3.10 Grenzwerte von Reihen

a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} \Rightarrow \lim_{b\to\infty} \sum_{n=0}^{b} \frac{(-1)^n}{2^n} \Rightarrow \lim_{b\to\infty} \sum_{n=0}^{b} \left(\frac{-1}{2}\right)^n \Rightarrow \frac{\lim_{b\to\infty} 1-\left(\frac{-1}{2}\right)^{b+1}}{1-\left(\frac{-1}{2}\right)} \Rightarrow \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$.

b) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2} - \frac{n}{n+1}\right) = \sum_{n=1}^{\infty} \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)} = \sum_{n=1}^{\infty} \frac{n^2+2n+1-n^2-2n}{(n+2)(n+1)} = \sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)}$

Partialbruchzerlegung:

$\frac{A}{n+2} + \frac{B}{n+1} = \frac{(n+1)A + B(n+2)}{(n+2)(n+1)} \Rightarrow A=-1, B=1$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n+2} - \frac{1}{n+1} \Rightarrow \frac{1}{3} - \frac{1}{2} + \frac{1}{4} - \frac{1}{3} + \frac{1}{5} - \frac{1}{6} \pm \ldots \Rightarrow \frac{1}{2} + \left(\frac{1}{3}-\frac{1}{3}\right) + \left(\frac{1}{4}-\frac{1}{4}\right) + \ldots \Rightarrow \frac{1}{2}$

c) $\sum_{n=2}^{\infty} \frac{1}{3^{n-1}} \Rightarrow \lim_{b\to\infty} \sum_{n=2}^{b} \frac{1}{3^{n-1}} \Rightarrow \lim_{b\to\infty} \sum_{n=0}^{b-2} \frac{1}{3^{n+1}} \Rightarrow \frac{1}{3} \lim_{b\to\infty} \sum_{n=0}^{b-2} \frac{1}{3^n} \Rightarrow \frac{1}{3} \lim_{b\to\infty} \frac{1-\frac{1}{3^{b-1}}}{1-\frac{1}{3}}$

$\Rightarrow \frac{1}{3} \cdot \frac{1}{\frac{2}{3}} = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$

d) $\sum_{n=1}^{\infty} \frac{1}{n^3+3n^2+3n+1} - \frac{1}{n^3} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{(n+1)^3} - \frac{1}{n^3} \Rightarrow \frac{1}{2^3} - 1 + \frac{1}{3^3} - \frac{1}{2^3} + \frac{1}{4^3} - \frac{1}{3^3} + \ldots \Rightarrow 1 + \frac{1}{2^3} - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{3^3} + \ldots$

$\Rightarrow -1$

e) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \Rightarrow \sum_{n=1}^{\infty} \frac{A}{n} + \frac{B}{n+1} \Rightarrow \sum_{n=1}^{\infty} \frac{A(n+1)}{n} + \frac{Bn}{n+1} \Rightarrow A=1, B=-1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} \Rightarrow 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} \ldots$

$\Rightarrow 1 - \frac{1}{2} + \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \ldots \Rightarrow 1$

f) $\sum_{n=1}^{\infty} \frac{3}{4^n} \Rightarrow \sum_{n=0}^{\infty} \frac{3}{4^n} - 3 \Rightarrow 3 \sum_{n=0}^{\infty} \frac{1}{4^n} \cdot 1 \Rightarrow 3 \lim_{b\to\infty} \sum_{n=0}^{b} \frac{1}{4^n} \cdot 1 \Rightarrow 3 \lim_{b\to\infty} \frac{1-\frac{1}{4^{b+1}}}{1-\frac{1}{4}} - 1 \Rightarrow 3\left(\frac{1}{\frac{3}{4}} - 1\right) = 4 - 3 = 1$