Universitat Politècnica de Catalunya
Facultat de Matemàtiques i Estadística

Master in Advanced Mathematics and Mathematical Engineering
Master's thesis

Why the non-monotonicity of excellence f***** up my life

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Thanks to...

Abstract

This should be an abstract in english, up to 1000 characters.

Keywords

regularity, stable graphs, graph theory, ...

1. Introduction

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2. Section 2

All this work holds on the idea that a sufficient condition for a graph to be stable is the absence of a certain kind of structure called *half-graph*. We now proceed to formalize this property using model theory notation: the *order property*.

Definition 2.1. Let G be a graph. We say that G has the *k-order property* if there exist two sequences of vertices $\langle a_i \mid i \in [1, k]\rangle$ and $\langle b_i \mid i \in [1, k]\rangle$ such that for all $i, j \leq k$, $a_i R b_j$ if and only if $i \neq j$. Otherwise, we say that G has the *non-k-order property*.

Add visual example of a half-graph

Remark 2.2. Notice that G having k-order property implies G having k'-order property for all $k' \leq k$. Conversely, G having the non-k-order property implies G having non-k'-order property for all $k' \geq k$.

Definition 2.3 (Truth value). Let G be a graph. For any $A, B \subseteq G$, we say that

$$t(A, B) = \begin{cases} 0 & \text{if } |\{(a, b) \in A \times B \mid \neg aRb\}| < |\{(a, b) \in A \times B \mid aRb\}| \\ 1 & \text{otherwise} \end{cases}$$

is the *truth value* of the pair (A, B).

When $B = \{b\}$, we write $t(A, b)$ instead of $t(A, \{b\})$, and we say that it is the truth value of A with respect to b.

I don't like how this looks

Extra notation:

- $B_{A,b} = \{a \in A \mid aRb \equiv t(A, b)\}$.
- $\overline{B}_{A,b} = \{a \in A \mid aRb \not\equiv t(A, b)\}$.
- $B^+_{A,b} = \{a \in A \mid aRb\}$.
- $B^-_{A,b} = \{a \in A \mid \neg aRb\}$.

With this notation, notice that either $t(A, b) = 1$ and thus $B_{A,b} = B^+_{A,b}$, or $t(A, b) = 0$ and $B_{A,b} = B^-_{A,b}$.

Large sets $B_{A,b}$, as we will see in the next sections, are closely related with lack of regularity in the graph. A useful tool to deal with them is Claim 2.9, which gives a bound on the number of such sets under the non-k-order property. In order to prove it, we first need to introduce the *VC dimension* of a family of sets, and relate it to the k-order property. This, together with Lemma 2.6, will give us the desired result.

Definition 2.4. Let $S = \{S_i \mid i \in I\}$ be a family of sets. A set A is said to be *shattered* by S (and S is said to *shatter* A) if for every $B \subseteq A$, there exists $S_i \in S$ such that $S_i \cap A = B$.

Definition 2.5. Let $S = \{S_i \mid i \in I\}$ be a family of sets. The *VC dimension* of S is the size of the largest set A that is shattered by S.

Lemma 2.6 (Sauer-Shelah). *Let $S = \{S_i \mid i \in I\}$ be a family of sets. If the VC dimension of S is at most k, and the union of all sets in S has n elements, then S consists of at most $\sum_{i=0}^{k} \binom{n}{i} \leq n^k$ sets.*

In order to prove the previous lemma, we first prove a stronger version of this lemma from Pajor.

Lemma 2.7 (Sauer-Shelah-Pajor). *Let S be a finite family of sets. Then S shatters at least $|S|$ sets.*

Proof. We will prove this by induction on the cardinality of S. If $|S| = 1$, then S consists of a single set, which only can shutter the empty set. If $|S| > 1$, we may choose an element $x \in S$ such that some sets of S contain x and some do not. Let $S^+ = \{s \in S \mid x \in S\}$ and $S^- = \{s \in S \mid x \notin S\}$. Then $S = S^+ \sqcup S^-$, and both S^+ and S^- are non-empty. By induction hypothesis, we know that $S^+ \subsetneq S$ shatters at least $|S^+|$ sets, and $S^- \subsetneq S$ shatters at least $|S^-|$ sets. Let T, T^+, T^- be the families of sets shattered by S, S^+ and S^- respectively. To conclude the proof, we just need to show that for each element in T^+ and T^-, there is a corresponding one in T. If a set is shattered by only one of the two families S^+ and S^-, then it only contributes by one unit to $|T^+| + |T^-|$ and one unit to $|T|$. Notice that no set shattered by S^+ or S^- may contain x, otherwise all or none of the intersections will contain this element. Thus, if a set s is shattered by both T^+ and T^-, it will contribute by two units to $|T^+| + |T^-|$ and one unit to $|T|$. But then, for each such set, we can consider $s \cup \{x\}$ which is not in T^+ or T^-, but it is in T. This follows the fact that for each subset of s, if it does not contain x it is the intersection with some set in $S^- \subsetneq S$, and if does contain x it is the intersection with some set in $S^+ \subsetneq S$. All in all, we conclude that

$$|T| \geq |T^+| + |T^-| \geq |S^+| + |S^-| \geq |S|$$

□

Proof. (of Lemma 2.6) Suppose that the union of S has n elements. Since there are at most $\sum_{i=0}^{k} \binom{n}{i}$ subsets of S of size at most k, if $|S| > \sum_{i=0}^{k} \binom{n}{i}$, by Lemma 2.7 S shatters at least $|S|$ subsets, and thus at least one of them has cardinality larger than k. □

Lemma 2.8. *Let G be a graph and $A \subseteq G$. Let $S = \{B_{A,b} \mid b \in G \setminus A\}$. If S has VC dimension (at least) k, then G has the k-order property.*

Proof. If S has VC dimension k, then it shatters a set $A' \subseteq A$ of size k. Now, choose any order of the vertices of $A' = \langle a_1, \dots, a_k \rangle$. Consider now the increasing sequence of subsets $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k = A'$, where $A_i = \{a_j \mid j \in [1, i]\}$. Since A' is shattered by S, for each $i \in [1, k]$ there exists a $b_i \in G$ such that $b_i R a$ if and only if $a \in A_i$. In particular, the two sequences $\langle a_i \mid i \in [1, k] \rangle$ and $\langle b_i \mid i \in [1, k] \rangle$ satisfy

$$a_i R b_j \Leftrightarrow i \leq j$$

and thus G has the k-order property. □

Lemma 2.9 (Claim 2.6). *Let G be a graph with the non-k-order property. Then, for any finite $A \subseteq G$,*

$$|\{\{a \in A \mid aRb\} \mid b \in G\}| \leq |A|^k$$

Proof. By Lemma 2.8, if G has the non-k-order property, then the family $\{B_{A,b} \mid b \in G \setminus A\}$ has VC dimension at most $k - 1$, so by the Sauer-Shelah lemma 2.6 we have $\{B_{A,b} \mid b \in G \setminus A\} \leq \sum_{i=0}^{k-1} \binom{|A|}{i}$. Since $\{B_{A,b} \mid b \in A\} \leq |A|$, we conclude that

$$|S| = |B_{A,b} \mid b \in G| \leq \sum_{i=0}^{k-1} \binom{|A|}{i} + |A|$$

This conditions should be set at some point of the tfm

Finally, when $|A| = n$, $k > 1$:

- if $n \leq k$, then $|S| \leq 2^n \leq 2^k \leq n^k$.

- if $n > k$, then $|S| \leq \sum_{i=0}^{k-1} + n \leq n^{k-1} + n \leq 2n^{k-1} \leq n^k$.

We conclude that $|S| \leq n^k$. □

We now prove the following equivalent versions of the lemma, which will be useful in the next sections.

Corollary 2.10 (Claim 2.6.1). *Let G be a graph with the non-k-order property. Then:*

1. *For any finite $A \subseteq G$*

$$|\{\{a \in A \mid \neg aRb\} \mid b \in G\}| \leq \sum_{i \leq k} \binom{|A|}{i} \leq |A|^k$$

2. *For any finite $A \subseteq G$*

$$|\{\{a \in A \mid \neg aRb \equiv t(A, b)\} \mid b \in G\}| \leq \sum_{i \leq k} \binom{|A|}{i} \leq |A|^k$$

Proof. 1. First of all, notice that $B^+_{A,b} = B - B^- A, b$, since by definition they are complementary. Thus, for any $b, b' \in G$, $B^+_{A,b} = B^+_{A,b'} \Leftrightarrow B^-_{A,b} = B^-_{A,b'}$. It follows that

$$\left|\left\{B^-_{A,b} \mid b \in G\right\}\right| = \left|\left\{B^+_{A,b} \mid b \in G\right\}\right| \leq \sum_{i \leq k} \binom{|A|}{i} \leq |A|^k$$

where the last inequalities come from Lemma 2.9.

2. Consider the following map:

$$\pi : \{B_{A,b} \mid b \in G\} \longrightarrow \left\{B^+_{A,b} \mid b \in G\right\}$$
$$B_{A,b} \longmapsto B^+_{A,b}$$

We show that the map π is injective. Let $b, b' \in G$ such that $B_{A,b} = B_{A,b'}$. Then, $t(A, b) = t(A, b')$, otherwise (suppose wlog that $t(A, b) = 1$ and $t(A, b') = 0$), we would have

$$\left|B^-_{A,b'}\right| > \left|B^+_{A,b'}\right| = \left|B^+_{A,b}\right| \geq \left|B^-_{A,b}\right| = \left|B^-_{A,b'}\right|$$

which is a contradiction. Then:

- if $t(A, b) = t(A, b') = 1$, we have that $B_{A,b} = B^+_{A,b} = B^+_{A,b'} = B_{A,b'}$.
- if $t(A, b) = t(A, b') = 0$, we have that $B_{A,b} = B^-_{A,b} = A \setminus B^+_{A,b} = A \setminus B^+_{A,b'} = B^-_{A,b'} = B_{A,b'}$.

This proves that π is injective. To conclude,

$$|\{B_{A,b} \mid b \in G\}| \leq \left|\left\{B^+_{A,b} \mid b \in G\right\}\right| \leq \sum_{i \leq k} \binom{|A|}{i} \leq |A|^k$$

This concludes the proof. Notice that in particular π is a bijection, but this is not needed for the proof.

□

Definition 2.11 (Definition 2.11). Suppose G is a finite graph with the non-k_*-property. We denote the *tree bound* $k_{**} = k_{**}(G) < \omega$ as the minimal value such that there do not exist sequences $\bar{a} = \left\langle a_\eta \mid \eta \in [2]^{k_{**}} \right\rangle$ and $\bar{b} = \left\langle b_\rho \mid \rho \in [2]^{<k_{**}} \right\rangle$ of elements of G satisfying that if $\rho \frown \langle \ell \rangle \triangleleft \eta$, then $(a_\eta R b_\rho) \equiv (\ell = 1)$.

Short introduction to the idea of the tree bound and how it is connected

3. Section 3

4. Section 4

Definition 4.1 (Definition 4.2(a)). Let $\epsilon \in (0,1)$. We say that $A \subseteq G$ is *ϵ-indivisible* if for every $B \in G$, for some truth value $t = t(b, A)$ we have that

$$|\{a \in A \mid aRb \not\equiv t\}| < |A|^{\epsilon}$$

Definition 4.2 (Definition 4.2(b)). Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a non-decreasing function. We say that $A \subseteq G$ is *f-indivisible* if for every $B \in G$, for some truth value $t = t(b, A)$ we have that

$$|\{a \in A \mid aRb \not\equiv t\}| < f(|A|)$$

Remark 4.3. Notice that the previous two definitions become meaningful only if $|A|^{\epsilon} \leq \frac{|A|}{2}$ and $f(|A|) \leq \frac{|A|}{2}$ respectively. In this context, the definition of the truth value t is compatible with that of Definition 2.3. In particular, Definition 4.1 would require that $|A| \geq (2)^{\frac{1}{1-\epsilon}}$.

Remark 4.4. If $f(n) = \epsilon n$, then f-indivisible $\equiv$ ϵ-good.

Remark 4.5. ϵ-indivisible is a much stronger condition then ϵ-good.

Lemma 4.6 (Claim 4.3). *Let G be a finite graph with the non-k_*-property. Assume $m_0 > \cdots > m_{k_{**}}$ is a sequence of non-zero natural numbers and for all $l \in [k_{**}]$, $f(m_{l-1}) \geq m_l$. If $A \subseteq G$, $|A| = m_0$, then for some $l < k_{**}$ there is a subset $B \subseteq A$ of size m_l which is f-indivisible.*

Proof. Suppose not. Then we can construct the sequences $\langle b_\eta \mid \eta \in [2]^{<k} \rangle$ and $\langle A_\eta \mid \eta \in [2]^{\leq k} \rangle$ on induction over $k < k_{**}$, where $k = |\eta|$, satisfying:

1. $A_{\eta^\frown\langle i \rangle} \subseteq A_\eta$, $\forall i \in \{0,1\}$, $\forall k < k_{**}$
2. $A_{\eta^\frown\langle 0 \rangle} \cap A_{\eta^\frown\langle 1 \rangle} = \emptyset$, $\forall k < k_{**}$
3. $|A_\eta| = m_k$, $\forall k \leq k_{**}$
4. $b_\eta \in G$ witnessing that A_η is not f-indivisible, $\forall k < k_{**}$
5. $A_{\eta^\frown\langle i \rangle} \subseteq A_\eta^{(i)} = \{a \in A_\eta \mid aRb_\eta \equiv (i = 1)\}$, $\forall \in \{0,1\}$, $\forall k < k_{**}$

Let's prove the induction:

- $\underline{k = 0}$. Consider $A_{\langle \cdot \rangle} = A$, which satisfies $|A_{\langle \cdot \rangle}| = m_0$ and $|b_{\langle \cdot \rangle}|$ witnessing the non-f-indivisibility of $A_{\langle \cdot \rangle}$.
- $\underline{k \Rightarrow k+1}$. We can assume $|A_\eta| = m_k$ and by hypothesis A_η is not f-indivisible. So, there exists b_η such that $A_\eta^{(i)} \geq f(m_k) \geq m_{k+1}$ (4), and we can choose $A_{\eta^\frown\langle i \rangle} \subseteq A_\eta^{(i)}$ (5), such that $|A_{\eta^\frown\langle i \rangle}| = m_{k+1} \forall i \in \{0,1\}$ (3). (1) and (2) are satisfied by the definition of $A_\eta^{(i)}$.

Now, for all η such that $|\eta| = k_{**}$, consider some element $a_\eta \in A_\eta$. Then, we have two sequences $\langle b_\eta \mid \eta \in [2]^{<k_{**}} \rangle$ and $\langle A_\eta \mid \eta \in [2]^{k_{**}} \rangle$ with the property:

$$\forall \rho \in [2]^{<k_{**}} \forall \eta \in [2]^{k_{**}} \text{ such that } \rho^\frown \langle i \rangle \trianglelefteq \eta, \ (a_\eta R b_\rho)$$

since $a_\eta \in A_\eta \subseteq A_{\rho^\frown\langle i \rangle}$. This contradicts the k_{**} tree bound. $\square$

Make an intrduction explaining what is the goal of this section, and how we will reach it. Explain what is the purpose of each Claim

Change the "if for some truth value we have" by "if the truth values satisfies"

From here on, change all l's with ℓ

Change all sequences of m's as in section 5

Lemma 4.7 (Claim 4.4). *Let G be a finite graph with the non-k_*-order property. Assume $m_0 > \cdots > m_{k_{**}}$ is a sequence of non-zero natural numbers and for all $l \in [k_{**}]$, $f(m_{l-1}) \geq m_l$. If $A \subseteq G$ with $|A| = n$, then we can find a sequence $\overline{A} = \langle A_j \mid j \in [j(*)] \rangle$ and reminder $B = A \setminus \bigcup \overline{A}$ such that:*

since we changed $l \in [k_**]$ to $l < k_**$, you should adapt consequently all that follows this point...

1. *For each $j \in [j(*)]$, A_j is f-indivisible*
2. *For each $j \in [j(*)]$, $|A_j| \in \{m_0, \dots, m_{k_{**}-1}\}$*
3. *$A_j \subseteq A \setminus \bigcup\{A_i \mid i < j\}$, in particular $A_i \cap A_j = \emptyset\ \forall i \neq j$*
4. *$|B| < m_0$*

Proof. Iteratively, apply Claim 4.6 to the remainder $A \setminus \bigcup\{A_i \mid i < j\}$ (3) to get an f-indivisible A_j (1) of size m_l, $l \in \{0, \dots, k_{**} - 1\}$ (2) until less then m_0 vertices are available (4). □

Lemma 4.8 (Claim 4.5). *Let G be a graph with the non-k_*-order property. Assume $n \geq m_0 > \cdots > m_{k_{**}}$ is a sequence of non-zero natural numbers satisfying that for all $l \in [k_{**}]$ $\lfloor (m_{l-1})^\epsilon \rfloor = m_l$, for $\epsilon \in (0, \frac{1}{2})$. If $A \subseteq G$, $|A| = n$, then we can find $\overline{A} = \langle A_i \mid i \in [i(*)] \rangle$ with remainder $B = A \setminus \bigcup \overline{A}$ such that:*

1. *For each $j \in [j(*)]$, A_j is ϵ-indivisible*
2. *For each $j \in [j(*)]$, $|A_j| \in \{m_0, \dots, m_{k_{**}-1}\}$*
3. *$A_i \cap A_j = \emptyset\ \forall i \neq j$*
4. *$|B| < m_0$*
5. *$\overline{A}$ is $\leq$-increasing*

Proof. The first four clauses are direct consequence of applying Claim 4.7 with $f(n) = n^\epsilon$. By renaming the A_i's in ascending-size order, we get (5). □

Remark 4.9. In this context, if $m_{k_{**}} > k_{**}$

$$n^{\epsilon^{k_{**}}} \geq m_0^{\epsilon^{k_{**}}} \geq m_1^{\epsilon^{k_{**}-1}} \geq \cdots \geq m_{k_{**}} > k_{**}$$

So, $n^{\epsilon^{k_{**}}} > k_{**}$.

Lemma 4.10 (Claim 4.6)). *Let G be a finite graph. Suppose $A, B \subseteq G$ such that A is f-indivisible, B is g-indivisible, and $f(|A|)g(|B|) < \frac{1}{2}|B|$. Then, for some truth value $t = t(A, B)$ for all but $< f(|A|)$ of the $a \in A$ for all but $< g(|B|)$ of the $b \in B$ we have that $aRb \equiv t$.*

Proof. Since B is g-indivisible, for each $a \in A$ there is a truth value $t_a = t(a, B)$ such that $\{b \in B \mid aRb \not\equiv t_a\} < g(|B|)$. Let $U_i = \{a \in A \mid t_a = i\}$ for $i \in \{0, 1\}$. If either U_i satisfies $|U_i| < f(|A|)$ then the statement is true. Suppose not. Then, there are $W_i \subseteq U_i$ with $|W_i| = f(|A|)$ for $i \in \{0, 1\}$. Now, let $V = \{b \in B \mid (\exists a \in W_0 \mid aRb) \vee (\exists a \in W_1 \mid \neg aRb)\}$, i.e. the b's which are an exception for some $a \in W_0 \cup W_1$. Then, $|V| < |W_0|g(|B|) + |W_1|g(|B|) = 2f(|A|)g(|B|) < |B|$, where the first inequality follows the g-indivisibility of B. Finally, there is a $b_* \in B \setminus V$ such that $\forall a \in W_0\ \neg aRb_*$ and $\forall a \in W_1$ aRb_* with $|W_0| = |W_1| = f(|A|)$, which contradicts the f-indivisibility of A. □

Definition 4.11. We say that the pair (A, B) with A f-indivisible and B g-indivisible satisfies the *average condition* if $f(|A|)g(|B|) < \frac{1}{2}|B|$ and thus the statement of Claim 4.10 is true for the pair (A, B).

Remark 4.12. The condition $f(|A|)g(|B|) < \frac{1}{2}|B|$ makes ordering of the pair (A, B) matter. Thus,

$$(A, B) \text{ has the average condition } \nRightarrow (B, A) \text{ has the average condition}$$

Remark 4.13 (Remark 4.7). When $f(n) = n^\epsilon$ and $g(n) = n^\zeta$, the average condition is $|A|^\epsilon|V|^\zeta < \frac{1}{2}|B|$.

Lemma 4.14 (Claim 4.8). *Let A be ϵ-indivisible, B ζ-indivisible and let the pair (A, B) satisfy the average condition. Then, for all $\epsilon_1 \in (0, 1-\epsilon)$, $\zeta_1 \in (0, 1-\zeta)$, $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| \geq |A|^{\epsilon+\epsilon_1}$ and $|B'| \geq |B|^{\zeta+\zeta_1}$, we have that:*

$$\frac{|\{(a, b) \in (A', B') \mid aRb \equiv \neg t(A, B)\}|}{|A' \times B'|} \leq \frac{1}{|A|}^{\epsilon_1} + \frac{1}{|B|}^{\zeta_1}$$

Proof. Notice:

- There are at most $|A|^\epsilon$ elements of A (hence in $A' \subseteq A$) which are exceptional (in the sense of the average condition).
- For each $a \in A$ (hence in $A' \subseteq A$) not exceptional, there are at most $|B|^\zeta$ elements $b \in B$ such that (a, b) does not satisfy the truth value $t(A, B)$, i.e. that are exceptional.

Putting it all together:

$$\begin{aligned}
\frac{|\{(a, b) \in (A', B') \mid aRb \equiv \neg t(A, B)\}|}{|A' \times B'|} &\leq \frac{|A|^\epsilon|B'| + (|A'| - |A|^\epsilon)|B|^\zeta}{|A'||B'|} \\
&= \frac{|A|^\epsilon}{|A'|} + \frac{|A'| - |A|^\epsilon}{|A'|}\frac{|B|^\zeta}{|B'|} \\
&\leq \frac{|A|^\epsilon}{|A'|} + \frac{|B|^\zeta}{|B'|} \\
&\leq \frac{|A|^\epsilon}{|A|^{\epsilon+\epsilon_1}} + \frac{|B|^\zeta}{|B|^{\zeta+\zeta_1}} \\
&= \frac{1}{|A|_1^\epsilon} + \frac{1}{|B|_1^\zeta}
\end{aligned}$$

□

Lemma 4.15 (f-indivisible version). *Let A be f-indivisible, B g-indivisible and let the pair (A, B) satisfy the average condition. Then, for all $\epsilon_1 \in \left(0, 1 - \frac{f(|A|)}{|A|}\right)$, $\zeta_1 \in \left(0, 1 - \frac{g(|B|)}{|B|}\right)$, $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| \geq f(|A|)|A|^{\epsilon_1}$ and $|B'| \geq g(|B|)|B|^{\zeta_1}$, we have that:*

$$\frac{|\{(a, b) \in (A', B') \mid aRb \equiv \neg t(A, B)\}|}{|A' \times B'|} \leq \frac{1}{|A|}^{\epsilon_1} + \frac{1}{|B|}^{\zeta_1}$$

Proof. Notice:

- There are at most $f(|A|)$ elements of A (hence in $A' \subseteq A$) which are exceptional (in the sense of the average condition).

- For each $a \in A$ (hence in $A' \subseteq A$) not exceptional, there are at most $g(|B|)$ elements $b \in B$ such that (a, b) does not satisfy the truth value $t(A, B)$, i.e. that are exceptional.

Putting it all together:

$$\begin{aligned}
\frac{|\{(a,b) \in (A', B') \mid aRb \equiv \neg t(A,B)\}|}{|A' \times B'|} &\leq \frac{f(|A|)|B'| + (|A'| - f(|A|))g(|B|)}{|A'||B'|} \\
&= \frac{f(|A|)}{|A'|} + \frac{|A'| - f(|A|)}{|A'|}\frac{g(|B|)}{|B'|} \\
&\leq \frac{f(|A|)}{|A'|} + \frac{g(|B|)}{|B'|} \\
&\leq \frac{f(|A|)}{f(|A|)|A|^{\epsilon_1}} + \frac{g(|B|)}{g(|B|)|B|^{\zeta_1}} \\
&= \frac{1}{|A|_1^{\epsilon}} + \frac{1}{|B|_1^{\zeta}}
\end{aligned}$$

$\square$

Corollary 4.16 (Corollary 4.9). *Let A and B be f-indivisible with $f(n) = c$ and (A, B) satisfy the average condition. Then, for all $\epsilon_1 \in (0, 1 - \frac{c}{|A|})$, $\zeta_1 \in (0, 1 - \frac{c}{|B|})$, $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq c|A|^{\epsilon_1}$ and $|B'| \geq c|B|^{\zeta_1}$, we have:*

$$\frac{|\{(a,b) \in (A', B') \mid aRb \equiv \neg t(A,B)\}|}{|A' \times B'|} \leq \frac{1}{|A|}^{\epsilon_1} + \frac{1}{|B|}^{\zeta_1}$$

Proof. Use Claim 4.15 with $f(n) = c$. $\square$

Lemma 4.17 (Claim 4.10). *Let G be a finite graph wit the non-k_*-order property. Assume $n \geq m_0 > \cdots > m_{k_{**}}$ is a sequence of non-zero natural numbers and for all $l \in [k_{**}]$, $\lfloor (m_{l-1})^{\epsilon} \rfloor = m_l$, for some $\epsilon \in (0, \frac{1}{2})$ such that $2 < (m_{k_{**}})^{1-2\epsilon}$. If $A \subseteq G$ with $|A| = n$, then we can find a sequence $\overline{A} = \langle A_i \mid i \in [i(*)] \rangle$ and reminder $B = A \setminus \bigcup \overline{A}$ satisfying:*

1. *For each $i \in [i*)]$, A_i is ϵ-indivisible*
2. *For each $i \in [i(*)]$, $|A_i| \in \{m_0, \ldots, m_{k_{**}-1}\}$*
3. *$A_i \cap A_j = \emptyset \; \forall i \neq j$*
4. *$|B| < m_0$*
5. *$\overline{A}$ is $\leq$-increasing*
6. *If $\zeta \in \left(0, \epsilon^{k_{**}}\right)$ then for every $i, j \in [i(*)]$ with $i < j$, $A \subseteq A_i$ ad $B \subseteq A_j$ such that $|A| \geq |A_i|^{\epsilon+\zeta}$ and $|B| \geq |A_j|^{\epsilon+\zeta}$ we have that:*

$$\begin{aligned}
\frac{|\{(a,b) \in (A, B) \mid aRb \equiv \neg t(A_i, A_j)\}|}{|A \times B|} &\leq \frac{1}{|A_i|}\zeta + \frac{1}{|A_j|}\zeta \\
&\leq \frac{1}{|A|}\zeta + \frac{1}{|B|}\zeta
\end{aligned}$$

Proof. The five points are direct consequence of Claim 4.8. Now, for any $A_i, A_j \in \overline{A}$ with $i < j$. By (2), there is some $l < k_{**}$ such that $|A_i| \leq |A_j| = m_l$ for some $l < k_{**}$. Then, it follows the condition $2 < (m_{k_{**}})^{1-2\epsilon}$ that:

$$\frac{|A|^\epsilon |B|^\epsilon}{|B|} \leq \frac{|B|^{2\epsilon}}{|B|} = \frac{1}{|B|^{1-2\epsilon}} = \frac{1}{m_l^{1-2\epsilon}} \leq \frac{1}{m_{k_{**}}} < \frac{1}{2}$$

i.e. $|A|^\epsilon |B|^\epsilon < \frac{1}{2}|B|$ and by Claim 4.13 the average condition is satisfied. Finally, notice that $\epsilon^{k_{**}} < \epsilon < 1-\epsilon$ since $\epsilon \in (0, \frac{1}{2})$, so that $\zeta \in (0, \epsilon^{k_{**}}) \subseteq (0, 1-\epsilon)$ and the condition for Claim 4.14 is satisfied. This gives us (6) and concludes the proof of the statement. □

Definition 4.18. Let A, B be f-indivisible sets with $f(A) \times f(B) < \frac{1}{2}|B|$. Let $\langle A_i \mid i < i_A \rangle$ be a partition of A with $|A_i| = m \forall i < i_A$ and $\langle B_i \mid i < i_B \rangle$ be a partition of B with $|B_i| = m \forall i < i_B$. $\varepsilon^+_{A_i, A_j, m}$ is the event:

$$\forall a \in A_i \forall b \in B_i, aRb = t(A, B)$$

Lemma 4.19 (Claim 4.13). *Let G be a finite graph wit the non-k_*-order property. Assume $n \geq m_0 > \cdots > m_{k_{**}}$ is a sequence of non-zero natural numbers and for all $l \in [k_{**}]$, $\lfloor (m_{l-1})^\epsilon \rfloor = m_l$, for some $\epsilon \in (0, \frac{1}{2})$ such that $2 < (m_{k_{**}})^{1-2\epsilon}$. Let $A_1, A_2 \subseteq G$ two ϵ-indivisible subsets such that $|A_1| = m_{l_1}$ and $|A_2| = m_{l_2}$ for some $l_1, l_2 < k_{**}$ and $|A_1| \leq |A_2|$. We will assume some approximation error by supposing $m_l = (m_{l-1})^\epsilon$. Suppose that, for some $c \in (0, 1-\epsilon)$ and $\zeta \leq \frac{1-\epsilon-c}{3}\epsilon^{k_{**}}$, $m = n^\zeta$ divides $|A_1|$ and $|A_2|$. Then, let $\left\langle A_{1,s} \mid s \in \left[\frac{|A_1|}{m}\right] \right\rangle$ and $\left\langle A_{2,t} \mid t \in \left[\frac{|A_2|}{m}\right] \right\rangle$ be random partitions of A_1 and A_2 respectively, with pieces of size m. We have that*

$$P(\varepsilon^+_{A_{1,s}, A_{2,t}, m}) \geq 1 - \frac{2}{n^{c\epsilon^{k_{**}}}}$$

Proof. Fix $s \in \frac{|A_1|}{m}$, $t \in \frac{|A_2|}{m}$.

UPS, something is missing here

...and thus the average condition is satisfied. Let $U_1 = \{a \in A_1 \mid |\{b \in A_2 \mid aRb \equiv \neg t(A_1, A_2)\}| \geq |A_2|^\epsilon\}$ and for each $a \in A_1 \setminus U_1$ let $U_{2,a} = \{b \in A_j \mid aRb \equiv \neg t(A_1, A_2)\}$ By Claim 4.10, $|U_1| \leq |A_1|^\epsilon$ and $\forall a \in A_1 \setminus U_1$, $|U_2| \leq |A_2|^\epsilon$. Now, we can bound the probability P_1 that $A_{1,s} \cap U_1 \neq \emptyset$ as follows:

$$\begin{aligned}
P_1 &\leq \frac{|U_1|}{|A_1|} + \cdots + \frac{|U_1|}{|A_1| - m + 1} < \frac{m|U_1|}{|A_1| - m} \leq \frac{m|A_1|^\epsilon}{|A_1| - m} \\
&\leq \frac{m^2|A_1|^\epsilon}{|A_1|} = \frac{m^2}{|A_1|^{1-\epsilon}} = \frac{n^{2\zeta}}{n^{(1-\epsilon)\epsilon^{l_1}}} \\
&\leq \frac{n^{2\frac{1-\epsilon-c}{3}\epsilon^{k_{**}}}}{n^{(1-\epsilon)\epsilon^{k_{**}}}} \leq \frac{n^{(1-\epsilon-c)\epsilon^{k_{**}}}}{n^{(1-\epsilon)\epsilon^{k_{**}}}} = \frac{1}{n^{c\epsilon^{k_{**}}}}
\end{aligned}$$

The forth inequality comes from the fact that $\frac{(|A_i|-m)m}{|A_i|} \geq 1$. Then, if $A_{1,s} \cap U_1 = \emptyset$, we have that $|\bigcup_{a \in A_{1,s}} U_{2,a}| \leq |A_{1,s}||A_2|^\epsilon$. So we can bound P_2, the probability that $A_{2,t} \cap \bigcup_{a \in A_{1,s}} U_{2,a} = \emptyset$, by:

$$\begin{aligned}
P_2 &\leq \frac{|\bigcup_{a \in A_{1,s}} U_{2,a}|}{|A_2|} + \cdots + \frac{|\bigcup_{a \in A_{1,s}} U_{2,a}|}{|A_2| - m + 1} < \frac{m|\bigcup_{a \in A_{1,s}} U_{2,a}|}{|A_2| - m} \leq \frac{mm|A_2|^\epsilon}{|A_2| - m} \\
&\leq \frac{m^3|A_2|^\epsilon}{|A_2|} = \frac{m^3}{|A_2|^{1-\epsilon}} = \frac{n^{3\zeta}}{n^{(1-\epsilon)\epsilon^{l_2}}} \\
&\leq \frac{n^{3\frac{1-\epsilon-c}{3}\epsilon^{k_{**}}}}{n^{(1-\epsilon)\epsilon^{k_{**}}}} \leq \frac{n^{(1-\epsilon-c)\epsilon^{k_{**}}}}{n^{(1-\epsilon)\epsilon^{k_{**}}}} = \frac{1}{n^{c\epsilon^{k_{**}}}}
\end{aligned}$$

Putting it all together:

$$P(\varepsilon^+_{A_{1,s},A_{2,t},m}) \geq (1-P_1)(1-P_2) \geq \left(1-\frac{1}{n^{c\epsilon^{k_{**}}}}\right)^2 \geq 1-\frac{2}{n^{c\epsilon^{k_{**}}}}$$

□

Remark 4.20. Since $\epsilon < \frac{1}{2}$, we can take $c = 1-2\epsilon$. In this context, $\zeta \leq \frac{\epsilon^{k_{**}+1}}{3}$.

Lemma 4.21 (Claim 4.14). *Let G be a finite graph wit the non-k_*-order property. Assume $n \geq m_0 > \cdots > m_{k_{**}}$ is a sequence of non-zero natural numbers and for all $l \in [k_{**}]$, $\lfloor (m_{l-1})^\epsilon \rfloor = m_l$, for some $\epsilon \in (0, \frac{1}{2})$ such that $2 < (m_{k_{**}})^{1-2\epsilon}$. Also, let m_0 be small enough to satisfy $m_0 < \frac{n}{n^{(1-2\epsilon)\epsilon^{k_{**}}}}$ and $m_0 \leq \frac{\sqrt{2}-1}{\sqrt{2}} n$. Finally, let m_{**} be a divisor of m_l for all $l < k_{**}$ and $m_{**} \leq n^{\frac{\epsilon^{k_{**}+1}}{3}}$. If $A \subseteq G$ with $|A| = n$, then we can find a partition $\overline{A} = \langle A_i \mid i \in [r]\rangle$ with reminder $B = A \setminus \bigcup \overline{A}$ such that:*

1. *$|A_i| = m_{**} \forall i \in [r]$*
2. *For all but $\frac{2r^2}{n^{(1-2\epsilon)\epsilon^{k_{**}}}}$ of the pairs (A_i, A_j) with $i < j$ there are no exceptional edges, i.e.*

$$\{(a,b) \in A_i \times A_j \mid aRb \not\equiv t(A_i, A_j)\} = \emptyset$$

3. *$|B| < m_0$*

Proof. We can use Claim 4.8 to get a partition $\overline{A'} = \langle A'_i \mid i \in [i(*)]\rangle$ and remainder $B' = A \setminus \bigcup A'$. We can refine the partition by randomly splitting each A'_i into pieces of size m_{**} (1). Consider the resulting partition $\overline{A} = \langle A_i \mid i \in [r]\rangle$ with remainder $B = B'$ (3). First of all, notice that for each pair (A_i, A_j) such that $A_i \subseteq A'_{i_1}$ and $A_j \subseteq A'_{j_1}$ with $i_1 \neq j_1$, the probability of the pair having exceptional edges is upper bounded by $\frac{2}{n^{(1-2\epsilon)\epsilon^{k_{**}}}}$. This follows Claim 4.19 in the context of Remark 4.20. Thus, given X the random variable counting the number of exceptional pairs of this kind, we have

$$E(X) = \sum_{\substack{A_i,A_j \text{ s.t.} \\ A_i \subseteq A'_{i_1}, A_j \subseteq A'_{j_1} \\ i_1 \neq j_1}} E(X_{A_i,A_j}) = \sum_{\substack{A_i,A_j \text{ s.t.} \\ A_i \subseteq A'_{i_1}, A_j \subseteq A'_{j_1} \\ i_1 \neq j_1}} P(\varepsilon_{A_i,A_j,m_{**}}) \leq \frac{r^2}{2}\frac{2}{n^{(1-2\epsilon)\epsilon^{k_{**}}}}$$

where X_{A_i,A_j} is the random variable giving 1 if (A_i, A_j) is exceptional, and 0 otherwise. Now, we have no control if $i_1 = j_1$, so let's bound how many of these we have:

$$\begin{aligned}
|\{\text{Esceptional } (A_i,A_j) \mid A_i, A_j \subseteq A'_{i_1}, i_1 \in [i(*)]\}| &\leq \binom{\frac{m_0}{m_{**}}}{2}\frac{n}{m_0} \\
&\leq \frac{\left(\frac{m_0}{m_{**}}\right)^2}{2}\frac{n}{m_0} = \frac{m_0 n}{2m_{**}^2} = \frac{m_0}{n}\left(\frac{n}{\sqrt{2}m_{**}}\right)^2 \\
&\leq \frac{m_0}{n}\left(\frac{n-m_0}{m_{**}}\right)^2 \leq \frac{m_0}{n}r^2 < \frac{r^2}{n^{(1-2\epsilon)\epsilon^{k_{**}}}}
\end{aligned}$$

Putting it all together, we see that the number of exceptional pairs is upper bounded by $\frac{2r^2}{n^{(1-2\epsilon)\epsilon^{k_{**}}}}$ satisfying (2). □

Remark 4.22 (Remark 4.15). Notice that, in the previous proof, the condition $m_0 < \frac{n}{n^{(1-2\epsilon)\epsilon^{k_{**}}}}$ can be weakened at the cost of increasing the number of exceptional pairs. More specifically, since this condition is only used to bound the exceptional sub-pairs in the same pair (the second part of the proof), the number of exceptional pairs can be generally bounded by

$$|\{\text{Exceptional pairs}\}| \leq \left(\frac{m_0}{n} + \frac{2}{n^{(1-2\epsilon)\epsilon^{k_{**}}}}\right) r^2$$

Theorem 4.23 (Theorem 4.16). *Let $\epsilon = \frac{1}{r} \in \left(0, \frac{1}{2}\right)$ with $r \in \mathbb{N}$ (this avoids rounding error) and k_* be given. Let G be a finite graph with the non-k_*-order property. Let $A \subseteq G$ with $|A| = n$. Then, for any $m_{**} \leq n^{\frac{\epsilon^{k_{**}+1}}{3}}$, there is a partition $\overline{A} = \langle A_i \mid i \in [m] \rangle$ of A with remainder $B = A \setminus \bigcup \overline{A}$ such that:*

1. $|A_i| = m_{**} \forall i \in [m]$
2. $|B| < n^{\frac{\epsilon}{3}}$
3. $|\{(i,j) \mid i,j \in [m], i < j$ *and* $\{(a,b) \in A_i \times A_j \mid aRb\} \notin \{A_i \times A_j, \emptyset\}\}| \leq \frac{2}{n^{(1-2\epsilon)\epsilon^{k_{**}}}} m^2$

Proof. Let $m_{k_{**}}$ be the smaller multiple of m_{**} such that $2 < (m_{k_{**}})^{1-2\epsilon}$. Then, consider the sequence

$$m_{**} \leq m_{k_{**}} < \cdots < m_0$$

such that for all $l \in [k_{**}]$ we have that $m_{l-1} = m_l^r$. Notice that:

1. m_{**} divides m_l for all $l \in [0, k_{**}]$ since the m_l's are powers of $m_{k_{**}}$ and m_{**} divides $m_{k_{**}}$ by construction.
2. $(m_{l-1})^{\epsilon} = m_l \forall l \in [k_{**}]$
3. $$\begin{aligned} \underline{m_0} = m_{k_{**}}^{r^{k_{**}}} &\leq m_{**}^{r^{k_{**}}} \leq n^{\frac{\epsilon}{3}\epsilon^{k_{**}} r^{k_{**}}} = \underline{n^{\frac{\epsilon}{3}}} \\ &< n^{\frac{1}{6}} < n^{1-\frac{1}{2}\epsilon^{k_{**}}} = \frac{n}{\underline{n^{\frac{1}{2}\epsilon^{k_{**}}}}} < \underline{n} \end{aligned}$$

So, all the conditions are satisfied to apply Claim 4.21, which gives us the partition $\overline{A}$ with remainder B satisfying the statement. Notice that (2) is satisfied by the fact that $|B| < m_0 \leq n^{\left(\frac{1}{6}-\frac{\epsilon}{3}\right)}$. □

Remark 4.24. Let $n^{\frac{\epsilon^{k_{**}+1}}{3}}$ be an integer and let m_{**} take this value. Then, the number of pieces of the partition is at most n^c with $c = 1 - \frac{\epsilon^{k_{**}+1}}{3}$.

Definition 4.25 (Definition 4.18). For $n, c \in \mathbb{N}$ and $\epsilon, \zeta, \xi \in \mathbb{R}$, let $\oplus[n, \epsilon, \zeta, \xi, c]$ be the statement: For any set A and $P \subseteq \mathcal{P}(A)$ such that $|A| = n$, $|P| \leq n^{\frac{1}{\zeta}}$ and for all $B \in P$ $|B| \leq n^{\epsilon}$, there exists $U \subseteq A$ with $|U| = \lfloor n^{\xi} \rfloor$ such that for all $B \in P$ $|U \cap B| \leq c$.

Lemma 4.26 (Lemma 4.19). *If the reals ϵ, ζ, ξ and the natural numbers n, c satisfy:*

- $\epsilon \in (0, 1)$
- $\zeta > 0$

- $0 < \xi < \min(1-\epsilon, \frac{1}{2})$
- *n sufficiently large ($n > n(\epsilon, \zeta, \xi, c)$) to satisfy the equation:*

$$\frac{1}{2n^{1-2\xi}} + \frac{1}{n^{(1-\xi-\epsilon)c-\frac{1}{\zeta}}} < 1$$

- $c > \frac{1}{\zeta(1-\xi-\epsilon)}$

then $\oplus[n, \epsilon, \zeta, \xi, c]$ holds.

Proof. Let $m = \lfloor n^{\xi} \rfloor$ the size of the set U we want to build, and let $\mathcal{F}_* = [A]^m$ the set of sequences of elements of A with length m. Let μ be a probability distribution on $\mathcal{F}_*$ such that for all $F \in \mathcal{F}_*$ $\mu(F) = \frac{|F|}{|\mathcal{F}_*|}$. We want to prove that the probability that a random U satisfies:

1. All elements of U are distinct
2. For all $B \in P$ $|U \cap B| < K$

is not trivial. First of all let's bound the converse (1) i.e. the probability that there are two equal elements in U:

$$P_1 = P(\exists s < t \in [m] \mid U_s = U_t) \leq \binom{m}{2}\frac{n}{n^2} \leq \frac{m^2}{2n} \leq \frac{n^{2\xi}}{2n} < \frac{1}{2n^{1-2\xi}}$$

Now, in order to bound (2), let's first bound the probability that at least c elements of U are in a given $B \in P$:

$$P_B = P(\exists^{\geq c} t \in [m] \mid U_t \in B) \leq \binom{m}{c}\left(\frac{|B|}{n}\right)^c \leq \frac{m^c |B|^c}{n^c} \leq \frac{n^{\xi c} n^{\epsilon c}}{n^c} = \frac{1}{n^{c(1-\xi-\epsilon)}}$$

Then, we can bound the converse of (2), i.e. the probability that this happens for some $B \in P$, by:

$$P_2 = P(\exists B \in P \mid \exists^{\geq c} t \in [m], U_t \in B) \leq \sum_{B \in P} P_B = \frac{|P|}{n^{c(1-\xi-\epsilon)}} \leq \frac{1}{n^{c(1-\xi-\epsilon)-\frac{1}{\xi}}}$$

Putting it all together, we have that

$$P((1) \cup (2)) \leq P_1 + P_2 < \frac{1}{2n^{1-2\xi}} + \frac{1}{n^{c(1-\xi-\epsilon)-\frac{1}{\xi}}}$$

Notice that

- Since $\xi < \frac{1}{2}$ we have that $1 - 2\xi > 0$
- Since $\xi < 1-\epsilon$, we have that $1-\epsilon-\xi > 0$ and given that c is natural $c(1-\xi-\epsilon) > 0$

so, the n-large enough condition of the forth point of the statement is well defined and

$$P((1) \cup (2)) < \frac{1}{2n^{1-2\xi}} + \frac{1}{n^{c(1-\xi-\epsilon)-\frac{1}{\xi}}} < 1$$

Thus, the probability that there exists a $U \subseteq A$ satisfying the condition is non-trivial, and $\oplus[n, \epsilon, \zeta, \xi, c]$ holds. $\square$

Lemma 4.27 (Claim 4.21). *Let $k_*, k, c \in \mathbb{N}$ and $\epsilon, \xi \in \mathbb{R}$ such that:*

1. *G is a graph with the non-k_*-order property*
2. *$A \subseteq G$ implies $|\{\{a \in A \mid aRb \equiv t(a,b)\} \mid b \in G\}| \leq |A|^k$*
3. *$\epsilon \in \left(0, \frac{1}{2}\right)$*
4. *$\xi \in \left(0, \frac{\epsilon^{k_{**}}}{2}\right)$*
5. *c satisfies*

$$c > \frac{1}{\frac{1}{k}(1 - \frac{\xi}{\epsilon^{k_{**}}} - \epsilon)}$$

Then, for every sufficiently large $n \in \mathbb{N}$ ($n^{\epsilon^{k_{**}}} > n\left(\epsilon, \frac{1}{k}, \frac{\xi}{\epsilon^{k_{**}}}, c\right)$ in the sense of Lemma 4.26 (d)), if $A \subseteq G$ with $|A| = n$, then there is $Z \subseteq A$ such that

(a) $|Z| = \lfloor n^{\xi} \rfloor$

(b) *Z is ϵ-indivisible in G*

Proof. In order to simplify the calculation, we will assume that $n^{\epsilon^l} \in \mathbb{N} \forall l \leq k_{**}$. Notice that can be easily achieved by setting ϵ as $\epsilon = \frac{1}{r}$ with $r \in \mathbb{N}$. Let $n = m_0 > m_1 > \cdots > m_{k_{**}}$ with $m_l = n^{\epsilon^l}$. So $m_{l+1} = m_l^{\epsilon} = \lfloor (m_l)^{\epsilon} \rfloor$ and we can use Claim 4.6 to get $A_1 \subseteq A$ with $|A_1| = m_l$ for some $l \leq k_{**}$ and A_1 ϵ-indivisible. By (2) we have that $|P_1| \leq |A_1|^k = m_l^k$. Notice that:

- $\epsilon \in (0, 1)$ by (3)
- $\zeta := \frac{1}{k} > 0$
- since $\epsilon \in \left(0, \frac{1}{2}\right)$ by (3), then by (4) $\frac{\xi}{\epsilon^l} \leq \frac{\xi}{\epsilon^{k_{**}}} < \frac{1}{2} < 1 - \epsilon$ and thus $0 < \xi < \min(1 - \epsilon, \frac{1}{2})$
- m_l sufficiently large: $m_l = n^{\epsilon^l} \geq n^{\epsilon^{k_{**}}} > n\left(\epsilon, \frac{1}{k}, \frac{\xi}{\epsilon^{k_{**}}}, c\right) > n\left(\epsilon, \zeta, \frac{\xi}{\epsilon^l}, c\right)$
- $c > \frac{1}{\frac{1}{k}(1 - \frac{\xi}{\epsilon^{k_{**}}} - \epsilon)} \geq \frac{1}{\zeta(1 - \frac{\xi}{\epsilon^{k_{**}}} - \epsilon)}$

By Lemma 4.26 then, $\oplus \left[m_l, \epsilon, \zeta, \frac{\xi}{\epsilon^l}\right]$ holds, and by taking $A_{(4.25)} := A_1$ and $P_{(4.25)} := P_1$ we have that:

- $|A_1| = m_l$
- $|P_1| \leq m_l^k = m_l^{\frac{1}{\zeta}}$
- $\forall B \in P_1$, $|B| \leq |A_1|^{\epsilon}$ by ϵ-indivisibility of A_1

Thus, by Definition 4.25 we have that there exists $Z \subseteq A_1$ such that:

- $|U| = \lfloor m_l^{\frac{\xi}{\epsilon^l}} \rfloor = \lfloor n^{\epsilon^l \frac{\xi}{\epsilon^l}} \rfloor \lfloor n^{\xi} \rfloor$ satisfying (a)
- Z is c-indivisible since $|B \cap Z| \leq c \forall B \in P_1$, satisfying (b)

This proves the statement. $\square$

Lemma 4.28 (Remark 4.22). *Notice that if $k = k_*$, the condition (2) will be satisfied by Claim ??? and the non-k_*-order of G.*

Theorem 4.29 (Theorem 4.23). *Let G be a graph with the non-k_*-property. For any $c \in \mathbb{N}$, $\epsilon, \xi \in \mathbb{R}$ satisfying the hypothesis of Claim 4.27 (with $k = k_*$ and $\zeta = \frac{1}{k_*}$), any $\theta \in (0,1)$ and $A \subseteq G$ with $A = n > n(c, \epsilon, \zeta, \xi, \theta)$ (i.e. n large enough in the sense of Claim 4.26), there is a partition $\overline{A} = \langle A_i \mid i \in [i(*)] \rangle$ of A with remainder $B = A \setminus \bigcup \overline{A}$ satisfying:*

- $|A_i| = \lfloor \lfloor n^\theta \rfloor^\zeta \rfloor \forall i \in [i(*)]$
- *A_i is c-indivisible $\forall i \in [i(*)]$ where c is the constant function $f(x) = c$*
- $|B| < \lfloor n^{\frac{\theta}{\epsilon^{k_{**}}}} \rfloor$

Proof. Let $n > \left(n\left(\epsilon, \frac{1}{k_*}, \frac{\xi}{\epsilon^{k_{**}}}, c\right)^{\frac{1}{\epsilon^{k_{**}}}+1} \right)^{\frac{1}{\theta}}$ in the sense of Lemma 4.26, so that $\lfloor n^\theta \rfloor$ satisfies the large enough condition of Claim 4.27:

$$\left(\lfloor n^\theta \rfloor \right)^{\epsilon^{k_{**}}} > n\left(\epsilon, \frac{1}{k_*}, \frac{\xi}{\epsilon^{k_{**}}}, c\right)$$

Notice that condition (2) in Claim 4.27 is satisfied by Remark 4.28. Now, we define a decreasing sequence $m_0 > m_1 > \cdots > m_{k_{**}}$ with $m_{k_{**}} = \lfloor n^\theta \rfloor$ and $m_{k_{**}-j} = \lceil (m_{k_{**}-j+1})^{\frac{1}{\epsilon}} \rceil \forall j \in [1, k_{**}]$. This sequence satisfies the condition of Claim 4.6 for $f(n) = n^\epsilon$. We will build a sequence of disjoint c-indivisible subsets A_i by induction on i as follows. Let $R_i = A \setminus \bigcup_{j<i} A_j$ (so $R_1 = A$). If $R_i < \lfloor n^{\frac{\theta}{\epsilon^{k_{**}}}} \rfloor$, then $\overline{A} = \langle A_j \mid j < i = i(*) \rangle$ and $B = R_i$, and we are done. Otherwise, we can apply Claim 4.6 to R_i with the sequence $\langle m_l \rangle_{l \leq k_{**}}$, to obtain an ϵ-indivisible subset $B_i \subseteq R_i$ of size $m_{k_{**}-l}$. Then, since $|B_i| = m_{k_{**}-l} \geq m_{k_{**}} = \lfloor n^\theta \rfloor$ by the n-large-enough assumption, we can apply Claim 4.27 and get a c-indivisible subset Z_i of size $|Z_i| = \lfloor m^\zeta_{k_{**}-l} \rfloor \geq \lfloor \lfloor n^{\frac{\theta}{\epsilon^l}} \rfloor^\zeta \rfloor \geq \lfloor \lfloor n^\theta \rfloor^\zeta \rfloor$. Since c-indivisible is preserved when taking subsets, we can choose $A_i \subseteq Z_i$ c-indivisible of size $\lfloor \lfloor n^\theta \rfloor^\zeta \rfloor$. $\square$

5. Section 5

Definition 5.1 (Definition 5.2(a)). Let G be a finite graph with the non-k_*-property. We say that $A \subseteq G$ is a *ϵ-good* when for every $b \in G$ for some truth value $t = t(b, A) \in \{0, 1\}$ we have $|\{a \in A \mid (aRb) \not\equiv t\}| < \epsilon|A|$.

Definition 5.2 (Definition 5.2(b)). Let G be a finite graph with the non-k_*-property. We say that $A \subseteq G$ is *(ϵ, ζ)-excellent* when A is ϵ-good and, if B is ζ-good, then for some truth value $t = t(B, A)$, $|\{a \in A \mid t(a, B) \neq t(A, B)\}| < \epsilon|A|$.

Remark 5.3. Notice that, if A is (ϵ, ϵ')-excellent and $B \subseteq G$ an is ϵ'-good set, then the number of exceptional edges between A and B, i.e. these vertex pairs that do not follow $t(A, B)$, is relatively small:

$$|\{\text{Exceptional edges between } A \text{ and } B\}| < \epsilon|A||B| + (1-\epsilon)|A|\epsilon'|B| = \left(\epsilon + (1-\epsilon)\,\epsilon'\right)|A||B|$$

Lemma 5.4 (Claim 5.4). *Let G be a finite graph with the non-k_*-order property. Let $\zeta < \frac{1}{2^{k_{**}}}$, $\epsilon \in \left\{0, \frac{1}{2}\right\}$. Then, for every $A \subseteq G$ with $|A| \geq \frac{1}{\epsilon^{k_{**}}}$ there exists (ϵ, ζ)-excellent subset $A' \subseteq A$ such that $|A'| \geq \epsilon^{k_{**}-1}|A|$.*

Proof. Suppose the converse. We will use this fact to build sets $\left\{b_\eta \mid \eta \in [2]^{<k_{**}}\right\}$ and $\left\{A_\eta \mid \eta \in [2]^{\leq k_{**}}\right\}$ on induction over $k < k_{**}$, where $k = |\eta|$, satisfying:

1. $A_{\langle\cdot\rangle} = A$.
2. B_η is an η-good set witnessing that A_η is not (ϵ, ζ)-excellent, for $k < k_{**}$.
3. $A_{\eta\frown\langle i\rangle} = \{a \in A_\eta \mid t(a, B_\eta) \equiv i\}$ for all $i \in \{0, 1\}$ and $k < k_{**}$.
4. $|A_{\eta\frown\langle i\rangle}| \geq \epsilon|A_\eta|$ for all $i \in \{0, 1\}$ and $k < k_{**}$.
5. $|A_\eta| \geq \epsilon^k|A|$, for $k \leq k_{**}$.
6. $A_\eta = A_{\eta\frown\langle 0\rangle} \sqcup A_{\eta\frown\langle 1\rangle}$, for $k < k_{**}$.
7. $\overline{A_k} = \left\{A_\eta \mid \eta \in [2]^k\right\}$ is a partition of A, for $k \leq k_{**}$.

First of all, notice that at each step, the non-(ϵ, ζ)-excellence of A_η comes by IH from (1) or (5). This allows the existence of B_η in (2). Notice that $t(a, B_\eta)$ in (3) is well-defined since B_η is ζ-good. Also, the non-(ϵ, ζ)-excellence of A_η allows (4). Finally, by definition (3), we have the disjoint union (6) which by itself ensures the partition (7). Now, our goal is to build two sequences $\left\{b_\eta \mid \eta \in [2]^{<k_{**}}\right\}$ and $\left\{a_\eta \mid \eta \in [2]^{k_{**}}\right\}$ to contradict the tree bound k_{**}. First of all, notice that, for $\eta \in [2]^{k_{**}}$

$$|A_\eta| \geq \epsilon^{k_{**}}|A| \geq \epsilon^{k_{**}}\frac{1}{\epsilon^{k_{**}}} = 1$$

So $A_\eta \neq \emptyset$. For each $\eta \in [2]^{k_{**}}$ we may choose an $a_\eta \in A_\eta$. Now, for $\nu \in [2]^{<k_{**}}$ and $\eta \in [2]^{k_{**}}$ such that $\nu \triangleleft \eta$, let

$$U_{\nu,\eta} = \{b \in B_\nu \mid (a_\eta Rb) \not\equiv t(a_\eta, B_\nu)\}$$

be the subset of elements of B_ν that do not relate with a_η in the expected way. By ζ-goodness of B_ν, $|U_{\nu,\eta}| < \zeta|B_\nu|$, and thus for every $\eta \in [2]^{k_{**}}$,

$$\left|\bigcup\left\{U_{\nu,\eta} \mid \nu \triangleleft \eta \in [2]^{k_{**}}\right\}\right| < 2^{k_{**}}\zeta|B_\nu| < |B_\nu|$$

We may choose $b_\nu \in B_\nu \backslash \bigcup \left\{ U_{\nu,\eta} \mid \nu \triangleleft \eta \in [2]^{k_{**}} \right\}$, for all $\nu \in [2]^{<k_{**}}$. Finally, the sequences $\left\langle a_\eta \mid \eta \in [2]^{k_{**}} \right\rangle$ and $\left\langle b_\nu \mid \nu \in [2]^{<k_{**}} \right\rangle$ satisfy that $\forall \eta, \nu$ such that $\nu \frown \langle i \rangle \triangleleft \eta$, $(a_\eta R b_\nu)^i$ by (3) and (6). This contradicts the definition of tree bound k_{**} (2.11). $\square$

Lemma 5.5 (Claim 5.4.1). *Let G be a finite graph with the non-k_*-order property. Let $\zeta < \frac{1}{2^{k_{**}}}$, $\epsilon \in \left\{0, \frac{1}{2}\right\}$. Let $\langle m\ell \mid \ell \in [0, k_{**}] \rangle$ be a decreasing sequence of natural numbers such that $\epsilon m_\ell \geq m_{\ell+1}$ for all $\ell \in [0, k_{**} - 1]$, $m_{k_{**}} \geq 1$, and $m_{k_{**}-1} > k_{**}$. Then, for every $A \subseteq G$ with $|A| \geq m_0$ there exists $\left(\frac{m_{\ell+1}}{m_\ell}, \zeta\right)$-excellent subset $A' \subseteq A$ such that $|A'| = m_\ell$ for some $\ell \in [0, k_{**} - 1]$.*

Proof. Suppose the converse. We will use this fact to build sets $\left\{ b_\eta \mid \eta \in [2]^{<k_{**}} \right\}$ and $\left\{ A_\eta \mid \eta \in [2]^{\leq k_{**}} \right\}$ on induction over $k < k_{**}$, where $k = |\eta|$, satisfying:

1. $A_{\langle \cdot \rangle} \subseteq A$, with $|A|_{\langle \cdot \rangle} = m_0$.
2. B_η is an η-good set witnessing that A_η is not $\left(\frac{m_{k+1}}{m_k}, \zeta\right)$-excellent, for all $k < k_{**}$.
3. $A_{\eta \frown \langle i \rangle} = \{a \in A_\eta \mid t(a, B_\eta) \equiv i\}$ for all $i \in \{0, 1\}$ and $k < k_{**}$.
4. $|A_\eta| = m_k$, for all $k \leq k_{**}$.
5. $A_{\eta \frown \langle 0 \rangle} \sqcup A_{\eta \frown \langle 1 \rangle} \subseteq A_\eta$, for all $k < k_{**}$.
6. $\overline{A_k} = \left\{ A_\eta \mid \eta \in [2]^k \right\}$ is a partition of a subset of A, for all $k \leq k_{**}$.

Notice that, by (1) and (4), the size of A_η is m_k, so by IH none of the sets A_η is $\left(\frac{m_{k+1}}{m_k}, \zeta\right)$-excellent. Then, B_η in (2) is well-defined. Also, by η-goodness of B_η, $t(a, B_\eta)$ in (3) is well-defined. Then, since B_η is witnessing the non-$\left(\frac{m_{k+1}}{m_k}, \zeta\right)$-excellence of A_η, we have that $|A_{\eta \frown \langle i \rangle}| \geq \frac{m_{k+1}}{m_k} m_k = m_{k+1}$ for all $i \in \{0, 1\}$, satisfying (4). Finally, by definition (3), we have the disjoint union (5) which by itself ensures (6). Now, our goal is to build two sequences $\left\{ b_\eta \mid \eta \in [2]^{<k_{**}} \right\}$ and $\left\{ a_\eta \mid \eta \in [2]^{k_{**}} \right\}$ to contradict the tree bound k_{**}. First of all, notice that, for $\eta \in [2]^{k_{**}}$

$$|A_\eta| = m_k \geq m_{k_{**}} \geq 1$$

so $A_\eta \neq \emptyset$. For each $\eta \in [2]^{k_{**}}$ we may choose an $a_\eta \in A_\eta$. Now, for $\nu \in [2]^{<k_{**}}$ and $\eta \in [2]^{k_{**}}$ such that $\nu \triangleleft \eta$, let

$$U_{\nu,\eta} = \{b \in B_\nu \mid (a_\eta R b) \not\equiv t(a_\eta, B_\nu)\}$$

be the subset of elements of B_ν that do not relate with a_η in the expected way. By ζ-goodness of B_ν, $|U_{\nu,\eta}| < \zeta |B_\nu|$, and thus for every $\eta \in [2]^{k_{**}}$,

$$\left| \bigcup \left\{ U_{\nu,\eta} \mid \nu \triangleleft \eta \in [2]^{k_{**}} \right\} \right| < 2^{k_{**}} \zeta |B_\nu| < |B_\nu|$$

We may choose $b_\nu \in B_\nu \backslash \bigcup \left\{ U_{\nu,\eta} \mid \nu \triangleleft \eta \in [2]^{k_{**}} \right\}$, for all $\nu \in [2]^{<k_{**}}$. Finally, the sequences $\left\langle a_\eta \mid \eta \in [2]^{k_{**}} \right\rangle$ and $\left\langle b_\nu \mid \nu \in [2]^{<k_{**}} \right\rangle$ satisfy that $\forall \eta, \nu$ such that $\nu \frown \langle i \rangle \triangleleft \eta$, $(a_\eta R b_\nu)^i$, which follows (3). This contradicts the definition of tree bound k_{**} (2.11). $\square$

Lemma 5.6 (Fact 5.9). *Let $p, q \in (0,1)$. Let A be a set of size n, $B \subseteq A$ a subset of size $p|A|$, and $A' \subseteq A$ a random subset of size $\geq q|A|$. Then, for $\zeta > 0$,*

$$P\left(\frac{|A' \cap B|}{|A'|} \in \left(\frac{|B|}{|A|} - \zeta, \frac{|B|}{|A|} + \zeta\right)\right)$$

can be modeled by a random variable which is asymptotically normally distributed when $n \to +\infty$.

Lemma 5.7 (Fact 5.10). *Let A be a set of events measured with a probability P_A. Let S a family of subsets of A, which are measurable with P_A. Let $A_r = \{a_1, \dots, a_r\} \subseteq A$ be a random sample of size r. For each $B \in S$, we may define $v_B^{A_r}$ the relative frequency of events of B in A_r, i.e.,*

$$v_B^{A_r} = \frac{P_A(A_r \cap B)}{P_A(A_r)}$$

Let

$$\pi^{A_r} = \sup_{B \in S} \left| v_B^{A_r} - P_A(B) \right|$$

i.e. the upperbound of error of $v_B^{A_r}$ as an approximation of $P_A(B)$. Also let $\Delta^s(A_r)$ be the number of subsets of A_r induced by sets of S ($B \in S$ induces $B \cap A_r \subseteq A_r$), i.e.

$$\Delta^s(A_r) = |\{B \cap A_r \mid B \in S\}|$$

Finally, let

$$m^S(r) = \max_{C \in \binom{A}{r}} \Delta^s(C)$$

Then, if there exists a finite $k > 0$ such that $m^s(r) \leq r^k + 1$ for all $r > 0$, we have that, for all $\epsilon > 0$,

$$\lim_{r \to +\infty} P_A\left(\pi^{A_r} > \epsilon\right) = 0$$

Remark 5.8 (Fact 5.12). If there exists $k > 0$ such that $m^s(r) \leq r^k + 1$ and r satisfies:

$$r \geq \frac{16}{\zeta^2}\left(k \log \frac{16k}{\zeta^2} - \log \frac{\eta}{4}\right)^{k+1}$$

for some $\eta > 0$, then

$$P\left(\pi^{A_r} < \zeta\right) \geq 1 - \eta$$

In particular, if we suppose that all events in A are equiprobable and sampled without replacement, then

$$P\left(\forall B \in S,\ \frac{|A_r \cap B|}{|A_r|} \in \left(\frac{|B|}{|A|} - \zeta, \frac{|B|}{|A|} + \zeta\right)\right) \geq 1 - \eta$$

or in other words, for all but a fraction η of all possible choices of A_r, we have that

$$\forall B \in S,\ \frac{|A_r \cap B|}{|A_r|} \in \left(\frac{|B|}{|A|} - \zeta, \frac{|B|}{|A|} + \zeta\right)$$

Lemma 5.9 (Claim 5.13). *Let G be a finite graph with the non-k_*-order property. Then:*

(a) *For every $\epsilon \in \left(0, \frac{1}{2}\right)$, $\zeta \in \left(0, \frac{1}{2} - \epsilon\right)$ and $\xi \in (0, 1)$ there is $N_1 = N_1(\epsilon, \zeta, \xi)$ such that for all $n > N_1$, if $A \subseteq G$ is an ϵ-good subset of size n, and $n \geq m \geq \log\log n$, then if we choose a random subset $A' \subseteq A$ of size m, it is $(\epsilon + \zeta)$-good with probability $1 - \xi$.*

(b) *Moreover, such A' satisfies $t(b, A') = t(b, A)$ for all $b \in G$.*

(c) *For every $\zeta \in \left(0, \frac{1}{2}\right)$ and $\zeta' < \zeta$, there is $\epsilon_1 = \epsilon_1(\zeta, \zeta')$ such that for every $\epsilon < \epsilon' \leq \epsilon_1$, if*

- *$A \subseteq G$ is (ϵ, ϵ')-excellent.*
- *$A' \subseteq A$ is $(\epsilon + \zeta')$-good.*

then, A' is $(\epsilon + \zeta, \epsilon')$-excellent.

(d) *For all $\zeta \in \left(0, \frac{1}{2}\right)$, $\zeta' < \zeta$, $r \geq 1$ and for all $\epsilon < \epsilon'$ small enough (in the sense of the previous point) there exists $N_2 = N_2(\epsilon, \zeta', r)$ such that, if $|A| = n > N_2$, r divides n and A is (ϵ, ϵ')-excellent, there exists a partition into r disjoint pieces of equal size, each of which is $(\epsilon + \zeta, \epsilon')$-excellent.*

Proof. (a) For each $b \in G$ we say that $\overline{B}_{A,b}$ is *exceptional* if $\left|\overline{B}_{A,b}\right| \geq \epsilon |A'|$. Notice that, if we prove that, with probability $1 - \xi$, A' satisfies that for all exceptional $\overline{B}_{A,b}$:

$$\frac{\left|A' \cap \overline{B}_{A,b}\right|}{|A'|} \in \left(\frac{\left|\overline{B}_{A,b}\right|}{|A|} - \zeta, \frac{\left|\overline{B}_{A,b}\right|}{|A|} + \zeta\right)$$

then, with the same probability:

$$\left|A' \cap \overline{B}_{A,b}\right| < \left(\frac{\left|\overline{B}_{A,b}\right|}{|A|} + \zeta\right) |A'| < (\epsilon + \zeta) |A'| \tag{1}$$

and we are done.

By Lemma 5.6, for $n = |A|$ large enough, we can approximate sampling a set of size m from A, with m i.i.d. random variables $x_1, \ldots, x_m$, where each x_i picks a vertex uniformly at random from A. Let $S := \left\{\text{Exceptional } \overline{B}_{A,b}\right\}$. Since G has the non-k_*-order property, we can apply Lemma 2.10 to $G_{2.10} = A$ and $A_{2.10} = A'$, which gets us that:

$$|S| \leq \left|\left\{\left\{a \in A' \mid aRb \not\equiv t(A', b)\right\} \mid b \in G\right\}\right| \leq |A'|^{k_*}$$

Then,

$$m^s(\ell) \leq |S| \leq |A'|^{k_*} \leq \ell^{k_*} \leq \ell^{k_*} + 1 \quad \forall \ell \geq |A'|$$

Notice that this is enough to satisfy the conditions of Lemma 5.7: For each $\ell < |A'|$, let k_l be the smallest integer such that $m^s(\ell) \leq \ell^{k_l} + 1$. Since there are finitely many of them, we can take the maximum $k_{\max} = \max\left\{k_1, \ldots, k_{|A'|-1}, k_*\right\}$, which satisfies

$$m^s(\ell) \leq \ell^{k_{\max}} + 1 \quad \forall \ell$$

So we conclude equation (1), which by itself is sufficient to prove A' is $(\epsilon + \zeta)$-good.

(b) We proved in (a) that, with probability $1 - \xi$, A' satisfies that for all $b \in G$:

$$|S| \leq \left|\left\{\left\{a \in A' \mid aRb \not\equiv t(A', b)\right\} \mid b \in G\right\}\right| \leq |A'|^{k_*}$$

We want to prove that these sets A' that satisfy the previous equation also satisfy $t(b, A') = t(b, A)$ for all $b \in G$. Now, for any $b \in G$:

- if $|B_{A,b}| < \epsilon |A'|$, then

$$\{a \in A' \mid aRb \not\equiv t(b, A)\} \leq |B_{A,b}| < \epsilon \left|A'\right| (\epsilon + \zeta) \left|A'\right|$$

- if $|B_{A,b}| \geq \epsilon |A'|$, then by (a) we have that:

$$\{A' \cap B_{A,b}\} < \frac{|B_{A,b}| \, |A'|}{|A|} + \zeta \left|A'\right| = (\epsilon + \zeta) \left|A'\right|$$

  so $\{a \in A' \mid aRb \not\equiv t(b, A)\} < (\epsilon + \zeta) |A'|$.

So, in both cases we have that $t(b, A') = t(b, A)$.

(c) Let $B \subseteq G$ be an ϵ'-good set. We first upperbound the number of exceptional vertices of B with respect to A':

$$\begin{aligned} \left|\{b \in B \mid t(b, A') \not\equiv t(B, A)\}\right| &= |\{b \in B \mid t(b, A) \not\equiv t(B, A)\}| \\ &\leq \frac{(\epsilon + (1 - \epsilon)\, \epsilon')\, |A| \, |B|}{(1 - \epsilon)\, |A|} \\ &= \left(\epsilon' + \frac{\epsilon}{1 - \epsilon}\right) |B| \end{aligned}$$

The first equality follows (b), and the first inequality follows from remark (5.3) for the numerator, and taking the worst case of only $(1 - \epsilon)\, |A|$ exceptional edges per exceptional $b \in B$ (considering that A is ϵ-good).

Now, let Q be the set of exceptional vertices of A' with respect to B, i.e.:

$$Q = \{a \in A' \mid t(a, B) \not\equiv t(A, B)\}$$

We want to double-count the number of exceptional edges between Q and B. On one hand, we have that:

$$|\{(a, b) \in Q \times B \mid t(a, b) \not\equiv t(A, B)\}| < \left(\epsilon' + \frac{\epsilon}{1 - \epsilon}\right) |B| \, |Q| + \left(1 - \epsilon' - \frac{\epsilon}{1 - \epsilon}\right) |B| \, (\epsilon + \zeta') \left|A'\right|$$

The first term is the maximum number of exceptional edges associated to exceptional $b \in B$ (considering all edges exceptional), while the second term bounds the number of exceptional edges of non-exceptional $b \in B$, using the fact that A' is $(\epsilon + \zeta')$-good.

On the other hand, we have that:

$$|\{(a, b) \in Q \times B \mid t(a, b) \not\equiv t(A, B)\}| \geq |Q| \, (1 - \epsilon') \, |B|$$

which follows B being ϵ'-good.

Putting it all together:

$$\left(1 - \epsilon' - \epsilon' - \frac{\epsilon}{1 - \epsilon}\right) |B| \, |Q| < \left(1 - \epsilon' + \frac{\epsilon}{1 - \epsilon}\right) (\epsilon + \zeta') \, |B| \left|A'\right|$$

So, we have that:

$$|Q| < \frac{\left(1-\epsilon'-\frac{\epsilon}{1-\epsilon}\right)}{\left(1-\epsilon'-\frac{\epsilon}{1-\epsilon}\right)-\epsilon'}\left(\epsilon+\zeta'\right)|A'|$$
$$= \left(1+\frac{\epsilon'}{1-2\epsilon'-\frac{\epsilon}{1-\epsilon}}\right)(\epsilon+\zeta')|A'|$$

Notice that $f(\epsilon,\epsilon') := \frac{\epsilon'}{1-2\epsilon'-\frac{\epsilon}{1-\epsilon}}$ decreases with ϵ and ϵ'. In particular,

$$f(\epsilon,\epsilon') \xrightarrow{\epsilon'\to 0} 0$$

and $\epsilon' > \epsilon$. Then,

$$|Q| < \left(\epsilon + \left(\underbrace{\epsilon f(\epsilon,\epsilon')}_{\to 0} + \underbrace{(1+f(\epsilon,\epsilon'))}_{\to 1}\right)\zeta'\right)|A'| \xrightarrow{\epsilon'\to 0} \left(\epsilon+\zeta'\right)|A'|$$

So, there exists an $\epsilon_1 = \epsilon_1(\zeta,\zeta')$ small enough such that for all $(\epsilon <)\epsilon' \le \epsilon_1$, we have that $|Q| < (\epsilon+\zeta)|A'|$, and since A' is $(\epsilon+\zeta')$-good, and thus $(\epsilon+\zeta)$-good, we conclude that A' is $(\epsilon+\zeta,\epsilon')$-excellent.

(d) Let $\zeta,\zeta',\epsilon,\epsilon'$ and r be given satisfying the conditions of the statement. Set $\xi = \frac{1}{r+1}$. We will see that the condition $n > N_2 := N_1\left(\epsilon,\zeta',\frac{1}{r+1}\right)$ is sufficient. First of all, randomly choose a function $h : A \longrightarrow \{1,\dots,r-1\}$ such that for all $s < n$ we have that $|\{a \in A \mid h(a) = s\}| = \frac{n}{r}$. Since h is random, each $A' \in [A]^{\frac{n}{r}}$ has the same probability of being part of the partition induced by h, i.e. to satisfy $A' = h^{-1}(s)$ for some $s \in \{1,\dots,r-1\}$. For each element of the partition A', we can apply (a) to get that

$$P\left(A' \text{ is not } \left(\epsilon+\zeta'\right)\text{-good}\right) < \xi$$

In particular, since A is (ϵ,ϵ')-excellent, it follows (c) that if A' is $(\epsilon+\zeta')$-good then it is also $(\epsilon+\zeta,\epsilon')$-excellent, so:

$$P\left(A' \text{ is not } \left(\epsilon+\zeta,\epsilon'\right)\text{-excellent}\right) < \xi$$

To conclude, by the union bound, we have that:

$$P\left(\bigcup_{s<r} h^{-1}(s) \text{ is not } \left(\epsilon+\zeta,\epsilon'\right)\text{-excellent}\right) \le \sum_{s<r} P\left(h^{-1}(s) \text{ is not } \left(\epsilon+\zeta,\epsilon'\right)\text{-excellent}\right)$$
$$< r\xi = \frac{r}{r+1} < 1$$

All in all, there is a non-zero chance that the partition satisfies the statement, i.e. there exists at least one.

□

Remark 5.10 (Remark 5.13.1). For following applications, we would like to use Lemma 5.9 (d) with $\epsilon' > k(\epsilon+\zeta)$, for an arbitrarily large $k \in \mathbb{N}$. Notice that if $\epsilon,\zeta < \frac{1}{t}, \epsilon' < \frac{1}{t'}$ and $t > t' \ge 5$, then:

(a) $\frac{\epsilon}{1-\epsilon} < \frac{\frac{1}{t}}{1-\frac{1}{t}} = \frac{\frac{1}{t}}{\frac{t-1}{t}} = \frac{1}{t-1}$

(b) $1 - 2\epsilon' - \frac{\epsilon}{1-\epsilon} > 1 - \frac{2}{t'} - \frac{1}{t-1} > 1 - \frac{3}{t'-1} = \frac{t'-4}{t'-1}$

(c) $\left(1 + \frac{\epsilon'}{1-2\epsilon'-\frac{\epsilon}{1-\epsilon}}\right) < \left(1 + \frac{t'-1}{t'-4}\epsilon'\right)(\epsilon + \zeta')$

Then, by requiring $\frac{1}{t} < \frac{1}{4k}\epsilon'$ we have that

$$\begin{aligned}\epsilon + \zeta' &< \frac{2}{t} < 2\left(\frac{1}{4k}\epsilon'\right) < \frac{1}{2}\left(\frac{1}{k}\epsilon'\right)\\ &< \frac{t'-4}{t'-3}\frac{1}{k}\epsilon' = \frac{1}{k}\frac{\epsilon'}{1+\frac{1}{t'-4}}\\ &< \frac{1}{k}\frac{\epsilon'}{1+\frac{t'-1}{t'}\frac{1}{t'-4}} = \frac{1}{k}\frac{\epsilon'}{1+\frac{t'-1}{t'-4}\frac{1}{t'}}\\ &< \frac{1}{k}\frac{\epsilon'}{1+\frac{t'-1}{t'-4}\epsilon'}\end{aligned}$$

i.e., we have:

$$\left(1 + \frac{t'-1}{t'-4}\epsilon'\right)(\epsilon + \zeta') < \frac{1}{k}\epsilon'$$

which by (c) gives us:

$$\left(1 + \frac{\epsilon'}{1-2\epsilon'-\frac{\epsilon}{1-\epsilon}}\right) < \frac{1}{k}\epsilon'$$

All in all, a sufficient condition, for the lemma to hold under the constraint $\epsilon' > k(\epsilon + \zeta)$, is:

$$\epsilon, \zeta' < \frac{1}{4k} \quad \text{and} \quad \epsilon' < \frac{1}{5}$$

We use this fact to reformulate point (d) of Lemma 5.9 as:

Lemma 5.11 (Claim 5.13.2(3)). *For all $k, r \geq 1$, $\epsilon' \leq \frac{1}{5}$ and $\epsilon < \frac{1}{4k}\epsilon'$, there exists $N_3 = N_3(\epsilon, \epsilon', r)$ large enough such that, for all $n > N_3$ and r dividing n, if $A \subseteq G$ is (ϵ, ϵ')-excellent, with $|A| = n$, then there exists a partition into r disjoint pieces of equal size, each of which is $\left(\frac{\epsilon'}{k}, \epsilon'\right)$-excellent.*

Proof. Choose any $\zeta' < \frac{1}{4k}\epsilon'$ and set $N_3 := N_2(\epsilon, \zeta', r)$. Remark 5.10 sufficiency condition is satisfied, Claim 5.9 (d) holds and we are done. □

Lemma 5.12 (Claim 5.14.1). *Let G be a finite graph with the non-k_*-order property. Let $\epsilon \in \left(0, \frac{1}{2}\right)$ and $\epsilon' < \frac{1}{2^{k_{**}}}$. Let $A \subseteq G$ such that $|A| = n$. Let $\langle m\ell \mid \ell \in [0, k_{**}]\rangle$ be a decreasing sequence of natural numbers such that $\epsilon m_\ell \geq m_{\ell+1}$ for all $\ell \in [0, k_{**} - 1]$, $m_{k_{**}} \geq 1$, and $m_{k_{**}-1} > k_{**}$. Denote $m_* := m_0$ and $m_{**} := m_{k_{**}}$. Then, there is a partition $\overline{A} = \langle A_j \mid j \in [0, j(*)]\rangle$ with remainder $B = A \setminus \bigcup_{j<j(*)} A_j$ such that:*

(a) For all $j \in [0, j(*)]$, $|A_j| \in \langle m\ell \mid \ell \in [0, k_{**} - 1]\rangle$.

(b) For all $i \neq j \in [0, j(*)]$, $A_i \cap A_j = \emptyset$.

(c) *For all $j \in [0, j(*)]$, A_j is (ϵ, ϵ')-excellent.*

(d) $|B| < \epsilon m_*$.

Proof. Apply Lemma 5.5 recursively to the remainder $A \setminus \bigcup_{i<j} A_i$, to obtain A_j at each step. The process stops at $j(*)$ when the remainder is smaller than m_0, and thus the lemma cannot be applied. Notice that, since $\frac{m_\ell}{m_{\ell-1}} \leq \epsilon$, $\left(\frac{m_\ell}{m_{\ell-1}}, \epsilon'\right)$-excellence implies (ϵ, ϵ')-excellence. □

Lemma 5.13 (Claim 5.14.1a). *Let G be a finite graph with the non-k_*-order property. Let $\epsilon \in \left(0, \frac{1}{2}\right)$ and $\epsilon' < \frac{1}{2^{k_{**}}}$. Let $A \subseteq G$ such that $|A| = n$. Let $\langle m\ell \mid \ell \in [0, k_{**}]\rangle$ be a decreasing sequence of natural numbers such that $\epsilon m_\ell \geq m_{\ell+1}$ for all $\ell \in [0, k_{**} - 1]$, $m_{k_{**}} \geq 1$, $m_{**} := m_{k_{**}} \mid m_l$ for all $l \in [0, k_{**}]$ and $m_{k_{**}-1} > N_3\left(\epsilon, \epsilon', \frac{m_*}{m_{**}}\right)$ (in the sense of Claim 5.11), where $m_* := m_0$. Then, for some $i(*) \leq \frac{n}{m_{**}}$, there is a partition $\overline{A} = \langle A_i \mid i \in [1, i(*)]\rangle$ with remainder $B = A \setminus \bigcup_{i \in [1, i(*)]}$ such that:*

(a) *For all $i \in [1, i(*)]$, $|A_i| = m_{**}$.*

(b) *For all $i \neq j \in [1, i(*)]$, $A_i \cap A_j = \emptyset$.*

(c) *For all $i \in [1, i(*)]$, A_i is $\left(\frac{\epsilon'}{k}, \epsilon'\right)$-excellent.*

(d) $|B| < \epsilon m_*$.

Proof. Use Claim 5.12 to obtain a partition $\overline{A}' = \left\langle A'_j \mid j \in [0, j(*)]\right\rangle$ and remainder B with $|B| < m_*$. Then, we can apply Claim 5.11 with $r = m_{**}$ to each of the parts A'_j. Putting together all the new subparts, we obtain a new partition $\overline{A} = \langle A_i \mid i \in [0, i(*)]\rangle$ with remainder B, satisfying all the conditions of the statement. □

Lemma 5.14 (Claim 5.14.2). *Under the same condition of Lemma 5.13, we can get a partition $\overline{A} = \langle A_i \mid i \in [1, i(*)]\rangle$ with no remainder, such that:*

(a) *For all $i \neq j \in [1, i(*)]$, $||A_i| - |A_j|| \leq 1$.*

(b) *For all $i \neq j \in [1, i(*)]$, $A_i \cap A_j = \emptyset$.*

(c) *For all $i \in [1, i(*)]$, A_i is (ϵ'', ϵ')-excellent, where*

$$\epsilon'' \leq \frac{\frac{\epsilon'}{k} m_{**} + \left\lceil \frac{m_*}{i(*)} \right\rceil}{m_{**} + \left\lceil \frac{m_*}{i(*)} \right\rceil}$$

(d) $A = \bigcup \overline{A}$.

Proof. Let $\overline{A}' = \langle A'_i \mid i \in [1, i(*)]\rangle$ and B from Claim 5.13. We can partition B into $\overline{B} = \langle B_i \mid i \in [1, i(*)]\rangle$ in such a way that for all $i \in [1, i(*)]$,

$$|B_i| \in \left\{ \left\lfloor \frac{|B|}{i(*)} \right\rfloor, \left\lceil \frac{|B|}{i(*)} \right\rceil \right\}$$

Notice that we are allowing $B_i = \emptyset$. Then, the new partition $\overline{A} = \langle A'_i \cup B_i \mid i \in [1, i(*)]\rangle$ satisfies (a), (b) and (d) by construction. To conclude, notice that for each ϵ'-good set B, the number of exceptions is bounded by

$$\begin{aligned}|\{a \in A_i \mid t(a, B) \not\equiv t(A_i, B)\}| &\leq \frac{\epsilon'}{k}\left|A'_i\right| + |B_i| \\ &= \frac{\frac{\epsilon'}{k}\left|A'_i\right| + |B_i|}{\left|A'_i\right| + |B_i|}\left(\left|A'_i\right| + |B_i|\right) \\ &\leq \frac{\frac{\epsilon'}{k} m_{**} + \left\lceil \frac{m_*}{i(*)} \right\rceil}{m_{**} + \left\lceil \frac{m_*}{i(*)} \right\rceil} |A_i|\end{aligned}$$

which proves that (c) can be satisfied. $\square$

Remark 5.15 (Remark 5.14.3). In the context of Lemma 5.14, if:

(a) $m_{**} > \frac{1}{\frac{\epsilon'}{k}}$

(b) $m_* \leq \frac{\frac{\epsilon'}{k} n+1}{\frac{\epsilon'}{k}+1}$

then $\epsilon'' < \frac{3\epsilon'}{k}$.

Proof. Notice that, if $|B_i| \leq 2\frac{\epsilon'}{k}|A_i|$ for all $i \in [1, i(*)]$, then ϵ'' can be bounded by:

$$\epsilon'' \leq \frac{\frac{\epsilon'}{k}|A_i| + |B_i|}{|A_i| + |B_i|} \leq \frac{\frac{\epsilon'}{k}|A_i| + 2\frac{\epsilon'}{k}|A_i|}{|A_i|} = \frac{3\epsilon'}{k}$$

Let's now prove that $|B_i| \leq \frac{2\epsilon'}{k}|A_i|$ is satisfied. Notice that, by construction:

$$|B_i| \leq \left\lceil \frac{|B|}{i(*)} \right\rceil \leq \left\lceil \frac{m_* - 1}{i(*)} \right\rceil \leq \frac{m_* - 1}{i(*)} + 1$$

Also we can bound $i(*)$ by:

$$\frac{n}{m_{**}} \geq i(*) \geq \frac{n - |B|}{m_{**}} \geq \frac{n - m_* + 1}{m_{**}} > \frac{n - m_*}{m_{**}}$$

Thus, $|B_i| - 1 \leq \frac{m_* - 1}{i(*)} \leq \frac{(m_* - 1)m_{**}}{n - m_*}$, then $\frac{|B_i| - 1}{m_{**}} < \frac{m_* - 1}{n - m_*}$, and since $|A_i| = m_{**}$ we get:

$$\frac{|B_i|}{|A_i|} < \frac{m_* - 1}{n - m_*} + \frac{1}{m_{**}}$$

Finally, notice that condition (a) implies:

$$\frac{\epsilon'}{k} > \frac{1}{m_{**}}$$

and condition (b) implies:

$$\frac{\epsilon'}{k} \geq \frac{m_* - 1}{n - m_*}$$

We conclude:

$$\frac{|B_i|}{|A_i|} < \frac{m_* - 1}{n - m_*} + \frac{1}{m_{**}} < 2\frac{\epsilon'}{k}$$

completing the proof. ☐

Lemma 5.16 (Corollary 5.15). *Let G be a graph with the non-k_*-order property. Suppose that we are given:*

1. $\epsilon < \min\left(\frac{1}{5}, \frac{1}{2^{k_{**}}}\right)$.
2. *A sequence of positive integers $\langle m\ell \mid \ell \in [0, k_{**}]\rangle$ such that:*
   - (a) $m_\ell \leq \frac{\epsilon}{12} m_{\ell+1}$.
   - (b) $m_{**} := m_{k_{**}} \geq \frac{\epsilon}{3}$.
   - (c) $m_{**} \mid m_\ell$ *for all* $\ell \in [0, k_{**}]$.
   - (d) $m_{k_{**}-1} > N_3\left(\frac{\epsilon}{12}, \epsilon, \frac{m_*}{m_{**}}\right)$ *(in the sense of Claim 5.11).*
3. $A \subseteq G$ *such that* $|A| = n$, *where n satisfies:*
   - (a') $n \geq m_0$.
   - (b') $m_* \leq \frac{1+\frac{\epsilon}{3}n}{1+\frac{\epsilon}{3}}$.

Then, there exists $i(*) \leq \frac{n}{m_{**}}$ and a partition of A into disjoint pieces $\overline{A} = \langle A_i \mid i \in [1, i(*)]\rangle$ such that:

- (i) *For all* $i \neq j \in [1, i(*)]$, $||A_i| - |A_j|| \leq 1$.
- (ii) *For all* $i \in [1, i(*)]$, A_i *is* ϵ*-excellent,*
- (iii) *For all* $i \neq j \in [1, i(*)]$, (A_i, A_j) *is* ϵ*-uniform.*

Proof. Simply apply Lemma 5.14 in the context of Remark 5.15 with $k > 3$, $\epsilon'_{5.14} = \epsilon$ and $\epsilon_{5.14} = \frac{1}{12}\epsilon$. This results in a partition of A into disjoint pieces that satisfy (i) and that are $(\epsilon''_{5.14}, \epsilon'_{5.14})$-excellent, with $\epsilon''_{5.14} < \frac{3\epsilon'_{5.14}}{k}$. But since $k > 3$, $\epsilon''_{5.14} < \epsilon'_{5.14}$, they are also $\epsilon'_{5.14}$-excellent, satisfying (ii) and (iii). ☐

Theorem 5.17 (Theorem 5.18). *Let k_* and therefore k_{**} be given. Then, for all $\epsilon < \min\left(\frac{1}{5}, \frac{1}{2^{k_{**}}}\right)$, there is $m = m(\epsilon, k_*)$ and $N = N(\epsilon, k_*)$ such that, for every finite graph G with the non-k_*-order property, and every $A \subseteq G$ with $|A| \geq N$, there exists a partition $\overline{A} = \langle A_i \mid i \in [1, i(*)]\rangle$ of A into at most m pieces, such that:*

1. *For all* $i, j \in [1, i(*)]$, $||A_i| - |A_j|| \leq 1$.
2. *For all* $i \in [1, i(*)]$, A_i *is* ϵ*-excellent.*
3. *For all* $i \neq j \in [1, i(*)]$, (A_i, A_j) *is* ϵ*-uniform.*
4. $m \leq (3 + \epsilon)\left(\frac{12}{\epsilon}\right)^{k_{**}}$.

Proof. Suppose $N = N(\epsilon, k_*)$ is large enough. We will state at the end of the proof the required size of N. Our goal is to apply Lemma 5.16. Let $q = \left\lceil \frac{12}{\epsilon} \right\rceil$. For n large enough (H1), we can choose m_{**} satisfying:

(a) $m_{**} \in \left(\frac{\epsilon n}{(3+\epsilon)q^{k_{**}-1}} - 1, \frac{\epsilon n}{(3+\epsilon)q^{k_{**}-1}}\right)$.

(b) $m_{**} \geq \frac{3}{\epsilon}$.

(c) $m_{**} > \frac{N_3\left(\frac{\epsilon}{12}, \epsilon, q^{k_{**}-1}\right)}{q}$.

Then, setting $m_{k_{**}} = m_{**}$ we can build recursively a sequence of integers $\langle m_\ell \mid \ell \in [0, k_{**}]\rangle$ such that $m_\ell = qm_{\ell+1}$ for all $\ell \in [0, k_{**} - 1]$. Finally, let $m_* = q^{k_{**}-1}m_{**}$. Notice that by (a) we have that $m_* = m_{**}q^{k_{**}-1} \leq \frac{\epsilon n}{3+\epsilon}$. This sequence satisfies all the conditions of Lemma 5.16:

(2.a) $m_{\ell-1} = qm_\ell \leq \frac{\epsilon}{12}m_\ell$.

(2.b) $m_{**} \geq \frac{3}{\epsilon}$.

(2.c) $m_\ell = q^{k_{**}-\ell}m_{**}$ so that $m_{**} \mid m_\ell$ for all $\ell \in [0, k_{**}]$.

(2.d) $m_{k_{**}-1} = qm_{**} < q\frac{N_3\left(\frac{\epsilon}{12}, \epsilon, q^{k_{**}-1}\right)}{q} = N_3\left(\frac{\epsilon}{12}, \epsilon, \frac{m_*}{m_{**}}\right)$.

(3.b) $m_* \leq \frac{\epsilon n}{3+\epsilon} \leq \frac{1+\frac{\epsilon}{3}n}{1+\frac{\epsilon}{3}}$.

Finally, we need $N > m_0$ to satisfy (3.a) (H2). Now, we can apply Lemma 5.16 to obtain a partition satisfying (1), (2) and (3). We just need to check that (4) holds. By (a), we have that $m_{**} \geq \frac{1}{2}\frac{\epsilon n}{(3+\epsilon)q^{k_{**}-1}}$. Thus, we can bound the number of pieces by:

$$m \leq \frac{n}{m_{**}} \leq \frac{2(3+\epsilon)q^{k_{**}-1}}{\epsilon} \leq (3+\epsilon)\left(\frac{2}{\epsilon}\right)\left(\frac{12}{\epsilon}\right)^{k_{**}-1} < (3+\epsilon)\left(\frac{12}{\epsilon}\right)^{k_{**}-1}$$

Notice that the bound on m only depends on ϵ and k_{**}.

To conclude, let's put together all requirements on N:

(H1) N needs to be large enough to satisfy

(H2) $N > N_3\left(\frac{\epsilon}{12}, \epsilon, \frac{m_*}{m_{**}}\right)$.

□

Lemma 5.18 (Lemma 5.17). *Suppose that $\epsilon_1, \epsilon_2, \epsilon_3 \in \left(0, \frac{1}{2}\right)$ with $\frac{\epsilon_1+\epsilon_2}{\epsilon_3} < \frac{1}{2}$ and the pair (A, B) is (ϵ_1, ϵ_2)uniform. By uniformity, there is a truth value $t(A, B) \in \{0, 1\}$. Let $A' \subseteq A$ with $|A'| \geq \epsilon_3|A|$, $B' \subseteq B$ with $|B'| \geq \epsilon_3|B|$ and denote $Z = \{(a, b) \in (A \times B) \mid aRb \not\equiv t(A, B)\}$ and $Z' = \{(a, b) \in (A' \times B') \mid aRb \not\equiv t(A, B)\}$. Then, we have:*

1. $\frac{|Z|}{|A||B|} < \epsilon_1 + \epsilon_2$.

2. $\frac{|Z'|}{|A||B|} < \frac{\epsilon_1+\epsilon_2}{\epsilon_3}$.

In particular, if for some $\epsilon_0, \epsilon \in \left(0, \frac{1}{2}\right)$, the pair (A, B) is ϵ_0-uniform, for $\epsilon_0 \leq \frac{\epsilon^2}{2}$, then:

a. *(A, B) is ϵ-regular.*

b. *If $A' \in [A]^{\geq \epsilon |A|}$ and $B' \in [B]^{\geq \epsilon |B|}$, then $d(A', B') < \epsilon$ or $d(A', B') \geq 1 - \epsilon$.*

Proof. Let $U = \left\{a \in A \mid \left|\overline{B}_{a,B}\right| > \epsilon_1 |A|\right\}$, i.e. the set of exceptional vertices $a \in A$. Then,

$$Z \subseteq U \times B \cup \bigcup_{a \in A \setminus U} \{a\} \times W_a$$

and

$$Z' \subseteq U \times B' \cup \bigcup_{a \in A' \setminus U} \{a\} \times W_a$$

Notice that, if $a \in A \setminus U$, then $|W_a| < \epsilon_2 |B|$, so

$$|Z| \leq \epsilon_1 |A| |B| + |A| \epsilon_2 |B|$$

which can be written as

$$\frac{|Z|}{|A| |B| \leq \epsilon_1 + \epsilon_2}$$

which proves (1). On the other hand,

$$\begin{aligned} \left|Z'\right| &\leq |U| \left|B'\right| + \left|A'\right| \max\left\{|W_a| \mid a \notin U\right\} \\ &< \epsilon_1 |A| \left|B'\right| + \left|A'\right| \epsilon_2 |B| \end{aligned}$$

By dividing both sides by $|A'| |B'|$ we conclude

$$\frac{|Z'|}{|A'| |B'|} < \epsilon_1 \frac{|A|}{|A'|} + \epsilon_2 \frac{|B|}{|B'|} \leq \frac{\epsilon_1 |A|}{\epsilon_3 |A|} + \frac{\epsilon_2 |B|}{\epsilon_3 |B|} = \frac{\epsilon_1 + \epsilon_2}{\epsilon_3}$$

proving (2). Let's now prove (a) and (b). First of all, notice that:

- if $t(A, B) = 1$,then $d(A, B) > 1 - (\epsilon_1 + \epsilon_2)$ and $d(A', B') > 1 - \frac{\epsilon_1 + \epsilon_2}{\epsilon_3}$ by (1) and (2). Thus,

$$\begin{aligned} \left|d(A, B) - d(A', B')\right| &\leq \max\left\{d(A, B) - d(A', B'), d(A', B) - d(A, B)\right\} \\ &< \max\left\{1 - \left(1 - \frac{\epsilon_1 + \epsilon_2}{\epsilon_3}\right), 1 - (1 - \epsilon_1 - \epsilon_2)\right\} \\ &= \frac{\epsilon_1 + \epsilon_2}{\epsilon_3} \end{aligned}$$

- if $t(A, B) = 0$,then $d(A, B) < (\epsilon_1 + \epsilon_2)$ and $d(A', B') < \frac{\epsilon_1 + \epsilon_2}{\epsilon_3}$, again by (1) and (2). Thus,

$$\begin{aligned} \left|d(A, B) - d(A', B')\right| &\leq \max\left\{d(A, B) - d(A', B'), d(A', B) - d(A, B)\right\} \\ &< \max\left\{(\epsilon_1 + \epsilon_2), \frac{\epsilon_1 + \epsilon_2}{\epsilon_3}\right\} \\ &= \frac{\epsilon_1 + \epsilon_2}{\epsilon_3} \end{aligned}$$

In both cases, we have that $|d(A, B) - d(A', B')|$ is bounded by $\frac{\epsilon_1 + \epsilon_2}{\epsilon_3} < \frac{1}{2}$. Also, $d(A', B')$ may only differ by $\frac{\epsilon_1 + \epsilon_2}{\epsilon_3}$ with either 0 or 1. In particular, we may choose $\epsilon_3 = \epsilon$ and $\epsilon_1 = \epsilon_2 = \epsilon_0 \leq \frac{\epsilon^2}{2}$. This way, the condition $\frac{\epsilon_1 + \epsilon_2}{\epsilon_3} \leq \epsilon < \frac{1}{2}$ is satisfied. We conclude that (A, B) is ϵ-regular (a) and that $d(A', B')$ is either $< \epsilon$ or $\geq 1 - \epsilon$ (b). $\square$

Theorem 5.19 (Theorem 5.19). *For every $k_* \in \mathbb{N}$ and $\epsilon \in \left(0, \frac{1}{2}\right)$, there exist $N = N(\epsilon, k_*)$ and $m = m(\epsilon, k_*)$ such that, for every finite graph G with the non-k_*-order property, and every $A \subseteq G$ with $|A| \geq N$, there is $\ell < m$ and a partition $\overline{A} = \langle A_i \mid i \in [1, \ell] \rangle$ of A such that each A_i is $\frac{\epsilon^2}{2}$-excellent, and for every $i \neq j \in [1, \ell]$,*

1. $||A_i| - |A_j|| \leq 1$.
2. (A_i, A_j) *is ϵ-uniform, and moreover if* $B_i \in [A_i]^{\geq \epsilon |A_i|}$ *and* $B_j \in [A_j]^{\geq \epsilon |A_j|}$, *then either* $d(B_i, B_j) < \epsilon$ *or* $d(B_i, B_j) \geq 1 - \epsilon$.
3. *If* $\epsilon < \min\left(\frac{1}{5}, \frac{1}{2^{k_{**}}}\right)$, *then* $m \leq \left(3 + \frac{\epsilon^2}{2}\right)\left(\frac{24}{\epsilon}\right)^{k_{**}}$.

Proof. If $\epsilon < \min\left(\frac{1}{5}, \frac{1}{2^{k_{**}}}\right)$, then we can apply Theorem 5.17 to A with $\frac{\epsilon^2}{2}$, and then use Lemma 5.18 to replace the $\frac{\epsilon^2}{2}$-uniformity of pairs by ϵ-regularity. Otherwise, to get (1) and (2), just do the same process for some $\epsilon' \leq \min\left(\frac{1}{5}, \frac{1}{2^{k_{**}}}\right) \leq \epsilon$. Then, since regularity is monotone, we get the wanted ϵ-regularity from the resulting ϵ'-regularity. Notice that a bound for m can be still obtained by replacing $\epsilon = \min\left(\frac{1}{5}, \frac{1}{2^{k_{**}}}\right)$ in $m \leq \left(3 + \frac{\epsilon^2}{2}\right)\left(\frac{24}{\epsilon}\right)^{k_{**}}$. $\square$

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A. Title of the appendix

You can include here an appendix with details that can not be included in the core of the document. You should reference the sections in this appendix in the core document.

B. Title of the appendix

Second appendix.