Universitat Politècnica de Catalunya
Facultat de Matemàtiques i Estadística

Master in Advanced Mathematics and Mathematical Engineering
Master's thesis

Why the non-monotonicity of excellence f***** up my life

Severino Da Dalt

Supervised by (Lluis Vena Cros)

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Thanks to...

Abstract

This should be an abstract in english, up to 1000 characters.

Keywords

regularity, stable graphs, graph theory, ...

1. Introduction

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2. Section 3

3. Section 4

Definition 3.1 (Definition 4.2(a))**.** Let $\epsilon \in (0,1)$. We say that $A \subseteq G$ is *ϵ-indivisible* if for every $B \in G$, for some truth value $t = t(b, A)$ we have that

$$|\{a \in A \mid aRb \not\equiv t\}| < |A|^{\epsilon}$$

Definition 3.2 (Definition 4.2(b))**.** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a non-decreasing function. We say that $A \subseteq G$ is *f-indivisible* if for every $B \in G$, for some truth value $t = t(b, A)$ we have that

$$|\{a \in A \mid aRb \not\equiv t\}| < f(|A|)$$

Remark 3.3. If $f(n) = \epsilon n$, then f-indivisible $\equiv$ ϵ-good.

Remark 3.4. ϵ-indivisible is a much stronger condition then ϵ-good.

Lemma 3.5 (Claim 4.3)*. Let G be a finite graph with the non-k_*-property. Assume $m_0 > \cdots > m_{k_{**}}$ is a sequence of non-zero natural numbers and for all $l \in [k_{**}]$, $f(m_{l-1}) \geq m_l$. If $A \subseteq G$, $|A| = m_0$, then for some $l < k_{**}$ there is a subset $B \subseteq A$ of size m_l which is f-indivisible.*

Proof. Suppose not. Then we can construct the sequences $\left\langle b_\eta \mid \eta \in [2]^{<k} \right\rangle$ and $\left\langle A_\eta \mid \eta \in [2]^{\leq k} \right\rangle$ on induction over $k < k_{**}$, where $k = |\eta|$, satisfying:

1. $A_{\eta^\frown\langle i\rangle} \subseteq A_\eta$, $\forall i \in \{0,1\}$, $\forall k < k_{**}$

2. $A_{\eta^\frown\langle 0\rangle} \cap A_{\eta^\frown\langle 1\rangle} = \emptyset$, $\forall k < k_{**}$

3. $|A_\eta| = m_k$, $\forall k \leq k_{**}$

4. $b_\eta \in G$ witnessing that A_η is not f-indivisible, $\forall k < k_{**}$

5. $A_{\eta^\frown\langle i\rangle} \subseteq A_\eta^{(i)} = \{a \in A_\eta \mid aRb_\eta \equiv (i = 1)\}$, $\forall \in \{0, 1\}$, $\forall k < k_{**}$

Let's prove the induction:

- $\underline{k = 0}$. Consider $A_{\langle\cdot\rangle} = A$, which satisfies $|A_{\langle\cdot\rangle}| = m_0$ and $|b_{\langle\cdot\rangle}|$ witnessing the non-f-indivisibility of $A_{\langle\cdot\rangle}$.
- $\underline{k \Rightarrow k+1}$. We can assume $|A_\eta| = m_k$ and by hypothesis A_η is not f-indivisible. So, there exists b_η such that $A_\eta^{(i)} \geq f(m_k) \geq m_{k+1}$ (4), and we can choose $A_{\eta^\frown\langle i\rangle} \subseteq A_\eta^{(i)}$ (5), such that $|A_{\eta^\frown\langle i\rangle}| = m_{k+1} \forall i \in \{0, 1\}$ (3). (1) and (2) are satisfied by the definition of $A_\eta^{(i)}$.

Now, for all η such that $|\eta| = k_{**}$, consider some element $a_\eta \in A_\eta$. Then, we have two sequences $\left\langle b_\eta \mid \eta \in [2]^{<k_{**}} \right\rangle$ and $\left\langle A_\eta \mid \eta \in [2]^{k_{**}} \right\rangle$ with the property:

$$\forall \rho \in [2]^{<k_{**}} \forall \eta \in [2]^{k_{**}} \text{ such that } \rho^\frown \langle i\rangle \trianglelefteq \eta, (a_\eta R b_\rho)$$

since $a_\eta \in A_\eta \subseteq A_{\rho^\frown\langle i\rangle}$. This contradicts the k_{**} tree bound. $\square$

Lemma 3.6 (Claim 4.4). *Let G be a finite graph wit the non-k_*-order property. Assume $m_0 > \cdots > m_{k_{**}}$ is a sequence of non-zero natural numbers and for all $l \in [k_{**}]$, $f(m_{l-1}) \geq m_l$. If $A \subseteq G$ with $|A| = n$, then we can find a sequence $\overline{A} = \langle A_j \mid j \in [j(*)]\rangle$ and reminder $B = A \setminus \bigcup \overline{A}$ such that:*

1. *For each $j \in [j(*)]$, A_j is f-indivisible*
2. *For each $j \in [j(*)]$, $|A_j| \in \{m_0, \ldots, m_{k_{**}-1}\}$*
3. *$A_j \subseteq A \setminus \bigcup\{A_i \mid i < j\}$, in particular $A_i \cap A_j = \emptyset \; \forall i \neq j$*
4. *$|B| < m_0$*

Proof. Iteratively, apply Claim 3.5 to the remainder $A \setminus \bigcup\{A_i \mid i < j\}$ (3) to get an f-indivisible A_j (1) of size m_l, $l \in \{0, \ldots, k_{**} - 1\}$ (2) until less then m_0 vertices are available (4). $\square$

Lemma 3.7 (Claim 4.5). *Let G be a graph with the non-k_*-order property. Assume $n \geq m_0 > \cdots > m_{k_{**}}$ is a sequence of non-zero natural numbers satisfying that for all $l \in [k_{**}]$ $\lfloor (m_{l-1})^\epsilon \rfloor = m_l$, for $\epsilon \in (0, \frac{1}{2})$. If $A \subseteq G$, $|A| = n$, then we can find $\overline{A} = \langle A_i \mid i \in [i(*)]\rangle$ with remainder $B = A \setminus \bigcup \overline{A}$ such that:*

1. *For each $j \in [j(*)]$, A_j is ϵ-indivisible*
2. *For each $j \in [j(*)]$, $|A_j| \in \{m_0, \ldots, m_{k_{**}-1}\}$*
3. *$A_i \cap A_j = \emptyset \; \forall i \neq j$*
4. *$|B| < m_0$*
5. *$\overline{A}$ is $\leq$-increasing*

Proof. The first four clauses are direct consequence of applying Claim 3.6 with $f(n) = n^{\epsilon}$. By renaming the A_i's in ascending-size order, we get (5). $\square$

Remark 3.8. In this context, if $m_{k_{**}} > k_{**}$

$$n^{\epsilon^{k_{**}}} \geq m_0^{\epsilon^{k_{**}}} \geq m_1^{\epsilon^{k_{**}-1}} \geq \cdots \geq m_{k_{**}} > k_{**}$$

So, $n^{\epsilon^{k_{**}}} > k_{**}$.

Lemma 3.9 (Claim 4.6)). *Let G be a finite graph. Suppose $A, B \subseteq G$ such that A is f-indivisible, B is g-indivisible, and $f(|A|)g(|B|) < \frac{1}{2}|B|$. Then, for some truth value $t = t(A, B)$ for all but $< f(|A|)$ of the $a \in A$ for all but $< g(|B|)$ of the $b \in B$ we have that $aRb \equiv t$.*

Proof. Since B is g-indivisible, for each $a \in A$ there is a truth value $t_a = t(a, B)$ such that $\{b \in B \mid aRb \not\equiv t_a\} < g(|B|)$. Let $U_i = \{a \in A \mid t_a = i\}$ for $i \in \{0, 1\}$. If either U_i satisfies $|U_i| < f(|A|)$ then the statement is true. Suppose not. Then, there are $W_i \subseteq U_i$ with $|W_i| = f(|A|)$ for $i \in \{0, 1\}$. Now, let $V = \{b \in B \mid (\exists a \in W_0 \mid aRb) \vee (\exists a \in W_1 \mid \neg aRb)\}$, i.e. the b's which are an exception for some $a \in W_0 \cup W_1$. Then, $|V| < |W_0|g(|B|) + |W_1|g(|B|) = 2f(|A|)g(|B|) < |B|$, where the first inequality follows the g-indivisibility of B. Finally, there is a $b_* \in B \setminus V$ such that $\forall a \in W_0$ $\neg aRb_*$ and $\forall a \in W_1$ aRb_* with $|W_0| = |W_1| = f(|A|)$, which contradicts the f-indivisibility of A. $\square$

Definition 3.10. We say that the pair (A, B) with A f-indivisible and B g-indivisible satisfies the *average condition* if $f(|A|)g(|B|) < \frac{1}{2}|B|$ and thus the statement of Claim 3.9 is true for the pair (A, B).

Remark 3.11. The condition $f(|A|)g(|B|) < \frac{1}{2}|B|$ makes ordering of the pair (A, B) matter. Thus,

$$(A, B) \text{ has the average condition } \not\Rightarrow (B, A) \text{ has the average condition}$$

Remark 3.12 (Remark 4.7). When $f(n) = n^{\epsilon}$ and $g(n) = n^{\zeta}$, the average condition is $|A|^{\epsilon}|V|^{\zeta} < \frac{1}{2}|B|$.

Lemma 3.13 (Claim 4.8). *Let A be ϵ-indivisible, B ζ-indivisible and let the pair (A, B) satisfy the average condition. Then, for all $\epsilon_1 \in (0, 1-\epsilon)$, $\zeta_1 \in (0, 1-\zeta)$, $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| \geq |A|^{\epsilon+\epsilon_1}$ and $|B'| \geq |B|^{\zeta+\zeta_1}$, we have that:*

$$\frac{|\{(a, b) \in (A', B') \mid aRb \equiv \neg t(A, B)\}|}{|A' \times B'|} \leq \frac{1}{|A|}\epsilon_1 + \frac{1}{|B|}\zeta_1$$

Proof. Notice:

- There are at most $|A|^{\epsilon}$ elements of A (hence in $A' \subseteq A$) which are exceptional (in the sense of the average condition).
- For each $a \in A$ (hence in $A' \subseteq A$) not exceptional, there are at most $|B|^{\zeta}$ elements $b \in B$ such that (a, b) does not satisfy the truth value $t(A, B)$, i.e. that are exceptional.

Putting it all together:

$$
\begin{aligned}
\frac{|\{(a,b)\in(A',B')\mid aRb\equiv\neg t(A,B)\}|}{|A'\times B'|} &\leq \frac{|A|^{\epsilon}|B'|+(|A'|-|A|^{\epsilon})|B|^{\zeta}}{|A'||B'|}\\
&=\frac{|A|^{\epsilon}}{|A'|}+\frac{|A'|-|A|^{\epsilon}}{|A'|}\frac{|B|^{\zeta}}{|B'|}\\
&\leq \frac{|A|^{\epsilon}}{|A'|}+\frac{|B|^{\zeta}}{|B'|}\\
&\leq \frac{|A|^{\epsilon}}{|A|^{\epsilon+\epsilon_1}}+\frac{|B|^{\zeta}}{|B|^{\zeta+\zeta_1}}\\
&=\frac{1}{|A|_1^{\epsilon}}+\frac{1}{|B|_1^{\zeta}}
\end{aligned}
$$

□

Lemma 3.14 (f-indivisible version). *Let A be f-indivisible, B g-indivisible and let the pair (A,B) satisfy the average condition. Then, for all $\epsilon_1 \in \left(0, 1-\frac{f(|A|)}{|A|}\right)$, $\zeta_1 \in \left(0, 1-\frac{g(|B|)}{|B|}\right)$, $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| \geq f(|A|)|A|^{\epsilon_1}$ and $|B'| \geq g(|B|)|B|^{\zeta_1}$, we have that:*

$$\frac{|\{(a,b)\in(A',B')\mid aRb\equiv\neg t(A,B)\}|}{|A'\times B'|} \leq \frac{1}{|A|}\epsilon_1+\frac{1}{|B|}\zeta_1$$

Proof. Notice:

- There are at most $f(|A|)$ elements of A (hence in $A' \subseteq A$) which are exceptional (in the sense of the average condition).
- For each $a \in A$ (hence in $A' \subseteq A$) not exceptional, there are at most $g(|B|)$ elements $b \in B$ such that (a,b) does not satisfy the truth value $t(A,B)$, i.e. that are exceptional.

Putting it all together:

$$
\begin{aligned}
\frac{|\{(a,b)\in(A',B')\mid aRb\equiv\neg t(A,B)\}|}{|A'\times B'|} &\leq \frac{f(|A|)|B'|+(|A'|-f(|A|))g(|B|)}{|A'||B'|}\\
&=\frac{f(|A|)}{|A'|}+\frac{|A'|-f(|A|)}{|A'|}\frac{g(|B|)}{|B'|}\\
&\leq \frac{f(|A|)}{|A'|}+\frac{g(|B|)}{|B'|}\\
&\leq \frac{f(|A|)}{f(|A|)|A|^{\epsilon_1}}+\frac{g(|B|)}{g(|B|)|B|^{\zeta_1}}\\
&=\frac{1}{|A|_1^{\epsilon}}+\frac{1}{|B|_1^{\zeta}}
\end{aligned}
$$

□

Corollary 3.15 (Corollary 4.9). *Let A and B be f-indivisible with $f(n) = c$ and (A, B) satisfy the average condition. Then, for all $\epsilon_1 \in (0, 1 - \frac{c}{|A|})$, $\zeta_1 \in (0, 1 - \frac{c}{|B|})$, $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq c|A|^{\epsilon_1}$ and $|B'| \geq c|B|^{\zeta_1}$, we have:*

$$\frac{|\{(a, b) \in (A', B') \mid aRb \equiv \neg t(A, B)\}|}{|A' \times B'|} \leq \frac{1}{|A|}\epsilon_1 + \frac{1}{|B|}\zeta_1$$

Proof. Use Claim 3.14 with $f(n) = c$. □

Lemma 3.16 (Claim 4.10). *Let G be a finite graph wit the non-k_*-order property. Assume $n \geq m_0 > \cdots > m_{k_{**}}$ is a sequence of non-zero natural numbers and for all $l \in [k_{**}]$, $\lfloor (m_{l-1})^\epsilon \rfloor = m_l$, for some $\epsilon \in (0, \frac{1}{2})$ such that $2 < (m_{k_{**}})^{1-2\epsilon}$. If $A \subseteq G$ with $|A| = n$, then we can find a sequence $\overline{A} = \langle A_i \mid i \in [i(*)]\rangle$ and reminder $B = A \setminus \bigcup \overline{A}$ satisfying:*

1. *For each $i \in [i*)]$, A_i is ϵ-indivisible*
2. *For each $i \in [i(*)]$, $|A_i| \in \{m_0, \dots, m_{k_{**}-1}\}$*
3. *$A_i \cap A_j = \emptyset \ \forall i \neq j$*
4. *$|B| < m_0$*
5. *$\overline{A}$ is $\leq$-increasing*
6. *If $\zeta \in (0, \epsilon^{k_{**}})$ then for every $i, j \in [i(*)]$ with $i < j$, $A \subseteq A_i$ ad $B \subseteq A_j$ such that $|A| \geq |A_i|^{\epsilon+\zeta}$ and $|B| \geq |A_j|^{\epsilon+\zeta}$ we have that:*

$$\begin{aligned}\frac{|\{(a, b) \in (A, B) \mid aRb \equiv \neg t(A_i, A_j)\}|}{|A \times B|} &\leq \frac{1}{|A_i|}\zeta + \frac{1}{|A_j|}\zeta \\ &\leq \frac{1}{|A|}\zeta + \frac{1}{|B|}\zeta\end{aligned}$$

Proof. The five points are direct consequence of Claim 3.7. Now, for any $A_i, A_j \in \overline{A}$ with $i < j$. By (2), there is some $l < k_{**}$ such that $|A_i| \leq |A_j| = m_l$ for some $l < k_{**}$. Then, it follows the condition $2 < (m_{k_{**}})^{1-2\epsilon}$ that:

$$\frac{|A|^\epsilon |B|^\epsilon}{|B|} \leq \frac{|B|^{2\epsilon}}{|B|} = \frac{1}{|B|^{1-2\epsilon}} = \frac{1}{m_l^{1-2\epsilon}} \leq \frac{1}{m_{k_{**}}} < \frac{1}{2}$$

i.e. $|A|^\epsilon |B|^\epsilon < \frac{1}{2}|B|$ and by Claim 3.12 the average condition is satisfied. Finally, notice that $\epsilon^{k_{**}} < \epsilon < 1-\epsilon$ since $\epsilon \in (0, \frac{1}{2})$, so that $\zeta \in (0, \epsilon^{k_{**}}) \subseteq (0, 1-\epsilon)$ and the condition for Claim 3.13 is satisfied. This gives us (6) and concludes the proof of the statement. □

Definition 3.17. Let A, B be f-indivisible sets with $f(A) \times f(B) < \frac{1}{2}|B|$. Let $\langle A_i \mid i < i_A\rangle$ be a partition of A with $|A_i| = m \forall i < i_A$ and $\langle B_i \mid i < i_B\rangle$ be a partition of B with $|B_i| = m \forall i < i_B$. $\epsilon^+_{A_i, A_j, m}$ is the event:

$$\forall a \in A_i \forall b \in B_i aRb = t(A, B)$$

Lemma 3.18 (Claim 4.13). *Let G be a finite graph wit the non-k_*-order property. Assume $n \geq m_0 > \cdots > m_{k_{**}}$ is a sequence of non-zero natural numbers and for all $l \in [k_{**}]$, $\lfloor (m_{l-1})^\epsilon \rfloor = m_l$, for some $\epsilon \in (0, \frac{1}{2})$ such that $2 < (m_{k_{**}})^{1-2\epsilon}$. Let $A_1, A_2 \subseteq G$ two ϵ-indivisible subsets such that $|A_1| = m_{l_1}$ and*

$|A_2| = m_{l_2}$ for some $l_1, l_2 < k_{**}$ and $|A_1| \le |A_2|$. We will assume some approximation error by supposing $m_l = (m_{l-1})^\epsilon$. Suppose that, for some $c \in (0, 1-\epsilon)$ and $\zeta \le \frac{1-\epsilon-c}{3}\epsilon^{k_{**}}$, $m = n^\zeta$ divides $|A_1|$ and $|A_2|$. Then, let $\left\langle A_{1,s} \mid s \in \left[\frac{|A_1|}{m}\right]\right\rangle$ and $\left\langle A_{2,t} \mid t \in \left[\frac{|A_2|}{m}\right]\right\rangle$ be random partitions of A_1 and A_2 respectively, with pieces of size m. We have that

$$P(\epsilon^+_{A_{1,s},A_{2,t},m}) \ge 1 - \frac{2}{n^{c\epsilon^{k_{**}}}}$$

Proof. Fix $s \in \frac{|A_1|}{m}$, $t \in \frac{|A_2|}{m}$.

UPS, something is missing here

...and thus the average condition is satisfied. Let $U_1 = \{a \in A_1 \mid |\{b \in A_2 \mid aRb \equiv \neg t(A_1, A_2)\}| \ge |A_2|^\epsilon\}$ and for each $a \in A_1 \setminus U_1$ let $U_{2,a} = \{b \in A_j \mid aRb \equiv \neg t(A_1, A_2)\}$ By Claim 3.9, $|U_1| \le |A_1|^\epsilon$ and $\forall a \in A_1 \setminus U_1$, $|U_2| \le |A_2|^\epsilon$. Now, we can bound the probability P_1 that $A_{1,s} \cap U_1 \ne \emptyset$ as follows:

$$\begin{aligned} P_1 &\le \frac{|U_1|}{|A_1|} + \cdots + \frac{|U_1|}{|A_1| - m + 1} < \frac{m|U_1|}{|A_1| - m} \le \frac{m|A_1|^\epsilon}{|A_1| - m} \\ &\le \frac{m^2|A_1|^\epsilon}{|A_1|} = \frac{m^2}{|A_1|^{1-\epsilon}} = \frac{n^{2\zeta}}{n^{(1-\epsilon)\epsilon^{l_1}}} \\ &\le \frac{n^{2\frac{1-\epsilon-c}{3}\epsilon^{k_{**}}}}{n^{(1-\epsilon)\epsilon^{k_{**}}}} \le \frac{n^{(1-\epsilon-c)\epsilon^{k_{**}}}}{n^{(1-\epsilon)\epsilon^{k_{**}}}} = \frac{1}{n^{c\epsilon^{k_{**}}}} \end{aligned}$$

The forth inequality comes from the fact that $\frac{(|A_i|-m)m}{|A_i|} \ge 1$. Then, if $A_{1,s} \cap = \emptyset$, we have that $|\bigcup_{a \in A_{1,s}} U_{2,a}| \le |A_{1,s}|$ ☐

Remark 3.19.

Lemma 3.20 (Claim 4.14). *Proof.* ☐

Lemma 3.21 (Claim 4.15). *Proof.* ☐

Theorem 3.22 (Theorem 4.16). *Proof.* ☐

Lemma 3.23 (Claim 4.8). *Proof.* ☐

Remark 3.24.

Definition 3.25 (Definition 4.18). *Proof.* ☐

Lemma 3.26 (Lemma 4.19). *Proof.* ☐

Lemma 3.27 (Claim 4.21). *Proof.* ☐

Lemma 3.28 (Remark 4.22). *Proof.* ☐

Theorem 3.29 (Theorem 4.23). *Proof.* ☐

4. Section 5

References

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A. Title of the appendix

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B. Title of the appendix

Second appendix.