## 5.3] Rechenregeln für den Betrag komplexer Zahlen

a)

$$|z| = |-z| = |\bar{z}| = \sqrt{z \cdot \bar{z}}$$

$$\left.\begin{aligned} |z| &= |x+iy| = \sqrt{x^2+y^2} \\ |-z| &= |z| = |x+iy| = \sqrt{x^2+y^2} \\ |\bar{z}| &= |x-iy| = \sqrt{x^2+(-y)^2} = \sqrt{x^2+y^2} \\ \sqrt{z \cdot \bar{z}} &= \sqrt{(x+iy)(x-iy)} = \sqrt{x^2+ixy-ixy+y^2} = \sqrt{x^2+y^2} \end{aligned}\right\} =$$

b)

$|\mathrm{Re}\, z| \overset{!}{\leq} |z|$

$\Leftrightarrow |x| \leq |x+iy|$

$\Leftrightarrow \sqrt{x^2} \leq \sqrt{x^2+y^2}$

$\Leftrightarrow x^2 \leq x^2+y^2$

$\Leftrightarrow 0 \leq y^2$ ☐

$|\mathrm{Im}\, z| \overset{!}{\leq} |z|$

$\Leftrightarrow |iy| \leq |x+iy|$

$\Leftrightarrow \sqrt{y^2} \leq \sqrt{x^2+y^2}$

$\Leftrightarrow y^2 \leq x^2+y^2$

$\Leftrightarrow 0 \leq x^2$ ☐

c)

$|z| = 0 \Leftrightarrow z = 0$

$\sqrt{x^2+y^2} = 0 \Rightarrow x = 0 \wedge y = 0$, da $x^2 \geq 0 \wedge y^2 \geq 0 \Rightarrow z = 0$, d $x = 0 \wedge y = 0 \Rightarrow x + i \cdot y = z = 0 + i \cdot 0 = 0$ ✓

d)

$|z_1 \cdot z_2| \overset{!}{=} |z_1| \cdot |z_2|$

$\Leftrightarrow |(x_1, y_1)(x_2, y_2)| = |(x_1, y_1)| \cdot |(x_2, y_2)|$

$\Leftrightarrow |(x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)| = |(x_1, y_1)| \cdot |(x_2, y_2)|$

$\Leftrightarrow \sqrt{(x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2} = \sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 + y_2^2}$

$\Leftrightarrow \sqrt{x_1^2 x_2^2 - 2 x_1 x_2 y_1 y_2 + y_1^2 y_2^2 + x_1^2 y_2^2 + 2 x_1 x_2 y_1 y_2 + x_2^2 y_1^2} = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$

$\Leftrightarrow \sqrt{x_1^2 x_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2 + y_1^2 y_2^2} = \sqrt{x_1^2 x_2^2 + x_1^2 y_2^2 + x_2^2 y_1^2 + y_1^2 y_2^2}$

e)

$|z_1 + z_2| \overset{!}{\leq} |z_1| + |z_2|$

$\Leftrightarrow |x_1 + i y_1 + x_2 + i y_2| \leq |x_1 + i y_1| + |x_2 + i y_2|$

$\Leftrightarrow \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \leq \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$

$\Leftrightarrow \sqrt{x_1^2 + 2 x_1 x_2 + x_2^2 + y_1^2 + 2 y_1 y_2 + y_2^2} \leq \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$

$\Leftrightarrow x_1^2 + 2 x_1 x_2 + x_2^2 + y_1^2 + 2 y_1 y_2 + y_2^2 \leq x_1^2 + y_1^2 + 2\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} + x_2^2 + y_2^2$

$\Leftrightarrow 2 x_1 x_2 + 2 y_1 y_2 \leq 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$

$\Leftrightarrow 2 x_1^2 x_2^2 + 2 y_1^2 y_2^2 \leq 4(x_1^2 x_2^2 + x_1^2 y_2^2 + y_1^2 x_2^2 + y_1^2 y_2^2)$

$\Leftrightarrow x_1^2 x_2^2 + y_1^2 y_2^2 \leq 2 x_1^2 x_2^2 + 2 x_1^2 y_2^2 + 2 x_2^2 y_1^2 + 2 y_1^2 y_2^2$

$\Leftrightarrow 0 \leq 2(x_1^2 y_2^2 + x_2^2 y_1^2) + x_1^2 x_2^2 + y_1^2 y_2^2$ ✓