3.9 Grenzwerte von Folgen

a) $\lim\limits_{n\to\infty} \frac{n^2+n+2}{4n^3+1} \Rightarrow \lim\limits_{n\to\infty} \frac{\frac{1}{n}+\frac{1}{n^2}+2\frac{1}{n^3}}{4+\frac{1}{n^3}} \Rightarrow \frac{\lim \frac{1}{n}+\lim\frac{1}{n^2}+2\lim\frac{1}{n^3}}{4+\lim\frac{1}{n^3}} \Rightarrow \frac{0}{4}=0$

b) $\lim\limits_{n\to\infty} \frac{(n+1)^2-n^2}{n} = \lim\limits_{n\to\infty} \frac{n^2+2n+1-n^2}{n} = \lim\limits_{n\to\infty} \frac{2n+1}{n} = \lim\limits_{n\to\infty} \frac{2+\frac{1}{n}}{\frac{1}{n}} \Rightarrow \frac{\lim\frac{1}{n}+2}{\lim\frac{1}{n}} \Rightarrow \infty$

c) $\lim\limits_{n\to\infty} \frac{4n^4-n+2}{2n^4+2n^2+n} = \lim\limits_{n\to\infty} \frac{4-\frac{1}{n^3}-\frac{1}{n^2}}{2+2\frac{1}{n^2}+\frac{1}{n^3}} \Rightarrow \frac{4-\lim\frac{1}{n^3}-\lim\frac{1}{n^3}}{2+2\lim\frac{1}{n^2}+\lim\frac{1}{n^3}} \Rightarrow \frac{4}{2}=2$

d) $\lim\limits_{n\to\infty} \frac{n+n^2}{1+n+n^2+3n^3+4n^4} = \lim\limits_{n\to\infty} \frac{\cancel{5}+\frac{1}{n}}{\frac{1}{n^4}+\frac{1}{n^3}+\frac{1}{n^2}+3\frac{1}{n}+4} \Rightarrow \frac{0}{4}=0$

e) $\lim\limits_{n\to\infty} \frac{-4n^2+3n^3+7}{2n^3+5n} = \lim\limits_{n\to\infty} \frac{-4\frac{1}{n}+3+7\frac{1}{n^3}}{2+5\frac{1}{n^2}} \Rightarrow \frac{3}{2}$

f) $\lim\limits_{n\to\infty} \frac{2n+(-1)^n}{n} \Rightarrow \lim\limits_{n\to\infty} \frac{2+\frac{1}{n}(-1)^n}{1} \Rightarrow 2$

g) $\lim\limits_{n\to\infty} \frac{2n^4-3n^2+17}{1000n^3+n^2+n} = \lim\limits_{n\to\infty} \frac{2-3\frac{1}{n^2}+17\frac{1}{n^4}}{1000\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}} \Rightarrow \infty$

h) $\lim\limits_{n\to\infty} \left(\frac{2-3n}{1+4n}\right)^2 \Rightarrow \left(\lim\limits_{n\to\infty} \frac{2-3n}{1+4n}\right)^2 = \left(\lim\limits_{n\to\infty} \frac{2\frac{1}{n}-3}{\frac{1}{n}+4}\right)^2 \Rightarrow \left(\frac{-3+\lim 2\frac{1}{n}}{4+\lim\frac{1}{n}}\right)^2 \Rightarrow \left(-\frac{3}{4}\right)^2 = \frac{9}{16}$

i) $\lim\limits_{n\to\infty} \frac{3^{2n}-19}{9^n+12} = \lim\limits_{n\to\infty} \frac{3^{2n}-19}{3^{2n}+12} = \lim\limits_{n\to\infty} \frac{1-19\frac{1}{3^{2n}}}{1+12\frac{1}{3^{2n}}} \Rightarrow \frac{1-\lim 19\frac{1}{3^{2n}}}{1+\lim 12\frac{1}{3^{2n}}} \Rightarrow \frac{1}{1}=1$

j) $\lim\limits_{n\to\infty} \frac{4(n+1)^4}{3n^4+3n+5} = \lim\limits_{n\to\infty} \frac{4(n^2+2n+1)(n^2+2n+1)}{3n^4+3n+5} = \lim\limits_{n\to\infty} \frac{4(n^4+2n^3+n^2+2n^3+2n^2+2n+n^2+2n+1)}{3n^4+3n+5}$
$= \lim\limits_{n\to\infty} \frac{4n^4+16n^3+16n^2+16n+4}{3n^4+3n+5} \Rightarrow \frac{4}{3}$

k) $\lim\limits_{n\to\infty} \frac{3n^2+4n}{\sqrt{n^6+n^4+1}} = \lim\limits_{n\to\infty} \frac{n^2(3+4\frac{1}{n})}{\sqrt{n^6}\cdot\sqrt{\frac{1}{n^2}+\frac{1}{n^6}}} = \lim\limits_{n\to\infty} \frac{3+4\frac{1}{n}}{\sqrt{\frac{1}{n^2}+\frac{1}{n^6}}} \Rightarrow \frac{3+4\lim\frac{1}{n}}{\sqrt{\lim\frac{1}{n}+\lim\frac{1}{n^6}}} \Rightarrow \frac{3}{1}=3$

l) $\lim\limits_{n\to\infty} \sqrt[3]{\frac{n+5n+2n^2}{16n^2+12n+4}} = \sqrt[3]{\frac{5\frac{1}{n}+5\frac{1}{n}+2}{16+12\frac{1}{n}+4\frac{1}{n}}} \Rightarrow \sqrt[3]{\frac{1}{8}} = \frac{1}{2}$

m) $\lim\limits_{n\to\infty} \frac{1+2+\dots+n}{n^2} = \lim\limits_{n\to\infty} \frac{\frac{n(n+1)}{2}}{n^2} = \lim\limits_{n\to\infty} \frac{n^2+n}{2n^2} = \lim\limits_{n\to\infty} \frac{1+\frac{1}{n}}{2} \Rightarrow \frac{1+\lim\frac{1}{n}}{2} = \frac{1}{2}$

n) $\lim\limits_{n\to\infty} \sqrt{n+1}-\sqrt{n} = \lim\limits_{n\to\infty} n^2\sqrt{\frac{1}{n}+\frac{1}{n^2}} - n^2\sqrt{\frac{1}{n}} \Rightarrow n^2\cdot 0 - 0 = 0$

1) $\lim\limits_{n\to\infty} \sqrt[n]{a} \; (a \geq 1) = \lim\limits_{n\to\infty} a^{\frac{1}{n}} \Rightarrow a^{\lim \frac{1}{n}} \Rightarrow a^0 = 1$

2) $\lim\limits_{n\to\infty} \sqrt{9n^2+2n+1} - 3n = \lim\limits_{n\to\infty} \frac{(\sqrt{9n^2+2n+1}-3n)(\sqrt{9n^2+2n+1}+3n)}{\sqrt{9n^2+2n+1}+3n} = \lim\limits_{n\to\infty} \frac{9n^2+2n+1-9n^2}{n(\sqrt{9+2\frac{1}{n}+\frac{1}{n^2}}+3)} = \lim\limits_{n\to\infty} \frac{2+\frac{1}{n}}{\sqrt{9+2\frac{1}{n}+\frac{1}{n^2}}+3}$

$= \frac{2+\lim \frac{1}{n}}{\sqrt{9+\lim 2\frac{1}{n}+\lim\frac{1}{n^2}}+3} = \frac{2}{3+3} = \frac{1}{3}$

3) $\lim\limits_{n\to\infty} (1-\frac{1}{2})(1-\frac{1}{3})\dots(1-\frac{1}{n}) = \lim\limits_{n\to\infty} (\frac{2}{2}-\frac{1}{2})(\frac{3}{3}-\frac{1}{3})\cdots(\frac{n}{n}-\frac{1}{n}) = \lim\limits_{n\to\infty} (\frac{2-1}{2})(\frac{3-1}{3})\cdots(\frac{n-1}{n}) \Rightarrow \lim\limits_{n\to\infty} \frac{1}{n} \Rightarrow 0$

4) $\lim\limits_{n\to\infty} \frac{1-\sqrt{\frac{n-1}{n}}}{1-\frac{n-1}{n}} = \lim\limits_{n\to\infty} \frac{1-\frac{n-1}{n}}{(1-\frac{n-1}{n})(1+\sqrt{\frac{n-1}{n}})} = \lim\limits_{n\to\infty} \frac{1}{1+\sqrt{\frac{n-1}{n}}} = \lim\limits_{n\to\infty} \frac{1}{1+\sqrt{1-\frac{1}{n}}} \Rightarrow \frac{1}{1+1} = \frac{1}{2}$

5) $\lim\limits_{n\to\infty} (1+\frac{1}{n^2})^{n^2} \overset{\text{Teilmenge}}{\Rightarrow} \lim\limits_{n\to\infty} (1+\frac{1}{n})^n = e$

6) $\lim\limits_{n\to\infty} (1+\frac{1}{n^2})^n \Rightarrow \lim\limits_{n\to\infty} \sqrt[n]{(1+\frac{1}{n^2})^{n^2}} \Rightarrow \lim\limits_{n\to\infty} \sqrt[n]{e} \Rightarrow 1$

7) $\lim\limits_{n\to\infty} \frac{n!}{n^n} = \lim\limits_{n\to\infty} \frac{n(n-1)(n-2)\dots}{n\cdot n\cdot n\cdot n\cdot\dots} = \lim\limits_{n\to\infty} \frac{n^{n-1}(1-\frac{1}{n})(1-2\frac{1}{n})\dots}{n^n} = \lim\limits_{n\to\infty} \frac{1}{n}(1-\frac{1}{n})(1-2\frac{1}{n})\dots$

$\Rightarrow \lim\limits_{n\to\infty} \frac{1}{n}(1-\lim\limits_{n\to\infty}\frac{1}{n})(1-2\lim\limits_{n\to\infty}\frac{1}{n})\dots \Rightarrow 0$