3.14 | Konvergenz von Reihen

a) $\frac{1}{2} + \frac{3}{4} + \frac{5}{6} + \frac{7}{8} + \dots$

$a_n = \frac{2n-1}{2n}$

Quotientenkriterium:

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2(n+1)-1}{2(n+1)} \cdot \frac{2n}{2n-1}\right| = \left|\frac{2n+2-1}{2n+1} \cdot \frac{2n}{2n-1}\right| = \left|\frac{2n}{2n-1}\right| > 1 \ \forall n \in \mathbb{N} \Rightarrow$ divergent

b) $\frac{2}{3} + \frac{4}{9} + \frac{6}{27} + \frac{8}{81} + \dots$

$a_n = \frac{2n}{3^n}$

Quotientenkriterium:

$\alpha = \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2(n+1)}{3^{n+1}} \cdot \frac{3^n}{2n}\right| = \left|\frac{2n+2}{3 \cdot 2n}\right| = \left|\frac{n+1}{3n}\right| = \left|\frac{1}{3} + \frac{1}{3n}\right| < 1$, da $\frac{1}{3} \geq \frac{1}{3n} \ \forall n \in \mathbb{N} \Rightarrow$ konvergent

c) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$

divergente Minorante: $\frac{1}{n}$, da $n > \sqrt{n^2-1} \Rightarrow \frac{1}{n} < \frac{1}{\sqrt{n^2-1}} \Rightarrow \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$ ist divergent

d) $\sum_{n=1}^{\infty} (\sqrt[n]{a} - 1)^n \quad (a \geq 1)$

Wurzelkriterium: $\alpha = \lim_{n\to\infty} \sqrt[n]{a_n} \Rightarrow \lim_{n\to\infty} \sqrt[n]{(\sqrt[n]{a} - 1)^n} = \lim_{n\to\infty} \sqrt[n]{a} - 1 \Rightarrow 1 - 1 = 0 < 1 \Rightarrow$ konvergent

e) $1 + \frac{1 \cdot 2}{1 \cdot 3} + \frac{1 \cdot 2 \cdot 3}{1 \cdot 3 \cdot 5} + \dots$

$a_n = \sum_{n=1}^{\infty} \frac{\prod_{n=1}^{\infty} n}{\prod_{n=1}^{\infty} 2n-1}$

Quotientenkriterium: $\alpha = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \Rightarrow \lim_{n\to\infty} \frac{\prod_{n=1}^{\infty} n+1}{\prod_{n=1}^{\infty} 2(n+1)-1} \cdot \frac{\prod_{n=1}^{\infty} 2n-1}{\prod_{n=1}^{\infty} n} \Rightarrow \lim_{n\to\infty} \frac{\prod_{n=0}^{\infty} n}{\prod_{n=0}^{\infty} 2n-1} \cdot \frac{\prod_{n=1}^{\infty} 2n-1}{\prod_{n=1}^{\infty} n}$

$= \lim_{n\to\infty} \frac{0}{-1} \Rightarrow 0 \Rightarrow$ konvergiert

f) $\frac{1}{100} + \frac{1}{200} + \frac{1}{300} + \dots$

$a_n = \sum_{n=1}^{\infty} \frac{1}{100n} \Rightarrow \frac{1}{100} \sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow$ harmonische Reihe $\Rightarrow$ divergent.