Universitat Politècnica de Catalunya
Facultat de Matemàtiques i Estadística

Master in Advanced Mathematics and Mathematical Engineering

Master's thesis

Why the non-monotonicity of excellence f***** up my life

Severino Da Dalt

Supervised by (Lluis Vena Cros)

February, 2025

Thanks to...

Abstract

This should be an abstract in english, up to 1000 characters.

Keywords

regularity, stable graphs, graph theory, ...

1. Introduction

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2. Section 2

Definition 2.1 (Truth value)**.** Let G be a graph. For any $A \subseteq G$ and $b \in G$, we say that

$$t(A,b) = \begin{cases} 0 & \text{if } |\{a \in A \mid aRb\}| < |\{a \in A \mid \neg aRb\}| \\ 1 & \text{otherwise} \end{cases}$$

is the *truth value* of A with respect to b.

Extra notation:

- $B_{A,b} = \{a \in A \mid aRb \equiv t(A,b)\}$.
- $\overline{B}_{A,b} = \{a \in A \mid aRb \not\equiv t(A,b)\}$.
- $B^{+}_{A,b} = \{a \in A \mid aRb\}$.
- $B^{-}_{A,b} = \{a \in A \mid \neg aRb\}$.

With this notation, notice that either $t(A,b) = 1$ and thus $B_{A,b} = B^{+}_{A,b}$, or $t(A,b) = 0$ and $B_{A,b} = B^{-}_{A,b}$.

Lemma 2.2 (Claim 2.6)**.** *Let G be a graph with the non-k-order property. Then, for any finite $A \subseteq G$,*

$$|\{\{a \in A \mid aRb\} \mid b \in G\}| \leq \sum_{i \leq k} \binom{|A|}{i} \leq |A|^k$$

Corollary 2.3 (Claim 2.6.1). *Let G be a graph with the non-k-order property. Then:*

1. *For any finite $A \subseteq G$*
$$|\{\{a \in A \mid \neg aRb\} \mid b \in G\}| \leq \sum_{i \leq k} \binom{|A|}{i} \leq |A|^k$$

2. *For any finite $A \subseteq G$*
$$|\{\{a \in A \mid \neg aRb \equiv t(A, b)\} \mid b \in G\}| \leq \sum_{i \leq k} \binom{|A|}{i} \leq |A|^k$$

Proof. 1. First of all, notice that $B^+_{A,b} = B - B^- A, b$, since by definition they are complementary. Thus, for any $b, b' \in G$, $B^+_{A,b} = B^+_{A,b'} \Leftrightarrow B^-_{A,b} = B^-_{A,b'}$. It follows that
$$\left|\left\{B^-_{A,b} \mid b \in G\right\}\right| = \left|\left\{B^+_{A,b} \mid b \in G\right\}\right| \leq \sum_{i \leq k} \binom{|A|}{i} \leq |A|^k$$

where the last inequalities come from Lemma 2.2.

2. Consider the following map:
$$\begin{aligned} \pi : \{B_{A,b} \mid b \in G\} &\longrightarrow \left\{B^+_{A,b} \mid b \in G\right\} \\ B_{A,b} &\longmapsto B^+_{A,b} \end{aligned}$$

We show that the map π is injective. Let $b, b' \in G$ such that $B_{A,b} = B_{A,b'}$. Then, $t(A, b) = t(A, b')$, otherwise (suppose wlog that $t(A, b) = 1$ and $t(A, b') = 0$), we would have
$$\left|B^-_{A,b'}\right| > \left|B^+_{A,b'}\right| = \left|B^+_{A,b}\right| \geq \left|B^-_{A,b}\right| = \left|B^-_{A,b'}\right|$$

which is a contradiction. Then:

- if $t(A, b) = t(A, b') = 1$, we have that $B_{A,b} = B^+_{A,b} = B^+_{A,b'} = B_{A,b'}$.
- if $t(A, b) = t(A, b') = 0$, we have that $B_{A,b} = B^-_{A,b} = A \setminus B^+_{A,b} = A \setminus B^+_{A,b'} = B^-_{A,b'} = B_{A,b'}$.

This proves that π is injective. To conclude,
$$|\{B_{A,b} \mid b \in G\}| \leq \left|\left\{B^+_{A,b} \mid b \in G\right\}\right| \leq \sum_{i \leq k} \binom{|A|}{i} \leq |A|^k$$

This concludes the proof. Notice that in particular π is a bijection, but this is not needed for the proof.

□

Definition 2.4 (Definition 2.11). Suppose G is a finite graph with the non-k_*-property. We denote the *tree bound* $k_{**} = k_{**}(G) < \omega$ as the minimal value such that there do not exist sequences $\bar{a} = \left\langle a_\eta \mid \eta \in [2]^{k_{**}} \right\rangle$ and $\bar{b} = \left\langle b_\rho \mid \rho \in [2]^{<k_{**}} \right\rangle$ of elements of G satisfying that if $\rho \frown \langle \ell \rangle \triangleleft \eta$, then $(a_\eta R b_\rho) \equiv (\ell = 1)$.

3. Section 3

4. Section 5

Definition 4.1 (Definition 5.2(a))**.** Let G be a finite graph with the non-k_*-property. We say that $A \subseteq G$ is a ϵ-*good* when for every $b \in G$ for some truth value $t = t(b, A) \in \{0, 1\}$ we have $|\{a \in A \mid (aRb) \not\equiv t\}| < \epsilon|A|$.

Definition 4.2 (Definition 5.2(b))**.** Let G be a finite graph with the non-k_*-property. We say that $A \subseteq G$ is (ϵ, ζ)-*excellent* when A is ϵ-good and, if B is ζ-good, then for some truth value $t = t(B, A)$, $|\{a \in A \mid t(a, B) \neq t(A, B)\}| < \epsilon|A|$.

Remark 4.3. Notice that, if A is (ϵ, ϵ')-excellent and $B \subseteq G$ an is ϵ'-good set, then the number of exceptional edges between A and B, i.e. these vertex pairs that do not follow $t(A, B)$, is relatively small:

$$|\{\text{Exceptional edges between } A \text{ and } B\}| < \epsilon|A||B| + (1-\epsilon)|A|\epsilon'|B| = \left(\epsilon + (1-\epsilon)\,\epsilon'\right)|A||B|$$

Lemma 4.4 (Claim 5.4)*. Let G be a finite graph with the non-k_*-order property. Let $\zeta < \frac{1}{2^{k_{**}}}$, $\epsilon \in \left\{0, \frac{1}{2}\right\}$. Then, for every $A \subseteq G$ with $|A| \geq \frac{1}{\epsilon^{k_{**}}}$ there exists (ϵ, ζ)-excellent subset $A' \subseteq A$ such that $|A'| \geq \epsilon^{k_{**}-1}|A|$.*

Proof. Suppose the converse. We will use this fact to build sets $\left\{b_\eta \mid \eta \in [2]^{<k_{**}}\right\}$ and $\left\{A_\eta \mid \eta \in [2]^{\leq k_{**}}\right\}$ on induction over $k < k_{**}$, where $k = |\eta|$, satisfying:

1. $A_{\langle\cdot\rangle} = A$.
2. B_η is an η-good set witnessing that A_η is not (ϵ, ζ)-excellent, for $k < k_{**}$.
3. $A_{\eta\frown\langle i\rangle} = \{a \in A_\eta \mid t(a, B_\eta) \equiv i\}$ for all $i \in \{0, 1\}$ and $k < k_{**}$.
4. $|A_{\eta\frown\langle i\rangle}| \geq \epsilon|A_\eta|$ for all $i \in \{0, 1\}$ and $k < k_{**}$.
5. $|A_\eta| \geq \epsilon^k|A|$, for $k \leq k_{**}$.
6. $A_\eta = A_{\eta\frown\langle 0\rangle} \sqcup A_{\eta\frown\langle 1\rangle}$, for $k < k_{**}$.
7. $\overline{A_k} = \left\{A_\eta \mid \eta \in [2]^k\right\}$ is a partition of A, for $k \leq k_{**}$.

First of all, notice that at each step, the non-(ϵ, ζ)-excellence of A_η comes by IH from (1) or (5). This allows the existence of B_η in (2). Notice that $t(a, B_\eta)$ in (3) is well-defined since B_η is ζ-good. Also, the non-(ϵ, ζ)-excellence of A_η allows (4). Finally, by definition (3), we have the disjoint union (6) which by itself ensures the partition (7). Now, our goal is to build two sequences $\left\{b_\eta \mid \eta \in [2]^{<k_{**}}\right\}$ and $\left\{a_\eta \mid \eta \in [2]^{k_{**}}\right\}$ to contradict the tree bound k_{**}. First of all, notice that, for $\eta \in [2]^{k_{**}}$

$$|A_\eta| \geq \epsilon^{k_{**}}|A| \geq \epsilon^{k_{**}}\frac{1}{\epsilon^{k_{**}}} = 1$$

So $A_\eta \neq \emptyset$. For each $\eta \in [2]^{k_{**}}$ we may choose an $a_\eta \in A_\eta$. Now, for $\nu \in [2]^{<k_{**}}$ and $\eta \in [2]^{k_{**}}$ such that $\nu \triangleleft \eta$, let

$$U_{\nu,\eta} = \{b \in B_\nu \mid (a_\eta R b) \not\equiv t(a_\eta, B_\nu)\}$$

be the subset of elements of B_ν that do not relate with a_η in the expected way. By ζ-goodness of B_ν, $|U_{\nu,\eta}| < \zeta|B_\nu|$, and thus for every $\eta \in [2]^{k_{**}}$,

$$\left|\bigcup\left\{U_{\nu,\eta} \mid \nu \triangleleft \eta \in [2]^{k_{**}}\right\}\right| < 2^{k_{**}}\zeta|B_\nu| < |B_\nu|$$

We may choose $b_\nu \in B_\nu \backslash \bigcup\left\{U_{\nu,\eta} \mid \nu \triangleleft \eta \in [2]^{k_{**}}\right\}$, for all $\nu \in [2]^{<k_{**}}$. Finally, the sequences $\left\langle a_\eta \mid \eta \in [2]^{k_{**}}\right\rangle$ and $\left\langle b_\nu \mid \nu \in [2]^{<k_{**}}\right\rangle$ satisfy that $\forall \eta, \nu$ such that $\nu \frown \langle i \rangle \triangleleft \eta$, $(a_\eta R b_\nu)^i$ by (3) and (6). This contradicts the definition of tree bound k_{**} (2.4). □

Lemma 4.5 (Claim 5.4.1). *Let G be a finite graph with the non-k_*-order property. Let $\zeta < \frac{1}{2^{k_{**}}}$, $\epsilon \in \left\{0, \frac{1}{2}\right\}$. Let $\langle m\ell \mid \ell \in [0, k_{**}]\rangle$ be a decreasing sequence of natural numbers such that $\epsilon m_\ell \geq m_{\ell+1}$ for all $\ell \in [0, k_{**}-1]$, $m_{k_{**}} \geq 1$, and $m_{k_{**}-1} > k_{**}$. Then, for every $A \subseteq G$ with $|A| \geq m_0$ there exists $\left(\frac{m_{\ell+1}}{m_\ell}, \zeta\right)$-excellent subset $A' \subseteq A$ such that $|A'| = m_\ell$ for some $\ell \in [0, k_{**}-1]$.*

Proof. Suppose the converse. We will use this fact to build sets $\left\{b_\eta \mid \eta \in [2]^{<k_{**}}\right\}$ and $\left\{A_\eta \mid \eta \in [2]^{\leq k_{**}}\right\}$ on induction over $k < k_{**}$, where $k = |\eta|$, satisfying:

1. $A_{\langle\cdot\rangle} \subseteq A$, with $|A|_{\langle\cdot\rangle} = m_0$.
2. B_η is an η-good set witnessing that A_η is not $\left(\frac{m_{k+1}}{m_k}, \zeta\right)$-excellent, for all $k < k_{**}$.
3. $A_{\eta\frown\langle i\rangle} = \{a \in A_\eta \mid t(a, B_\eta) \equiv i\}$ for all $i \in \{0,1\}$ and $k < k_{**}$.
4. $|A_\eta| = m_k$, for all $k \leq k_{**}$.
5. $A_{\eta\frown\langle 0\rangle} \sqcup A_{\eta\frown\langle 1\rangle} \subseteq A_\eta$, for all $k < k_{**}$.
6. $\overline{A_k} = \left\{A_\eta \mid \eta \in [2]^k\right\}$ is a partition of a subset of A, for all $k \leq k_{**}$.

Notice that, by (1) and (4), the size of A_η is m_k, so by IH none of the sets A_η is $\left(\frac{m_{k+1}}{m_k}, \zeta\right)$-excellent. Then, B_η in (2) is well-defined. Also, by η-goodness of B_η, $t(a, B_\eta)$ in (3) is well-defined. Then, since B_η is witnessing the non-$\left(\frac{m_{k+1}}{m_k}, \zeta\right)$-excellence of A_η, we have that $|A_{\eta\frown\langle i\rangle}| \geq \frac{m_{k+1}}{m_k} m_k = m_{k+1}$ for all $i \in \{0,1\}$, satisfying (4). Finally, by definition (3), we have the disjoint union (5) which by itself ensures (6). Now, our goal is to build two sequences $\left\{b_\eta \mid \eta \in [2]^{<k_{**}}\right\}$ and $\left\{a_\eta \mid \eta \in [2]^{k_{**}}\right\}$ to contradict the tree bound k_{**}. First of all, notice that, for $\eta \in [2]^{k_{**}}$

$$|A_\eta| = m_k \geq m_{k_{**}} \geq 1$$

so $A_\eta \neq \emptyset$. For each $\eta \in [2]^{k_{**}}$ we may choose an $a_\eta \in A_\eta$. Now, for $\nu \in [2]^{<k_{**}}$ and $\eta \in [2]^{k_{**}}$ such that $\nu \triangleleft \eta$, let

$$U_{\nu,\eta} = \{b \in B_\nu \mid (a_\eta R b) \not\equiv t(a_\eta, B_\nu)\}$$

be the subset of elements of B_ν that do not relate with a_η in the expected way. By ζ-goodness of B_ν, $|U_{\nu,\eta}| < \zeta|B_\nu|$, and thus for every $\eta \in [2]^{k_{**}}$,

$$\left|\bigcup\left\{U_{\nu,\eta} \mid \nu \triangleleft \eta \in [2]^{k_{**}}\right\}\right| < 2^{k_{**}}\zeta|B_\nu| < |B_\nu|$$

We may choose $b_\nu \in B_\nu \backslash \bigcup\left\{U_{\nu,\eta} \mid \nu \triangleleft \eta \in [2]^{k_{**}}\right\}$, for all $\nu \in [2]^{<k_{**}}$. Finally, the sequences $\left\langle a_\eta \mid \eta \in [2]^{k_{**}}\right\rangle$ and $\left\langle b_\nu \mid \nu \in [2]^{<k_{**}}\right\rangle$ satisfy that $\forall \eta, \nu$ such that $\nu \frown \langle i \rangle \triangleleft \eta$, $(a_\eta R b_\nu)^i$, which follows (3). This contradicts the definition of tree bound k_{**} (2.4). □

Lemma 4.6 (Fact 5.9). *Let $p, q \in (0,1)$. Let A be a set of size n, $B \subseteq A$ a subset of size $p|A|$, and $A' \subseteq A$ a random subset of size $\geq q|A|$. Then, for $\zeta > 0$,*

$$P\left(\frac{|A' \cap B|}{|A'|} \in \left(\frac{|B|}{|A|} - \zeta, \frac{|B|}{|A|} + \zeta\right)\right)$$

can be modeled by a random variable which is asymptotically normally distributed when $n \to +\infty$.

Lemma 4.7 (Fact 5.10). *Let A be a set of events measured with a probability P_A. Let S a family of subsets of A, which are measurable with P_A. Let $A_r = \{a_1, \dots, a_r\} \subseteq A$ be a random sample of size r. For each $B \in S$, we may define $v_B^{A_r}$ the relative frequency of events of B in A_r, i.e.,*

$$v_B^{A_r} = \frac{P_A(A_r \cap B)}{P_A(A_r)}$$

Let

$$\pi^{A_r} = \sup_{B \in S} \left| v_B^{A_r} - P_A(B) \right|$$

i.e. the upperbound of error of $v_B^{A_r}$ as an approximation of $P_A(B)$. Also let $\Delta^s(A_r)$ be the number of subsets of A_r induced by sets of S ($B \in S$ induces $B \cap A_r \subseteq A_r$), i.e.

$$\Delta^s(A_r) = |\{B \cap A_r \mid B \in S\}|$$

Finally, let

$$m^S(r) = \max_{C \in \binom{A}{r}} \Delta^s(C)$$

Then, if there exists a finite $k > 0$ such that $m^s(r) \leq r^k + 1$ for all $r > 0$, we have that, for all $\epsilon > 0$,

$$\lim_{r \to +\infty} P_A\left(\pi^{A_r} > \epsilon\right) = 0$$

Remark 4.8 (Fact 5.12). If there exists $k > 0$ such that $m^s(r) \leq r^k + 1$ and r satisfies:

$$r \geq \frac{16}{\zeta^2}\left(k \log \frac{16k}{\zeta^2} - \log \frac{\eta}{4}\right)^{k+1}$$

for some $\eta > 0$, then

$$P\left(\pi^{A_r} < \zeta\right) \geq 1 - \eta$$

In particular, if we suppose that all events in A are equiprobable and sampled without replacement, then

$$P\left(\forall B \in S,\ \frac{|A_r \cap B|}{|A_r|} \in \left(\frac{|B|}{|A|} - \zeta, \frac{|B|}{|A|} + \zeta\right)\right) \geq 1 - \eta$$

or in other words, for all but a fraction η of all possible choices of A_r, we have that

$$\forall B \in S,\ \frac{|A_r \cap B|}{|A_r|} \in \left(\frac{|B|}{|A|} - \zeta, \frac{|B|}{|A|} + \zeta\right)$$

Lemma 4.9 (Claim 5.13). *Let G be a finite graph with the non-k_*-order property. Then:*

(a) For every $\epsilon \in \left(0, \frac{1}{2}\right)$, $\zeta \in \left(0, \frac{1}{2} - \epsilon\right)$ and $\xi \in (0, 1)$ there is $N_1 = N_1(\epsilon, \zeta, \xi)$ such that for all $n > N_1$, if $A \subseteq G$ is an ϵ-good subset of size n, and $n \geq m \geq \log\log n$, then if we choose a random subset $A' \subseteq A$ of size m, it is $(\epsilon + \zeta)$-good with probability $1 - \xi$.

(b) Moreover, such A' satisfies $t(b, A') = t(b, A)$ for all $b \in G$.

(c) For every $\zeta \in \left(0, \frac{1}{2}\right)$ and $\zeta' < \zeta$, there is $\epsilon_1 = \epsilon_1(\zeta, \zeta')$ such that for every $\epsilon < \epsilon' \leq \epsilon_1$, if

- *$A \subseteq G$ is (ϵ, ϵ')-excellent.*
- *$A' \subseteq A$ is $(\epsilon + \zeta')$-good.*

then, A' is $(\epsilon + \zeta, \epsilon')$-excellent.

(d) For all $\zeta \in \left(0, \frac{1}{2}\right)$, $\zeta' < \zeta$, $r \geq 1$ and for all $\epsilon < \epsilon'$ small enough (in the sense of the previous point) there exists $N_2 = N_2(\epsilon, \zeta', r)$ such that, if $|A| = n > N_2$, r divides n and A is (ϵ, ϵ')-excellent, there exists a partition into r disjoint pieces of equal size, each of which is $(\epsilon + \zeta, \epsilon')$-excellent.

Proof. (a) For each $b \in G$ we say that $\overline{B}_{A,b}$ is *exceptional* if $\left|\overline{B}_{A,b}\right| \geq \epsilon |A'|$. Notice that, if we prove that, with probability $1 - \xi$, A' satisfies that for all exceptional $\overline{B}_{A,b}$:

$$\frac{\left|A' \cap \overline{B}_{A,b}\right|}{|A'|} \in \left(\frac{\left|\overline{B}_{A,b}\right|}{|A|} - \zeta, \frac{\left|\overline{B}_{A,b}\right|}{|A|} + \zeta\right)$$

then, with the same probability:

$$\left|A' \cap \overline{B}_{A,b}\right| < \left(\frac{\left|\overline{B}_{A,b}\right|}{|A|} + \zeta\right)|A'| < (\epsilon + \zeta)|A'| \tag{1}$$

and we are done.

By Lemma 4.6, for $n = |A|$ large enough, we can approximate sampling a set of size m from A, with m i.i.d. random variables $x_1, \dots, x_m$, where each x_i picks a vertex uniformly at random from A. Let $S := \left\{\text{Exceptional } \overline{B}_{A,b}\right\}$. Since G has the non-k_*-order property, we can apply Lemma 2.3 to $G_{2.3} = A$ and $A_{2.3} = A'$, which gets us that:

$$|S| \leq \left|\left\{\left\{a \in A' \mid aRb \not\equiv t(A', b)\right\} \mid b \in G\right\}\right| \leq |A'|^{k_*}$$

Then,

$$m^s(\ell) \leq |S| \leq |A'|^{k_*} \leq \ell^{k_*} \leq \ell^{k_*} + 1 \quad \forall \ell \geq |A'|$$

Notice that this is enough to satisfy the conditions of Lemma 4.7: For each $\ell < |A'|$, let k_l be the smallest integer such that $m^s(\ell) \leq \ell^{k_l} + 1$. Since there are finitely many of them, we can take the maximum $k_{\max} = \max\left\{k_1, \dots, k_{|A'|-1}, k_*\right\}$, which satisfies

$$m^s(\ell) \leq \ell^{k_{\max}} + 1 \quad \forall \ell$$

So we conclude equation (1), which by itself is sufficient to prove A' is $(\epsilon + \zeta)$-good.

(b) We proved in (a) that, with probability $1 - \xi$, A' satisfies that for all $b \in G$:

$$|S| \leq \left|\left\{\left\{a \in A' \mid aRb \not\equiv t(A', b)\right\} \mid b \in G\right\}\right| \leq |A'|^{k_*}$$

We want to prove that these sets A' that satisfy the previous equation also satisfy $t(b, A') = t(b, A)$ for all $b \in G$. Now, for any $b \in G$:

- if $|B_{A,b}| < \epsilon |A'|$, then
$$\left\{a \in A' \mid aRb \not\equiv t(b,A)\right\} \leq |B_{A,b}| < \epsilon \left|A'\right| (\epsilon + \zeta) \left|A'\right|$$
- if $|B_{A,b}| \geq \epsilon |A'|$, then by (a) we have that:
$$\left\{A' \cap B_{A,b}\right\} < \frac{|B_{A,b}|\,|A'|}{|A|} + \zeta \left|A'\right| = (\epsilon + \zeta) \left|A'\right|$$
so $\{a \in A' \mid aRb \not\equiv t(b,A)\} < (\epsilon + \zeta)\,|A'|$.

So, in both cases we have that $t(b,A') = t(b,A)$.

(c) Let $B \subseteq G$ be an ϵ'-good set. We first upperbound the number of exceptional vertices of B with respect to A':
$$\begin{aligned}\left|\left\{b \in B \mid t(b,A') \not\equiv t(B,A)\right\}\right| &= |\{b \in B \mid t(b,A) \not\equiv t(B,A)\}| \\ &\leq \frac{(\epsilon + (1-\epsilon)\,\epsilon')\,|A|\,|B|}{(1-\epsilon)\,|A|} \\ &= \left(\epsilon' + \frac{\epsilon}{1-\epsilon}\right)|B|\end{aligned}$$
The first equality follows (b), and the first inequality follows from remark (4.3) for the numerator, and taking the worst case of only $(1-\epsilon)\,|A|$ exceptional edges per exceptional $b \in B$ (considering that A is ϵ-good).

Now, let Q be the set of exceptional vertices of A' with respect to B, i.e.:
$$Q = \left\{a \in A' \mid t(a,B) \not\equiv t(A,B)\right\}$$
We want to double-count the number of exceptional edges between Q and B. On one hand, we have that:
$$|\{(a,b) \in Q \times B \mid t(a,b) \not\equiv t(A,B)\}| < \left(\epsilon' + \frac{\epsilon}{1-\epsilon}\right)|B|\,|Q| + \left(1 - \epsilon' - \frac{\epsilon}{1-\epsilon}\right)|B|\,(\epsilon + \zeta')\,\left|A'\right|$$
The first term is the maximum number of exceptional edges associated to exceptional $b \in B$ (considering all edges exceptional), while the second term bounds the number of exceptional edges of non-exceptional $b \in B$, using the fact that A' is $(\epsilon + \zeta')$-good.

On the other hand, we have that:
$$|\{(a,b) \in Q \times B \mid t(a,b) \not\equiv t(A,B)\}| \geq |Q|\,(1-\epsilon')\,|B|$$
which follows B being ϵ'-good.

Putting it all together:
$$\left(1 - \epsilon' - \epsilon' - \frac{\epsilon}{1-\epsilon}\right)|B|\,|Q| < \left(1 - \epsilon' + \frac{\epsilon}{1-\epsilon}\right)(\epsilon + \zeta')\,|B|\,\left|A'\right|$$

So, we have that:

$$|Q| < \frac{\left(1-\epsilon'-\frac{\epsilon}{1-\epsilon}\right)}{\left(1-\epsilon'-\frac{\epsilon}{1-\epsilon}\right)-\epsilon'}\left(\epsilon+\zeta'\right)|A'|$$
$$= \left(1+\frac{\epsilon'}{1-2\epsilon'-\frac{\epsilon}{1-\epsilon}}\right)(\epsilon+\zeta')|A'|$$

Notice that $f(\epsilon,\epsilon') := \frac{\epsilon'}{1-2\epsilon'-\frac{\epsilon}{1-\epsilon}}$ decreases with ϵ and ϵ'. In particular,

$$f(\epsilon,\epsilon') \overset{\epsilon'\to 0}{\longrightarrow} 0$$

and $\epsilon' > \epsilon$. Then,

$$|Q| < \left(\epsilon+\left(\underbrace{\epsilon f(\epsilon,\epsilon')}_{\to 0}+\underbrace{(1+f(\epsilon,\epsilon'))}_{\to 1}\right)\zeta'\right)|A'| \overset{\epsilon'\to 0}{\longrightarrow} \left(\epsilon+\zeta'\right)|A'|$$

So, there exists an $\epsilon_1 = \epsilon_1(\zeta,\zeta')$ small enough such that for all $(\epsilon <)\epsilon' \leq \epsilon_1$, we have that $|Q| < (\epsilon+\zeta)|A'|$, and since A' is $(\epsilon+\zeta')$-good, and thus $(\epsilon+\zeta)$-good, we conclude that A' is $(\epsilon+\zeta,\epsilon')$-excellent.

(d) Let $\zeta,\zeta',\epsilon,\epsilon'$ and r be given satisfying the conditions of the statement. Set $\xi = \frac{1}{r+1}$. We will see that the condition $n > N_2 := N_1\left(\epsilon,\zeta',\frac{1}{r+1}\right)$ is sufficient. First of all, randomly choose a function $h : A \longrightarrow \{1,\dots,r-1\}$ such that for all $s < n$ we have that $|\{a \in A \mid h(a) = s\}| = \frac{n}{r}$. Since h is random, each $A' \in [A]^{\frac{n}{r}}$ has the same probability of being part of the partition induced by h, i.e. to satisfy $A' = h^{-1}(s)$ for some $s \in \{1,\dots,r-1\}$. For each element of the partition A', we can apply (a) to get that

$$P\left(A' \text{ is not } \left(\epsilon+\zeta'\right)\text{-good}\right) < \xi$$

In particular, since A is (ϵ,ϵ')-excellent, it follows (c) that if A' is $(\epsilon+\zeta')$-good then it is also $(\epsilon+\zeta,\epsilon')$-excellent, so:

$$P\left(A' \text{ is not } \left(\epsilon+\zeta,\epsilon'\right)\text{-excellent}\right) < \xi$$

To conclude, by the union bound, we have that:

$$P\left(\bigcup_{s<r} h^{-1}(s) \text{ is not } \left(\epsilon+\zeta,\epsilon'\right)\text{-excellent}\right) \leq \sum_{s<r} P\left(h^{-1}(s) \text{ is not } \left(\epsilon+\zeta,\epsilon'\right)\text{-excellent}\right)$$
$$< r\xi = \frac{r}{r+1} < 1$$

All in all, there is a non-zero chance that the partition satisfies the statement, i.e. there exists at least one.

□

Remark 4.10 (Remark 5.13.1). For following applications, we would like to use Lemma 4.9 (d) with $\epsilon' > k(\epsilon+\zeta)$, for an arbitrarily large $k \in \mathbb{N}$. Notice that if $\epsilon,\zeta < \frac{1}{t}, \epsilon' < \frac{1}{t'}$ and $t > t' \geq 5$, then:

(a) $\frac{\epsilon}{1-\epsilon} < \frac{\frac{1}{t}}{1-\frac{1}{t}} = \frac{\frac{1}{t}}{\frac{t-1}{t}} = \frac{1}{t-1}$

(b) $1 - 2\epsilon' - \frac{\epsilon}{1-\epsilon} > 1 - \frac{2}{t'} - \frac{1}{t-1} > 1 - \frac{3}{t'-1} = \frac{t'-4}{t'-1}$

(c) $\left(1 + \frac{\epsilon'}{1-2\epsilon'-\frac{\epsilon}{1-\epsilon}}\right) < \left(1 + \frac{t'-1}{t'-4}\epsilon'\right)(\epsilon + \zeta')$

Then, by requiring $\frac{1}{t} < \frac{1}{4k}\epsilon'$ we have that

$$\begin{aligned}
\epsilon + \zeta' &< \frac{2}{t} < 2\left(\frac{1}{4k}\epsilon'\right) < \frac{1}{2}\left(\frac{1}{k}\epsilon'\right) \\
&< \frac{t'-4}{t'-3}\frac{1}{k}\epsilon' = \frac{1}{k}\frac{\epsilon'}{1+\frac{1}{t'-4}} \\
&< \frac{1}{k}\frac{\epsilon'}{1+\frac{t'-1}{t'}\frac{1}{t'-4}} = \frac{1}{k}\frac{\epsilon'}{1+\frac{t'-1}{t'-4}\frac{1}{t'}} \\
&< \frac{1}{k}\frac{\epsilon'}{1+\frac{t'-1}{t'-4}\epsilon'}
\end{aligned}$$

i.e., we have:

$$\left(1 + \frac{t'-1}{t'-4}\epsilon'\right)(\epsilon + \zeta') < \frac{1}{k}\epsilon'$$

which by (c) gives us:

$$\left(1 + \frac{\epsilon'}{1-2\epsilon'-\frac{\epsilon}{1-\epsilon}}\right) < \frac{1}{k}\epsilon'$$

All in all, a sufficient condition, for the lemma to hold under the constraint $\epsilon' > k(\epsilon + \zeta)$, is:

$$\epsilon, \zeta' < \frac{1}{4k} \quad \text{and} \quad \epsilon' < \frac{1}{5}$$

We use this fact to reformulate point (d) of Lemma 4.9 as:

Lemma 4.11 (Claim 5.13.2(3)). *For all $k, r \geq 1$, $\epsilon' \leq \frac{1}{5}$ and $\epsilon < \frac{1}{4k}\epsilon'$, there exists $N_3 = N_3(\epsilon, \epsilon', r)$ large enough such that, for all $n > N_3$ and r dividing n, if $A \subseteq G$ is (ϵ, ϵ')-excellent, with $|A| = n$, then there exists a partition into r disjoint pieces of equal size, each of which is $\left(\frac{\epsilon'}{k}, \epsilon'\right)$-excellent.*

Proof. Choose any $\zeta' < \frac{1}{4k}\epsilon'$ and set $N_3 := N_2(\epsilon, \zeta', r)$. Remark 4.10 sufficiency condition is satisfied, Claim 4.9 (d) holds and we are done. ☐

Lemma 4.12 (Claim 5.14.1). *Let G be a finite graph with the non-k_*-order property. Let $\epsilon \in \left(0, \frac{1}{2}\right)$ and $\epsilon' < \frac{1}{2^{k_{**}}}$. Let $A \subseteq G$ such that $|A| = n$. Let $\langle m\ell \mid \ell \in [0, k_{**}]\rangle$ be a decreasing sequence of natural numbers such that $\epsilon m_\ell \geq m_{\ell+1}$ for all $\ell \in [0, k_{**} - 1]$, $m_{k_{**}} \geq 1$, and $m_{k_{**}-1} > k_{**}$. Denote $m_* := m_0$ and $m_{**} := m_{k_{**}}$. Then, there is a partition $\overline{A} = \langle A_j \mid j \in [0, j(*)]\rangle$ with remainder $B = A \setminus \bigcup_{j<j(*)} A_j$ such that:*

(a) For all $j \in [0, j(*)]$, $|A_j| \in \langle m\ell \mid \ell \in [0, k_{**} - 1]\rangle$.

(b) For all $i \neq j \in [0, j(*)]$, $A_i \cap A_j = \emptyset$.

(c) *For all $j \in [0, j(*)]$, A_j is (ϵ, ϵ')-excellent.*

(d) $|B| < \epsilon m_*$.

Proof. Apply Lemma 4.5 recursively to the remainder $A \setminus \bigcup_{i<j} A_i$, to obtain A_j at each step. The process stops at $j(*)$ when the remainder is smaller than m_0, and thus the lemma cannot be applied. Notice that, since $\frac{m_\ell}{m_{\ell-1}} \leq \epsilon$, $\left(\frac{m_\ell}{m_{\ell-1}}, \epsilon'\right)$-excellence implies (ϵ, ϵ')-excellence. $\square$

Lemma 4.13 (Claim 5.14.1a). *Let G be a finite graph with the non-k_*-order property. Let $\epsilon \in \left(0, \frac{1}{2}\right)$ and $\epsilon' < \frac{1}{2^{k_{**}}}$. Let $A \subseteq G$ such that $|A| = n$. Let $\langle m\ell \mid \ell \in [0, k_{**}]\rangle$ be a decreasing sequence of natural numbers such that $\epsilon m_\ell \geq m_{\ell+1}$ for all $\ell \in [0, k_{**} - 1]$, $m_{k_{**}} \geq 1$, $m_{**} := m_{k_{**}} \mid m_l$ for all $l \in [0, k_{**}]$ and $m_{k_{**}-1} > N_3\left(\epsilon, \epsilon', \frac{m_*}{m_{**}}\right)$ (in the sense of Claim 4.11), where $m_* := m_0$. Then, for some $i(*) \leq \frac{n}{m_{**}}$, there is a partition $\overline{A} = \langle A_i \mid i \in [1, i(*)]\rangle$ with remainder $B = A \setminus \bigcup_{i \in [1,i(*)]}$ such that:*

(a) *For all $i \in [1, i(*)]$, $|A_i| = m_{**}$.*

(b) *For all $i \neq j \in [1, i(*)]$, $A_i \cap A_j = \emptyset$.*

(c) *For all $i \in [1, i(*)]$, A_i is $\left(\frac{\epsilon'}{k}, \epsilon'\right)$-excellent.*

(d) $|B| < \epsilon m_*$.

Proof. Use Claim 4.12 to obtain a partition $\overline{A}' = \left\langle A'_j \mid j \in [0, j(*)]\right\rangle$ and remainder B with $|B| < m_*$. Then, we can apply Claim 4.11 with $r = m_{**}$ to each of the parts A'_j. Putting together all the new subparts, we obtain a new partition $\overline{A} = \langle A_i \mid i \in [0, i(*)]\rangle$ with remainder B, satisfying all the conditions of the statement. $\square$

Lemma 4.14 (Claim 5.14.2). *Under the same condition of Lemma 4.13, we can get a partition $\overline{A} = \langle A_i \mid i \in [1, i(*)]\rangle$ with no remainder, such that:*

(a) *For all $i \neq j \in [1, i(*)]$, $||A_i| - |A_j|| \leq 1$.*

(b) *For all $i \neq j \in [1, i(*)]$, $A_i \cap A_j = \emptyset$.*

(c) *For all $i \in [1, i(*)]$, A_i is (ϵ'', ϵ')-excellent, where*

$$\epsilon'' \leq \frac{\frac{\epsilon'}{k} m_{**} + \left\lceil \frac{m_*}{i(*)} \right\rceil}{m_{**} + \left\lceil \frac{m_*}{i(*)} \right\rceil}$$

(d) $A = \bigcup \overline{A}$.

Proof. Let $\overline{A}' = \langle A'_i \mid i \in [1, i(*)]\rangle$ and B from Claim 4.13. We can partition B into $\overline{B} = \langle B_i \mid i \in [1, i(*)]\rangle$ in such a way that for all $i \in [1, i(*)]$,

$$|B_i| \in \left\{ \left\lfloor \frac{|B|}{i(*)} \right\rfloor, \left\lceil \frac{|B|}{i(*)} \right\rceil \right\}$$

Notice that we are allowing $B_i = \emptyset$. Then, the new partition $\overline{A} = \langle A'_i \cup B_i \mid i \in [1, i(*)] \rangle$ satisfies (a), (b) and (d) by construction. To conclude, notice that for each ϵ'-good set B, the number of exceptions is bounded by

$$\begin{aligned}
|\{a \in A_i \mid t(a, B) \not\equiv t(A_i, B)\}| &\leq \frac{\epsilon'}{k} |A'_i| + |B_i| \\
&= \frac{\frac{\epsilon'}{k} |A'_i| + |B_i|}{|A'_i| + |B_i|} \left(|A'_i| + |B_i|\right) \\
&\leq \frac{\frac{\epsilon'}{k} m_{**} + \left\lceil \frac{m_*}{i(*)} \right\rceil}{m_{**} + \left\lceil \frac{m_*}{i(*)} \right\rceil} |A_i|
\end{aligned}$$

which proves that (c) can be satisfied. $\square$

Remark 4.15 (Remark 5.14.3). In the context of Lemma 4.14, if:

(a) $m_{**} > \frac{1}{\frac{\epsilon'}{k}}$

(b) $m_* \leq \frac{\frac{\epsilon'}{k} n + 1}{\frac{\epsilon'}{k} + 1}$

then $\epsilon'' < \frac{3\epsilon'}{k}$.

Proof. Notice that, if $|B_i| \leq 2\frac{\epsilon'}{k} |A_i|$ for all $i \in [1, i(*)]$, then ϵ'' can be bounded by:

$$\epsilon'' \leq \frac{\frac{\epsilon'}{k} |A_i| + |B_i|}{|A_i| + |B_i|} \leq \frac{\frac{\epsilon'}{k} |A_i| + 2\frac{\epsilon'}{k} |A_i|}{|A_i|} = \frac{3\epsilon'}{k}$$

Let's now prove that $|B_i| \leq \frac{2\epsilon'}{k} |A_i|$ is satisfied. Notice that, by construction:

$$|B_i| \leq \left\lceil \frac{|B|}{i(*)} \right\rceil \leq \left\lceil \frac{m_* - 1}{i(*)} \right\rceil \leq \frac{m_* - 1}{i(*)} + 1$$

Also we can bound $i(*)$ by:

$$\frac{n}{m_{**}} \geq i(*) \geq \frac{n - |B|}{m_{**}} \geq \frac{n - m_* + 1}{m_{**}} > \frac{n - m_*}{m_{**}}$$

Thus, $|B_i| - 1 \leq \frac{m_* - 1}{i(*)} \leq \frac{(m_* - 1) m_{**}}{n - m_*}$, then $\frac{|B_i| - 1}{m_{**}} < \frac{m_* - 1}{n - m_*}$, and since $|A_i| = m_{**}$ we get:

$$\frac{|B_i|}{|A_i|} < \frac{m_* - 1}{n - m_*} + \frac{1}{m_{**}}$$

Finally, notice that condition (a) implies:

$$\frac{\epsilon'}{k} > \frac{1}{m_{**}}$$

and condition (b) implies:

$$\frac{\epsilon'}{k} \geq \frac{m_* - 1}{n - m_*}$$

We conclude:

$$\frac{|B_i|}{|A_i|} < \frac{m_* - 1}{n - m_*} + \frac{1}{m_{**}} < 2\frac{\epsilon'}{k}$$

completing the proof. $\square$

Lemma 4.16 (Corollary 5.15). *Let G be a graph with the non-k_*-order property. Suppose that we are given:*

1. $\epsilon < \min\left(\frac{1}{5}, \frac{1}{2^{k_{**}}}\right)$.
2. *A sequence of positive integers $\langle m\ell \mid \ell \in [0, k_{**}]\rangle$ such that:*
   (a) $m_\ell \leq \frac{\epsilon}{12} m_{\ell+1}$.
   (b) $m_{**} := m_{k_{**}} \geq \frac{\epsilon}{3}$.
   (c) $m_{**} \mid m_\ell$ *for all* $\ell \in [0, k_{**}]$.
   (d) $m_{k_{**}-1} > N_3\left(\frac{\epsilon}{12}, \epsilon, \frac{m_*}{m_{**}}\right)$ *(in the sense of Claim 4.11).*
3. *$A \subseteq G$ such that $|A| = n$, where n satisfies:*
   (a') $n \geq m_0$.
   (b') $m_* \leq \frac{1+\frac{\epsilon}{3}n}{1+\frac{\epsilon}{3}}$.

Then, there exists $i(*) \leq \frac{n}{m_{**}}$ and a partition of A into disjoint pieces $\overline{A} = \langle A_i \mid i \in [1, i(*)]\rangle$ such that:

(i) *For all $i \neq j \in [1, i(*)]$, $||A_i| - |A_j|| \leq 1$.*

(ii) *For all $i \in [1, i(*)]$, A_i is ϵ-excellent,*

(iii) *For all $i \neq j \in [1, i(*)]$, (A_i, A_j) is ϵ-uniform.*

Proof. Simply apply Lemma 4.14 in the context of Remark 4.15 with $k > 3$, $\epsilon'_{4.14} = \epsilon$ and $\epsilon_{4.14} = \frac{1}{12}\epsilon$. This results in a partition of A into disjoint pieces that satisfy (i) and that are $(\epsilon''_{4.14}, \epsilon'_{4.14})$-excellent, with $\epsilon''_{4.14} < \frac{3\epsilon'_{4.14}}{k}$. But since $k > 3$, $\epsilon''_{4.14} < \epsilon'_{4.14}$, they are also $\epsilon'_{4.14}$-excellent, satisfying (ii) and (iii). $\square$

Theorem 4.17 (Theorem 5.18). *Let k_* and therefore k_{**} be given. Then, for all $\epsilon < \min\left(\frac{1}{5}, \frac{1}{2^{k_{**}}}\right)$, there is $m = m(\epsilon, k_*)$ and $N = N(\epsilon, k_*)$ such that, for every finite graph G with the non-k_*-order property, and every $A \subseteq G$ with $|A| \geq N$, there exists a partition $\overline{A} = \langle A_i \mid i \in [1, i(*)]\rangle$ of A into at most m pieces, such that:*

1. *For all $i, j \in [1, i(*)]$, $||A_i| - |A_j|| \leq 1$.*
2. *For all $i \in [1, i(*)]$, A_i is ϵ-excellent.*
3. *For all $i \neq j \in [1, i(*)]$, (A_i, A_j) is ϵ-uniform.*
4. $m \leq (3 + \epsilon)\left(\frac{12}{\epsilon}\right)^{k_{**}}$.

Proof. Suppose $N = N(\epsilon, k_*)$ is large enough. We will state at the end of the proof the required size of N. Our goal is to apply Lemma 4.16. Let $q = \left\lceil \frac{12}{\epsilon} \right\rceil$. For n large enough (H1), we can choose m_{**} satisfying:

(a) $m_{**} \in \left(\frac{\epsilon n}{(3+\epsilon)q^{k_{**}-1}} - 1, \frac{\epsilon n}{(3+\epsilon)q^{k_{**}-1}}\right)$.

(b) $m_{**} \geq \frac{3}{\epsilon}$.

(c) $m_{**} > \frac{N_3\left(\frac{\epsilon}{12}, \epsilon, q^{k_{**}-1}\right)}{q}$.

Then, setting $m_{k_{**}} = m_{**}$ we can build recursively a sequence of integers $\langle m_\ell \mid \ell \in [0, k_{**}] \rangle$ such that $m_\ell = q m_{\ell+1}$ for all $\ell \in [0, k_{**} - 1]$. Finally, let $m_* = q^{k_{**}-1} m_{**}$. Notice that by (a) we have that $m_* = m_{**} q^{k_{**}-1} \leq \frac{\epsilon n}{3+\epsilon}$. This sequence satisfies all the conditions of Lemma 4.16:

(2.a) $m_{\ell-1} = q m_\ell \leq \frac{\epsilon}{12} m_\ell$.

(2.b) $m_{**} \geq \frac{3}{\epsilon}$.

(2.c) $m_\ell = q^{k_{**}-\ell} m_{**}$ so that $m_{**} \mid m_\ell$ for all $\ell \in [0, k_{**}]$.

(2.d) $m_{k_{**}-1} = q m_{**} < q \frac{N_3\left(\frac{\epsilon}{12}, \epsilon, q^{k_{**}-1}\right)}{q} = N_3\left(\frac{\epsilon}{12}, \epsilon, \frac{m_*}{m_{**}}\right)$.

(3.b) $m_* \leq \frac{\epsilon n}{3+\epsilon} \leq \frac{1+\frac{\epsilon}{3} n}{1+\frac{\epsilon}{3}}$.

Finally, we need $N > m_0$ to satisfy (3.a) (H2). Now, we can apply Lemma 4.16 to obtain a partition satisfying (1), (2) and (3). We just need to check that (4) holds. By (a), we have that $m_{**} \geq \frac{1}{2} \frac{\epsilon n}{(3+\epsilon) q^{k_{**}-1}}$. Thus, we can bound the number of pieces by:

$$m \leq \frac{n}{m_{**}} \leq \frac{2\,(3+\epsilon)\,q^{k_{**}-1}}{\epsilon} \leq (3+\epsilon)\left(\frac{2}{\epsilon}\right)\left(\frac{12}{\epsilon}\right)^{k_{**}-1} < (3+\epsilon)\left(\frac{12}{\epsilon}\right)^{k_{**}-1}$$

Notice that the bound on m only depends on ϵ and k_{**}.

To conclude, let's put together all requirements on N:

(H1) N needs to be large enough to satisfy

(H2) $N > N_3\left(\frac{\epsilon}{12}, \epsilon, \frac{m_*}{m_{**}}\right)$.

□

References

[1] S.K. Agrawal, J. Yan. 'A three-wheel vehicle with expanding wheels: differential flatness, trajectory planning, and control', Proc. of the 2003 IEEWRSJ, Intl. Conference on Intelligent Robots and Systems, Las Vegas, 2003.

[2] L. Ahlfors. Complex analysis. An introduction to the theory of analytic functions of one complex variable, *3rd ed. McGraw-Hill, 1978.*

[3] L. Ahlfors. Lectures on quasiconformal mappings, *2nd ed. University Lecture series* **38**, *American Mathematical Society, 2006.*

[4] *L. Ahlfors and L. Bers. Riemann mapping's theorem for variable metrics,* Annals of Math. **72** *(1960), 385–404.*

[5] *B. Charlet, J. Lévine, R. Marino. On dynamic feedback linearization*, System and Control Letters **13** *(1989), 143–151.*

A. Title of the appendix

You can include here an appendix with details that can not be included in the core of the document. You should reference the sections in this appendix in the core document.

B. Title of the appendix

Second appendix.