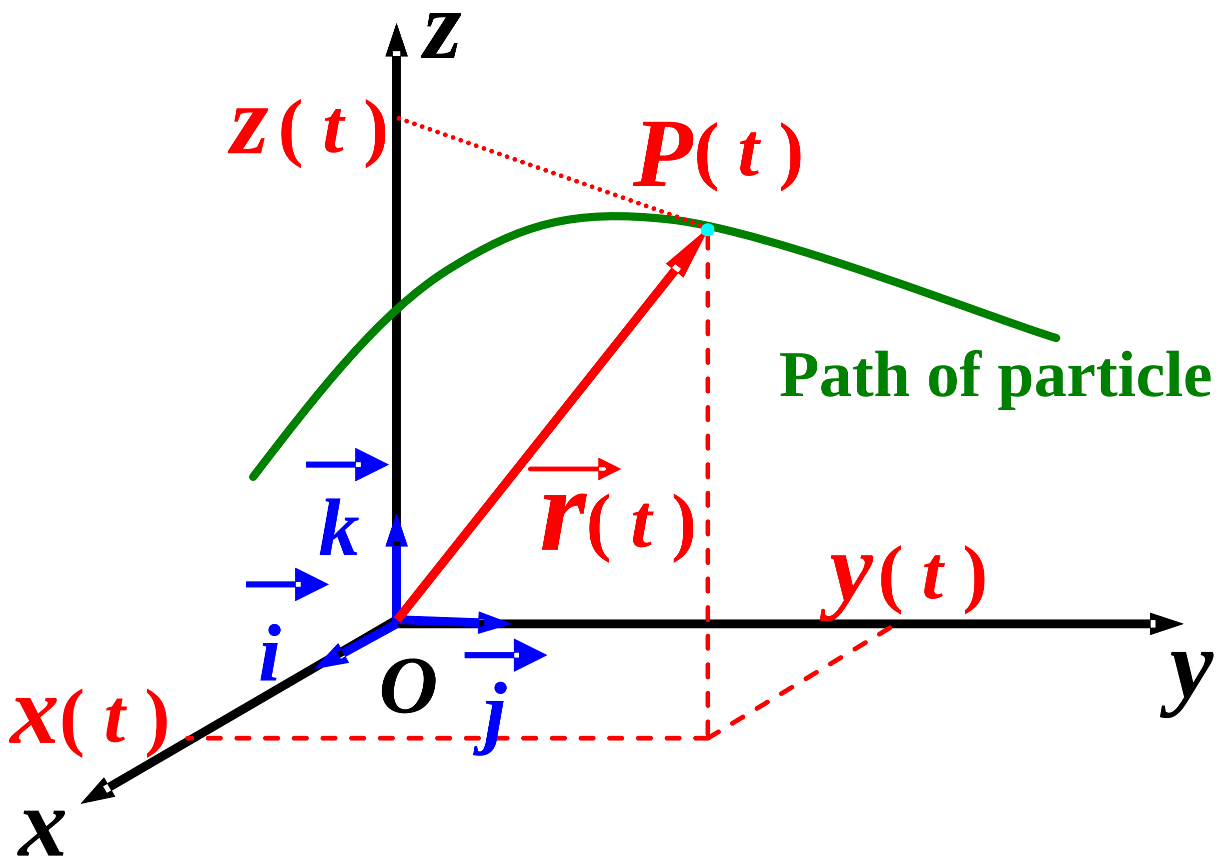
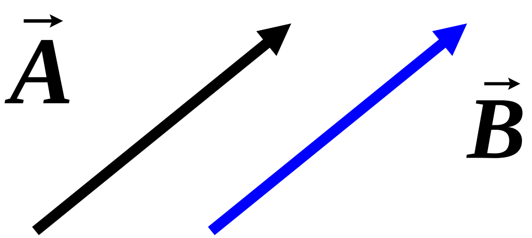
How to describe the motion in 2, 3, ….., N dimensions?



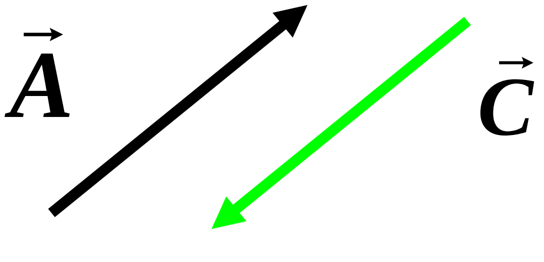
Chapter 3: Vector (矢量)

Arrows are used to represent vectors.

- **The length of the arrow signifies magnitude**
- **The head of the arrow signifies direction**



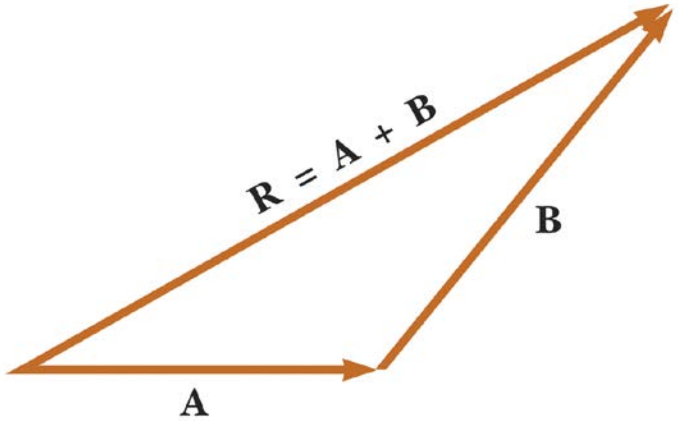
$$\vec{B} = \vec{A}$$



$$\vec{C} = -\vec{A}$$

Additions of vectors :

Triangle method of addition
（三角形法则）



Commutative law (交换律):

$$\vec{A}+\vec{B}=\vec{B}+\vec{A}$$

Associative law (结合律):

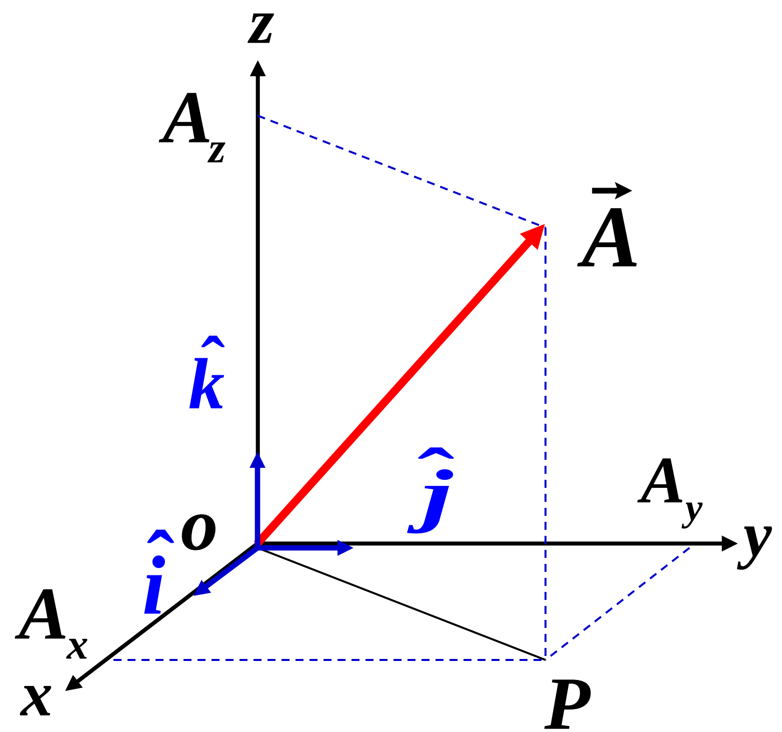
$$\vec{A}+(\vec{B}+\vec{C})=(\vec{A}+\vec{B})+\vec{C}$$

Resolution of a Vector: （矢量分解）

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

单位矢量： $\hat{i},\hat{j},\hat{k}$

矢量分量：A_x, A_y, A_z



正交直角坐标系

Addition by components:

$$\vec{R} = \vec{A} + \vec{B}$$

$$= (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$$

Multiplying Vectors:

1. Multiplying a vector by a scalar

$$\vec{C} = \lambda \vec{A}$$

Direction: $\vec{C} \,//\, \vec{A}$

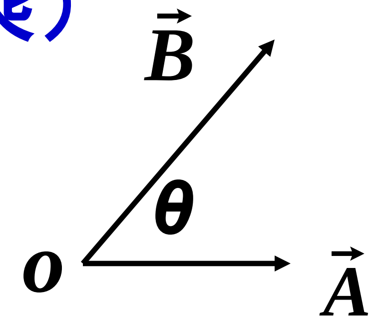
Magnitude: $|\vec{C}| = |\lambda|\,|\vec{A}|$

Associative: $\lambda(\mu\vec{A}) = (\lambda\mu)\vec{A}$

Distributive: $\lambda(\vec{A}+\vec{B}) = \lambda\vec{A}+\lambda\vec{B}$

$$(\lambda+\mu)\vec{A} = \lambda\vec{A}+\mu\vec{A}$$

2. Scalar product （标量积，点乘）



$$\vec{A}\cdot\vec{B} = AB\cos\theta \qquad A^2 = \vec{A}\cdot\vec{A}$$

Commutative: $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

Associative: $\lambda(\vec{A}\cdot\vec{B}) = (\lambda\vec{A})\cdot\vec{B}$

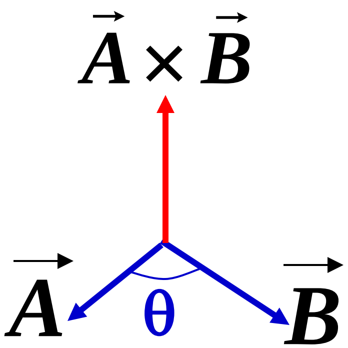
Distributive: $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$

In Cartesian coordinate system

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \qquad \hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0$$

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

3. Vector product （矢量积，叉乘）



Right-hand rule
(右手定则)

$$\left|\vec{A}\times\vec{B}\right| = AB\sin\theta$$
$$(0<\theta<\pi)$$

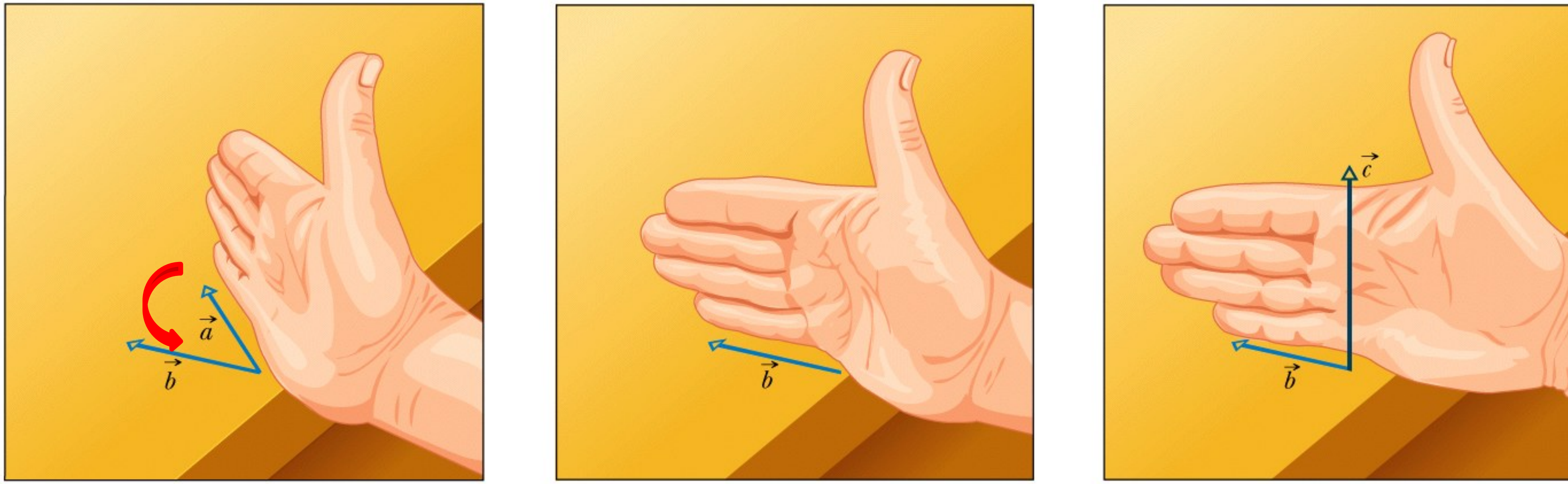
$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

$$\vec{A}\times\vec{A} = 0$$

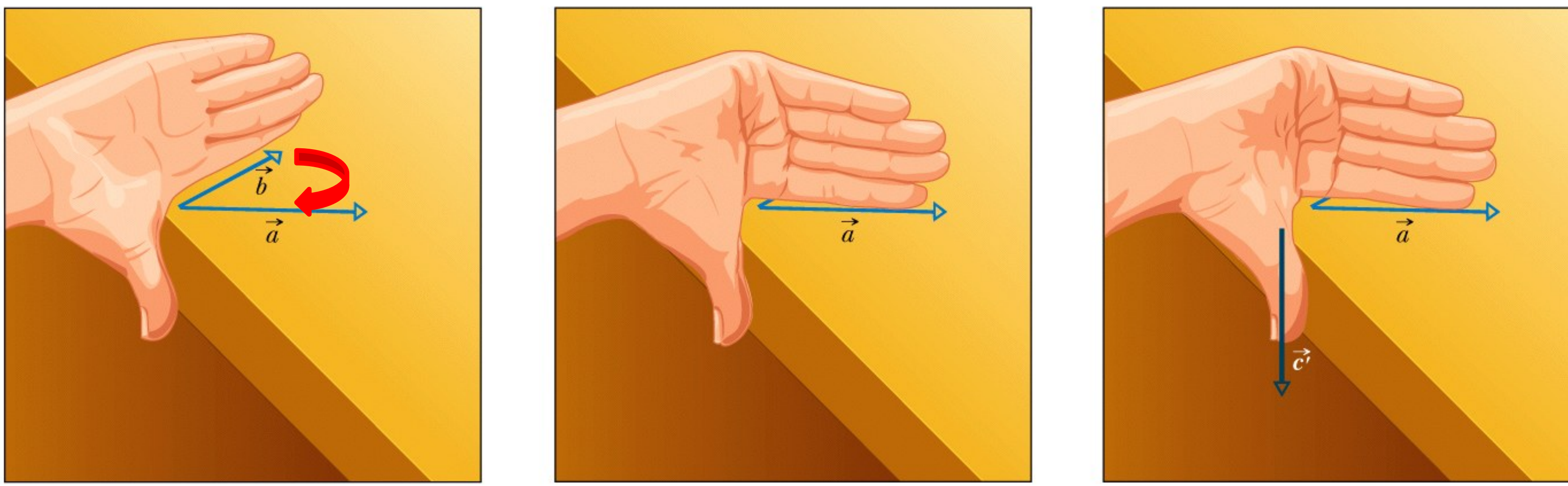
不满足交换率

Distributive: $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$

Associative: $(\lambda\vec{A})\times\vec{B} = \lambda(\vec{A}\times\vec{B})$

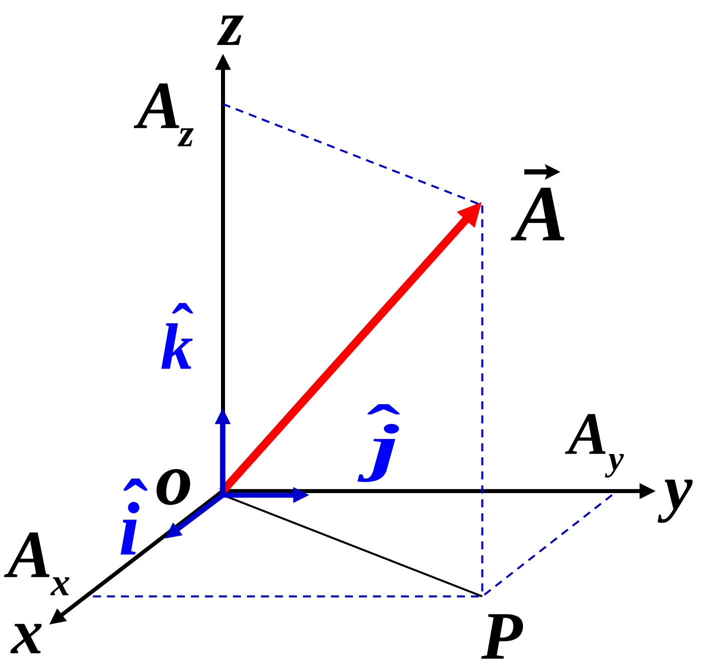


(a)



(b)

In Cartesian coordinate system （笛卡尔坐标系）



$$\hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\hat{j}$$

$$\hat{j}\times\hat{i}=-\hat{k},\ \hat{k}\times\hat{j}=-\hat{i},\ \hat{i}\times\hat{k}=-\hat{j}$$

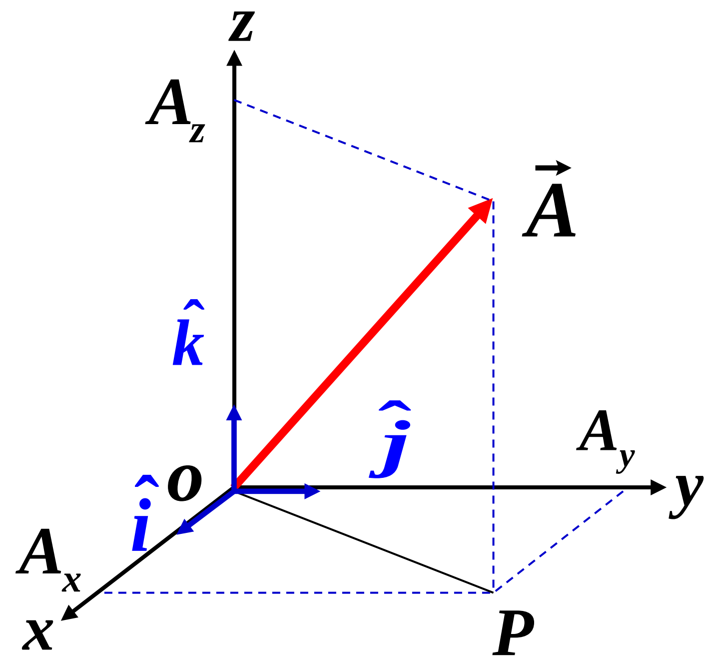
$$\hat{i}\times\hat{i}=0,\ \hat{j}\times\hat{j}=0,\ \hat{k}\times\hat{k}=0$$

In Cartesian coordinate system （笛卡尔坐标系）

$$\hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\hat{j}$$

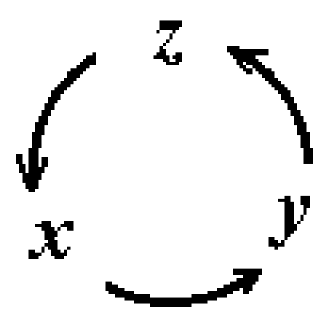
$$\hat{j}\times\hat{i}=-\hat{k},\ \hat{k}\times\hat{j}=-\hat{i},\ \hat{i}\times\hat{k}=-\hat{j}$$

$$\hat{i}\times\hat{i}=0,\ \hat{j}\times\hat{j}=0,\ \hat{k}\times\hat{k}=0$$



$$\vec{A}\times\vec{B}=\hat{i}\left(A_yB_z-A_zB_y\right)+\hat{j}\left(A_zB_x-A_xB_z\right)+\hat{k}\left(A_xB_y-A_yB_x\right)$$

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\hat{i}\left(A_yB_z-A_zB_y\right)+\hat{j}\left(A_zB_x-A_xB_z\right)+\hat{k}\left(A_xB_y-A_yB_x\right)$$



Homework 2: 2.1-2.2

2.1 Prove the following law of scalar product:

Commutative: $$\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}$$

Associative: $$\lambda(\vec{A}\cdot\vec{B})=(\lambda\vec{A})\cdot\vec{B}$$

Distributive: $$\vec{A}\cdot(\vec{B}+\vec{C})=\vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$$

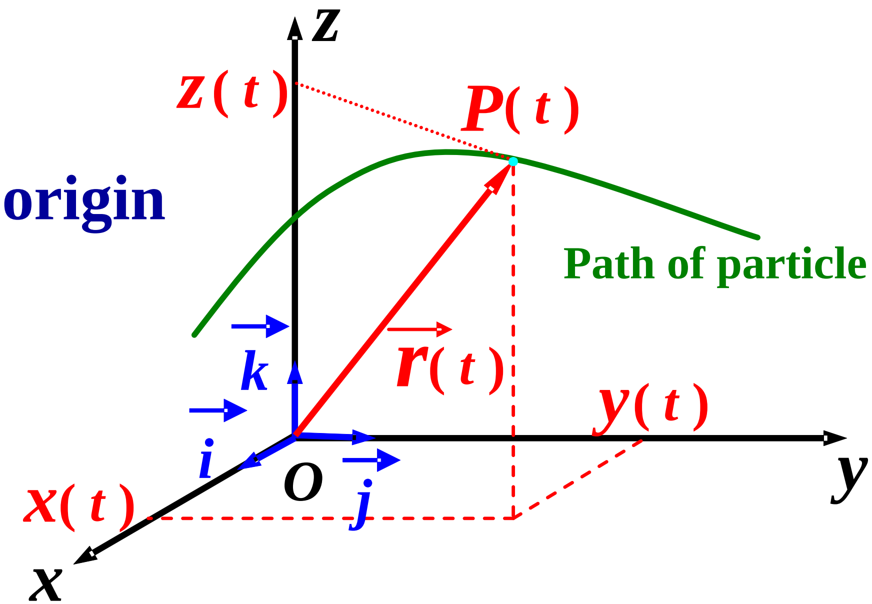
2.2 Prove the following formula

$$\boxed{\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})}$$

Chapter 4 : Kinematics-Motion in Two and Three Dimensions

§ 4.2-4.4 basic quantities of motion

1. Position vector and motion equation



- **Position vector of a particle: (origin → particle)**

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

- **Motion equation:**

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

- **Path equation: (cancel the time t)**

$$f(x, y, z) = 0$$

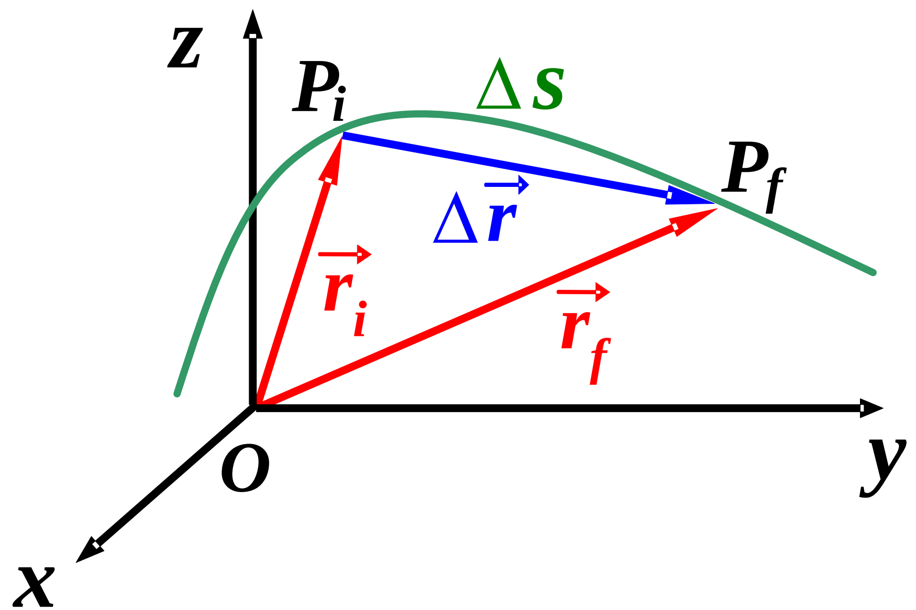
2. Displacement:

the difference between the final position and initial position during the time $t_i \rightarrow t_f$ and time interval Δt

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i$$

Path (路程): Δs

$$|\Delta\vec{r}| \neq \Delta s$$

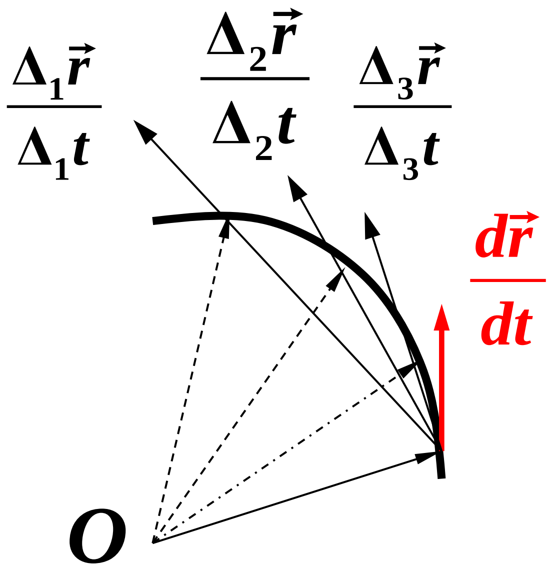


3. Velocity and speed

Average velocity (平均速度): $\bar{\vec{v}} = \dfrac{\Delta \vec{r}}{\Delta t}$

Instantaneous velocity (瞬时速度):

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$



The velocity is always tangent to the path .

In Cartesian coordinate system

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

$$\Rightarrow \vec{v} = \frac{d\vec{r}(t)}{dt} = \frac{dx(t)}{dt}\hat{i} + \frac{dy(t)}{dt}\hat{j} + \frac{dz(t)}{dt}\hat{k}$$

In the same coordinate system,the velocity can be decomposed as

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

Then, we obtain

$$v_x = \frac{dx(t)}{dt}, v_y = \frac{dy(t)}{dt}, v_z = \frac{dz(t)}{dt}$$

Average speed (平均速率): $$\bar{v} = \frac{\Delta s}{\Delta t}$$

Instantaneous speed (瞬时速率):

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

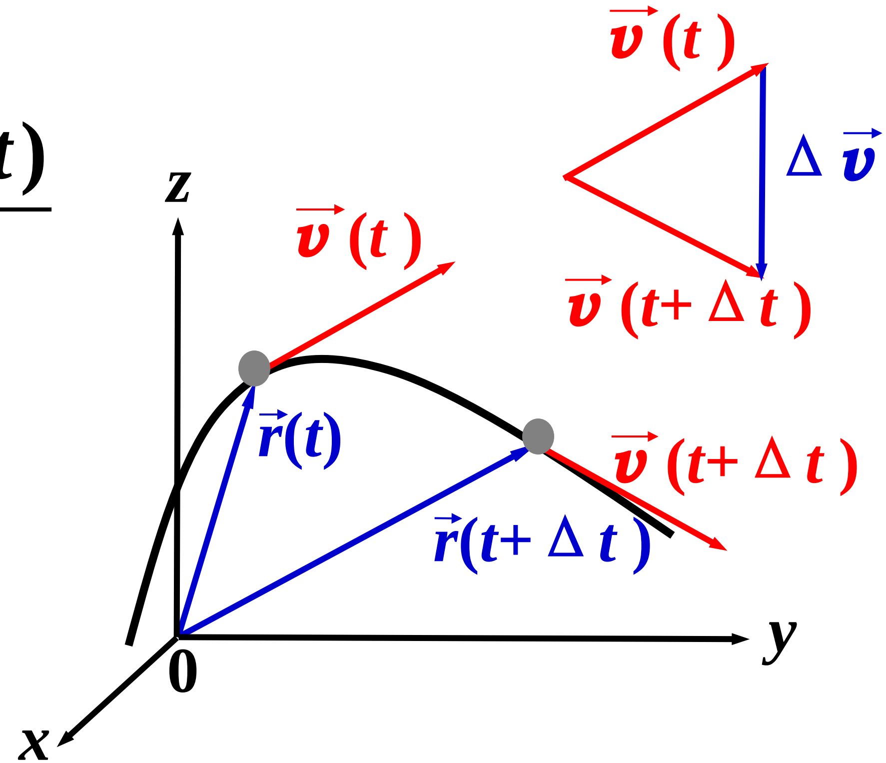
For infinitesimal time interval

$$|d\vec{r}| = ds \qquad \therefore v = \left|\frac{d\vec{r}}{dt}\right| = |\vec{v}|$$

Instantaneous speed is the magnitude of instantaneous velocity.

4. Acceleration

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}(t)}{dt} = \frac{d^2\vec{r}(t)}{dt^2}$$



$$\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

$$\left.\begin{aligned} a_x(t) &= \frac{dv_x(t)}{dt} = \frac{d^2x(t)}{dt^2} \\ a_y(t) &= \frac{dv_y(t)}{dt} = \frac{d^2y(t)}{dt^2} \\ a_z(t) &= \frac{dv_z(t)}{dt} = \frac{d^2z(t)}{dt^2} \end{aligned}\right\}$$

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

Example: A particle moves in an *xy* plane , the motion function is $x=R\cos\omega t$, $y=R\sin\omega t$, where R and ω are positive constants。Find

(1) the path equation, (2) the position, (3) velocity, and (4) acceleration 。

Solution:

(1) path equation

The motion function is
$$\begin{cases} x(t) = R\cos\omega t \\ y(t) = R\sin\omega t \end{cases}$$

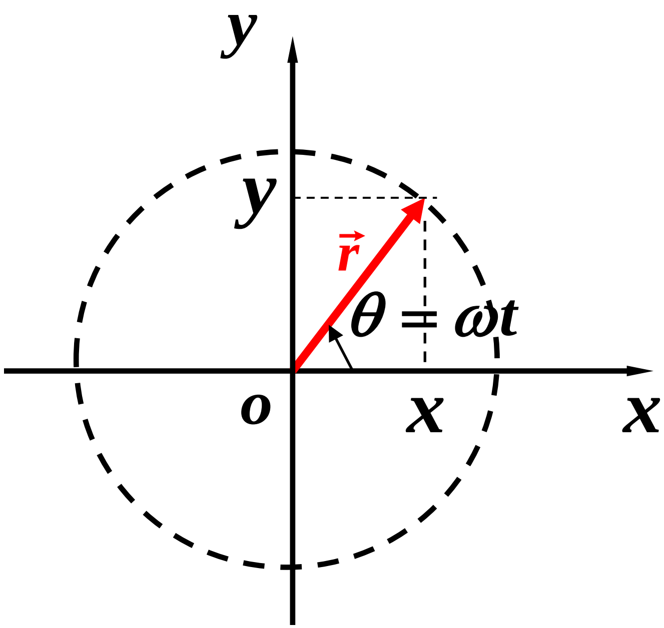
By canceling the time t, we obtain the path equation,

$$x^2 + y^2 = R^2$$ **Circular motion**

(2) Position

At any time, the position vector is

$$\vec{r}(t) = R\cos\omega t\hat{i} + R\sin\omega t\hat{j}$$



Magnitude: $r = \sqrt{x^2 + y^2} = R$

Direction: θ

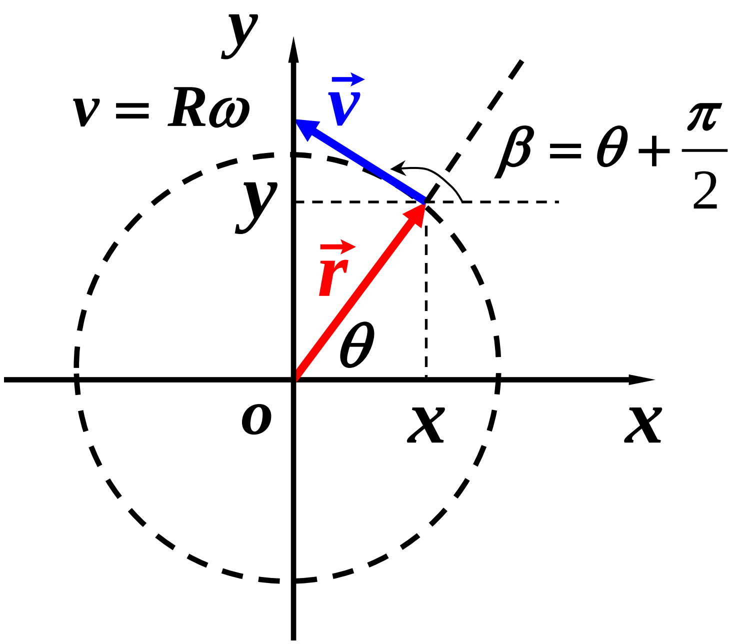
measured counterclockwise from the $+x$ axis

$$\tan\theta = \frac{y}{x} = \frac{R\sin\omega t}{R\cos\omega t} = \tan\omega t \qquad \theta = \omega t$$

(3) Velocity

$$\vec{v} = \frac{d\vec{r}}{dt} = -R\omega\sin\omega t\hat{i} + R\omega\cos\omega t\hat{j}$$

$$\begin{cases} v_x = -R\omega\sin\omega t = R\omega\cos(\omega t + \frac{\pi}{2}) \\ v_y = R\omega\cos\omega t = R\omega\sin(\omega t + \frac{\pi}{2}) \end{cases}$$



Magnitude:

$$v = \sqrt{v_x{}^2 + v_y{}^2} = R\omega$$

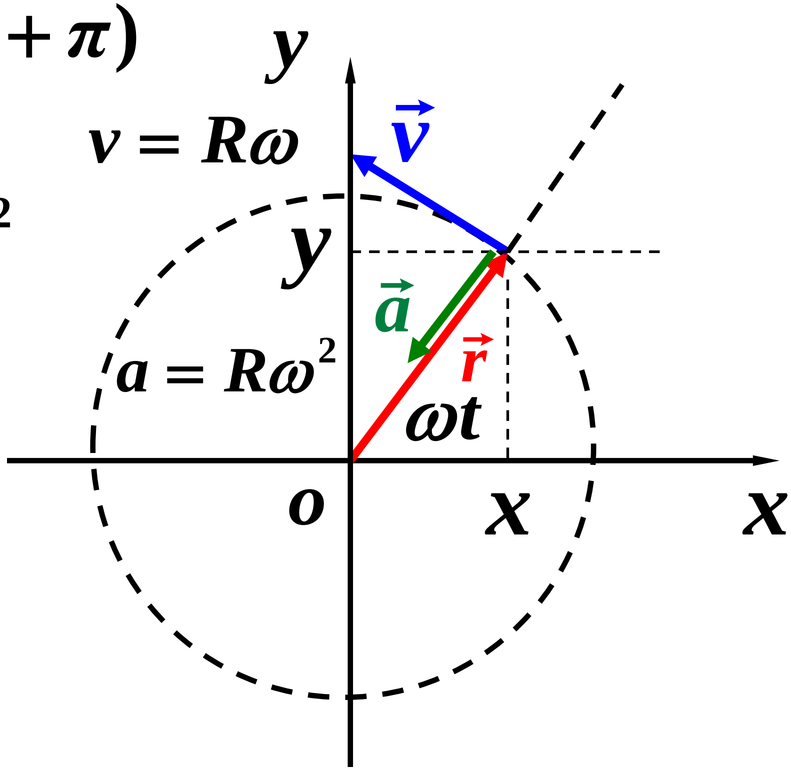
Uniform circular motion

Direction: $\beta = \omega t + \frac{\pi}{2} = \theta + \frac{\pi}{2}$

(4) Acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = -R\omega^2 \cos\omega t\hat{i} - R\omega^2 \sin\omega t\hat{j}$$

$$\begin{cases} a_x = -R\omega^2 \cos\omega t = R\omega^2 \cos(\omega t + \pi) \\ a_y = -R\omega^2 \sin\omega t = R\omega^2 \sin(\omega t + \pi) \end{cases}$$



Magnitude: $a = \sqrt{a_x{}^2 + a_y{}^2} = R\omega^2$

Direction: $\gamma = \omega t + \pi = \theta + \pi$

$$\vec{a} = -\omega^2 \vec{r}$$

Centripetal acceleration

(向心加速度)

§ 4.5 Projectile motion

For two and three dimension motion with constant acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \quad \text{constant}$$

$$d\vec{v} = \vec{a}dt$$

Initial velocity $t = 0 \qquad \vec{v} = \vec{v}_0$

By integration, $\int_{\vec{v}_0}^{\vec{v}} d\vec{v} = \int_0^t \vec{a}dt$

we obtain

$$\vec{v} = \vec{v}_0 + \vec{a}t$$

$$\vec{v} = \frac{d\vec{r}}{dt} \Rightarrow d\vec{r} = \vec{v}dt = (\vec{v}_0 + \vec{a}t)dt$$

$t = 0$， $\vec{r} = \vec{r}_0$

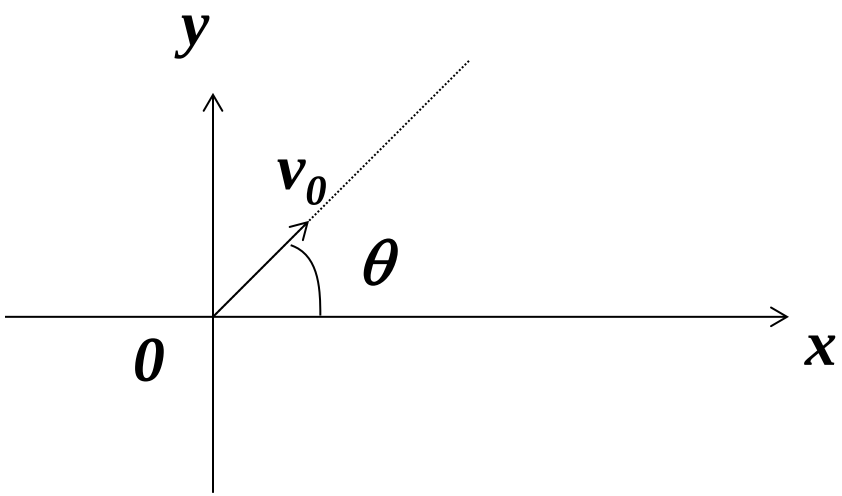
$$\Rightarrow \int_{\vec{r}_0}^{\vec{r}} d\vec{r} = \int_0^t (\vec{v}_0 + \vec{a}t)dt$$

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$$

For Cartesian coordinate system,

$$\begin{cases} v_x = v_{0x} + a_x t \\ v_y = v_{0y} + a_y t \\ v_z = v_{0z} + a_z t \end{cases} \qquad \begin{cases} x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \\ y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \\ z = z_0 + v_{0z}t + \frac{1}{2}a_z t^2 \end{cases}$$

Projectile motion is a two dimension constant acceleration motion



$$\vec{a} = \vec{g}$$

$$a_x = 0, \qquad a_y = -g$$

Initial condition (t=0)

$$x_0 = y_0 = 0,$$

$$v_{0x} = v_0 \cos\theta, \qquad v_{0y} = v_0 \sin\theta$$

$$\vec{v} = \vec{v}_0 + \vec{a}t \quad \Rightarrow \begin{cases} v_x = v_0 \cos\theta, \\ v_y = v_0 \sin\theta - gt \end{cases}$$

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 \quad \Rightarrow \begin{cases} x = v_0 t \cos\theta \\ y = v_0 t \sin\theta - \frac{1}{2} g t^2 \end{cases}$$

By canceling the time *t*, we obtain the path equation, or trajectory of the projectile

$$y = x \tan\theta - \frac{g x^2}{2 v_0^2 \cos^2\theta}$$

§ 4.6 Motion with varying acceleration

For two- and three-dimension motion with varying acceleration

$$\vec{a} = \vec{a}(t)$$

$$d\vec{v} = \vec{a}dt$$

Initial velocity $t = 0 \qquad \vec{v} = \vec{v}_0$

By integration, $\int_{\vec{v}_0}^{\vec{v}} d\vec{v} = \int_0^t \vec{a}(t)dt$

we obtain $\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(t)dt$

$$\vec{v}(t)=\frac{d\vec{r}}{dt} \Rightarrow d\vec{r}=\vec{v}(t)dt=[\vec{v}_0+\int_0^t \vec{a}(t)dt]dt$$

$$t=0\ , \quad \vec{r}=\vec{r}_0$$

$$\Rightarrow \int_{\vec{r}_0}^{\vec{r}} d\vec{r}=\int_0^t [\vec{v}_0+\int_0^t \vec{a}(t)dt]dt$$

$$\vec{r}(t)=\vec{r}_0+\vec{v}_0 t+\int_0^t [\int_0^t \vec{a}(t)dt]dt$$

By integrating the x, y, and z components separately, the result can be obtained, i.e.,

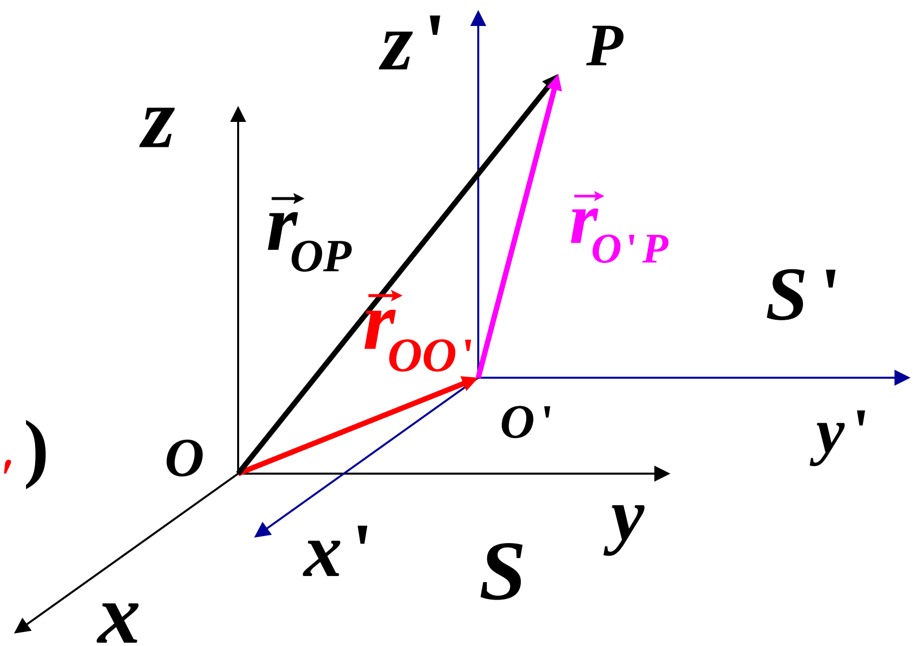
$$r_i(t)=r_{0i}+v_{0i}t+\int_0^t [\int_0^t a_i(t)dt]dt$$

$$i = x \text{ or } y \text{ or } z$$

§ 4.8-4.9 Relative motion

The position vectors of the same point in different frame:

$$\vec{r}_{OP} = \vec{r}_{O'P} + \vec{r}_{OO'}$$



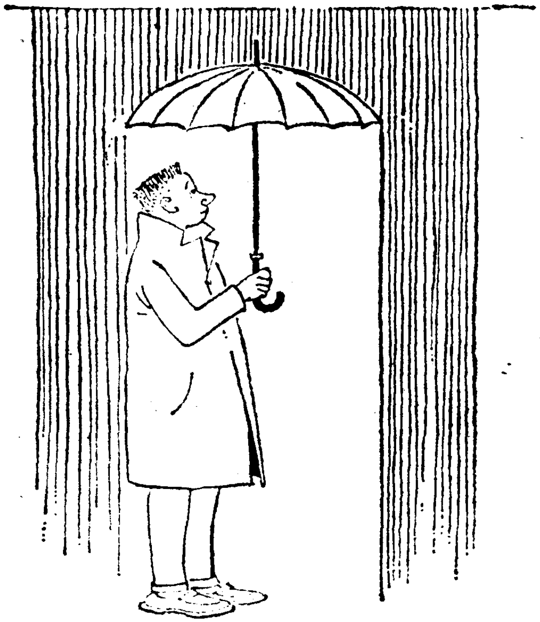
$$\frac{d}{dt}(\vec{r}_{OP}) = \frac{d}{dt}(\vec{r}_{O'P}) + \frac{d}{dt}(\vec{r}_{OO'})$$

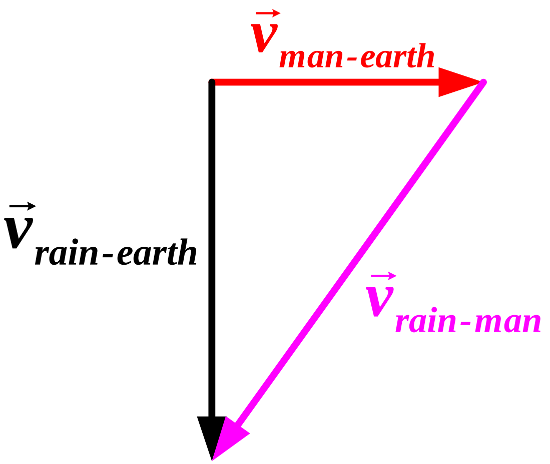
$$\vec{v}_{OP} = \vec{v}_{O'P} + \vec{v}_{OO'}$$

$\vec{v}_{OO'} =$ constant

$$\frac{d}{dt}(\vec{v}_{OP}) = \frac{d}{dt}(\vec{v}_{O'P}) + \frac{d}{dt}(\vec{v}_{OO'}) \Rightarrow \vec{a}_{OP} = \vec{a}_{O'P}$$

Example: The man in the rain.

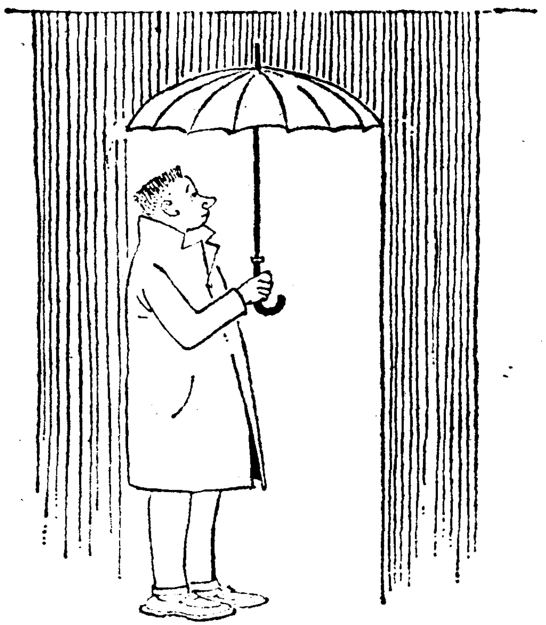


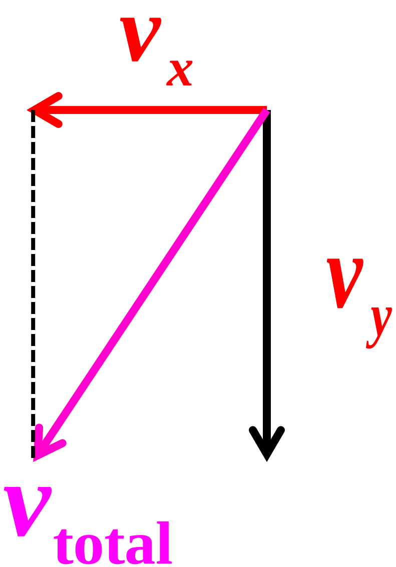


$$\vec{v}_{rain-earth} = \vec{v}_{rain-man} + \vec{v}_{man-earth}$$

Example: The man in the rain.

In the reference system of the running man





In the reference system of the running man, the velocity of the rain

$$\vec{v}_{rain-man} = \vec{v}_x + \vec{v}_y$$

Classwork 1: **a particle moves in an *xy* plane, its motion function is given by**

$$x = 2t^3 - 5t$$

$$y = 6 - 7t^4$$

with *x*, and *y* in meters and *t* in seconds.

Please find its position, velocity and acceleration at *t*=2.0s.

Homework 3: 3.1-3.5

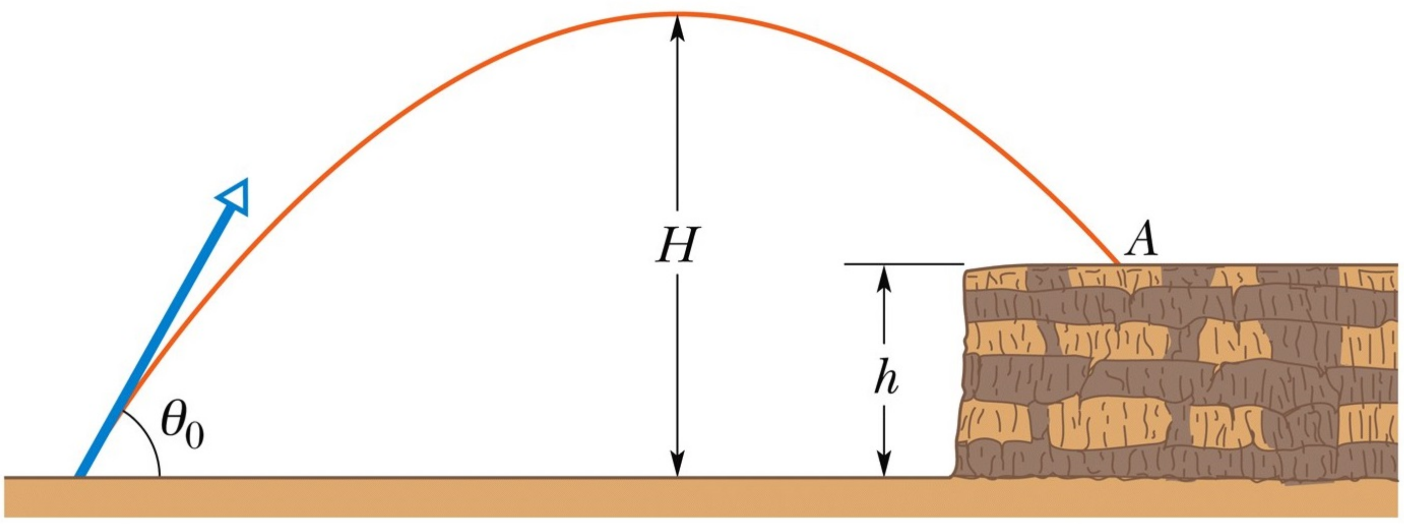
3.1 An electron's position is given by $\hat{r} = 3.00t\hat{\mathrm{i}} - 4.00t^2\hat{\mathrm{j}} + 2.00\hat{\mathrm{k}}$, with t in seconds and $\vec{r}$ in meters. (a) In unit-vector notation, what is the electron's velocity $\vec{v}(t)$? At $t = 3.00$ s , what is $\vec{v}$ (b) in unit-vector notation and as (c) a magnitude and (d) an angle relative to the positive direction of the x axis?

3.2. In a particle accelerator, the position vector of a particle is initially estimated as $\hat{r} = 6.0\hat{\mathrm{i}} - 7.0\hat{\mathrm{j}} + 3.0\hat{\mathrm{k}}$ and after 10 s, it is estimated to be $\hat{r} = -3.0\hat{\mathrm{i}} + 9.0\hat{\mathrm{j}} - 3.0\hat{\mathrm{k}}$, all in meters. In unit vector notation, what is the average velocity of the particle?

3.3. A proton initially has $\vec{v} = 4.0\hat{\mathrm{i}} - 2.0\hat{\mathrm{j}} + 3.0\hat{\mathrm{k}}$ and then 4.0 s later has $\vec{v} = -2.0\hat{\mathrm{i}} - 2.0\hat{\mathrm{j}} + 5.0\hat{\mathrm{k}}$ (in meters per second). For that 4.0 s, what are (a) the proton's average acceleration $\vec{a}_{\mathrm{avg}}$, in unit-vector notation, (b) the magnitude of $\vec{a}_{\mathrm{avg}}$, and (c) the angle between $\vec{a}_{\mathrm{avg}}$ and the positive direction of the *x* axis?

3.4 The acceleration of a particle moving only on a horizontal plane is given by $\vec{a} = 3t\hat{\mathrm{i}} + 4t\hat{\mathrm{j}}$, where $\vec{a}$ is in meters per second-squared and t is in seconds. At $t = 0$, the position vector $\vec{r} = (20.0\ \mathrm{m})\hat{\mathrm{i}} + (40.0\ \mathrm{m})\hat{\mathrm{j}}$ locates the particle, which then has the velocity vector $\vec{v} = (5.00\ \mathrm{m/s})\hat{\mathrm{i}} + (2.00\ \mathrm{m/s})\hat{\mathrm{j}}$. At $t = 4.00$ s, what are (a) its position vector in unit-vector notation and (b) the angle between its direction of travel and the positive direction of the x axis?

3.5. In the figure, a stone is projected at a cliff of height h with an initial speed of 42.0 m/s directed at angle $\boldsymbol{\theta_0} =$ 60.0° above the horizontal. The stone strikes at A, 5.50 s after launching. Find (a) the height *h* of the cliff, (b) the speed of the stone just before impact at A, and (c) the maximum height *H* reached above the ground.



Back up

Classwork 2: **A particle moves along a circular track with constant angular speed $\omega = \frac{\pi}{6}$ s^{-1}. Initial position of the particle is (0, 1m).**

Calculate:

(a) The position of the particle at t = 2s.

(b) The velocity and acceleration of the particle at t = 2s.