Lecture 1

A complex number z can be represented by an ordered pair (x, y) of real numbers x and y with operations of addition and multiplication defined by the equations

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \qquad (1)$$

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) \qquad (2)$$

We identify x by $(x,0)$ meaning that *the set of real numbers is a subset of the set of complex numbers*.

A complex number of the form $(0, y)$ is called a *pure imaginary number*. In particular,

$$(x,0) + (0, y) = (x, y) \text{ and } (0,1)(y,0) = (0, y).$$

Hence, $(x, y) = (x,0) + (0,1)(y,0).$

The operations defined by equations (1) and (2) become the usual operations of addition and multiplication when real numbers are considered:

$$(x_1,0) + (x_2,0) = (x_1 + x_2,0) \text{ and } (x_1,0)(x_2,0) = (x_1x_2,0). \qquad (3)$$

Therefore, *the complex number system is a natural extension of the real number system*. The real numbers x and y in the expression $z = (x, y)$ are known as the real and imaginary parts of z, respectively; and we denote them by $\operatorname{Re} z = x$, $\operatorname{Im} z = y$.

Two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are said to be equal whenever

$$(x_1, y_1) = (x_2, y_2) \Leftrightarrow x_1 = x_2 \text{ and } y_1 = y_2.$$

Let i denotes the pure imaginary number $(0,1)$. So $(x, y) = x + iy$. Also, with the convention $z^2 = zz, z^3 = zz^2$, etc., we can write $i^2 = ii = (0,1)(0,1) = (-1,0) = -1$, i.e., $i^2 = -1$. Hence, we have

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2).$$

Graphical Representation of Complex Numbers

Consider two mutually perpendicular axes XOX' and YOY' (called the x-axis and y-axis, respectively).

Since a complex number $z = x + iy$ can be considered as an ordered pair of real numbers, we can represent such numbers by points in the xy-plane, called the *complex plane* or *Argand diagram*.

Example. We can represent complex numbers $P = (x, y) = x + iy$, $Q = (4,3) = 4 + 3i$ and $R = (-3,4) = -3 + 4i$.

Sometimes we refer to the x-axis and y-axis as the $\mathrm{Re}(z)$-axis and $\mathrm{Im}(z)$-axis, respectively and to the complex plane as the z-plane.

The *complex conjugate* (or simply the *conjugate*) of a complex number $z = x + iy$ is defined by the complex number $\bar{z} = x - iy$ and is denoted by $\bar{z}$, that is, $\bar{z} = x - iy = (x, -y)$.

Any circle with centre at z_0 and radius R is given by $|z - z_0| = R$. For example, if C is the circle $|z - 1 + 3i| = 2$, then the centre of the circle is $(1, -3)$ and radius 2.

The real numbers $|z|$, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are connected by the relation

$$|z|^2 = [\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2$$

So, we have $|z|^2 = z\bar{z}$ and $|z| = \sqrt{x^2 + y^2}$.

Functions of a complex variable

Let S be a set of complex numbers. A function f defined on S is a rule that assigns to each $z \in S$, there is a complex number w. The number w is called the value of f at z, denoted by $f(z)$ and is defined as $w = f(z)$. The set S is called the domain of f.

Example 1

Consider $w = f(z) = \dfrac{1}{z}$, $z \neq 0$

Suppose that $w = u + iv = f(z) = f(x + iy)$. So, each of the real numbers u and v depends on the real variables x and y. It follows that $f(z)$ can be expressed in terms of a pair of real-valued functions u and v of real variables x and y. We can write

$$f(z) = \frac{1}{x+iy} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

So that $u(x, y) = \frac{x}{x^2+y^2}$ and $v(x, y) = -\frac{y}{x^2+y^2}$.

Example 2

Consider $w = f(z) = z^2$.

We can write $w = u(x, y) + iv(x, y) = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$.

Therefore, we have $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

In polar coordinates, we have $u+iv=f(re^{i\theta})$, where $w=u+iv$ and $z=re^{i\theta}$. We write $f(z)=u(r,\theta)+iv(r,\theta)$.

Example 3

Consider $w=f(z)=z+\dfrac{1}{z}.$

We write $f(re^{i\theta})=re^{i\theta}+\dfrac{1}{r}e^{-i\theta}=r(\cos\theta+i\sin\theta)+\dfrac{1}{r}(\cos\theta-i\sin\theta)$

$$=(r+\frac{1}{r})\cos\theta+i(r-\frac{1}{r})\sin\theta.$$

Hence, we have $u(r,\theta)=(r+\dfrac{1}{r})\cos\theta$ and $v(r,\theta)=(r-\dfrac{1}{r})\sin\theta.$

Example 4

Consider $w=f(z)=|z|^2$.

We write $w=|z|^2=x^2+y^2+i0$. So $f(z)$ is a real-valued function of a complex variable z.

If $n=0$, $n\in\mathbf{N}$ and if $a_0,a_1,a_2,\ldots,a_n$ are complex constants, then the function

$$p(z)=a_0+a_1z+a_2z^2+\cdots+a_nz^n,\ (a_n\neq 0)$$

is a polynomial in z of degree n. The domain of $p(z)$ is the entire z-plane. The rational (quotient) function is $r(z)=\dfrac{p(z)}{q(z)}$ if $q(z)\neq 0$.

Polynomials and rational functions play a vital role in the theory of functions of complex variables.