6.14) Regel von Sarrus und Gram'sche Determinante

a) Regel von Sarrus:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Herleitung:

$$\begin{array}{ccccc} & & & - & + \\ a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot a_{33} + \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot (-a_{32}) + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \cdot a_{31}$$

$$= (a_{11}a_{22} - a_{12}a_{21})a_{33} + (a_{11}a_{23} - a_{21}a_{13})(-a_{32}) + (a_{12}a_{23} - a_{13}a_{22})a_{31}$$

$$= a_{11}a_{22}a_{33} + a_{21}a_{13}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \quad \blacksquare$$

b) $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ $\underline{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

i) $[\underline{a}, \underline{b}, \underline{c}] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Spatprodukt: $(a \times b) \cdot c = (b \times c) \cdot a = (c \times a) \cdot b$

$$\underline{a} \times \underline{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

$$(\underline{a} \times \underline{b}) \cdot \underline{c} = \left| \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right| = |c_1(a_2b_3 - a_3b_2) + c_2(a_3b_1 - a_1b_3) + c_3(a_1b_2 - a_2b_1)|$$

Determinante nach Sarrus:

$$a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - b_1a_2c_3 - a_1c_2b_3 - c_1b_2a_3$$

$$= c_1(a_2b_3 - a_3b_2) + c_2(a_3b_1 - a_1b_3) + c_3(a_1b_2 - a_2b_1) = (a \times b) \cdot c$$ wenn die Determinante positiv ist