Problem 7. *Let X and Y be disjoint sets of vertices with $e(X,Y)=0$. Prove that*

$$\frac{|X||Y|}{(n-|X|)(n-|Y|)} \leq \left(\frac{\mu_n-\mu_2}{\mu_n+\mu_2}\right)^2$$

[Hint: consider the matrix $M = \begin{pmatrix} 0 & L+\mu I \\ L+\mu I & 0 \end{pmatrix}$ where $\mu = -\frac{1}{2}(\mu_n+\mu_2)$ and use interlacing with partition $(X, V\setminus X, Y, V\setminus Y)$.]

Solution. *Proof.* Consider the matrix $M = \begin{pmatrix} 0 & L+\mu I \\ L+\mu I & 0 \end{pmatrix}$. Also, index the vertices of the graph such that $\{v_1,\dots,v_{|X|}\} = X$, $\{v_{|X|+1},\dots,v_{n-|Y|}\} = V\setminus X\setminus Y$ and $\{v_{n-|Y|+1},\dots,v_n\} = Y$.

This way,

$$\underbrace{v_1,\dots,v_{|X|}}_{X}, \underbrace{v_{|X|+1},\dots,v_{n-|Y|},v_{n-|Y|+1},\dots,v_n}_{V\setminus X}$$

and

$$\underbrace{v_1,\dots,v_{|X|},v_{|X|+1},\dots,v_{n-|Y|}}_{V\setminus Y}, \underbrace{v_{n-|Y|+1},\dots,v_n}_{Y}$$

Now, we want to calculate the quotient matrix of M over the partition $(X, V\setminus X, Y, V\setminus Y)$.

Observation. *To help with the intuition of what this partition means, consider the bipartite graph $G^{\times 2} = (V^1, V^2, E)$, where V^1 and V^2 are copies of the vertices V, such that:*

- *For all $i \in \{1,2\}$ and $j,k \in \{1,\dots,n\}$, $v_j^i \nsim v_k^i$*
- *For all $j,k \in \{1,\dots,n\}$, $v_j^1 \sim v_k^2 \Leftrightarrow v_j \sim v_k$*

Then the partition $(X, V\setminus X, Y, V\setminus Y)$ can be thought as the partition $(X^1, V^1\setminus X^1, Y^2, V^2\setminus Y^2)$ of $G^{\times 2}$, where X^1 and Y^2 are the analogous of X and Y in V^1 and V^2.

Let's rewrite M to fit the partition:

$$M = \begin{array}{c} \begin{array}{cccc} X & V\setminus X & V\setminus Y & Y \end{array} \\ \begin{pmatrix} 0 & 0 & M_{X\times V\setminus Y} & M_{X\times Y} \\ 0 & 0 & M_{V\setminus X\times V\setminus Y} & M_{V\setminus X\times Y} \\ M_{V\setminus Y\times X} & M_{V\setminus Y\times V\setminus X} & 0 & 0 \\ M_{Y\times X} & M_{Y\times V\setminus X} & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} X \\ V\setminus X \\ V\setminus Y \\ Y \end{array}$$

Notice that $L+\mu I = \begin{pmatrix} M_{X\times V\setminus Y} & 0 \\ M_{V\setminus X\times V\setminus Y} & M_{V\setminus X\times Y} \end{pmatrix} = \begin{pmatrix} M_{V\setminus Y\times X} & M_{V\setminus Y\times V\setminus X} \\ 0 & M_{Y\times V\setminus X} \end{pmatrix}$, since $e(X,Y) = 0$ and $M_{X\times Y}$, $M_{Y\times X}$ do not cross the diagonal of $L+\mu I$.

Call B the quotient matrix over the partition $(X, V\setminus X, Y, V\setminus Y)$. B is a 4×4 matrix:

$$B = \begin{pmatrix} 0 & 0 & b_{X\times V\setminus Y} & 0 \\ 0 & 0 & b_{V\setminus X\times V\setminus Y} & b_{V\setminus X\times Y} \\ b_{V\setminus Y\times X} & b_{V\setminus Y\times V\setminus X} & 0 & 0 \\ 0 & b_{Y\times V\setminus X} & 0 & 0 \end{pmatrix}$$

Claim. $B = \begin{pmatrix} 0 & 0 & \mu & 0 \\ 0 & 0 & \mu - \mu\frac{|Y|}{n-|X|} & \mu\frac{|Y|}{n-|X|} \\ \mu\frac{|X|}{n-|Y|} & \mu - \mu\frac{|X|}{n-|Y|} & 0 & 0 \\ 0 & \mu & 0 & 0 \end{pmatrix}$

Proof. We only need to prove that:

- $b_{X\times V\setminus Y} = b_{Y\times V\setminus X} = \mu$. Since the rows of L sum zero, the rows of $L+\mu I$ sum μ. Then, the average these sums is μ.
- $b_{V\setminus X\times Y} = \mu\frac{|Y|}{n-|X|}$. Again, the columns of $L+\mu I$ sum μ, so the sum of all the elements of $M_{V\setminus X\times Y}$ is $\mu|Y|$. We average this over the number of rows, resulting in $\mu\frac{|Y|}{n-|X|}$. (analogously for $b_{V\setminus Y\times X}$).
- $b_{V\setminus X\times V\setminus Y} = \mu - \mu\frac{|Y|}{n-|X|}$. This comes directly from the fact that the rows of $M_{V\setminus X\times Y}$ on average sum $\mu\frac{|Y|}{n-|X|}$, and the rows of $L-\mu I$ sum μ. (analogously for $b_{V\setminus Y\times V\setminus X}$).

$\square$

Before calculating the eigenvalues of M and B, let us prove the following claim:

Claim. *Consider N a square matrix such that $N = \begin{pmatrix} 0 & E \\ F & 0 \end{pmatrix}$ where E and F are square matrices. Then, the eigenvalues of N are the square roots of the eigenvalues of FE.*

Proof. λ is an eigenvalue of N if and only if $det\begin{pmatrix} D & E \\ F & D \end{pmatrix} = 0$, where $D = -\lambda I$. Since D is invertible, we can use Schur complement:

$$\begin{pmatrix} D & E \\ F & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ FD^{-1} & I \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & D - FD^{-1}E \end{pmatrix}\begin{pmatrix} I & D^{-1}E \\ 0 & I \end{pmatrix}$$

Also,

$$det\begin{pmatrix} I & 0 \\ FD^{-1} & I \end{pmatrix} = det\begin{pmatrix} I & D^{-1}E \\ 0 & I \end{pmatrix} = 1$$

since they are triangular matrices with 1s in the diagonal, and

$$det\begin{pmatrix} D & 0 \\ 0 & D - FD^{-1}E \end{pmatrix} = det(D)det(D - FD^{-1}E) = det(D^2 - FE)$$

Thus, $det(N-\lambda I) = det(D^2-FE) = det(\lambda^2 I-FE)$ and $det(N-\lambda I) = 0 \Leftrightarrow det(FE-\lambda^2 I) = 0$. We conclude that λ is an eigenvalue of N if and only if λ^2 is an eigenvalue of FE. $\square$

Recall, the eigenvalues of L are

$$0 = \mu_1 \le \mu_2 \le \cdots \le \mu_n$$

Then, the eigenvalues of $L+\mu I$ are

$$\mu \le \mu_2 + \mu = \frac{\mu_2 - \mu_n}{2} \le \cdots \le \mu_n + \mu = \frac{\mu_n - \mu_2}{2} \tag{1}$$

and the eigenvalues of $(L+\mu I)^2$ are

$$\mu^2, \left(\frac{\mu_2-\mu_n}{2}\right)^2, \dots, \left(\frac{\mu_n-\mu_2}{2}\right)^2$$

Thus, using the second claim, we conclude that the eigenvalues of M are the positive and negative square roots of the eigenvalues of $(L+\mu I)^2$. Also, from 1 we can conclude that $\pm\mu$ and $\pm\frac{\mu_n-\mu_2}{2}$ are the first and second eigenvalues with higher absolute values. So, the eigenvalues of M are

$$\lambda_1 = -\mu \geq \lambda_2 = \frac{\mu_n-\mu_2}{2} \geq \dots \geq \lambda_{2n-1} = \frac{\mu_2-\mu_n}{2} \geq \lambda_{2n} = \mu$$

Similarly, the eigenvalues of B are the square roots of the eigenvalues of B_2B_1, where $B_1 = \mu\begin{pmatrix} 1 & E \\ 1-b & b\end{pmatrix}$, $B_2 = \mu\begin{pmatrix} a & 1-a \\ 0 & 1\end{pmatrix}$, $a = \frac{|X|}{n-|Y|}$ and $b = \frac{|Y|}{n-|X|}$. Let's calculate B_2B_1:

$$B_2B_1 = \mu^2\begin{pmatrix} a & 1-a \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & E \\ 1-b & b\end{pmatrix} = \mu^2\begin{pmatrix} 1-b+ab & b-ab \\ 1-b & b\end{pmatrix}$$

The determinant of $B_2B_1/\mu^2 - \delta I$ is

$$\begin{aligned} det(B_2B_1/\mu^2-\delta I) &= (1-b-ab-\delta)(b-\delta) - b(1-a)(1-b) = \\ &= \delta^2 - \delta(1-b+ab+b) + b - b^2 + ab^2 - b + ab + b^2 - ab^2 = \\ &= \delta^2 - (1+ab)\delta + ab = (\delta-1)(\delta-ab) \end{aligned}$$

Thus, the eigenvalues of B_2B_1 are μ^2 and $\mu^2\frac{|X||Y|}{(n-|X|)(n-|Y|)}$. Again, using the second claim, we conclude that the eigenvalues of B are

$$\delta_1 = -\mu \geq \delta_2 = -\mu\sqrt{\frac{|X||Y|}{(n-|X|)(n-|Y|)}} \geq \delta_3 = \mu\sqrt{\frac{|X||Y|}{(n-|X|)(n-|Y|)}} \geq \delta_4 = \mu$$

Finally, we can use interlacing between the eigenvalues of M and B:

- $\lambda_2 \geq \delta_2 \geq 0$
- $0 > \delta_3 \geq \lambda_{2n-4+3} = \lambda_{2n-1}$

Thus,

$$\begin{aligned} \frac{|X||Y|}{(n-|X|)(n-|Y|)} &= -\frac{\delta_2\delta_3}{\mu^2} \leq -\frac{\lambda_2\lambda_{2n-1}}{\mu^2} = \\ &= \left(-\frac{2}{\mu_n+\mu_2}\right)^2\left(-\frac{\mu_n-\mu_2}{2}\right)^2 = \left(\frac{\mu_n-\mu_2}{\mu_n+\mu_2}\right)^2 \end{aligned}$$

This proves the statement of the problem.

$\square$