Aufgabe 7.8 (Regel von L'Hospital)

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$
$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$$

Berechnen Sie:

a) $\lim\limits_{x\to\pi} \dfrac{\sin 3x}{\tan 5x}$ b) $\lim\limits_{x\to 0} \dfrac{2x\sin 2x}{\sinh^2 x}$ c) $\lim\limits_{x\to 0} \dfrac{2-\sqrt{4-x^2}}{3-\sqrt{9-x^2}}$ d) $\lim\limits_{x\to 0} \dfrac{e^x - 2x - e^{-x}}{x - \sin x}$

e) $\lim\limits_{x\to\infty} \dfrac{x^5}{e^{3x}}$ f) $\lim\limits_{x\downarrow 0} x^x$ g) $\lim\limits_{x\uparrow 0} (1+\sin x)^{\frac{1}{x}}$ h) $\lim\limits_{x\uparrow \frac{\pi}{2}} (\tan x)^{\cot x}$

a) $\lim\limits_{x\to\pi} \dfrac{\sin 3x}{\tan 5x}$

$$\left.\begin{aligned}&\lim_{x\to\pi} \sin 3x \Rightarrow \sin 3\pi = 0\\ &\lim_{x\to\pi} \tan 5x \Rightarrow \tan 5\pi = 0\end{aligned}\right\} \lim_{x\to\pi} \sin 3x = \lim_{x\to\pi} \tan 5x = 0$$

$$\lim_{x\to\pi} \frac{(\sin 3x)'}{(\tan 5x)'} = \frac{\lim\limits_{x\to\pi} 3\cos 3x \cdot \cos^2 5x}{\lim\limits_{x\to\pi} 5\cos^2 5x + 5\sin^2 5x}$$

$$= \frac{3\cos 3\pi \cdot \cos^2 5\pi}{5\cos^2 5\pi + 5\sin^2 5\pi}$$

$$= \frac{3\cdot(-1)\cdot 1}{5\cdot 1 + 5\cdot 0}$$

$$= -\frac{3}{5}$$

$$\tan 5x = \frac{\sin 5x}{\cos 5x} \Rightarrow (\tan 5x)' = \left(\frac{\sin 5x}{\cos 5x}\right)'$$

$$= \frac{5\cos 5x \cdot \cos 5x - \sin 5x \cdot (-5)\sin 5x}{\cos^2 5x}$$

$$= \frac{5\cos^2 5x + 5\sin^2 5x}{\cos^2 5x}$$

Aus der Regel von L'Hospital folgt $\lim\limits_{x\to\pi} \dfrac{(\sin 3x)'}{(\tan 5x)'} = \lim\limits_{x\to\pi} \dfrac{\sin 3x}{\tan 5x} = -\dfrac{3}{4}$ //

b) $\lim\limits_{x\to 0} \dfrac{2x\sin 2x}{\sinh^2 x}$

$$\left.\begin{aligned}&\lim_{x\to 0} 2x \cdot \sin 2x = 2\cdot 0 \cdot \sin(2\cdot 0) = 0\\ &\lim_{x\to 0} \sinh^2 x = \lim_{x\to 0}\left(\frac{1}{2}\left(e^x - e^{-x}\right)\right)^2 = \left(\frac{1}{2}\left(e^0 - e^{-0}\right)\right)^2 = \left(\frac{1}{2}(1-1)\right)^2 = 0\end{aligned}\right\} \lim_{x\to 0} 2x\sin 2x = \lim_{x\to 0} \sinh^2 x = 0$$

$$2\cdot(x\cdot\sin 2x)' = 2\cdot(\sin 2x + x\cdot\cos 2x\cdot 2)$$
$$= 2\sin 2x + 4x\cos 2x$$

$$\lim_{x\to 0} \frac{(2x\sin 2x)'}{(\sinh^2 x)'} = \lim_{x\to 0} \frac{4x\cos 2x + 2\sin 2x}{\frac{e^{2x}-e^{-2x}}{2}}$$

$$= \lim_{x\to 0} \frac{8x\cos 2x + 4\sin 2x}{e^{2x} - e^{-2x}}$$

$$= \lim_{x\to 0} \frac{8x\cos 2x + 4\sin 2x}{(e^x + e^{-x})(e^x - e^{-x})}$$

$$\Rightarrow \lim_{x\to 0} \frac{(8x\cos 2x + 4\sin 2x)(e^x + e^{-x})}{(e^x + e^{-x})(e^x - e^{-x})(e^x + e^{-x})}$$

$$(2x\sin 2x)' = 2x\cdot 2\cos 2x + 2\sin 2x$$

$$= 4x\cos 2x + 2\sin 2x$$

$$(\sinh^2 x)' = \left(\left(\frac{1}{2}\left(e^x - e^{-x}\right)\right)^2\right)' = \left(\frac{1}{4}\left(e^x - e^{-x}\right)^2\right)'$$

$$= \left(\frac{1}{4}\left(e^{2x} - 2e^x\cdot e^{-x} + e^{-2x}\right)\right)'$$

$$= \left(\frac{e^{2x}}{4} - \frac{1}{2} + \frac{e^{-2x}}{4}\right)'$$

$$= \frac{e^{2x}}{2} - \frac{e^{-2x}}{2}$$

$\lim_{x\to0}(2x\sin 2x)' = \lim_{x\to0} 4x\cos 2x + 2\cdot\sin 2x = 4\cdot0\cos(2\cdot0)+2\cdot\sin(2\cdot0) = 0$

$\lim_{x\to0}(\sinh^2 x)' = \frac{e^{2x}-e^{-2x}}{2} = \frac{1-1}{2} = 0$

$\lim_{x\to0}(2x\sin 2x)' = \lim_{x\to0}(\sinh^2 x)' = 0$

$\lim_{x\to0}\frac{(2x\sin 2x)''}{(\sinh^2 x)''} = \lim_{x\to0}\frac{(4x\cos 2x + 2\cdot\sin 2x)'}{\left(\frac{e^{2x}-e^{-2x}}{2}\right)'}$

$(4x\cos 2x + 2\sin 2x)' = -8x\sin 2x + 8\cos 2x$

$\left(\frac{e^{2x}-e^{-2x}}{2}\right)' = e^{2x}+e^{-2x}$

$= \lim_{x\to0}\frac{-8x\sin 2x + 8\cos 2x}{e^{2x}+e^{-2x}}$

$= \frac{-8\cdot0\cdot\sin(2\cdot0)+8\cdot\cos(2\cdot0)}{e^{2\cdot0}+e^{-2\cdot0}}$

$= \frac{8}{2} = 4$

Aus dem Satz von L'Hospital folgt $\lim_{x\to0}\frac{(2x\sin 2x)''}{(\sinh^2 x)''} = \lim_{x\to0}\frac{(2x\sin 2x)'}{(\sinh^2 x)'} = \lim_{x\to0}\frac{2x\sin 2x}{\sinh^2 x} = 4$ //

c) $\lim_{x\to0}\frac{2-\sqrt{4-x^2}}{3-\sqrt{9-x^2}}$

$\lim_{x\to0} 2-\sqrt{4-0^2} = 2-\sqrt{4} = 2-2 = 0$

$\lim_{x\to0} 3-\sqrt{9-0^2} = 3-\sqrt{9} = 3-3 = 0$

$\lim_{x\to0} 2-\sqrt{4-x^2} = \lim_{x\to0} 3-\sqrt{9-x^2} = 0$

1. Ableitung:

$(2-\sqrt{4-x^2})' = \frac{2x}{2\sqrt{4-x^2}} = \frac{x}{\sqrt{4-x^2}}$

$(3-\sqrt{9-x^2})' = \frac{2x}{2\sqrt{9-x^2}} = \frac{x}{\sqrt{9-x^2}}$

$\lim_{x\to0}\frac{x}{\sqrt{4-x^2}} = \frac{0}{\sqrt{4-0^2}} = 0$

$\lim_{x\to0}\frac{x}{\sqrt{9-x^2}} = \frac{0}{\sqrt{9-0^2}} = 0$

$\lim_{x\to0}\frac{\frac{x}{\sqrt{4-x^2}}}{\frac{x}{\sqrt{9-x^2}}} = \lim_{x\to0}\frac{x\sqrt{9-x^2}}{x\sqrt{4-x^2}} = \lim_{x\to0}\frac{\sqrt{9-x^2}}{\sqrt{4-x^2}} = \frac{\sqrt{9-0^2}}{\sqrt{4-0^2}} = \frac{\sqrt{9}}{\sqrt{4}} = \frac{3}{2}$

Aus der Regel von L'Hospital folgt $\lim_{x\to0}\frac{(2-\sqrt{4-x^2})'}{(3-\sqrt{9-x^2})'} = \lim_{x\to0}\frac{2-\sqrt{4-x^2}}{3-\sqrt{9-x^2}} = \frac{3}{2}$ //.

e) $\lim_{x\to\infty}\frac{x^5}{e^{3x}}$

$\lim_{x\to\infty} x^5 \Rightarrow \infty$

$\lim_{x\to\infty} e^{3x} \Rightarrow \infty$

$\lim_{x\to\infty} x^5 = \lim_{x\to\infty} e^{3x} = \infty$

1. Ableitung:

$(x^5)' = 5x^4$

$(e^{3x})' = 3e^{3x}$

$$\left.\begin{array}{l} \lim_{x\to\infty} 5x^4 \Rightarrow \infty \\ \lim_{x\to\infty} 3e^{3x} \Rightarrow \infty \end{array}\right\} \lim_{x\to\infty} 5x^4 = \lim_{x\to\infty} 3e^{3x} = \infty$$

2. Ableitung:

$(5x^4)' = 20x^3$

$(3e^{3x})' = 9e^{3x}$

$$\left.\begin{array}{l} \lim_{x\to\infty} 20x^3 \Rightarrow \infty \\ \lim_{x\to\infty} 27e^{3x} \Rightarrow \infty \end{array}\right\} \lim_{x\to\infty} 20x^3 = \lim_{x\to\infty} 27e^{3x} = \infty$$

3. Ableitung:

$(20x^3)' = 60x^2$

$(27e^{3x})' = 81e^{3x}$

$$\left.\begin{array}{l} \lim_{x\to\infty} 60x^2 \Rightarrow \infty \\ \lim_{x\to\infty} 81e^{3x} \Rightarrow \infty \end{array}\right\} \lim_{x\to\infty} 60x^2 = \lim_{x\to\infty} 81e^{3x} = \infty$$

4. Ableitung:

$(60x^2)' = 120x$

$(81e^{3x})' = 243e^{3x}$

$$\left.\begin{array}{l} \lim_{x\to\infty} 120x \Rightarrow \infty \\ \lim_{x\to\infty} 243e^{3x} \Rightarrow \infty \end{array}\right\} \lim_{x\to\infty} 120x = \lim_{x\to\infty} 243e^{3x} = \infty$$

5. Ableitung:

$(120x)' = 120$

$(243e^{3x})' = 731e^{3x}$

$$\lim_{x\to\infty} \frac{120}{731e^{3x}} \Rightarrow \frac{120}{\lim_{x\to\infty} 731e^{3x}} \Rightarrow 0$$

Nach dem Satz von L'Hospital folgt $\lim_{x\to\infty} \frac{(5x^4)^{(5)}}{(e^{3x})^{(5)}} = \lim_{x\to\infty} \frac{(5x)^{(4)}}{(e^{3x})^{(4)}} = \lim_{x\to\infty} \frac{(5x)'''}{(e^{3x})'''} = \lim_{x\to\infty} \frac{(5x)''}{(e^{3x})''} = \lim_{x\to\infty} \frac{(5x)'}{(e^{3x})'}$

$= \lim_{x\to\infty} \frac{5x}{e^{3x}} = 0$

d) $\lim_{x\to 0} \frac{e^x - 2x - e^{-x}}{x - \sin x}$

$$\left.\begin{aligned} &\lim_{x\to 0} e^x - 2x - e^{-x} = e^0 - 2\cdot 0 - e^{"-0"} = 1 - 0 - 1 = 0 \\ &\lim_{x\to 0} x - \sin x = 0 - \sin 0 = 0 \end{aligned}\right\} \lim_{x\to 0} e^x - 2x - e^{-x} = \lim_{x\to 0} x - \sin x = 0$$

1. Ableitung: (führt zu Division mit 0)

$(e^x - 2x - e^{-x})' = e^x - 2 + e^{-x}$

$(x - \sin x)' = 1 - \cos x$

$$\left.\begin{aligned} &\lim_{x\to 0} e^x - 2 + e^{-x} = e^0 - 2 + e^{"-0"} = 1 - 2 + 1 = 0 \\ &\lim_{x\to 0} 1 - \cos x = 1 - \cos 0 = 1 - 1 = 0 \end{aligned}\right\} \lim_{x\to 0} e^x - 2 + e^{-x} = \lim_{x\to 0} 1 - \cos x = 0$$

2. Ableitung: (führt zu Division mit 0)

$(e^x - 2x - e^{-x})'' = (e^x - 2 + e^{-x})' = e^x - e^{-x}$

$(x - \sin x)'' = (1 - \cos x)' = \sin x$

$$\left.\begin{aligned} &\lim_{x\to 0} e^x - e^{-x} = e^0 - e^{"-0"} = 1 - 1 = 0 \\ &\lim_{x\to 0} \sin x = \sin 0 = 0 \end{aligned}\right\} \lim_{x\to 0} e^x - e^{-x} = \lim_{x\to 0} \sin x = 0$$

3. Ableitung:

$(e^x - 2x - e^{-x})''' = (e^x - e^{-x})' = e^x + e^{-x}$

$(x - \sin x)''' = (\sin x)' = \cos x$

$$\lim_{x\to 0} = \frac{(e^x - 2x - e^{-x})'''}{(x - \sin x)'''} = \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x} = \frac{e^0 + e^{"-0"}}{\cos 0} = \frac{1+1}{1} = \frac{2}{1} = 2$$

Aus der Regel von L'Hospital folgt $\lim_{x\to 0} \frac{(e^x - 2x - e^{-x})'''}{(x - \sin x)'''} = \lim_{x\to 0} \frac{(e^x - 2x - e^{-x})''}{(x - \sin x)''} = \lim_{x\to 0} \frac{(e^x - 2x - e^{-x})'}{(x - \sin)'}$

$= \lim_{x\to 0} \frac{e^x - 2x - e^{-x}}{x - \sin x} = 2$ //.