6.15 Invertierbarkeit von Matrizen

a)

i) $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ sei $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A \cdot A^{-1} = E$

$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2a+c & a+c \\ 2b+d & b+d \end{pmatrix}$

$2a+c = 1 \Leftrightarrow 2a-a = 1 \Leftrightarrow a = 1$
$a+c = 0 \Leftrightarrow c = -a \Leftrightarrow c = -1$
$2b+d = 0 \Leftrightarrow d = -2b \Leftrightarrow d = 2$
$b+d = 1 \Leftrightarrow -b = 1 \Leftrightarrow b = -1$

$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ //

ii) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ sei $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A \cdot A^{-1} = E$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+c & a+c \\ b+d & b+d \end{pmatrix}$

In ein LGS übertragen:

$\left.\begin{array}{l} a+c = 1 \\ a+c = 0 \end{array}\right\}$ Widerspruch
$\left.\begin{array}{l} b+d = 1 \\ b+d = 0 \end{array}\right\}$ Widerspruch

Die Matrix A ist nicht invertierbar.

iii) $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 1 & -2 \\ 1 & 2 & 0 \end{pmatrix}$ sei $A^{-1} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$A \cdot A^{-1} = E$

$\begin{pmatrix} 3 & 1 & 4 \\ 0 & 1 & -2 \\ 1 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3a+d+4g & d-2g & a+2d \\ 3b+e+4h & e-2h & b+2e \\ 3c+f+4i & f-2i & c+2f \end{pmatrix}$

$3a+d+4g = 1 \Rightarrow 3\cdot(-2d)+d+4\cdot\frac{1}{2}d = -6d+d+2d \Leftrightarrow -3d = 1 \Leftrightarrow d = -\frac{1}{3}$
$d-2g = 0 \Leftrightarrow g = \frac{1}{2}d \Rightarrow g = \frac{1}{2}\cdot\left(-\frac{1}{3}\right) = -\frac{1}{6}$
$a+2d = 0 \Leftrightarrow a = -2d \Rightarrow a = -2\cdot\left(-\frac{1}{3}\right) = \frac{2}{3}$

$3b+e+4h = 0 \Leftrightarrow 0 = 3\cdot(-2e)+e+4\cdot\left(-\frac{1}{2}\right)(e-1) = -6e+e-2e+2 \Leftrightarrow e = \frac{2}{7}$
$e-2h = 1 \Leftrightarrow h = -\frac{1}{2}(e-1) \Rightarrow h = -\frac{1}{2}\cdot\frac{2}{7}+\frac{1}{2} = -\frac{1}{7}+\frac{1}{2} = \frac{5}{14}$
$b+2e = 0 \Leftrightarrow b = -2e \Rightarrow b = -2\cdot\frac{2}{7} = -\frac{4}{7}$

$3c+f+4i = 0 \Leftrightarrow 0 = 3-6f+f+2f = 3-3f \Leftrightarrow f = 1$
$f-2i = 0 \Leftrightarrow i = \frac{1}{2}f \Rightarrow i = \frac{1}{2}$
$c+2f = 1 \Leftrightarrow c = 1-2f \Rightarrow c = 1-2 = -1$

$\begin{pmatrix} \frac{2}{3} & -\frac{4}{7} & -1 \\ -\frac{1}{3} & \frac{2}{7} & 1 \\ -\frac{1}{6} & \frac{5}{14} & \frac{1}{2} \end{pmatrix}$ // $= \frac{1}{6}\begin{pmatrix} 4 & -\frac{24}{7} & -6 \\ -2 & \frac{12}{7} & 6 \\ -1 & \frac{15}{7} & 3 \end{pmatrix}$ Im 2ten LGS verrechnet!

ii)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{pmatrix} \quad \text{sei } A^{-1} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$A \cdot A^{-1} = E$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{pmatrix} \cdot \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In ein LGS übertragen:

$$\begin{aligned} a+2d+3g &= 1 \\ 2a+3d+4g &= 0 \\ a+5d+7g &= 0 \end{aligned} \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 0 \\ 1 & 5 & 7 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 0 \\ 0 & 3 & 4 & -1 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 1 & 2 & 3 & 1 \\ 0 & 3 & 4 & -1 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 0 & 2 & 3 & \frac{1}{2} \\ 0 & 3 & 4 & -1 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 0 & 2 & 3 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & -2 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & -11{,}5 \\ 0 & 0 & -1 & -4 \end{array}\right)$$

$2a = 1 \Leftrightarrow a = \frac{1}{2} = 0{,}5$
$2d = -11{,}5 \Leftrightarrow d = 5{,}75$
$-g = -4 \Leftrightarrow g = 4$

$$\begin{aligned} b+2e+3h &= 0 \\ 2b+3e+4h &= 1 \\ b+5e+7h &= 0 \end{aligned} \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 1 \\ 1 & 5 & 7 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 1 \\ 0 & 3 & 4 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 1 & 2 & 3 & 0 \\ 0 & 3 & 4 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 0 & 2 & 3 & -\frac{1}{2} \\ 0 & 3 & 4 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 0 & 2 & 3 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{3}{4} \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & -5 \\ 0 & 0 & -1 & \frac{3}{2} \end{array}\right)$$

$2b = 1 \Leftrightarrow b = \frac{1}{2}$
$2e = -5 \Leftrightarrow e = -\frac{5}{2}$
$-h = \frac{3}{2} \Leftrightarrow h = -\frac{3}{2}$

$$\begin{aligned} c+2f+3i &= 0 \\ 2c+3f+4i &= 0 \\ c+5f+7i &= 1 \end{aligned} \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 0 \\ 1 & 5 & 7 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 0 \\ 0 & 3 & 4 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 0 & 3 & 4 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & -1 \\ 0 & 2 & 3 & \frac{1}{2} \\ 0 & 3 & 4 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & -1 \\ 0 & 2 & 3 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{4} \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & -1 & \frac{1}{2} \end{array}\right)$$

$2c = -1 \Leftrightarrow c = -\frac{1}{2}$
$2f = -1 \Leftrightarrow f = -\frac{1}{2}$
$-i = \frac{1}{2} \Leftrightarrow i = -\frac{1}{2}$

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{23}{4} & -\frac{5}{2} & -\frac{1}{2} \\ 4 & \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ \frac{23}{2} & -5 & -1 \\ 8 & -3 & -1 \end{pmatrix}$$

verrechnet!

iii)

$$\begin{pmatrix} 3 & 1 & 4 \\ 0 & 1 & -2 \\ 1 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow A^{-1} = \begin{pmatrix} 3a+d+4g & d-2g & a+2d \\ 3b+e+h & e-2h & d+2e \\ 3c+f+i & f-2i & c+2f \end{pmatrix}$$

adg:

$$\left(\begin{array}{ccc|c} 3 & 1 & 4 & 1 \\ 0 & 1 & -2 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 3 & 2 & 0 & 1 \\ 0 & 1 & -2 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & -\frac{1}{2} \\ 0 & 1 & -2 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & -\frac{1}{2} \\ 0 & 0 & -2 & \frac{1}{4} \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{8} \end{array}\right)$$

beh:

$$\left(\begin{array}{ccc|c} 3 & 1 & 4 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 2 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 3 & 2 & 0 & 2 \\ 0 & 1 & -2 & 1 \\ 1 & 2 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 1 & -2 & 1 \\ 1 & 2 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & -\frac{1}{2} \\ 0 & 1 & -2 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & -\frac{1}{2} \\ 0 & 0 & -2 & \frac{1}{4} \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{8} \end{array}\right)$$

cfi:

$$\left(\begin{array}{ccc|c} 3 & 1 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} 3 & 2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & -1 \\ 0 & 1 & -2 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & \frac{1}{2} \\ 0 & 1 & -2 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & \frac{1}{2} \\ 0 & 0 & -2 & -\frac{1}{4} \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{8} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \end{pmatrix} = \begin{pmatrix} 4 & 8 & -4 \\ -2 & -2 & 2 \\ -1 & -1 & -1 \end{pmatrix} \frac{1}{8}$$

noch mal verrechnet!