Seminar on Algebra, Geometry and Discrete Mathematics

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1 Proof of asymptotic behaviour of eigenvalues in families of expanders

The main topic of the seminar is the study of families of expanders in the context of connected, k-regular and finite graphs. A family of expanders consists in a sequence of graphs of increasing size such that the isoperimetric number $h(X)$ satisfies $h(X) \geq \epsilon$ for some fixed positive ϵ and for all graphs X in the sequence. The isoperimetric number, in the context of finite graphs, can be defined as:

$$h(x) = \min \left\{ \frac{|\partial F|}{|F|} \mid F \subseteq V, |F| \leq \frac{n}{2} \right\}$$

Earlier in the seminar, we established that the isoperimetric number is bounded by the spectral gap, in particular,

$$\frac{k - \mu_1}{2} \leq h(x) \leq \sqrt{2k(k - \mu_1)}$$

where μ_1 denotes the first nonzero eigenvalue. Moreover, we showed that the spectral gap cannot grow too much, i.e.

$$\liminf_{m \to +\infty} \mu_1 \geq 2\sqrt{k-1}$$

In this session, we aim to strengthen this result by proving that a non-trivial fraction of the eigenvalues of the graph lies within the interval $[(2-\epsilon)\sqrt{k-1}, k]$ for any given $\epsilon > 0$.

To prove this, we introduce a set of matrices A_r, which are polynomials in the incidence matrix A. The entries of A_r describe the number of backtracking-free paths of length r between two vertices.

After establishing the necessary properties of these polynomials, we compute their generating function which is closely related to the generating function of Chebyshev polynomials $(U_m)_{m\in\mathbb{N}}$. This relationship between the two polynomials leads to the following key result:

$$\sum_{x\in V}\sum_{0\leq r\leq \frac{m}{2}} (A_{m-2r})_{xx} = (k-1)^{\frac{m}{2}} \sum_{j=0}^{n-1} U_m\left(\frac{\mu_j}{2\sqrt{k-1}}\right) \tag{1}$$

for all $m \in \mathbb{N}$.

We then introduce the measure:

$$\nu = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{\frac{\mu_j}{\sqrt{k-1}}}$$

where δ_a is the Dirac measure at a, and $\mu_0, \dots, \mu_{n-1}$ are the eigenvalues of the graph. The Dirac measure δ_a acts as an indicator function of whether a lies in a given interval. Hence, this measure is in some way counting the number of eigenvalues within a specific range, which is central to the argument.

Using equation (1), we show that, for an appropriate choice of $L \geq 2$, ν satisfies:

$$\int_{-L}^{L} U_m\left(\frac{x}{2}\right) d\nu \geq 0$$

for all m. Finally, we will see that this condition is sufficient to prove that the measure ν has a positive support in the interval $[2-\epsilon, L]$. In other words,

$$\nu[2-\epsilon, L] \geq C$$

where C is a constant that only depends on ϵ and L, and is strictly positive. This proves the main result.

Additionally, we show a similar conclusion for the negative eigenvalues of the graph, generalizing an earlier result.