Universitat Politècnica de Catalunya
Facultat de Matemàtiques i Estadística

Master in Advanced Mathematics and Mathematical Engineering

Master's thesis

Why the non-monotonicity of excellence f***** up my life

Severino Da Dalt

Supervised by (Lluis Vena Cros)

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Thanks to...

Abstract

This should be an abstract in english, up to 1000 characters.

Keywords

regularity, stable graphs, graph theory, ...

1. Introduction

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2. Section 3

3. Section 4

Definition 3.1 (Definition 4.2(a))**.** Let $\epsilon \in (0,1)$. We say that $A \subseteq G$ is *ϵ-indivisible* if for every $B \in G$, for some truth value $t = t(b, A)$ we have that

$$|\{a \in A \mid aRb \not\equiv t\}| < |A|^\epsilon$$

Definition 3.2 (Definition 4.2(b))**.** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a non-decreasing function. We say that $A \subseteq G$ is *f-indivisible* if for every $B \in G$, for some truth value $t = t(b, A)$ we have that

$$|\{a \in A \mid aRb \not\equiv t\}| < f(|A|)$$

Remark 3.3. If $f(n) = \epsilon n$, then f-indivisible $\equiv$ ϵ-good.

Remark 3.4. ϵ-indivisible is a much stronger condition then ϵ-good.

Lemma 3.5 (Claim 4.3). *Let G be a finite graph with the non-k_*-property. Assume $m_0 > \cdots > m_{k_{**}}$ is a sequence of non-zero natural numbers and for all $l \in [k_{**}]$, $f(m_{l-1}) \geq m_l$. If $A \subseteq G$, $|A| = m_0$, then for some $l < k_{**}$ there is a subset $B \subseteq A$ of size m_l which is f-indivisible.*

Proof. Suppose not. Then we can construct the sequences $\left\langle b_\eta \mid \eta \in [2]^{<k} \right\rangle$ and $\left\langle A_\eta \mid \eta \in [2]^{\leq k} \right\rangle$ on induction over $k < k_{**}$, where $k = |\eta|$, satisfying:

1. $A_{\eta^\frown\langle i \rangle} \subseteq A_\eta$, $\forall i \in \{0,1\}$, $\forall k < k_{**}$

2. $A_{\eta^\frown\langle 0\rangle} \cap A_{\eta^\frown\langle 1\rangle} = \emptyset$, $\forall k < k_{**}$

3. $|A_\eta| = m_k$, $\forall k \leq k_{**}$

4. $b_\eta \in G$ witnessing that A_η is not f-indivisible, $\forall k < k_{**}$

5. $A_{\eta^\frown\langle i\rangle} \subseteq A_\eta^{(i)} = \{a \in A_\eta \mid aRb_\eta \equiv (i = 1)\}$, $\forall \in \{0, 1\}$, $\forall k < k_{**}$

Let's prove the induction:

- $\underline{k = 0}$. Consider $A_{\langle\cdot\rangle} = A$, which satisfies $|A_{\langle\cdot\rangle}| = m_0$ and $|b_{\langle\cdot\rangle}|$ witnessing the non-f-indivisibility of $A_{\langle\cdot\rangle}$.
- $\underline{k \Rightarrow k+1}$. We can assume $|A_\eta| = m_k$ and by hypothesis A_η is not f-indivisible. So, there exists b_η such that $A_\eta^{(i)} \geq f(m_k) \geq m_{k+1}$ (4), and we can choose $A_{\eta^\frown\langle i\rangle} \subseteq A_\eta^{(i)}$ (5), such that $|A_{\eta^\frown\langle i\rangle}| = m_{k+1} \forall i \in \{0, 1\}$ (3). (1) and (2) are satisfied by the definition of $A_\eta^{(i)}$.

Now, for all η such that $|\eta| = k_{**}$, consider some element $a_\eta \in A_\eta$. Then, we have two sequences $\langle b_\eta \mid \eta \in [2]^{<k_{**}}\rangle$ and $\langle A_\eta \mid \eta \in [2]^{k_{**}}\rangle$ with the property:

$$\forall \rho \in [2]^{<k_{**}} \forall \eta \in [2]^{k_{**}} \text{ such that } \rho^\frown \langle i\rangle \trianglelefteq \eta, \ (a_\eta R b_\rho)$$

since $a_\eta \in A_\eta \subseteq A_{\rho^\frown\langle i\rangle}$. This contradicts the k_{**} tree bound. □

Lemma 3.6 (Claim 4.4). *Let G be a finite graph wit the non-k_*-order property. Assume $m_0 > \cdots > m_{k_{**}}$ is a sequence of non-zero natural numbers and for all $l \in [k_{**}]$, $f(m_{l-1}) \geq m_l$. If $A \subseteq G$ with $|A| = n$, then we can find a sequence $\overline{A} = \langle A_j \mid j \in [j(*)]\rangle$ and reminder $B = A \setminus \bigcup \overline{A}$ such that:*

1. *For each $j \in [j(*)]$, A_j is f-indivisible*
2. *For each $j \in [j(*)]$, $|A_j| \in \{m_0, \dots, m_{k_{**}-1}\}$*
3. *$A_j \subseteq A \setminus \bigcup\{A_i \mid i < j\}$, in particular $A_i \cap A_j = \emptyset \ \forall i \neq j$*
4. *$|B| < m_0$*

Proof. Iteratively, apply Claim 3.5 to the remainder $A \setminus \bigcup\{A_i \mid i < j\}$ (3) to get an f-indivisible A_j (1) of size m_l, $l \in \{0, \dots, k_{**} - 1\}$ (2) until less then m_0 vertices are available (4). □

Lemma 3.7 (Claim 4.5). *Let G be a graph with the non-k_*-order property. Assume $n \geq m_0 > \cdots > m_{k_{**}}$ is a sequence of non-zero natural numbers satisfying that for all $l \in [k_{**}]$ $\lfloor (m_{l-1})^\epsilon \rfloor = m_l$, for $\epsilon \in (0, \frac{1}{2})$. If $A \subseteq G$, $|A| = n$, then we can find $\overline{A} = \langle A_i \mid i \in [i(*)]\rangle$ with remainder $B = A \setminus \bigcup \overline{A}$ such that:*

1. *For each $j \in [j(*)]$, A_j is ϵ-indivisible*
2. *For each $j \in [j(*)]$, $|A_j| \in \{m_0, \dots, m_{k_{**}-1}\}$*
3. *$A_i \cap A_j = \emptyset \ \forall i \neq j$*
4. *$|B| < m_0$*
5. *$\overline{A}$ is $\leq$-increasing*

Proof. The first four clauses are direct consequence of applying Claim 3.6 with $f(n) = n^{\epsilon}$. By renaming the A_i's in ascending-size order, we get (5). $\square$

Remark 3.8. In this context, if $m_{k_{**}} > k_{**}$

$$n^{\epsilon^{k_{**}}} \geq m_0^{\epsilon^{k_{**}}} \geq m_1^{\epsilon^{k_{**}-1}} \geq \cdots \geq m_{k_{**}} > k_{**}$$

So, $n^{\epsilon^{k_{**}}} > k_{**}$.

Lemma 3.9 (Claim 4.6)). *Let G be a finite graph. Suppose $A, B \subseteq G$ such that A is f-indivisible, B is g-indivisible, and $f(|A|)g(|B|) < \frac{1}{2}|B|$. Then, for some truth value $t = t(A, B)$ for all but $< f(|A|)$ of the $a \in A$ for all but $< g(|B|)$ of the $b \in B$ we have that $aRb \equiv t$.*

Proof. Since B is g-indivisible, for each $a \in A$ there is a truth value $t_a = t(a, B)$ such that $\{b \in B \mid aRb \not\equiv t_a\} < g(|B|)$. Let $U_i = \{a \in A \mid t_a = i\}$ for $i \in \{0, 1\}$. If either U_i satisfies $|U_i| < f(|A|)$ then the statement is true. Suppose not. Then, there are $W_i \subseteq U_i$ with $|W_i| = f(|A|)$ for $i \in \{0, 1\}$. Now, let $V = \{b \in B \mid (\exists a \in W_0 \mid aRb) \vee (\exists a \in W_1 \mid \neg aRb)\}$, i.e. the b's which are an exception for some $a \in W_0 \cup W_1$. Then, $|V| < |W_0|g(|B|) + |W_1|g(|B|) = 2f(|A|)g(|B|) < |B|$, where the first inequality follows the g-indivisibility of B. Finally, there is a $b_* \in B \setminus V$ such that $\forall a \in W_0\ \neg aRb_*$ and $\forall a \in W_1\ aRb_*$ with $|W_0| = |W_1| = f(|A|)$, which contradicts the f-indivisibility of A. $\square$

Definition 3.10. We say that the pair (A, B) with A f-indivisible and B g-indivisible satisfies the *average condition* if $f(|A|)g(|B|) < \frac{1}{2}|B|$ and thus the statement of Claim 3.9 is true for the pair (A, B).

Remark 3.11. The condition $f(|A|)g(|B|) < \frac{1}{2}|B|$ makes ordering of the pair (A, B) matter. Thus,

$$(A, B) \text{ has the average condition } \not\Rightarrow (B, A) \text{ has the average condition}$$

Remark 3.12 (Remark 4.7). When $f(n) = n^{\epsilon}$ and $g(n) = n^{\zeta}$, the average condition is $|A|^{\epsilon}|V|^{\zeta} < \frac{1}{2}|B|$.

Lemma 3.13 (Claim 4.8). *Let A be ϵ-indivisible, B ζ-indivisible and let the pair (A, B) satisfy the average condition. Then, for all $\epsilon_1 \in (0, 1-\epsilon)$, $\zeta_1 \in (0, 1-\zeta)$, $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| \geq |A|^{\epsilon+\epsilon_1}$ and $|B'| \geq |B|^{\zeta+\zeta_1}$, we have that:*

$$\frac{|\{(a, b) \in (A', B') \mid aRb \equiv \neg t(A, B)\}|}{|A' \times B'|} \leq \frac{1}{|A|}\epsilon_1 + \frac{1}{|B|}\zeta_1$$

Proof. Notice:

- There are at most $|A|^{\epsilon}$ elements of A (hence in $A' \subseteq A$) which are exceptional (in the sense of the average condition).
- For each $a \in A$ (hence in $A' \subseteq A$) not exceptional, there are at most $|B|^{\zeta}$ elements $b \in B$ such that (a, b) does not satisfy the truth value $t(A, B)$, i.e. that are exceptional.

Putting it all together:

$$\begin{aligned}
\frac{|\{(a,b)\in(A',B')\mid aRb\equiv\neg t(A,B)\}|}{|A'\times B'|} &\le \frac{|A|^{\epsilon}|B'|+(|A'|-|A|^{\epsilon})|B|^{\zeta}}{|A'||B'|}\\
&= \frac{|A|^{\epsilon}}{|A'|}+\frac{|A'|-|A|^{\epsilon}}{|A'|}\frac{|B|^{\zeta}}{|B'|}\\
&\le \frac{|A|^{\epsilon}}{|A'|}+\frac{|B|^{\zeta}}{|B'|}\\
&\le \frac{|A|^{\epsilon}}{|A|^{\epsilon+\epsilon_1}}+\frac{|B|^{\zeta}}{|B|^{\zeta+\zeta_1}}\\
&= \frac{1}{|A|^{\epsilon}_1}+\frac{1}{|B|^{\zeta}_1}
\end{aligned}$$

□

Lemma 3.14 (f-indivisible version). *Let A be f-indivisible, B g-indivisible and let the pair (A,B) satisfy the average condition. Then, for all $\epsilon_1 \in \left(0, 1-\frac{f(|A|)}{|A|}\right)$, $\zeta_1 \in \left(0, 1-\frac{g(|B|)}{|B|}\right)$, $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| \ge f(|A|)|A|^{\epsilon_1}$ and $|B'| \ge g(|B|)|B|^{\zeta_1}$, we have that:*

$$\frac{|\{(a,b)\in(A',B')\mid aRb\equiv\neg t(A,B)\}|}{|A'\times B'|} \le \frac{1}{|A|}\epsilon_1+\frac{1}{|B|}\zeta_1$$

Proof. Notice:

- There are at most $f(|A|)$ elements of A (hence in $A' \subseteq A$) which are exceptional (in the sense of the average condition).
- For each $a \in A$ (hence in $A' \subseteq A$) not exceptional, there are at most $g(|B|)$ elements $b \in B$ such that (a,b) does not satisfy the truth value $t(A,B)$, i.e. that are exceptional.

Putting it all together:

$$\begin{aligned}
\frac{|\{(a,b)\in(A',B')\mid aRb\equiv\neg t(A,B)\}|}{|A'\times B'|} &\le \frac{f(|A|)|B'|+(|A'|-f(|A|))g(|B|)}{|A'||B'|}\\
&= \frac{f(|A|)}{|A'|}+\frac{|A'|-f(|A|)}{|A'|}\frac{g(|B|)}{|B'|}\\
&\le \frac{f(|A|)}{|A'|}+\frac{g(|B|)}{|B'|}\\
&\le \frac{f(|A|)}{f(|A|)|A|^{\epsilon_1}}+\frac{g(|B|)}{g(|B|)|B|^{\zeta_1}}\\
&= \frac{1}{|A|^{\epsilon}_1}+\frac{1}{|B|^{\zeta}_1}
\end{aligned}$$

□

Corollary 3.15 (Corollary 4.9). *Let A and B be f-indivisible with $f(n) = c$ and (A, B) satisfy the average condition. Then, for all $\epsilon_1 \in (0, 1 - \frac{c}{|A|})$, $\zeta_1 \in (0, 1 - \frac{c}{|B|})$, $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq c|A|^{\epsilon_1}$ and $|B'| \geq c|B|^{\zeta_1}$, we have:*

$$\frac{|\{(a, b) \in (A', B') \mid aRb \equiv \neg t(A, B)\}|}{|A' \times B'|} \leq \frac{1}{|A|}\epsilon_1 + \frac{1}{|B|}\zeta_1$$

Proof. Use Claim 3.14 with $f(n) = c$. ☐

Lemma 3.16 (Claim 4.10). *Let G be a finite graph wit the non-k_*-order property. Assume $n \geq m_0 > \cdots > m_{k_{**}}$ is a sequence of non-zero natural numbers and for all $l \in [k_{**}]$, $\lfloor (m_{l-1})^\epsilon \rfloor = m_l$, for some $\epsilon \in (0, \frac{1}{2})$ such that $2 < (m_{k_{**}})^{1-2\epsilon}$. If $A \subseteq G$ with $|A| = n$, then we can find a sequence $\overline{A} = \langle A_i \mid i \in [i(*)] \rangle$ and reminder $B = A \setminus \bigcup \overline{A}$ satisfying:*

1. *For each $i \in [i*)]$, A_i is ϵ-indivisible*
2. *For each $i \in [i(*)]$, $|A_i| \in \{m_0, \ldots, m_{k_{**}-1}\}$*
3. *$A_i \cap A_j = \emptyset \ \forall i \neq j$*
4. *$|B| < m_0$*
5. *$\overline{A}$ is $\leq$-increasing*
6. *If $\zeta \in (0, \epsilon^{k_{**}})$ then for every $i, j \in [i(*)]$ with $i < j$, $A \subseteq A_i$ ad $B \subseteq A_j$ such that $|A| \geq |A_i|^{\epsilon+\zeta}$ and $|B| \geq |A_j|^{\epsilon+\zeta}$ we have that:*

$$\begin{aligned}\frac{|\{(a, b) \in (A, B) \mid aRb \equiv \neg t(A_i, A_j)\}|}{|A \times B|} &\leq \frac{1}{|A_i|}\zeta + \frac{1}{|A_j|}\zeta \\ &\leq \frac{1}{|A|}\zeta + \frac{1}{|B|}\zeta\end{aligned}$$

Proof. The five points are direct consequence of Claim 3.7. Now, for any $A_i, A_j \in \overline{A}$ with $i < j$. By (2), there is some $l < k_{**}$ such that $|A_i| \leq |A_j| = m_l$ for some $l < k_{**}$. Then, it follows the condition $2 < (m_{k_{**}})^{1-2\epsilon}$ that:

$$\frac{|A|^\epsilon |B|^\epsilon}{|B|} \leq \frac{|B|^{2\epsilon}}{|B|} = \frac{1}{|B|^{1-2\epsilon}} = \frac{1}{m_l^{1-2\epsilon}} \leq \frac{1}{m_{k_{**}}} < \frac{1}{2}$$

i.e. $|A|^\epsilon |B|^\epsilon < \frac{1}{2}|B|$ and by Claim 3.12 the average condition is satisfied. Finally, notice that $\epsilon^{k_{**}} < \epsilon < 1 - \epsilon$ since $\epsilon \in (0, \frac{1}{2})$, so that $\zeta \in (0, \epsilon^{k_{**}}) \subseteq (0, 1 - \epsilon)$ and the condition for Claim 3.13 is satisfied. This gives us (6) and concludes the proof of the statement. ☐

Definition 3.17. Let A, B be f-indivisible sets with $f(A) \times f(B) < \frac{1}{2}|B|$. Let $\langle A_i \mid i < i_A \rangle$ be a partition of A with $|A_i| = m \forall i < i_A$ and $\langle B_i \mid i < i_B \rangle$ be a partition of B with $|B_i| = m \forall i < i_B$. $\varepsilon^+_{A_i, A_j, m}$ is the event:

$$\forall a \in A_i \forall b \in B_i, aRb = t(A, B)$$

Lemma 3.18 (Claim 4.13). *Let G be a finite graph wit the non-k_*-order property. Assume $n \geq m_0 > \cdots > m_{k_{**}}$ is a sequence of non-zero natural numbers and for all $l \in [k_{**}]$, $\lfloor (m_{l-1})^\epsilon \rfloor = m_l$, for some $\epsilon \in (0, \frac{1}{2})$ such that $2 < (m_{k_{**}})^{1-2\epsilon}$. Let $A_1, A_2 \subseteq G$ two ϵ-indivisible subsets such that $|A_1| = m_{l_1}$ and*

$|A_2| = m_{l_2}$ for some $l_1, l_2 < k_{**}$ and $|A_1| \leq |A_2|$. We will assume some approximation error by supposing $m_l = (m_{l-1})^{\epsilon}$. Suppose that, for some $c \in (0, 1-\epsilon)$ and $\zeta \leq \frac{1-\epsilon-c}{3}\epsilon^{k_{**}}$, $m = n^{\zeta}$ divides $|A_1|$ and $|A_2|$. Then, let $\left\langle A_{1,s} \mid s \in \left[\frac{|A_1|}{m}\right]\right\rangle$ and $\left\langle A_{2,t} \mid t \in \left[\frac{|A_2|}{m}\right]\right\rangle$ be random partitions of A_1 and A_2 respectively, with pieces of size m. We have that

$$P(\varepsilon^{+}_{A_{1,s},A_{2,t},m}) \geq 1 - \frac{2}{n^{c\epsilon^{k_{**}}}}$$

Proof. Fix $s \in \frac{|A_1|}{m}$, $t \in \frac{|A_2|}{m}$.

UPS, something is missing here

...and thus the average condition is satisfied. Let $U_1 = \{a \in A_1 \mid |\{b \in A_2 \mid aRb \equiv \neg t(A_1, A_2)\}| \geq |A_2|^{\epsilon}\}$ and for each $a \in A_1 \setminus U_1$ let $U_{2,a} = \{b \in A_j \mid aRb \equiv \neg t(A_1, A_2)\}$ By Claim 3.9, $|U_1| \leq |A_1|^{\epsilon}$ and $\forall a \in A_1 \setminus U_1$, $|U_2| \leq |A_2|^{\epsilon}$. Now, we can bound the probability P_1 that $A_{1,s} \cap U_1 \neq \emptyset$ as follows:

$$\begin{aligned}
P_1 &\leq \frac{|U_1|}{|A_1|} + \cdots + \frac{|U_1|}{|A_1| - m + 1} < \frac{m|U_1|}{|A_1| - m} \leq \frac{m|A_1|^{\epsilon}}{|A_1| - m} \\
&\leq \frac{m^2|A_1|^{\epsilon}}{|A_1|} = \frac{m^2}{|A_1|^{1-\epsilon}} = \frac{n^{2\zeta}}{n^{(1-\epsilon)\epsilon^{l_1}}} \\
&\leq \frac{n^{2\frac{1-\epsilon-c}{3}\epsilon^{k_{**}}}}{n^{(1-\epsilon)\epsilon^{k_{**}}}} \leq \frac{n^{(1-\epsilon-c)\epsilon^{k_{**}}}}{n^{(1-\epsilon)\epsilon^{k_{**}}}} = \frac{1}{n^{c\epsilon^{k_{**}}}}
\end{aligned}$$

The forth inequality comes from the fact that $\frac{(|A_i|-m)m}{|A_i|} \geq 1$. Then, if $A_{1,s} \cap U_1 = \emptyset$, we have that $|\bigcup_{a \in A_{1,s}} U_{2,a}| \leq |A_{1,s}||A_2|^{\epsilon}$. So we can bound P_2, the probability that $A_{2,t} \cap \bigcup_{a \in A_{1,s}} U_{2,a} = \emptyset$, by:

$$\begin{aligned}
P_2 &\leq \frac{|\bigcup_{a \in A_{1,s}} U_{2,a}|}{|A_2|} + \cdots + \frac{|\bigcup_{a \in A_{1,s}} U_{2,a}|}{|A_2| - m + 1} < \frac{m|\bigcup_{a \in A_{1,s}} U_{2,a}|}{|A_2| - m} \leq \frac{mm|A_2|^{\epsilon}}{|A_2| - m} \\
&\leq \frac{m^3|A_2|^{\epsilon}}{|A_2|} = \frac{m^3}{|A_2|^{1-\epsilon}} = \frac{n^{3\zeta}}{n^{(1-\epsilon)\epsilon^{l_2}}} \\
&\leq \frac{n^{3\frac{1-\epsilon-c}{3}\epsilon^{k_{**}}}}{n^{(1-\epsilon)\epsilon^{k_{**}}}} \leq \frac{n^{(1-\epsilon-c)\epsilon^{k_{**}}}}{n^{(1-\epsilon)\epsilon^{k_{**}}}} = \frac{1}{n^{c\epsilon^{k_{**}}}}
\end{aligned}$$

Putting it all together:

$$P(\varepsilon^{+}_{A_{1,s},A_{2,t},m}) \geq (1 - P_1)(1 - P_2) \geq \left(1 - \frac{1}{n^{c\epsilon^{k_{**}}}}\right)^2 \geq 1 - \frac{2}{n^{c\epsilon^{k_{**}}}}$$

□

Remark 3.19. Since $\epsilon < \frac{1}{2}$, we can take $c = 1 - 2\epsilon$. In this context, $\zeta \leq \frac{\epsilon^{k_{**}+1}}{3}$.

Lemma 3.20 (Claim 4.14). *Let G be a finite graph wit the non-k_*-order property. Assume $n \geq m_0 > \cdots > m_{k_{**}}$ is a sequence of non-zero natural numbers and for all $l \in [k_{**}]$, $\lfloor (m_{l-1})^{\epsilon} \rfloor = m_l$, for some $\epsilon \in (0, \frac{1}{2})$ such that $2 < (m_{k_{**}})^{1-2\epsilon}$. Also, let m_0 be small enough to satisfy $m_0 < \frac{n}{n^{(1-2\epsilon)\epsilon^{k_{**}}}}$ and $m_0 \leq \frac{\sqrt{2}-1}{\sqrt{2}}n$. Finally, let m_{**} be a divisor of m_l for all $l < k_{**}$ and $m_{**} \leq n^{\frac{\epsilon^{k_{**}+1}}{3}}$. If $A \subseteq G$ with $|A| = n$, then we can find a partition $\overline{A} = \langle A_i \mid i \in [r] \rangle$ with reminder $B = A \setminus \bigcup \overline{A}$ such that:*

1. *$|A_i| = m_{**} \forall i \in [r]$*

2. *For all but* $\frac{2r^2}{n^{(1-2\epsilon)\epsilon^{k_{**}}}}$ *of the pairs* (A_i, A_j) *with* $i < j$ *there are no exceptional edges, i.e.*

$$\{(a, b) \in A_i \times A_j \mid aRb \not\equiv t(A_i, A_j)\} = \emptyset$$

3. $|B| < m_0$

Proof. We can use Claim 3.7 to get a partition $\overline{A'} = \langle A'_i \mid i \in [i(*)]\rangle$ and remainder $B' = A \setminus \bigcup A'$. We can refine the partition by randomly splitting each A'_i into pieces of size m_{**} (1). Consider the resulting partition $\overline{A} = \langle A_i \mid i \in [r]\rangle$ with remainder $B = B'$ (3). First of all, notice that for each pair (A_i, A_j) such that $A_i \subseteq A'_{i_1}$ and $A_j \subseteq A'_{j_1}$ with $i_1 \neq j_1$, the probability of the pair having exceptional edges is upper bounded by $\frac{2}{n^{(1-2\epsilon)\epsilon^{k_{**}}}}$. This follows Claim 3.18 in the context of Remark 3.19. Thus, given X the random variable counting the number of exceptional pairs of this kind, we have

$$E(X) = \sum_{\substack{A_i, A_j \text{ s.t.} \\ A_i \subseteq A'_{i_1}, A_j \subseteq A'_{j_1} \\ i_1 \neq j_1}} E(X_{A_i, A_j}) = \sum_{\substack{A_i, A_j \text{ s.t.} \\ A_i \subseteq A'_{i_1}, A_j \subseteq A'_{j_1} \\ i_1 \neq j_1}} P(\varepsilon_{A_i, A_j, m_{**}}) \leq \frac{r^2}{2} \frac{2}{n^{(1-2\epsilon)\epsilon^{k_{**}}}}$$

where X_{A_i, A_j} is the random variable giving 1 if (A_i, A_j) is exceptional, and 0 otherwise. Now, we have no control if $i_1 = j_1$, so let's bound how many of these we have:

$$\begin{aligned}
|\left\{\text{Exceptional } (A_i, A_j) \mid A_i, A_j \subseteq A'_{i_1}, i_1 \in [i(*)]\right\}| &\leq \binom{\frac{m_0}{m_{**}}}{2} \frac{n}{m_0} \\
&\leq \frac{\left(\frac{m_0}{m_{**}}\right)^2}{2} \frac{n}{m_0} = \frac{m_0 n}{2m_{**}^2} = \frac{m_0}{n} \left(\frac{n}{\sqrt{2} m_{**}}\right)^2 \\
&\leq \frac{m_0}{n} \left(\frac{n - m_0}{m_{**}}\right)^2 \leq \frac{m_0}{n} r^2 < \frac{r^2}{n^{(1-2\epsilon)\epsilon^{k_{**}}}}
\end{aligned}$$

Putting it all together, we see that the number of exceptional pairs is upper bounded by $\frac{2r^2}{n^{(1-2\epsilon)\epsilon^{k_{**}}}}$ satisfying (2). □

Remark 3.21 (Remark 4.15). Notice that, in the previous proof, the condition $m_0 < \frac{n}{n^{(1-2\epsilon)\epsilon^{k_{**}}}}$ can be weakened at the cost of increasing the number of exceptional pairs. More specifically, since this condition is only used to bound the exceptional sub-pairs in the same pair (the second part of the proof), the number of exceptional pairs can be generally bounded by

$$|\{\text{Exceptional pairs}\}| \leq \left(\frac{m_0}{n} + \frac{2}{n^{(1-2\epsilon)\epsilon^{k_{**}}}}\right) r^2$$

Theorem 3.22 (Theorem 4.16). *Let* $\epsilon = \frac{1}{r} \in \left(0, \frac{1}{2}\right)$ *with* $r \in \mathbb{N}$ *(this avoids rounding error) and* k_* *be given. Let* G *be a finite graph with the non-*k_**-order property. Let* $A \subseteq G$ *with* $|A| = n$*. Then, for any* $m_{**} \leq n^{\frac{\epsilon^{k_{**}+1}}{3}}$*, there is a partition* $\overline{A} = \langle A_i \mid i \in [m]\rangle$ *of* A *with remainder* $B = A \setminus \bigcup \overline{A}$ *such that:*

1. $|A_i| = m_{**} \forall i \in [m]$
2. $|B| < n^{\frac{\epsilon}{3}}$
3. $|\{(i, j) \mid i, j \in [m], i < j$ *and* $\{(a, b) \in A_i \times A_j \mid aRb\} \notin \{A_i \times A_j, \emptyset\}\}| \leq \frac{2}{n^{(1-2\epsilon)\epsilon^{k_{**}}}} m^2$

Proof. Let $m_{k_{**}}$ be the smaller multiple of m_{**} such that $2 < (m_{k_{**}})^{1-2\epsilon}$. Then, consider the sequence

$$m_{**} \leq m_{k_{**}} < \cdots < m_0$$

such that for all $l \in [k_{**}]$ we have that $m_{l-1} = m_l^r$. Notice that:

1. m_{**} divides m_l for all $l \in [0, k_{**}]$ since the m_l's are powers of $m_{k_{**}}$ and m_{**} divides $m_{k_{**}}$ by construction.

2. $(m_{l-1})^\epsilon = m_l \forall l \in [k_{**}]$

3. $$\begin{aligned}\underline{m_0} = m_{k_{**}}^{r^{k_{**}}} \leq m_{**}^{r^{k_{**}}} \leq n^{\frac{\epsilon}{3}\epsilon^{k_{**}} r^{k_{**}}} = \underline{n^{\frac{\epsilon}{3}}} \\ < n^{\frac{1}{6}} < n^{1-\frac{1}{2}\epsilon^{k_{**}}} = \underline{\frac{n}{n^{\frac{1}{2}\epsilon^{k_{**}}}}} < \underline{n}\end{aligned}$$

So, all the conditions are satisfied to apply Claim 3.20, which gives us the partition $\overline{A}$ with remainder B satisfying the statement. Notice that (2) is satisfied by the fact that $|B| < m_0 \leq n^{\left(\frac{1}{6}-\frac{\epsilon}{3}\right)}$. □

Remark 3.23. Let $n^{\frac{\epsilon^{k_{**}+1}}{3}}$ be an integer and let m_{**} take this value. Then, the number of pieces of the partition is at most n^c with $c = 1 - \frac{\epsilon^{k_{**}+1}}{3}$.

Definition 3.24 (Definition 4.18). For $n, c \in \mathbb{N}$ and $\epsilon, \zeta, \xi \in \mathbb{R}$, let $\oplus[n, \epsilon, \zeta, \xi, c]$ be the statement: For any set A and $P \subseteq \mathcal{P}(A)$ such that $|A| = n$, $|P| \leq n^{\frac{1}{\zeta}}$ and for all $B \in P$ $|B| \leq n^\epsilon$, there exists $U \subseteq A$ with $|U| = \lfloor n^\xi \rfloor$ such that for all $B \in P$ $|U \cap B| \leq c$.

Lemma 3.25 (Lemma 4.19). *If the reals ϵ, ζ, ξ and the natural numbers n, c satisfy:*

- $\epsilon \in (0, 1)$
- $\zeta > 0$
- $0 < \xi < \min(1 - \epsilon, \frac{1}{2})$
- *n sufficiently large ($n > n(\epsilon, \zeta, \xi, c)$) to satisfy the equation:*

$$\frac{1}{2n^{1-2\xi}} + \frac{1}{n^{(1-\xi-\epsilon)c-\frac{1}{\zeta}}} < 1$$

- $c > \frac{1}{\zeta(1-\xi-\epsilon)}$

then $\oplus[n, \epsilon, \zeta, \xi, c]$ holds.

Proof. Let $m = \lfloor n^\xi \rfloor$ the size of the set U we want to build, and let $\mathcal{F}_* = [A]^m$ the set of sequences of elements of A with length m. Let μ be a probability distribution on $\mathcal{F}_*$ such that for all $F \in \mathcal{F}_*$ $\mu(F) = \frac{|F|}{|\mathcal{F}_*|}$. We want to prove that the probability that a random U satisfies:

1. All elements of U are distinct

2. For all $B \in P$ $|U \cap B| < K$

is not trivial. First of all let's bound the converse (1) i.e. the probability that there are two equal elements in U:

$$P_1 = P(\exists s < t \in [m] \mid U_s = U_t) \leq \binom{m}{2}\frac{n}{n^2} \leq \frac{m^2}{2n} \leq \frac{n^{2\xi}}{2n} < \frac{1}{2n^{1-2\xi}}$$

Now, in order to bound (2), let's first bound the probability that at least c elements of U are in a given $B \in P$:

$$P_B = P(\exists^{\geq c} t \in [m] \mid U_t \in B) \leq \binom{m}{c}\left(\frac{|B|}{n}\right)^c \leq \frac{m^c|B|^c}{n^c} \leq \frac{n^{\xi c}n^{\epsilon c}}{n^c} = \frac{1}{n^{c(1-\xi-\epsilon)}}$$

Then, we can bound the converse of (2), i.e. the probability that this happens for some $B \in P$, by:

$$P_2 = P(\exists B \in P \mid \exists^{\geq c} t \in [m], U_t \in B) \leq \sum_{B \in P} P_B = \frac{|P|}{n^{c(1-\xi-\epsilon)}} \leq \frac{1}{n^{c(1-\xi-\epsilon)-\frac{1}{\xi}}}$$

Putting it all together, we have that

$$P((1) \cup (2)) \leq P_1 + P_2 < \frac{1}{2n^{1-2\xi}} + \frac{1}{n^{c(1-\xi-\epsilon)-\frac{1}{\xi}}}$$

Notice that

- Since $\xi < \frac{1}{2}$ we have that $1 - 2\xi > 0$
- Since $\xi < 1 - \epsilon$, we have that $1 - \epsilon - \xi > 0$ and given that c is natural $c(1 - \xi - \epsilon) > 0$

so, the n-large enough condition of the forth point of the statement is well defined and

$$P((1) \cup (2)) < \frac{1}{2n^{1-2\xi}} + \frac{1}{n^{c(1-\xi-\epsilon)-\frac{1}{\xi}}} < 1$$

Thus, the probability that there exists a $U \subseteq A$ satisfying the condition is non-trivial, and $\oplus[n, \epsilon, \zeta, \xi, c]$ holds. $\square$

Lemma 3.26 (Claim 4.21). *Let $k_*, k, c \in \mathbb{N}$ and $\epsilon, \xi \in \mathbb{R}$ such that:*

1. *G is a graph with the non-k_*-order property*
2. *$A \subseteq G$ implies $|\{\{a \in A \mid aRb \equiv t(a, b)\} \mid b \in G\}| \leq |A|^k$*
3. *$\epsilon \in \left(0, \frac{1}{2}\right)$*
4. *$\xi \in \left(0, \frac{\epsilon^{k_{**}}}{2}\right)$*
5. *c satisfies*

$$c > \frac{1}{\frac{1}{k}(1 - \frac{\xi}{\epsilon^{k_{**}}} - \epsilon)}$$

Then, for every sufficiently large $n \in \mathbb{N}$ ($n^{\epsilon^{k_{**}}} > n\left(\epsilon, \frac{1}{k}, \frac{\xi}{\epsilon^{k_{**}}}, c\right)$ in the sense of Lemma 3.25 (d)), if $A \subseteq G$ with $|A| = n$, then there is $Z \subseteq A$ such that

(a) $|Z| = \lfloor n^{\xi} \rfloor$

(b) Z is ϵ-indivisible in G

Proof. In order to simplify the calculation, we will assume that $n^{\epsilon^l} \in \mathbb{N} \forall l \leq k_{**}$. Notice that can be easily achieved by setting ϵ as $\epsilon = \frac{1}{r}$ with $r \in \mathbb{N}$. Let $n = m_0 > m_1 > \cdots > m_{k_{**}}$ with $m_l = n^{\epsilon^l}$. So $m_{l+1} = m_l^{\epsilon} = \lfloor (m_l)^{\epsilon} \rfloor$ and we can use Claim 3.5 to get $A_1 \subseteq A$ with $|A_1| = m_l$ for some $l \leq k_{**}$ and A_1 ϵ-indivisible. By (2) we have that $|P_1| \leq |A_1|^k = m_l^k$. Notice that:

- $\epsilon \in (0, 1)$ by (3)
- $\zeta := \frac{1}{k} > 0$
- since $\epsilon \in \left(0, \frac{1}{2}\right)$ by (3), then by (4) $\frac{\xi}{\epsilon^l} \leq \frac{\xi}{\epsilon^{k_{**}}} < \frac{1}{2} < 1 - \epsilon$ and thus $0 < \xi < \min(1 - \epsilon, \frac{1}{2})$
- m_l sufficiently large: $m_l = n^{\epsilon^l} \geq n^{\epsilon^{k_{**}}} > n\left(\epsilon, \frac{1}{k}, \frac{\xi}{\epsilon^{k_{**}}}, c\right) > n\left(\epsilon, \zeta, \frac{\xi}{\epsilon^l}, c\right)$
- $c > \frac{1}{\frac{1}{k}(1 - \frac{\xi}{\epsilon^{k_{**}}} - \epsilon)} \geq \frac{1}{\zeta(1 - \frac{\xi}{\epsilon^{k_{**}}} - \epsilon)}$

By Lemma 3.25 then, $\oplus \left[m_l, \epsilon, \zeta, \frac{\xi}{\epsilon^l}\right]$ holds, and by taking $A_{(3.24)} := A_1$ and $P_{(3.24)} := P_1$ we have that:

- $|A_1| = m_l$
- $|P_1| \leq m_l^k = m_l^{\frac{1}{\zeta}}$
- $\forall B \in P_1$, $|B| \leq |A_1|^{\epsilon}$ by ϵ-indivisibility of A_1

Thus, by Definition 3.24 we have that there exists $Z \subseteq A_1$ such that:

- $|U| = \lfloor m_l^{\frac{\xi}{\epsilon^l}} \rfloor = \lfloor n^{\epsilon^l \frac{\xi}{\epsilon^l}} \rfloor \lfloor n^{\xi} \rfloor$ satisfying (a)
- Z is c-indivisible since $|B \cap Z| \leq c \forall B \in P_1$, satisfying (b)

This proves the statement. □

Lemma 3.27 (Remark 4.22). *Notice that if $k = k_*$, the condition (2) will be satisfied by Claim ??? and the non-k_*-order of G.*

Theorem 3.28 (Theorem 4.23). *Let G be a graph with the non-k_*-property. For any $c \in \mathbb{N}$, $\epsilon, \xi \in \mathbb{R}$ satisfying the hypothesis of Claim 3.26 (with $k = k_*$ and $\zeta = \frac{1}{k_*}$), any $\theta \in (0, 1)$ and $A \subseteq G$ with $A = n > n(c, \epsilon, \zeta, \xi, \theta)$ (i.e. n large enough in the sense of Claim 3.25), there is a partition $\overline{A} = \langle A_i \mid i \in [i(*)] \rangle$ of A with remainder $B = A \setminus \bigcup \overline{A}$ satisfying:*

- $|A_i| = \lfloor \lfloor n^{\theta} \rfloor^{\zeta} \rfloor \forall i \in [i(*)]$
- *A_i is c-indivisible $\forall i \in [i(*)]$ where c is the constant function $f(x) = c$*
- $|B| < \lfloor n^{\frac{\theta}{\epsilon^{k_{**}}}} \rfloor$

Proof. Let $n > \left(n \left(\epsilon, \frac{1}{k_*}, \frac{\xi}{\epsilon^{k_{**}}}, c \right)^{\frac{1}{\epsilon^{k_{**}}}+1} \right)^{\frac{1}{\theta}}$ in the sense of Lemma 3.25, so that $\lfloor n^\theta \rfloor$ satisfies the large enough condition of Claim 3.26:

$$\left(\lfloor n^\theta \rfloor \right)^{\epsilon^{k_{**}}} > n \left(\epsilon, \frac{1}{k_*}, \frac{\xi}{\epsilon^{k_{**}}}, c \right)$$

Notice that condition (2) in Claim 3.26 is satisfied by Remark 3.27. Now, we define a decreasing sequence $m_0 > m_1 > \cdots > m_{k_{**}}$ with $m_{k_{**}} = \lfloor n^\theta \rfloor$ and $m_{k_{**}-j} = \lceil (m_{k_{**}-j+1})^{\frac{1}{\epsilon}} \rceil \forall j \in [1, k_{**}]$. This sequence satisfies the condition of Claim 3.5 for $f(n) = n^\epsilon$. We will build a sequence of disjoint c-indivisible subsets A_i by induction on i as follows. Let $R_i = A \setminus \bigcup_{j<i} A_j$ (so $R_1 = A$). If $R_i < \lfloor n^{\frac{\theta}{\epsilon^{k_{**}}}} \rfloor$, then $\overline{A} = \langle A_j \mid j < i = i(*) \rangle$ and $B = R_i$, and we are done. Otherwise, we can apply Claim 3.5 to R_i with the sequence $\langle m_l \rangle_{l \leq k_{**}}$, to obtain an ϵ-indivisible subset $B_i \subseteq R_i$ of size $m_{k_{**}-l}$. Then, since $|B_i| = m_{k_{**}-l} \geq m_{k_{**}} = \lfloor n^\theta \rfloor$ by the n-large-enough assumption, we can apply Claim 3.26 and get a c-indivisible subset Z_i of size $|Z_i| = \lfloor m^{\zeta}_{k_{**}-l} \rfloor \geq \lfloor \lfloor n^{\frac{\theta}{\epsilon^l}} \rfloor^\zeta \rfloor \geq \lfloor \lfloor n^\theta \rfloor^\zeta \rfloor$. Since c-indivisible is preserved when taking subsets, we can choose $A_i \subseteq Z_i$ c-indivisible of size $\lfloor \lfloor n^\theta \rfloor^\zeta \rfloor$. □

4. Section 5

References

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A. Title of the appendix

You can include here an appendix with details that can not be included in the core of the document. You should reference the sections in this appendix in the core document.

B. Title of the appendix

Second appendix.