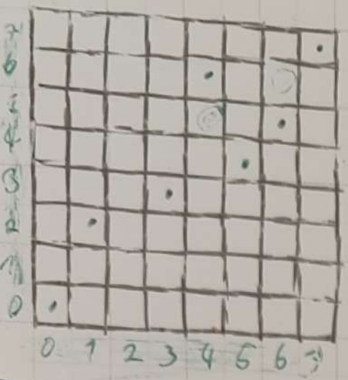
Altklausur (Die offizielle)

1)

a) Schachfeld



$$\underline{s} = \begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}-1\\2\end{pmatrix} + \begin{pmatrix}2\\-1\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix}$$

b) Laut GPT: min   Warum?

c) Punktnormalenform: $0 = \underline{n}(\underline{x} \times \underline{a})$

$$0 = \begin{pmatrix}1\\2\\3\end{pmatrix}\left(\underline{x} \times \begin{pmatrix}3\\2\\1\end{pmatrix}\right)$$

d) $f(x) = |(-x^2+1)|$

e)

arccos x

Umkehrfunktion:

$h = \arccos x \Leftrightarrow x = \cos h$

$$f^{-1\prime}(x) = \frac{1}{f'(f^{-1}(x))}$$

$$t'(h) = (\cos h)' = -\sin h$$

$$f^{-1\prime}(x) = \frac{1}{-\sin(\arccos x)} = \frac{1}{-\sqrt{\sin^2(\arccos x)}} = \frac{1}{-\sqrt{1-\cos^2(\arccos x)}} = \frac{1}{-\sqrt{1-x^2}} //$$

f) Die Feinheit einer Zerlegung ist definiert als die Breite des größten Teilintervals.

g) $f(x) = 7e^x //$

h) $f(x) = \sqrt{R^2 - x^2}$ $[-R, R]$

$$V = \pi\int_{-R}^{R} f(x)^2 dx = \pi\int_{-R}^{R} (\sqrt{R^2-x^2})^2 dx = \pi\int_{-R}^{R} R^2 - x^2 dx = \pi\left(xR^2 - \frac{1}{3}x^3 \Big|_{-R}^{R}\right)$$

$$= \pi\left(R \cdot R^2 - \frac{1}{3}R^3 - (-R)\cdot R^2 + \frac{1}{3}(-R^3)\right) = \pi\left(R^3 - \frac{1}{3}R^3 + R^3 - \frac{1}{3}R^3\right) = \pi\left(\frac{6R^3}{3} - \frac{2}{3}R^3\right) = \frac{4}{3}R^3 \cdot \pi$$

3) $A \cdot B = 0,\ A \neq 0,\ B \neq 0 \Rightarrow |A| = |B| = 0$

Widerspruchsbeweis:

Sei $|A| \neq 0$: -> A ist invertierbar

$A \cdot B = 0 \Leftrightarrow B = 0 \cdot A^{-1} \Leftrightarrow B = 0$ > Widerspruch zu der Vorraussetzung!

=> $|A| \neq 0$ muss gelten, damit die Formel stimmt

Dies kann analog auf B angewandt werden. ■

5) x sei die Gesamtanzahl der Passagiere

m sei die Anzahl der männlichen Passagiere

w sei die Anzahl der weiblichen Passagiere

$1: x = (w - \frac{1}{3}w) + (m + \frac{1}{3}w) = \frac{2}{3}w + (m + \frac{1}{3}w)$

$2: x = (\frac{2}{3}w + \frac{1}{3}(m + \frac{1}{3}w)) + ((m + \frac{1}{3}w) - \frac{1}{3}(m + \frac{1}{3}w))$

$= (\frac{6}{9}w + \frac{1}{3}m + \frac{1}{9}w) + (m + \frac{3}{9}w - \frac{1}{3}m - \frac{1}{9}w)$

$= (\frac{7}{9}w + \frac{1}{3}m) + (\frac{2}{3}m + \frac{2}{9}w)$

$3: (\frac{7}{9}w + \frac{1}{3}m) - (\frac{2}{3}m + \frac{2}{9}w) = 2 \Leftrightarrow \frac{7}{9}w - \frac{2}{9}w + \frac{1}{3}m - \frac{2}{3}m = 2 \Leftrightarrow \frac{5}{9}w - \frac{1}{3}m = 2$

$4: \frac{2}{3}m + \frac{2}{9}w = w \Leftrightarrow \frac{2}{3}m - \frac{7}{9}w = 0$

Matrix aufstellen:

$$\begin{pmatrix} \frac{5}{9} & -\frac{1}{3} \\ -\frac{7}{9} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{5}{9} & -\frac{3}{9} \\ -\frac{7}{9} & \frac{6}{9} \end{pmatrix} = G$$

Determinante:

$$|G| = \begin{vmatrix} \frac{5}{9} & -\frac{3}{9} \\ -\frac{7}{9} & \frac{6}{9} \end{vmatrix} = \frac{5}{9} \cdot \frac{6}{9} - (-\frac{3}{9} \cdot (-\frac{7}{9})) = \frac{30}{81} - \frac{21}{81} - \frac{9}{81} = \frac{1}{9}$$

Gleichung aufstellen:

$G \cdot x = a$

$$\begin{pmatrix} \frac{5}{9} & -\frac{3}{9} \\ -\frac{7}{9} & \frac{6}{9} \end{pmatrix} \cdot \begin{pmatrix} w \\ m \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

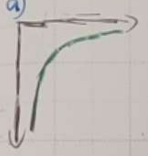
Menge berechnen:

$|G_w| = \begin{vmatrix} 2 & -\frac{3}{9} \\ 0 & \frac{6}{9} \end{vmatrix} = 2 \cdot \frac{6}{9} - 0 = \frac{12}{9}$ $\quad w = \frac{12}{9} : \frac{1}{9} = \frac{12}{9} \cdot \frac{9}{1} = 12$

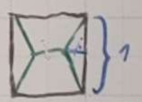
$|G_m| = \begin{vmatrix} \frac{5}{9} & 2 \\ -\frac{7}{9} & 0 \end{vmatrix} = 0 - 2 \cdot \frac{-7}{9} = \frac{14}{9}$ $\quad m = \frac{14}{9} : \frac{1}{9} = 14$

3) $f<0,\ f'<0,\ f''<0$

a)



b)



$m = 1-2x$

$k = \sqrt{\frac{1}{4}+x^2}$

$g(x) = 1-2x+4\sqrt{\frac{1}{4}+x^2}$

Ableitungen: g muss minimal werden. $g'(x)=0$, $g''(x)>0$:

$g'(x) = (1-2x)' + \left(4\sqrt{\frac{1}{4}+x^2}\right)' = -2+2\frac{1}{\sqrt{\frac{1}{4}+x^2}}\cdot 2x = -2+4x\frac{1}{\sqrt{\frac{1}{4}+x^2}}$

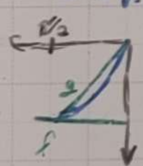
$g''(x) = \left(-2+4x\frac{1}{\sqrt{\frac{1}{4}+x^2}}\right)' = \frac{4\sqrt{\frac{1}{4}+x^2} - 4x\frac{1}{2\sqrt{\frac{1}{4}+x^2}}\cdot 2x}{\frac{1}{4}+x^2} = \frac{4\sqrt{\frac{1}{4}+x^2} - 4x^2\frac{1}{\sqrt{\frac{1}{4}+x^2}}}{\frac{1}{4}+x^2}$

$g'(x) = 0 = -2+4x\frac{1}{\sqrt{\frac{1}{4}+x^2}} \Leftrightarrow 2 = 4x\frac{1}{\sqrt{\frac{1}{4}+x^2}} \Leftrightarrow \frac{1}{2x} = \frac{1}{\sqrt{\frac{1}{4}+x^2}} \Leftrightarrow 4x^2 = \frac{1}{4}+x^2 \Leftrightarrow 3x^2 = \frac{1}{4} \Leftrightarrow \frac{1}{12} = x^2$

$\Leftrightarrow \sqrt{\frac{1}{12}} = x$

$g''\left(\sqrt{\tfrac{1}{12}}\right) = \frac{4\sqrt{\frac{1}{4}+\left(\frac{1}{\sqrt{12}}\right)^2} - 4\left(\frac{1}{\sqrt{12}}\right)^2\frac{1}{\sqrt{\frac{1}{4}+\left(\frac{1}{\sqrt{12}}\right)^2}}}{\frac{1}{4}+\left(\frac{1}{\sqrt{12}}\right)^2} = \frac{4\sqrt{\frac{1}{4}+\frac{1}{12}} - \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{12}}}}{\frac{1}{4}+\frac{1}{12}} = \frac{4\sqrt{\frac{4}{12}} - \frac{1}{\sqrt{\frac{4}{12}}}}{\frac{4}{12}} = \frac{4\sqrt{3}\cdot\frac{1}{\sqrt{3}}}{3} > 0$

4) $-\frac{\pi}{2} \leq \int_{-\pi/2}^{0} \sin x \, dx \leq -\frac{\pi}{4}$



$g(x) = \frac{2}{\pi}x \qquad f(x) = -1$

$\int_{-\pi/2}^{0} -1\, dx \Rightarrow -x\Big|_{-\pi/2}^{0} \Rightarrow 0-\left(-\left(-\frac{\pi}{2}\right)\right) = -\frac{\pi}{2}$

$\int_{-\pi/2}^{0} \frac{2}{\pi}x\, dx \Rightarrow \frac{1}{\pi}x^2\Big|_{-\pi/2}^{0} \Rightarrow 0-\frac{1}{\pi}\left(\frac{\pi}{2}\right)^2 = -\frac{\pi}{4}$

5) $\sim := \{(a,b)\in\mathbb{R}^2 \mid \int_a^b f(x)dx \quad f(x) = 1-x^2$

Reflexivität: $hRh\ (x\in\mathbb{R})$

$\int_h^h f(x)dx = \int_h^h 1-x^2 dx \Rightarrow x-\frac{1}{3}x^3\Big|_h^h = \left(h-\frac{1}{3}h^3\right)-\left(h-\frac{1}{3}h^3\right) = 0$

Symmetrie: $hRt \Rightarrow tRh$:

$\int_h^t f(x)dx \overset{\text{Satz}}{=} -\int_t^h f(x)dx$, wenn $\int_h^t f(x)dx = 0$ gilt $\int_t^h f(x)dx = 0$, da $0=-0$.

Transitiv: $hRt \wedge tRn \Rightarrow hRn$:

$\int_h^t f(x)dx + \int_t^n f(x)dx \overset{\text{Satz}}{=} \int_h^n f(x)dx$

$0+0=0$ ■

Äquivalenzklassen:

$F(x) = x - \frac{1}{3}x^3$

$\int_0^h 1-x^2 \, dx = 0 \Leftrightarrow x - \frac{1}{3}x^3 \Big|_0^h = 0 \Leftrightarrow 0 = h - \frac{1}{3}h^3 - 0 \Leftrightarrow 0 = \underbrace{h}_{h=0}\left(1 - \frac{1}{3}h^2\right)$

$0 = 1 - \frac{1}{3}h^2 \Leftrightarrow -1 = -\frac{1}{3}h^2 \Leftrightarrow 3 = h^2 \Leftrightarrow \pm\sqrt{3} = h$

$[0]_\sim = -\sqrt{3}, 0, \sqrt{3}$

$\int_2^h 1-x^2 \, dx = 0 \Leftrightarrow x - \frac{1}{3}x^3 \Big|_2^h = 0 \Leftrightarrow 0 = h - \frac{1}{3}h^3 - \left(2 - \frac{1}{3}2^3\right) \Leftrightarrow 0 = h - \frac{1}{3}h^3 - \left(-\frac{2}{3}\right) \Leftrightarrow -\frac{2}{3} = h - \frac{1}{3}h^3$

$-\frac{1}{3}h^3 + h + \frac{2}{3} = -h^3 + 3h + 2 : (h+1) = -h^2 + h + 2$

$+(h^3 + h^2)$

$h^2 + 3h + 2$

$-(h^2 + h)$

$2h + 2$

$-(2h + 2)$

0

$0 = -h^2 + h + 2 \Leftrightarrow 0 = h^2 - h - 2$

$h_{1/2} = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + 2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2} = \frac{1}{2} \pm \sqrt{\frac{9}{4}} = \frac{1}{2} \pm \frac{3}{2} = \begin{cases} 2 \\ -1 \end{cases}$

$[2]_\sim = 2, -1$