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The Regularity Lemma for Stable Graphs and its applications in Property Testing

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Thanks to...

Abstract

Szemerédi's Regularity Lemma is a cornerstone of modern graph theory, asserting that any graph can be partitioned into a bounded number of vertex sets, where the connections between most pairs of sets behave quasi-randomly. Despite its wide-ranging applications in areas like number theory, combinatorics and computer science, the lemma suffers from two major limitations: a partition size bounded by a tower of exponentials, and the presence of irregular pairs, both unavoidable in the general case.

This work focuses on a specific subclass of graphs, the *stable graphs*, where these limitations can be overcome. By avoiding a bipartite substructure known as the half-graph, stable graphs admit a much stronger regularity lemma. This specialized lemma, originally developed by Malliaris and Shelah, guarantees a partition where all pairs are regular and the number of parts is bounded by a polynomial, a significant improvement over the general tower-type bound.

This thesis first presents a self-contained, combinatorial, and complete presentation of the proof of the stable regularity lemma, developing a unified notational framework to bridge concepts from extremal graph theory, stability, and property testing. Building on this theoretical foundation, we then construct an efficient algorithm for testing *H-freeness* (the property of not containing an induced copy of a fixed graph H) for stable graphs. This application leverages the lemma's superior properties to achieve a query complexity with significantly improved bounds compared to testers for general graphs.

Keywords

Graph Theory, Stable Graphs, Stability, VC-dimension, Szemerédi Regularity Lemma, Property Testing

Contents

1. Graphs and the SzRL

1.1 Graphs and basic notation

In all this work we will consider only simple graphs, that is, unweighted, undirected graphs with no loops or multiple edges. The following definition accounts for this.

Definition 1.1. A (simple) *graph* is a pair $G = (V, E)$ where V is a finite set whose elements are called *vertices* and $E \subseteq \{\{v_1, v_2\} \mid v_1, v_2 \in V \text{ and } v_1 \neq v_2\}$ is a set of unordered pairs of distinct vertices, whose elements are called *edges*. If $\{v_1, v_2\} \in E$, then v_1 and v_2 are said to be *the endpoints* of the edge.

By abuse of notation, we will often denote a graph $G = (V, E)$ simply by G and write $v \in G$ to mean $v \in V$. Similarly, we will write $uv \in G$ to mean $\{u, v\} \in E$.

As most of this work is inspired by model theory and logic results (see the use of k-trees in Section 2.3), it is useful to note that vertices adjacency (whether two vertices are the endpoints of an edge) is a symmetric and irreflexive binary relation on the vertex set. With this perspective, to denote vertex adjacency between two vertices v_1 and v_2 we will often use the notation $v_1 R v_2$, where R is the adjacency relation in V. Also, in order to simplify future notation, we will assume that a logical true statement and the value 1 are equivalent, and similarly a false statement and the value 0. As an example, if two vertices v_1 and v_2 are not adjacent, we say that $\neg v_1 R v_2 \equiv \neg 1 \equiv 0$.

Now, a class of graphs of particular relevance in this work is that of bipartite graphs, which we define as follows.

Definition 1.2. A graph G is *bipartite* if there exists a partition of its vertex set into two disjoint sets L and R such that every edge in G connects a vertex in L to a vertex in R. That is, no edge connects vertices within the same set of the partition.

Also, it is often useful to be able to restrict a graph to a subset of its vertices.

Definition 1.3. Let $G = (V, E)$ be a graph and let $S \subseteq V$ be a subset of its vertices. The *subgraph of G induced by S*, denoted by $G[S]$, is the graph whose vertex set is S and whose edge set consists of all edges in E that have both endpoints in S. Formally, $G[S] = (S, E_S)$ where $E_S = \{\{v_1, v_2\} \in E \mid v_1, v_2 \in S\}$.

A similar restriction can defined for bipartite graphs, but only controlling edges between the two disjoint sets.

Definition 1.4. We say that a bipartite graph H with disjoint sets L and R is *bi-induced* in a graph G if there exist two injective homomorphisms $\phi_L : L \longrightarrow G$ and $\phi_R : R \longrightarrow G$ such that, for all $u \in L$ and $v \in G$, $uv \in H \Leftrightarrow uv \in G$.

Notice that this definition does not require the two sets $\phi_L(L)$ and $\phi_R(R)$ to be disjoint (as defined in [6, pg. 417] and [1, pg. 2]). This is important for the arguments used in this thesis, and needs to be noted that other works define such condition without this relaxation [9, pg. 3].

1.2 Regular pairs and partitions

We now want to formalize the concept of regular pairs of vertex sets, which is central to Szemerédi's Regularity Lemma. The idea is that a pair of vertex sets is regular if the edges between them are "randomly" distributed, an idea that we can formalize using edge density.

Definition 1.5. Let G be a graph and let $X, Y \subseteq G$ be two (not necessarily disjoint) non-empty subsets of its vertices. The *edge density* between X and Y is defined as

$$d(X, Y) = \frac{|e(X, Y)|}{|X||Y|},$$

where $e(X, Y) = \{(x, y) \in E \mid x \in X, y \in Y\}$ is the set of edges with one endpoint in X and the other in Y.

When X and Y are disjoint, the edge density $d(X, Y)$ measures the proportion of possible edges between X and Y that are actually present in the graph. If X and Y are not disjoint, this is not the case. On one hand, because simple graphs do not allow loops, and so edges between the same vertex are never present in $e(X, Y)$, but they are counted in the denominator as "possible edges". On the other hand, edges between vertices in the intersection $X \cap Y$ are counted twice both in $e(X, Y)$ and $|X||Y|$. However, we will only be interested in knowing the exact proportion of edges in a pair in two specific cases: either when X and Y are disjoint, or when they are equal. The first case has no problems, while for the second case we note the following.

Remark 1.6. If X is a subset of vertices of a graph G such that $|X| \geq 2$, then the proportion of possible edges between vertices in X that are actually present in G is at most twice the density $d(X, X)$. That is,

$$\frac{|E_X|}{\binom{|X|}{2}} = \frac{|e(X, X)|/2}{(|X| - 1)|X|/2} = \frac{|X|}{|X| - 1}\frac{|e(X, X)|}{|X|^2},$$

where first equality follows from the fact that E_X counts each edge in $e(X, X)$ twice. So,

$$d(X, X) \leq \frac{|E_X|}{\binom{|X|}{2}} \leq 2d(X, X).$$

This also implies that the proportion of possible edges between vertices in X that are actually not present in G lies between $1 - 2d(X, X)$ and $1 - d(X, X)$.

Definition 1.7. Given $\epsilon > 0$ and a graph G, a pair of (not necessarily disjoint) subsets of vertices $A, B \subseteq G$ is said to be *ϵ-regular* if for all $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| \geq \epsilon|A|$ and $|B'| \geq \epsilon|B|$, we have

$$|d(A', B') - d(A, B)| \leq \epsilon.$$

Intuitively, this means that the edges of the pair are fairly uniformly distributed, and the pair behaves similarly to a random bipartite graph with edge density $d(A, B)$.

Now, this notion of regularity can be used in the context of a partition of a graph's vertex set.

Definition 1.8. Given a graph G, we say that $\{A_1, \dots, A_k\}$ is a partition of the vertex set of G with *remainder* set B, if $G = A_1 \cup \cdots \cup A_k \cup B$, and $A_1, \dots, A_k$ are non-empty sets. Implicitly, we allow the remainder to be empty.

The partition we want to study needs to satisfy that most pairs of parts are regular, but we allow a small number of such pairs to be irregular.

Definition 1.9. Let G be a graph and let $\epsilon > 0$. An *ϵ-regular partition* of G is a partition of its vertex set into k parts $\{A_1, \dots, A_k\}$ with remainder set B such that:

- $|B| \leq \epsilon|G|$, and may be empty.
- All but at most ϵk^2 of the pairs (A_i, A_j) with $1 \leq i < j \leq k$ are ϵ-regular.

Also, we want the partition's sets to be roughly of the same size, which can be formalized in two different ways.

Definition 1.10. A partition $\{A_1, \dots, A_k\}$ of the vertex set of a graph G is said to be *equitable* if for all $1 \leq i \leq j \leq k$, we have that $||A_i| - |A_j|| \leq 1$. On the other hand, a partition $\{A_1, \dots, A_k\}$ with remainder B of the vertex set of a graph G is said to be *even* if $|A_1| = |A_2| = \cdots = |A_k|$.

Remark 1.11. The two previous definitions, although very close in concept, have a key difference that needs to be noted. As most of the results requires the partition property (such as regularity) to be satisfied only by parts in the partition, and not necessarily by the reminder, in even partitions the behaviour of a (not necessarily trivial) fraction of vertices is unknown. Thus, results with equitable partitions are generally preferable over those with even partitions, but require some extra arguments. For example, in the context of regular partitions, one can make an even partition into an equitable one by distributing the remainder (which by definition is small) evenly between all the parts (with some extra arguments). The resulting partition is equitable with a 1-vertex difference between parts[1], and with a small increase in the regularity error. In other cases, such as the results of **??**, the remainder is much larger, and such a strategy does not work. These (secondary) results will be presented with even partitions. The more relevant Stable Regularity Lemma in Section 3 presents an equitable one.

1.3 Szemerédi's Regularity Lemma

The following is the celebrated Szemerédi's Regularity Lemma. The statement and proof we provide in this thesis follows the one given in [4], with minor notation modifications.

Theorem 1.12 (Szemerédi's Regularity Lemma, [14])**.** *For every $\epsilon > 0$ and every positive integer m, there exists a positive integer $M = M(\epsilon, m)$ such that every graph with at least m vertices admits an even ϵ-regular partition $\{A_1, \dots, A_k\}$ and reminder B with $m \leq k \leq M$.*

The principal strength of this lemma lies in the fact that it guarantees the existence of a regular partition whose number of parts is independent of the size of the graph, and only depends on the regularity parameter ϵ and the minimum number of parts (and thus vertices) m[2].

The proof of the regularity lemma uses a density-increment argument. There is a quantity that we shall call *energy* of the partition (Definition 1.13) that is upper bounded by a constant (Equation (2)) and which is non-decreasing by partition refinement (Lemma 1.14). Also, we prove that if an even partition is not ϵ-regular, then one could refine the partition in such a way that the energy increases by a constant depending only on ϵ, and the number of parts in the new partition only depends on the size of the previous partition (Lemma 1.16). Thus, one can iteratively refine until reaching a regular partition, a process that must culminate in finitely many steps (Theorem 1.12).

[1] This ± 1 size difference is a simple consequence of the number of vertices possibly not being divisible by the number of parts. It has no major consequences, since it becomes proportionally more trivial as the size of the parts gets larger.

[2] The dependency of M on m has more to do with practical and applicability purposes (in this form of the result we do not control de edges within each part) than conceptual ones. Since we want to be able to choose a minimal number of parts m, the upper bound on the number of pairs will also depend on such value.

The following inequality will be useful during the proof. For any $\mu_1, \dots, \mu_k > 0$ and for all $e_1, \dots, e_k \geq 0$:

$$\sum_{i=1}^{k} \frac{e_i^2}{\mu_i} \geq \frac{(\sum_{i=1}^{k} e_i)^2}{\sum_{i=1}^{k} \mu_i}. \tag{1}$$

This is a direct consequence of applying the Cauchy-Schwarz inequality $\sum a_i^2 \sum b_i^2 \geq (\sum a_i b_i)^2$ with the sequences $a_i = \sqrt{\mu_i}$ and $b_i = e_i/\sqrt{\mu_i}$.

We now formalize the concept of the *energy* of a partition.

Definition 1.13. Let G be a graph with n vertices and let A_1, A_2 be two disjoint subset of its vertex set. Then, we define

$$q(A_1, A_2) = \frac{|A_1||A_2|}{n^2} d(A_1, A_2)^2 = \frac{e(A_1, A_2)^2}{n^2 |A_1||A_2|}.$$

For a partition $\overline{A_1}$ of A_1 and $\overline{A_2}$ of A_2, we define

$$q(\overline{A_1}, \overline{A_2}) = \sum_{A_1' \in \overline{A_1}, A_2' \in \overline{A_2}} q(A_1', A_2').$$

Finally, we define the *energy* of a partition $\overline{A} = \{A_1, \dots, A_k\}$ of the vertex set of G as

$$q(\overline{A}) = \sum_{1 \leq i < j \leq k} q(A_i, A_j).$$

Let $\overline{A}$ be a partition with reminder set B, we define $\widetilde{A} := \overline{A} \cup \overline{B}$, and we use $\overline{B}$ to denote the set of singletons of the remainder set, $\overline{B} := \{\{b\} \mid b \in B\}$. Then, $q(\widetilde{A}) = q(\overline{A} \cup \overline{B})$

As promised, we see that the energy of a partition is upper bounded by a constant:

$$\begin{aligned} q(\widetilde{A}) &= \sum_{\substack{C_1, C_2 \in \widetilde{A} \\ C_1 \neq C_2}} q(C_1, C_2) \\ &= \sum_{\substack{C_1, C_2 \in \widetilde{A} \\ C_1 \neq C_2}} \frac{|C_1||C_2|}{n^2} d(C_1, C_2)^2 \\ &\leq \frac{\sum |C_1||C_2|}{n^2} \leq 1. \end{aligned} \tag{2}$$

We now prove that refining a pair of parts or a whole partition does not decrease its energy.

Lemma 1.14. *Let G be a graph.*

1. *Let $A_1, A_2 \subseteq G$ be disjoint. If $\overline{A}_1$ is a partition of A_1 and $\overline{A}_2$ is a partition of A_2, then $q(\overline{A}_1, \overline{A}_2) \geq q(A_1, A_2)$.*

2. *If $\overline{A}, \overline{A}'$ are partitions of G and $\overline{A}'$ is a refinement of $\overline{A}$, then $q(\overline{A}') \geq q(\overline{A})$.*

Proof. 1. Let $\overline{A}_1 = \{A_{1,1}, \dots, A_{1,k}\}$ and $\overline{A}_2 = \{A_{2,1}, \dots, A_{2,\ell}\}$. Then

$$\begin{aligned} q(\overline{A}_1, \overline{A}_2) &= \sum_{i=1}^{k}\sum_{j=1}^{\ell} q(A_{1,i}, A_{2,j}) \\ &= \frac{1}{n^2}\sum_{i=1}^{k}\sum_{j=1}^{\ell} \frac{e(A_{1,i}, A_{2,j})^2}{|A_{1,i}||A_{2,j}|} \\ &\overset{(1)}{\geq} \frac{1}{n^2}\frac{\left(\sum_{i=1}^{k}\sum_{j=1}^{\ell} e(A_{1,i}, A_{2,j})\right)^2}{\sum_{i=1}^{k}\sum_{j=1}^{\ell} |A_{1,i}||A_{2,j}|} \\ &= \frac{1}{n^2}\frac{e(A_1, A_2)^2}{(\sum_{i=1}^{k}|A_{1,i}|)(\sum_{j=1}^{\ell}|A_{2,j}|)} \\ &= q(A_1, A_2). \end{aligned}$$

2. Let $\overline{A} = \{A_1, \dots, A_k\}$, and for all $i \in \{1, \dots, k\}$ let $\overline{A}_i$ be the partition of A_i induced by $\overline{A}'$. Then,

$$\begin{aligned} q(\overline{A}) &= \sum_{1 \leq i < j \leq k} q(A_i, A_j) \\ &\overset{1.}{\leq} \sum_{1 \leq i < j \leq k} q(\overline{A}_i, \overline{A}_j) \\ &\leq q(\overline{A}'), \end{aligned}$$

where last inequality follows from the fact that $q(\overline{A}') = \sum_{1 \leq i \leq k} q(\overline{A}_i) + \sum_{1 \leq i < j \leq k} q(\overline{A}_i, \overline{A})$.

□

Next, we show that refining an irregular pair results in a significant increase in energy. This amount, does not yet depend only on ϵ, but it will when applied to all irregular pairs at the same time.

Lemma 1.15. *Let G be a graph with n vertices, $A_1, A_2 \subseteq G$ be disjoint subsets and $\epsilon > 0$. If the pair (A_1, A_2) is not ϵ-regular, then there exist partitions $\overline{A}_1 = \{A_{1,1}, A_{1,2}\}$ of A_1 and $\overline{A}_2 = \{A_{2,1}, A_{2,2}\}$ of A_2 such that*

$$q(\overline{A}_1, \overline{A}_2) \geq q(A_1, A_2) + \epsilon^4 \frac{|A_1||A_2|}{n^2}.$$

Proof. Suppose that (A_1, A_2) is not ϵ-regular. Then there are subsets $A_{1,1} \subseteq A_1$ and $A_{2,1} \subseteq A_2$ with $|A_{1,1}| \geq \epsilon|A_1|$ and $|A_{2,1}| \geq \epsilon|A_2|$ such that

$$|\eta| > \epsilon, \tag{3}$$

where $\eta = d(A_{1,1}, A_{2,1}) - d(A_1, A_2)$. We now show that $\overline{A}_1 = \{A_{1,1}, A_{1,2}\}$ and $\overline{A}_2 = \{A_{2,1}, A_{2,2}\}$, where $A_{1,2} := A_1 \setminus A_{1,1}$ and $A_{2,2} := A_2 \setminus A_{2,1}$, satisfy the statement.

For ease of notation, we write $c_i := |A_{1,i}|$, $d_i := |A_{2,i}|$, $e_{ij} := e(A_{1,i}, A_{2,j})$, $c := |A_1|$, $d := |A_2|$ and

$e = e(A_1, A_2)$. Then, we have

$$\begin{aligned} q(\overline{A}_1, \overline{A}_2) &= \frac{1}{n^2} \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{e_{ij}^2}{c_i d_j} \\ &= \frac{1}{n^2} \left(\frac{e_{11}^2}{c_1 d_1} + \sum_{i+j>2} \frac{e_{ij}^2}{c_i d_j} \right) \\ &\overset{(1)}{\geq} \frac{1}{n^2} \left(\frac{e_{11}^2}{c_1 d_1} + \frac{(e - e_{11})^2}{cd - c_1 d_1} \right). \end{aligned}$$

By definition of η, in the new notation we have that $e_{11} = \frac{c_1 d_1 e}{cd} + \eta c_1 d_1$, and so

$$\begin{aligned} n^2 q(\overline{A}_1, \overline{A}_2) &\geq \frac{1}{c_1 d_1} \left(\frac{c_1 d_1 e}{cd} + \eta c_1 d_1 \right)^2 + \frac{1}{cd - c_1 d_1} \left(e - \frac{c_1 d_1 e}{cd} - \eta c_1 d_1 \right)^2 \\ &\geq \frac{1}{c_1 d_1} \left(\frac{c_1 d_1 e}{cd} + \eta c_1 d_1 \right)^2 + \frac{1}{cd - c_1 d_1} \left(\frac{cd - c_1 d_1}{cd} e - \eta c_1 d_1 \right)^2 \\ &= \frac{c_1 d_1 e^2}{c^2 d^2} + \frac{2e\eta c_1 d_1}{cd} + \eta^2 c_1 d_1 + \frac{(cd - c_1 d_1)e^2}{c^2 d^2} - \frac{2e\eta c_1 d_1}{cd} + \frac{\eta^2 c_1^2 d_1^2}{cd - c_1 d_1} \\ &\geq \frac{e^2}{cd} + \eta^2 c_1 d_1 \\ &\overset{(3)}{\geq} \frac{e^2}{cd} + \epsilon^4 cd = n^2 q(A_1, A_2) + \epsilon^4 cd \end{aligned}$$

and we obtain the inequality from the statement by simply dividing by n^2 at each side of the inequality. □

The next lemma shows that applying the previous lemma to all irregular pairs of a partition achieves the desired constant increase in energy.

Lemma 1.16. *Let $0 < \epsilon \leq \frac{1}{4}$, let G be a graph with n vertices, and let $\overline{A} = \{A_1, \dots, A_k\}$ be an even partition of its vertex set with remainder set B such that $|B| \leq \epsilon n$ and $|A_1| = \cdots = |A_k| =: c$. If the partition $\overline{A}$ is not ϵ-regular, then there is an even refinement $\overline{A}' = \{A_1', \dots, A_\ell'\}$ of $\overline{A}$ with remainder set B' such that $k \leq \ell \leq k4^{k+1}$, $|A_0'| \leq |A_0| + \frac{n}{2^k}$, and*

$$q(\widetilde{A}') \geq q(\widetilde{A}) + \frac{\epsilon^5}{2}.$$

Proof. For all $1 \leq i < j \leq k$, let $\overline{A}_{ij}$ be a partition of A_i and $\overline{A}_{ji}$ a partition of A_j as follows. If the pair (A_i, A_j) is ϵ-regular, then $\overline{A}_{ij} := \{A_i\}$ and $\overline{A}_{ji} := \{A_j\}$. Otherwise, we can apply Lemma 1.15 to obtain a partition $\overline{A}_{ij}$ of A_i and a partition $\overline{A}_{ji}$ of A_j with $|\overline{A}_{ij}| = |\overline{A}_{ji}| = 2$ such that

$$q(\overline{A}_{ij}, \overline{A}_{ji}) \geq q(A_i, A_j) + \epsilon^4 \frac{c^2}{n^2}. \tag{4}$$

Now, consider two vertices $u, v \in A_i$ to be equivalent if for every $j \neq i$ they belong to the same set of the partition $\overline{A}_{ij}$. We can define $\overline{A}_i$ to be the set of such equivalence classes. Then, since each partition $\overline{A}_{ij}$

may at most double the number of parts that end up in $\overline{A}_i$, we have that $|\overline{A}_i| \le 2^{k-1}$. Putting all of this together, we have a new (not necessarily even) partition

$$\overline{A}'' := \bigcup_{i=1}^{k} \overline{A}_i$$

of G with reminder set still B. Note that $\overline{A}''$ refines $\overline{A}$, and that

$$k \le |\overline{A}''| \le k2^{k-1} \le k2^k. \tag{5}$$

By hypothesis, we know that $\overline{A}$ is not ϵ-regular, and so there are at least ϵk^2 pairs (A_i, A_j), with $1 \le i < j \le k$, such that the partition $\overline{A}_{ij}$ is non-trivial. Thus,

$$\begin{aligned}
q(\widetilde{A}'') &= \sum_{1\le i<j\le k} q(\overline{A}_i, \overline{A}_j) + \sum_{1\le i\le k} q(\overline{A}_i, \overline{B}) + \sum_{1\le i\le k} q(\overline{A}_i) + q(\overline{B}) \\
&\ge \sum_{1\le i<j\le k} q(\overline{A}_{ij}, \overline{A}_{ji}) + \sum_{1\le i\le k} q(\{A_i\}, \overline{B}) + q(\overline{B}) \\
&\overset{(4)}{\ge} \sum_{1\le i<j\le k} q(A_i, A_j) + \epsilon k^2 \epsilon^4 \frac{c^2}{n^2} + \sum_{1\le i\le k} q(\{A_i\}, \overline{B}) + q(\overline{B}) \\
&= q(\widetilde{A}) + \epsilon^5 (\frac{ck}{n})^2 \\
&\ge q(\widetilde{A}) + \frac{\epsilon^5}{2}.
\end{aligned}$$

First equality follows from the definition of energy, first inequality uses 1. from Lemma 1.14, and last inequality follows from the fact that $|B| \le \epsilon n \le \frac{1}{4}$, so kc is necessarily at least $\frac{3}{4}n$.

Finally, we need to turn $\overline{A}''$ into an even partition. In order to achieve this, we split each part into pieces of equal size, and move the remaining vertices to the reminder set. We need to separate two cases, as we may not have enough vertices to make substantially sized parts.

If $c < 4^k$, we just consider all the parts to be singletons, and keep the reminder set B as it is. Since there are at most k parts in $\overline{A}$, we have that the resulting partition $\overline{A}'$ of size ℓ satisfies $k \le \ell = kc < k4^k$.

Otherwise, if $c \ge 4^k$, consider $A'_1, \dots, A'_\ell$ to be a maximal collection of disjoint sets of size $d := \lfloor \frac{c}{4^k} \rfloor \ge 1$ such that each A'_i is contained in some part of $\overline{A}''$. Then, the remainder set B' is obtained by adding to B all the remaining vertices from all the parts of $\overline{A}''$, or simply $B' = G \setminus \bigcup_{i=1}^{\ell} A'_i$.

The resulting partition $\overline{A}' = \{A'_1, \dots, A'_\ell\}$ is a refinement of $\overline{A}''$ and, following 2. from Lemma 1.14, satisfies

$$q(\widetilde{A}') \ge q(\widetilde{A}'') \ge q(\widetilde{A}) + \frac{\epsilon^5}{2}.$$

Now, no more than $\frac{c}{d} \le 4^{k+1}$ sets A'_i can lie within the same part of $\overline{A}$, so the condition $k \le \ell \le k4^{k+1}$

is satisfied. Also, no more than d vertices are left out from each part of $\overline{A}''$, and so

$$\begin{aligned}
|B'| &\leq |B| + d|\overline{A}''| \\
&\overset{(5)}{\leq} |B| + \frac{c}{4^k}k2^k \\
&= |B| + \frac{kc}{2^k} \\
&\leq |B| + \frac{n}{2^k}.
\end{aligned}$$

Thus, the partition $\overline{A}'$ with remainder set B' satisfies all the conditions in the statement, and we are done. □

We now have all the tools required to prove Szemerédi's Regularity Lemma. The idea will be to start with an arbitrary even partition, with a large enough number of parts and small enough reminder set, and then keep refining it until we reach a regular partition. Then, reaching regularity is inevitable, as the previous result guarantees a constant increase in energy which we previously proved to be upper bounded.

Proof of Theorem 1.12. Let $\epsilon > 0$, $m \geq 1$ and assume without loss of generality that $\epsilon \leq \frac{1}{4}$. This is possible by monotonicity of the regularity condition[3]. Also, set $s := \frac{2}{\epsilon^5}$.

While refining repeatedly the partition using Lemma 1.16, (s times) we need to make sure that the remainder set does not grow too large, as the lemma requires it to be at most ϵn. At each refinement, the size of the reminder set increases by at most $\frac{n}{2^k}$, where k is the number of parts of the partition before refining. Since at each iteration the number of parts can only increase, at most $\frac{n}{2^k}$ vertices are added to the reminder set. By choosing k and n large enough, we can ensure that the initial size of the remainder set and the total growth of it over all the s steps are at most $\frac{\epsilon n}{2}$ each.

With this in mind, we choose k large enough to satisfy $\frac{s}{2^k} \leq \frac{\epsilon}{2}$, and n large enough so that $k \leq \frac{\epsilon n}{2}$. Then,

$$k + \frac{sn}{2^k} \leq \frac{\epsilon n}{2} + \frac{\epsilon n}{2} = \epsilon n. \tag{6}$$

Now, let's bound the number of parts of the partition at the end of the process. Since at each step the number of parts goes from r up to at most $r4^{r+1}$, starting with k parts we can simply set $M := \max(f^s(k), 2\frac{k}{\epsilon})$, where $f(r) = r4^{r+1}$. The second term ensures that if n is sufficiently large (in particular when $n \geq M$) then (6) holds.

Now, given a graph G with $n \geq m$ vertices, we can build a partition into k', with $m \leq k' \leq M$ parts, and with remainder B as follows. If $n \leq M$, simply take the partition to be all the vertices as singletons, and the remainder set to be empty. The resulting partition is trivially ϵ-regular, as pairs of singletons are always either complete or empty. Suppose now that $n > M$. We randomly partition the vertex set of G into $k := m$ maximal parts of equal size, and put the remaining vertices in the remainder set. This remainder set has size at most $k - 1 < \epsilon n$ by (6). We now can apply Lemma 1.16 repeatedly, as the choice of k and $n \geq M$ in (6) ensures that the reminder is at most ϵn during s steps. But this process must stop in at most s steps, as the energy of the partition increases by at least $\frac{\epsilon^5}{2}$ at each step, so after s steps the energy would be at least 1, which is the theoretical maximum as shown earlier. □

[3] By monotonicity of the regularity property we mean that, if a partition is ϵ-regular, than it is also ϵ'-regular for any $\epsilon' \geq \epsilon$. This follows the Definition 1.9, as both the allowed error in regular pairs and the number of irregular ones permitted increase with the regularity parameter.

For the matters of this thesis, it is important to note that it is actually known that:

- The remainder set can be avoided in the resulting partition of Szemerédi's Regularity Lemma, moving from an even partition to an equitable one (Definition 1.10). This is done by evenly distributing the leftover vertices evenly throughout the large clusters of the part, and overserving that energy lost in this operation is smaller than the gains from the former.
- It can be ensured that not only (most) pairs of different parts are regular, but also (most) parts with themselves (self-pairs) satisfy this property.

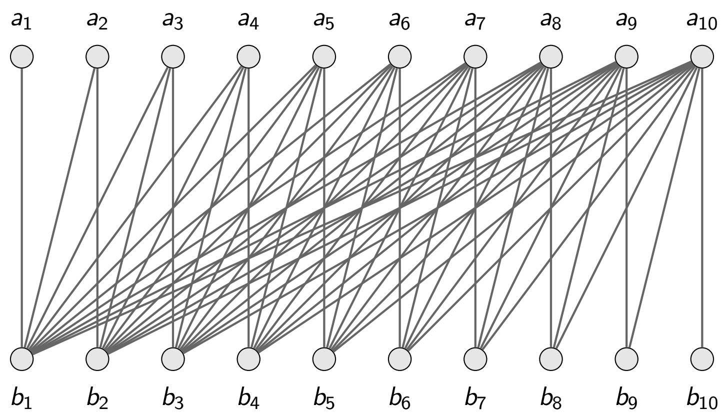
In this work we have focused our attention to the case of the Stable Regularity Lemma, but we have opted to include a proof of a (less technically involved but conceptually complete) version of the SzRL for completeness.

The interested reader is redirected to [3, 16][4] for more detailed proofs on how to obtain such partitions.

[4] In [16], authors show how to obtain a regular parition that includes regularity within pairs themselves, but omit the details on how to get an equitable partition. [3] proves the existence of a partition has regular self-pairs and no reminder, but the proof of the critical lemma to refine a partition into an equitable one (at the loss of a small amount of energy) is hinted at but omitted.

2. Stable graphs

In this section we introduce the class of *stable* graphs. A graph is considered stable, if it does not contain bi-induced (see Definition 1.4) large half-graphs, a particularly non-quasi-random structure in graphs. See Figure 1 for an example of such a graph.



	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
a_1	1	0	0	0	0	0	0	0	0	0
a_2	1	1	0	0	0	0	0	0	0	0
a_3	1	1	1	0	0	0	0	0	0	0
a_4	1	1	1	1	0	0	0	0	0	0
a_5	1	1	1	1	1	0	0	0	0	0
a_6	1	1	1	1	1	1	0	0	0	0
a_7	1	1	1	1	1	1	1	0	0	0
a_8	1	1	1	1	1	1	1	1	0	0
a_9	1	1	1	1	1	1	1	1	1	0
a_{10}	1	1	1	1	1	1	1	1	1	1

Figure 1: *On the left*, a half-graph with 2×10 vertices. *On the right*, the corresponding bi-adjacency matrix.

First, stability implies a bounded *Vapnik-Chervonenkis (VC) dimension*, which limits the variety of neighborhoods of vertices within the graph. While stability implies a bounded VC-dimension for the entire graph (See [6]), our work primarily focuses on bounding the VC-dimension restricted to a subset of vertices. This is formalized in Lemma 2.10.

Second, stability implies a finite *tree bound*. This property is the foundational tool we use to prove the existence of parts that are quasi-random with respect to the rest of the graph. We use this to establish the existence of indivisible parts in **??** (**??**) and excellent parts in Section 3 (Lemma 3.6).

2.1 The k-order property

First, we formally define stability as the non-k-order property, where k determines the size of the excluded half-graphs.

Definition 2.1. Let G be a graph. We say that G has the *k-order property*[5] if there exist two sequences of vertices $\langle a_i \mid i \in \{1, \dots, k\} \rangle$ and $\langle b_i \mid i \in \{1, \dots, k\} \rangle$ such that for all $i, j \leq k$, $a_i R b_j$ if and only if $i \geq j$. Otherwise, we say that G has the *non-k-order property* or that G is *k-stable*. Additionally, we say that G has the *disjoint k-order property* if $\langle a_i \rangle_i \cap \langle b_i \rangle_i = \emptyset$.

Remark 2.2. Notice that the vertices within each sequence $\langle a_i \rangle_i$, $\langle b_i \rangle_i$ must be distinct, as their neighborhoods within the other sequence differ, which makes this definition equivalent to "the graph not containing a bi-induced copy of a k-half-graph", as defined in Definition 1.4. However, the sequences themselves need not be disjoint. One may have $a_i = b_j$, provided $i < j$ (so that $\neg(a_i R b_j)$). Furthermore, the definition does not specify the presence or absence of edges within the same sequence. Consequently, the non-k-order

[5]Note that the vertex tuples $\langle a_i \rangle_i$ and $\langle b_i \rangle_i$ are "ordered" according to their neighbourhood, and thus the name *k-order* comes very naturally.

property requires avoiding not only the k-half-graph, but a whole family of induced subgraphs (the ones resulting by adding edges in the independent sets $\langle a_i \rangle_i$, $\langle b_i \rangle_i$, and possibly identifying some pairs of vertices (a_i, b_j)).

Possibly add visual example of this too.

Remark 2.3. G having the k-order property implies that G has the k'-order property for all $k' \leq k$. Conversely, G having the non-k-order property implies that G has the non-k'-order property for all $k' \geq k$.

An important concept used all over the thesis is that of *exceptional edges* and *exceptional vertices*. That is, edges and vertices that, in the context of a pair of sets of vertices, do not "behave" as the rest. In order to classify what is the expected behaviour in a graph, or more specifically, in a pair of sets of vertices, we define the *truth value*.

Echarle un ojo.

Definition 2.4 (Truth value)**.** Let G be a graph. For any (not necessarily disjoint) $A, B \subseteq G$, we say that

$$t(A, B) = \begin{cases} 0 & \text{if } |\{(a, b) \in A \times B \mid aRb, a \neq b\}| < |\{(a, b) \in A \times B \mid \neg aRb, a \neq b\}| \\ 1 & \text{otherwise} \end{cases}$$

is the *truth value* of the pair (A, B). That is, $t(A, B) = 0$ if A and B are mostly disconnected, and $t(A, B) = 1$ if they are mostly connected. When $B = \{b\}$, we write $t(A, b)$ instead of $t(A, \{b\})$, and we say that it is the truth value of A with respect to b.

In this context, we say that a vertex $a \in A$ is *exceptional* with respect to $B \subseteq G$ if $t(a, B) \not\equiv t(A, B)$, or that it is exceptional with respect to $b \in G$ if $aRb \not\equiv t(A, b)$. On the other hand, we say that an edge ab with $a \in A$ and $b \in B$ is exceptional in (A, B) if $aRb \not\equiv t(A, B)$. Also, it is useful to define the following set of vertices.

- $B_{A,b} = \{a \in A \mid aRb \equiv t(A, b)\}$, i.e. the set of non-exceptional vertices of A with respect to b.
- $\overline{B}_{A,b} = \{a \in A \mid aRb \not\equiv t(A, b)\}$, the set of exceptional vertices of A with respect to b.
- $B^+_{A,b} = \{a \in A \mid aRb\}$, the vertices of A connected to b.
- $B^-_{A,b} = \{a \in A \mid \neg aRb\}$, the vertices of A that are not connected to b.

With this notation, notice that either $t(A, b) = 1$ and thus $B_{A,b} = B^+_{A,b}$, or $t(A, b) = 0$ and $B_{A,b} = B^-_{A,b}$. Naturally, sets of vertices A with a large number of large $\overline{B}_{A,b}$ are a great obstacle towards creating (close to) full or empty pairs.

2.2 VC-dimension and the Sauer-Shelah Lemma

Recall from the introduction, that graphs with the non-k-order property have bounded VC-dimension. We proceed to formally define the concept of VC-dimension in Definition 2.6, and some of its properties (Lemma 2.7) to bound the number of $\overline{B}_{A,b}$ under the non-k-order property (Lemma 2.10).

Definition 2.5. Let G be a set and $S = \{S_i \subseteq G \mid i \in I\}$ be a family of sets. A set $A \subseteq G$ is said to be *shattered* by S (and S is said to *shatter* A) if for every $B \subseteq A$, there exists $S_i \in S$ such that $S_i \cap A = B$.

Definition 2.6. Let G be a set and $S = \{S_i \subseteq G \mid i \in I\}$ be a family of sets. The *VC-dimension* of S is the size of the largest set $A \subseteq G$ that is shattered by S.

Lemma 2.7 (Sauer-Shelah (-Perles -Vapnik-Chervonenkis) Lemma, [11], [12])**.** *Let G be a set and $S = \{S_i \subseteq G \mid i \in I\}$ be a family of sets. If the VC-dimension of S is at most k, and the union of all the sets in S has n elements, then S consists of at most $\sum_{i=0}^{k} \binom{n}{i} \leq n^k$ sets.*

We'll begin by proving a stronger version of this lemma from Pajor, for which Sauer-Shelah will be a straightforward consequence.

Lemma 2.8 (Pajor's variant, [10])**.** *Let G be a set and S be a finite family of sets in G. Then S shatters at least $|S|$ sets.*

Proof. We will prove this by induction on the cardinality of S. If $|S| = 1$, then S consists of a single set, which only shatters the empty set. If $|S| > 1$, we may choose an element $x \in S$ such that some sets of S contain x and some do not. Let $S^+ = \{s \in S \mid x \in S\}$ and $S^- = \{s \in S \mid x \notin S\}$. Then $S = S^+ \sqcup S^-$, and both S^+ and S^- are non-empty. By induction hypothesis, we know that $S^+ \subsetneq S$ shatters at least $|S^+|$ sets, and $S^- \subsetneq S$ shatters at least $|S^-|$ sets. Let T, T^+, T^- be the families of sets shattered by S, S^+ and S^- respectively. To conclude the proof, we just need to show that for each element in T^+ and T^-, there is a corresponding one in T. If a set is shattered by only one of the two families S^+ and S^-, then it only contributes by one unit to $|T^+| + |T^-|$ and one unit to $|T|$. Notice that no set shattered by S^+ or S^- may contain x, otherwise all or none of the intersections will contain this element. Thus, if a set s is shattered by both S^+ and S^-, it will contribute by two units to $|T^+| + |T^-|$ and one unit to $|T|$. But then, for each such set, we can consider $s \cup \{x\}$ which is not in T^+ or T^-, but it is in T. Indeed, for each subset of s, if it does not contain x it is the intersection with some set in $S^- \subsetneq S$, and if it does contain x it is the intersection with some set in $S^+ \subsetneq S$. All in all, we conclude that

$$|T| \geq |T^+| + |T^-| \geq |S^+| + |S^-| \geq |S|,$$

which proves the statement. $\square$

Proof of Lemma 2.7. Suppose that $\bigcup_{s \in S} s$ has n elements. By Lemma 2.8, S shatters at least $|S|$ subsets, and since there are at most $\sum_{i=0}^{k} \binom{n}{i}$ subsets of S of size at most k, if $|S| > \sum_{i=0}^{k} \binom{n}{i}$, at least one of the shattered sets has cardinality larger than k, and hence the VC-dimension of S is larger than k. $\square$

Next, we want to prove that if G has the non-k-order property, then the size of the family of exceptional sets of A, relative to each vertex $b \in G$, is bounded by $|A|^k$. Instead, we prove a stronger result, that is we prove this same bound with only the condition that G has the disjoint non-k-order property (recall that then the two sequences of vertices in the Definition 2.1 are disjoint). Even though results in this thesis use the weaker non-disjoint version of Lemma 2.10, we prove it in this form to highlight that the non-disjointness of the sequences (and thus the broadening of the excluded structures in stable graphs) is not crucial to obtain this value of the bound, but later on[6].

HERE!

Lemma 2.9. *Let G be a graph and $A \subseteq G$. Let $S = \{B^+_{A,b} \mid b \in G \setminus A\}$. If S has VC-dimension at least k, then G has the (disjoint) k-order property.*

Proof. If S has VC-dimension k, then it shatters a set $A' \subseteq A$ of size k. Now, choose any order of the vertices of $A' = \langle a_1, \ldots, a_k \rangle$. Then, consider the increasing sequence of subsets $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k = A'$, where $A_i = \{a_j \mid j \in \{1, \ldots, i\}\}$. Since A' is shattered by S, for each $i \in \{1, \ldots, k\}$ there exists a

[6]Specifically, the non-disjointness of the sequences beceomes relevant in **??** of **??** and in Lemma 3.6 of Section 3, as it allows to prove the existence of quasi-randompairs.

$b_i \in G$ such that $b_i R a$ if and only if $a \in A_i$. In particular, the two sequences $\langle a_i \mid i \in \{1, \dots, k\}\rangle$ and $\langle b_i \mid i \in \{1, \dots, k\}\rangle$ satisfy

$$a_i R b_j \Leftrightarrow i \leq j,$$

and thus G has the k-order property. □

Lemma 2.10 (Claim 2.6 in [8]). *Let G be a graph with the (disjoint) non-k-order property. Then, for any finite non-trivial $A \subseteq G$,*

$$|\{B^+_{A,b} \mid b \in G\}| \leq |A|^k.$$

Proof. By Lemma 2.9, if G has the non-k-order property, then the family $\{B^+_{A,b} \mid b \in G \setminus A\}$ has VC-dimension at most $k-1$, so by the Sauer-Shelah Lemma 2.7 we have $\{B^+_{A,b} \mid b \in G \setminus A\} \leq \sum_{i=0}^{k-1} \binom{|A|}{i}$. Since $\{B^+_{A,b} \mid b \in A\} \leq |A|$, we conclude that

$$|S| = |\{B^+_{A,b} \mid b \in G\}| \leq \sum_{i=0}^{k-1} \binom{|A|}{i} + |A|.$$

Finally, when $|A| = n, k > 1$:

This conditions should be set at some point of the tfm. Specify that if they are not met, the problem becomes trivial.

- if $n \leq k$, then $|S| \leq 2^n \leq 2^k \leq n^k$.
- if $n > k$, then $|S| \leq \sum_{i=0}^{k-1} \binom{n}{i} + n \leq n^{k-1} + n \leq 2n^{k-1} \leq n^k$.

We conclude that $|S| \leq n^k$. □

Remark 2.11. The condition $n, k > 1$ is trivial. If $n = 1$ then A is the trivial graph with a single vertex. If $k = 1$ we are not allowing even a single edge, so G is the empty graph.

The following equivalent versions of Lemma 2.10 will be useful in the different sections of the thesis. The idea is that any choice, of either the exceptional or the non-exceptional vertices set of A with respect to each vertex $b \in G$, has the same bound. The proof is given in Appendix A.

Corollary 2.12. *Let G be a graph with the non-k-order property. Then:*

1. *For any finite $A \subseteq G$*

$$|\{B^-_{A,b} \mid b \in G\}| \leq |A|^k.$$

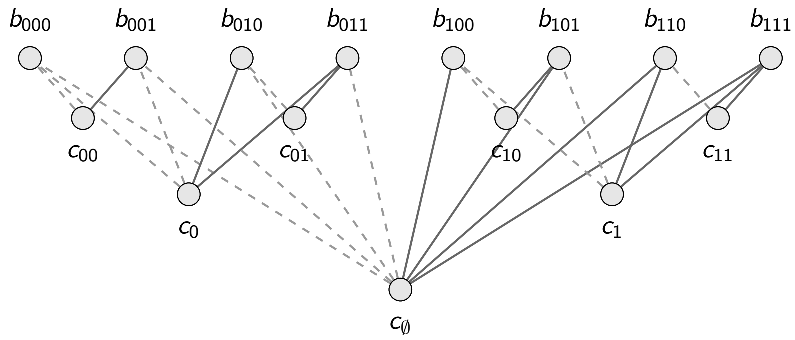
2. *For any finite $A \subseteq G$*

$$|\{\overline{B}_{A,b} \mid b \in G\}| \leq |A|^k.$$

2.3 Tree bound

During the next sections, it will be a key point proving that some sort of "regular" subgraphs (*independent* in **??** and *excellent* in Section 3) exist in a given stable graph. A useful structure strongly related to the k-order property is the k-*tree*.

Defining such concept requires us to introduce some tuple notation. First of all, we use $\langle a_1, \dots, a_n\rangle$ to denote an n-tuple which is an ordered list of objects (in this work, such objects will be integers). When using such tuples as a subscript of a variable and the tuples are sequences of 0's and 1's, we may skip



	c_{00}	c_0	c_{01}	$c_\emptyset$	c_{10}	c_1	c_{11}
b_{000}	0	0	-	0	-	-	-
b_{001}	1	0	-	0	-	-	-
b_{010}	-	1	0	0	-	-	-
b_{011}	-	1	1	0	-	-	-
b_{100}	-	-	-	1	0	0	-
b_{101}	-	-	-	1	1	0	-
b_{110}	-	-	-	1	-	1	0
b_{111}	-	-	-	1	-	1	1

Figure 2: *On the left*, example of a 3-tree. Solid lines show adjacent vertices, and dashed lines show non-adjacent vertices. Pairs of vertices without a line may or may not be connected. In particular, notice that connections between disjoint sub-trees are not defined, and may be edges or non-edges in any combination (e.g. the pair (c_1, c_{01})). *On the right*, the corresponding bi-adjacency matrix.

the $\langle\rangle$ and commas for ease of read (for example, $\langle 0,0,1\rangle$ would be written as 001). The empty tuple is denoted as $\langle\cdot\rangle$ and occasionally in subscripts as $\emptyset$. A useful operation is the concatenation of tuples, which we denote with the symbol $\frown$. Finally, we say that $\eta_1 \triangleleft \eta_2$ if for some tuple η_3 we have that $\eta_1 \frown \eta_3 = \eta_2$. We now have all the notation to formally define the concept of k-tree.

Definition 2.13. A *k-tree* in G is an ordered pair $H = (\overline{c}, \overline{b})$ comprising two (not necessarily disjoint) sequences:

- $\overline{c} = \{c_\eta \in G \mid \eta \in \{0,1\}^{<k}\}$, the set of *nodes* and
- $\overline{b} = \{b_\rho \in G \mid \rho \in \{0,1\}^{k}\}$, the set of *branches*,

satisfying that for all $\eta \in \{0,1\}^{<k}$ and $\rho \in \{0,1\}^k$, if for some $\ell \in \{0,1\}$ we have $\eta \frown \langle\ell\rangle \triangleleft \rho$, then $b_\rho R c_\eta \equiv \ell$. Nothing else is said about the adjacency of the rest of pairs of vertices. See Figure 2 for an example of such a structure.

Similarly to stability, we can define the *tree bound* of a graph to measure the level of freeness from k-trees of graph.

Definition 2.14 (Definition 2.11 in [8])**.** Suppose G is a finite graph. We denote the *tree bound* $k_{**} = k_{**}(G)$ as the minimal positive integer such that there is no k_{**}-tree $H = (\overline{c}, \overline{b})$ in G.

As mentioned earlier, the tree bound is closely related to the k-order property. The following theorem states that if a graph has a sufficiently large tree bound, then it has the k-order property and vice versa.

Theorem 2.15 (Lemma 6.7.9 in [5])**.** *If a graph G has the $2^{k_{**}}$-order property, then the tree bound of G is at least $k_{**}+1$. On the other hand, if a graph G has tree bound at least $k_{**} = 2^{k_*+1} - 1$, then it has the k_*-order property.*

Proof. For the first implication, just consider $\langle a_i \mid i \in \{1, \dots, 2^{k_{**}} - 1\}\rangle$ and $\langle b_i \mid i \in \{0, \dots, 2^{k_{**}} - 1\}\rangle$ to be the two sequences of vertices witnessing the $2^{k_{**}}$-order property in G, and thus for all $i, j \le k$, $a_i R b_j$ if and only if $i \ge j$. It is straightforward to build a k_{**}-tree using these vertices. Take $\langle b_i \mid i \in \{0, \dots, 2^{k_{**}} - 1\}\rangle$ to be the branches of the tree, indexing them by the binary decomposition of their index, and run the following construction for the nodes:

- Initiate $C_\emptyset = \langle a_i \mid i \in \{0, \dots, 2^{k_{**}} - 2\} \rangle$.
- At each step $k \in \{0, k_{**} - 1\}$, for each $\eta \in \{0,1\}^k$, take the middle element of the sequence C_η and set it to be the node c_η. Then, the remaining first half of C_η becomes the sequence $C_{\eta \frown \langle 0 \rangle}$ and the second half is $C_{\eta \frown \langle 1 \rangle}$.

Notice that at each step, the sequence C_η has an odd number of elements. The resulting two sequences of nodes and branches form a k_{**}-tree. See Figure 3 for a visual example of this construction. This finishes the argument for the first part of the argument.

During the proof of the second implication, we say that a set of nodes N of a k-tree $H = (\bar{c}, \bar{b})$ *contains* a k'-tree H', if there exists a map $f \colon \{0,1\}^{<k'} \longrightarrow \{0,1\}^{<k}$ such that for all $\eta, \eta' \in \{0,1\}^{<k'}$, $c_{f(\eta)}$ and $c_{f(\eta')}$ are in N, and if $\eta \frown \langle i \rangle = \eta'$ then $f(\eta) \frown \langle i \rangle \triangleleft f(\eta')$, for all $i \in \{0,1\}$. This clearly implies that there is a k'-tree H' with nodes in N and branches in $\bar{b}$. Simply, for each $\eta \in \{0,1\}^{k'-1}$, pick exactly two branches b_{ρ_0} and b_{ρ_1} such that $f(\eta) \frown \langle i \rangle \triangleleft \rho_i$ for $i \in \{0,1\}$.

Also, we will use H'_i to denote the subtree of H' consisting of the nodes $c_{f(\eta)}$ and branches $b_{f(\rho)}$ such that $\langle i \rangle \triangleleft \eta$ and $\langle i \rangle \triangleleft \rho$, with $\eta \in \{0,1\}^{<k'}$ and $\rho \in \{0,1\}^{k'}$. Notice that, if H is an h-tree, H_0 and H_1 are $(h-1)$-trees, and together with the root node singleton $\{c_{f(\emptyset)}\}$, they partition H.

Next, we prove the following claim, which shows that we can always find a tree in one of the parts of a bipartition of the nodes of a larger tree.

Claim 2.16. *For all $n, k \geq 0$, if H is a $(n+k)$-tree and the nodes of H are partitioned into two sets N and P, then either N contains an n-tree or P contains a k-tree.*

Proof of Claim 2.16. We prove this by induction on $n+k$. Clearly, the statement is true for the trivial case $n = k = 0$. Suppose $n + k > 0$. Without loss of generality, we may assume that the root node $c_\emptyset$ is in N. Let Z_i be the set of nodes of H_i, which is an $(n+k-1)$-tree. By I.H., for each $i \in \{0,1\}$, either $N \cap Z_i$ contains an $(n-1)$-tree or $P \cap Z_i$ contains a k-tree. If either $P \cap Z_0$ or $P \cap Z_1$ contains a k-tree, then P contains a k-tree, and we are done. Otherwise, both $N \cap Z_0$ and $N \cap Z_1$ contain an $(n-1)$-tree. Since $c_\emptyset$ is in N, the root with the two $(k-1)$-tree are in N and make an n-tree. Thus, N contains an n-tree. ☐

Suppose that G has a tree bound of at least $2^{k_*+1} - 1$, and thus contains a $(2^{k_*+1} - 2)$-tree. We show by induction on $k_* - r$, with $1 \leq r \leq k_*$, that the following scenario S_r holds. There are

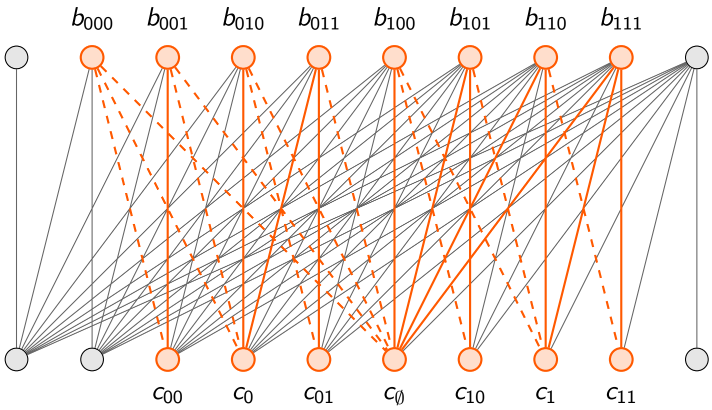
$$b_0, c_0, \dots, b_{q-1}, c_{q-1}, H, b_q, c_q, \dots, b_{k_*-r-1}, c_{k_*-r-1} \tag{7}$$

such that:

1. for all $i \in \{0, \dots, k_* - r - 1\}$, b_i and c_i are vertices in G, and H is a $(2^{r+1} - 2)$-tree in G.
2. for all $i, j \in \{0, \dots, k_* - r - 1\}$, $b_i R c_j \Leftrightarrow i \geq j$.
3. if c is a node of H, $b_i R c \Leftrightarrow i \geq q$.
4. if b is a branch of H, $b R c_i \Leftrightarrow i < q$.

The initial case S_{k_*} only requires the existence of a $(2^{k_*+1} - 2)$-tree in G, which is the premise. If the final case S_1 is true, then we are done: this case assumes that H is a 2-tree, hence there is a node c_* and branch b_* in H that are adjacent. These vertices satisfy conditions 3. and 4., so the sequence resulting from replacing H in (7) by b_*, c_* implies that G has the k_*-order property.

specify k-order



			c_{00}	c_0	c_{01}	$c_\emptyset$	c_{10}	c_1	c_{11}	
	1	0	0	0	0	0	0	0	0	0
b_{000}	1	1	0	0	0	0	0	0	0	0
b_{001}	1	1	1	0	0	0	0	0	0	0
b_{010}	1	1	1	1	0	0	0	0	0	0
b_{011}	1	1	1	1	1	0	0	0	0	0
b_{100}	1	1	1	1	1	1	0	0	0	0
b_{101}	1	1	1	1	1	1	1	0	0	0
b_{110}	1	1	1	1	1	1	1	1	0	0
b_{111}	1	1	1	1	1	1	1	1	1	0
	1	1	1	1	1	1	1	1	1	1

Figure 3: *On the left*, example of a 3-tree in a half-graph with 2×10 vertices. Orange lines and nodes highlight the 3-tree structure, with dashed orange lines remarking the relevant non-edges. *On the right* is the corresponding bi-adjacency matrix. Again, orange cells highlight edges relative to the 3-tree structure.

To conclude the proof it remains to show that if S_r holds, then so does S_{r-1} for $r > 1$. Assume S_r. Fixing $h = 2^r - 2$, by 1. we have that H is a $(2h+2)$-tree. For each branch b of H we denote with $Z(b)$ the set of nodes c of H such that bRc.

We have two cases:

- *Case 1.* There is a branch b_* such that $Z(b_*)$ contains an $(h+1)$-tree H'. In that case, we can take c_* to be the top node of the $(h+1)$-tree, and H_* to be the h-subtree H'_0. Replacing H in (7) with H_*, b_*, c_* in this order, the conditions for S_{r-1} are satisfied.

- *Case 2.* There is no branch b such that $Z(b)$ contains an $(h+1)$-tree. Now, let c_* be the top node of H, Z_1 the set of nodes of H_1, and b_* any branch of H_1. By the case assumption, $Z(b) \cap Z_1$ contains no $(h+1)$-tree, so by the claim and the fact that Z_1 is the set of nodes of a $(2h+1)$-tree, $Z_1 \setminus Z(b)$ contains an h-tree H_*. Finally, replacing H in (7) by b_*, c_*, H_* in this order, the conditions for S_{r-1} are satisfied.

In any case, S_{r-1} is satisfied, and the proof is complete. $\square$

Remark 2.17. The key point of the proof of the second implication of Theorem 2.15 is that the found bi-induced half-graph copy does not only utilize edges and non-edges of the k-tree structure itself. Instead, it relies on the fact that, for a tall enough tree, a k-order must appear in some way, leveraging some "unknown" edges, independently of disposition of the edges.

The second implication of this theorem is of special interest in the next sections, as it proves that in the context of a k-stable graph no $2^{k+1} - 2$-trees can be found.

Given that the stability of the studied graphs is fixed for all proofs in the next sections, from now on we will use k_* as the value of the non-k-property of the studied graphs, and k_{**} for the associated tree bound.

3. The Stable Regularity Lemma

This section focuses in leveraging the stability of a graph to create a stable partition whose maximum number of parts only depends on the error and stability parameters; hence it does not grow with the size of the graph. In order to do so, we first prove the existence of a partition whose parts satisfy a property which we prove stronger than regularity: *excellence*.

3.1 Goodness and excellence

We proceed to formalize this concept.

Definition 3.1 (Definition 5.2 (1) in [8])**.** Let G be a finite graph with the non-k_*-property. We say that $A \subseteq G$ is *ϵ-good* when for every $b \in G$ the truth value $t = t(b, A) \in \{0, 1\}$ satisfies $|\overline{B}_{A,b}| = |\{a \in A \mid aRb \not\equiv t\}| < \epsilon|A|$.

Definition 3.2 (Definition 5.2 (2) in [8])**.** Let G be a finite graph with the non-k_*-property. We say that $A \subseteq G$ is *(ϵ, ζ)-excellent* when A is ϵ-good and, if B is ζ-good, then the truth value $t = t(B, A)$ satisfies $|\{a \in A \mid t(a, B) \neq t(A, B)\}| < \epsilon|A|$. In particular, we say A is *ϵ-excellent* if A is (ϵ, ϵ)-excellent.

We now make some observations about these two properties.

Remark 3.3. For comparison with the properties studied in the previous section, a set being ϵ-good is equivalent to the set being f-indivisible with $f(n) = \epsilon n$, while ϵ-indivisibility is a much stronger condition then ϵ-goodness, as for large enough n, we have that $n^\epsilon < \epsilon n$.

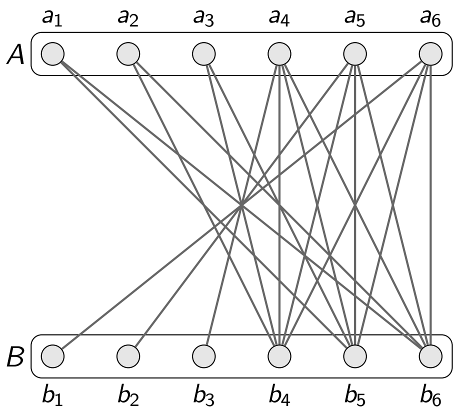
On the other hand, ϵ-excellence carries some kind of reciprocity with other good (and in particular, excellent) sets, which makes it particularly suitable for studying quasi-randomness between pairs of sets. While independence and goodness only bound the number of exceptions with each vertex of the graph independently, excellence of a set A also ensures that the truth values of each of its vertex with respect to each good set in the graph remains mostly the same. **??** shows an example of an ϵ-good set that, as it does not satisfy this reciprocity condition with another good set, it is not ϵ-excellent.

Remark 3.4. If $A, B \subseteq G$ are two (not necessarily disjoint) subsets of vertices with A (ϵ, ϵ')-excellent and B ϵ'-good set, then the number of exceptional edges between A and B, i.e. these vertex pairs that do not follow $t(A, B)$, is relatively small:

$$|\{\text{Exceptional edges between } A \text{ and } B\}| < \epsilon|A||B| + (1-\epsilon)|A|\epsilon'|B| = (\epsilon + (1-\epsilon)\epsilon')|A||B|.$$

A relevant example is that of two (not necessarily disjoint) ϵ-excellent sets, in which case we have that the density (as defined in Definition 1.5) of exceptional edges between them is less than 2ϵ. See Remark 1.6 to see the relation of this density and the real fraction of exceptional edges of the pair.

Remark 3.5. A final important remark, is the fact that differently then most quasi-random properties, ϵ-excellence is not "monotonic". That is, in general, for $\epsilon < \epsilon'$, a set being ϵ-excellent does not imply it being also ϵ'-excellent (and trivially neither the converse). See Figure 4 for a counterexample to the monotonicity of this property. More precisely, each of the two variables in the (ϵ, ϵ')-excellence are oppositely monotonic. That is, if a given set is $(\epsilon_1, \epsilon'_1)$-excellent, then it is also $(\epsilon_2, \epsilon'_2)$-excellent for all $\epsilon_1 \leq \epsilon_2$ and $\epsilon'_1 \geq \epsilon'_2$, since restricting the condition on the goodness of the relevant good sets (ϵ'_1 to ϵ'_2) takes less of such sets into account, and relaxing the condition on the "exceptional truth values" (ϵ_1 to ϵ_2) only enlarges the error accepted.



	b_1	b_2	b_3	b_4	b_5	b_6
a_1	0	0	0	0	1	1
a_2	0	0	0	1	0	1
a_3	0	0	0	1	1	0
a_4	0	0	1	1	1	1
a_5	0	1	0	1	1	1
a_6	1	0	0	1	1	1

Figure 4: Example of the ϵ-excellence property not being monotonic. *On the left*, a bipartite graph with two independent sets A and B. A simple exhaustive check shows that A is $\frac{1}{5}$-excellent. On the other hand, raising the value of ϵ up to $\frac{2}{5}$ introduces a new $\frac{2}{5}$-good set B witnessing that A is not $\frac{2}{5}$-excellent, as half of the vertices of A have one truth value, and half the other. *On the right* is the corresponding bi-adjacency matrix.

3.2 Excellent partitions

The first step towards constructing a partition of excellent sets is to prove the existence of such sets under the stability condition. Similar to **??** in **??**, we prove their existence by assuming the converse and getting to contradiction with the tree bound.

We actually show two versions of the same lemma on existence of excellent sets. Lemma 3.6 is slightly more readable, while Lemma 3.8 is the one we will be using in further proofs, as it fixes the possibles sizes of the resulting set. For that reason, in this section we only prove the first one, and leave the proof of the latter in Appendix A.

Lemma 3.6 (Claim 5.4 (I) in [8]). *Let G be a finite graph with the non-k_*-order property. Let $\zeta \le \frac{1}{2^{k_{**}}}$, $\epsilon \in (0, \frac{1}{2})$. Then, for every $A \subseteq G$ with $|A| \ge \frac{1}{\epsilon^{k_{**}}}$ there exists an (ϵ, ζ)-excellent subset $A' \subseteq A$ such that $|A'| \ge \epsilon^{k_{**}-1}|A|$.*

Proof. Suppose the converse. We use this fact to build two family of sets $\{B_\eta \mid \eta \in \{0,1\}^{<k_{**}}\}$ and $\{A_\eta \mid \eta \in \{0,1\}^{\le k_{**}}\}$ inductively over $k \le k_{**}$, where $k = |\eta|$, satisfying:

1. $A_{\langle\cdot\rangle} = A$.
2. B_η is a ζ-good set witnessing that A_η is not (ϵ, ζ)-excellent, for $k < k_{**}$.
3. $A_{\eta^\frown\langle i\rangle} = \{a \in A_\eta \mid t(a, B_\eta) \equiv i\}$ for all $i \in \{0,1\}$ and $k < k_{**}$.
4. $|A_{\eta^\frown\langle i\rangle}| \ge \epsilon|A_\eta|$ for all $i \in \{0,1\}$ and $k < k_{**}$.
5. $|A_\eta| \ge \epsilon^k|A|$, for $k \le k_{**}$.
6. $A_\eta = A_{\eta^\frown\langle 0\rangle} \sqcup A_{\eta^\frown\langle 1\rangle}$, for $k < k_{**}$.
7. $\overline{A_k} = \{A_\eta \mid \eta \in \{0,1\}^k\}$ is a partition of A, for $k \le k_{**}$.

First of all, notice that at each step, the non-(ϵ,ζ)-excellence of A_η comes by the initial supposition (negation of the statement's thesis) and by IH from 1. or 5.. This allows the existence of the B_η claimed in 2.. Condition 4. follows from the definition of $A_{\eta\frown\langle i\rangle}$ in 3. and the fact that B_η is witnessing that A_η is not (ϵ,ζ)-excellent. Applying recursively this last point we obtain 5.. Finally, by definition 3., we have the disjoint union 6. which ensures the partition 7..

Now, our goal is to build two sequences $\{b_\eta \mid \eta\in\{0,1\}^{<k_{**}}\}$ and $\{a_\eta \mid \eta\in\{0,1\}^{k_{**}}\}$ to contradict the tree bound k_{**}. First of all, notice that, for $\eta\in\{0,1\}^{k_{**}}$

$$|A_\eta|\geq \epsilon^{k_{**}}|A|\geq \epsilon^{k_{**}}\frac{1}{\epsilon^{k_{**}}}=1.$$

So, for each $\eta\in\{0,1\}^{k_{**}}$, $A_\eta\neq\emptyset$ and we may choose an $a_\eta\in A_\eta$. Now, for $\nu\in\{0,1\}^{<k_{**}}$ and $\eta\in\{0,1\}^{k_{**}}$ such that $\nu\triangleleft\eta$, let

$$U_{\nu,\eta}=\{b\in B_\nu \mid a_\eta Rb\not\equiv t(a_\eta,B_\nu)\}$$

be the subset of elements of B_ν that do not relate with a_η in the expected way. By ζ-goodness of B_ν, $|U_{\nu,\eta}|<\zeta|B_\nu|$, and thus for every $\nu\in\{0,1\}^{<k_{**}}$,

$$|\bigcup\{U_{\nu,\eta}\mid \nu\triangleleft\eta\in\{0,1\}^{k_{**}}\}|<2^{k_{**}}\zeta|B_\nu|\leq|B_\nu|.$$

Therefore, for all $\nu\in\{0,1\}^{<k_{**}}$, we may choose $b_\nu\in B_\nu\setminus\bigcup\{U_{\nu,\eta}\mid\nu\triangleleft\eta\in\{0,1\}^{k_{**}}\}$. Finally, by 3. and 6. the sequences $\langle a_\eta\mid\eta\in\{0,1\}^{k_{**}}\rangle$ and $\langle b_\nu\mid\nu\in\{0,1\}^{<k_{**}}\rangle$ satisfy:

$$\forall\eta,\nu \text{ such that } \nu\frown\langle i\rangle\triangleleft\eta,\ a_\eta Rb_\nu\equiv i,$$

which forms a k_{**}-tree. This contradicts the tree bound k_{**} (see Definition 2.14). □

Remark 3.7. The two sequences $\langle a_\eta\mid\eta\in\{0,1\}^{k_{**}}\rangle$ and $\langle b_\nu\mid\nu\in\{0,1\}^{<k_{**}}\rangle$, constructed in the proof of Lemma 3.6, are not necessarily disjoint. This is the reason why, for this to work, the Definition 2.13, and consequently Definition 2.1, do not take this condition. Although it makes the non-k-order assumption on the graph stricter, this also allows the definition of excellence to work with respect to the set itself (as *excellent* sets are *good* by definition). Thus, the resulting partition will not only satisfy a quasi-random property between different parts, but actually ensures that the parts are quasi-random within themsleves.

Lemma 3.8 (Claim 5.4 (II) in [8]). *Let G be a finite graph with the non-k_*-order property. Let $\zeta<\frac{1}{2^{k_{**}}}$, $\epsilon\in(0,\frac{1}{2})$. Let $\langle m_\ell\mid\ell\in\{0,\dots,k_{**}\}\rangle$ be a decreasing sequence of natural numbers such that $\epsilon m_\ell\geq m_{\ell+1}$ for all $\ell\in\{0,\dots,k_{**}-1\}$ and $m_{k_{**}}\geq 1$. Then, for every $A\subseteq G$ with $|A|\geq m_0$ there exists $(\frac{m_{\ell+1}}{m_\ell},\zeta)$-excellent subset $A'\subseteq A$ such that $|A'|=m_\ell$ for some $\ell\in\{0,\dots,k_{**}-1\}$.*

Proof in Appendix A. □

Now, we can get a first (not necessarily even) partition by applying the previous lemma recursively, until the remainder is too small to further apply the previous statement.

Lemma 3.9 (Claim 5.14 (1) in [8]). *Let G be a finite graph with the non-k_*-order property. Let $\epsilon\in(0,\frac{1}{2})$ and $\epsilon'\leq\frac{1}{2^{k_{**}}}$. Let $A\subseteq G$ such that $|A|=n\geq m_0$. Let $\langle m_\ell\mid\ell\in\{0,\dots,k_{**}\}\rangle$ be a decreasing sequence of natural numbers such that $\epsilon m_\ell\geq m_{\ell+1}$ for all $\ell\in\{0,\dots,k_{**}-1\}$ and $m_{k_{**}}\geq 1$. Denote $m_*:=m_0$ and $m_{**}:=m_{k_{**}}$. Then, there is a partition $\overline{A}=\langle A_j\mid j\in\{1,\dots,j(*)\}\rangle$ with remainder $B=A\setminus\bigcup_{j<j(*)}A_j$ such that:*

(a) *For all* $j \in \{1, \ldots, j(*)\}$, $|A_j| \in \langle m_\ell \mid \ell \in \{0, \ldots, k_{**} - 1\}\rangle$.

(b) *For all* $i \neq j \in \{1, \ldots, j(*)\}$, $A_i \cap A_j = \emptyset$.

(c) *For all* $j \in \{1, \ldots, j(*)\}$, A_j *is* (ϵ, ϵ')*-excellent.*

(d) $|B| < m_*$.

Proof. Apply Lemma 3.8 recursively to the remainder $A \setminus \bigcup_{i<j} A_i$, to obtain A_j at each step. The process stops at $j(*)$ when the remainder is smaller than $m_0 = m_*$, and thus the lemma cannot be further applied. Notice that, since $\frac{m_\ell}{m_{\ell-1}} \leq \epsilon$, then by Remark 3.5 the pair being $(\frac{m_\ell}{m_{\ell-1}}, \epsilon')$-excellence implies it is also (ϵ, ϵ')-excellence. □

Say that if A is smaller than m_0, then the partition is empty and $B = A$.

Note that, in Lemma 3.9, if $n < m_0$ then the statement holds for an empty partition where the reminder is the whole set A.

The next step is refining this partition to obtain an even partition. In order to do so, we first show that any random sample of a given size from an excellent set is still excellent with high probability, at the cost of a slightly reduced excellence (condition c. in Lemma 3.11). Then, we use this result in a union-bound argument to show that we can actually fully partition the excellent set into pieces of equal size (condition d. in Lemma 3.11), which still are excellent. Finally, Lemma 3.15 applies this result to the partition from Lemma 3.9 to get an even excellent partition.

Before getting to it, we prove the following calculus result, which we use in the proof of Claim 3.10. The statement is inspired by [15, page 272] and, for completeness, we provide here a short proof.

Claim 3.10. *For* $k > 1$, $\zeta, \eta \in (0, 1)$ *the function* $f(m) = m^k \cdot e^{-2\zeta^2 m}$ *satisfies* $f(m) \leq \eta$ *for all* $m \geq \frac{1}{\zeta^2}(k \log(\frac{1}{\zeta^2}k) - \log(\eta))$.

Proof. First of all, notice that for $m = \frac{1}{\zeta^2}(k \log(\frac{1}{\zeta^2}k) - \log(\eta))$,

$$f(m) = \frac{m^k}{e^{2\zeta^2 m}} = \frac{(\frac{1}{\zeta^2}(k\log(\frac{1}{\zeta^2}k) - \log(\eta)))^k}{(\frac{k}{\zeta^2})^{2k}\eta^{-2}} \leq \frac{k^k(\log(\frac{k}{\zeta^2}(\frac{1}{\eta})^{\frac{1}{k}}))^k}{k^k(\frac{k}{\zeta^2}(\frac{1}{\eta})^{\frac{1}{k}})^k}\eta < \eta.$$

To conclude, we prove that f is decreasing for larger values of m:

$$f'(m) = \frac{km^{k-1}e^{2\zeta^2 m} - 2\zeta^2 m^k e^{2\zeta^2 m}}{(e^{2\zeta^2 m})^2} = (k - 2m\zeta^2)\frac{m^{k-1}}{e^{2\zeta^2 m}}.$$

The second factor is always positive, and $m > \frac{k}{\zeta^2} > \frac{k}{2\zeta^2}$, proving that $f'(m) < 0$ and thus f is decreasing. □

Lemma 3.11 (Claim 5.13 in [8])**.** *Let* G *be a finite graph with the non-*k_**-order property. Then:*

(a) *For every* $\epsilon \in (0, \frac{1}{2})$, $\zeta \in (0, \frac{1}{2} - \epsilon)$, $\xi \in (0, 1)$ *and* $m \geq \frac{1}{\zeta^2}(k_* \log \frac{1}{\zeta^2} k_* - \log \xi)$, *if* $A \subseteq G$ *is an* ϵ*-good subset of size* $n \geq m$, *then a subset* $A' \subseteq A$ *of size* m, *sampled uniformly at random, is* $(\epsilon + \zeta)$*-good with probability* $1 - \xi$.

(b) *Moreover, such* A' *satisfies* $t(b, A') = t(b, A)$ *for all* $b \in G$.

(c) *For every* $\zeta \in \{0, \frac{1}{2}\}$ *and* $\zeta' < \zeta$, *there is* $\epsilon_1 = \epsilon_1(\zeta, \zeta')$ *such that for every* $\epsilon < \epsilon' \leq \epsilon_1$, *if*

- *$A \subseteq G$ is $\{\epsilon, \epsilon'\}$-excellent and*
- *$A' \subseteq A$ is $\{\epsilon + \zeta'\}$-good,*

then, A' is $(\epsilon + \zeta, \epsilon')$-excellent.

(d) *For all $\zeta \in (0, \frac{1}{2})$, $\zeta' < \zeta$, $r \geq 1$ and for all $\epsilon < \epsilon'$ small enough ($\leq \epsilon_1(\zeta, \zeta')$ from the previous point) there exists $N = N(k_*, \zeta', r)$ such that: if $|A| = n > N$, r divides n and A is (ϵ, ϵ')-excellent, then there exists a partition into r disjoint pieces of equal size, each of which is $(\epsilon + \zeta, \epsilon')$-excellent.*

Proof. (a) For each $b \in G$, we say that $B_{A,b}$ is *bad* if $|B_{A,b}| \geq \epsilon m$. For each bad $B_{A,b}$, let $X_{A,b,A'}$ be the event that $|B_{A,b} \cap A'| \geq (\epsilon + \zeta)m$ for a subset $A' \subseteq A$ of size m, sampled uniformly at random. Notice that $|B_{A,b} \cap A'|$ is modelled by a hypergeometric distribution, and so the probability of it upperly deviating from the mean by ζ, the event $X_{A,b,A'}$, can be bounded (see [13, 2]) by

$$P(X_{A,b,A'}) \leq e^{-2\zeta^2 m}.$$

Now we want to study the random variable X counting the number of events in $X_{A,b,A'}$ that are satisfied. That is, $X = \sum_{\text{bad } B_{A,b}} \mathbb{1}_{X_{A,b,A'}}$, where $\mathbb{1}_Y$ is the indicator random variable of the event Y. We compute the expectation

$$\mathbb{E}[X] = \sum_{\text{bad } B_{A,b}} \mathbb{E}[\mathbb{1}_{X_{A,b,A'}}] = \sum_{\text{bad } B_{A,b}} P(X_{A,b,A'}) \leq \sum_{\text{bad } B_{A,b}} e^{-2\zeta^2 m}.$$

Following 2. in Corollary 2.12, the number of intersections of bad $B_{A,b}$'s with A', can be bounded by m^{k_*}. Thus, using the First Moment Method, we have that:

$$P(X \geq 1) \leq \mathbb{E}[X] \leq m^{k_*} \cdot e^{-2\zeta^2 m} \leq \xi.$$

Last inequality follows Claim 3.10 using the lower bound $m \geq \frac{1}{\zeta^2}(k_* \log \frac{1}{\zeta^2} k_* - \log \xi)$. Thus, with probability at least $1 - \xi$, we have that A' is $(\epsilon + \zeta)$-good.

(b) Suppose that A' is the subset described in a.. We proved that such set satisfies

$$|A' \cap B_{A,b}| < (\epsilon + \zeta)|A'|$$

for all $b \in G$ such that $|B_{A,b}| \geq \epsilon m$. Thus, given $b \in G$, we have that:

- If $|B_{A,b}| < \epsilon m$, then $|\{a \in A' \mid aRb \not\equiv t(b, A)\}| \leq |B_{A,b}| < \epsilon m < (\epsilon + \zeta)m$.
- If $|B_{A,b}| \geq \epsilon m$, then $|\{a \in A' \mid aRb \not\equiv t(b, A)\}| = |A' \cap B_{A,b}| < (\epsilon + \zeta)m$.

We conclude that $t(b, A) = t(b, A')$ for all $b \in G$.

(c) Let $B \subseteq G$ be an ϵ'-good set. We first upperbound the number of exceptional vertices of B with respect to A':

$$\begin{aligned} |\{b \in B \mid t(b, A') \not\equiv t(B, A)\}| &= |\{b \in B \mid t(b, A) \not\equiv t(B, A)\}| \\ &\leq \frac{(\epsilon + (1 - \epsilon)\epsilon')|A||B|}{(1 - \epsilon)|A|} \\ &= (\epsilon' + \frac{\epsilon}{1 - \epsilon})|B|. \end{aligned}$$

The first equality follows b.. First inequality uses Remark 3.4 for the numerator, as it upper bounds the number of exceptional edges we can and taking the worst case of only $(1-\epsilon)|A|$ exceptional edges per exceptional $b \in B$ (considering that A is ϵ-good).

HERE!

Now, let Q be the set of exceptional vertices of A' with respect to B, i.e.:

$$Q = \{a \in A' \mid t(a,B) \not\equiv t(A,B)\}.$$

We want to double-count the number of exceptional edges between Q and B. On one hand, we have that:

$$|\{(a,b) \in Q \times B \mid t(a,b) \not\equiv t(A,B)\}| < (\epsilon' + \frac{\epsilon}{1-\epsilon})|B||Q| + (1 - \epsilon' - \frac{\epsilon}{1-\epsilon})|B|(\epsilon + \zeta')|A'|.$$

The first term is the maximum number of exceptional edges associated to exceptional $b \in B$ (considering all edges exceptional), while the second term bounds the number of exceptional edges of non-exceptional $b \in B$, using the fact that A' is $(\epsilon + \zeta')$-good.

On the other hand, we have that:

$$|\{(a,b) \in Q \times B \mid t(a,b) \not\equiv t(A,B)\}| \geq |Q|(1-\epsilon')|B|,$$

which follows B being ϵ'-good.

Putting it all together:

$$(1 - \epsilon' - \epsilon' - \frac{\epsilon}{1-\epsilon})|B||Q| < (1 - \epsilon' + \frac{\epsilon}{1-\epsilon})(\epsilon + \zeta')|B||A'|.$$

So, we have that:

$$\begin{aligned} |Q| &< \frac{(1-\epsilon'-\frac{\epsilon}{1-\epsilon})}{(1-\epsilon'-\frac{\epsilon}{1-\epsilon}) - \epsilon'}(\epsilon+\zeta')|A'| \\ &= (1 + \frac{\epsilon'}{1 - 2\epsilon' - \frac{\epsilon}{1-\epsilon}})(\epsilon + \zeta')|A'|. \end{aligned}$$

Notice that $f(\epsilon, \epsilon') := \frac{\epsilon'}{1-2\epsilon'-\frac{\epsilon}{1-\epsilon}}$ decreases with ϵ and ϵ'. In particular,

$$f(\epsilon, \epsilon') \overset{\epsilon' \to 0}{\longrightarrow} 0,$$

and $\epsilon' > \epsilon$. Then,

$$|Q| < (\epsilon + (\underbrace{\epsilon f(\epsilon,\epsilon')}_{\to 0} + \underbrace{(1 + f(\epsilon,\epsilon'))}_{\to 1})\zeta')|A'| \overset{\epsilon' \to 0}{\longrightarrow} (\epsilon + \zeta')|A'|.$$

So, there exists an $\epsilon_1 = \epsilon_1(\zeta, \zeta')$ small enough such that for all $(\epsilon <)\ \epsilon' \leq \epsilon_1$, we have that $|Q| < (\epsilon + \zeta)|A'|$, and since A' is $(\epsilon + \zeta')$-good, and thus $(\epsilon + \zeta)$-good, we conclude that A' is $(\epsilon + \zeta, \epsilon')$-excellent.

Mention that in the next claim we show valid values for this.

(d) Let $\zeta, \zeta', \epsilon, \epsilon'$ and r be given satisfying the conditions of the statement. Set $\xi = \frac{1}{r+1}$. We will see that the condition $n > N = N(k_*, \zeta', r) := r\frac{1}{\zeta'^2}(k_* \log \frac{1}{\zeta'^2} k_* - \log \frac{1}{r+1})$ is sufficient. First of all, randomly choose a function $h : A \longrightarrow \{1, \dots, r-1\}$ such that for all $s < n$ we have that

$|\{a \in A \mid h(a) = s\}| = \frac{n}{r}$. Since h is random, each $A' \in [A]^{\frac{n}{r}}$ has the same probability of being part of the partition induced by h, i.e. to satisfy $A' = h^{-1}(s)$ for some $s \in \{1, \dots, r-1\}$. Since each element of the partition A' has size $\frac{n}{r} > \frac{N}{r} = \frac{1}{\zeta'^2}(k_* \log \frac{1}{\zeta'^2} k_* - \log \xi)$, we can apply a. to get that

$$P(A' \text{ is not } (\epsilon + \zeta')\text{-good}) < \xi.$$

In particular, since A is (ϵ, ϵ')-excellent, it follows c. that if A' is $(\epsilon + \zeta')$-good then it is also $(\epsilon + \zeta, \epsilon')$-excellent, so:

$$P(A' \text{ is not } (\epsilon + \zeta, \epsilon')\text{-excellent}) < \xi.$$

To conclude, by the union bound, we have that:

$$\begin{aligned} P(\bigcup_{s<r} h^{-1}(s) \text{ is not } (\epsilon + \zeta, \epsilon')\text{-excellent}) &\leq \sum_{s<r} P(h^{-1}(s) \text{ is not } (\epsilon + \zeta, \epsilon')\text{-excellent}) \\ &< r\xi = \frac{r}{r+1} < 1. \end{aligned}$$

All in all, there is a non-zero chance that the partition satisfies the statement, i.e. there exists at least one.

$\square$

Remark 3.12. For following applications, we would like to use d. from Lemma 3.11 with $\epsilon' > k(\epsilon + \zeta)$, for an arbitrarily large $k \in \mathbb{N}$. Notice that if $\epsilon, \zeta' \leq \frac{1}{t}, \epsilon' \leq \frac{1}{t'}$ and $t > t' \geq 5$, then:

(a) $\frac{\epsilon}{1-\epsilon} \leq \frac{\frac{1}{t}}{1-\frac{1}{t}} = \frac{\frac{1}{t}}{\frac{t-1}{t}} = \frac{1}{t-1}$.

(b) $1 - 2\epsilon' - \frac{\epsilon}{1-\epsilon} \geq 1 - \frac{2}{t'} - \frac{1}{t-1} > 1 - \frac{3}{t'-1} = \frac{t'-4}{t'-1}$.

(c) $(1 + \frac{\epsilon'}{1-2\epsilon'-\frac{\epsilon}{1-\epsilon}}) < 1 + \frac{\epsilon'}{1-\frac{3}{t'-1}} = (1 + \frac{t'-1}{t'-4}\epsilon')(\epsilon + \zeta')$.

Then, by requiring $\frac{1}{t} \leq \frac{1}{4k}\epsilon'$ we have that

$$\begin{aligned} \epsilon + \zeta' &\leq \frac{2}{t} \leq 2(\frac{1}{4k}\epsilon') = \frac{1}{2}(\frac{1}{k}\epsilon') \\ &< \frac{t'-4}{t'-3}\frac{1}{k}\epsilon' = \frac{1}{k}\frac{\epsilon'}{1+\frac{1}{t'-4}} \\ &< \frac{1}{k}\frac{\epsilon'}{1+\frac{t'-1}{t'}\frac{1}{t'-4}} = \frac{1}{k}\frac{\epsilon'}{1+\frac{t'-1}{t'-4}\frac{1}{t'}} \\ &\leq \frac{1}{k}\frac{\epsilon'}{1+\frac{t'-1}{t'-4}\epsilon'} \end{aligned}$$

i.e., we have:

$$(1 + \frac{t'-1}{t'-4}\epsilon')(\epsilon + \zeta') < \frac{1}{k}\epsilon',$$

which by c. gives us:

$$(1 + \frac{\epsilon'}{1 - 2\epsilon' - \frac{\epsilon}{1-\epsilon}}) < \frac{1}{k}\epsilon'.$$

All in all, a sufficient condition, for the lemma to hold under the constraint $\epsilon' \geq k(\epsilon + \zeta)$, is:

$$\epsilon, \zeta' \leq \frac{1}{4k}\epsilon' \quad \text{and} \quad \epsilon' \leq \frac{1}{5}.$$

We use this fact to reformulate point d. of Lemma 3.11 as:

Lemma 3.13. *Let G be a finite graph with the non-k_*-property. For all $k, r \geq 1$, $\epsilon' \leq \frac{1}{5}$ and $\epsilon \leq \frac{1}{4k}\epsilon'$, there exists $N = N(k, k_*, \epsilon', r)$ large enough such that, for all $n > N$ and r dividing n, if $A \subseteq G$ is (ϵ, ϵ')-excellent, with $|A| = n$, then there exists a partition into r disjoint pieces of equal size, each of which is $(\frac{\epsilon'}{k}, \epsilon')$-excellent.*

Proof. Choose any $\zeta' \leq \frac{1}{4k}\epsilon'$ and set $N := N_{3.11}(k_*, \zeta', r)$. Remark 3.12 sufficiency condition is satisfied, d. from Lemma 3.11 holds and we are done. □

Remark 3.14. A sufficient condition for $N_{3.13}$ to be large enough is to choose $\zeta' = \frac{1}{4k}\epsilon'$ in which case $N_{3.13}(k, k_*, \epsilon', r) := N_{3.11}(k_*, \frac{1}{4k}\epsilon', r)$

Now we proceed to refine the partition from Lemma 3.9 into an even one.

Lemma 3.15 (Claim 5.14 (1A) in [8]). *Let G be a finite graph with the non-k_*-order property. Let ϵ' and ϵ be two real numbers such that $\epsilon' \leq \min(\frac{1}{5}, \frac{1}{2^{k_{**}}})$ and $\epsilon \leq \frac{1}{4k}\epsilon'$ for some $k > 1$. Also, let m_*, m_{**} and q be natural numbers such that $q \geq \lceil \frac{1}{\epsilon} \rceil$, $m_{**} > \frac{N_{3.13}(k, k_*, \epsilon', \frac{m_*}{m_{**}})}{q}$ and $m_* := q^{k_{**}} m_{**}$. Then, for any $A \subseteq G$ with $|A| = n \geq m_*$ there exists a partition $\overline{A} = \langle A_i \mid i \in \{1, \dots, i(*)\} \rangle$ with remainder $B = A \setminus \bigcup \overline{A}$ such that:*

(a) $i(*) \leq \frac{n}{m_{**}}$.

(b) For all $i \in \{1, \dots, i(*)\}$, $|A_i| = m_{**}$.

(c) For all $i \in \{1, \dots, i(*)\}$, A_i is $(\frac{\epsilon'}{k}, \epsilon')$-excellent.

(d) $|B| < m_*$.

Proof. Consider the decreasing sequence of natural numbers

$$m_0 \geq m_1 \geq \dots \geq m_{k_{**}} = m_{**}$$

defined by $m_\ell = q m_{\ell+1}$, so that it satisfies $m_\ell \geq \frac{m_{\ell+1}}{\epsilon}$ for all $\ell \in \{0, \dots, k_{**} - 1\}$. Then $m_0 = q^{k_{**}} m_{**} = m_* \leq n$, and $m_{k_{**}-1} = q m_{**} > N_{3.13}(k, k_*, \epsilon', \frac{m_*}{m_{**}})$. With such a sequence, we can apply Lemma 3.9 to A to obtain a partition $\overline{A}' = \left\langle A'_j \mid j \in \{1, \dots, j(*)\} \right\rangle$ and remainder B with $|B| < m_*$. Then, we can apply Lemma 3.13 to each of the parts A'_j with $r = \frac{m_*}{m_{**}}$, as $m_{**} \mid m_\ell$ for all $\ell \in \{0, \dots, k_{**} - 1\}$. Putting together all the new subparts, we obtain a new partition $\overline{A} = \langle A_i \mid i \in \{1, \dots, i(*)\} \rangle$ with remainder B, satisfying all the conditions of the statement. □

Notice that our partition is even with a small reminder. We can turn it into an equitable one, as the next lemma proves, at the cost of another slight increase of the excellence parameter.

Lemma 3.16 (Claim 5.14 (2) in [8]). *Under the same condition of Lemma 3.15, we can get a partition $\overline{A} = \langle A_i \mid i \in \{1, \dots, i(*)\} \rangle$ with no remainder, such that:*

(a) For all $i, j \in \{1, \dots, i(*)\}$, $||A_i| - |A_j|| \leq 1$.

(b) For all $i, j \in \{1, \dots, i(*)\}$, $A_i \cap A_j = \emptyset$.

(c) *For all $i \in \{1, \dots, i(*)\}$, A_i is (ϵ'', ϵ')-excellent, where*

$$\epsilon'' \leq \frac{\frac{\epsilon'}{k} m_{**} + \left\lceil \frac{m_*}{i(*)} \right\rceil}{m_{**} + \left\lceil \frac{m_*}{i(*)} \right\rceil}.$$

(d) $A = \bigcup \overline{A}$.

Proof. Let $\overline{A}' = \langle A'_i \mid i \in \{1, \dots, i(*)\} \rangle$ and B from Lemma 3.15. We can partition B into $\overline{B} = \langle B_i \mid i \in \{1, \dots, i(*)\} \rangle$ in such a way that for all $i \in \{1, \dots, i(*)\}$,

$$|B_i| \in \left\{ \left\lfloor \frac{|B|}{i(*)} \right\rfloor, \left\lceil \frac{|B|}{i(*)} \right\rceil \right\}.$$

Notice that we are allowing $B_i = \emptyset$. Then, the new partition $\overline{A} = \langle A'_i \cup B_i \mid i \in \{1, \dots, i(*)\} \rangle$ satisfies a., b. and d. by construction. To conclude, notice that for each ϵ'-good set B, the number of exceptions is bounded by

$$\begin{aligned} |\{a \in A_i \mid t(a, B) \not\equiv t(A_i, B)\}| &\leq \frac{\epsilon'}{k}|A'_i| + |B_i| \\ &= \frac{\frac{\epsilon'}{k}|A'_i| + |B_i|}{|A'_i| + |B_i|}(|A'_i| + |B_i|) \\ &\leq \frac{\frac{\epsilon'}{k} m_{**} + \left\lceil \frac{m_*}{i(*)} \right\rceil}{m_{**} + \left\lceil \frac{m_*}{i(*)} \right\rceil}|A_i|, \end{aligned}$$

which proves that c. can be satisfied. □

We now have an (ϵ'', ϵ')-excellent equitable partition. Also ϵ'' is bounded by something very close to $\frac{\epsilon'}{k}$, where k is a settable parameter which only affects the large-enough condition on the size of the graph. It is reasonable to assume that, under some conditions of m_* and m_{**}, and under an appropriate choice of k, we can upper bound ϵ'' by ϵ', thus ensuring that the partition is ϵ'-excellent.

Remark 3.17 (Remark 5.14 (3) in [8]). In the context of Lemma 3.16, if:

(a) $m_{**} \geq \frac{1}{\frac{\epsilon'}{k}}$ and

(b) $m_* \leq \frac{\frac{\epsilon'}{k} n + 1}{\frac{\epsilon'}{k} + 1}$,

then $\epsilon'' \leq \frac{3\epsilon'}{k}$.

Proof. Notice that, if $|B_i| \leq 2\frac{\epsilon'}{k}|A_i|$ for all $i \in \{1, \dots, i(*)\}$, then ϵ'' can be bounded by:

$$\epsilon'' \leq \frac{\frac{\epsilon'}{k}|A_i| + |B_i|}{|A_i| + |B_i|} \leq \frac{\frac{\epsilon'}{k}|A_i| + 2\frac{\epsilon'}{k}|A_i|}{|A_i|} = \frac{3\epsilon'}{k}.$$

Let's now prove that $|B_i| \leq \frac{2\epsilon'}{k}|A_i|$ is satisfied. Notice that, by construction:

$$|B_i| \leq \left\lceil \frac{|B|}{i(*)} \right\rceil \leq \left\lceil \frac{m_* - 1}{i(*)} \right\rceil \leq \frac{m_* - 1}{i(*)} + 1.$$

Also, we can bound $i(*)$ by:

$$\frac{n}{m_{**}} \geq i(*) \geq \frac{n - |B|}{m_{**}} \geq \frac{n - m_* + 1}{m_{**}} > \frac{n - m_*}{m_{**}}.$$

Is the lower bound needed?

Thus, $|B_i| - 1 \leq \frac{m_*-1}{i(*)} \leq \frac{(m_*-1)m_{**}}{n-m_*}$, then $\frac{|B_i|-1}{m_{**}} \leq \frac{m_*-1}{n-m_*}$, and since $|A_i| = m_{**}$ we get:

$$\frac{|B_i|}{|A_i|} \leq \frac{m_* - 1}{n - m_*} + \frac{1}{m_{**}}.$$

Finally, notice that condition a. implies:

$$\frac{\epsilon'}{k} \geq \frac{1}{m_{**}},$$

and condition b. implies:

$$\frac{\epsilon'}{k} \geq \frac{m_* - 1}{n - m_*}.$$

We conclude:

$$\frac{|B_i|}{|A_i|} \leq \frac{m_* - 1}{n - m_*} + \frac{1}{m_{**}} \leq 2\frac{\epsilon'}{k},$$

completing the proof. □

We now resume all the conditions necessaries for the previous result to hold in the context of the values m_* and m_{**} given by the previous remark.

Lemma 3.18 (Corollary 5.15 in [8]). *Let G be a graph with the non-k_*-order property. Suppose that we are given:*

1. *A real value $\epsilon \leq \min(\frac{1}{5}, \frac{1}{2^{k_{**}}})$.*
2. *Three natural numbers m_*, m_{**} and q such that:*
   (a) $q \geq \lceil \frac{1}{\epsilon} \rceil$.
   (b) $m_{**} > \frac{N_{3.13}(3, k_*, \epsilon, \frac{m_*}{m_{**}})}{q}$.
   (c) $m_* := q^{k_{**}} m_{**}$.
3. *$A \subseteq G$ such that $|A| = n$, where n is large enough to satisfy $m_* \leq \frac{1+\frac{\epsilon}{3}n}{1+\frac{\epsilon}{3}}$.*

Then, there exists $i(*) \leq \frac{n}{m_{**}}$ and a partition of A into disjoint pieces $\overline{A} = \langle A_i \mid i \in \{1, \dots, i(*)\} \rangle$ such that:

(i) *For all $i, j \in \{1, \dots, i(*)\}$, $||A_i| - |A_j|| \leq 1$.*

(ii) *For all $i \in \{1, \dots, i(*)\}$, A_i is ϵ-excellent.*

(iii) For all $i, j \in \{1, \ldots, i(*)\}$, (A_i, A_j) is ϵ-uniform.

Proof. First of all, notice that condition 2.b. is a tighter bound then $m_{**} \geq \frac{3}{\epsilon}$. To prove the statement, we simply apply Lemma 3.16 in the context of Remark 3.17 with $k = 3$, $\epsilon'_{3.16} = \epsilon$ and $\epsilon_{3.16} \leq \frac{1}{12}\epsilon$. This results in a partition of A into disjoint pieces that satisfy i. and that are $(\epsilon''_{3.16}, \epsilon'_{3.16})$-excellent, with $\epsilon''_{3.16} \leq \frac{3\epsilon'_{3.16}}{k}$. But since $k \geq 3$, $\epsilon''_{3.16} \leq \epsilon'_{3.16}$, they are also $\epsilon'_{3.16}$-excellent, satisfying ii. and iii.. □

To conclude, we prove that the conditions of the previous lemma can be satisfied, under some minimal conditions of the two parameters ϵ (the excellence parameter) and m (the minimum number of parts in the resulting partition), and rewrite the statement accordingly.

Theorem 3.19 (Theorem 5.18 in [8])**.** *Let k_* and therefore k_{**} be given. Then, for all $\epsilon \leq \min(\frac{1}{5}, \frac{1}{2^{k_{**}}})$ and $m > 1$, there is $M = M(\epsilon, m, k_*)$ and $N = N(\epsilon, m, k_*)$ such that, for every finite graph G with the non-k_*-order property, and every $A \subseteq G$ with $|A| \geq N$, there exists a partition $\overline{A} = \langle A_i \mid i \in \{1, \ldots, i(*)\}\rangle$ of A, such that:*

1. *The number of parts is bounded by $m \leq i(*) \leq M := \max(\lceil \frac{12}{\epsilon} \rceil^{k_{**}+1}, 4m)$.* Move the bound on M to another point?
2. *For all $i, j \in \{1, \ldots, i(*)\}$, $||A_i| - |A_j|| \leq 1$.*
3. *For all $i \in \{1, \ldots, i(*)\}$, A_i is ϵ-excellent.*
4. *For all $i, j \in \{1, \ldots, i(*)\}$, (A_i, A_j) is ϵ-uniform.* Redundant?

Proof. Our goal is to apply Lemma 3.18. Let $q = \lceil \frac{12}{\epsilon} \rceil$. For $N(\epsilon, m, k_*)$, and thus n, large enough, we can then choose the smallest m_{**} satisfying:

(a) $m_{**} \in [\delta n - 1, \delta n]$, where $\delta = \min(\frac{\epsilon}{(3+\epsilon)q^{k_{**}}}, \frac{1}{m+q^{k_{**}}})$.

(b) $m_{**} > \frac{3}{\epsilon}$.

(c) $m_{**} > \frac{N_{3.13}(3, k_*, \epsilon, q^{k_{**}})}{q}$.

By a. we have that $m_* \leq \frac{\epsilon n}{3+\epsilon}$. This sequence satisfies all the conditions of Lemma 3.18:

- 2.a. $q \geq \lceil \frac{1}{\epsilon} \rceil$, and in particular defined it to be equal.
- 2.b. $m_{**} > \frac{N_{3.13}(3, k_*, \epsilon, \frac{m_*}{m_{**}})}{q}$ by choice of m_{**}.
- 2.c. $m_* := q^{k_{**}} m_{**}$.
- 3. $m_{k_{**}-1} = q m_{**} > q\frac{N_{3.13}(3, k_*, \epsilon, q^{k_{**}})}{q} = N_{3.13}(3, k_*, \epsilon, \frac{m_*}{m_{**}})$.

We can apply Lemma 3.18 to obtain a partition satisfying 2., 3. and 4..
We proceed to bound the number of parst $i(*)$. First, the upper bound follows from the fact that $m_{**} \geq \frac{1}{2}\min(\frac{\epsilon}{3+\epsilon}, \frac{1}{m+q^{k_{**}}})n$:

$$i(*) \leq \frac{n}{m_{**}} \leq \frac{2\max(\frac{3+\epsilon}{\epsilon}q^{k_{**}}, m + q^{k_{**}})n}{n} < 2\max(\frac{3+\epsilon}{\epsilon}q^{k_{**}}, 2m) \leq \max(\left\lceil \frac{12}{\epsilon} \right\rceil^{k_{**}+1}, 4m).$$

In the last inequality, we used that if $m < q^{k_{**}}$, then $m + q^{k_{**}} \leq 2q^{k_{**}} < \frac{3+\epsilon}{\epsilon} q^{k_{**}}$, which is dealt in the first argument of the maximum, so we may assume that $m \geq q^{k_{**}}$. We also show that the lower bound is satisfied:

$$i(*) \geq \frac{n - m_*}{m_{**}} \geq \frac{n - m_{**} q^{k_{**}}}{m_{**}} = \frac{n}{m_{**}} - q^{k_{**}} \geq \frac{m + q^{k_{**}}}{n} n - q^{k_{**}} = m.$$

□

Remark 3.20. We now see how large N, and thus n, actually needs to be. First of all, we see that:

$$\begin{aligned}\frac{1}{q} N_{3.13}(4, k_*, \epsilon, q^{k_{**}}) &= \frac{1}{q} N_{3.11}(k_*, \frac{1}{4 \cdot 3}\epsilon, q^{k_{**}}) \\ &= \frac{1}{q} q^{k_{**}} (\frac{12}{\epsilon})^2 (k_* \log (\frac{12}{\epsilon})^2 k_* - \log \frac{1}{q^{k_{**}} + 1}) \\ &< k_*^2 q^{2k_{**}+3}.\end{aligned}$$

Also, $\frac{3}{\epsilon}$ is clearly smaller than this value. Then, since m_{**} is the smallest integer larger than both values, we conclude:

$$\begin{aligned}\frac{m_{**}}{\delta} &\leq \frac{k_*^2 q^{2k_{**}+3}}{\min(\frac{\epsilon}{(3+\epsilon)q^{k_{**}}}, \frac{1}{m+q^{k_{**}}})} \\ &= k_*^2 q^{2k_{**}+3} \max(\frac{3+\epsilon}{\epsilon} q^{k_{**}}, m + q^{k_{**}}) \\ &\leq \max(q^{k_{**}+1}, 4m) k_*^2 q^{2k_{**}+3}.\end{aligned}$$

Define or remove uniformity.

3.3 The Stable Regularity Lemma

Lluis: is it ok to call a subsection as the section?

As mentioned in the beginning of the section, it can be proven that excellence is a stronger condition then regularity. In fact, as shown in the following lemma, excellence of a pair not only implies some level of regularity, but also it ensures that the pair is mostly full or empty of edges.

Lemma 3.21 (Lemma 5.17 in [8]). *Suppose that $\epsilon_1, \epsilon_2, \epsilon_3 \in (0, \frac{1}{2})$ with $\frac{\epsilon_1+\epsilon_2}{\epsilon_3} < \frac{1}{2}$ and the (not necessarily disjoint) pair (A, B) satisfies that A is ϵ_1-excellent and B is ϵ_2-good. Let $A' \subseteq A$ with $|A'| \geq \epsilon_3 |A|$, $B' \subseteq B$ with $|B'| \geq \epsilon_3 |B|$ and denote $Z = \{(a, b) \in (A \times B) \mid aRb \not\equiv t(A, B)\}$ and $Z' = \{(a, b) \in (A' \times B') \mid aRb \not\equiv t(A, B)\}$. Then, we have:*

1. $\frac{|Z|}{|A||B|} < \epsilon_1 + \epsilon_2$.

2. $\frac{|Z'|}{|A||B|} < \frac{\epsilon_1+\epsilon_2}{\epsilon_3}$.

In particular, if for some $\epsilon_0, \epsilon \in (0, \frac{1}{2})$, and A, B are ϵ_0-excellent, for $\epsilon_0 \leq \frac{\epsilon^2}{2}$, then:

a. *(A, B) is ϵ-regular.*

b. *If $A' \in [A]^{\geq \epsilon |A|}$ and $B' \in [B]^{\geq \epsilon |B|}$, then $d(A', B') < \epsilon$ or $d(A', B') \geq 1 - \epsilon$.*

Proof. Let $U = \{a \in A \mid t(a, B) \not\equiv t(A, B)\}$, i.e. the set of exceptional vertices $a \in A$. Then,

$$Z \subseteq U \times B \cup \bigcup_{a \in A \setminus U} \{a\} \times \overline{B}_{B,a}$$

and

$$Z' \subseteq U \times B' \cup \bigcup_{a \in A' \setminus U} \{a\} \times \overline{B}_{B,a}.$$

Notice that, by ϵ_1-excellence of A, $|U| < \epsilon_1 |A|$. Furthermore, by ϵ_2-goodness of B, if $a \in A \setminus U$, then $|\overline{B}_{B,a}| < \epsilon_2 |B|$. So,

$$|Z| < \epsilon_1 |A||B| + |A|\epsilon_2 |B|,$$

which can be written as

$$\frac{|Z|}{|A||B|} < \epsilon_1 + \epsilon_2,$$

proving 1.. Similarly,

$$\begin{aligned}|Z'| &\leq |U||B'| + |A'| \max\{|\overline{B}_{B,a}| \mid a \notin U\} \\ &< \epsilon_1 |A||B'| + |A'|\epsilon_2 |B|.\end{aligned}$$

By dividing both sides by $|A'||B'|$ we conclude

$$\frac{|Z'|}{|A'||B'|} < \epsilon_1 \frac{|A|}{|A'|} + \epsilon_2 \frac{|B|}{|B'|} \leq \frac{\epsilon_1 |A|}{\epsilon_3 |A|} + \frac{\epsilon_2 |B|}{\epsilon_3 |B|} = \frac{\epsilon_1 + \epsilon_2}{\epsilon_3},$$

proving 2.. Let's now prove a. and b.. First of all, notice that:

- if $t(A, B) = 1$, then $d(A, B) > 1 - (\epsilon_1 + \epsilon_2)$ and $d(A', B') > 1 - \frac{\epsilon_1 + \epsilon_2}{\epsilon_3}$, which follows 1. and 2. respectively. Thus,

$$\begin{aligned}|d(A, B) - d(A', B')| &= \max\{d(A, B) - d(A', B'), d(A', B') - d(A, B)\} \\ &< \max\{1 - (1 - \frac{\epsilon_1 + \epsilon_2}{\epsilon_3}), 1 - (1 - \epsilon_1 - \epsilon_2)\} \\ &= \frac{\epsilon_1 + \epsilon_2}{\epsilon_3}.\end{aligned}$$

- if $t(A, B) = 0$, similarly $d(A, B) < (\epsilon_1 + \epsilon_2)$ and $d(A', B') < \frac{\epsilon_1 + \epsilon_2}{\epsilon_3}$. Thus,

$$\begin{aligned}|d(A, B) - d(A', B')| &= \max\{d(A, B) - d(A', B'), d(A', B) - d(A, B)\} \\ &< \max\{(\epsilon_1 + \epsilon_2), \frac{\epsilon_1 + \epsilon_2}{\epsilon_3}\} \\ &= \frac{\epsilon_1 + \epsilon_2}{\epsilon_3}.\end{aligned}$$

In both cases, we have that $|d(A, B) - d(A', B')|$ is bounded by $\frac{\epsilon_1 + \epsilon_2}{\epsilon_3} < \frac{1}{2}$. Also, $d(A', B')$ may only differ by $\frac{\epsilon_1 + \epsilon_2}{\epsilon_3}$ with either 0 or 1. In particular, we may choose $\epsilon_3 = \epsilon$ and $\epsilon_1 = \epsilon_2 = \epsilon_0 \leq \frac{\epsilon^2}{2}$. This way, the condition $\frac{\epsilon_1 + \epsilon_2}{\epsilon_3} \leq \epsilon < \frac{1}{2}$ is satisfied. We conclude that (A, B) is ϵ-regular (a.) and that $d(A', B')$ is either $< \epsilon$ or $\geq 1 - \epsilon$ (b.). $\square$

We finally prove the Stable Regularity Lemma using the previous lemma to reformulate Theorem 3.19 in the context of regularity.

Theorem 3.22 (Theorem 5.19 in [8]). *For every $k_* \in \mathbb{N}$ and $\epsilon \in (0, \frac{1}{2})$ and $m > 1$, there exist $N = N(\epsilon, m, k_*)$ and $M = M(\epsilon, m, k_*)$ such that, for every finite graph G with the non-k_*-order property, and every $A \subseteq G$ with $|A| \geq N$, there is $m < \ell < M$ and a partition $\overline{A} = \langle A_i \mid i \in \{1, \ldots, \ell\}\rangle$ of A such that each A_i is $\frac{\epsilon^2}{2}$-excellent, and for every $i, j \in \{1, \ldots, \ell\}$,*

1. $||A_i| - |A_j|| \leq 1$.
2. *(A_i, A_j) is ϵ-regular, and moreover if $B_i \in [A_i]^{\geq \epsilon |A_i|}$ and $B_j \in [A_j]^{\geq \epsilon |A_j|}$, then either $d(B_i, B_j) < \epsilon$ or $d(B_i, B_j) \geq 1 - \epsilon$.*
3. *If $\epsilon \leq \min(\frac{1}{5}, \frac{1}{2^{k_{**}}})$, then $M \leq \max(\lceil \frac{12}{\epsilon} \rceil^{k_{**}+1}, 4m)$.*

Proof. If $\epsilon \leq \min(\frac{1}{5}, \frac{1}{2^{k_{**}}})$, then we can apply Theorem 3.19 to A with $\frac{\epsilon^2}{2}$, and then use Lemma 3.21 to replace the $\frac{\epsilon^2}{2}$-uniformity of pairs by ϵ-regularity. Otherwise, to get 1. and 2., just do the same process for some $\epsilon' = \min(\frac{1}{5}, \frac{1}{2^{k_{**}}}) \leq \epsilon$. Then, since regularity is monotone, we get the wanted ϵ-regularity from the resulting ϵ'-regularity. In this last case, the bound on M is $M \leq \max(\lceil \frac{12}{\epsilon'} \rceil^{k_{**}+1}, 4m)$. $\square$

Remark 3.23. By Theorem 2.15, we have that $k_{**} \leq 2^{k_*+1} - 2$ in the context of the non-k_*-order property. Thus, the bound on the number of parts M can clearly be reformulated as a function of only k_*, ϵ and m:

$$M \leq \max\left(\left\lceil \frac{12}{\epsilon} \right\rceil^{2^{k_*+1}-1}, 4m\right).$$

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A. Other proofs

For completeness, here we leave secondary proofs we skipped in the thesis.

Proof of Corollary 2.12.

1. First of all, notice that $B^+_{A,b} = A - B^-_{A,b}$, since by definition they are complementary. Thus, for any $b, b' \in G$, $B^+_{A,b} = B^+_{A,b'} \Leftrightarrow B^-_{A,b} = B^-_{A,b'}$. It follows that
$$|\{B^-_{A,b} \mid b \in G\}| = |\{B^+_{A,b} \mid b \in G\}| \leq |A|^k,$$
where the last inequality follows from Lemma 2.10.

2. Consider the following map:
$$\begin{aligned} \pi : \{B^+_{A,b} \mid b \in G\} &\longrightarrow \{\overline{B}_{A,b} \mid b \in G\}. \\ B^+_{A,b} &\longmapsto \overline{B}_{A,b} \end{aligned}$$
We first prove that the map π is well-defined. If $B^+_{A,b}$ and $B^+_{A,b'}$ are equal, then they have the same size, and thus the same truth value. Then,
   - if $t(A, b) = t(A, b') = 1$, we have that $\overline{B}_{A,b} = B^+_{A,b} = B^+_{A,b'} = \overline{B}_{A,b'}$.
   - if $t(A, b) = t(A, b') = 0$, we have that $\overline{B}_{A,b} = B^-_{A,b} = A \setminus B^+_{A,b} = A \setminus B^+_{A,b'} = B^-_{A,b'} = \overline{B}_{A,b'}$.

   which proves that the map is well-defined. The map π is also surjective, since for each $b \in G$, and thus for each $\overline{B}_{A,b}$, the set $B^+_{A,b}$ is mapped to $\overline{B}_{A,b}$ by construction. Hence,
$$|\{\overline{B}_{A,b} \mid b \in G\}| \leq |\{B^+_{A,b} \mid b \in G\}| \leq \sum_{i \leq k} \binom{|A|}{i} \leq |A|^k.$$
This concludes the proof. Notice that, actually, the map π is a not necessarily a bijection, since (at most) two b's with different truth value with respect to A may induce the same set $\overline{B}_{A,b}$. □

Proof of **??**. Notice that, by the average condition of the pair (A, B):

- there are at most $f(|A|)$ vertices of A (hence in $A' \subseteq A$), say S, which are exceptional with respect to B, so the number of edges $(a, b) \in S \times B'$ which are exceptional is at most $|S| \cdot |B'|$, and

- for each $a \in A$ (hence in $A' \subseteq A$) not in S, there are at most $g(|B|)$ elements $b \in B$ such that (a, b) does not satisfy the truth value $t(A, B)$, i.e. that are exceptional. Thus, we have at most $(a, b) \in (A' \setminus S) \times B'$ is at most $(|A'| - |S|)g(|B|)$.

The overall worse case in this scenario is when S is maximum ($|S| = f(|A|)$), and thus we have at most

$f(|A|)|B'| + (|A'| - f(|A|))g(|B|)$ exceptional edges in $A' \times B'$, as $|B'| \geq g(|B|)$. Putting it all together:

$$\begin{aligned}
\frac{|\{(a,b) \in (A',B') \mid aRb \equiv \neg t(A,B)\}|}{|A' \times B'|} &\leq \frac{f(|A|)|B'| + (|A'| - f(|A|))g(|B|)}{|A'||B'|} \\
&= \frac{f(|A|)}{|A'|} + \frac{|A'| - f(|A|)}{|A'|}\frac{g(|B|)}{|B'|} \\
&\leq \frac{f(|A|)}{|A'|} + \frac{g(|B|)}{|B'|} \\
&\leq \frac{f(|A|)}{f(|A|)|A|^{\epsilon_1}} + \frac{g(|B|)}{g(|B|)|B|^{\zeta_1}} \\
&= \frac{1}{|A|^{\epsilon_1}} + \frac{1}{|B|^{\zeta_1}}.
\end{aligned}$$

This finishes the proof. □

Proof of Lemma 3.8. Suppose the converse. We use this fact to build sets $\{b_\eta \mid \eta \in \{0,1\}^{<k_{**}}\}$ and $\{A_\eta \mid \eta \in \{0,1\}^{\leq k_{**}}\}$ on induction over $k < k_{**}$, where $k = |\eta|$, satisfying:

1. $A_{\langle\cdot\rangle} \subseteq A$, with $|A|_{\langle\cdot\rangle} = m_0$.
2. B_η is an ζ-good set witnessing that A_η is not $(\frac{m_{k+1}}{m_k}, \zeta)$-excellent, for all $k < k_{**}$.
3. $A_{\eta\frown\langle i\rangle} = \{a \in A_\eta \mid t(a, B_\eta) \equiv i\}$ for all $i \in \{0,1\}$ and $k < k_{**}$.
4. $|A_\eta| = m_k$, for all $k \leq k_{**}$.
5. $A_{\eta\frown\langle 0\rangle} \sqcup A_{\eta\frown\langle 1\rangle} \subseteq A_\eta$, for all $k < k_{**}$.
6. $\overline{A_k} = \{A_\eta \mid \eta \in \{0,1\}^k\}$ is a partition of a subset of A, for all $k \leq k_{**}$.

Notice that, by 1. and 4., the size of A_η is m_k, so by IH none of the sets A_η is $(\frac{m_{k+1}}{m_k}, \zeta)$-excellent. Then, B_η in 2. is well-defined. Also, by ζ-goodness of B_η, $t(a, B_\eta)$ in 3. is well-defined. Then, since B_η is witnessing the non-$(\frac{m_{k+1}}{m_k}, \zeta)$-excellence of A_η, we have that $|A_{\eta\frown\langle i\rangle}| \geq \frac{m_{k+1}}{m_k} m_k = m_{k+1}$ for all $i \in \{0,1\}$, satisfying 4.. Finally, by definition 3., we have the disjoint union 5. which by itself ensures 6..

Now, our goal is to build two sequences $\{b_\eta \mid \eta \in \{0,1\}^{<k_{**}}\}$ and $\{a_\eta \mid \eta \in \{0,1\}^{k_{**}}\}$ to contradict the tree bound k_{**}. First of all, notice that, for $\eta \in \{0,1\}^{k_{**}}$

$$|A_\eta| = m_k \geq m_{k_{**}} \geq 1$$

So, for each $\eta \in \{0,1\}^{k_{**}}$, $A_\eta \neq \emptyset$ and we may choose an $a_\eta \in A_\eta$. Now, for $\nu \in \{0,1\}^{<k_{**}}$ and $\eta \in \{0,1\}^{k_{**}}$ such that $\nu \triangleleft \eta$, let

$$U_{\nu,\eta} = \{b \in B_\nu \mid (a_\eta R b) \not\equiv t(a_\eta, B_\nu)\}$$

be the subset of elements of B_ν that do not relate with a_η in the expected way. By ζ-goodness of B_ν, $|U_{\nu,\eta}| < \zeta|B_\nu|$, and thus for every $\nu \in \{0,1\}^{<k_{**}}$,

$$|\bigcup\{U_{\nu,\eta} \mid \nu \triangleleft \eta \in \{0,1\}^{k_{**}}\}| < 2^{k_{**}}\zeta|B_\nu| \leq |B_\nu|$$

We may choose $b_\nu \in B_\nu \setminus \bigcup\{U_{\nu,\eta} \mid \nu \triangleleft \eta \in \{0,1\}^{k_{**}}\}$, for all $\nu \in \{0,1\}^{<k_{**}}$. Finally, the sequences $\langle a_\eta \mid \eta \in \{0,1\}^{k_{**}}\rangle$ and $\langle b_\nu \mid \nu \in \{0,1\}^{<k_{**}}\rangle$ satisfy that $\forall \eta, \nu$ such that $\nu \frown \langle i\rangle \triangleleft \eta$, $a_\eta R b_\nu \equiv i$, which follows 3.. This contradicts Definition 2.14 of tree bound k_{**}. □

B. Main changes

This section of the appendix is dedicated at showing the main changes this thesis applies to the original results of [8] and [7].

- Definition 2.3 of the k-order property in [8] does not specify adjacency (or not) of vertices with the same index.
- In order for the arguments of Section 4 to work, most results require that the function f (of the f-indivisibility) satisfies $x \geq f(x)$, instead of the *non-decreasing* condition given in Definition 4.2 in [8], which is redundant.
- In order for the average condition to be satisfied, and thus being able to apply Claim 4.8 in the proof of Claim 4.10 in [8], something like the extra condition provided by **??** needs to be added to the claim statement.
- Second to last inequality in the equation of P_1 at page 1569 of [8] is actually opposite (the $<$ should be a $>$). The same occurs, with last inequality of P_2 equation at the same page. This breaks the proof's argument, requiring extra conditions and some (non-trivial) changes in the argument. The most important change in the result is the extra condition $m_0 \geq n^\epsilon$ in **??**, which strongly reduces the interval of possible choices of parts size in the result, and needs to be carried until the end of the subsection.
- Condition $m_{**} > k_{**}$, which is persistent in results of Section 4 in [8] can be relaxed into $m_{**} \geq 1$.
- Proof of Theorem 4.16 in [8] is unclear, even more when previous points are noted. **??** provides a complete proof of a weaker (but coherent) version of the same result.
- Theorem 4.23 proof construction first finds an ϵ-indivisible set, and then applies Claim 4.21 to find a c-indivisible set. But Claim 4.21 itself does not require an ϵ-indivisible set as input, as it is constructed in its own proof. Noticing this allows to fully rewrite the theorem for a stronger (and more interpretable) result (**??**).
- Non-monotonicity

To do...

Also, we note that...

Section 3 argument does not work because...

C. Excellence is not monotonic.

Here give more details on the counterexample to the monotonicity of the excellence property given in Figure 4. We see that this example is in fact the smallest bipartite graph of a family of counterexamples. Each element of the family can be described by the following adjacency matrix, defined by blocks:

$$G_r = \left[\begin{array}{c|c} 0 & H_r \\ \hline H_r^T & 0 \end{array}\right], \text{ with} \quad H_r = \left[\begin{array}{c|c} 0 & \mathbb{1}_r - \mathbb{I}_r \\ \hline \mathbb{J}_r & \mathbb{I}_r \end{array}\right]$$

where $\mathbb{1}_r$ is the $r \times r$ matrix of all 1's, $\mathbb{I}_r$ is the $r \times r$ diagonal matrix, and $\mathbb{J}_r$ is the $r \times r$ anti-diagonal matrix. Also, we use H^T to refer to the transpose of H.

By calling A the set of the first (as indices) $2r$ vertices of G_r, and B the last $2r$, we have the desired counterexample: $A \subseteq G$ is $\frac{1}{2r-1}$-excellent, but B witnesses that A is not $\frac{1}{2r-1}$-excellent. The example in Figure 4 shows G_r for $r = 3$.

A sufficient proof of this is a simple exhaustive check, and code for this precise purpose is provided with all the material of this thesis in a GitHub repository[7]. There are two main relevant scripts in the repository. One allows to check whether G_r for a given value of r is in fact $\frac{1}{2r-1}$-excellent and not $\frac{1}{2r-1}$-excellent. The other allows for an exhaustive search of *possible* counterexamples under some given parameters, which is how this counterexamples were found in the first place. Read the documentation for more information on how to run the code.

[7] See `https://github.com/SeverinoDaDalt/tfm_severino_da_dalt/`