Information-Outage Analysis of Finite-Length Codes

Thesis Defense

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Outline

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Motivation

- Shannon Channel Capacity
  - Gives reliable rate for communication at specific SNR.
  - Requires infinite-length codewords.
  - In practice, codes are restricted to finite lengths.
- Predicting Performance of Finite-Length Codes
  - Several bounds based on blocklength exist, but are non-trivial to calculate.
  - Want information-theoretic metric which is a function of blocklength.
  - Information-outage probability used previously with block fading channels.
  - Apply information-outage probability based on the mutual-information rate.
  - Compare with coded performance and alternative bounds.

Mutual Information

- Let X and Y be random variables associated with the input and output, respectively, of a channel.
- We can write the *mutual information* between the events, $\{X = x\}$ and $\{Y = y\}$ as

$$i(x; y) \quad = \quad \log \frac{p_{X,Y}(x, y)}{p_X(x) p_Y(y)}$$

- The *average mutual information* is the expectation of the mutual information random variable

$$I(X; Y) \quad = \quad E\left[i(X; Y)\right]$$

Relationship between Capacity and MI

- Average mutual information used to find *channel capacity*

$$\mathcal{C} = \max_{p_X(x)} I(X;Y)$$

- Gaussian input maximizes average mutual information.
- Inifinitely long codewords required to achieve capacity.
- Can we use mutual information to describe achievable rates of finite-length codes?

Mutual-Information Rate

- Let $\mathbf{X}$ be a vector of n i.i.d. random variables representing the input codeword
- The mutual information between the events $\{\mathbf{X} = \mathbf{x}\}$ and $\{\mathbf{Y} = \mathbf{y}\}$ averaged over the vector length, n, is

$$\begin{aligned}\frac{1}{n} i(\mathbf{x};\mathbf{y}) &= \frac{1}{n} \log \frac{p_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y})}{p_{\mathbf{X}}(\mathbf{x}) p_{\mathbf{Y}}(\mathbf{y})} \\ &= \frac{1}{n} \sum_{k=1}^{n} i\left(x_k; y_k\right)\end{aligned}$$

- Also known as *mutual-information rate*.
- Expressions equivalent due to i.i.d. channel inputs.
- We let a sample of the mutual-information rate be the *instantaneous capacity* of a given codeword.
- Rather than find mutual-information rate directly, only need to average n samples of mutual information random variable.

Outline

Mutual Information of AWGN Channel

- Let X_k be input complex Gaussian random variable to AWGN channel

$$Y_k = X_k + Z_k$$

where Z_k is complex AWGN noise, and Y_k is channel output for $1 \leq k \leq n$.

- Mutual information can also be expressed as

$$i(x;y) = \log \frac{p_{Y_k|X_k}(y|x)}{p_{Y_k}(y)}$$

- Substituting pdf for $Y_k|X_k$ and Y_k

$$i(x;y) = \log\left(1 + \frac{\mathcal{E}_s}{N_o}\right) + \frac{|y|^2}{\mathcal{E}_s + N_0} - \frac{|y-x|^2}{N_0}$$

Finding Capacity from Mutual Information

- The channel capacity is found by taking the expectation of the mutual information random variable

$$\begin{aligned} I(X;Y) &= E\left[i(X;Y)\right] \\ &= \log\left(1+\frac{\mathcal{E}_s}{N_0}\right)+\frac{E\left[|y|^2\right]}{\mathcal{E}_s+N_0}-\frac{E\left[|y-x|^2\right]}{N_0} \\ &= \log\left(1+\frac{\mathcal{E}_s}{N_0}\right) \end{aligned}$$

- This is the ergodic capacity of a 2-dimensional channel.

Alternative Representation

- An alternative representation of the mutual information for the AWGN channel [Laneman, 2006] is

$$i(x;y) \quad = \quad \log\left(1+\frac{\mathcal{E}_s}{N_o}\right)+W$$

where W is a Laplacian random variable with zero mean and variance

$$\sigma_W^2 \quad = \quad \frac{2\mathcal{E}_s}{\mathcal{E}_s+N_0}.$$

- We can now find the mutual-information rate by averaging n samples of the mutual information.

Distribution of Mutual-Information Rate

- Let Z_n be the mutual information rate between the channel input and output vectors, $\mathbf{x}$ and $\mathbf{y}$

$$Z_n = \log\left(1 + \frac{\mathcal{E}_s}{N_0}\right) + W_n$$

where W_n is the average of n i.i.d. Laplacian random variables [Laneman, 2006].

- W_n is a Bessel-K random variable with pdf given by:

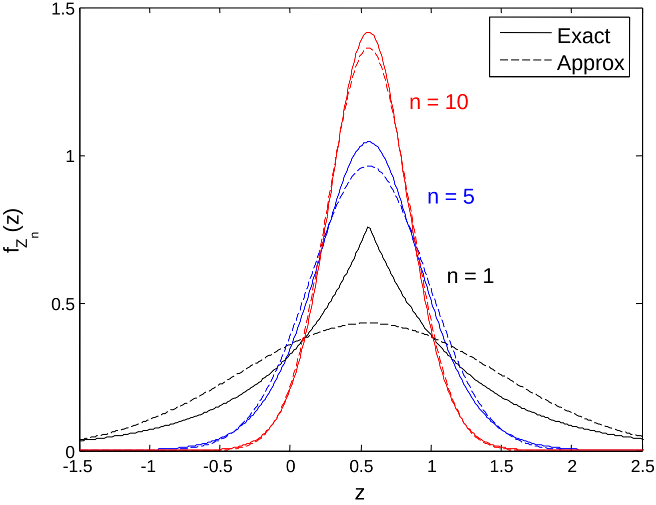
$$p_{W_n}(w) = \frac{2^{1-n}}{\sqrt{\pi}\Gamma(n)\sigma_W}\left(\frac{\sqrt{2}|w|}{\sigma_W}\right)^{n-\frac{1}{2}} K_{n-\frac{1}{2}}\left(\frac{\sqrt{2}|w|}{\sigma_W}\right)$$

Approximating the Mutual Information

- The Central Limit Theorem tells us that the sum of n i.i.d. random variables with finite mean and variance will approach a Gaussian distribution as $n \rightarrow \infty$
- Exact Bessel-K distribution will approach a Gaussian distribution as blocklength increases.
- We introduce a Gaussian approximation to the mutual-information rate.

$$\tilde{Z}_n \sim \mathcal{N}\left(\log\left(1 + \frac{\mathcal{E}_s}{N_0}\right), \frac{2\mathcal{E}_s}{n\left(\mathcal{E}_s + N_0\right)}\right)$$

Distribution Comparison



Mutual Information of Fading Channel

- Let X_k now be input to i.i.d. fading channel

$$Y_k = \mathsf{a}_k X_k + Z_k$$

where a_k is a complex Gaussian random variable representing the fading

- Mutual information conditioned on $\{\mathsf{a} = a\}$ can be manipulated to the form [Laneman, 2006]

$$i(x; y|\mathsf{a}) = \log(1 + \lambda) + \sqrt{\frac{\lambda}{\lambda + 1}} w$$

where

  - $\lambda = |\mathsf{a}|^2 \mathcal{E}_s / N_o$ is an exponential random variable representing the SNR.
  - w is Laplacian with zero mean and variance 2.

Finding Capacity from Mutual Information

- The ergodic channel capacity is found by taking the expectation of the mutual information random variable

$$
\begin{aligned}
I(X;Y|\mathsf{a}) &= E\left[i(X;Y|\mathsf{a})\right] \\
&= E\left[\log\left(1+\lambda\right)\right] + E\left[\sqrt{\frac{\lambda}{\lambda+1}}w\right] \\
&= E\left[\log\left(1+\lambda\right)\right] + E\left[\sqrt{\frac{\lambda}{\lambda+1}}\right]E\left[w\right] \\
&= E\left[\log\left(1+\lambda\right)\right] \\
&= e^{N_o/\mathcal{E}_s}E_1\left(\frac{N_o}{\mathcal{E}_s}\right)
\end{aligned}
$$

- This is the ergodic capacity of a 2-dimensional channel with i.i.d. fading known to the receiver.

Distribution of Mutual-Information Rate

- Let Z_n be the mutual-information rate between the channel input and output vectors, $\mathbf{x}$ and $\mathbf{y}$

$$Z_n = \frac{1}{n}\sum_{k=1}^{n}\left[\log(1+\lambda_k) + \sqrt{\frac{\lambda_k}{\lambda_k+1}}w_k\right]$$

- Exact distribution for the mutual-information rate cannot be readily found.
- Monte-Carlo simulation can be used to generate individual samples.

Approximating the Mutual Information

- Similar to AWGN, we can approximate the mutual-information rate with a Gaussian random variable.
- Mean is known, but variance remains to be computed.

$$\begin{aligned}
\sigma_i^2 &= E\left[\left(i(x;y|\mathsf{a}) - E\left[i(x;y|\mathsf{a})\right]\right)^2\right] \\
&= E\left[\left(\log(1+\lambda) + \sqrt{\frac{\lambda}{\lambda+1}}w - e^{N_o/\mathcal{E}_s}E_1\left(\frac{N_o}{\mathcal{E}_s}\right)\right)^2\right] \\
&= E\left[\log^2(1+\lambda)\right] + E\left[\frac{\lambda}{\lambda+1}w^2\right] - e^{2N_o/\mathcal{E}_s}\left[E_1\left(\frac{N_o}{\mathcal{E}_s}\right)\right]^2
\end{aligned}$$

Approximating the Mutual Information

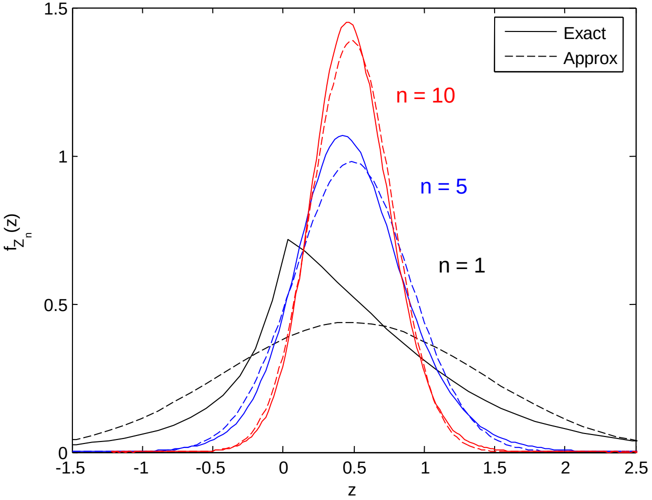
- Gaussian approximation to the mutual-information rate of a fading channel is given by

$$\tilde{Z}_n \sim \mathcal{N}\left(e^{N_o/\mathcal{E}_s} E_1\left(\frac{N_o}{\mathcal{E}_s}\right), \frac{\sigma_i^2}{n}\right)$$

where

$$\begin{aligned}\sigma_i^2 &= e^{N_o/\mathcal{E}_s}\left[\frac{\pi^2}{6} + \left(\mathbf{C} + \log\left(\frac{N_o}{\mathcal{E}_s}\right)\right)^2\right] - \\ &\quad 2e^{N_o/\mathcal{E}_s}\left(\frac{N_o}{\mathcal{E}_s}\right) {}_3F_3\left([1,1,1],[2,2,2],-\frac{N_o}{\mathcal{E}_s}\right) + \\ &\quad 2 - 2\left(\frac{N_o}{\mathcal{E}_s}\right) e^{N_o/\mathcal{E}_s} E_1\left(\frac{N_o}{\mathcal{E}_s}\right) - e^{2N_o/\mathcal{E}_s}\left[E_1\left(\frac{N_o}{\mathcal{E}_s}\right)\right]^2\end{aligned}$$

Distribution Comparison



Outline

Defining Information-Outage Probability

- Let $R_2 = k/n$ represent the code rate in bits per symbol.
  - $R_e = \log(2)R_2$ is the equivalent rate in nats per symbol.
- Mutual-information rate is random.
- Each sample of the mutual-information rate we view as the instantaneous rate at which a codeword can be reliably transmitted.
- An *outage* occurs when the mutual-information rate is less than the code rate.
- Therefore the *outage probability* is defined as

$$P_o = P\left[Z_n \le R_e\right] = F_{Z_n}\left(R_e\right)$$

where F_{Z_n} is the CDF of the mutual-information rate random variable, Z_n.

Exact Outage Probability for Each Channel

- For AWGN channel, we can write P_o as

$$\begin{aligned} P_o &= P\left[Z_n \leq R_e\right] \\ &= P\left[\log\left(1+\frac{\mathcal{E}_s}{N_0}\right)+W_n \leq R_e\right] \\ &= F_{W_n}\left(R_e-\log\left(1+\frac{\mathcal{E}_s}{N_0}\right)\right) \end{aligned}$$

  where F_{W_n} is the CDF of the random variable W_n
- For fading channel, we will require simulation to determine P_o.

Exact and Approximate CDFs

- Recall that W_n is a Bessel-K random variable, which has CDF

$$F_{W_n}(w) \quad = \quad 1 - \sum_{l=0}^{n-1} \frac{\Gamma(n+l)\Gamma(n-l,\sqrt{2}w/\sigma)}{\Gamma(n)\Gamma(n-l)\Gamma(l+1)} 2^{-n-l}$$

for $w \geq 0$ and

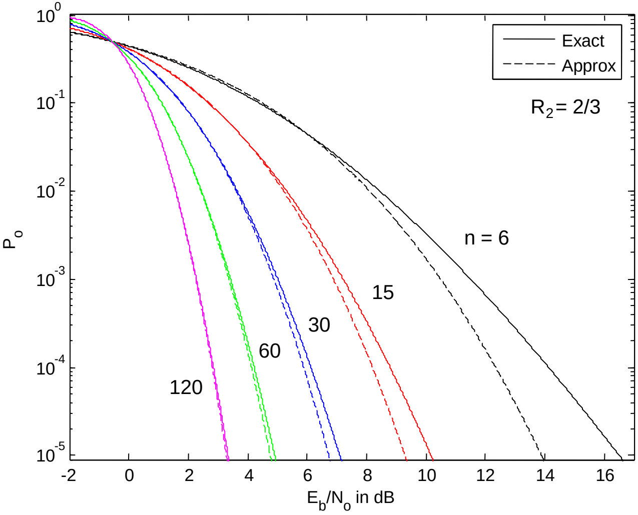
$$F_{W_n}(w) \quad = \quad \sum_{l=0}^{n-1} \frac{\Gamma(n+l)\Gamma(n-l,-\sqrt{2}w/\sigma)}{\Gamma(n)\Gamma(n-l)\Gamma(l+1)} 2^{-n-l}$$

for $w < 0$.

- Alternatively, the CDF of the Gaussian approximation can be found by using the Q-function

$$F_{\tilde{Z}_n}(z) \quad = \quad Q\left(\frac{\log\left(1+\frac{\mathcal{E}_s}{N_0}\right) - z}{\sqrt{\frac{2\mathcal{E}_s}{n(\mathcal{E}_s+N_0)}}}\right).$$

IOP Comparison



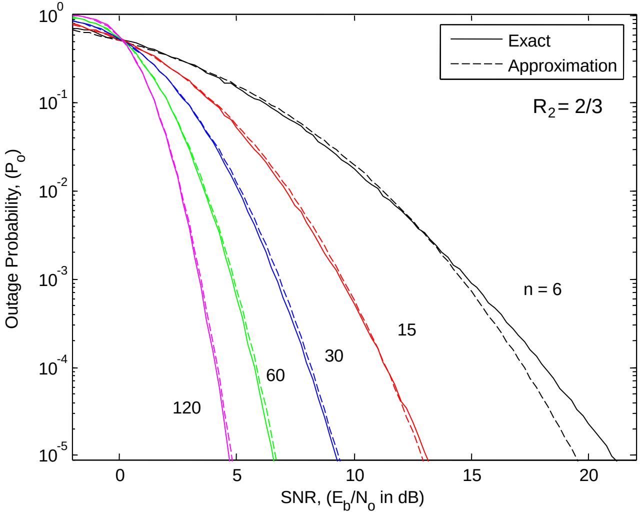
Exact and Approximate CDFs

- Recall that an exact distribution cannot be readily found for mutual-information rate.
- To calculate the exact information-outage probabilities, we rely on simulation.
- Alternatively, the CDF of the Gaussian approximation can be found by using the Q-function

$$F_{\tilde{Z}_n}(z) = Q\left(\frac{e^{N_o/\mathcal{E}_s} E_1\left(\frac{N_o}{\mathcal{E}_s}\right) - z}{\sqrt{\frac{\sigma_i^2}{n}}}\right)$$

where σ_i^2 is the variance of the mutual information, defined earlier.

IOP Comparison



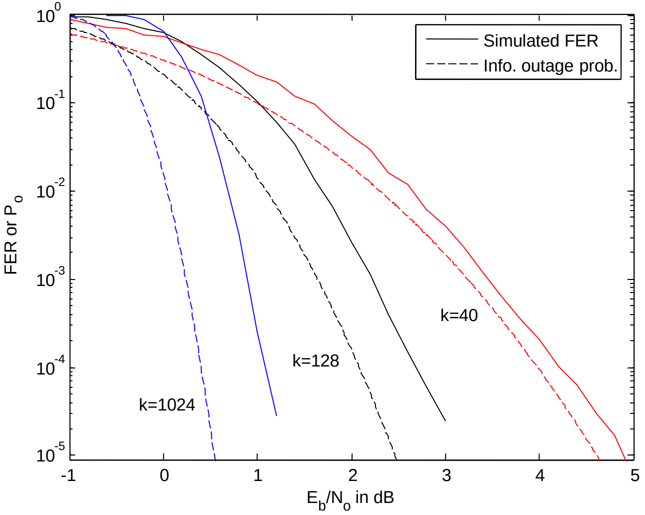
Outline

Turbo Code Performance

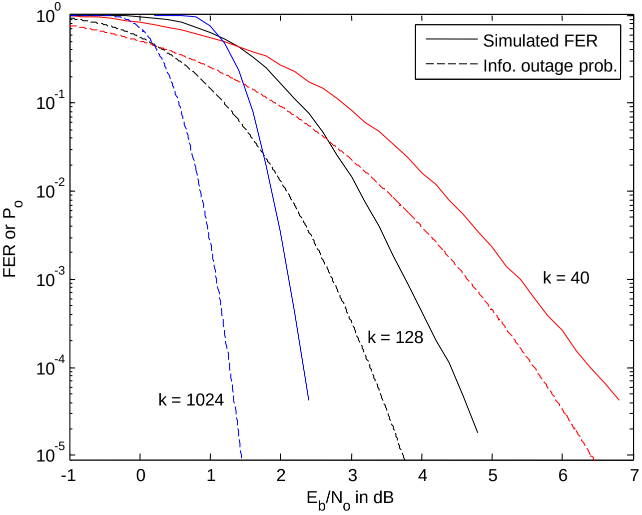
- Information-outage probability as a predictor of performance.
  - How does IOP compare to an existing capacity-approaching code?
- UMTS LTE (long term evolution) turbo code
  - Supports 188 distinct values of information block size, k, in bits.
  - Codeword blocklength defined as $n = 3k + 12$.
  - Simulated with QPSK, binary rate given by

$$R_2 = 2\frac{k}{n} = \frac{2k}{3k + 12} \approx 2/3$$

IOP vs. LTE - AWGN



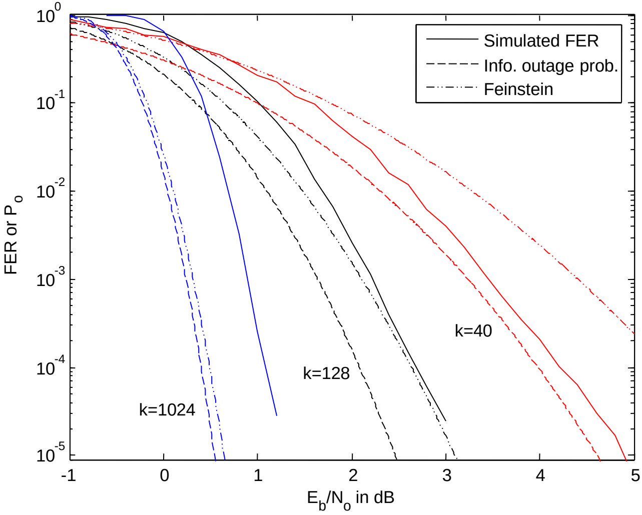
IOP vs. LTE - Fading



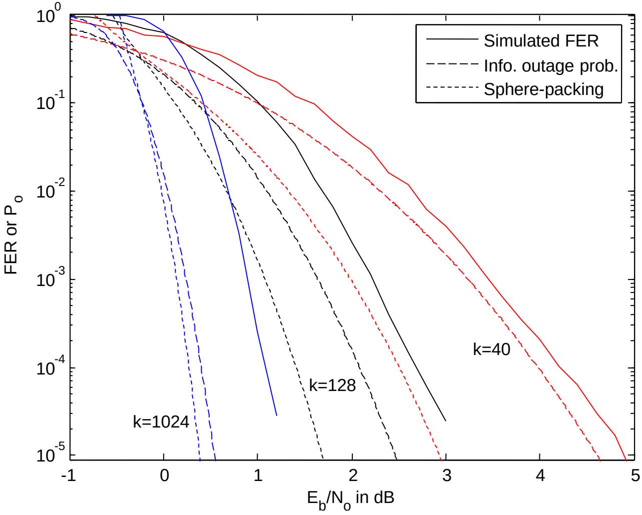
Alternative Bounds

- Feinstein's Lemma, [Feinstein, 1954].
  - Bound on maximal codeword error rate, based on mutual information rate.
  - States that a code exists that can achieve a specific codeword error probability.
  - Codes may exist that perform better than bound.
- Sphere-Packing Bound, [Shannon, 1959].
  - Lower bound on codeword-error probability based on n-dimensional Euclidian space.
  - Sphere in n-dimensional space is packed with $M = 2^k$ cones.
- Random Coding Bound, [Shannon, 1959].
  - Bound on the ensemble average word-error probability.
  - Averaged over all possible (n, k) codes from randomly selected set of 2^k codewords.

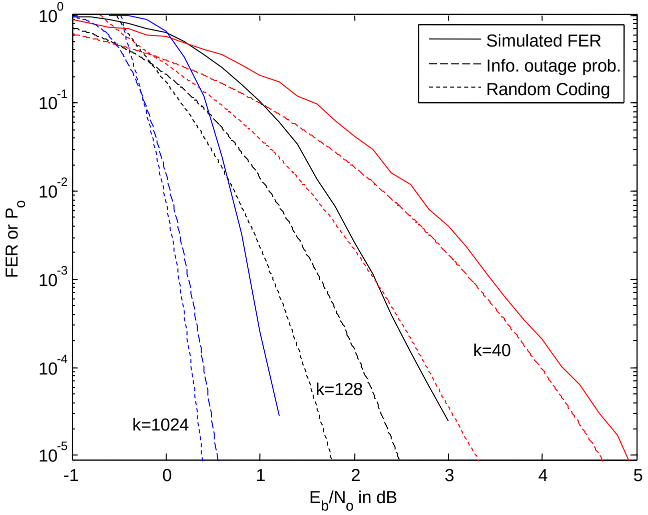
IOP vs. Feinstein



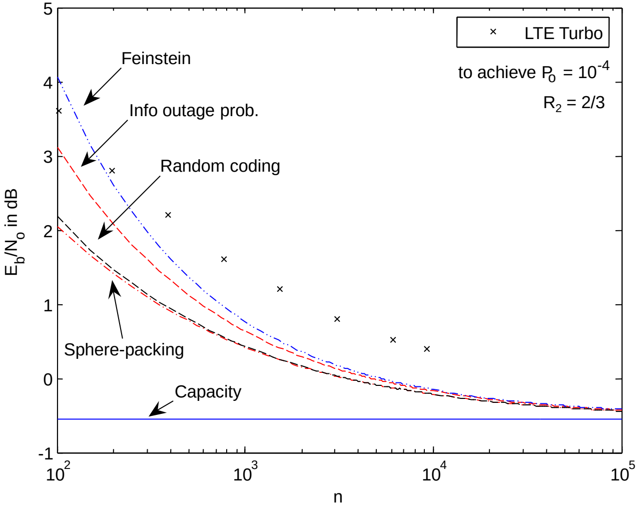
IOP vs. Sphere-Packing Bound



IOP vs. Random Coding Bound



Bound Comparison



Outline

Conclusion

- Distribution of Mutual-Information Rate
  - Exact
    - AWGN: Mean-shifted Bessel-K distribution.
    - Fading: Not Defined. Computed via simulation.
  - Approximation
    - Both: Gaussian distributions introduced.
  - As blocklength increases,
    - AWGN: Exact distribution becomes increasingly difficult to calculate as numerical stability becomes a factor.
    - Both: Exact distributions approach Gaussian approximations.
- Information-Outage Probability
  - Both: Useful predictor of error performance.
  - AWGN: Calculation using Gaussian approximation is trivial compared to other previously derived bounds.

Conclusion

- Distribution of Mutual-Information Rate
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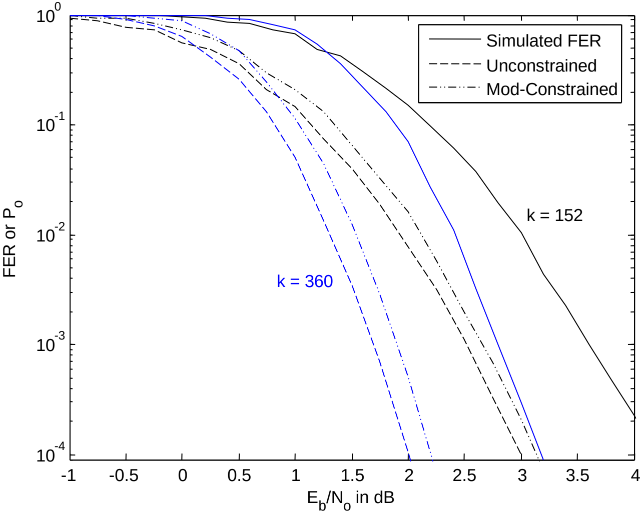
Future Work

- Quantifying Individual Constraint Losses
  - Loss Due to Modulation Constraint.
  - Loss Due to Channel Coding.
- Applying information-outage methodology to more sophisticated channels
  - Frequency-Selective Fading Channel

Bridging the Coded Performance Gap



Contributions

- D. Buckingham and M. C. Valenti, "The information-outage probability of finite-length codes over AWGN channels," in *Proc. Conf. on Information Sciences and Systems (CISS)*, Princeton, NJ, Mar. 2008.
- Coded Modulation Library (CML)
    - 1-D and 2-D Unconstrained Gaussian Input (UGI) information-outage simulator.

Thank You.