Seminar on Algebra, Geometry and Discrete Mathematics – MAMME
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Proof of asymptotic behaviour of eigenvalues in families of expanders

The main topic of the seminar is the study of families of expanders in the context of connected, k-regular and finite graphs. A family of expanders consists in a sequence of graphs of increasing size such that the isoperimetric number $h(X)$ satisfies $h(X) \geq \epsilon$ for some fixed positive ϵ and for all graphs X in the sequence.

Earlier in the seminar, we established that the isoperimetric number is bounded by the spectral gap, in particular,

$$\frac{k-\mu_1}{2} \leq h(x) \leq \sqrt{2k(k-\mu_1)}$$

where μ_1 denotes the first nonzero eigenvalue. Moreover, we showed that the spectral gap cannot grow too much, i.e.

$$\liminf_{m\to+\infty} \mu_1 \geq 2\sqrt{k-1}$$

In this session, we aim to strengthen this result by proving that a non-trivial fraction of the eigenvalues of the graph lies within the interval $[(2-\epsilon)\sqrt{k-1}, k]$ for any given $\epsilon > 0$.

To prove this, we introduce a set of matrices A_r, which are polynomials in the incidence matrix A. The entries of A_r describe the number of backtracking-free paths of length r between two vertices.

After establishing the necessary properties of these polynomials, we compute their generating function, which is closely related to the generating function of Chebyshev polynomials $(U_m)_{m\in\mathbb{N}}$. This allows us to relate the two polynomials:

$$T_m = (k-1)^{\frac{m}{2}} U_m \left(\frac{A}{2\sqrt{k-1}}\right) \tag{1}$$

where $T_m = \sum_{0\leq r\leq \frac{m}{2}} A_{m-2r}$.

This way, we can compute:

$$\sum_{x\in V} \sum_{0\leq r\leq \frac{m}{2}} (A_{m-2r})_{xx} = Tr(T_m) = (k-1)^{\frac{m}{2}} \sum_{j=0}^{n-1} U_m \left(\frac{\mu_j}{2\sqrt{k-1}}\right) \tag{2}$$

which holds for all $m \in \mathbb{N}$. This result is crucial, as it proves that the right-hand side of the equation is positive.

We then introduce the measure:

$$v = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{\frac{\mu_j}{\sqrt{k-1}}}$$

where δ_a is the Dirac measure at a, and $\mu_0, \dots, \mu_{n-1}$ are the eigenvalues of the graph. The Dirac measure δ_a acts as an indicator function of whether a lies in a given interval. Thus, ν satisfies:

$$\nu[x_1, x_2] = \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{[x_1, x_2]} \left(\frac{\mu_j}{\sqrt{k-1}} \right) = \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{[x_1\sqrt{k-1}, x_2\sqrt{k-1}]} (\mu_j) \tag{3}$$

that is, ν counts the number of eigenvalues in the interval $[x_1\sqrt{k-1}, x_2\sqrt{k-1}]$.

To conclude the proof, equation (2) is used to prove that $\nu[2-\epsilon, L] \geq C$, where $C = C(\epsilon, k)$ is a strictly positive constant. We first define polynomials $X_m = U_m(\frac{x}{2})$ and $Y_m(x) = \frac{X_m(x)^2}{x - \alpha_m}$ where α_m is the largest root of X_m. Then we show some properties concerning these polynomials. In particular, we see that, for all $m \geq 0$, Y_m is a linear combination with non-negative coefficients of $X_0, X_1, \dots, X_{2m-1}$.

It follows that, if

$$\int_{-L}^{L} X_m(x) d\nu(x) \geq 0$$

then

$$\int_{-L}^{L} Y_m(x) d\nu(x) \geq 0$$

Since $Y_m(x) \leq 0$ when $x \leq \alpha_m$, and $\alpha_m > 2 - \epsilon$ for m large enough, it can be proved that the measure of ν in the interval $[2-\epsilon, L]$ is strictly positive. Moreover, using compactness in the weak topology, we establish that $\nu[2-\epsilon, L] \geq C(\epsilon, k) > 0$.

The next step is to prove that ν indeed satisfies $\int_{-L}^{L} X_m(x) d\nu(x) \geq 0$. By definition of ν and Dirac measure:

$$\int_{-L}^{L} U_m\left(\frac{x}{2}\right) d\nu(x) = \frac{1}{n} \sum_{j=0}^{n-1} \int_{-L}^{L} U_m\left(\frac{x}{2}\right) d\delta_{\frac{\mu_j}{\sqrt{k-1}}} = \frac{1}{n} \sum_{j=0}^{n-1} U_m\left(\frac{\mu_j}{\sqrt{k-1}}\right)$$

which is non-negative by equation (2).

The assumption is satisfied, therefore there exists $C = C(\epsilon, k) > 0$ such that $\nu[2-\epsilon] \geq C$. Finally, when $L = \frac{k}{\sqrt{k-1}}$, equation (3) shows that ν counts the number of eigenvalues in the interval $[(2-\epsilon)\sqrt{k-1}, k]$, and hence:

$$\left| \left\{ \text{Eigenvalues of } X \text{ in } \left[(2-\epsilon)\sqrt{k-1}, k \right] \right\} \right| = \nu[2-\epsilon, L] * n \geq C * n$$

which proves the main result.

Additionally, we show a similar conclusion for the negative eigenvalues of the graph. Let $(X_m)_{m \geq 1}$ a family of connected, k-regular and finite graphs, with $g(X_m) \to \infty$ as $m \to \infty$. For every $\epsilon > 0$, there exists $C = C(\epsilon, k) > 0$ such that the number of eigenvalues of X_m in the interval $[-k, -(2-\epsilon)\sqrt{k-1}]$ is at least $C|X_m|$.