Aufgabe 7.8 (Regel von L'Hospital)

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$
$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$$

Berechnen Sie:

a) $\lim\limits_{x\to\pi} \dfrac{\sin 3x}{\tan 5x}$ b) $\lim\limits_{x\to 0} \dfrac{2x\sin 2x}{\sinh^2 x}$ c) $\lim\limits_{x\to 0} \dfrac{2-\sqrt{4-x^2}}{3-\sqrt{9-x^2}}$ d) $\lim\limits_{x\to 0} \dfrac{e^x - 2x - e^{-x}}{x - \sin x}$

e) $\lim\limits_{x\to\infty} \dfrac{x^5}{e^{3x}}$ f) $\lim\limits_{x\downarrow 0} x^x$ g) $\lim\limits_{x\uparrow 0}(1+\sin x)^{\frac{1}{x}}$ h) $\lim\limits_{x\uparrow \frac{\pi}{2}} (\tan x)^{\cot x}$

a) $\lim\limits_{x\to\pi} \dfrac{\sin 3x}{\tan 5x}$

$\lim\limits_{x\to\pi} \sin 3x \Rightarrow \sin 3\pi = 0$

$\lim\limits_{x\to\pi} \tan 5x \Rightarrow \tan 5\pi = 0$

$\lim\limits_{x\to\pi} \sin 3x = \lim\limits_{x\to\pi} \tan 5x = 0$

$$\lim_{x\to\pi} \frac{(\sin 3x)'}{(\tan 5x)'} = \frac{\lim\limits_{x\to\pi} 3\cos 3x \cdot \cos^2 5x}{\lim\limits_{x\to\pi} 5\cos^2 5x + 5\sin^2 5x}$$
$$= \frac{3\cos 3\pi \cdot \cos^2 5\pi}{5\cos^2 5\pi + 5\sin^2 5\pi}$$
$$= \frac{3\cdot(-1)\cdot 1}{5\cdot 1 + 5\cdot 0}$$
$$= -\frac{3}{5}$$

$\tan 5x = \dfrac{\sin 5x}{\cos 5x} \Rightarrow (\tan 5x)' = \left(\dfrac{\sin 5x}{\cos 5x}\right)'$

$$= \frac{5\cos 5x \cdot \cos 5x - \sin 5x \cdot (-5)\sin 5x}{\cos^2 5x}$$
$$= \frac{5\cos^2 5x + 5\sin^2 5x}{\cos^2 5x}$$

Aus der Regel von L'Hospital folgt $\lim\limits_{x\to\pi} \dfrac{(\sin 3x)'}{(\tan 5x)'} = \lim\limits_{x\to\pi} \dfrac{\sin 3x}{\tan 5x} = -\dfrac{3}{4}$

b) $\lim\limits_{x\to 0} \dfrac{2x \sin 2x}{\sinh^2 x}$

$\lim\limits_{x\to 0} 2x \cdot \sin 2x = 2\cdot 0 \cdot \sin(2\cdot 0) = 0$

$\lim\limits_{x\to 0} \sinh^2 x = \lim\limits_{x\to 0}\left(\frac{1}{2}\left(e^x - e^{-x}\right)\right)^2 = \left(\frac{1}{2}\left(e^0 - e^{-0}\right)\right)^2 = \left(\frac{1}{2}(1-1)\right)^2 = 0$

$\lim\limits_{x\to 0} 2x \sin 2x = \lim\limits_{x\to 0} \sinh^2 x = 0$

$2\cdot(x\cdot\sin 2x)' = 2\cdot(\sin 2x + x\cdot\cos 2x \cdot 2)$
$= 2\sin 2x + 4x\cos 2x$

$$\lim_{x\to 0} \frac{(2x\sin 2x)'}{(\sinh^2 x)'} = \lim_{x\to 0} \frac{4x\cos 2x + 2\sin 2x}{\frac{e^{2x} - e^{-2x}}{2}}$$
$$= \lim_{x\to 0} \frac{8x\cos 2x + 4\sin 2x}{e^{2x} - e^{-2x}}$$
$$= \lim_{x\to 0} \frac{8x\cos 2x + 4\sin 2x}{(e^x + e^{-x})(e^x - e^{-x})}$$
$$\Rightarrow \lim_{x\to 0} \frac{(8x\cos 2x + 4\sin 2x)(e^x + e^{-x})}{(e^x + e^{-x})(e^x - e^{-x})(e^x + e^{-x})}$$

$(2x\sin 2x)' = 2x\cdot 2\cos 2x + 2\sin 2x$
$= 4x\cos 2x + 2\sin 2x$

$$(\sinh^2 x)' = \left(\left(\frac{1}{2}\left(e^x - e^{-x}\right)\right)^2\right)' = \left(\frac{1}{4}\left(e^x - e^{-x}\right)^2\right)'$$
$$= \left(\frac{1}{4}\left(e^{2x} - 2e^x \cdot e^{-x} + e^{-2x}\right)\right)'$$
$$= \left(\frac{e^{2x}}{4} - \frac{1}{2} + \frac{e^{-2x}}{4}\right)'$$
$$= \frac{e^{2x}}{2} - \frac{e^{-2x}}{2}$$

$\lim_{x\to 0}(2x\sin 2x)' = \lim_{x\to 0} 4x\cos 2x + 2\cdot\sin 2x = 4\cdot 0\cos(2\cdot 0) + 2\cdot\sin(2\cdot 0) = 0$

$\lim_{x\to 0}(\sinh^2 x)' = \frac{e^{2x}-e^{-2x}}{2} = \frac{1-1}{2} = 0$

$\lim_{x\to 0}(2x\sin 2x)' = \lim_{x\to 0}(\sinh^2 x)' = 0$

$$\lim_{x\to 0}\frac{(2x\sin 2x)''}{(\sinh^2 x)''} = \lim_{x\to 0}\frac{(4x\cos 2x + 2\cdot\sin 2x)'}{\left(\frac{e^{2x}-e^{-2x}}{2}\right)'}$$

$$= \lim_{x\to 0}\frac{-8x\sin 2x + 8\cos 2x}{e^{2x}+e^{-2x}}$$

$$= \frac{-8\cdot 0\cdot\sin(2\cdot 0) + 8\cdot\cos(2\cdot 0)}{e^{2\cdot 0}+e^{-2\cdot 0}}$$

$$= \frac{8}{2} = 4$$

$(4x\cos 2x + 2\sin 2x)' = -8x\sin 2x + 8\cos 2x$

$\left(\frac{e^{2x}-e^{-2x}}{2}\right)' = e^{2x}+e^{-2x}$

Aus dem Satz von L'Hospital folgt $\lim_{x\to 0}\frac{(2x\sin 2x)''}{(\sinh^2 x)''} = \lim_{x\to 0}\frac{(2x\sin 2x)'}{(\sinh^2 x)'} = \lim_{x\to 0}\frac{2x\sin 2x}{\sinh^2 x} = 4$

d) $\lim_{x\to 0} \frac{e^x - 2x - e^{-x}}{x - \sin x}$

$\lim_{x\to 0} e^x - 2x - e^{-x} = e^0 - 2\cdot 0 - e^{-0} = 1 - 0 - 1 = 0$

$\lim_{x\to 0} x - \sin x = 0 - \sin 0 = 0$

$\left.\right\} \lim_{x\to 0} e^x - 2x - e^{-x} = \lim_{x\to 0} x - \sin x = 0$

1. Ableitung: (führt zu Division mit 0)

$(e^x - 2x - e^{-x})' = e^x - 2 + e^{-x}$

$(x - \sin x)' = 1 - \cos x$

$\lim_{x\to 0} e^x - 2 + e^{-x} = e^0 - 2 + e^{-0} = 1 - 2 + 1 = 0$

$\lim_{x\to 0} 1 - \cos x = 1 - \cos 0 = 1 - 1 = 0$

$\left.\right\} \lim_{x\to 0} e^x - 2 + e^{-x} = \lim_{x\to 0} 1 - \cos x = 0$

2. Ableitung: (führt zu Division mit 0)

$(e^x - 2x - e^{-x})'' = (e^x - 2 + e^{-x})' = e^x - e^{-x}$

$(x - \sin x)'' = (1 - \cos x)' = \sin x$

$\lim_{x\to 0} e^x - e^{-x} = e^0 - e^{-0} = 1 - 1 = 0$

$\lim_{x\to 0} \sin x = \sin 0 = 0$

$\left.\right\} \lim_{x\to 0} e^x - e^{-x} = \lim_{x\to 0} \sin x = 0$

3. Ableitung:

$(e^x - 2x - e^{-x})''' = (e^x - e^{-x})' = e^x + e^{-x}$

$(x - \sin x)''' = (\sin x)' = \cos x$

$\lim_{x\to 0} = \frac{(e^x - 2x - e^{-x})'''}{(x - \sin x)'''} = \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x} = \frac{e^0 + e^{-0}}{\cos 0} = \frac{1+1}{1} = \frac{2}{1} = 2$

Aus der Regel von L'Hospital folgt $\lim_{x\to 0} \frac{(e^x - 2x - e^{-x})'''}{(x - \sin x)'''} = \lim_{x\to 0} \frac{(e^x - 2x - e^{-x})''}{(x - \sin x)''} = \lim_{x\to 0} \frac{(e^x - 2x - e^{-x})'}{(x - \sin)'}$

$= \lim_{x\to 0} \frac{e^x - 2x - e^{-x}}{x - \sin x} = 2$ //.