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Master in Advanced Mathematics and Mathematical Engineering

Master's thesis

On the importance of details

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Thanks to...

Abstract

This should be an abstract in english, up to 1000 characters.

Keywords

regularity, stable graphs, graph theory, ...

1. Introduction

Things to talk about: - Szemerédi's regularity lemma.
- Half-graphs and stable regularity lemma.
- Property testing.
- Stable regularity lemma for testing whether a graph has the property of not containing a fixed graph as a subgraph. (Specify this is a ∀ P first order property)

Lluis: is there a better prior cite for this?

Szemerédi's regularity lemma is a powerful tool in graph theory, stating that any sufficiently large graph can be decomposed into an equitable partition of its vertices such that most pairs of parts are *regular*. A regular pair is one whose edge distribution resembles that of a random bipartite graph, a powerful property with many applications in extremal graph theory. The primary drawback of the lemma, however, is the immense bound on the required number of parts, which grows as a tower of exponentials whose height depends on the regularity parameter.

The source of this combinatorial complexity can be traced to the presence of specific induced subgraphs. As demonstrated by Malliaris and Shelah in their seminal work [2], a key structure responsible for irregularity is the *half-graph*. For graphs that exclude large half-graphs, a class known as *stable graphs*, they proved that a much stronger form of regularity is achievable. Their *stable regularity lemma* not only yield vastly improved bounds on the partition size but, remarkably, can guarantee a decomposition entirely free of irregular pairs.

Regularity lemmas are particularly useful in the field of *property testing*. A property testing algorithm for a decision problem P is a randomized algorithm that, by querying only a small portion of its input, can distinguish with high probability between objects that satisfy P and those that are "far" from satisfying it. For instance, in [1] the authors use Szemerédi's regularity lemma to prove that it is possible to test the property of a graph G being H-free (for a fixed graph H) using an algorithm which query complexity is independent on the size of the input graph G.

The query complexity of such testers, however, is intrinsically linked to the number of parts in the underlying regular partition. Consequently, the power-tower bounds of the standard regularity lemma lead to prohibitively large, although constant, query counts. This raises a natural question: can the superior bounds of the stable regularity lemma be exploited to create more efficient property testers for graphs in a half-graph-restricted setting?

In this thesis, we present an algorithm for testing H-freeness in stable graphs, thereby providing a concrete application that highlights the practical strength and utility of stable regularity partitions.

The main contributions of this thesis are:

- **A rigorous reformulation and correction of the central proofs** in [2]. Our contribution provides a self-contained, combinatorial framework for these results, systematically resolving foundational gaps and inaccuracies in the original arguments to ensure their validity. This reworking also makes the associated combinatorial bounds fully explicit for the first time.

- **The construction of an efficient property testing algorithm** for H-freeness tailored to stable graphs. The algorithm's analysis leverages the stable regularity lemma to achieve a query complexity with significantly improved bounds compared to the general case.

- **The development of a unified notational framework** that cohesively integrates the concepts from extremal graph theory, stability, and property testing used throughout the thesis.

The remainder of this thesis is organized as follows. Section 2 reviews fundamental concepts from graph theory, culminating in a formal statement of Szemerédi's Regularity Lemma. Section 3 introduces the graph-theoretic notion of stability and proves some basic results in this context. Section 4 presents and analyzes a weaker variant of the stable regularity lemma, and illustrate both its strengths and its inherent limitations. Section 5 dedicated to the proof of the main Stable Regularity Lemma, which forms the technical core of this work. Finally, Section 6 applies this previous results to prove our property testing algorithm for H-freeness in stable graphs works, providing explicit bounds on its query complexity.

2. Section 2

Things that should be included in this section:
- General notation.
- Definition of a graph.
- Probably, also present edges as a relation on vertices, mentioning its properties, and explain that this is the bridge with model theory.
- Define density of a (non necessarily disjoint) pair of sets of vertices.
- Definition of a bipartite graph.
- Reglarity definitions.
- Szemerédi's regularity lemma.

Notation:
- By abuse of notation aRb is a value in $\{0, 1\}$.
- Abuse of notation: $a \in G$ to say that $a \in V(G)$.
- $\langle \cdot \rangle$ to represent tuples.

Mention that during the thesis, a lot of results carry many conditions most of which seem almost trivial, but are necessary for the computations to work. In the final result of each section, the results are cleaned out and tried to be delivered in a more readable form.

3. Section 3

In this section we introduce the class of graphs we will be working on, the *stable* graphs. Stable graphs are graphs which do not contain "quasi-induced" large half-graphs, a particularly "irregular" structure in graphs. See **??** for an example of such graph. We formally define the stability as the non-k-order, where k establishes how large are the half-graphs we are excluding.

Lluis: what is the right term here?

Mention how different papers call this differently.

Add visual example of a half-graph

Definition 3.1. Let G be a graph. We say that G has the *k-order property* if there exist two sequences of vertices $\langle a_i \mid i \in \{1, \dots, k\}\rangle$ and $\langle b_i \mid i \in \{1, \dots, k\}\rangle$ such that for all $i, j \leq k$, $a_i R b_j$ if and only if $i \geq j$. Otherwise, we say that G has the *non-k-order property*.

Remark 3.2. It is important to note what is left unspecified in Definition 3.1. First, the vertices within each sequence must be distinct, as their neighborhoods within the other sequence differ. However, the sequences themselves need not be disjoint. One may have $a_i = b_j$, provided $i < j$ (so that $\neg(a_i R b_j)$). Furthermore, the definition does not specify the presence or absence of edges within the same sequence. Consequently, the non-k-order property requires the containment of a subgraph from a broad class of structures, not merely a k-half-graph.

Possibly add visual example of this too.

Remark 3.3. G having k-order property implies G having k'-order property for all $k' \leq k$. Conversely, G having the non-k-order property implies G having non-k'-order property for all $k' \geq k$.

An important concept used all over the thesis is that of *exceptional edges* and *exceptional vertices*. That is, edges and vertices that in some sense are not regular, and do not behave as the rest of the graph. In order to classify what is the expected behaviour in a graph, or more specifically, in a pair of sets of vertices, we define the *truth value*.

Definition 3.4 (Truth value)**.** Let G be a graph. For any (not necessarily disjoint) $A, B \subseteq G$, we say that

$$t(A, B) = \begin{cases} 0 & \text{if } |\{(a, b) \in A \times B \mid aRb, a \neq b\}| < |\{(a, b) \in A \times B \mid \neg aRb, a \neq b\}| \\ 1 & \text{otherwise} \end{cases}$$

is the *truth value* of the pair (A, B). That is, $t(A, B) = 0$ if A and B are mostly disconnected, and $t(A, B) = 1$ if they are mostly connected. When $B = \{b\}$, we write $t(A, b)$ instead of $t(A, \{b\})$, and we say that it is the truth value of A with respect to b.

In this context, we say that a vertex $a \in A$ is *exceptional* with respect to $B \subseteq G$ if $t(a, B) \not\equiv t(A, B)$, or that it is exceptional with respect to $b \in G$ if $aRb \not\equiv t(A, b)$. On the other hand, we say that an edge ab with $a \in A$ and $b \in B$ is exceptional in (A, B) if $aRb \not\equiv t(A, B)$. Also, it is useful to define the following set of vertices.

- $B_{A,b} = \{a \in A \mid aRb \equiv t(A, b)\}$, i.e. the set of non-exceptional vertices of A with respect to B.
- $\overline{B}_{A,b} = \{a \in A \mid aRb \not\equiv t(A, b)\}$, the set of exceptional vertices of A with respect to B.
- $B^+_{A,b} = \{a \in A \mid aRb\}$, the vertices of A connected to b.
- $B^-_{A,b} = \{a \in A \mid \neg aRb\}$, the vertices of A that are not connected to b.

Maybe, move these two to the lemma where they are needed.

With this notation, notice that either $t(A, b) = 1$ and thus $B_{A,b} = B^+_{A,b}$, or $t(A, b) = 0$ and $B_{A,b} = B^-_{A,b}$. Large sets $B_{A,b}$, as we will see in the next sections, are closely related with lack of regularity in the graph.

A useful tool to deal with them is Lemma 3.10, which gives a bound on the number of such sets under the non-k-order property. In order to prove it, we first need to introduce the *VC dimension* of a family of sets, and relate it to the k-order property. This, together with Lemma 3.7, will give us the desired result.

Definition 3.5. Let G be a set and $S = \{S_i \subseteq G \mid i \in I\}$ be a family of sets. A set $A \subseteq G$ is said to be *shattered* by S (and S is said to *shatter* A) if for every $B \subseteq A$, there exists $S_i \in S$ such that $S_i \cap A = B$.

Definition 3.6. Let G be a set and $S = \{S_i \subseteq G \mid i \in I\}$ be a family of sets. The *VC dimension* of S is the size of the largest set $A \subseteq G$ that is shattered by S.

Lemma 3.7 (Sauer-Shelah)**.** *Let G be a set and $S = \{S_i \subseteq G \mid i \in I\}$ be a family of sets. If the VC dimension of S is at most k, and the union of all sets in S has n elements, then S consists of at most $\sum_{i=0}^{k} \binom{n}{i} \leq n^k$ sets.*

We'll begin by proving a stronger version of this lemma from Pajor, for which Sauer-Shelah will be a straightforward consequence.

Lemma 3.8 (Sauer-Shelah-Pajor)**.** *Let G be a set and S be a finite family of sets in G. Then S shatters at least $|S|$ sets.*

Proof. We will prove this by induction on the cardinality of S. If $|S| = 1$, then S consists of a single set, which only shatters the empty set. If $|S| > 1$, we may choose an element $x \in S$ such that some sets of S contain x and some do not. Let $S^+ = \{s \in S \mid x \in S\}$ and $S^- = \{s \in S \mid x \notin S\}$. Then $S = S^+ \sqcup S^-$, and both S^+ and S^- are non-empty. By induction hypothesis, we know that $S^+ \subsetneq S$ shatters at least $|S^+|$ sets, and $S^- \subsetneq S$ shatters at least $|S^-|$ sets. Let T, T^+, T^- be the families of sets shattered by S, S^+ and S^- respectively. To conclude the proof, we just need to show that for each element in T^+ and T^-, there is a corresponding one in T. If a set is shattered by only one of the two families S^+ and S^-, then it only contributes by one unit to $|T^+| + |T^-|$ and one unit to $|T|$. Notice that no set shattered by S^+ or S^- may contain x, otherwise all or none of the intersections will contain this element. Thus, if a set s is shattered by both S^+ and S^-, it will contribute by two units to $|T^+| + |T^-|$ and one unit to $|T|$. But then, for each such set, we can consider $s \cup \{x\}$ which is not in T^+ or T^-, but it is in T. Indeed, for each subset of s, if it does not contain x it is the intersection with some set in $S^- \subsetneq S$, and if it does contain x it is the intersection with some set in $S^+ \subsetneq S$. All in all, we conclude that

$$|T| \geq |T^+| + |T^-| \geq |S^+| + |S^-| \geq |S|$$

□

Proof. (of Lemma 3.7) Suppose that $\bigcup S$ has n elements. By Lemma 3.8, S shatters at least $|S|$ subsets, and since there are at most $\sum_{i=0}^{k} \binom{n}{i}$ subsets of S of size at most k, if $|S| > \sum_{i=0}^{k} \binom{n}{i}$, at least one of the shattered sets has cardinality larger than k, and hence the VC dimension of S is larger than k. □

Next, we want to prove that if G has the non-k-order property, then the size of the family of exceptional sets of A, relative to each vertex $b \in G$, is bounded by $|A|^k$. Instead, we prove a stronger result, that is we prove this same bound with only the condition that G has the "disjoint" non-k-order property, in which the two sequences of vertices in the Definition 3.1 are in fact disjoint. This stronger version is neither more useful nor easier to prove, but remarks that the non-disjointness of the sequences, and thus the broadening of the excluded structures, is not needed to obtain the bound, but later on.

Lluis: pot semblar raro això, pero crec que aixi deixa clar que és més endavant on de veritat necessitem que puguin ser non disjoint.

Lemma 3.9. *Let G be a graph and $A \subseteq G$. Let $S = \{B^+_{A,b} \mid b \in G \setminus A\}$. If S has VC dimension (at least) k, then G has the k-order property.*

Proof. If S has VC dimension k, then it shatters a set $A' \subseteq A$ of size k. Now, choose any order of the vertices of $A' = \langle a_1, \dots, a_k \rangle$. Then, consider the increasing sequence of subsets $A_1 \subseteq A_2 \subseteq \dots \subseteq A_k = A'$, where $A_i = \{a_j \mid j \in \{1, \dots, i\}\}$. Since A' is shattered by S, for each $i \in \{1, \dots, k\}$ there exists a $b_i \in G$ such that $b_i R a$ if and only if $a \in A_i$. In particular, the two sequences $\langle a_i \mid i \in \{1, \dots, k\} \rangle$ and $\langle b_i \mid i \in \{1, \dots, k\} \rangle$ satisfy

$$a_i R b_j \Leftrightarrow i \leq j$$

and thus G has the k-order property. □

Lemma 3.10 (Claim 2.6)**.** *Let G be a graph with the non-k-order property. Then, for any finite non-trivial $A \subseteq G$,*

$$|\{B^+_{A,b} \mid b \in G\}| \leq |A|^k$$

Proof. By Lemma 3.9, if G has the non-k-order property, then the family $\{B^+_{A,b} \mid b \in G \setminus A\}$ has VC dimension at most $k-1$, so by the Sauer-Shelah Lemma 3.7 we have $\{B^+_{A,b} \mid b \in G \setminus A\} \leq \sum_{i=0}^{k-1} \binom{|A|}{i}$. Since $\{B^+_{A,b} \mid b \in A\} \leq |A|$, we conclude that

$$|S| = |\{B^+_{A,b} \mid b \in G\}| \leq \sum_{i=0}^{k-1} \binom{|A|}{i} + |A|$$

Finally, when $|A| = n, k > 1$:

This conditions should be set at some point of the tfm. Specify that if they are not met, the problem becomes trivial.

- if $n \leq k$, then $|S| \leq 2^n \leq 2^k \leq n^k$.
- if $n > k$, then $|S| \leq \sum_{i=0}^{k-1} + n \leq n^{k-1} + n \leq 2n^{k-1} \leq n^k$.

We conclude that $|S| \leq n^k$. □

Remark 3.11. The condition $n, k > 1$ is trivial. If $n = 1$ then A is the trivial graph with a single vertex. If $k = 1$ we are not allowing even a single edge, so G is the empty graph.

We now prove the following equivalent versions of the lemma, which will be useful in the different sections of the thesis. The idea is that any choice of either the exceptional or the non-exceptional vertices set of A with respect to each vertex $b \in G$, have the same bound.

Corollary 3.12 (Claim 2.6.1)**.** *Let G be a graph with the non-k-order property. Then:*

1. *For any finite $A \subseteq G$*

$$|\{B^-_{A,b} \mid b \in G\}| \leq |A|^k$$

2. *For any finite $A \subseteq G$*

$$|\{\overline{B}_{A,b} \mid b \in G\}| \leq |A|^k$$

Proof. 1. First of all, notice that $B^+_{A,b} = B - B^-_{A,b}$, since by definition they are complementary. Thus, for any $b, b' \in G$, $B^+_{A,b} = B^+_{A,b'} \Leftrightarrow B^-_{A,b} = B^-_{A,b'}$. It follows that

$$|\{B^-_{A,b} \mid b \in G\}| = |\{B^+_{A,b} \mid b \in G\}| \leq |A|^k$$

where the last inequality follows Lemma 3.10.

2. Consider the following map:

$$\pi : \{\overline{B}_{A,b} \mid b \in G\} \longrightarrow \{B^+_{A,b} \mid b \in G\}$$
$$\overline{B}_{A,b} \longmapsto B^+_{A,b}$$

We show that the map π is injective. Let $b, b' \in G$ such that $\overline{B}_{A,b} = \overline{B}_{A,b'}$. Then, $t(A, b) = t(A, b')$, otherwise (w.l.o.g., suppose that $t(A, b) = 1$ and $t(A, b') = 0$), we would have

$$|B^-_{A,b'}| > |B^+_{A,b'}| = |B^+_{A,b}| \geq |B^-_{A,b}| = |B^-_{A,b'}|$$

which is a contradiction. Then:

- if $t(A, b) = t(A, b') = 1$, we have that $B_{A,b} = B^+_{A,b} = B^+_{A,b'} = B_{A,b'}$.
- if $t(A, b) = t(A, b') = 0$, we have that $B_{A,b} = B^-_{A,b} = A \setminus B^+_{A,b} = A \setminus B^+_{A,b'} = B^-_{A,b'} = B_{A,b'}$.

This proves that π is injective. Hence, we have that

$$|\{\overline{B}_{A,b} \mid b \in G\}| \leq |\{B^+_{A,b} \mid b \in G\}| \leq \sum_{i \leq k} \binom{|A|}{i} \leq |A|^k$$

This concludes the proof. Notice that in particular π is a bijection, but this is not needed for the proof.

□

During the next sections, it will be a key point proving that some sort of "regular" subgraphs (*independent* in Section 4 and *excellent* in Section 5) exist in a given stable graph. In order to do so, a useful structure strongly related to the k-order property is the k-*tree*.

Definition 3.13. A k-*tree* is an ordered pair $H = (\overline{c}, \overline{b})$ comprising:

- $\overline{c} = \{c_\eta \mid \eta \in \{0,1\}^{<k_{**}}\}$, the set of *nodes*.
- $\overline{b} = \{b_\rho \mid \rho \in \{0,1\}^{k_{**}}\}$, the set of *branches*.

satisfying that, for all $\eta \in \{0,1\}^{<k_{**}}$ and $\rho \in \{0,1\}^{k_{**}}$, if given $\ell \in \{0,1\}$ we have $\eta \frown \langle \ell \rangle \triangleleft \rho$, then $(b_\rho R c_\eta) \equiv (\ell = 1)$.

Make a visual example of the k-tree.

Similarly to stability, we can define the *tree bound* of a graph to measure the level of freeness from k-trees of graph.

Definition 3.14 (Definition 2.11)**.** Suppose G is a finite graph. We denote the *tree bound* $k_{**} = k_{**}(G)$ as the minimal positive integer such that there is no k_{**}-tree $H = (\overline{c}, \overline{b})$, where $\overline{b}$ and $\overline{c}$ are two sets of vertices of G.

As mentioned earlier, the tree bound is closely related to the k-order property. The following theorem states that if a graph has a sufficiently large tree bound, then it has the k-order property and vice versa.

Theorem 3.15. *If a graph G has the $2^{k_{**}}$-order property, then the tree bound of G is at least $k_{**} + 1$. On the other hand, if a graph G has tree bound at least $k_{**} = 2^{k_*+1} - 3$, then it has the k_*-order property.*

Proof. For the first implication, just consider $\langle a_i \mid i \in \{1, \dots, 2^{k_{**}} - 1\}\rangle$ and $\langle b_i \mid i \in \{0, \dots, 2^{k_{**}} - 1\}\rangle$ to be the two sequences of vertices witnessing the $2^{k_{**}}$-order property in G, and thus for all $i, j \leq k$, $a_i R b_j$ if and only if $i \geq j$. It is straightforward to build a k_{**}-tree using these vertices. Take $\langle b_i \mid i \in \{0, \dots, 2^{k_{**}} - 1\}\rangle$ to be the branches of the tree, indexing them by the binary decomposition of their index, and run the following construction for the nodes:

- Initiate $C_= \langle a_i \mid i \in \{0, \dots, 2^{k_{**}} - 2\}\rangle$.
- At each step $k \in \{0, k_{**} - 1\}$, for each $\eta \in \{0,1\}^k$, take the middle element of the sequence C_η and set it to be the node c_η. Then, the remaining first half of C_η becomes the sequence $C_{\eta \frown \langle 0 \rangle}$ and the second half is $C_{\eta \frown \langle 1 \rangle}$.

Notice that at each step, the sequence C_η has an odd number of elements. The resulting two sequences of nodes and branches form a k_{**}-tree. See **??** for a visual example of this construction.

Add visual example of order implies tree.

During the proof of the second implication, we will say that a set of nodes N of a k-tree $H = (\overline{c}, \overline{b})$ *contains* a k'-tree, if there exists a map $f \colon \{0,1\}^{<k'} \longrightarrow \{0,1\}^{<k}$ such that for all $\eta, \eta' \in \{0,1\}^{<k'}$, $c_{f(\eta)}$ and $c_{f(\eta')}$ are in N, and if $\eta \frown \langle i \rangle = \eta'$ then $f(\eta) \frown \langle i \rangle \triangleleft f(\eta')$, for all $i \in \{0,1\}$.

This clearly implies that there is a k'-tree H' with nodes in N and branches in $\overline{b}$. Simply, for each $\eta \in \{0,1\}^{k'-1}$, pick exactly two branches b_{ρ_0} and b_{ρ_1} such that $f(\eta) \frown \langle i \rangle \triangleleft \rho_i$ for $i \in \{0,1\}$.

Also, we will use H_i to denote the subtree of H consisting of the nodes $c_{f(\eta)}$ and branches $b_{f(\rho)}$ such that $\langle i \rangle \triangleleft \eta$ and $\langle i \rangle \triangleleft \rho$. Notice that, if H is an h-tree, H_0 and H_1 are $(h-1)$-trees, and together with the root node $c_{f(\emptyset)}$, they partition H.

Next, we prove the following claim, which shows that we can always find a tree in one of the parts of a bipartition of the nodes of a larger tree.

Claim 3.16. For all $n, k \geq 0$, if H is a $(n+k)$-tree and the nodes of H are partitioned into two sets N and P, then either N contains an n-tree or P contains a k-tree.

Proof. (of claim) We prove this by induction on $n+k$. Clearly, the statement is true for the trivial case $n = k = 0$. Suppose $n + k > 0$. Without loss of generality, we may assume that the root node $c_\emptyset$ is in N. Let Z_i be the set of nodes of H_i. By H.I., for each $i \in \{0,1\}$, either $N \cap Z_i$ contains an $(n-1)$-tree or $P \cap Z_i$ contains a k-tree. If either $P \cap Z_0$ or $P \cap Z_1$ contains a k-tree, then P contains a k-tree, and we are done. Otherwise, both $N \cap Z_0$ and $N \cap Z_1$ contain an $(n-1)$-tree. Since $c_\emptyset$ is in N, the root with the two $(k-1)$-tree are in N and make an n-tree. Thus, N contains an n-tree. □

Suppose that G has tree bound at least $2^{k_*+1} - 3$, and thus contains a $(2^{k_*+1} - 2)$-tree. We show by induction on $k_* - r$, with $1 \leq r \leq k_*$, that the following scenario S_r holds:

1. There are
$$b_0, c_0, \dots, b_{q-1}, c_{q-1}, H, b_q, c_q, \dots, b_{k_*-r-1}, c_{k_*-r-1}$$
such that:
2. for all $i \in \{0, \dots, k_* - r - 1\}$, b_i and c_i are vertices in G, and H is a $(2^{r+1} - 2)$-tree in G.
3. for all $i, j \in \{0, \dots, k_* - r - 1\}$, $b_i R c_j \Leftrightarrow i \geq j$.
4. if c is a node of H, $b_i R c \Leftrightarrow i \geq q$.

5. if b is a branch of H, $bRc_i \Leftrightarrow i < q$.

The initial case S_{k_*} only requires the existence of a $(2^{k_*+1} - 2)$-tree in G, which is the premise. If the final case S_1 is true, then we are done: this case assumes that H is a 2-tree, in which case there is a node c_* and branch b_* in H which are connected. These vertices satisfy conditions 4. and 5., so the sequence resulting by replacing H in 1. by b_*, c_* implies that G has the k_*-order property.

To conclude the proof it remains to prove that if S_r holds, then so does S_{r-1} for $r > 1$. Assume S_r. Fixing $h = 2^r - 2$, by 2. we have that H is a $(2h+2)$-tree. For each branch b of H we denote $Z(b)$ the set of nodes c of H such that bRc.

We have two cases:

- *Case 1.* There is a branch b_* such that $Z(b_*)$ contains an $(h+1)$-tree H'. In that case, we can take c_* to be the top node of the $(h+1)$-tree, and H_* to be the h-subtree H'_0. Replacing H in 1. with H_*, b_*, c_* in this order, the conditions for S_{r-1} are satisfied.

- *Case 2.* There is no branch b such that $Z(b)$ contains an $(h+1)$-tree. Now, let c_* be the top node of H, Z_1 the set of nodes of H_1, and b_* any branch of H_1. By the case assumption, $Z(b) \cap Z_1$ contains no $(h+1)$-tree, so by the claim, $Z_1 \setminus Z(b)$ contains an h-tree H_*. Finally, replacing H in 1. by b_*, c_*, H_* in this order, the conditions for S_{r-1} are satisfied.

In any case, S_{r-1} is satisfied, and the proof is complete. □

Remark 3.17. The key point of the proof of the second implication of Theorem 3.15 is that the found k-order does not only utilize edges and non-edges of the k-tree structure itself. Instead, it relies on the fact that, for a tall enough tree, a k-order must appear in some way, leveraging some "unknown" edges, independently on the choice of those.

Given that stability of the studied graph is fixed for all proofs in the next sections, from now on we will use k_* as the value of the non-k-property of the studied graphs, and k_{**} for the associated tree bound.

4. Section 4

This section works around the concept of ϵ-*indivisible* sets, a strong condition on the regularity of a subset respect to all the vertices of the graph. This condition results in pairs of sufficiently large subsets of vertices satisfying the *average condition*, which strictly bounds the number of exceptional edges in the pair. Using these tools we obtain the first result in Lemma 4.14, which proves the existence of a partition of highly regular pairs with no exceptions, at the cost of a non-homogeneous partition. Next, we improve the results obtaining an equitable partition in Theorem 4.20, but this time with a small number of exceptional pairs, and a tradeoff between a non-negligible remainder set and exponentially small parts. The final result, Theorem 4.26, achieves removing irregular pairs and reduce the size of the remainder set. All in all, even though the partitions of this section present a very strong form of "regularity", they all share the same drawback: a large number of parts that grows with the size of the graph, something that we will be dealing with in the next section.

First step is defining *indivisibility*. The general definition is for any function f, but for the rest of the section we are mostly interested in the case of $f(n) = n^\epsilon$, which we call ϵ-indivisible, and at the end in the constant case $f(n) = c$.

Definition 4.1 (Definition 4.2(b))**.** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. We say that $A \subseteq G$ is f-*indivisible* if for every $b \in G$,

$$|\overline{B}_{A,b}| < f(|A|)$$

Definition 4.2 (Definition 4.2(a))**.** Let $\epsilon \in (0, 1)$. We say that $A \subseteq G$ is ϵ-*indivisible* if for every $b \in G$,

$$|\overline{B}_{A,b}| < |A|^\epsilon$$

Remark 4.3. An ϵ-indivisible set is f-indivisible for $f(n) = n^\epsilon$.

Redundant.

A natural follow-up question, is how strongly bounded are exceptions in the context of two indivisible sets. The following lemma measures precisely that, although doing so in a "directed" way.

Lemma 4.4 (Claim 4.6))**.** *Let G be a finite graph. Suppose $A, B \subseteq G$ such that A is f-indivisible, B is g-indivisible, and $f(|A|)g(|B|) < \frac{1}{2}|B|$. Then, the truth value $t = t(A, B)$ satisfies that for all but $< f(|A|)$ of the $a \in A$ for all but $< g(|B|)$ of the $b \in B$ we have that $aRb \equiv t$.*

Proof. Since B is g-indivisible, for each $a \in A$ we have that $|B_{B,a}| < g(|B|)$. Let $U_i = \{a \in A \mid t(a, B) \equiv i\}$ for $i \in \{0, 1\}$. If either U_i satisfies $|U_i| < f(|A|)$ then the statement is true. Suppose not. Then, there are $W_i \subseteq U_i$ with $|W_i| = f(|A|)$ for $i \in \{0, 1\}$. Now, let $V = \{b \in B \mid (\exists a \in W_0 \mid aRb) \vee (\exists a \in W_1 \mid \neg aRb)\}$, i.e. the b's which are an exception for some $a \in W_0 \cup W_1$. Then, $|V| < |W_0|g(|B|) + |W_1|g(|B|) = 2f(|A|)g(|B|) < |B|$, where the first inequality follows the g-indivisibility of B. Finally, there is a $b_* \in B \setminus V$ such that $\forall a \in W_0\ \neg aRb_*$ and $\forall a \in W_1\ aRb_*$ with $|W_0| = |W_1| = f(|A|)$, which contradicts the f-indivisibility of A. $\square$

Definition 4.5. We say that the pair (A, B) with A f-indivisible and B g-indivisible satisfies the *average condition* if $f(|A|)g(|B|) < \frac{1}{2}|B|$ and thus the statement of Lemma 4.4 is true for the pair (A, B).

Remark 4.6. The condition $f(|A|)g(|B|) < \frac{1}{2}|B|$ makes ordering of the pair (A, B) matter, that is,

$$(A, B) \text{ has the average condition } \not\Leftrightarrow (B, A) \text{ has the average condition}$$

Remark 4.7 (Remark 4.7). When $f(n) = n^\epsilon$ and $g(n) = n^\zeta$, the average condition is $|A|^\epsilon |B|^\zeta < \frac{1}{2}|B|$.

I don't think this is useful in any way.

Next, we are interested in studying how the average condition of an indivisible pair controls the homogeneity of large enough subpairs, in the sense of bounding exceptional edges. We study the f and ϵ case separately, as the specific case of ϵ gives a slightly better condition on the range of the size of the subpair.

Lemma 4.8 (Claim 4.8). *Let A be ϵ-indivisible, B ζ-indivisible and let the pair (A, B) satisfy the average condition. Then, for all $\epsilon_1 \in (0, 1-\epsilon)$, $\zeta_1 \in (0, 1-\zeta)$, $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| \geq |A|^{\epsilon+\epsilon_1}$ and $|B'| \geq |B|^{\zeta+\zeta_1}$, we have that:*

$$\frac{|\{(a,b) \in (A', B') \mid aRb \equiv \neg t(A,B)\}|}{|A' \times B'|} \leq \frac{1}{|A|^{\epsilon_1}} + \frac{1}{|B|^{\zeta_1}}$$

Proof. Notice:

This is not the same use of the word exceptional as defined in Section 3.

- There are at most $|A|^\epsilon$ vertices of A (hence in $A' \subseteq A$) which are exceptional (in the sense of the average condition).
- For each $a \in A$ (hence in $A' \subseteq A$) not exceptional, there are at most $|B|^\zeta$ elements $b \in B$ such that (a, b) does not satisfy the truth value $t(A, B)$, i.e. that are exceptional.

This makes my eyes bleed.

Putting it all together:

$$\begin{aligned}
\frac{|\{(a,b) \in (A', B') \mid aRb \equiv \neg t(A,B)\}|}{|A' \times B'|} &\leq \frac{|A|^\epsilon |B'| + (|A'| - |A|^\epsilon)|B|^\zeta}{|A'||B'|} \\
&= \frac{|A|^\epsilon}{|A'|} + \frac{|A'| - |A|^\epsilon}{|A'|} \frac{|B|^\zeta}{|B'|} \\
&\leq \frac{|A|^\epsilon}{|A'|} + \frac{|B|^\zeta}{|B'|} \\
&\leq \frac{|A|^\epsilon}{|A|^{\epsilon+\epsilon_1}} + \frac{|B|^\zeta}{|B|^{\zeta+\zeta_1}} \\
&= \frac{1}{|A|^{\epsilon_1}} + \frac{1}{|B|^{\zeta_1}}
\end{aligned}$$

□

This next generalization may seem to make previous lemma redundant. But they actually prove different results. But is it worth to keep both?

Lemma 4.9 (f-indivisible version). *Let A be f-indivisible, B g-indivisible and let the pair (A, B) satisfy the average condition. Then, for all $\epsilon_1 \in (0, 1 - \frac{f(|A|)}{|A|})$, $\zeta_1 \in (0, 1 - \frac{g(|B|)}{|B|})$, $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| \geq f(|A|)|A|^{\epsilon_1}$ and $|B'| \geq g(|B|)|B|^{\zeta_1}$, we have that:*

$$\frac{|\{(a,b) \in (A', B') \mid aRb \equiv \neg t(A,B)\}|}{|A' \times B'|} \leq \frac{1}{|A|^{\epsilon_1}} + \frac{1}{|B|^{\zeta_1}}$$

Proof. Notice:

- There are at most $f(|A|)$ elements of A (hence in $A' \subseteq A$) which are exceptional (in the sense of the average condition).

- For each $a \in A$ (hence in $A' \subseteq A$) not exceptional, there are at most $g(|B|)$ elements $b \in B$ such that (a, b) does not satisfy the truth value $t(A, B)$, i.e. that are exceptional.

Putting it all together:

$$\begin{aligned}
\frac{|\{(a,b) \in (A', B') \mid aRb \equiv \neg t(A,B)\}|}{|A' \times B'|} &\leq \frac{f(|A|)|B'| + (|A'| - f(|A|))g(|B|)}{|A'||B'|} \\
&= \frac{f(|A|)}{|A'|} + \frac{|A'| - f(|A|)}{|A'|}\frac{g(|B|)}{|B'|} \\
&\leq \frac{f(|A|)}{|A'|} + \frac{g(|B|)}{|B'|} \\
&\leq \frac{f(|A|)}{f(|A|)|A|^{\epsilon_1}} + \frac{g(|B|)}{g(|B|)|B|^{\zeta_1}} \\
&= \frac{1}{|A|^{\epsilon_1}} + \frac{1}{|B|^{\zeta_1}}
\end{aligned}$$

□

For later use, we are particularly interested in the case when $f(n) = c$.

This may be skipped, and be directly commented in the appropiate remark following the theorem.

Corollary 4.10 (Corollary 4.9)**.** *Let A and B be f-indivisible with $f(n) = c$ and (A, B) satisfy the average condition. Then, for all $\epsilon_1 \in (0, 1 - \frac{c}{|A|})$, $\zeta_1 \in (0, 1 - \frac{c}{|B|})$, $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq c|A|^{\epsilon_1}$ and $|B'| \geq c|B|^{\zeta_1}$, we have:*

$$\frac{|\{(a,b) \in (A', B') \mid aRb \equiv \neg t(A,B)\}|}{|A' \times B'|} \leq \frac{1}{|A|^{\epsilon_1}} + \frac{1}{|B|^{\zeta_1}}$$

Proof. Use Lemma 4.9 with $f(n) = c$. □

Remark 4.11. Notice that the average condition is easily satisfied if a large enough condition is met by the pair. If $f(n) = n^\epsilon$, A and B are f-indivisible, and $|B| \geq |A| \geq m$, then $m^{1-2\epsilon} > 2$ is sufficient for the average condition to hold for the pair (A, B):

$$\frac{|A|^\epsilon |B|^\epsilon}{|B|} \leq \frac{|B|^{2\epsilon}}{|B|} = \frac{1}{|B|^{1-2\epsilon}} = \frac{1}{m^{1-2\epsilon}} < \frac{1}{2}$$

We will be using this fact in the context of a sequence of non-zero natural numbers $\{m_\ell \mid \ell \in \{0, \dots, k_{**}\}\}$ where $\lfloor m_\ell^\epsilon \rfloor = m_{\ell+1}$ for some $\epsilon \in (0, \frac{1}{2})$ and for all $\ell \in \{0, \dots, k_{**} - 1\}$. Here, $2 < (m_{k_{**}-1})^{1-2\epsilon}$ is sufficient for any f-indivisible A and B, with $|A|, |B| \in \{m_0, \dots, m_{k_{**}-1}\}$, to satisfy the average condition.

Now that we have proven the nice properties of indivisible sets, we are actually interested in whether they can be found in a graph. It turns out that the non-k-order property, or more specifically the associated tree bound, is sufficient for proving it. The proof resumes in assuming that there is no indivisible set to recursively refine a "semi-partition" which by construction contains a k_{**}-tree.

Lluis: hi ha alguna manera de dir una partició que no cubreix tots els vertex amb una paraula?

Lemma 4.12 (Claim 4.3)**.** *Let G be a finite graph with the non-k_*-property and $f : \mathbb{R} \longrightarrow \mathbb{R}$ a function such that $x \geq f(x)$. Let $\{m_\ell \mid \ell \in \{0, \dots, k_{**}\}\}$ be a sequence of non-zero natural numbers such that for all $\ell \in \{0, \dots, k_{**} - 1\}$, $f(m_\ell) \geq m_{\ell+1}$. If $A \subseteq G$, $|A| = m_0$, then for some $\ell \in \{0, \dots, k_{**} - 1\}$ there is a subset $B \subseteq A$ of size m_ℓ which is f-indivisible.*

Proof. Suppose not. Then we can construct the sequences $\langle b_\eta \mid \eta \in \{0,1\}^{<k} \rangle$ and $\langle A_\eta \mid \eta \in \{0,1\}^{\leq k} \rangle$ on induction over $k = |\eta|$, satisfying:

1. $A_{\eta ^\frown \langle i \rangle} \subseteq A_\eta$, $\forall i \in \{0,1\}$, $\forall k \in \{0, \dots, k_{**} - 1\}$
2. $A_{\eta ^\frown \langle 0 \rangle} \cap A_{\eta ^\frown \langle 1 \rangle} = \emptyset$, $\forall k \in \{0, \dots, k_{**} - 1\}$
3. $|A_\eta| = m_k$, $\forall k \in \{0, \dots, k_{**}\}$
4. $b_\eta \in G$ witnessing that A_η is not f-indivisible, $\forall k \in \{0, \dots, k_{**} - 1\}$
5. $A_{\eta ^\frown \langle i \rangle} \subseteq A_\eta^{(i)} = \{a \in A_\eta \mid aRb_\eta \equiv (i = 1)\}$, $\forall \in \{0,1\}$, $\forall k \in \{0, \dots, k_{**} - 1\}$

Let's prove the induction. For $k = 0$, we consider $A_{\langle \cdot \rangle} = A$, which satisfies $|A_{\langle \cdot \rangle}| = m_0$ and $b_{\langle \cdot \rangle}$ is witnessing the non-f-indivisibility of $A_{\langle \cdot \rangle}$. For $k > 0$ we can assume by hypothesis that A_η, with $|A_\eta| = m_k$, is not f-indivisible. Thus, there exists b_η such that $A_\eta^{(i)} \geq f(m_k) \geq m_{k+1}$ (4.), and we can choose $A_{\eta ^\frown \langle i \rangle} \subseteq A_\eta^{(i)}$ (5.), such that $|A_{\eta ^\frown \langle i \rangle}| = m_{k+1}$ $\forall i \in \{0,1\}$ (3.). 1. and 2. are satisfied by the definition of $A_\eta^{(i)}$. Now, for all η such that $|\eta| = k_{**}$, consider some element $a_\eta \in A_\eta$, which exists since $m_\ell > 0$ for all ℓ. Then, we have two sequences $\langle b_\eta \mid \eta \in \{0,1\}^{<k_{**}} \rangle$ and $\langle a_\eta \mid \eta \in \{0,1\}^{k_{**}} \rangle$ satisfying the k_{**}-tree property: for all $\rho \in \{0,1\}^{<k_{**}}$ and $\eta \in \{0,1\}^{k_{**}}$ if given $\ell \in \{0,1\}$ we have $\rho ^\frown \langle \ell \rangle \trianglelefteq \eta$ then $(a_\eta R b_\rho) \equiv (\ell = 1)$ since $a_\eta \in A_\eta \subseteq A_{\rho ^\frown \langle i \rangle}$. This contradicts the k_{**} tree bound. □

The previous proof can be iteratively be used to partition the graph into indivisible parts with a small reminder. As the average condition cares about the ordering, we define the partition as a tuple instead of a family of sets, and fix an ascending order on the size of the parts.

Lemma 4.13 (Claim 4.4 + 4.5)**.** *Let G be a finite graph with the non-k_*-order property and $f : \mathbb{R} \longrightarrow \mathbb{R}$ a function such that $x \geq f(x)$. Let $\{m_\ell \mid \ell \in \{0, \dots, k_{**}\}\}$ be a sequence of non-zero natural numbers such that for all $\ell \in \{0, \dots, k_{**} - 1\}$, $f(m_\ell) \geq m_{\ell+1}$. If $A \subseteq G$ with $|A| = n$, then we can find a sequence $\overline{A} = \langle A_j \mid j \in \{1, \dots, j(*)\} \rangle$ and reminder $B = A \setminus \bigcup \overline{A}$ such that:*

1. *For each $j \in \{1, \dots, j(*)\}$, A_j is f-indivisible.*
2. *For each $j \in \{1, \dots, j(*)\}$, $|A_j| \in \{m_0, \dots, m_{k_{**}-1}\}$.*
3. *$A_j \subseteq A \setminus \bigcup \{A_i \mid i < j\}$, in particular $A_i \cap A_j = \emptyset$ $\forall i \neq j$.*
4. *$|B| < m_0$.*
5. *$\overline{A}$ is $\leq$-increasing.*

Proof. Iteratively, apply Lemma 4.12 to the remainder $A \setminus \bigcup \{A_i \mid i < j\}$ (3.) to get an f-indivisible A_j (1.) of size m_ℓ, $\ell \in \{0, \dots, k_{**} - 1\}$ (2.) until less then m_0 vertices are available (4.). To conclude, reorder the indices of the A_j's in ascending size order (5.). □

Finally, we can ensure the pairs satisfy the average condition by simply ensuring a minimal size of the parts, which can be easily implemented in the sequence of integers.

Lemma 4.14 (Claim 4.10)**.** *Let G be a finite graph with the non-k_*-order property. Let $\{m_\ell \mid \ell \in \{0, \dots, k_{**}\}\}$ be a sequence of non-zero natural numbers such that $n \geq m_0$ and for all $\ell \in \{0, \dots, k_{**} - 1\}$, $\lfloor m_\ell^\epsilon \rfloor = m_{\ell+1}$, for some $\epsilon \in (0, \frac{1}{2})$ such that $2 < (m_{k_{**}-1})^{1-2\epsilon}$. If $A \subseteq G$ with $|A| = n$, then we can find a sequence $\overline{A} = \langle A_i \mid i \in \{1, \dots, i(*)\} \rangle$ and reminder $B = A \setminus \bigcup \overline{A}$ satisfying:*

Last condition can be changed (and probably should) for a condition on $m_{k_{**}}$

1. *For each $i \in \{1, \dots, i(*)\}$, A_i is ϵ-indivisible.*

2. *For each $i \in \{1, \dots, i(*)\}$, $|A_i| \in \{m_0, \dots, m_{k_{**}-1}\}$.*

3. *$A_i \cap A_j = \emptyset$ for all $i \neq j$.*

4. *$|B| < m_0$.*

5. *$\overline{A}$ is $\leq$-increasing.*

6. *If $\zeta \in (0, \epsilon^{k_{**}})$ then for every $i, j \in \{1, \dots, i(*)\}$ with $i < j$, $A \subseteq A_i$ ad $B \subseteq A_j$ such that $|A| \geq |A_i|^{\epsilon+\zeta}$ and $|B| \geq |A_j|^{\epsilon+\zeta}$ we have that:*

$$\begin{aligned} \frac{|\{(a,b) \in (A,B) \mid aRb \equiv \neg t(A_i, A_j)\}|}{|A \times B|} &\leq \frac{1}{|A_i|^\zeta} + \frac{1}{|A_j|^\zeta} \\ &\leq \frac{1}{|A|^\zeta} + \frac{1}{|B|^\zeta} \end{aligned}$$

Proof. The five points are direct consequence of Lemma 4.13, setting $f(x) = x^\epsilon$. Now, by 2., for any $A_i, A_j \in \overline{A}$ with $i < j$ there is some $\ell \in \{0, \dots, k_{**} - 1\}$ such that $|A_i| \leq |A_j| = m_\ell$. Also, it follows the condition $2 < (m_{k_{**}-1})^{1-2\epsilon}$ and Remark 4.11 that the pair (A_i, A_j) satisfies the average condition. Finally, notice that $\epsilon^{k_{**}} < \epsilon < 1 - \epsilon$ since $\epsilon \in (0, \frac{1}{2})$, so that $\zeta \in (0, \epsilon^{k_{**}}) \subseteq (0, 1-\epsilon)$ and the condition for Lemma 4.8 is satisfied. This gives us 6. and concludes the proof of the statement. $\square$

Remark 4.15. For sufficiently small ϵ, the condition $2 < (m_{k_{**}-1})^{1-2\epsilon}$ is almost, trivial. For example, if $\epsilon < \frac{1}{4}$, then we are just requiring that $m_{k_{**}-1} \geq 4$.

Maybe merge the last two lemmas?

TO BE CONTINUED... *(JoJo's Bizarre Adventure theme plays in the background)*

Definition 4.16. Let A, B be f-indivisible sets with $f(A) \times f(B) < \frac{1}{2}|B|$. Let $\langle A_i \mid i \in \{1, \dots, i_A\}\rangle$ be a partition of A with $|A_i| = m$ for all $i \in \{1, \dots, i_A\}$ and $\langle B_i \mid i \in \{1, \dots, i_B\}\rangle$ be a partition of B with $|B_i| = m$ for all $i \in \{1, \dots, i_B\}$. We define $\varepsilon^+_{A_i, A_j, m}$ as the event:

$$\forall a \in A_i \; \forall b \in B_i, aRb = t(A, B)$$

Lemma 4.17 (Claim 4.13)**.** *Let G be a finite graph with the non-k_*-order property. Let $\langle m_\ell \mid \ell \in \{0, \dots, k_{**}\}\rangle$ be a sequence of non-zero natural numbers such that $n \geq m_0 \geq n^\epsilon$ and for all $\ell \in \{0, \dots, k_{**} - 1\}$, $\lfloor m_\ell^\epsilon \rfloor = m_{\ell+1}$, for some $\epsilon \in (0, \frac{1}{2})$ such that $2 < (m_{k_{**}-1})^{1-2\epsilon}$. Let $A_1, A_2 \subseteq G$ be two ϵ-indivisible subsets such that $|A_1| = m_{\ell_1}$ and $|A_2| = m_{\ell_2}$ for some $\ell_1, \ell_2 \in \{0, \dots, k_{**} - 1\}$ and $|A_1| \leq |A_2|$. In order to simplify computation, we will assume some approximation error by supposing $m_{\ell+1} = (m_\ell)^\epsilon$. Let $c \in (0, 1-\epsilon)$ and $\zeta \leq \frac{1-\epsilon-c}{3}\epsilon^{k_{**}}$ such that $m := n^\zeta$ divides $|A_1|$ and $|A_2|$. Then, let $\left\langle A_{1,s} \mid s \in \{1, \dots, \frac{|A_1|}{m}\}\right\rangle$ and $\left\langle A_{2,t} \mid t \in \{1, \dots, \frac{|A_2|}{m}\}\right\rangle$ be random partitions of A_1 and A_2 respectively, with pieces of size m. We have that*

$$P(\varepsilon^+_{A_{1,s}, A_{2,t}, m}) \geq 1 - \frac{2}{n^{c\epsilon^{k_{**}}}}$$

Should be mentioned that this is a strong limitation which is not mentioned in the original paper, but required for the calculations (sin hacer trampa).

We should probably avoid the approximation error by enforcing this equality as a condition. It can be enforced because the only lemma that uses it actually suppose it in the construction.

Proof. Fix $s \in \frac{|A_1|}{m}$, $t \in \frac{|A_2|}{m}$. It follows the condition $2 < (m_{k_{**}-1})^{1-2\epsilon}$ and Remark 4.11 that the pair (A_1, A_2) satisfies the average condition. Let $U_1 = \{a \in A_1 \mid \{b \in A_2 \mid aRb \equiv \neg t(A_1, A_2)\}| \geq |A_2|^\epsilon\}$

and for each $a \in A_1 \setminus U_1$ let $U_{2,a} = \{b \in A_j \mid aRb \equiv \neg t(A_1, A_2)\}$. By Lemma 4.4, $|U_1| \leq |A_1|^\epsilon$ and $\forall a \in A_1 \setminus U_1$, $|U_{2,a}| \leq |A_2|^\epsilon$. Now, we can bound the probability P_1 that $A_{1,s} \cap U_1 \neq \emptyset$ as follows:

$$
\begin{aligned}
P_1 &\leq \frac{|U_1|}{|A_1|} + \cdots + \frac{|U_1|}{|A_1| - m + 1} < \frac{m|U_1|}{|A_1| - m} \leq \frac{m|A_1|^\epsilon}{|A_1| - m} \\
&\leq \frac{m^2|A_1|^\epsilon}{|A_1|} = \frac{m^2}{|A_1|^{1-\epsilon}} = \frac{m^2}{m_0^{(1-\epsilon)\epsilon^{\ell_1}}} \leq \frac{n^{2\zeta}}{n^{(1-\epsilon)\epsilon^{\ell_1+1}}} \\
&\leq \frac{n^{2\frac{1-\epsilon-c}{3}\epsilon^{k_{**}}}}{n^{(1-\epsilon)\epsilon^{k_{**}}}} \leq \frac{n^{(1-\epsilon-c)\epsilon^{k_{**}}}}{n^{(1-\epsilon)\epsilon^{k_{**}}}} = \frac{1}{n^{c\epsilon^{k_{**}}}}
\end{aligned}
$$

The forth inequality comes from the fact that $\frac{(|A_i|-m)m}{|A_i|} \geq 1$. Then, if $A_{1,s} \cap U_1 = \emptyset$, we have that $|\bigcup_{a \in A_{1,s}} U_{2,a}| \leq |A_{1,s}||A_2|^\epsilon$. So we can bound P_2, the probability that $A_{2,t} \cap \bigcup_{a \in A_{1,s}} U_{2,a} = \emptyset$, by:

$$
\begin{aligned}
P_2 &\leq \frac{|\bigcup_{a \in A_{1,s}} U_{2,a}|}{|A_2|} + \cdots + \frac{|\bigcup_{a \in A_{1,s}} U_{2,a}|}{|A_2| - m + 1} < \frac{m|\bigcup_{a \in A_{1,s}} U_{2,a}|}{|A_2| - m} \leq \frac{mm|A_2|^\epsilon}{|A_2| - m} \\
&\leq \frac{m^3|A_2|^\epsilon}{|A_2|} = \frac{m^3}{|A_2|^{1-\epsilon}} = \frac{m^3}{m_0^{(1-\epsilon)\epsilon^{\ell_2}}} \leq \frac{n^{3\zeta}}{n^{(1-\epsilon)\epsilon^{\ell_2+1}}} \\
&\leq \frac{n^{3\frac{1-\epsilon-c}{3}\epsilon^{k_{**}}}}{n^{(1-\epsilon)\epsilon^{k_{**}}}} \leq \frac{n^{(1-\epsilon-c)\epsilon^{k_{**}}}}{n^{(1-\epsilon)\epsilon^{k_{**}}}} = \frac{1}{n^{c\epsilon^{k_{**}}}}
\end{aligned}
$$

Putting it all together:

$$
P(\varepsilon^+_{A_{1,s}, A_{2,t}, m}) \geq (1 - P_1)(1 - P_2) \geq (1 - \frac{1}{n^{c\epsilon^{k_{**}}}})^2 \geq 1 - \frac{2}{n^{c\epsilon^{k_{**}}}}
$$

□

Lemma 4.18 (Claim 4.14). *Let G be a finite graph with the non-k_*-order property. Let $\langle m_\ell \mid \ell \in \{0, \ldots, k_{**}\}\rangle$ be a sequence of non-zero natural numbers such that for all $\ell \in \{0, \ldots, k_{**} - 1\}$, $\lfloor m_\ell^\epsilon \rfloor = m_{\ell+1}$, for some $\epsilon \in (0, \frac{1}{2})$ such that $2 < (m_{k_{**}-1})^{1-2\epsilon}$. Also, suppose m_0 satisfies $n^\epsilon \leq m_0 < \min(\frac{\sqrt{2}-1}{\sqrt{2}}n, \frac{n}{n^{c\epsilon^{k_{**}}}})$, with $c \in (0, 1-\epsilon)$. Finally, let m_{**} be a divisor of m_ℓ for all $\ell \in \{0, \ldots, k_{**} - 1\}$ and $m_{**} \leq n^{\frac{1-\epsilon-c}{3}\epsilon^{k_{**}}}$. If $A \subseteq G$ with $|A| = n$, then we can find a partition $\overline{A} = \langle A_i \mid i \in \{1, \ldots, r\}\rangle$ with reminder $B = A \setminus \bigcup \overline{A}$ such that:*

1. *$|A_i| = m_{**}$ for all $i \in \{1, \ldots, r\}$.*
2. *For all but $\frac{2}{n^{c\epsilon^{k_{**}}}}r^2$ of the pairs (A_i, A_j) with $i < j$ there are no exceptional edges, i.e.*

$$
\{(a, b) \in A_i \times A_j \mid aRb \not\equiv t(A_i, A_j)\} = \emptyset
$$

3. *$|B| < m_0$*

Proof. We can use Lemma 4.13 to get a partition $\overline{A'} = \langle A'_i \mid i \in \{1, \ldots, i(*)\}\rangle$ and remainder $B' = A \setminus \bigcup A'$. We can refine the partition by randomly splitting each A'_i into pieces of size m_{**} (1.). Consider the resulting partition $\overline{A} = \langle A_i \mid i \in \{1, \ldots, r\}\rangle$ with remainder $B = B'$ (3.). First of all, notice that for each pair (A_i, A_j) such that $A_i \subseteq A'_{i_1}$ and $A_j \subseteq A'_{j_1}$ with $i_1 \neq j_1$, the probability of the pair having exceptional edges is upper bounded by $\frac{2}{n^{c\epsilon^{k_{**}}}}$. This follows Lemma 4.17. Thus, given X the random variable counting the number of exceptional pairs of this kind, we have

$$E(X) = \sum_{\substack{A_i, A_j \text{ s.t.} \\ A_i \subseteq A'_{i_1}, A_j \subseteq A'_{j_1} \\ i_1 \neq j_1}} E(X_{A_i,A_j}) = \sum_{\substack{A_i, A_j \text{ s.t.} \\ A_i \subseteq A'_{i_1}, A_j \subseteq A'_{j_1} \\ i_1 \neq j_1}} P(\varepsilon_{A_i,A_j,m_{**}}) \leq \frac{r^2}{2} \frac{2}{n^{c\epsilon^{k_{**}}}}$$

where X_{A_i,A_j} is the random variable giving 1 if (A_i, A_j) is exceptional, and 0 otherwise. Since the expectation is an average, for some refinement $\overline{A}$ of $\overline{A'}$ we have that the number of exceptional pairs when $i_1 \neq j_1$ is at most $\frac{r^2}{n^{c\epsilon^{k_{**}}}}$. Now, we have no control if $i_1 = j_1$, so let's bound how many of these we have:

$$\begin{aligned}
|\{\text{Exceptional } (A_i, A_j) \mid A_i, A_j \subseteq A'_{i_1}, i_1 \in \{1, \dots, i(*)\}\}| &\leq \binom{\frac{m_0}{m_{**}}}{2} \frac{n}{m_0} \\
&\leq \frac{(\frac{m_0}{m_{**}})^2}{2} \frac{n}{m_0} = \frac{m_0 n}{2m_{**}^2} = \frac{m_0}{n} (\frac{n}{\sqrt{2}m_{**}})^2 \\
&\leq \frac{m_0}{n} (\frac{n - m_0}{m_{**}})^2 \leq \frac{m_0}{n} r^2 < \frac{r^2}{n^{c\epsilon^{k_{**}}}}
\end{aligned}$$

Notice that the third inequality comes after the condition $m_0 \leq \frac{\sqrt{2}-1}{\sqrt{2}} n$. Putting it all together, we see that the number of exceptional pairs is upper bounded by $\frac{2r^2}{n^{c\epsilon^{k_{**}}}}$ satisfying 2.. $\square$

Remark 4.19 (Remark 4.15). In the previous proof, the condition $m_0 < \frac{n}{n^{c\epsilon^{k_{**}}}}$ can be weakened at the cost of increasing the number of exceptional pairs. More specifically, since this condition is only used to bound the exceptional sub-pairs in the same pair (the second part of the proof), the number of exceptional pairs can be generally bounded by

$$|\{\text{Exceptional pairs}\}| \leq (\frac{m_0}{n} + \frac{2}{n^{c\epsilon^{k_{**}}}}) r^2$$

Theorem 4.20 (Theorem 4.16). *Let $\epsilon = \frac{1}{r} \in (0, \frac{1}{2})$ with $r \in \mathbb{N}$ (this avoids rounding error), $c \in (0, 1-\epsilon)$ and k_* be given. Let G be a finite graph with the non-k_*-order property. Let $A \subseteq G$ with $|A| = n$, and $n > 2^{\frac{r^{k_{**}}}{1-2\epsilon}}$. Then, for any $m_{**} \in [n^{\frac{(1-\epsilon-c)}{3}\epsilon^{k_{**}+1}}, (\frac{\sqrt{2}-1}{\sqrt{2}})^{\frac{1-\epsilon-c}{3}\epsilon^{k_{**}}} n^{\frac{(1-\epsilon-c)}{3}\epsilon^{k_{**}} - \frac{(1-\epsilon-c)c}{3}\epsilon^{2k_{**}}}]$, there is a partition $\overline{A} = \langle A_i \mid i \in \{1, \dots, m\} \rangle$ of A with remainder $B = A \setminus \bigcup \overline{A}$ such that:*

Notation here is confusing. r is another thing, and m becomes the number of parts.

1. *$|A_i| = m_{**}$ for all $i \in \{1, \dots, m\}$.*
2. *$|B| < m_{**}^{\frac{3}{(1-\epsilon-c)} r^{k_{**}}}$.*
3. *$|\{(i,j) \mid i,j \in \{1, \dots, m\}, i < j \text{ and } \{(a,b) \in A_i \times A_j \mid aRb\} \notin \{A_i \times A_j, \emptyset\}\}| \leq \frac{2}{n^{c\epsilon^{k_{**}}}} m^2$*

Proof. Let $m_{k_{**}} = m_{**}^{\frac{3}{1-\epsilon-c}}$, and consider the sequence

$$m_{**} \leq m_{k_{**}} < \cdots < m_0$$

such that for all $\ell \in \{1, \dots, k_{**}\}$ we have that $m_{\ell-1} = m_\ell^r$. Notice that:

1. m_{**} divides m_ℓ for all $\ell \in \{0, \dots, k_{**}\}$ since the m_ℓ's are powers of $m_{k_{**}}$ and m_{**} divides $m_{k_{**}}$ by construction.

Probably it is not needed that m_{**} divides $m_{k_{**}}$, with $m_{k_{**}-1}$ is enough, but it comes for free.

2. $(m_{\ell-1})^\epsilon = m_\ell$ for all $\ell \in \{1, \dots, k_{**}\}$.

3. $m_{**} \le n^{\frac{1-\epsilon-c}{3}\epsilon^{k_{**}}}$.

4. $m_0 = m_{**}^{\frac{3}{1-\epsilon-c}r^{k_{**}}}$, so on one hand

$$m_0 = m_{**}^{\frac{3}{1-\epsilon-c}r^{k_{**}}} \ge n^{\frac{1-\epsilon-c}{3}\epsilon^{k_{**}+1}\frac{3}{1-\epsilon-c}r^{k_{**}}} \ge n^\epsilon$$

and on the other hand,

$$m_0 = m_{**}^{\frac{3}{1-\epsilon-c}r^{k_{**}}} \le \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)n^{1-c\epsilon^{k_{**}}}$$

and thus n is both smaller than $(\frac{\sqrt{2}-1}{\sqrt{2}})n$ and smaller than $n^{1-c\epsilon^{k_{**}}}$.

5. $m_{k_{**}-1} = m_{**}^{\frac{3}{1-\epsilon-c}r} \ge n^{\epsilon^{k_{**}}} > 2^{\frac{1}{1-2\epsilon}}$.

So, all the conditions of Lemma 4.18 are satisfied, and we can use it to get a partition $\overline{A}$ with remainder B satisfying the statement. Notice that 2. is satisfied by the fact that $|B| < m_0 \le m_{**}^{\frac{3}{(1-\epsilon-c)}r^{k_{**}}}$. □

Remark 4.21. TO BE CONTINUED... *(JoJo's Bizarre Adventure theme plays in the background)*

Things to be discussed in the remark:
- the upperbound on m_{**} is close to a more reasonable value.
- in any case, the window of choice for m_{**} is not that large.
- c closer to 0 increases the number of exceptional pairs but decreases size of the remainder, and viceversa.

In what follows, c should be another letter, it collides with previous definition.

Definition 4.22 (Definition 4.18)**.** For $n, c \in \mathbb{N}$ and $\epsilon, \zeta, \xi \in \mathbb{R}$, let $\oplus[n, \epsilon, \zeta, \xi, c]$ be the statement: For any set A and family of subsets $P \subseteq \mathcal{P}(A)$ such that $|A| = n$ and $|P| \le n^{\frac{1}{\zeta}}$, and for all $B \in P$ with $|B| \le n^\epsilon$, there exists $U \subseteq A$ with $|U| = \lfloor n^\xi \rfloor$ such that for all $B \in P$, $|U \cap B| \le c$.

Lemma 4.23 (Lemma 4.19)**.** *If the reals ϵ, ζ, ξ and the natural numbers n, c satisfy:*

- $\epsilon \in (0, 1)$
- $\zeta > 0$
- $0 < \xi < \frac{1}{2}$
- *n sufficiently large ($n > n(\epsilon, \zeta, \xi, c)$) to satisfy the equation:*

$$\frac{1}{2n^{1-2\xi}} + \frac{1}{n^{(1-\xi-\epsilon)c-\frac{1}{\zeta}}} < 1$$

- $c > \frac{1}{\zeta(1-\xi-\epsilon)}$

then $\oplus[n, \epsilon, \zeta, \xi, c]$ holds.

Proof. First of all, notice that the last condition implies that $(1 - \xi - \epsilon) > 0$, and thus $\xi < 1 - \epsilon$. Let $m = \lfloor n^\xi \rfloor$ be the size of the set U we want to build, and let $\mathcal{F}_* = [A]^m$ the set of sequences of elements of A with length m. Let μ be a probability distribution on $\mathcal{F}_*$ such that for all $F \in \mathcal{F}_*$, $\mu(F) = \frac{|F|}{|\mathcal{F}_*|}$. We want to prove that the probability that a random U satisfies:

1. All elements of U are distinct.

2. For all $B \in P$ $|U \cap B| < c$.

is non-zero. First of all let's bound the converse of 1., i.e. the probability that there are two equal elements in U:

$$P_1 = P(\exists s < t \in [m] \mid U_s = U_t) \leq \binom{m}{2}\frac{n}{n^2} \leq \frac{m^2}{2n} \leq \frac{n^{2\xi}}{2n} < \frac{1}{2n^{1-2\xi}}$$

Now, in order to bound 2., let's first bound the probability that at least c elements of U are in a given $B \in P$:

$$P_B = P(\exists^{\geq c} t \in [m] \mid U_t \in B) \leq \binom{m}{c}(\frac{|B|}{n})^c \leq \frac{m^c|B|^c}{n^c} \leq \frac{n^{\xi c} n^{\epsilon c}}{n^c} = \frac{1}{n^{c(1-\xi-\epsilon)}}$$

Then, we can bound the converse of 2., i.e. the probability that this happens for some $B \in P$, by:

$$P_2 = P(\exists B \in P \mid \exists^{\geq c} t \in [m], U_t \in B) \leq \sum_{B \in P} P_B = \frac{|P|}{n^{c(1-\xi-\epsilon)}} \leq \frac{1}{n^{c(1-\xi-\epsilon)-\frac{1}{\zeta}}}$$

Putting it all together, we have that

$$P((1.) \cup (2.)) \leq P_1 + P_2 < \frac{1}{2n^{1-2\xi}} + \frac{1}{n^{c(1-\xi-\epsilon)-\frac{1}{\zeta}}}$$

Notice that

- Since $\xi < \frac{1}{2}$ we have that $1 - 2\xi > 0$.
- $c(1-\xi-\epsilon) - \frac{1}{\zeta} > 0$.

so, the n-large enough condition of the forth point of the statement is well defined and

$$P((1.) \cup (2.)) < \frac{1}{2n^{1-2\xi}} + \frac{1}{n^{c(1-\xi-\epsilon)-\frac{1}{\xi}}} < 1$$

Thus, the probability that there exists a $U \subseteq A$ satisfying the condition is non-trivial, and $\oplus[n, \epsilon, \zeta, \xi, c]$ holds. □

Lemma 4.24 (Claim 4.21)**.** *Let $k_*, k, c \in \mathbb{N}$ and $\epsilon, \xi \in \mathbb{R}$ such that:*

1. *G is a graph with the non-k_*-order property.*
2. *$A \subseteq G$ implies $|\{\{a \in A \mid aRb \equiv t(a, b)\} \mid b \in G\}| \leq |A|^k$.*
3. *$\epsilon \in (0, \frac{1}{2})$.*
4. *$\xi \in (0, \frac{\epsilon^{k_{**}}}{2})$.*
5. *c satisfies*

$$c > \frac{1}{\frac{1}{k}(1 - \frac{\xi}{\epsilon^{k_{**}}} - \epsilon)}$$

Then, for every sufficiently large $n \in \mathbb{N}$ ($n^{\epsilon^{k_{**}}} > n(\epsilon, \frac{1}{k}, \frac{\xi}{\epsilon^{k_{**}}}, c)$ in the sense of Lemma 4.23 (d)), if $A \subseteq G$ with $|A| = n$, there is $Z \subseteq A$ such that

(a) $|Z| = \lfloor n^{\xi} \rfloor$.

(b) Z is ϵ-indivisible in G.

This should be coherent with previous sections.

Proof. In order to simplify the calculation, we will assume that $n^{\epsilon^{\ell}} \in \mathbb{N}$ for all $\ell \in \{0, \dots, k_{**}\}$. Notice that can be easily achieved by setting ϵ as $\epsilon = \frac{1}{r}$ with $r \in \mathbb{N}$. Let $n = m_0 > m_1 > \cdots > m_{k_{**}}$ with $m_{\ell} = n^{\epsilon^{\ell}}$. So $m_{\ell+1} = m_{\ell}^{\epsilon} = \lfloor (m_{\ell})^{\epsilon} \rfloor$ and we can use Lemma 4.12 to get $A_1 \subseteq A$ with $|A_1| = m_{\ell}$ for some $\ell \in \{0, \dots, k_{**} - 1\}$ and A_1 ϵ-indivisible. By 2. we have that $|P| \leq |A_1|^k = m_{\ell}^k$. Notice that:

Define P in this context.

- $\epsilon \in (0, 1)$ by 3..
- $\zeta := \frac{1}{k} > 0$.
- since $\epsilon \in (0, \frac{1}{2})$ by 3., then by 4. $\frac{\xi}{\epsilon^{\ell}} \leq \frac{\xi}{\epsilon^{k_{**}}} < \frac{1}{2} < 1 - \epsilon$ and thus $0 < \xi < \min(1 - \epsilon, \frac{1}{2})$.
- m_{ℓ} sufficiently large: $m_{\ell} = n^{\epsilon^{\ell}} \geq n^{\epsilon^{k_{**}}} > n(\epsilon, \frac{1}{k}, \frac{\xi}{\epsilon^{k_{**}}}, c) > n(\epsilon, \zeta, \frac{\xi}{\epsilon^{\ell}}, c)$.
- $c > \frac{1}{\frac{1}{k}(1 - \frac{\xi}{\epsilon^{k_{**}}} - \epsilon)} = \frac{1}{\zeta(1 - \frac{\xi}{\epsilon^{k_{**}}} - \epsilon)}$.

By Lemma 4.23 then, $\oplus[m_{\ell}, \epsilon, \zeta, \frac{\xi}{\epsilon^{\ell}}]$ holds, and we can take $A_{(4.22)} := A_1$ and $P_{(4.22)} := P$ which satisfy the conditions:

- $|A_1| = m_{\ell}$.
- $|P| \leq m_{\ell}^k = m_{\ell}^{\frac{1}{\zeta}}$.
- $\forall B \in P$, $|B| \leq |A_1|^{\epsilon}$ by ϵ-indivisibility of A_1.

Thus, by Definition 4.22 we have that there exists $Z \subseteq A_1$ such that:

- $|U| = \lfloor m_{\ell}^{\frac{\xi}{\epsilon^{\ell}}} \rfloor = \lfloor n^{\epsilon^{\ell} \frac{\xi}{\epsilon^{\ell}}} \rfloor \lfloor n^{\xi} \rfloor$ satisfying a..
- Z is c-indivisible since $|B \cap Z| \leq c$ for all $B \in P$, satisfying b..

This proves the statement. □

Remark 4.25 (Remark 4.22). Notice that if $k = k_*$, the condition 2. will be satisfied by Corollary 3.12 and the non-k_*-order of G.

Theorem 4.26 (Theorem 4.23). *Let G be a graph with the non-k_*-property. For any $c \in \mathbb{N}$, $\epsilon, \xi \in \mathbb{R}$ satisfying the hypothesis of Lemma 4.24 (with $k = k_*$ and $\zeta = \frac{1}{k_*}$), any $\theta \in (0, 1)$ and $A \subseteq G$ with $A = n > n(c, \epsilon, \zeta, \xi, \theta)$ (i.e. n large enough in the sense of Lemma 4.23), there is a partition $\overline{A} = \langle A_i \mid i \in \{1, \dots, i(*)\} \rangle$ of A with remainder $B = A \setminus \bigcup \overline{A}$ satisfying:*

Change this notation to use $N_{4.23}$.

- *$|A_i| = \lfloor \lfloor n^{\theta} \rfloor^{\zeta} \rfloor$ for all $i \in \{1, \dots, i(*)\}$.*
- *A_i is c-indivisible for all $i \in \{1, \dots, i(*)\}$ where c is the constant function $f(x) = c$.*
- *$|B| < \lfloor n^{\frac{\theta}{\epsilon^{k_{**}}}} \rfloor$.*

Proof. Let $n > (n(\epsilon, \frac{1}{k_*}, \frac{\xi}{\epsilon^{k_{**}}}, c)^{\frac{1}{\epsilon^{k_{**}}}} + 1)^{\frac{1}{\theta}}$ in the sense of Lemma 4.23, so that $\lfloor n^\theta \rfloor$ satisfies the large enough condition of Lemma 4.24:

$$(\lfloor n^\theta \rfloor)^{\epsilon^{k_{**}}} > n(\epsilon, \frac{1}{k_*}, \frac{\xi}{\epsilon^{k_{**}}}, c)$$

Notice that condition 2. in Lemma 4.24 is satisfied by Remark 4.25. Now, we define a decreasing sequence $m_0 > m_1 > \cdots > m_{k_{**}}$ with $m_{k_{**}} = \lfloor n^\theta \rfloor$ and $m_\ell = \lceil (m_{\ell+1})^{\frac{1}{\epsilon}} \rceil$ for all $\ell \in \{0, \dots, k_{**} - 1\}$. This sequence satisfies the condition of Lemma 4.12 for $f(n) = n^\epsilon$. We will build a sequence of disjoint c-indivisible subsets A_i by induction on i as follows. Let $R_i = A \setminus \bigcup_{j<i} A_j$ (so $R_1 = A$). If $R_i < \lfloor n^{\frac{\theta}{\epsilon^{k_{**}}}} \rfloor$, then $\overline{A} = \langle A_j \mid j < i = i(*) \rangle$ and $B = R_i$, and we are done. Otherwise, we can apply Lemma 4.12 to R_i with the sequence $\langle m_\ell \rangle_{\ell \leq k_{**}}$, to obtain an ϵ-indivisible subset $B_i \subseteq R_i$ of size $m_{k_{**}-\ell}$. Then, since $|B_i| = m_{k_{**}-\ell} \geq m_{k_{**}} = \lfloor n^\theta \rfloor$ by the n-large-enough assumption, we can apply Lemma 4.24 and get a c-indivisible subset Z_i of size $|Z_i| = \lfloor m^\zeta_{k_{**}-\ell} \rfloor \geq \lfloor \lfloor n^{\frac{\theta}{\epsilon^\ell}} \rfloor^\zeta \rfloor \geq \lfloor \lfloor n^\theta \rfloor^\zeta \rfloor$. Since c-indivisible is preserved when taking subsets, we can choose $A_i \subseteq Z_i$ c-indivisible of size $\lfloor \lfloor n^\theta \rfloor^\zeta \rfloor$. ☐

Something is odd here.

Make some remark on the fact that θ needs to be smaller then $\epsilon^{k_{**}}$ for this to make sense.

5. Section 5

Definition 5.1 (Definition 5.2(a))**.** Let G be a finite graph with the non-k_*-property. We say that $A \subseteq G$ is *ϵ-good* when for every $b \in G$ the truth value $t = t(b, A) \in \{0, 1\}$ satisfies $|\{a \in A \mid aRb \not\equiv t\}| < \epsilon|A|$.

Remark 5.2. The property of a set being ϵ-good is equivalent to the set being f-indivisible with $f(n) = \epsilon n$, and ϵ-indivisible is a much stronger condition then ϵ-good.

Definition 5.3 (Definition 5.2(b))**.** Let G be a finite graph with the non-k_*-property. We say that $A \subseteq G$ is *(ϵ, ζ)-excellent* when A is ϵ-good and, if B is ζ-good, then the truth value $t = t(B, A)$ satisfies $|\{a \in A \mid t(a, B) \neq t(A, B)\}| < \epsilon|A|$.

In particular, we say A is *ϵ-excellent* if A is (ϵ, ϵ)-excellent.

Remark 5.4. Notice that, if $A, B \subseteq G$ are two (not necessarily disjoint) subsets of vertices with A (ϵ, ϵ')-excellent and B ϵ'-good set, then the number of exceptional edges between A and B, i.e. these vertex pairs that do not follow $t(A, B)$, is relatively small:

$$|\{\text{Exceptional edges between } A \text{ and } B\}| < \epsilon|A||B| + (1-\epsilon)|A|\epsilon'|B| = (\epsilon + (1-\epsilon)\epsilon')|A||B|$$

A relevant example is that of two disjoint ϵ-excellent sets, in which case we have that the fraction of exceptional edges between them is less than 2ϵ. If they are not disjoint, we can still use the same reasoning to conclude that the fraction of exceptional edges is less than $2\epsilon\frac{|A||B|}{e(A,B)} < 8\epsilon$, since $e(A, B) > \frac{|A||B|}{4}$.

Discuss with Luis, this may be reduced but I am not sure.

Lemma 5.5 (Claim 5.4)**.** *Let G be a finite graph with the non-k_*-order property. Let $\zeta \leq \frac{1}{2^{k_{**}}}$, $\epsilon \in (0, \frac{1}{2})$. Then, for every $A \subseteq G$ with $|A| \geq \frac{1}{\epsilon^{k_{**}}}$ there exists an (ϵ, ζ)-excellent subset $A' \subseteq A$ such that $|A'| \geq \epsilon^{k_{**}-1}|A|$.*

Proof. Suppose the converse. We use this fact to build sets $\{b_\eta \mid \eta \in \{0, 1\}^{<k_{**}}\}$ and $\{A_\eta \mid \eta \in \{0, 1\}^{\leq k_{**}}\}$ on induction over $k < k_{**}$, where $k = |\eta|$, satisfying:

1. $A_{\langle \cdot \rangle} = A$.
2. B_η is a ζ-good set witnessing that A_η is not (ϵ, ζ)-excellent, for $k < k_{**}$.
3. $A_{\eta\frown\langle i\rangle} = \{a \in A_\eta \mid t(a, B_\eta) \equiv i\}$ for all $i \in \{0, 1\}$ and $k < k_{**}$.
4. $|A_{\eta\frown\langle i\rangle}| \geq \epsilon|A_\eta|$ for all $i \in \{0, 1\}$ and $k < k_{**}$.
5. $|A_\eta| \geq \epsilon^k|A|$, for $k \leq k_{**}$.
6. $A_\eta = A_{\eta\frown\langle 0\rangle} \sqcup A_{\eta\frown\langle 1\rangle}$, for $k < k_{**}$.
7. $\overline{A_k} = \{A_\eta \mid \eta \in \{0, 1\}^k\}$ is a partition of A, for $k \leq k_{**}$.

First of all, notice that at each step, the non-(ϵ, ζ)-excellence of A_η comes by IH from 1. or 5., and thus allows the existence of B_η in 2.. 4. follows the definition of $A_{\eta\frown\langle i\rangle}$ in 3. and the fact B_η is witnessing that A_η is not (ϵ, ζ)-excellent. Applying recursively this last point we obtain 5.. Finally, by definition 3., we have the disjoint union 6. which ensures the partition 7..

Now, our goal is to build two sequences $\{b_\eta \mid \eta \in \{0,1\}^{<k_{**}}\}$ and $\{a_\eta \mid \eta \in \{0,1\}^{k_{**}}\}$ to contradict the tree bound k_{**}. First of all, notice that, for $\eta \in \{0,1\}^{k_{**}}$

$$|A_\eta| \geq \epsilon^{k_{**}}|A| \geq \epsilon^{k_{**}}\frac{1}{\epsilon^{k_{**}}} = 1$$

So, for each $\eta \in \{0,1\}^{k_{**}}$, $A_\eta \neq \emptyset$ and we may choose an $a_\eta \in A_\eta$. Now, for $\nu \in \{0,1\}^{<k_{**}}$ and $\eta \in \{0,1\}^{k_{**}}$ such that $\nu \triangleleft \eta$, let

$$U_{\nu,\eta} = \{b \in B_\nu \mid a_\eta R b \not\equiv t(a_\eta, B_\nu)\}$$

be the subset of elements of B_ν that do not relate with a_η in the expected way. By ζ-goodness of B_ν, $|U_{\nu,\eta}| < \zeta|B_\nu|$, and thus for every $\nu \in \{0,1\}^{<k_{**}}$,

$$|\bigcup\{U_{\nu,\eta} \mid \nu \triangleleft \eta \in \{0,1\}^{k_{**}}\}| < 2^{k_{**}}\zeta|B_\nu| \leq |B_\nu|$$

We may choose $b_\nu \in B_\nu \setminus \bigcup\{U_{\nu,\eta} \mid \nu \triangleleft \eta \in \{0,1\}^{k_{**}}\}$, for all $\nu \in \{0,1\}^{<k_{**}}$. Finally, the sequences $\langle a_\eta \mid \eta \in \{0,1\}^{k_{**}}\rangle$ and $\langle b_\nu \mid \nu \in \{0,1\}^{<k_{**}}\rangle$ satisfy that $\forall \eta, \nu$ such that $\nu \frown \langle i\rangle \triangleleft \eta$, $(a_\eta R b_\nu)^i$ by 3. and 6.. This contradicts Definition 3.14 of tree bound k_{**}. □

Lemma 5.6 (Claim 5.4.1)**.** *Let G be a finite graph with the non-k_*-order property. Let $\zeta < \frac{1}{2^{k_{**}}}$, $\epsilon \in (0, \frac{1}{2})$. Let $\langle m\ell \mid \ell \in \{0,\dots,k_{**}\}\rangle$ be a decreasing sequence of natural numbers such that $\epsilon m_\ell \geq m_{\ell+1}$ for all $\ell \in \{0,\dots,k_{**}-1\}$ and $m_{k_{**}} \geq 1$. Then, for every $A \subseteq G$ with $|A| \geq m_0$ there exists $(\frac{m_{\ell+1}}{m_\ell}, \zeta)$-excellent subset $A' \subseteq A$ such that $|A'| = m_\ell$ for some $\ell \in \{0,\dots,k_{**}-1\}$.*

Proof. Suppose the converse. We use this fact to build sets $\{b_\eta \mid \eta \in \{0,1\}^{<k_{**}}\}$ and $\{A_\eta \mid \eta \in \{0,1\}^{\leq k_{**}}\}$ on induction over $k < k_{**}$, where $k = |\eta|$, satisfying:

1. $A_{\langle\cdot\rangle} \subseteq A$, with $|A|_{\langle\cdot\rangle} = m_0$.
2. B_η is an ζ-good set witnessing that A_η is not $(\frac{m_{k+1}}{m_k}, \zeta)$-excellent, for all $k < k_{**}$.
3. $A_{\eta\frown\langle i\rangle} = \{a \in A_\eta \mid t(a, B_\eta) \equiv i\}$ for all $i \in \{0,1\}$ and $k < k_{**}$.
4. $|A_\eta| = m_k$, for all $k \leq k_{**}$.
5. $A_{\eta\frown\langle 0\rangle} \sqcup A_{\eta\frown\langle 1\rangle} \subseteq A_\eta$, for all $k < k_{**}$.
6. $\overline{A_k} = \{A_\eta \mid \eta \in \{0,1\}^k\}$ is a partition of a subset of A, for all $k \leq k_{**}$.

Notice that, by 1. and 4., the size of A_η is m_k, so by IH none of the sets A_η is $(\frac{m_{k+1}}{m_k}, \zeta)$-excellent. Then, B_η in 2. is well-defined. Also, by ζ-goodness of B_η, $t(a, B_\eta)$ in 3. is well-defined. Then, since B_η is witnessing the non-$(\frac{m_{k+1}}{m_k}, \zeta)$-excellence of A_η, we have that $|A_{\eta\frown\langle i\rangle}| \geq \frac{m_{k+1}}{m_k}m_k = m_{k+1}$ for all $i \in \{0,1\}$, satisfying 4.. Finally, by definition 3., we have the disjoint union 5. which by itself ensures 6..

Now, our goal is to build two sequences $\{b_\eta \mid \eta \in \{0,1\}^{<k_{**}}\}$ and $\{a_\eta \mid \eta \in \{0,1\}^{k_{**}}\}$ to contradict the tree bound k_{**}. First of all, notice that, for $\eta \in \{0,1\}^{k_{**}}$

$$|A_\eta| = m_k \geq m_{k_{**}} \geq 1$$

So, for each $\eta \in \{0,1\}^{k_{**}}$, $A_\eta \neq \emptyset$ and we may choose an $a_\eta \in A_\eta$. Now, for $\nu \in \{0,1\}^{<k_{**}}$ and $\eta \in \{0,1\}^{k_{**}}$ such that $\nu \triangleleft \eta$, let

$$U_{\nu,\eta} = \{b \in B_\nu \mid (a_\eta R b) \not\equiv t(a_\eta, B_\nu)\}$$

be the subset of elements of B_ν that do not relate with a_η in the expected way. By ζ-goodness of B_ν, $|U_{\nu,\eta}| < \zeta|B_\nu|$, and thus for every $\nu \in \{0,1\}^{<k_{**}}$,

$$|\bigcup\{U_{\nu,\eta} \mid \nu \triangleleft \eta \in \{0,1\}^{k_{**}}\}| < 2^{k_{**}}\zeta|B_\nu| \leq |B_\nu|$$

We may choose $b_\nu \in B_\nu \setminus \bigcup\{U_{\nu,\eta} \mid \nu \triangleleft \eta \in \{0,1\}^{k_{**}}\}$, for all $\nu \in \{0,1\}^{<k_{**}}$. Finally, the sequences $\langle a_\eta \mid \eta \in \{0,1\}^{k_{**}} \rangle$ and $\langle b_\nu \mid \nu \in \{0,1\}^{<k_{**}} \rangle$ satisfy that $\forall \eta, \nu$ such that $\nu \frown \langle i \rangle \triangleleft \eta$, $(a_\eta R b_\nu)^i$, which follows 3.. This contradicts Definition 3.14 of tree bound k_{**}. $\square$

Lemma 5.7. *For $k > 1$, $\zeta, \eta \in (0,1)$ the function $f(m) = m^k \cdot e^{-2\zeta^2 m}$ satisfies $f(m) \leq \eta$ for all $m \geq \frac{1}{\zeta^2}(k \log \frac{1}{\zeta^2} k - \log \eta)$.*

Proof. First of all, notice that for $m = \frac{1}{\zeta^2}(k \log \frac{1}{\zeta^2} k - \log \eta)$,

$$f(m) = \frac{m^k}{e^{2\zeta^2 m}} = \frac{(\frac{1}{\zeta^2}(k \log \frac{1}{\zeta^2} k - \log \eta))^k}{(\frac{k}{\zeta^2})^{2k}\eta^{-2}} \leq \frac{k^k(\log \frac{k}{\zeta^2}(\frac{1}{\eta})^{\frac{1}{k}})^k}{k^k(\frac{k}{\zeta^2}(\frac{1}{\eta})^{\frac{1}{k}})^k}\eta < \eta$$

To conclude, we prove that f is decreasing for larger values of m:

$$f'(m) = \frac{km^{k-1}e^{2\zeta^2 m} - 2\zeta^2 m^k e^{2\zeta^2 m}}{(e^{2\zeta^2 m})^2} = (k - 2m\zeta^2)\frac{m^{k-1}}{e^{2\zeta^2 m}}$$

The second factor is always positive, and $m > \frac{k}{\zeta^2} > \frac{k}{2\zeta^2}$, proving that $f'(m) < 0$ and thus f is decreasing. $\square$

Lemma 5.8 (Claim 5.13)**.** *Let G be a finite graph with the non-k_*-order property. Then:*

(a) For every $\epsilon \in (0, \frac{1}{2})$, $\zeta \in (0, \frac{1}{2} - \epsilon)$, $\xi \in (0,1)$ and $m \geq \frac{1}{\zeta^2}(k_* \log \frac{1}{\zeta^2} k_* - \log \xi)$, if $A \subseteq G$ is an ϵ-good subset of size $n \geq m$, then a random subset $A' \subseteq A$ of size m is $(\epsilon + \zeta)$-good with probability $1 - \xi$.

(b) Moreover, such A' satisfies $t(b, A') = t(b, A)$ for all $b \in G$.

(c) For every $\zeta \in \{0, \frac{1}{2}\}$ and $\zeta' < \zeta$, there is $\epsilon_1 = \epsilon_1(\zeta, \zeta')$ such that for every $\epsilon < \epsilon' \leq \epsilon_1$, if

- *$A \subseteq G$ is $\{\epsilon, \epsilon'\}$-excellent.*
- *$A' \subseteq A$ is $\{\epsilon + \zeta'\}$-good.*

then, A' is $(\epsilon + \zeta, \epsilon')$-excellent.

(d) For all $\zeta \in (0, \frac{1}{2})$, $\zeta' < \zeta$, $r \geq 1$ and for all $\epsilon < \epsilon'$ small enough (in the sense of the previous point) there exists $N = N(k_*, \zeta', r)$ such that, if $|A| = n > N$, r divides n and A is (ϵ, ϵ')-excellent, there exists a partition into r disjoint pieces of equal size, each of which is $(\epsilon + \zeta, \epsilon')$-excellent.

Proof. (a) For each $b \in G$, we say that $B_{A,b}$ is *bad* if $|B_{A,b}| \geq \epsilon|A'|$. For each bad $B_{A,b}$, let $X_{A,b}$ be the event that $|B_{A,b}| \geq (\epsilon+\zeta)|A'|$ for a random subset $A' \subseteq A$ of size m. Notice that $X_{A,b}$ is modelled by a hypergeometric distribution, and so the probability of upperly deviating from the mean by ζ, can be modeled by

$$P(X_{A,b}=1) \leq e^{-2\zeta^2 m}$$

Now we want to study the random variable X counting the number of events $X_{A,b}$ that are satisfied. That is, $X = \sum_{\text{bad } B_{A,b}} X_{A,b}$. We compute the expectation

$$\mathbb{E}[X] = \sum_{\text{bad } B_{A,b}} \mathbb{E}[X_{A,b}] = \sum_{\text{bad } B_{A,b}} P(X_{A,b}=1) \leq \sum_{\text{bad } B_{A,b}} e^{-2\zeta^2 m}$$

Following 2., the number of intersections of bad $B_{A,b}$'s with A', can be bounded by m^{k_*}. Thus, using the First Moment Method, we have that:

$$P(X \geq 1) \leq \mathbb{E}[X] \leq m^{k_*} \cdot e^{-2\zeta^2 m} \leq \xi$$

Last inequality follows Lemma 5.7 using the lower bound on m. Thus, with probability at least $1-\xi$, we have that A' is $(\epsilon+\zeta)$-good.

(b) Suppose that A' is the subset described in a.. We proved that, such set satisfies

$$|A' \cap B_{A,b}| < (\epsilon+\zeta)|A'|$$

for all $b \in G$ such that $|B_{A,b}| \geq \epsilon m$. Thus, we have that:

- If $|B_{A,b}| < \epsilon m$, then $|\{a \in A' \mid aRb \not\equiv t(b,A)\}| \leq |B_{A,b}| < \epsilon m < (\epsilon+\zeta)m$.
- If $|B_{A,b}| \geq \epsilon m$, then $|\{a \in A' \mid aRb \not\equiv t(b,A)\}| = |A' \cap B_{A,b}| < (\epsilon+\zeta)m$.

We conclude that $t(b,A) = t(b,A')$ for all $b \in G$.

(c) Let $B \subseteq G$ be an ϵ'-good set. We first upperbound the number of exceptional vertices of B with respect to A':

$$\begin{aligned} |\{b \in B \mid t(b,A') \not\equiv t(B,A)\}| &= |\{b \in B \mid t(b,A) \not\equiv t(B,A)\}| \\ &\leq \frac{(\epsilon+(1-\epsilon)\epsilon')|A||B|}{(1-\epsilon)|A|} \\ &= (\epsilon' + \frac{\epsilon}{1-\epsilon})|B| \end{aligned}$$

The first equality follows b., and the first inequality follows from Remark 5.4 for the numerator, and taking the worst case of only $(1-\epsilon)|A|$ exceptional edges per exceptional $b \in B$ (considering that A is ϵ-good).

Now, let Q be the set of exceptional vertices of A' with respect to B, i.e.:

$$Q = \{a \in A' \mid t(a,B) \not\equiv t(A,B)\}$$

We want to double-count the number of exceptional edges between Q and B. On one hand, we have that:

$$|\{(a,b) \in Q \times B \mid t(a,b) \not\equiv t(A,B)\}| < (\epsilon' + \frac{\epsilon}{1-\epsilon})|B||Q| + (1-\epsilon' - \frac{\epsilon}{1-\epsilon})|B|(\epsilon+\zeta')|A'|$$

The first term is the maximum number of exceptional edges associated to exceptional $b \in B$ (considering all edges exceptional), while the second term bounds the number of exceptional edges of non-exceptional $b \in B$, using the fact that A' is $(\epsilon + \zeta')$-good.

On the other hand, we have that:

$$|\{(a, b) \in Q \times B \mid t(a, b) \not\equiv t(A, B)\}| \geq |Q|(1 - \epsilon')|B|$$

which follows B being ϵ'-good.

Putting it all together:

$$(1 - \epsilon' - \epsilon' - \frac{\epsilon}{1-\epsilon})|B||Q| < (1 - \epsilon' + \frac{\epsilon}{1-\epsilon})(\epsilon + \zeta')|B||A'|$$

So, we have that:

$$\begin{aligned} |Q| &< \frac{(1 - \epsilon' - \frac{\epsilon}{1-\epsilon})}{(1 - \epsilon' - \frac{\epsilon}{1-\epsilon}) - \epsilon'}(\epsilon + \zeta')|A'| \\ &= (1 + \frac{\epsilon'}{1 - 2\epsilon' - \frac{\epsilon}{1-\epsilon}})(\epsilon + \zeta')|A'| \end{aligned}$$

Notice that $f(\epsilon, \epsilon') := \frac{\epsilon'}{1-2\epsilon'-\frac{\epsilon}{1-\epsilon}}$ decreases with ϵ and ϵ'. In particular,

$$f(\epsilon, \epsilon') \xrightarrow{\epsilon' \to 0} 0$$

and $\epsilon' > \epsilon$. Then,

$$|Q| < (\epsilon + (\underbrace{\epsilon f(\epsilon, \epsilon')}_{\to 0} + \underbrace{(1 + f(\epsilon, \epsilon'))}_{\to 1})\zeta')|A'| \xrightarrow{\epsilon' \to 0} (\epsilon + \zeta')|A'|$$

So, there exists an $\epsilon_1 = \epsilon_1(\zeta, \zeta')$ small enough such that for all $(\epsilon <)\ \epsilon' \leq \epsilon_1$, we have that $|Q| < (\epsilon + \zeta)|A'|$, and since A' is $(\epsilon + \zeta')$-good, and thus $(\epsilon + \zeta)$-good, we conclude that A' is $(\epsilon + \zeta, \epsilon')$-excellent.

Mention that in the next claim we show valid values for this.

(d) Let $\zeta, \zeta', \epsilon, \epsilon'$ and r be given satisfying the conditions of the statement. Set $\xi = \frac{1}{r+1}$. We will see that the condition $n > N = N(k_*, \zeta', r) := r\frac{1}{\zeta'^2}(k_* \log \frac{1}{\zeta'^2} k_* - \log \frac{1}{r+1})$ is sufficient. First of all, randomly choose a function $h : A \longrightarrow \{1, \dots, r-1\}$ such that for all $s < n$ we have that $|\{a \in A \mid h(a) = s\}| = \frac{n}{r}$. Since h is random, each $A' \in [A]^{\frac{n}{r}}$ has the same probability of being part of the partition induced by h, i.e. to satisfy $A' = h^{-1}(s)$ for some $s \in \{1, \dots, r-1\}$. Since each element of the partition A' has size $\frac{n}{r} > \frac{N}{r} = \frac{1}{\zeta'^2}(k_* \log \frac{1}{\zeta'^2} k_* - \log \xi)$, we can apply a. to get that

$$P(A' \text{ is not } (\epsilon + \zeta')\text{-good}) < \xi$$

In particular, since A is (ϵ, ϵ')-excellent, it follows c. that if A' is $(\epsilon + \zeta')$-good then it is also $(\epsilon + \zeta, \epsilon')$-excellent, so:

$$P(A' \text{ is not } (\epsilon + \zeta, \epsilon')\text{-excellent}) < \xi$$

To conclude, by the union bound, we have that:

$$\begin{aligned} P(\bigcup_{s<r} h^{-1}(s) \text{ is not } (\epsilon + \zeta, \epsilon')\text{-excellent}) &\leq \sum_{s<r} P(h^{-1}(s) \text{ is not } (\epsilon + \zeta, \epsilon')\text{-excellent}) \\ &< r\xi = \frac{r}{r+1} < 1 \end{aligned}$$

All in all, there is a non-zero chance that the partition satisfies the statement, i.e. there exists at least one.

$\square$

Remark 5.9 (Remark 5.13.1). For following applications, we would like to use d. from Lemma 5.8 with $\epsilon' > k(\epsilon + \zeta)$, for an arbitrarily large $k \in \mathbb{N}$. Notice that if $\epsilon, \zeta' \leq \frac{1}{t}, \epsilon' \leq \frac{1}{t'}$ and $t > t' \geq 5$, then:

(a) $\frac{\epsilon}{1-\epsilon} \leq \frac{\frac{1}{t}}{1-\frac{1}{t}} = \frac{\frac{1}{t}}{\frac{t-1}{t}} = \frac{1}{t-1}$

(b) $1 - 2\epsilon' - \frac{\epsilon}{1-\epsilon} \geq 1 - \frac{2}{t'} - \frac{1}{t-1} > 1 - \frac{3}{t'-1} = \frac{t'-4}{t'-1}$

(c) $(1 + \frac{\epsilon'}{1-2\epsilon'-\frac{\epsilon}{1-\epsilon}}) < 1 + \frac{\epsilon'}{1-\frac{3}{t'-1}} = (1 + \frac{t'-1}{t'-4}\epsilon')(\epsilon + \zeta')$

Then, by requiring $\frac{1}{t} \leq \frac{1}{4k}\epsilon'$ we have that

$$\begin{aligned}
\epsilon + \zeta' &\leq \frac{2}{t} \leq 2(\frac{1}{4k}\epsilon') = \frac{1}{2}(\frac{1}{k}\epsilon') \\
&< \frac{t'-4}{t'-3}\frac{1}{k}\epsilon' = \frac{1}{k}\frac{\epsilon'}{1+\frac{1}{t'-4}} \\
&< \frac{1}{k}\frac{\epsilon'}{1+\frac{t'-1}{t'}\frac{1}{t'-4}} = \frac{1}{k}\frac{\epsilon'}{1+\frac{t'-1}{t'-4}\frac{1}{t'}} \\
&\leq \frac{1}{k}\frac{\epsilon'}{1+\frac{t'-1}{t'-4}\epsilon'}
\end{aligned}$$

i.e., we have:

$$(1 + \frac{t'-1}{t'-4}\epsilon')(\epsilon + \zeta') < \frac{1}{k}\epsilon'$$

which by c. gives us:

$$(1 + \frac{\epsilon'}{1 - 2\epsilon' - \frac{\epsilon}{1-\epsilon}}) < \frac{1}{k}\epsilon'$$

All in all, a sufficient condition, for the lemma to hold under the constraint $\epsilon' \geq k(\epsilon + \zeta)$, is:

$$\epsilon, \zeta' \leq \frac{1}{4k}\epsilon' \quad \text{and} \quad \epsilon' \leq \frac{1}{5}$$

We use this fact to reformulate point d. of Lemma 5.8 as:

Lemma 5.10 (Claim 5.13.2(3))**.** *Let G be a finite graph with the non-k_*-property. For all $k, r \geq 1$, $\epsilon' \leq \frac{1}{5}$ and $\epsilon \leq \frac{1}{4k}\epsilon'$, there exists $N = N(k, k_*, \epsilon', r)$ large enough such that, for all $n > N$ and r dividing n, if $A \subseteq G$ is (ϵ, ϵ')-excellent, with $|A| = n$, then there exists a partition into r disjoint pieces of equal size, each of which is $(\frac{\epsilon'}{k}, \epsilon')$-excellent.*

Proof. Choose any $\zeta' \leq \frac{1}{4k}\epsilon'$ and set $N := N_{5.8}(k_*, \zeta', r)$. Remark 5.9 sufficiency condition is satisfied, d. from Lemma 5.8 holds and we are done. $\square$

Remark 5.11. A sufficient condition for $N_{5.10}$ to be large enough is to choose $\zeta' = \frac{1}{4k}\epsilon'$ in which case $N_{5.10}(k, k_*, \epsilon', r) := N_{5.8}(k_*, \frac{1}{4k}\epsilon', r)$

Lemma 5.12 (Claim 5.14.1). *Let G be a finite graph with the non-k_*-order property. Let $\epsilon \in (0, \frac{1}{2})$ and $\epsilon' \leq \frac{1}{2^{k_{**}}}$. Let $A \subseteq G$ such that $|A| = n$. Let $\langle m\ell \mid \ell \in \{0, \dots, k_{**}\}\rangle$ be a decreasing sequence of natural numbers such that $\epsilon m_\ell \geq m_{\ell+1}$ for all $\ell \in \{0, \dots, k_{**} - 1\}$ and $m_{k_{**}} \geq 1$. Denote $m_* := m_0$ and $m_{**} := m_{k_{**}}$. Then, there is a partition $\overline{A} = \langle A_j \mid j \in \{1, \dots, j(*)\}\rangle$ with remainder $B = A \setminus \bigcup_{j<j(*)} A_j$ such that:*

(a) *For all $j \in \{1, \dots, j(*)\}$, $|A_j| \in \langle m\ell \mid \ell \in \{0, \dots, k_{**} - 1\}\rangle$.*

(b) *For all $i \neq j \in \{1, \dots, j(*)\}$, $A_i \cap A_j = \emptyset$.*

(c) *For all $j \in \{1, \dots, j(*)\}$, A_j is (ϵ, ϵ')-excellent.*

(d) *$|B| < m_*$.*

Proof. Apply Lemma 5.6 recursively to the remainder $A \setminus \bigcup_{i<j} A_i$, to obtain A_j at each step. The process stops at $j(*)$ when the remainder is smaller than m_0, and thus the lemma cannot be applied. Notice that, since $\frac{m_\ell}{m_{\ell-1}} \leq \epsilon$, $(\frac{m_\ell}{m_{\ell-1}}, \epsilon')$-excellence implies (ϵ, ϵ')-excellence. □

Say that if A is smaller than m_0, then the partition is empty and $B = A$.

Lemma 5.13 (Claim 5.14.1a). *Let G be a finite graph with the non-k_*-order property. Let $\epsilon' \leq \min(\frac{1}{5}, \frac{1}{2^{k_{**}}})$ and $\epsilon \leq \frac{1}{4k}\epsilon'$ for some $k > 1$. Let $A \subseteq G$ such that $|A| = n$. Let $\langle m\ell \mid \ell \in \{0, \dots, k_{**}\}\rangle$ be a decreasing sequence of natural numbers such that $\epsilon m_\ell \geq m_{\ell+1}$ for all $\ell \in \{0, \dots, k_{**} - 1\}$, $m_{k_{**}} \geq 1$, $m_{**} := m_{k_{**}} \mid m_\ell$ for all $\ell \in \{0, \dots, k_{**}\}$, $m_{k_{**}-1} > N(k, k_*, \epsilon', \frac{m_*}{m_{**}})$ (in the sense of Lemma 5.10), and $n \geq m_0$. Let $m_* := m_0$. Then, for some $i(*) \leq \frac{n}{m_{**}}$, there is a partition $\overline{A} = \langle A_i \mid i \in \{1, \dots, i(*)\}\rangle$ with remainder $B = A \setminus \bigcup \overline{A}$ such that:*

(a) *For all $i \in \{1, \dots, i(*)\}$, $|A_i| = m_{**}$.*

(b) *For all $i \in \{1, \dots, i(*)\}$, A_i is $(\frac{\epsilon'}{k}, \epsilon')$-excellent.*

(c) *$|B| < m_*$.*

Proof. Use Lemma 5.12 to obtain a partition $\overline{A}' = \left\langle A'_j \mid j \in \{1, \dots, j(*)\}\right\rangle$ and remainder B with $|B| < m_*$. Then, we can apply Lemma 5.10 with $r = \frac{m_*}{m_{**}}$ to each of the parts A'_j. Putting together all the new subparts, we obtain a new partition $\overline{A} = \langle A_i \mid i \in \{1, \dots, i(*)\}\rangle$ with remainder B, satisfying all the conditions of the statement. □

Lemma 5.14 (Claim 5.14.2). *Under the same condition of Lemma 5.13, we can get a partition $\overline{A} = \langle A_i \mid i \in \{1, \dots, i(*)\}\rangle$ with no remainder, such that:*

(a) *For all $i, j \in \{1, \dots, i(*)\}$, $||A_i| - |A_j|| \leq 1$.*

(b) *For all $i, j \in \{1, \dots, i(*)\}$, $A_i \cap A_j = \emptyset$.*

(c) *For all $i \in \{1, \dots, i(*)\}$, A_i is (ϵ'', ϵ')-excellent, where*

$$\epsilon'' \leq \frac{\frac{\epsilon'}{k} m_{**} + \left\lceil \frac{m_*}{i(*)} \right\rceil}{m_{**} + \left\lceil \frac{m_*}{i(*)} \right\rceil}$$

(d) $A = \bigcup \overline{A}$.

Proof. Let $\overline{A}' = \langle A'_i \mid i \in \{1, \dots, i(*)\}\rangle$ and B from Lemma 5.13. We can partition B into $\overline{B} = \langle B_i \mid i \in \{1, \dots, i(*)\}\rangle$ in such a way that for all $i \in \{1, \dots, i(*)\}$,

$$|B_i| \in \{ \left\lfloor \frac{|B|}{i(*)} \right\rfloor, \left\lceil \frac{|B|}{i(*)} \right\rceil \}$$

Notice that we are allowing $B_i = \emptyset$. Then, the new partition $\overline{A} = \langle A'_i \cup B_i \mid i \in \{1, \dots, i(*)\}\rangle$ satisfies a., b. and d. by construction. To conclude, notice that for each ϵ'-good set B, the number of exceptions is bounded by

$$\begin{aligned}
|\{a \in A_i \mid t(a, B) \not\equiv t(A_i, B)\}| &\leq \frac{\epsilon'}{k}|A'_i| + |B_i| \\
&= \frac{\frac{\epsilon'}{k}|A'_i| + |B_i|}{|A'_i| + |B_i|}(|A'_i| + |B_i|) \\
&\leq \frac{\frac{\epsilon'}{k} m_{**} + \left\lceil \frac{m_*}{i(*)} \right\rceil}{m_{**} + \left\lceil \frac{m_*}{i(*)} \right\rceil}|A_i|
\end{aligned}$$

which proves that c. can be satisfied. $\square$

Remark 5.15 (Remark 5.14.3). In the context of Lemma 5.14, if:

(a) $m_{**} \geq \frac{1}{\frac{\epsilon'}{k}}$

(b) $m_* \leq \frac{\frac{\epsilon'}{k}n+1}{\frac{\epsilon'}{k}+1}$

then $\epsilon'' \leq \frac{3\epsilon'}{k}$.

Proof. Notice that, if $|B_i| \leq 2\frac{\epsilon'}{k}|A_i|$ for all $i \in \{1, \dots, i(*)\}$, then ϵ'' can be bounded by:

$$\epsilon'' \leq \frac{\frac{\epsilon'}{k}|A_i| + |B_i|}{|A_i| + |B_i|} \leq \frac{\frac{\epsilon'}{k}|A_i| + 2\frac{\epsilon'}{k}|A_i|}{|A_i|} = \frac{3\epsilon'}{k}$$

Let's now prove that $|B_i| \leq \frac{2\epsilon'}{k}|A_i|$ is satisfied. Notice that, by construction:

$$|B_i| \leq \left\lceil \frac{|B|}{i(*)} \right\rceil \leq \left\lceil \frac{m_* - 1}{i(*)} \right\rceil \leq \frac{m_* - 1}{i(*)} + 1$$

Also we can bound $i(*)$ by:

$$\frac{n}{m_{**}} \geq i(*) \geq \frac{n - |B|}{m_{**}} \geq \frac{n - m_* + 1}{m_{**}} > \frac{n - m_*}{m_{**}}$$

Thus, $|B_i| - 1 \leq \frac{m_* - 1}{i(*)} \leq \frac{(m_* - 1)m_{**}}{n - m_*}$, then $\frac{|B_i| - 1}{m_{**}} \leq \frac{m_* - 1}{n - m_*}$, and since $|A_i| = m_{**}$ we get: Is the lower bound needed?

$$\frac{|B_i|}{|A_i|} \leq \frac{m_* - 1}{n - m_*} + \frac{1}{m_{**}}$$

Finally, notice that condition a. implies:

$$\frac{\epsilon'}{k} \geq \frac{1}{m_{**}}$$

and condition b. implies:

$$\frac{\epsilon'}{k} \geq \frac{m_* - 1}{n - m_*}$$

We conclude:

$$\frac{|B_i|}{|A_i|} \leq \frac{m_* - 1}{n - m_*} + \frac{1}{m_{**}} \leq 2\frac{\epsilon'}{k}$$

completing the proof. □

Lemma 5.16 (Corollary 5.15)**.** *Let G be a graph with the non-k_*-order property. Suppose that we are given:*

1. $\epsilon \leq \min(\frac{1}{5}, \frac{1}{2^{k_{**}}})$.
2. *A sequence of positive integers $\langle m\ell \mid \ell \in \{0, \dots, k_{**}\}\rangle$, and values m_* and m_{**}, such that:*
   (a) $\frac{\epsilon}{12} m_\ell \geq m_{\ell+1}$.
   (b) $m_{**} := m_{k_{**}} > \frac{3}{\epsilon}$.
   (c) $m_{**} \mid m_\ell$ *for all* $\ell \in \{0, \dots, k_{**}\}$.
   (d) $m_{k_{**}-1} > N(3, k_*, \epsilon, \frac{m_*}{m_{**}})$ *(in the sense of Lemma 5.10).*
3. $A \subseteq G$ *such that* $|A| = n$, *where* n *is large enough to satisfy:*

This is implied by next condition.

   (a') $n \geq m_0$.
   (b') $m_* \leq \frac{1+\frac{\epsilon}{3}n}{1+\frac{\epsilon}{3}}$.

Then, there exists $i(*) \leq \frac{n}{m_{**}}$ and a partition of A into disjoint pieces $\overline{A} = \langle A_i \mid i \in \{1, \dots, i(*)\}\rangle$ such that:

(i) *For all $i, j \in \{1, \dots, i(*)\}$, $||A_i| - |A_j|| \leq 1$.*

(ii) *For all $i \in \{1, \dots, i(*)\}$, A_i is ϵ-excellent,*

(iii) *For all $i, j \in \{1, \dots, i(*)\}$, (A_i, A_j) is ϵ-uniform.*

Proof. Simply apply Lemma 5.14 in the context of Remark 5.15 with $k = 3$, $\epsilon'_{5.14} = \epsilon$ and $\epsilon_{5.14} \leq \frac{1}{12}\epsilon$. This results in a partition of A into disjoint pieces that satisfy i. and that are $(\epsilon''_{5.14}, \epsilon'_{5.14})$-excellent, with $\epsilon''_{5.14} \leq \frac{3\epsilon'_{5.14}}{k}$. But since $k \geq 3$, $\epsilon''_{5.14} \leq \epsilon'_{5.14}$, they are also $\epsilon'_{5.14}$-excellent, satisfying ii. and iii.. □

Theorem 5.17 (Theorem 5.18)**.** *Let k_* and therefore k_{**} be given. Then, for all $\epsilon \leq \min(\frac{1}{5}, \frac{1}{2^{k_{**}}})$ and $m > 1$, there is $M = M(\epsilon, m, k_*)$ and $N = N(\epsilon, m, k_*)$ such that, for every finite graph G with the non-k_*-order property, and every $A \subseteq G$ with $|A| \geq N$, there exists a partition $\overline{A} = \langle A_i \mid i \in \{1, \dots, i(*)\}\rangle$ of A, such that:*

Move the bound on M to another point?

1. *The number of parts is bounded by $m \leq i(*) \leq M := \max(\lceil \frac{12}{\epsilon} \rceil^{k_{**}+1}, 4m)$.*

2. *For all $i, j \in \{1, \dots, i(*)\}$, $||A_i| - |A_j|| \leq 1$.*

3. *For all $i \in \{1, \dots, i(*)\}$, A_i is ϵ-excellent.*

4. *For all $i, j \in \{1, \dots, i(*)\}$, (A_i, A_j) is ϵ-uniform.* Redundant?

Proof. Our goal is to apply Lemma 5.16. Let $q = \lceil \frac{12}{\epsilon} \rceil$. For $N(\epsilon, m, k_*)$, and thus n, large enough, we can then choose the smallest m_{**} satisfying:

(a) $m_{**} \in [\delta n - 1, \delta n]$, where $\delta = \min(\frac{\epsilon}{(3+\epsilon)q^{k_{**}}}, \frac{1}{m+q^{k_{**}}})$

(b) $m_{**} > \frac{3}{\epsilon}$.

(c) $m_{**} > \frac{N_{5.10}(3,k_*,\epsilon,q^{k_{**}})}{q}$.

We set $m_{k_{**}} = m_{**}$ and we build recursively a sequence of integers $\langle m_\ell \mid \ell \in \{0, \dots, k_{**}\} \rangle$ such that $m_\ell = qm_{\ell+1}$ for all $\ell \in \{0, \dots, k_{**} - 1\}$. Also, let $m_* := m_0 = q^{k_{**}} m_{**}$. By a. we have that $m_* \leq \frac{\epsilon n}{3+\epsilon}$. This sequence satisfies all the conditions of Lemma 5.16:

2.a. $m_{\ell+1} = \frac{1}{q} m_\ell \leq \frac{\epsilon}{12} m_\ell$.

2.b. $m_{**} \geq \frac{3}{\epsilon}$.

2.c. $m_{**} \mid m_\ell$ for all $\ell \in \{0, \dots, k_{**}\}$, since q is an integer.

2.d. $m_{k_{**}-1} = qm_{**} > q\frac{N_{5.10}(3,k_*,\epsilon,q^{k_{**}})}{q} = N_{5.10}(3, k_*, \epsilon, \frac{m_*}{m_{**}})$.

3.b. $m_* < \frac{\epsilon n}{3+\epsilon} < \frac{1+\frac{\epsilon}{3}n}{1+\frac{\epsilon}{3}}$.

3.a. $m_0 = m* < \frac{\epsilon n}{3+\epsilon} < n$

We can apply Lemma 5.16 to obtain a partition satisfying 2., 3. and 4..

We proceed to bound the number of parst $i(*)$. First, the upper bound follows from the fact that $m_{**} \geq \frac{1}{2} \min(\frac{\epsilon}{3+\epsilon}, \frac{1}{m+q^{k_{**}}})n$:

$$i(*) \leq \frac{n}{m_{**}} \leq \frac{2\max(\frac{3+\epsilon}{\epsilon}q^{k_{**}}, m + q^{k_{**}})n}{n} < 2\max(\frac{3+\epsilon}{\epsilon}q^{k_{**}}, 2m) \leq \max(\left\lceil \frac{12}{\epsilon} \right\rceil^{k_{**}+1}, 4m)$$

In the last inequality, we used that if $m < q^{k_{**}}$, then $m + q^{k_{**}} \leq 2q^{k_{**}} < \frac{3+\epsilon}{\epsilon}q^{k_{**}}$, which is dealt in the first argument of the maximum, so we may assume that $m \geq q^{k_{**}}$. We also show that the lower bound is satisfied:

$$i(*) \geq \frac{n - m_*}{m_{**}} \geq \frac{n - m_{**}q^{k_{**}}}{m_{**}} = \frac{n}{m_{**}} - q^{k_{**}} \geq \frac{m + q^{k_{**}}}{n} n - q^{k_{**}} = m$$

□

Remark 5.18. We now see how large N, and thus n, actually needs to be. First of all, we see that:

$$\begin{aligned}\frac{1}{q}N_{5.10}(4,k_*,\epsilon,q^{k_{**}}) &= \frac{1}{q}N_{5.8}(k_*,\frac{1}{4\cdot 3}\epsilon,q^{k_{**}}) \\ &= \frac{1}{q}q^{k_{**}}(\frac{12}{\epsilon})^2(k_*\log(\frac{12}{\epsilon})^2k_* - \log\frac{1}{q^{k_{**}}+1}) \\ &< k_*^2q^{2k_{**}+3}\end{aligned}$$

Also, $\frac{3}{\epsilon}$ is clearly smaller than this value. Then, since m_{**} is the smallest integer larger than both values, we conclude:

$$\begin{aligned}\frac{m_{**}}{\delta} &\leq \frac{k_*^2q^{2k_{**}+3}}{\min(\frac{\epsilon}{(3+\epsilon)q^{k_{**}}},\frac{1}{m+q^{k_{**}}})} \\ &= k_*^2q^{2k_{**}+3}\max(\frac{3+\epsilon}{\epsilon}q^{k_{**}},m+q^{k_{**}}) \\ &\leq \max(q^{k_{**}+1},4m)k_*^2q^{2k_{**}+3}\end{aligned}$$

Define or remove uniformity.

Lemma 5.19 (Lemma 5.17)**.** *Suppose that $\epsilon_1,\epsilon_2,\epsilon_3 \in (0,\frac{1}{2})$ with $\frac{\epsilon_1+\epsilon_2}{\epsilon_3} < \frac{1}{2}$ and the pair (A,B) is (ϵ_1,ϵ_2)-uniform. Let $A' \subseteq A$ with $|A'| \geq \epsilon_3|A|$, $B' \subseteq B$ with $|B'| \geq \epsilon_3|B|$ and denote $Z = \{(a,b) \in (A\times B) \mid aRb \not\equiv t(A,B)\}$ and $Z' = \{(a,b) \in (A'\times B') \mid aRb \not\equiv t(A,B)\}$. Then, we have:*

1. $\frac{|Z|}{|A||B|} < \epsilon_1+\epsilon_2$.
2. $\frac{|Z'|}{|A||B|} < \frac{\epsilon_1+\epsilon_2}{\epsilon_3}$.

In particular, if for some $\epsilon_0,\epsilon \in (0,\frac{1}{2})$, the pair (A,B) is ϵ_0-uniform, for $\epsilon_0 \leq \frac{\epsilon^2}{2}$, then:

a. *(A,B) is ϵ-regular.*

b. *If $A' \in [A]^{\geq \epsilon|A|}$ and $B' \in [B]^{\geq\epsilon|B|}$, then $d(A',B') < \epsilon$ or $d(A',B') \geq 1-\epsilon$.*

Proof. Let $U = \{a \in A \mid |\overline{B}_{B,a}| > \epsilon_1|A|\}$, i.e. the set of exceptional vertices $a \in A$. Then,

$$Z \subseteq U\times B \cup \bigcup_{a\in A\setminus U}\{a\}\times\overline{B}_{B,a}$$

and

$$Z' \subseteq U\times B' \cup \bigcup_{a\in A'\setminus U}\{a\}\times\overline{B}_{B,a}$$

Notice that, if $a \in A\setminus U$, then $|\overline{B}_{B,a}| < \epsilon_2|B|$, so

$$|Z| < \epsilon_1|A||B| + |A|\epsilon_2|B|$$

which can be written as

$$\frac{|Z|}{|A||B|} < \epsilon_1+\epsilon_2$$

which proves 1.. Similarly,

$$\begin{aligned}|Z'| &\leq |U||B'| + |A'| \max\{|\overline{B}_{B,a}| \mid a \notin U\} \\ &< \epsilon_1|A||B'| + |A'|\epsilon_2|B|\end{aligned}$$

By dividing both sides by $|A'||B'|$ we conclude

$$\frac{|Z'|}{|A'||B'|} < \epsilon_1\frac{|A|}{|A'|} + \epsilon_2\frac{|B|}{|B'|} \leq \frac{\epsilon_1|A|}{\epsilon_3|A|} + \frac{\epsilon_2|B|}{\epsilon_3|B|} = \frac{\epsilon_1+\epsilon_2}{\epsilon_3}$$

proving 2.. Let's now prove a. and b.. First of all, notice that:

This only works so nice when A and B are disjoint. Check what happens when they are not. Something more on the line of $d(A,B) > 1 - 4(\epsilon_1 + \epsilon_2)$

- if $t(A,B) = 1$, then $d(A,B) > 1-(\epsilon_1+\epsilon_2)$ and $d(A',B') > 1 - \frac{\epsilon_1+\epsilon_2}{\epsilon_3}$, which follows 1. and 2. respectively. Thus,

$$\begin{aligned}|d(A,B) - d(A',B')| &\leq \max\{d(A,B) - d(A',B'), d(A',B') - d(A,B)\} \\ &< \max\{1 - (1 - \frac{\epsilon_1+\epsilon_2}{\epsilon_3}), 1 - (1-\epsilon_1-\epsilon_2)\} \\ &= \frac{\epsilon_1+\epsilon_2}{\epsilon_3}\end{aligned}$$

- if $t(A,B) = 0$, similarly $d(A,B) < (\epsilon_1+\epsilon_2)$ and $d(A',B') < \frac{\epsilon_1+\epsilon_2}{\epsilon_3}$. Thus,

$$\begin{aligned}|d(A,B) - d(A',B')| &\leq \max\{d(A,B) - d(A',B'), d(A',B) - d(A,B)\} \\ &< \max\{(\epsilon_1+\epsilon_2), \frac{\epsilon_1+\epsilon_2}{\epsilon_3}\} \\ &= \frac{\epsilon_1+\epsilon_2}{\epsilon_3}\end{aligned}$$

In both cases, we have that $|d(A,B) - d(A',B')|$ is bounded by $\frac{\epsilon_1+\epsilon_2}{\epsilon_3} < \frac{1}{2}$. Also, $d(A',B')$ may only differ by $\frac{\epsilon_1+\epsilon_2}{\epsilon_3}$ with either 0 or 1. In particular, we may choose $\epsilon_3 = \epsilon$ and $\epsilon_1 = \epsilon_2 = \epsilon_0 \leq \frac{\epsilon^2}{2}$. This way, the condition $\frac{\epsilon_1+\epsilon_2}{\epsilon_3} \leq \epsilon < \frac{1}{2}$ is satisfied. We conclude that (A,B) is ϵ-regular (a.) and that $d(A',B')$ is either $< \epsilon$ or $\geq 1-\epsilon$ (b.). □

Theorem 5.20 (Theorem 5.19)**.** *For every $k_* \in \mathbb{N}$ and $\epsilon \in (0, \frac{1}{2})$ and $m > 1$, there exist $N = N(\epsilon, m, k_*)$ and $M = M(\epsilon, m, k_*)$ such that, for every finite graph G with the non-k_*-order property, and every $A \subseteq G$ with $|A| \geq N$, there is $m < \ell < M$ and a partition $\overline{A} = \langle A_i \mid i \in \{1,\dots,\ell\}\rangle$ of A such that each A_i is $\frac{\epsilon^2}{2}$-excellent, and for every $i,j \in \{1,\dots,\ell\}$,*

1. *$||A_i| - |A_j|| \leq 1$.*
2. *(A_i, A_j) is ϵ-regular, and moreover if $B_i \in [A_i]^{\geq\epsilon|A_i|}$ and $B_j \in [A_j]^{\geq\epsilon|A_j|}$, then either $d(B_i,B_j) < \epsilon$ or $d(B_i,B_j) \geq 1-\epsilon$.*
3. *If $\epsilon \leq \min(\frac{1}{5}, \frac{1}{2^{k_{**}}})$, then $M \leq \max(\lceil\frac{12}{\epsilon}\rceil^{k_{**}+1}, 4m)$.*

Proof. If $\epsilon \leq \min(\frac{1}{5}, \frac{1}{2^{k_{**}}})$, then we can apply Theorem 5.17 to A with $\frac{\epsilon^2}{2}$, and then use Lemma 5.19 to replace the $\frac{\epsilon^2}{2}$-uniformity of pairs by ϵ-regularity. Otherwise, to get 1. and 2., just do the same process for some $\epsilon' = \min(\frac{1}{5}, \frac{1}{2^{k_{**}}}) \leq \epsilon$. Then, since regularity is monotone, we get the wanted ϵ-regularity from the resulting ϵ'-regularity. In this last case, the bound on M is $M \leq \max(\lceil\frac{12}{\epsilon'}\rceil^{k_{**}+1}, 4m)$. □

Remark 5.21. By Theorem 3.15, we have that $k_{**} \leq 2^{k_*+1} - 2$ in the context of the non-k_*-order property. Thus, the bound on the number of parts M can clearly be reformulated as a function of only k_*, ϵ and m:

$$M \leq \max(\left\lceil \frac{12}{\epsilon} \right\rceil^{2^{k_*+1}-1}, 4m)$$

6. Section 6

Definition 6.1. A graph H is *γ-unavoidable* in a graph G if no adding or removing of up to $\epsilon\binom{|G|}{2}$ edges in G results in H not appearing as an induced subgraph of G.

Definition 6.2. A graph H is *η-abundant* in a graph G if G contains at least $\eta|G|^{|H|}$ induced copies of H.

Lemma 6.3 (Lemma 3.1 of "Efficient Testing of Large graphs", *Alon et al.*)**.** *Let $\epsilon \leq \epsilon' < \frac{1}{2}$ and $\delta \in (0,1)$. If (A,B) is an ϵ-regular pair with density δ, and $A' \in [A]^{\geq \epsilon'|A|}$, $B' \in [B]^{\geq \epsilon'|B|}$, then (A',B') is an $(\frac{\epsilon}{\epsilon'})$-regular pair with density at least $\delta - \epsilon$ and at most $\delta + \epsilon$.*

Proof. Let $A'' \subseteq A' \subseteq A$, $B'' \subseteq B' \subseteq B$ be such that

$$\begin{aligned}|A''| &\geq \frac{\epsilon}{\epsilon'}|A'| \geq \frac{\epsilon}{\epsilon'}\epsilon'|A| = \epsilon|A| \text{ and}\\ |B''| &\geq \frac{\epsilon}{\epsilon'}|B'| \geq \frac{\epsilon}{\epsilon'}\epsilon'|B| = \epsilon|B|\end{aligned}$$

By ϵ-regularity of (A,B), $|d(A,B) - d(A'',B'')| < \epsilon$. Thus,

$$\begin{aligned}|d(A',B') - d(A'',B'')| &= |d(A',B') - d(A,B) + d(A,B) - d(A'',B'')|\\ &\leq |d(A',B') - d(A,B)| + |d(A,B) - d(A'',B'')|\\ &< 2\epsilon \leq \frac{\epsilon}{\epsilon'}\end{aligned}$$

This proves the $(\frac{\epsilon}{\epsilon'})$-regularity of (A',B').

Also, since (A,B) is ϵ-regular, $|d(A,B) - d(A',B')| < \epsilon$, and thus,

$$\delta - \epsilon < d(A',B') < \delta + \epsilon$$

□

Lemma 6.4 (Lemma 3.2 of "Efficient Testing of Large graphs", *Alon et al.*)**.** *For every $\delta \in (0,1)$ and $\ell > 0$ there exist $\epsilon = \epsilon(\delta,\ell)$ and $\eta = \eta(\eta,\ell)$ satisfying the following property:*

Let H be a graph with vertices $v_1,\dots,v_\ell$ and let $V_1,\dots,V_\ell$ be an ℓ-tuple of disjoint sets of vertices of a graph G such that for every $1 \leq i < i' \leq \ell$, the pair $(V_i, V_{i'})$ is ϵ-regular, with density at least δ if $v_iv_{i'}$ is an edge of H, and at most $1-\delta$ if $v_iv_{i'}$ is not an edge of H. Then, at least $\eta\prod_{i=1}^{\ell}|V_i|$ of ℓ-tuples $w_1 \in V_1,\dots,w_\ell \in V_\ell$ span induced copies of H where w_i plays the role of v_i.

Proof. Without loss of generality, we assume that H is the complete graph, since we can simply replace each non-edge $v_iv_{i'}$ of H with an edge by exchanging all edges and non-edges between V_i and $V_{i'}$.

We prove the lemma by induction on ℓ. The case $k=1$ is trivial, and the number of induced copies of H is $|V_1|$, so $\eta(\delta,1) = 1$ and $\epsilon(\delta,1) = 1$ (No regularity needed if no pairs). The I.H. is that the values $\eta(\delta,\ell-1)$ and $\epsilon(\delta,\ell-1)$ exist and are known for all ℓ. We proceed to prove that the following values η and ϵ hold:

$$\begin{aligned}\epsilon = \epsilon(\delta,\ell) &= \min(\frac{1}{2\ell-2}, \frac{1}{2}\delta\epsilon(\frac{1}{2}\delta,\ell-1))\\ \eta = \eta(\delta,\ell) &= \frac{1}{2}(\delta-\epsilon)^{\ell-1}\eta(\frac{1}{2}\delta,\ell-1)\end{aligned}$$

For each $1 < i \leq \ell$, the number of vertices of V_1 which have less then $(\delta - \epsilon)|V_i|$ neighbors in V_i is less then $\epsilon|V_i|$. Otherwise, the set of such vertices, say $U \in [V_1]^{\geq \epsilon|V_1|}$ together with V_i would form a subpair (U, V_i) with density $< \delta - \epsilon$ which, by Lemma 6.3 contradicts the ϵ-regularity of the pair (V_1, V_i).

Therefore, at least $(1 - (\ell - 1)\epsilon)|V_1|$ of the vertices of V_1 have at least $(\delta - \epsilon)|V_i|$ neighbors in V_i for all $1 < i \leq \ell$. In particular, since $\epsilon \leq \frac{1}{2\ell - 2}$ we have that $(\ell - 1)\epsilon \leq \frac{1}{2}$ and then $1 - (\ell - 1)\epsilon \geq \frac{1}{2}$, so at least half of the vertices of V_1 satisfy the above condition.

For each such vertex $w_1 \in V_1$, let V_i' denote the subset of vertices of V_i which are neighbors of w_1. Since *epsilon* $\leq \frac{1}{2}\delta$, Lemma 6.3 implies that for all $1 < i < i' \leq \ell$, the pair $(V_i', V_{i'}')$ is $(\frac{\epsilon}{\delta - \epsilon})$-regular, and given that $(\frac{\epsilon}{\delta - \epsilon}) \leq (\frac{2\epsilon}{\delta}) \leq \epsilon(\frac{1}{2}\delta, \ell - 1)$, it is $\epsilon(\frac{1}{2}\delta, \ell - 1)$-regular. Also, it has density at least $\delta - \epsilon \geq \frac{1}{2}\delta$. By the induction hypothesis, we have at least

$$\eta(\frac{1}{2}\delta, \ell - 1) \prod_{i=2}^{\ell} |V_i'| \geq \eta(\frac{1}{2}\delta, \ell - 1) \prod_{i=2}^{\ell} (\delta - \epsilon)|V_i|$$

possible choices of $w_2 \in V_2, \dots, w_\ell \in V_\ell$ such that the induced subgraph spanned by $w_1, \dots, w_\ell$ is complete. Since there are at least $\frac{1}{2}|V_1|$ vertices w_1 which satisfy the above condition, the chosen values of η satisfies the lemma, and we are done. □

Remark 6.5. The non-recursive form of ϵ and η for $\ell > 1$ is:

$$\epsilon(\delta, \ell) = 2(\frac{\delta}{4})^{\ell - 1}$$

$$\eta(\delta, \ell) \geq \frac{1}{2^{\frac{(\ell+2)(\ell+1)}{2} - 4}} \delta^{\frac{\ell(\ell-1)}{2}}$$

Theorem 6.6. *For every k_*, γ, ℓ there is a $\delta(k_*, \gamma, \ell)$ such that if H is a graph with ℓ vertices, G has the non-k_*-order property and H is γ-unavoidable in G, then H is δ-abundant in G.*

Proof. Apply Theorem 5.20 to G with $\epsilon = \min(\frac{\sqrt{\gamma}}{2}, \frac{\epsilon_{6.4}(1 - \frac{\sqrt{\gamma}}{2}, \ell)}{\ell})$, k_* and $m = 0$. We have a partition $\overline{A} = \{A_i \mid i \in \{1, \dots, m_+\}\}$ into $m_* \leq M$ disjoint parts with,

$$M \leq \left\lceil 12 \max(\frac{2}{\sqrt{\gamma}}, \frac{\ell}{\epsilon_{6.4}(1 - \frac{\sqrt{\gamma}}{2}, \ell)}) \right\rceil^{2^{k_*+1} - 1}$$

such that all pairs of parts are ϵ-regular, and self-pairs are 4ϵ-regular. Also, by Remark 5.4 and $\frac{\epsilon^2}{2}$-excellence of the parts, pairs have density at most ϵ^2 or at least $1 - \epsilon^2$.

Maybe make a remark in Theorem 5.19

Now, we randomly partition each part A_i into ℓ equitable subparts $A_{i,j}$. By Lemma 6.3, each pair of such subparts is $\ell\epsilon$-regular. On the other hand, Theorem 5.20 guarantees that such pairs have density at most ϵ or at least $1 - \epsilon$.

Next, we modify the graph G into G' by only adding and removing no more than $\gamma\binom{|G|}{2}$ edges:

- For each pair of parts (A_{i_1}, A_{i_2}) with $i_1 \neq i_2$, if the pair's density is at most ϵ^2, we remove all edges between A_{i_1} and A_{i_2}. Otherwise, the pair's density is at least $1 - \epsilon^2$, and we add all remaining edges. This changes at most a fraction ϵ^2 of the edges between (disjoint) parts.
- For each self-pair (A_i, A_i), if the pair's density is at most $4\epsilon^2$ again we remove all edges in A_i. Otherwise, the pair's density is at least $1 - 4\epsilon^2$, and we add all remaining edges. This changes at most a fraction $4\epsilon^2$ of the edges in self-pairs.

The resulting graph G' differs from G in at most $4\epsilon^2\binom{|G|}{2} \leq \gamma\binom{|G|}{2}$ edges. Thus, the γ-unavoidability of H in G ensures that there is still a copy of H in G'. Denote its vertices $v_{i_1}, \dots, v_{i_\ell}$, choosing $i_1, \dots, i_\ell$ such that $v_{i_1} \in A_{i_1,1}, \dots, v_{i_\ell} \in A_{i_\ell,\ell}$. Notice that $A_{i_1,1}, \dots, A_{i_\ell,\ell}$ satisfy the conditions of Lemma 6.4 with $\delta_{6.4} = 1 - \frac{\sqrt{\gamma}}{2}$:

- Each subpair $(A_{i_j,j}, A_{i_{j'},j'})$ with $j \neq j'$ is $\ell\epsilon$-regular, and since $\epsilon \leq \frac{\epsilon_{6.4}(1-\frac{\sqrt{\gamma}}{2},\ell)}{\ell}$, in particular is $\epsilon_{6.4}(1 - \frac{\sqrt{\gamma}}{2}, \ell)$-regular.
- For each $i_j \neq i_{j'}$, if $v_{i_j}v_{i_{j'}}$ is an edge of G then, by construction of G', the subpair $(A_{i_j,j}, A_{i_{j'},j'})$ has density at least $1-\epsilon \leq 1-\frac{\sqrt{\gamma}}{2}$, and if $v_{i_j}v_{i_{j'}}$ is not an edge of G, the subpair $(A_{i_j,j}, A_{i_{j'},j'})$ has density at most $\epsilon \geq 1 - (1 - \frac{\sqrt{\gamma}}{2})$

The lemma guarantees that there are at least $\eta_{6.4}(1 - \frac{\sqrt{\gamma}}{2}, \ell)\prod_{j=1}^{\ell}\{A_{i_j}, j\}$ copies of H in G. The fraction of induced copies of H in G is at least

$$\frac{\eta_{6.4}(1 - \frac{\sqrt{\gamma}}{2}, \ell)\prod_{j=1}^{\ell}\{A_{i_j}, j\}}{n^\ell} \geq \eta_{6.4}(1 - \frac{\sqrt{\gamma}}{2}, \ell)(\frac{\frac{n}{M\cdot\ell}}{n})^\ell = \eta_{6.4}(1 - \frac{\sqrt{\gamma}}{2}, \ell)(M\cdot\ell)^{-\ell} =: \eta$$

and H is at least η-abundant in G. □

Remark 6.7. A more explicit lower bound for η only depending on γ, k_* and ℓ is:

$$\eta \geq \frac{1}{2^{\frac{(\ell+2)(\ell+1)}{2}-4}}(1 - \frac{\sqrt{\gamma}}{2})^{\frac{\ell(\ell-1)}{2}}(\frac{1}{24}\min(\frac{\sqrt{\gamma}}{2}, \frac{\epsilon(1 - \frac{\sqrt{\gamma}}{2}, \ell)}{\ell}))^{\ell(2^{k_*+1}-1)}(\frac{1}{\ell})^\ell$$

References

[1] Noga Alon et al. "Efficient testing of large graphs". In: *Combinatorica* 20.4 (2000), pp. 451–476.

[2] Maryanthe Malliaris and Saharon Shelah. "Regularity lemmas for stable graphs". In: *Transactions of the American Mathematical Society* 366.3 (2014), pp. 1551–1585.

A. Title of the appendix

You can include here an appendix with details that can not be included in the core of the document. You should reference the sections in this appendix in the core document.

B. Title of the appendix

Second appendix.