exercise 2 (based on Strang 1.6.17)

a. Show that the vector $[a, b, c]^T$ is perpendicular to the plane defined by the equation $ax + by + cz = d$. To show this, you could use the dot product of this vector with a vector in the plane.

b. Using a., derive the equation of a plane that is perpendicular to the vector $[1, 2, 3]^T$ and passes through the point $(x, y, z) = (1, 1, 1)$.

A vector in the plane $ax + by + cz = d$ can be defined by the vector connecting two points in that plane.



$$\underline{P} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} = 0$$

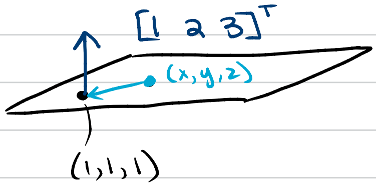
A: Show that $[a, b, c]^T$ is $\perp$ to $\underline{P}$:

ie: That $\begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} = 0$

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$$
$$a x_2 - a x_1 + b y_2 - b y_1 + c z_2 - c z_1 = 0$$
$$\underbrace{a x_2 + b y_2 + c z_2}_{d} - \underbrace{a x_1 - b y_1 - c z_1}_{d} = 0$$

because the plane is defined as $ax + by + cz = d$, two points in this plane, defined as $P_1 = x_1 + y_1 + z_1$ and $P_2 = x_2 + y_2 + z_2$ form a vector. A vector $[a \ b \ c]^T$ is shown to be $\perp$ to the plane because the dot product between $[a \ b \ c]^T$ and the vector in the plane equals zero.

B : A vector $[1 \;\; 2 \;\; 3]^T$ is $\perp$ to the plane that contains the point $(x,y,z) = (1,1,1)$. derive the equation for that plane.



$1+1+1 = d$ (a point on the plane)

To define the plane using the point $(1,1,1)$ we need a second point to form a vector in the plane which can be defined as $P_1 = (x_1, y_1, z_1)$

$P_1 = (x, y, z)$

$P_2 = (1, 1, 1)$

$P_2 - P_1 = 1-x + 1-y + 1-z = d-d = 0$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1-x \\ 1-y \\ 1-z \end{bmatrix} = 1(1-x) + 2(1-y) + 3(1-z) = 0$$

$$1-x + 2-2y + 3-3z = 0$$

$$+x \quad +2y \quad +3z \quad = +x + 2y + 3z$$

$$1 + 2 + 3 = x + 2y + 3z$$

$$6 = x + 2y + 3z$$

$$\boxed{x + 2y + 3z = 6}$$

Exercise 3

Derive the general formula for the inverse of the matrix $\underline{\underline{\mathbf{A}}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. What is the condition for the existence of the inverse based on this formula?

$$\underline{\underline{A}}^{-1} \cdot \underline{\underline{A}} = \underline{\underline{I}}$$

find $\underline{\underline{A}}^{-1}$: $\underline{\underline{A}}^{-1} = \langle \underline{A}_1^{-1}, \underline{A}_2^{-1} \rangle$

$$\underline{\underline{A}}^{-1} = \begin{bmatrix} x & w \\ y & z \end{bmatrix}$$

$$\underline{A}_1^{-1} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \underline{A}_2^{-1} = \begin{bmatrix} w \\ z \end{bmatrix}$$

$$\underline{\underline{A}} \times \underline{A}_1^{-1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{matrix} ax + by = 1 \\ cx + dy = 0 \end{matrix}$$

$$y = \frac{1-ax}{b} \quad x = -\frac{dy}{c}, \quad x = -\left(\frac{d - dax}{bc}\right) = \frac{-d}{bc} + \frac{dax}{bc}, \quad x - \frac{dax}{bc} = \frac{-d}{bc}, \quad x = -\left(\frac{\frac{d}{bc}}{1 - \frac{ad}{bc}}\right) = -\left(\frac{d}{bc\left(1 - \frac{ad}{bc}\right)}\right) = -\frac{d}{bc - ad} = \frac{d}{ad - bc} = x$$

$$y = \frac{1 - a\left(\frac{d}{ad-bc}\right)}{b} = \frac{1 - \frac{ad}{ad-bc}}{b} = \frac{ad - bc - ad}{(ad-bc)\,b} = -\frac{bc}{(ad-bc)\,b} = \frac{-c}{ad-bc} = y$$

$$\underline{\underline{A}} \times \underline{A}_2^{-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{matrix} aw + bz = 0 \\ cw + dz = 1 \end{matrix}$$

$$w = -\frac{bz}{a} \quad z = \frac{1 - cw}{d} \quad w = -\frac{b\left(\frac{1-cw}{d}\right)}{a} = -\frac{b - bcw}{ad} = \frac{bcw}{ad} - \frac{b}{ad} \Rightarrow w - \frac{bcw}{ad} = w\left(1 - \frac{bc}{ad}\right) = -\frac{b}{ad}$$

$$w = -\frac{b}{ad\left(1 - \frac{bc}{ad}\right)} = -\frac{b}{ad - bc} = \frac{-b}{ad-bc} = w \qquad z = \frac{1 - c\left(\frac{b}{bc - ad}\right)}{d} = \frac{bc - ad - bc}{(bc - ad)\,d} = \frac{a}{ad - bc} = z$$

$$\underline{\underline{A}}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Condition : The determinant of $\underline{\underline{A}}$ can not $= 0$

Exercise 4 (based on Strang 1.6.8)
Show that the conjugate transpose of the inverse of a matrix is the inverse of its conjugate transpose. Verify this (by hand) for $\underline{\underline{\mathbf{A}}} = \begin{bmatrix} i & 0 \\ 4 & 1 \end{bmatrix}$, where $i = \sqrt{-1}$.

given a matrix $\underline{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we find that the inverse of this matrix is $\frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and it's conjugate transpose is $\underline{A}^H = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ where any complex values will be of opposite sign. given that a matrix times its inverse $= \underline{I} = \underline{A}\underline{A}^{-1}$, we can show that the inverse of the conjugate transpose of $\underline{A}$, $(\underline{A}^H)$ is equal to the conjugate transpose of $\underline{A}^{-1}$.

$$\underline{A}^H = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \Rightarrow (\underline{A}^H)^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-c}{ad-bc} \\ \frac{-b}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

$$\underline{A}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \qquad (\underline{A}^{-1})^H = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-c}{ad-bc} \\ \frac{-b}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

we see that $(\underline{A}^H)^{-1} = (\underline{A}^{-1})^H$. We can also see that $(\underline{A}^{-1})^H(\underline{A}^H) = \underline{\underline{I}}$

$$\begin{bmatrix} \frac{d}{ad-bc} & \frac{-c}{ad-bc} \\ \frac{-b}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} \frac{ad}{ad-bc} + \frac{-bc}{ad-bc} & \frac{cd}{ad-bc} + \frac{-cd}{ad-bc} \\ \frac{-ab}{ad-bc} + \frac{ab}{ad-bc} & \frac{-bc}{ad-bc} + \frac{ad}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ and that

Likewise, $(\underline{A}^H)(\underline{A}^{-1})^H = \underline{\underline{I}} \Rightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} \frac{d}{ad-bc} & \frac{-c}{ad-bc} \\ \frac{-b}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} \frac{ad-bc}{ad-bc} & \frac{ac-ac}{ad-bc} \\ \frac{bd-db}{ad-bc} & \frac{ad-bc}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\underline{A} = \begin{bmatrix} i & 0 \\ 4 & 1 \end{bmatrix} \quad \underline{A}^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{i}\begin{bmatrix} 1 & 0 \\ -4 & i \end{bmatrix} = \begin{bmatrix} \frac{1}{i} & 0 \\ \frac{-4}{i} & 1 \end{bmatrix} = \underline{A}^{-1}$$

Conj transpose of $\underline{A}^{-1} = (A^{-1})^H = \begin{bmatrix} -\frac{1}{i} & \frac{+4}{i} \\ 0 & 1 \end{bmatrix}$

$\underline{A} = \begin{bmatrix} i & 0 \\ 4 & 1 \end{bmatrix} \quad \underline{A}^H = \begin{bmatrix} -i & 4 \\ 0 & 1 \end{bmatrix}$

The inverse of the complex Conj. of $\underline{A}$. $\quad (\underline{A}^H)^{-1} = \det(A^H)\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$(\underline{A}^H) = \frac{1}{-i}\begin{bmatrix} 1 & -4 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} -\frac{1}{i} & +\frac{4}{i} \\ 0 & 1 \end{bmatrix} = (A^H)^{-1}$$

$$(\underline{A}^{-1})^H(\underline{A}^H) = \begin{bmatrix} -\frac{1}{i} & \frac{4}{i} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -i & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{4}{i}+\frac{4}{i} \\ 0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{\underline{I}}$$

$$(\underline{A}^H)(\underline{A}^{-1})^H = \underline{\underline{I}} = \begin{bmatrix} -i & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{i} & \frac{4}{i} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4+4 \\ 0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{\underline{I}}$$

Exercise 5 (based on Strang 1.6.28)
Find (by hand) the singular value decomposition $\underline{\underline{U}}\,\underline{\underline{S}}\,\underline{\underline{V}}^H$ for the matrices $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ and $\begin{bmatrix}1 & 1\\1 & i\end{bmatrix}$, where $i=\sqrt{-1}$.

$\underline{\underline{A}} = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ $\quad \underline{\underline{A}}^H = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ $\quad AA^H = \underline{\underline{A}}^H\underline{\underline{A}} = \begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}$ $\quad \underline{\underline{A}}^H\underline{\underline{A}} - \lambda \underline{\underline{I}} = \begin{bmatrix}2-\lambda & 2\\2 & 2-\lambda\end{bmatrix}$

$\det(\underline{\underline{A}}^H\underline{\underline{A}} - \lambda \underline{\underline{I}}) = (2-\lambda)^2 - 4 = (2-\lambda)(2-\lambda) - 4 = 4 - 4\lambda + \lambda^2 - 4 = \lambda^2 - 4\lambda = 0 = \lambda(\lambda - 4)$

$\lambda_1 = 0 \quad \lambda_2 = 4$

$(\underline{\underline{A}}^H\underline{\underline{A}} - \lambda_1 \underline{\underline{I}})v_1 = 0 = \begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ $\quad \begin{matrix}2x_1 + 2x_2 = 0\\2x_1 + 2x_2 = 0\end{matrix}$ $\quad v_1 = \begin{bmatrix}1\\-1\end{bmatrix}$

make v_1 unitary, $\frac{1}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{2}}$ $\quad \frac{1}{\sqrt{2}} \times v_1 = \begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{bmatrix}$

$(\underline{\underline{A}}^H\underline{\underline{A}} - \lambda_2 \underline{\underline{I}})v_2 = 0$ $\quad \begin{bmatrix}-2 & 2\\2 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ $\quad \begin{matrix}-2x_1 \quad 2x_2 = 0\\2x_1 \quad -2x_2 = 0\end{matrix}$ $\quad v_2 = \begin{bmatrix}1\\1\end{bmatrix}$

make v_2 unitary $\Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

$\underline{\underline{V}} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$ because $A^HA = AA^H$ $\quad \underline{\underline{U}} = \underline{\underline{V}}$ $\quad \underline{\underline{U}} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$, $S_i = \sqrt{\lambda_i}$ $\quad \underline{\underline{S}} = \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}$

$\underline{\underline{A}} = \underline{\underline{U}}\,\underline{\underline{S}}\,\underline{\underline{V}}^H$ $\quad \underline{\underline{A}} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix}$

$\underline{\underline{A}} = \begin{bmatrix}1 & 1\\1 & i\end{bmatrix}$ $\quad \underline{\underline{A}}^H = \begin{bmatrix}1 & 1\\1 & -i\end{bmatrix}$ $\quad \underline{\underline{A}}\,\underline{\underline{A}}^H = \begin{bmatrix}2 & 1-i\\1+i & 2\end{bmatrix}$ $\quad (AA^H - \lambda \underline{\underline{I}}) = \begin{bmatrix}2-\lambda & 1-i\\1+i & 2-\lambda\end{bmatrix}$

$\det(AA^H - \lambda \underline{\underline{I}}) = (2-\lambda)(2-\lambda) - (1+i)(1-i) = 4 - 4\lambda + \lambda^2 - 2 = \lambda^2 - 4\lambda + 2 = 0$

$\lambda^2 - 4\lambda + 4 = 2 \quad = \quad (\lambda-2)^2 = 2 \Rightarrow \lambda - 2 = \sqrt{2}$ $\quad \lambda_1 = 2+\sqrt{2},\ \lambda_2 = 2-\sqrt{2}$

$\underline{\underline{A}}^H\underline{\underline{A}} = \begin{bmatrix}1 & 1\\1 & -i\end{bmatrix}\begin{bmatrix}1 & 1\\1 & i\end{bmatrix} = \begin{bmatrix}2 & 1+i\\1-i & 2\end{bmatrix}$ $\quad (A^HA - \lambda I) = \begin{bmatrix}2-\lambda & 1+i\\1-i & 2-\lambda\end{bmatrix}$

$\det(A^HA - \lambda I) = (2-\lambda)^2 - (1-i)(1+i) = 0$ $\quad (2-\lambda)(2-\lambda) - (1-i)(1+i) = 0$

$4 - 4\lambda + \lambda^2 - (1 \cancel{+i} \cancel{-i} \overset{+1}{\cancel{-i^2}}) = \lambda^2 - 4\lambda + 2 = 0$ $\quad \lambda_1 = 2+\sqrt{2}$ $\quad \lambda_2 = 2-\sqrt{2}$

$S_i = \sqrt{\lambda_i}$ $\quad \underline{\underline{S}} = \begin{bmatrix}\sqrt{2+\sqrt{2}} & 0\\0 & \sqrt{2-\sqrt{2}}\end{bmatrix}$

$2 - (2+\sqrt{2}) = -\sqrt{2}$

$(\underline{\underline{A}}\,\underline{\underline{A}}^H - \lambda_1 I)U_1 = \begin{bmatrix}-\sqrt{2} & 1-i\\1+i & -\sqrt{2}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\left[\begin{array}{cc|c}-\sqrt{2} & 1-i & 0\\1+i & -\sqrt{2} & 0\end{array}\right] \xrightarrow{(1)/-\sqrt{2}} \left[\begin{array}{cc|c}1 & \frac{(-1+i)\sqrt{2}}{2} & 0\\1+i & -\sqrt{2} & 0\end{array}\right] \xrightarrow{(2)-(1)(1+i)} \begin{bmatrix}1 & \frac{(-1+i)\sqrt{2}}{2}\\0 & 0\end{bmatrix}$ $\quad \begin{matrix}x_1 - \frac{(1-i)\sqrt{2}}{2}x_2 = 0\\x_2 = x_2\end{matrix}$ $\quad \underline{U}_1 = \begin{bmatrix}\frac{(1-i)\sqrt{2}}{2}\\1\end{bmatrix}$

$\frac{(-1+i)\sqrt{2}}{2}(1+i) = \frac{(-1+i)\sqrt{2}}{2} + \frac{(-i-1)\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(-1+i-i-1) = -\sqrt{2}$

make $\underline{u}_1$ unitary $\frac{1}{\sqrt{\left|\frac{(1-i)\sqrt{2}}{2}\right|^2 + |1|^2}} \times \underline{U} = \frac{1}{\sqrt{2}} \times \underline{U}$ $\quad$ * unitary $\underline{u}_1 = \begin{bmatrix}\frac{1-i}{2}\\\frac{\sqrt{2}}{2}\end{bmatrix}$

$\frac{(1-i)^2\,2}{4} = 1$

$$2-(2-\sqrt{2})$$

$$(\underline{A}\underline{A}^H - \lambda_2 I)u_2 = \begin{bmatrix} \sqrt{2} & 1-i \\ 1+i & \sqrt{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} \sqrt{2} & 1-i & 0 \\ 1+i & \sqrt{2} & 0 \end{array}\right] \xrightarrow{(1)/\sqrt{2}} \left[\begin{array}{cc|c} 1 & \frac{(1-i)\sqrt{2}}{2} & 0 \\ 1+i & \sqrt{2} & 0 \end{array}\right] \xrightarrow{(2)-(1)(1+i)} \left[\begin{array}{cc|c} 1 & \frac{(1-i)\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1 + \frac{(1-i)\sqrt{2}}{2} x_2 = 0 \qquad x_2 = x_2 \qquad \underline{u}_2 = \begin{bmatrix} \frac{(-1+i)\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

$$\frac{(1-i)\sqrt{2}}{2}(1+i) = \frac{(1+i)(1-i)\sqrt{2}}{2} = \frac{1-i+i+1}{2}\sqrt{2} = \sqrt{2}$$

make $\underline{u}_2$ unitary $\frac{1}{\sqrt{\left|\frac{(-1+i)\sqrt{2}}{2}\right|^2+1^2}} \times \underline{u}_2$

$$\frac{(-1+i)^2 2}{4} = \frac{1}{2}(-1+i)(-1+i) = \frac{1}{2}(1-i-i-1) = -i \quad \Rightarrow \frac{1}{\sqrt{-i^2+1^2}} = \frac{1}{\sqrt{2}} \qquad \frac{1}{\sqrt{2}}\underline{u}_2 = \begin{bmatrix} \frac{i-1}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

*unitary

$$\underline{U} = \begin{bmatrix} \frac{1-i}{2} & \frac{i-1}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

*unitary

Solve for v_i using $\underline{A}^H \underline{u}_i = s_i \underline{v}_i$ $\qquad \underline{v}_i = \frac{1}{s_i}\underline{A}^H \underline{u}_i$

for $s_1 = \sqrt{2+\sqrt{2}}$, $\underline{v}_1 = \frac{1}{\sqrt{2+\sqrt{2}}}\begin{bmatrix} 1 & 1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} \frac{1-i}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1-i+\sqrt{2}}{2\sqrt{2+\sqrt{2}}} \\ \frac{1-i-i\sqrt{2}}{2\sqrt{2+\sqrt{2}}} \end{bmatrix}$

for $s_2 = \sqrt{2-\sqrt{2}}$, $\underline{v}_2 = \frac{1}{\sqrt{2-\sqrt{2}}}\begin{bmatrix} 1 & 1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} \frac{i-1}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{i-1+\sqrt{2}}{2\sqrt{2-\sqrt{2}}} \\ \frac{i-1-\sqrt{2}i}{2\sqrt{2-\sqrt{2}}} \end{bmatrix}$

$$\underline{V} = \begin{bmatrix} \frac{1-i+\sqrt{2}}{2\sqrt{2+\sqrt{2}}} & \frac{i-1+\sqrt{2}}{2\sqrt{2-\sqrt{2}}} \\ \frac{1-i-i\sqrt{2}}{2\sqrt{2+\sqrt{2}}} & \frac{i-1-\sqrt{2}i}{2\sqrt{2-\sqrt{2}}} \end{bmatrix} \qquad \underline{V}^H = \begin{bmatrix} \frac{1+i+\sqrt{2}}{2\sqrt{2+\sqrt{2}}} & \frac{-i-1+\sqrt{2}}{2\sqrt{2-\sqrt{2}}} \\ \frac{1+i+i\sqrt{2}}{2\sqrt{2+\sqrt{2}}} & \frac{-i-1+\sqrt{2}i}{2\sqrt{2-\sqrt{2}}} \end{bmatrix}$$

SVD:

$$\underline{A} = \underline{U}\,\underline{\underline{S}}\,\underline{V}^H$$

$$\underline{A} = \begin{bmatrix} \frac{1-i}{2} & \frac{i-1}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} \sqrt{2+\sqrt{2}} & 0 \\ 0 & \sqrt{2-\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1+i+\sqrt{2}}{2\sqrt{2+\sqrt{2}}} & \frac{-i-1+\sqrt{2}}{2\sqrt{2-\sqrt{2}}} \\ \frac{1+i+i\sqrt{2}}{2\sqrt{2+\sqrt{2}}} & \frac{-i-1+\sqrt{2}i}{2\sqrt{2-\sqrt{2}}} \end{bmatrix}$$

Exercise 6

$$\underline{\underline{\mathbf{A}}} = \begin{bmatrix} 2 & 2 & 2 & 2 \\ \frac{17}{10} & \frac{1}{10} & -\frac{17}{10} & -\frac{1}{10} \\ \frac{3}{5} & \frac{9}{5} & -\frac{3}{5} & -\frac{9}{5} \end{bmatrix}$$

a. Find by hand the singular value decomposition of matrix $\underline{\underline{\mathbf{A}}}$.

b. Find the solution(s) to the equation $\underline{\underline{\mathbf{A}}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$ for $\underline{\mathbf{b}} = [1, 0, 0]^T$, using the pseudo-inverse of $\underline{\underline{\mathbf{A}}}$. Based on the rank of $\underline{\underline{\mathbf{A}}}$, is the solution unique? If not, provide all solutions.

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 2 & 2 & 2 \\ \frac{17}{10} & \frac{1}{10} & -\frac{17}{10} & -\frac{1}{10} \\ \frac{3}{5} & \frac{9}{5} & -\frac{3}{5} & -\frac{9}{5} \end{bmatrix} \qquad \underline{\underline{A}}^H = \begin{bmatrix} 2 & \frac{17}{10} & \frac{3}{5} \\ 2 & \frac{1}{10} & \frac{9}{5} \\ 2 & -\frac{17}{10} & -\frac{3}{5} \\ 2 & -\frac{1}{10} & -\frac{9}{5} \end{bmatrix}$$

Rank of $A = 2$

$$\underline{\underline{A}} \cdot \underline{\underline{A}}^H = \begin{bmatrix} 2 & 2 & 2 & 2 \\ \frac{17}{10} & \frac{1}{10} & -\frac{17}{10} & -\frac{1}{10} \\ \frac{3}{5} & \frac{9}{5} & -\frac{3}{5} & -\frac{9}{5} \end{bmatrix} \begin{bmatrix} 2 & \frac{17}{10} & \frac{3}{5} \\ 2 & \frac{1}{10} & \frac{9}{5} \\ 2 & -\frac{17}{10} & -\frac{3}{5} \\ 2 & -\frac{1}{10} & -\frac{9}{5} \end{bmatrix} = \begin{bmatrix} 2\cdot2+2\cdot2+2\cdot2+2\cdot2 & \frac{34}{10}+\frac{2}{10}-\frac{34}{10}-\frac{2}{10} & \frac{6}{5}+\frac{18}{5}-\frac{6}{5}-\frac{18}{5} \\ \frac{34}{10}+\frac{2}{10}-\frac{34}{10}-\frac{2}{10} & \left(\frac{17}{10}\right)^2+\left(\frac{1}{10}\right)^2+\left(-\frac{17}{10}\right)^2+\left(-\frac{1}{10}\right)^2 & \frac{51}{50}+\frac{9}{50}+\frac{51}{50}+\frac{9}{50} \\ \frac{6}{5}+\frac{18}{5}-\frac{6}{5}-\frac{18}{5} & \frac{51}{50}+\frac{9}{50}+\frac{51}{50}+\frac{9}{50} & \frac{9}{25}+\frac{81}{25}+\frac{9}{25}+\frac{81}{25} \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & \frac{29}{5} & \frac{12}{5} \\ 0 & \frac{12}{5} & \frac{36}{5} \end{bmatrix}$$

$$\underline{\underline{A}}^H \cdot \underline{\underline{A}} = \begin{bmatrix} 2 & \frac{17}{10} & \frac{3}{5} \\ 2 & \frac{1}{10} & \frac{9}{5} \\ 2 & -\frac{17}{10} & -\frac{3}{5} \\ 2 & -\frac{1}{10} & -\frac{9}{5} \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 & 2 \\ \frac{17}{10} & \frac{1}{10} & -\frac{17}{10} & -\frac{1}{10} \\ \frac{3}{5} & \frac{9}{5} & -\frac{3}{5} & -\frac{9}{5} \end{bmatrix} = \begin{bmatrix} \frac{29}{4} & \frac{21}{4} & \frac{3}{4} & \frac{11}{4} \\ \frac{21}{4} & \frac{29}{4} & \frac{11}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{11}{4} & \frac{29}{4} & \frac{21}{4} \\ \frac{11}{4} & \frac{3}{4} & \frac{21}{4} & \frac{29}{4} \end{bmatrix}$$

Compute λ for $\underline{A} \cdot \underline{A}^H$: $(\underline{A}\underline{A}^H - \lambda \underline{I}) = \begin{bmatrix} 16-\lambda & 0 & 0 \\ 0 & \frac{29}{5}-\lambda & \frac{12}{5} \\ 0 & \frac{12}{5} & \frac{36}{5}-\lambda \end{bmatrix}$

$$\det(\underline{A}\underline{A}^H - \lambda I) = 0 = \begin{vmatrix} 16-\lambda & 0 & 0 \\ 0 & \frac{29}{5}-\lambda & \frac{12}{5} \\ 0 & \frac{12}{5} & \frac{36}{5}-\lambda \end{vmatrix} = (16-\lambda) \times \left[\left(\frac{29}{5}-\lambda\right)\left(\frac{36}{5}-\lambda\right) - \left(\frac{12}{5}\right)^2\right]$$

$$\frac{1044}{25} - \frac{29}{5}\lambda - \frac{36}{5}\lambda + \lambda^2 - \frac{144}{25}$$

$$(16-\lambda) \quad (\lambda^2 - 13\lambda + 36)$$

$$16\lambda^2 - 208\lambda + 576 - \lambda^3 + 13\lambda^2 - 36\lambda$$

$$-\lambda^3 + 29\lambda^2 - 244\lambda + 576$$

$$\lambda_3 = 4, \quad \lambda_2 = 9, \quad \lambda_1 = 16$$

Calculate $\underline{V}_i$ using $\lambda_3 = 4$, $\lambda_2 = 9$, $\lambda_1 = 16$, $\lambda_4 = 0$

$(\underline{A}^H\underline{A} - \lambda_1 \underline{\underline{I}})\underline{v} = 0 = \begin{bmatrix} \frac{29}{4}-16 & \frac{1}{4} & \frac{3}{4} & \frac{11}{4} \\ \frac{21}{4} & \frac{29}{4}-16 & \frac{11}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{11}{4} & \frac{29}{4}-16 & \frac{21}{4} \\ \frac{11}{4} & \frac{3}{4} & \frac{21}{4} & \frac{29}{4}-16 \end{bmatrix}$ $\quad V_1 = \begin{bmatrix} -\frac{35}{4} & \frac{1}{4} & \frac{3}{4} & \frac{11}{4} \\ \frac{21}{4} & -\frac{35}{4} & \frac{11}{4} & \frac{3}{4} \\ \frac{8}{4} & \frac{11}{4} & -\frac{35}{4} & \frac{21}{4} \\ \frac{11}{4} & \frac{3}{4} & \frac{21}{4} & -\frac{35}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_1 = 16$

$$\left[\begin{array}{cccc|c} -\frac{7}{4} & \frac{1}{4} & \frac{3}{4} & \frac{11}{4} & 0 \\ \frac{21}{4} & -\frac{7}{4} & \frac{11}{4} & \frac{3}{4} & 0 \\ \frac{3}{4} & \frac{11}{4} & -\frac{7}{4} & \frac{21}{4} & 0 \\ \frac{11}{4} & \frac{3}{4} & \frac{21}{4} & -\frac{7}{4} & 0 \end{array}\right]$$

$(1)\cdot -\frac{4}{7}$ | $(2)\cdot -\frac{5}{28}$ | $(4)-(3)\cdot\frac{48}{7}$

$(2)-(1)\cdot\frac{21}{4}$ | $(3)-(2)\cdot\frac{16}{5}$ | $(2)+(3)\frac{4}{7}$

$(3)-(1)\cdot\frac{3}{4}$ | $(4)-(2)\cdot\frac{12}{5}$ | $(1)+(3)\frac{3}{35}$

$(4)-(1)\cdot\frac{11}{4}$ | $(3)\cdot -\frac{7}{48}$ | $(1)+(2)\frac{3}{5}$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = x_4$

$x_2 = x_4$

$x_3 = x_4$

$x_4 = x_4$

$$\underline{V}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

make $\underline{V}_1$ unitary $= \frac{1}{2}\underline{V}_1 \rightarrow$ $\quad {}^*\underline{V}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$*$ unitary

$$(\underline{A}^H\underline{A} - \lambda_2 \underline{I})\underline{V}_2 = 0 = \begin{bmatrix} \frac{29}{4}-9 & \frac{1}{4} & \frac{3}{4} & \frac{11}{4} \\ \frac{21}{4} & \frac{29}{4}-9 & \frac{11}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{11}{4} & \frac{29}{4}-9 & \frac{21}{4} \\ \frac{11}{4} & \frac{3}{4} & \frac{21}{4} & \frac{29}{4}-9 \end{bmatrix} V_2 = \begin{bmatrix} -\frac{7}{4} & \frac{1}{4} & \frac{3}{4} & \frac{11}{4} \\ \frac{21}{4} & -\frac{7}{4} & \frac{11}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{11}{4} & -\frac{7}{4} & \frac{21}{4} \\ \frac{11}{4} & \frac{3}{4} & \frac{21}{4} & -\frac{7}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_2 = 9$

$$\left[\begin{array}{cccc|c} -\frac{7}{4} & \frac{1}{4} & \frac{3}{4} & \frac{11}{4} & 0 \\ \frac{21}{4} & -\frac{7}{4} & \frac{11}{4} & \frac{3}{4} & 0 \\ \frac{3}{4} & \frac{11}{4} & -\frac{7}{4} & \frac{21}{4} & 0 \\ \frac{11}{4} & \frac{3}{4} & \frac{21}{4} & -\frac{7}{4} & 0 \end{array}\right] \begin{array}{l} (1)\cdot\frac{-4}{7} \\ (2)-(1)\cdot\frac{21}{4} \\ (3)-(1)\frac{3}{4} \\ (4)-(1)\frac{11}{4} \end{array} \begin{array}{l} (2)\cdot\frac{1}{14} \\ (3)-(2)\cdot 5 \\ (4)-(2)\cdot 9 \\ (3)\cdot\frac{-14}{45} \end{array} \begin{array}{l} (4)-(3)\cdot\frac{45}{14} \\ (2)-(3)\cdot\frac{5}{14} \\ (1)-(3)\cdot\frac{-3}{7} \\ (1)+(2)\cdot 3 \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = -x_4$
$x_2 = -x_4$
$x_3 = x_4$
$x_4 = x_4$

$$\underline{V}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

make $\underline{V}_2$ unitary $\rightarrow$ $\frac{1}{2}\underline{V}_2 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ * unitary

$$(\underline{A}^H\underline{A} - \lambda_3 \underline{I})\underline{V}_3 = 0 = \begin{bmatrix} \frac{29}{4}-4 & \frac{21}{4} & \frac{3}{4} & \frac{11}{4} \\ \frac{21}{4} & \frac{29}{4}-4 & \frac{11}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{11}{4} & \frac{29}{4}-4 & \frac{21}{4} \\ \frac{11}{4} & \frac{3}{4} & \frac{21}{4} & \frac{29}{4}-4 \end{bmatrix} V_3 = \begin{bmatrix} \frac{13}{4} & \frac{21}{4} & \frac{3}{4} & \frac{11}{4} \\ \frac{21}{4} & \frac{13}{4} & \frac{11}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{11}{4} & \frac{13}{4} & \frac{21}{4} \\ \frac{11}{4} & \frac{3}{4} & \frac{21}{4} & \frac{29}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_3 = 4$

$$\left[\begin{array}{cccc|c} \frac{13}{4} & \frac{21}{4} & \frac{3}{4} & \frac{11}{4} & 0 \\ \frac{21}{4} & \frac{13}{4} & \frac{11}{4} & \frac{3}{4} & 0 \\ \frac{3}{4} & \frac{11}{4} & \frac{13}{4} & \frac{21}{4} & 0 \\ \frac{11}{4} & \frac{3}{4} & \frac{21}{4} & \frac{29}{4} & 0 \end{array}\right] \begin{array}{l} (1)\cdot\frac{4}{13} \\ (2)-(1)\cdot\frac{21}{4} \\ (3)-(1)\cdot\frac{3}{4} \\ (4)-(1)\cdot\frac{11}{4} \end{array} \rightarrow \begin{array}{l} (2)\cdot -\frac{13}{68} \\ (3)-(2)\cdot\frac{20}{13} \\ (4)-(2)\cdot\frac{-48}{13} \\ (3)\cdot\frac{17}{60} \end{array} \begin{array}{l} (4)-(3)\cdot\frac{60}{17} \\ (2)-(3)\cdot\frac{-5}{17} \\ (1)-(3)\cdot\frac{3}{13} \\ (1)-(2)\cdot\frac{21}{13} \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = x_4$
$x_2 = -x_4$
$x_3 = -x_4$
$x_4 = x_4$

$$V_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

make $\underline{V}_3$ unitary $\frac{1}{\sqrt{1^2+1^2+1^2+1^2}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$ unitary ${}^*\underline{V}_3 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$ * unitary

$$(\underline{A}^H\underline{A} - \lambda_4 \underline{I})\underline{V}_4 = 0 = \begin{bmatrix} \frac{29}{4} & \frac{1}{4} & \frac{3}{4} & \frac{11}{4} \\ \frac{21}{4} & \frac{29}{4} & \frac{11}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{11}{4} & \frac{29}{4} & \frac{21}{4} \\ \frac{11}{4} & \frac{3}{4} & \frac{21}{4} & \frac{29}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_4 = 0$

$$\left[\begin{array}{cccc|c} \frac{29}{4} & \frac{1}{4} & \frac{3}{4} & \frac{11}{4} & 0 \\ \frac{21}{4} & \frac{29}{4} & \frac{11}{4} & \frac{3}{4} & 0 \\ \frac{3}{4} & \frac{11}{4} & \frac{29}{4} & \frac{21}{4} & 0 \\ \frac{11}{4} & \frac{3}{4} & \frac{21}{4} & \frac{29}{4} & 0 \end{array}\right] \begin{array}{l} (1)\cdot\frac{4}{29} \\ (2)-(1)\cdot\frac{21}{4} \\ (3)-(1)\cdot\frac{3}{4} \\ (4)-(1)\cdot\frac{11}{4} \end{array} \begin{array}{l} (2)\cdot\frac{29}{100} \\ (3)-(2)\frac{64}{29} \\ (4)+(2)\frac{36}{29} \\ (3)\cdot\frac{25}{144} \end{array} \begin{array}{l} (4)-(3)\frac{144}{25} \\ (2)-(3)\frac{16}{25} \\ (1)-(3)\cdot\frac{3}{29} \\ (1)-(2)\frac{21}{29} \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = -x_4$
$x_2 = x_4$
$x_3 = -x_4$
$x_4 = x_4$

$$\underline{V}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

make $\underline{V}_4$ unitary $\frac{1}{2}\underline{V}_4 \rightarrow {}^*\underline{V}_4 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$ * unitary

$$\underline{V} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \qquad \underline{S} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$\underline{u}_i = \frac{1}{S_i}\underline{A}\,\underline{V}_i$ from $\underline{A}\,\underline{V}_i = S_i\,\underline{u}_i$

u_1 for v_1

$$u_1 = \frac{1}{4}\cdot\begin{bmatrix} 2 & 2 & 2 & 2 \\ \frac{17}{10} & \frac{1}{10} & -\frac{17}{10} & -\frac{1}{10} \\ \frac{3}{5} & \frac{9}{5} & -\frac{3}{5} & -\frac{9}{5} \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1/4 + 1/4 + 1/4 + 1/4 \\ 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

u_2 for v_2

$$u_2 = \frac{1}{3}\cdot\begin{bmatrix} 2 & 2 & 2 & 2 \\ \frac{17}{10} & \frac{1}{10} & -\frac{17}{10} & -\frac{1}{10} \\ \frac{3}{5} & \frac{9}{5} & -\frac{3}{5} & -\frac{9}{5} \end{bmatrix}\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{17}{30} & \frac{1}{30} & -\frac{7}{30} & -\frac{1}{30} \\ \frac{1}{5} & \frac{3}{5} & -\frac{1}{5} & -\frac{3}{5} \end{bmatrix}\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} - \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \\ -\frac{17}{60} - \frac{1}{60} - \frac{17}{60} - \frac{1}{60} \\ -\frac{1}{10} - \frac{3}{10} - \frac{1}{10} - \frac{3}{10} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{3}{5} \\ -\frac{4}{5} \end{bmatrix}$$

u_3 for v_3

$$u_3 = \frac{1}{2}\cdot\begin{bmatrix} 2 & 2 & 2 & 2 \\ \frac{17}{10} & \frac{1}{10} & -\frac{17}{10} & -\frac{1}{10} \\ \frac{3}{5} & \frac{9}{5} & -\frac{3}{5} & -\frac{9}{5} \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{17}{20} & \frac{1}{20} & -\frac{17}{20} & -\frac{1}{20} \\ \frac{3}{10} & \frac{9}{10} & -\frac{3}{10} & -\frac{9}{10} \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{1}{2} \\ \frac{17}{40} - \frac{1}{40} + \frac{17}{40} - \frac{1}{40} \\ \frac{3}{20} - \frac{9}{20} + \frac{3}{20} - \frac{9}{20} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix}$$

$$\underline{\underline{U}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & -\frac{3}{5} \end{bmatrix} \qquad \underline{\underline{V}}^H = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\underline{\underline{A}} = \underline{\underline{U}}\,\underline{\underline{S}}\,\underline{\underline{V}}^H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & -\frac{3}{5} \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 & 2 \\ \frac{17}{10} & \frac{1}{10} & -\frac{17}{10} & -\frac{1}{10} \\ \frac{3}{5} & \frac{9}{5} & -\frac{3}{5} & -\frac{9}{5} \end{bmatrix}$$

$[3\times3][3\times4]$

$[3\times4][4\times4] = [4\times4]$

Part B Find the Solutions to the equation $\underline{A}x = b$ where $b = [1, 0, 0]^T$ using the Pseudo inverse of $\underline{A}$.

$$\underline{\underline{A}}^+ = \underline{\underline{V}}\,\underline{\underline{S}}\,\underline{\underline{U}}^H$$

$4\times4 \;\; 4\times3 \;\; 3\times3 = 4\times3$

$$\underline{\underline{V}} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \qquad \underline{\underline{S}} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \qquad \underline{\underline{U}}^H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$$

$$\underline{\underline{V}}\,\underline{\underline{S}} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & -\frac{1}{6} & \frac{1}{4} \\ \frac{1}{8} & -\frac{1}{6} & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{6} & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{6} & \frac{1}{4} \end{bmatrix}$$

$$(\underline{\underline{V}}\,\underline{\underline{S}})\underline{\underline{U}}^H = \begin{bmatrix} \frac{1}{8} & -\frac{1}{6} & \frac{1}{4} \\ \frac{1}{8} & -\frac{1}{6} & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{6} & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{6} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{5} & -\frac{4}{5} \\ 0 & \frac{4}{5} & -\frac{3}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{13}{40} & -\frac{1}{15} \\ \frac{1}{8} & -\frac{3}{40} & \frac{1}{3} \\ \frac{1}{8} & -\frac{13}{40} & \frac{1}{15} \\ \frac{1}{8} & \frac{3}{40} & -\frac{1}{3} \end{bmatrix}$$

$$\underline{\tilde{X}} = \underline{A}^{-1}\underline{b} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{13}{40} & -\frac{1}{15} \\ \frac{1}{8} & -\frac{3}{40} & \frac{1}{3} \\ \frac{1}{8} & -\frac{13}{40} & \frac{1}{15} \\ \frac{1}{8} & \frac{3}{40} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \end{bmatrix}$$

The rank of $\underline{\underline{A}}$ is 3, the unique solution is $\tilde{x} + x_0$ (the solution for the equation $\underline{\underline{A}}\underline{x} = 0$.) x_0 is the homogeneous solution associated with the nullspace of $\underline{\underline{A}}$. x_0 is All of the vectors spanned by the eigenvector associated with the singular value $\emptyset$.

So $x_0 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ and the unique Soln of $\underline{A} = \begin{bmatrix} \frac{9}{8} \\ -\frac{7}{8} \\ \frac{9}{8} \\ -\frac{7}{8} \end{bmatrix}$

Exercise 7

Identify two peer-reviewed articles in the fields of marine, atmospheric, or Earth sciences; (i) one that employs an analysis based on eigen decomposition, and (ii) one that employs an analysis based on singular value decomposition, both incorporating graphical representations. Include the DOIs of the articles and the respective sections of the articles where the decompositions are used.

Explain in your own words (no ChatGPT please) why and how the decomposition is applied in each study. Specifically, describe the matrices being decomposed and what they represent, the normalizations (if any) applied to the decomposed matrices and to the eigen and singular vectors. In the cases where the matrices decomposed represent physical qunatities, which quantities from the decompositions seem to hold the physical units (you can review my online lecture on eigen methods).

i) https://doi.org/10.1093/jge/gxab051 An optimised one-way wave migration method for complex media with arbitrarily varying velocity based on eigen-decomposition. Their methods for eigen decomposition are detailed and used in section 2.1 and the introduction yet the decomposition is still referenced in later sections.

The eigen decomposition is applied in this study to improve the imaging of wave propagation through “complex media”, specifically small scale and complex structures in geophysical contexts, like steep-dipping faults. *How:* They use the one-way depth migration (OWDM) method which is widely used for collecting practical seismic information because it utilizes the one-way wave equation which results in correct phase information compared to two-way wave methods. They use the eigenvalue and eigenvector decomposition of the Helmholtz matrix for OWDM. Their scheme computed the one-way propagation operator with the best polynomial approximation based on the infinite norm. They use the Helmholtz operator L which has two terms. The first is a diagonal matrix which consisted of all velocity points of horizon coordination in the same depth, and the second is a second-order finite-difference discretization which is for improving the precision of the calculation. Their decomposition does not include a normalization of their eigenvector matrix, V, but it does apply a square root to the Helmholtz operator matrix, L, and the respective matrix of eigenvalues, M.

$$L = \frac{\omega^2}{V(x,z)^2} + \frac{\partial^2}{\partial x^2} \quad \text{where} \quad \frac{\omega^2}{V(x,z)^2} = \begin{bmatrix} \frac{\omega^2}{V^2(x_1,z_i)} & 0 & \cdots & 0 \\ 0 & \frac{\omega^2}{V^2(x_2,z_i)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\omega^2}{V^2(x_N,z_i)} \end{bmatrix}$$

$$\text{and} \quad \frac{\partial^2}{\partial x^2} = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 0 \end{bmatrix}$$

$$L = VMV^T \qquad \text{with the Square root operator } \Lambda = \sqrt{L} = V\sqrt{M}V^T$$

It seems like the matrix L holds the units of m/s because it contains velocity information. Lambda will also have units of sqrt(m/s). Because of this, the matrix with eigenvalues will be the matrix in the decomposition with units.

ii) https://doi.org/10.1016/j.eswa.2023.121924 Deep learning combined with singular value decomposition to reconstruct databases in fluid dynamics. The SVD is applied to a matrix in section 4.1, but the beginning of section 4 and 4.2 also includes relevant methods with SVD for the full experiment, including the application of a neural network to the SVD. They deploy an SVD based method to reconstruct databases with three, four and five dimensions by extracting the main flow dynamics from a sparse database of sensor measurements. They use physical fluid dynamics equations in their SVD with neural networks in order to obtain a simpler way to solve fluid dynamics problems in a time efficient way by reducing the size of the matrices used in the SVD. They create reconstructions of sensor data known as “reduced order models”.

They apply SVD to a reduced tensor that represents one variable of data collected from a sensor, in their case, they apply this to three different spatial variables, streamwise and normal velocity and vorticity. They reduce the size of these matrices by making the resolution more coarse and they reshape the databases so that they are MxNxD and store the information on the rescaling so that the appropriate spatial positions in the original domain are retained and can be reapplied. They apply SVD to each of their rescaled, reshaped matrices. They call it the “reduced tensor” and they define it as: $\hat{X} = \hat{X}_{imnL}$, for $i = 1 \ldots N$, $m = 1 \ldots N_y$, $n = 1 \ldots n_x$ and $l = 1 \ldots N_{var}$

X hat, is the reduced tensor of X. X hat represents the real database of sparsely collected sensor measurements. They do use a scaling method on this reduced tensor before the SVD (on X hat) to retain relevant information of any variables with smaller magnitude by homogenizing the range values between all of the input variables. They test this by running the experiment twice, scaling the data or “adding the normalization” and again not scaling the data.

They use the python technique, numpy.linalg.svd to apply the SVD to X hat. They define the decomposition as

$$X_U^{DS} = U_{il} \quad X_S^{DS} = \Sigma_{il} \quad X_V^{DS} = V_{il} \quad \text{for } i = 1, \ldots, N_t \text{ and } l = 1, \ldots, N_{var}$$

It seems like for this SVD, the matrix L has the units, so the S matrix which is X_S ^DS, above is the matrix that is made of the square root of the eigenvalues, is the matrix in the decomposition with units.