Strategic Coalitions with Perfect Recall

Pavel Naumov
Department of Computer Science
Vassar College
Poughkeepsie, New York 12604
pnaumov@vassar.edu

Jia Tao
Department of Computer Science
Lafayette College
Easton, Pennsylvania 18042
taoj@lafayette.edu

Abstract

The paper proposes a bimodal logic that describes an interplay between distributed knowledge modality and coalition know-how modality. Unlike other similar systems, the one proposed here assumes perfect recall by all agents. Perfect recall is captured in the system by a single axiom. The main technical results are the soundness and the completeness theorems for the proposed logical system.

Introduction

Autonomous agents such as self-driving cars and robotic vacuum cleaners are facing the challenge of navigating without having complete information about the current situation. Such a setting could be formally captured by an *epistemic transition system* where an agent uses instructions to transition the system between states without being able to distinguish some of these states. In this paper we study properties of strategies in such systems. An example of such a system is the epistemic transition system T_1, depicted in Figure 1. It has six states named $w_0, w'_0, w_1, w'_1, w_2, w'_2$ and two instructions 0 and 1 that an agent a can use to transition the system from one state to another. For instance, if an instruction 0 is given in state w_0, then the system transitions into state w_1. The system is called *epistemic* because the agent cannot distinguish state w_i from state w'_i for each $i = 0, 1, 2$. The indistinguishability relation is shown in the figure using dashed lines. Atomic proposition p is true only in state w_2.
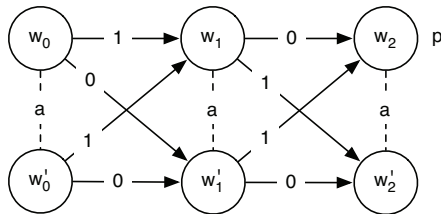

Figure 1: Epistemic Transition System T_1.

The logical system that we propose consists of two modalities. The first is the knowledge modality K. Imagine that

the system starts in state w_2. Since agent a cannot distinguish state w_2 from state w'_2 where statement p is not satisfied, the agent does not know if p is true or not. We write this as $(w_2) \nVdash \mathsf{K}_a p$. Next, suppose that the system started in state w_1 and the agent used instruction 0 to transition the system into state w_2. In this paper we assume that all agents have perfect recall, so in state w_2 the agent remembers *history* $(w_1, 0, w_2)$. However, such a history is indistinguishable from history $(w'_1, 0, w'_2)$ because the agent cannot distinguish state w_1 from state w'_1 and state w_2 from state w'_2. Thus, the agent does not know that proposition p is true in the state w_2 even with history $(w_1, 0, w_2)$. We denote this by $(w_1, 0, w_2) \nVdash \mathsf{K}_a p$. Finally, assume that the system started in state w_0 and the agent first used instruction 1 to transition it into state w_1 and later instruction 0 to transition it to state w_2. Thus, the history of the system is $(w_0, 1, w_1, 0, w_2)$. The only history that the agent cannot distinguish from this one is history $(w'_0, 1, w_1, 0, w_2)$. Since both of these histories end in a state where proposition p is satisfied, agent a *does* know that proposition p is true in state w_2, given history $(w_0, 1, w_1, 0, w_2)$. We write this as $(w_0, 1, w_1, 0, w_2) \Vdash \mathsf{K}_a p$.

The other modality that we consider is the strategic power. In system T_1, the agent can transition the system from state w_1 to state w_2 by using instruction 0. Similarly, the agent can transition the system from state w'_1 to state w_2 by using instruction 1. In other words, given either history (w_1) or history (w'_1) the agent can transition the system to a state in which atomic proposition p is satisfied. We say that, given either history, agent a has a *strategy* to achieve p. Histories (w_1) and (w'_1) are the only histories indistinguishable by agent a from history (w_1). Since she has a strategy to achieve p under all histories indistinguishable from history (w_1), we say that given history (w_1) the agent *knows that she has a strategy*. Similarly, given history (w'_1), she also knows that she has a strategy. However, since indistinguishable histories (w_1) and (w'_1) require different strategies to achieve p, given history (w_1) she *does not know what the strategy is*. We say that she does not have a know-how strategy. We denote this by $(w_1) \nVdash \mathsf{H}_a p$, where H stands for know-How. Of course, it is also true that $(w'_1) \nVdash \mathsf{H}_a p$.

The situation changes if the transition system starts in state w_0 instead of state w_1 and transitions to state w_1 under instruction 1. Now the history is $(w_0, 1, w_1)$ and the histo-

ries that the agent cannot distinguish from this one are histories $(w_0', 1, w_1)$ and history $(w_0, 1, w_1)$ itself. Given both of these two histories, agent a can achieve p using the same transition 0. Thus, $(w_0, 1, w_1) \Vdash \mathsf{H}_a p$.

Finally note that there are only two histories: (w_0) and (w_0') indistinguishable from (w_0). Given either history, agent a can achieve $\mathsf{H}_a p$ using instruction 1. Thus, $(w_0) \Vdash \mathsf{H}_a \mathsf{H}_a p$. That is, given history (w_0) agent a knows how to transition to a state in which formula $\mathsf{H}_a p$ is satisfied.

Multiagent Systems   Like many other autonomous agents, self-driving cars are expected to use vehicle-to-vehicle communication to share traffic information and to coordinate actions (Harding et al. 2014). Thus, it is natural to consider epistemic transition systems that have more than one agent. An example of such a system T_2 is depicted in Figure 2. This
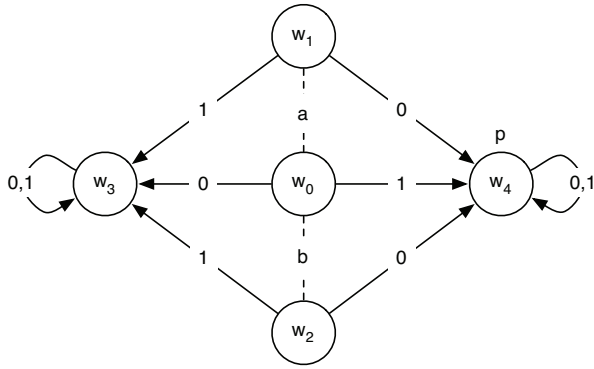

Figure 2: Epistemic Transition System T_2.

system has five epistemic states: w_0, w_1, w_2, w_3, and w_4 and three agents: a, b, and c. In each state the agents vote either 0 or 1 and the system transitions into the next state based on the majority vote. For example, since the directed edge from state w_0 to state w_4 is labelled with 1, if the majority of agents in state w_0 votes 1, then the system transitions into state w_4. Since coalition $\{a, b\}$ forms a majority, this coalition has a strategy to transition the system from state w_0 to state w_4 and, thus, to achieve p. Note that agent a cannot distinguish state w_0 from state w_1 and thus agent a does not know what she should vote for to achieve p. Similarly, agent b also does not know what she should vote for to achieve p because she cannot distinguish state w_0 from state w_2. In this paper, we assume that members of a coalition make the decisions based on combined (distributed) knowledge of the whole coalition. In our example, coalition $\{a, b\}$ can distinguish state w_0 from both state w_1 and state w_2. Thus, given history (w_0) the coalition $\{a, b\}$ knows how to achieve p. We denote this by $(w_0) \Vdash \mathsf{H}_{\{a,b\}} p$, or simply as $(w_0) \Vdash \mathsf{H}_{a,b} p$.

Universal Principles   We have discussed a statement being true or false given a certain history. This paper focuses on the logical principles that are true for each history in each epistemic transition system. An example of such a principle is the *strategic positive introspection*: $\mathsf{H}_C \varphi \rightarrow \mathsf{K}_C \mathsf{H}_C \varphi$. This principle says that if a coalition knows how to achieve φ, then the coalition knows that it knows how to achieve φ.

Informally, this principle is true because in order for statement $\mathsf{H}_C \varphi$ to be satisfied for a given history h, coalition C must have a strategy to achieve φ that works under any history h' indistinguishable from history h by the coalition. Thus, the same strategy must work for any history h'' indistinguishable from history h' by the coalition. In other words, it is also true that $h' \Vdash \mathsf{H}_C \varphi$. Recall that h' is an arbitrary history indistinguishable from history h by coalition C. Hence, $h \Vdash \mathsf{K}_C \mathsf{H}_C \varphi$ according to the standard semantics of the epistemic modality K_C. A similar argument can be used to justify the *strategic negative introspection*: $\neg \mathsf{H}_C \varphi \rightarrow \mathsf{K}_C \neg \mathsf{H}_C \varphi$.

Another universal principle is the *empty coalition* principle: $\mathsf{K}_\varnothing \varphi \rightarrow \mathsf{H}_\varnothing \varphi$. Indeed, $\mathsf{K}_\varnothing \varphi$ means that statement φ is true under any history indistinguishable from the given history by an empty coalition. Since an empty coalition cannot distinguish any two histories, the assumption $\mathsf{K}_\varnothing \varphi$ means that statement φ is true under any history. In particular, this statement is true after the next transition no matter how agents vote. Hence, $\mathsf{H}_\varnothing \varphi$.

The epistemic modality K_C also satisfies axioms of epistemic logic S5 for distributed knowledge. Know-how modality satisfies the *unachievability of falsehood* principle: $\neg \mathsf{H}_C \bot$, stating that no coalition can achieve $\bot$. Know-how modality also satisfies a form of *cooperation* principle (Pauly 2001; 2002):

$$\mathsf{H}_C(\varphi \rightarrow \psi) \rightarrow (\mathsf{H}_D \varphi \rightarrow \mathsf{H}_{C \cup D} \psi), \text{ where } C \cap D = \varnothing.$$

Perfect Recall   A complete trimodal logical system describing the interplay between distributed knowledge modality K_C, coalition know-how modality H_C, and standard (not know-how) strategic power modality in the *imperfect* recall setting was proposed by (Naumov and Tao 2017c). We provide a complete axiomatization of the interplay between modalities K_C and H_C in the *perfect* recall setting. Surprisingly, the assumption of perfect recall by all agents is captured by a single principle that we call the *perfect recall* principle: $\mathsf{H}_D \varphi \rightarrow \mathsf{H}_D \mathsf{K}_C \varphi$, where $D \subseteq C \neq \varnothing$. This principle says that if a sub-coalition $D \subseteq C$ can achieve φ, then after the vote the whole coalition will know that φ is true. Informally, this principle is true because coalition C is able to recall how sub-coalition D voted and, thus, will deduce that formula φ is true after the transition. As an empty coalition has no memory even in the perfect recall setting, it is essential for coalition C to be nonempty. However, the sub-coalition D can be empty.

Literature Review   Non-epistemic logics of coalition power were developed by (Pauly 2001; 2002), who also proved the completeness of the basic logic of coalition power. His approach has been widely studied in the literature (Goranko 2001; van der Hoek and Wooldridge 2005; Borgo 2007; Sauro et al. 2006; Ågotnes et al. 2010; Ågotnes, van der Hoek, and Wooldridge 2009; Belardinelli 2014; Goranko, Jamroga, and Turrini 2013). An alternative logical system for coalition power was proposed by (More and Naumov 2012).

(Alur, Henzinger, and Kupferman 2002) introduced Alternating-Time Temporal Logic (ATL) that combines

temporal and coalition modalities. (van der Hoek and Wooldridge 2003) proposed to combine ATL with epistemic modality to form Alternating-Time Temporal Epistemic Logic. They did not prove the completeness theorem for the proposed logical system. Aminof, Murano, Rubin and Zuleger (Aminof et al. 2016) studied model-checking problems of an extension of ATL with epistemic and "prompt eventually" modal operators.

(Ågotnes and Alechina 2012) proposed a complete logical system that combines the coalition power and epistemic modalities. Since their system does not have epistemic requirements on strategies, it does not contain any axioms describing the interplay of these modalities. In the extended version, (Ågotnes and Alechina 2016) added a complete axiomatization of an interplay between single-agent knowledge and know-how modalities.

Know-how strategies were studied before under different names. While (Jamroga and Ågotnes 2007) talked about "knowledge to identify and execute a strategy", (Jamroga and van der Hoek 2004) discussed "difference between an agent knowing that he has a suitable strategy and knowing the strategy itself". (van Benthem 2001) called such strategies "uniform". (Broersen 2008) investigated a related notion of "knowingly doing", while (Broersen, Herzig, and Troquard 2009) studied modality "know they can do". (Wang 2015; 2016) captured the "knowing how" as a binary modality in a complete logical system with a single agent and without the knowledge modality.

Coalition know-how strategies for enforcing a condition indefinitely were investigated by (Naumov and Tao 2017a). Such strategies are similar to (Pauly 2001, p. 80) "goal maintenance" strategies in "extended coalition logic". A similar complete logical system in a *single-agent* setting for know-how strategies to achieve a goal in multiple steps rather than to maintain a goal is developed by (Fervari et al. 2017).

(Naumov and Tao 2017c) also proposed a complete trimodal logical system describing an interplay between distributed knowledge modality K_C, coalition know-how modality H_C, and standard (not know-how) strategic power modality in the *imperfect* recall setting.

In this paper we provide a complete axiomatization of an interplay between modalities K_C and H_C in the *perfect* recall setting. The main challenge in proving the completeness, compared to (Ågotnes and Alechina 2016; Fervari et al. 2017; Naumov and Tao 2017c; 2017a), is the need to construct not only "possible worlds", but the entire "possible histories", see the proof of Lemma 15.

Outline First, we introduce the syntax and semantics of our logical system. Next, we list the axioms and give examples of proofs in the system. Then, we prove the completeness of this system. The proof of the soundness is available in the full version of this paper (Naumov and Tao 2017b).

Syntax and Semantics

Throughout the rest of the paper we assume a fixed set of agents $\mathcal{A}$. By X^Y we denote the set of all functions from set Y to set X, or in other words, the set of all tuples of elements from set X indexed by the elements of set Y. If $t \in X^Y$ is

such a tuple and $y \in Y$, then by $(t)_y$ we denote the y-th component of tuple t.

We now proceed to describe the formal syntax and semantics of our logical system starting with the definition of a transition system. Although our introductory examples used voting to decide on the next state of the system, in this paper we investigate universal properties of an arbitrary nondeterministic action aggregation mechanism.

Definition 1 *A tuple* $(W, \{\sim_a\}_{a \in \mathcal{A}}, V, M, \pi)$ *is called an epistemic transition system, if*

1. *W is a set of epistemic states,*
2. *$\sim_a$ is an indistinguishability equivalence relation on W for each $a \in \mathcal{A}$,*
3. *V is a nonempty set called domain of choices,*
4. *$M \subseteq W \times V^{\mathcal{A}} \times W$ is an aggregation mechanism,*
5. *π is a function that maps propositional variables into subsets of W.*

For example, in the transition system T_1 depicted in Figure 1, the set of states W is $\{w_0, w_1, w_2, w_0', w_1', w_2'\}$ and relation $\sim_a$ is a transitive reflexive closure of $\{(w_0, w_0'), (w_1, w_1'), (w_2, w_2')\}$.

Informally, an epistemic transition system is *regular* if there is at least one next state for each outcome of the vote.

Definition 2 *An epistemic transition system $(W, \{\sim_a\}_{a \in \mathcal{A}}, V, M, \pi)$ is regular if for each $w \in W$ and each $\mathbf{s} \in V^{\mathcal{A}}$, there is $w' \in W$ such that $(w, \mathbf{s}, w') \in M$.*

A *coalition* is a subset of $\mathcal{A}$. A *strategy profile* of coalition C is any tuple in the set V^C.

Definition 3 *For any states $w_1, w_2 \in W$ and any coalition C, let $w_1 \sim_C w_2$ if $w_1 \sim_a w_2$ for each agent $a \in C$.*

Lemma 1 *For each coalition C, relation $\sim_C$ is an equivalence relation on the set of epistemic states W.* ⊠

Definition 4 *For all strategy profiles $\mathbf{s}_1$ and $\mathbf{s}_2$ of coalitions C_1 and C_2 respectively and any coalition $C \subseteq C_1 \cap C_2$, let $\mathbf{s}_1 =_C \mathbf{s}_2$ if $(\mathbf{s}_1)_a = (\mathbf{s}_2)_a$ for each $a \in C$.*

Lemma 2 *For any coalition C, relation $=_C$ is an equivalence relation on the set of all strategy profiles of coalitions containing coalition C.* ⊠

Definition 5 *A history is an arbitrary sequence $h = (w_0, \mathbf{s}_1, w_1, \mathbf{s}_2, w_2, \ldots, \mathbf{s_n}, w_n)$ such that $n \geq 0$ and*

1. *$w_i \in W$ for each $i \leq n$,*
2. *$\mathbf{s_i} \in V^{\mathcal{A}}$ for each $i \leq n$,*
3. *$(w_i, \mathbf{s}_{i+1}, w_{i+1}) \in M$ for each $i < n$.*

In this paper we assume that votes of all agents are private. Thus, an individual agent only knows her own votes and the equivalence classes of the states that the system has been at. This is formally captured in the following definition of indistinguishability of histories by an agent.

Definition 6 *For any history $h = (w_0, \mathbf{s}_1, w_1, \ldots, \mathbf{s_n}, w_n)$, any history $h' = (w_0', \mathbf{s_1'}, w_1', \ldots, \mathbf{s_m'}, w_m')$, and any agent $a \in \mathcal{A}$, let $h \approx_a h'$ if $n = m$ and*

1. *$w_i \sim_a w_i'$ for each $i \leq n$,*

2. $(\mathbf{s_i})_a = (\mathbf{s'_i})_a$ for each $i \le n$.

Definition 7 *For any histories h_1, h_2 and any coalition C, let $h_1 \approx_C h_2$ if $h_1 \approx_a h_2$ for each agent $a \in C$.*

Lemma 3 *For any coalition C, relation $\approx_C$ is an equivalence relation on the set of histories.* ⊠

The length $|h|$ of a history $h = (w_0, \mathbf{s}_1, w_1, \ldots, \mathbf{s_n}, w_n)$ is the value of n. By Definition 7, the empty coalition cannot distinguish any two histories, even of different lengths.

Lemma 4 $|h_1| = |h_2|$ *for each histories h_1 and h_2 such that $h_1 \approx_C h_2$ for some nonempty coalition C.* ⊠

For any sequence $x = x_1, \ldots, x_n$ and an element y, by sequence $x :: y$ we mean $x_1, \ldots, x_n, y$. If sequence x is nonempty, then by $hd(x)$ we mean element x_n.

Lemma 5 *If $(h_1 :: \mathbf{s}_1 :: w_1) \approx_C (h_2 :: \mathbf{s}_2 :: w_2)$, then $h_1 \approx_C h_2$, $\mathbf{s}_1 =_C \mathbf{s}_2$, and $w_1 \sim_C w_2$.* ⊠

Definition 8 *Let Φ be the language specified as follows $\varphi := p \mid \neg\varphi \mid \varphi \to \varphi \mid \mathsf{K}_C\varphi \mid \mathsf{H}_C\varphi$, where $C \subseteq \mathcal{A}$.*

Boolean constants $\bot$ and $\top$ are defined as usual.

Definition 9 *For any history h of an epistemic transition system $(W, \{\sim_a\}_{a \in \mathcal{A}}, V, M, \pi)$ and any formula $\varphi \in \Phi$, let satisfiability relation $h \Vdash \varphi$ be defined as follows*

1. $h \Vdash p$ *if $hd(h) \in \pi(p)$ and p is a propositional variable,*
2. $h \Vdash \neg\varphi$ *if $h \nVdash \varphi$,*
3. $h \Vdash \varphi \to \psi$ *if $h \nVdash \varphi$ or $h \Vdash \psi$,*
4. $h \Vdash \mathsf{K}_C\varphi$ *if $h' \Vdash \varphi$ for each history h' s.t. $h \approx_C h'$,*
5. $h \Vdash \mathsf{H}_C\varphi$ *if there is a strategy profile $\mathbf{s} \in V^C$ such that for any history $h' :: \mathbf{s}' :: w'$, if $h \approx_C h'$ and $\mathbf{s} =_C \mathbf{s}'$, then $h' :: \mathbf{s}' :: w' \Vdash \varphi$.*

Axioms

In additional to propositional tautologies in language Φ, our logical system consists of the following axioms:

1. Truth: $\mathsf{K}_C\varphi \to \varphi$,

2. Negative Introspection: $\neg\mathsf{K}_C\varphi \to \mathsf{K}_C\neg\mathsf{K}_C\varphi$,

3. Distributivity: $\mathsf{K}_C(\varphi \to \psi) \to (\mathsf{K}_C\varphi \to \mathsf{K}_C\psi)$,

4. Monotonicity: $\mathsf{K}_C\varphi \to \mathsf{K}_D\varphi$, if $C \subseteq D$,

5. Strategic Positive Introspection: $\mathsf{H}_C\varphi \to \mathsf{K}_C\mathsf{H}_C\varphi$,

6. Cooperation: $\mathsf{H}_C(\varphi \to \psi) \to (\mathsf{H}_D\varphi \to \mathsf{H}_{C \cup D}\psi)$, where $C \cap D = \varnothing$,

7. Empty Coalition: $\mathsf{K}_\varnothing\varphi \to \mathsf{H}_\varnothing\varphi$,

8. Perfect Recall: $\mathsf{H}_D\varphi \to \mathsf{H}_D\mathsf{K}_C\varphi$, where $D \subseteq C \neq \varnothing$,

9. Unachievability of Falsehood: $\neg\mathsf{H}_C\bot$.

We write $\vdash \varphi$ if formula φ is provable from the axioms of our logical system using Necessitation, Strategic Necessitation, and Modus Ponens inference rules:

$$\frac{\varphi}{\mathsf{K}_C\varphi} \qquad \frac{\varphi}{\mathsf{H}_C\varphi} \qquad \frac{\varphi, \quad \varphi \to \psi}{\psi}.$$

We write $X \vdash \varphi$ if formula φ is provable from the theorems of our logical system and a set of additional axioms X using only Modus Ponens inference rule.

The next lemma follows from a well-known observation that the Positive Introspection axiom is provable from the other axioms of S5. The proof can be found in the full version of this paper (Naumov and Tao 2017b).

Lemma 6 $\vdash \mathsf{K}_C\varphi \to \mathsf{K}_C\mathsf{K}_C\varphi$. ⊠

The proof of the following soundness theorem is also given in (Naumov and Tao 2017b).

Theorem 1 *If $\vdash \varphi$, then $h \Vdash \varphi$ for each history of each regular epistemic transition system.*

Derivation Examples

This section contains examples of formal proofs in our logical system. The results obtained here are used in the proof of completeness. The proof of Lemma 7 is based on the one proposed to us by Natasha Alechina.

Lemma 7 (Alechina) $\vdash \neg\mathsf{H}_C\varphi \to \mathsf{K}_C\neg\mathsf{H}_C\varphi$.

Proof. By the Positive Strategic Introspection axiom, $\vdash \mathsf{H}_C\varphi \to \mathsf{K}_C\mathsf{H}_C\varphi$. Thus, $\vdash \neg\mathsf{K}_C\mathsf{H}_C\varphi \to \neg\mathsf{H}_C\varphi$ by the contrapositive. Hence, $\vdash \mathsf{K}_C(\neg\mathsf{K}_C\mathsf{H}_C\varphi \to \neg\mathsf{H}_C\varphi)$ by the Necessitation inference rule. Then, by the Distributivity axiom and the Modus Ponens inference rule $\vdash \mathsf{K}_C\neg\mathsf{K}_C\mathsf{H}_C\varphi \to \mathsf{K}_C\neg\mathsf{H}_C\varphi$. Thus, by the Negative Introspection axiom and the laws of propositional reasoning, $\vdash \neg\mathsf{K}_C\mathsf{H}_C\varphi \to \mathsf{K}_C\neg\mathsf{H}_C\varphi$. Note that $\neg\mathsf{H}_C\varphi \to \neg\mathsf{K}_C\mathsf{H}_C\varphi$ is the contrapositive of the Truth axiom. Therefore, by the laws of propositional reasoning, $\vdash \neg\mathsf{H}_C\varphi \to \mathsf{K}_C\neg\mathsf{H}_C\varphi$. ⊠

Lemma 8 $\vdash \mathsf{H}_C\varphi \to \mathsf{H}_D\varphi$, *where $C \subseteq D$.*

Proof. Note that $\varphi \to \varphi$ is a propositional tautology. Thus, $\vdash \varphi \to \varphi$. Hence, $\vdash \mathsf{H}_{D \setminus C}(\varphi \to \varphi)$ by the Strategic Necessitation inference rule. At the same time, by the Cooperation axiom, $\vdash \mathsf{H}_{D \setminus C}(\varphi \to \varphi) \to (\mathsf{H}_C\varphi \to \mathsf{H}_D\varphi)$ due to the assumption $C \subseteq D$. Therefore, $\vdash \mathsf{H}_C\varphi \to \mathsf{H}_D\varphi$ by the Modus Ponens inference rule. ⊠

Lemma 9 *If $\varphi_1, \ldots, \varphi_n \vdash \psi$, then*

1. $\mathsf{K}_C\varphi_1, \ldots, \mathsf{K}_C\varphi_n \vdash \mathsf{K}_C\psi$,
2. $\mathsf{H}_{C_1}\varphi_1, \ldots, \mathsf{H}_{C_n}\varphi_n \vdash \mathsf{H}_{\bigcup_{i=1}^{n} C_i}\psi$, *where sets $C_1, \ldots, C_n$ are pairwise disjoint.*

Proof. To prove the second statement, apply deduction lemma for propositional logic n time. Then, we have $\vdash \varphi_1 \to (\cdots \to (\varphi_n \to \psi))$. Thus, by the Strategic Necessitation inference rule, $\vdash \mathsf{H}_\varnothing(\varphi_1 \to (\cdots \to (\varphi_n \to \psi)))$ Hence, $\vdash \mathsf{H}_{C_1}\varphi_1 \to \mathsf{H}_{C_1}(\varphi_2 \cdots \to (\varphi_n \to \psi))$ by the Cooperation axiom and the Modus Ponens inference rule. Then, $\mathsf{H}_{C_1}\varphi_1 \vdash \mathsf{H}_{C_1}(\varphi_2 \cdots \to (\varphi_n \to \psi))$ by the Modus Ponens inference rule. Thus, again by the Cooperation axiom and the Modus Ponens inference rule we have $\mathsf{H}_{C_1}\varphi_1 \vdash \mathsf{H}_{C_2}\varphi_2 \to \mathsf{H}_{C_1 \cup C_2}(\varphi_3 \cdots \to (\varphi_n \to \psi))$. Therefore, by repeating the last two steps $n - 2$ times, $\mathsf{H}_{C_1}\varphi_1, \ldots, \mathsf{H}_{C_n}\varphi_n \vdash \mathsf{H}_{\bigcup_{i=1}^{n} C_i}\psi$. The proof of the first statement is similar, but it uses the Distributivity axiom instead of the Cooperation axiom. ⊠

Completeness

In the rest of this paper we focus on the completeness theorem for our logical system with respect to the class of regular epistemic transition systems. We start the proof of completeness by fixing a maximal consistent set of formulae X_0 and defining a canonical epistemic transition system $ETS(X_0) = (W, \{\sim_a\}_{a \in \mathcal{A}}, \Phi, M, \pi)$ using the "unravelling" technique (Sahlqvist 1975). Note that the domain of choices in the canonical model is the set of all formulae Φ.

Canonical Epistemic Transition System

Definition 10 *The set of epistemic states W consists of all sequences $X_0, C_1, X_1, \ldots, C_n, X_n$, such that $n \geq 0$ and*

1. *X_i is a maximal consistent subset of Φ for each $i \geq 1$,*
2. *$C_i \subseteq \mathcal{A}$ for each $i \geq 1$,*
3. *$\{\varphi \mid \mathsf{K}_{C_i}\varphi \in X_{i-1}\} \subseteq X_i$, for each $i \geq 1$.*

Definition 11 *Suppose that $w = X_0, C_1, X_1, \ldots, C_n, X_n$ and $w' = X_0, C_1', X_1' \ldots, C_m', X_m'$ are epistemic states. For any agent $a \in \mathcal{A}$, let $w \sim_a w'$ if there is a non-negative integer $k \leq \min\{n, m\}$ such that*

1. *$X_i = X_i'$ for each i such that $0 < i \leq k$,*
2. *$C_i = C_i'$ for each i such that $0 < i \leq k$,*
3. *$a \in C_i$ for each i such that $k < i \leq n$,*
4. *$a \in C_i'$ for each i such that $k < i \leq m$.*

The next lemma states the basic property of the indistinguishability relation $\sim_a$. The proof of this lemma is standard. See (Naumov and Tao 2017b) for details.

Lemma 10 *For any epistemic states $w, w' \in W$ such that $w \sim_C w'$, if $\mathsf{K}_C\varphi \in hd(w)$, then $\varphi \in hd(w')$.*

Next, we specify the aggregation mechanism of the canonical epistemic transition system. Informally, if a coalition has a know-how strategy to achieve φ and all members of the coalition vote for φ, then φ must be true after the transition.

Definition 12 *For any states $w, w' \in W$ and any complete strategy profile $\mathbf{s} \in \Phi^{\mathcal{A}}$, let $(w, \mathbf{s}, w') \in M$ if*

$$\{\varphi \mid (\mathsf{H}_D\varphi \in hd(w)) \wedge \forall a \in D((\mathbf{s})_a = \varphi)\} \subseteq hd(w').$$

Definition 13 $\pi(p) = \{w \in W \mid p \in hd(w)\}$.

This concludes the definition of the canonical epistemic transition system $ETS(X_0) = (W, \{\sim_a\}_{a \in \mathcal{A}}, \Phi, M, \pi)$. We prove that this system is regular in Lemma 18.

K-child Lemmas The following technical results (Lemmas 11–15) about the knowledge modality K are used in the proof of completeness.

Lemma 11 *For any epistemic state $w \in W$ if $\neg\mathsf{K}_C\varphi \in hd(w)$, then there is an epistemic state $w' \in W$ such that $w \sim_C w'$ and $\neg\varphi \in hd(w')$.*

Proof. Consider the set $X = \{\neg\varphi\} \cup \{\psi \mid \mathsf{K}_C\psi \in hd(w)\}$. First, we show that set X is consistent. Assume the opposite. Then, there exist formulae $\mathsf{K}_C\psi_1, \ldots, \mathsf{K}_C\psi_n \in hd(w)$ such that $\psi_1, \ldots, \psi_n \vdash \varphi$. Thus, $\mathsf{K}_C\psi_1, \ldots, \mathsf{K}_C\psi_n \vdash \mathsf{K}_C\varphi$ by Lemma 9. Therefore, $hd(w) \vdash \mathsf{K}_C\varphi$ by the choice of formulae $\mathsf{K}_C\psi_1, \ldots, \mathsf{K}_C\psi_n$, which contradicts the consistency of the set $hd(w)$ due to the assumption $\neg\mathsf{K}_C\varphi \in hd(w)$.

Let $\hat{X}$ be a maximal consistent extension of set X and let w' be sequence $w :: C :: X$. Note that $w' \in W$ by Definition 10 and the choice of set X. Furthermore, $w \sim_C w'$ by Definition 11. To finish the proof, note that $\neg\varphi \in X \subseteq \hat{X} = hd(w')$ by the choice of set X. $\boxtimes$

History h is a sequence whose last element $hd(h)$ is an epistemic state. Epistemic state $hd(h)$, by Definition 10, is also a sequence. Expression $hd(hd(h))$ denotes the last element of the sequence $hd(h)$.

Lemma 12 *For any history h, if $\mathsf{K}_C\varphi \in hd(hd(h))$, then $\varphi \in hd(hd(h'))$ for each history h' such that $h \approx_C h'$.*

Proof. Assumption $h \approx_C h'$ by Definition 6 implies that $hd(h) \sim_C hd(h')$. Therefore, $\varphi \in hd(hd(h'))$ by Lemma 10 and the assumption $\mathsf{K}_C\varphi \in hd(hd(h))$. $\boxtimes$
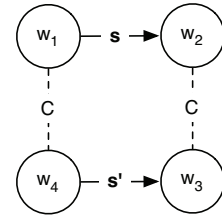

Figure 3: Illustration for Lemma 13.

Lemma 13 *For any nonempty coalition C, any states $w_1, w_2, w_3 \in W$, and any complete strategy profile $\mathbf{s}$ such that $(w_1, \mathbf{s}, w_2) \in M$ and $w_2 \sim_C w_3$, see Figure 3, there is a state w_4 and a complete strategy profile $\mathbf{s}'$ such that $w_1 \sim_C w_4$, $(w_4, \mathbf{s}', w_3) \in M$, and $\mathbf{s} =_C \mathbf{s}'$.*

Proof. Let $\mathbf{s}'$ be a complete strategy profile such that

$$(\mathbf{s}')_a = \begin{cases} (\mathbf{s})_a, & \text{if } a \in C, \\ \bot, & \text{otherwise.} \end{cases} \quad (1)$$

Consider the set of formulae

$$X = \{\varphi \mid \mathsf{K}_C\varphi \in hd(w_1)\} \cup$$
$$\{\neg\mathsf{H}_D\psi \mid \neg\psi \in hd(w_3) \wedge \forall a \in D((\mathbf{s}')_a = \psi)\}.$$

First, we show that set X is consistent. Indeed, set

$$\{\neg\mathsf{H}_D\psi \mid \neg\psi \in hd(w_3) \wedge \forall a \in D((\mathbf{s}')_a = \psi)\}$$

is equal to the union of the following two sets

$$\{\neg\mathsf{H}_D\psi \mid \neg\psi \in hd(w_3) \wedge D \subseteq C \wedge \forall a \in D((\mathbf{s}')_a = \psi)\},$$
$$\{\neg\mathsf{H}_D\psi \mid \neg\psi \in hd(w_3) \wedge D \nsubseteq C \wedge \forall a \in D((\mathbf{s}')_a = \psi)\}.$$

The second set is a subset of $\{\neg\mathsf{H}_D\bot \mid D \subseteq \mathcal{A}\}$ by the choice of strategy $\mathbf{s}'$, see (1). Thus, set X is a subset of

$$\{\varphi \mid \mathsf{K}_C\varphi \in hd(w_1)\} \cup \{\neg\mathsf{H}_D\bot \mid D \subseteq \mathcal{A}\} \cup$$
$$\{\neg\mathsf{H}_D\psi \mid \neg\psi \in hd(w_3) \wedge D \subseteq C \wedge \forall a \in D((\mathbf{s}')_a = \psi)\}.$$

Hence, by the Unachievability of Falsehood axiom, to show the consistency of set X it suffices to prove the consistency of the union of the set $\{\varphi \mid \mathsf{K}_C\varphi \in hd(w_1)\}$ and the set

$$\{\neg\mathsf{H}_D\psi \mid \neg\psi \in hd(w_3) \wedge D \subseteq C \wedge \forall a \in D((\mathbf{s}')_a = \psi)\}.$$

Suppose the opposite. In other words, assume that there are

formulae $\quad \mathsf{K}_C\varphi_1, \ldots, \mathsf{K}_C\varphi_n \in hd(w_1),$ $\qquad$ (2)

formulae $\quad \neg\psi_1, \ldots, \neg\psi_m \in hd(w_3),$ $\qquad$ (3)

and sets $\quad D_1, \ldots, D_m \subseteq C,$ $\qquad$ (4)

such that $\quad \forall i \leq m \, \forall a \in D_i \, ((\mathbf{s}')_a = \psi_i),$ $\qquad$ (5)

and $\quad \varphi_1, \ldots, \varphi_n, \neg\mathsf{H}_{D_1}\psi_1, \ldots, \neg\mathsf{H}_{D_m}\psi_m \vdash \bot.$

By Lemma 9 and the Truth axiom,

$$\mathsf{K}_C\varphi_1, \ldots, \mathsf{K}_C\varphi_n, \mathsf{K}_C\neg\mathsf{H}_{D_1}\psi_1, \ldots, \mathsf{K}_C\neg\mathsf{H}_{D_m}\psi_m \vdash \bot.$$

Hence, statement (2) and the consistency of the set $hd(w_1)$ imply that there exists $k \leq m$ such that $\mathsf{K}_C\neg\mathsf{H}_{D_k}\psi_k \notin hd(w_1)$. Thus, $\neg\mathsf{K}_C\neg\mathsf{H}_{D_k}\psi_k \in hd(w_1)$ due to the maximality of the set $hd(w_1)$. Then, $\neg\mathsf{K}_{D_k}\neg\mathsf{H}_{D_k}\psi_k \in hd(w_1)$ by statement (4) and the contrapositive of the Monotonicity axiom. Then, $hd(w_1) \vdash \mathsf{H}_{D_k}\psi_k$ by the contrapositive of Lemma 7. Thus, $hd(w_1) \vdash \mathsf{H}_{D_k}\mathsf{K}_C\psi_k$ by the Perfect Recall axiom, statement (4), and the assumption of the lemma that $C \neq \varnothing$. Hence, $\mathsf{H}_{D_k}\mathsf{K}_C\psi_k \in hd(w_1)$ due to the maximality of the set $hd(w_1)$. Note that statements (1) and (4) imply $\mathbf{s} =_{D_k} \mathbf{s}'$. Then, $(\mathbf{s})_a = \psi_k$ for each agent $a \in D_k$ by statement (5). Thus, $\mathsf{K}_C\psi_k \in hd(w_2)$ by assumption $(w_1, \mathbf{s}, w_2) \in M$, statement (4), and Definition 12. Hence, $\psi_k \in hd(w_3)$ by the assumption $w_2 \sim_C w_3$ of the lemma and Lemma 10. This contradicts statement (3) and the consistency of the set $hd(w_3)$. Therefore, set X is consistent.

Let $\hat{X}$ be any maximal consistent extension of set X. Define w_4 to be the sequence $w_1 :: C :: \hat{X}$. Note that $w_4 \in W$ by Definition 10 and the choice of set X. At the same time, $w_1 \sim_C w_4$ by Definition 11 and Definition 3.

Finally, let us show that $(w_4, \mathbf{s}', w_3) \in M$ using Definition 12. Consider any $D \subseteq \mathcal{A}$ and any $\mathsf{H}_D\psi \in hd(w_4) = \hat{X}$ such that $\mathbf{s}'(a) = \psi$ for each $a \in D$. We need to show that $\psi \in hd(w_3)$. Suppose the opposite. Then, $\neg\psi \in hd(w_3)$ by the maximality of set $hd(w_3)$. Thus, $\neg\mathsf{H}_D\psi \in X$ by the choice of set X. Hence, $\neg\mathsf{H}_D\psi \in X \subseteq \hat{X}$. Therefore, $\mathsf{H}_D\psi \notin \hat{X}$ due to the consistency of set $\hat{X}$, which contradicts the choice of formula $\mathsf{H}_D\psi$. $\qquad\boxtimes$

Lemma 14 *For any history h, if $\mathsf{K}_\varnothing\varphi \notin hd(hd(h))$, then there is a history h' s.t. $h \approx_\varnothing h'$ and $\neg\varphi \in hd(hd(h'))$.*

Proof. By Lemma 11, there is a state $w \in W$ such that $hd(h) \sim_C w$ and $\neg\varphi \in hd(w)$. Let h' be a one-element sequence w. Note that $h \approx_\varnothing h'$ by Definition 7. Finally, $\neg\varphi \in hd(w) = hd(hd(h'))$. $\qquad\boxtimes$

Lemma 15 *For any nonempty coalition C and any history h, if $\mathsf{K}_C\varphi \notin hd(hd(h))$, then there is a history h' such that $h \approx_C h'$ and $\neg\varphi \in hd(hd(h'))$.*

Proof. Let $h = (w_0, \mathbf{s}_1, w_1, \ldots, \mathbf{s_n}, w_n)$. We prove the lemma by induction on integer n.
Base Case. Let $n = 0$. By Lemma 11, there is $w_0' \in W$ such that $w_0 \sim_C w_0'$ and $\neg\varphi \in hd(w_0')$. Let h' be a one-element sequence w_0'. Note that $w_0 \sim_C w_0'$ implies $h \approx_C h'$ by Definition 6. Also, $\neg\varphi \in hd(w_0') = hd(hd(h'))$.
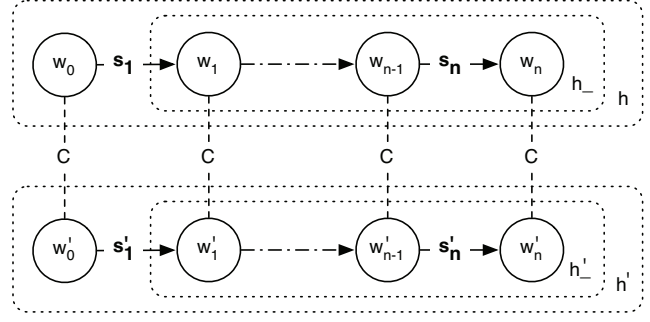

Figure 4: Histories h and h'.

Induction Step. Let $h_- = (w_1, \mathbf{s}_2, \ldots, \mathbf{s_n}, w_n)$, see Figure 4. By Definition 5, sequence h_- is a history. By the induction hypothesis there is a history h'_- such that $h_- \approx_C h'_-$ and $\neg\varphi \in hd(hd(h'_-))$. Let $h'_- = (w_1', \mathbf{s}_2', \ldots, \mathbf{s_n}', w_n')$. By Lemma 13, there is a state w_0' and a complete strategy profile $\mathbf{s_1}'$ such that $w_0 \sim_C w_0'$, $(w_0', \mathbf{s_1}', w_1') \in M$, and $\mathbf{s}_1 =_C \mathbf{s_1}'$.

Define $h' = (w_0', \mathbf{s_1}', w_1', \mathbf{s}_2', \ldots, \mathbf{s_n}', w_n')$. By Definition 5, statement $(w_0', \mathbf{s_1}', w_1') \in M$ implies that h' is a history. Note that $h \approx_C h'$ by Definition 6 because $h'_- \approx_C h_-$, $w_0 \sim_C w_0'$, and $\mathbf{s}_1 =_C \mathbf{s_1}'$. Finally, $\neg\varphi \in hd(hd(h'))$ because $hd(h') = hd(h'_-)$ and $\neg\varphi \in hd(hd(h'_-))$. $\qquad\boxtimes$

H-child Lemmas Lemmas 16 and 17 are about the know-how modality H. They are used later in the proof.

Lemma 16 *For any history h, if $\mathsf{H}_C\varphi \in hd(hd(h))$, then there is a strategy profile $\mathbf{s}$ of coalition C s.t. for any history $h' :: \mathbf{s}' :: w'$ if $h \approx_C h'$ and $\mathbf{s} =_C \mathbf{s}'$, then $\varphi \in hd(w')$.*

Proof. Consider a strategy profile $\mathbf{s}$ of coalition C such that $(\mathbf{s})_a = \varphi$ for each agent $a \in C$. Suppose that $\mathsf{H}_C\varphi \in hd(hd(h))$ and sequence $h' :: \mathbf{s}' :: w'$ is a history such that $h \approx_C h'$ and $\mathbf{s} =_C \mathbf{s}'$. It suffices to show that $\varphi \in hd(w')$.

By the Strategic Positive Introspection axiom, assumption $\mathsf{H}_C\varphi \in hd(hd(h))$ implies $hd(hd(h)) \vdash \mathsf{K}_C\mathsf{H}_C\varphi$. Thus, $\mathsf{K}_C\mathsf{H}_C\varphi \in hd(hd(h))$ by the maximality of set $hd(hd(h))$.

Assumption $h \approx_C h'$ implies $hd(h) \sim_C hd(h')$ by Definition 6. Thus, $\mathsf{H}_C\varphi \in hd(hd(h'))$ by Lemma 10 and because $\mathsf{K}_C\mathsf{H}_C\varphi \in hd(hd(h))$.

By Definition 5, assumption that sequence $h' :: \mathbf{s}' :: w'$ is a history implies that $(hd(h'), \mathbf{s}', w') \in M$. Thus, $\varphi \in hd(w')$ by Definition 12 because $\mathsf{H}_C\varphi \in hd(hd(h'))$ and $(\mathbf{s}')_a = (\mathbf{s})_a = \varphi$ for each agent $a \in C$. $\qquad\boxtimes$

Lemma 17 *For any history h and any strategy profile $\mathbf{s}$ of a coalition C, if $\neg\mathsf{H}_C\varphi \in hd(hd(h))$, then there is a history $h :: \mathbf{s}' :: w'$ such that $\mathbf{s} =_C \mathbf{s}'$ and $\neg\varphi \in hd(w')$.*

Proof. Let $w = hd(h)$. Consider a complete strategy profile $\mathbf{s}'$ and a set of formulae X such that

$$(\mathbf{s}')_a = \begin{cases} (\mathbf{s})_a, & \text{if } a \in C, \\ \top, & \text{otherwise,} \end{cases} \qquad (6)$$

and
$$X = \{\neg\varphi\} \cup \{\chi \mid \mathsf{K}_\varnothing \chi \in hd(w)\} \cup$$
$$\{\psi \mid \mathsf{H}_D\psi \in hd(w) \wedge \forall a \in D((\mathbf{s}')_a = \psi)\}.$$

First, we show that set X is consistent. Indeed, note that set $\{\psi \mid \mathsf{H}_D\psi \in hd(w) \wedge \forall a \in D((\mathbf{s}')_a = \psi)\}$ is equal to the union of the following two sets

$$\{\psi \mid \mathsf{H}_D\psi \in hd(w) \wedge D \subseteq C \wedge \forall a \in D((\mathbf{s}')_a = \psi)\},$$
$$\{\psi \mid \mathsf{H}_D\psi \in hd(w) \wedge D \nsubseteq C \wedge \forall a \in D((\mathbf{s}')_a = \psi)\}.$$

The second of the two sets is a subset of $\{\top\}$ by the choice (6) of strategy profile $\mathbf{s}'$. Thus, set X is a subset of

$$\{\top\} \cup \{\neg\varphi\} \cup \{\chi \mid \mathsf{K}_\varnothing \chi \in hd(w)\} \cup$$
$$\{\psi \mid \mathsf{H}_D\psi \in hd(w) \wedge D \subseteq C \wedge \forall a \in D((\mathbf{s}')_a = \psi)\}.$$

Hence, to show the consistency of set X, it suffices to prove the consistency of the union of $\{\neg\varphi\} \cup \{\chi \mid \mathsf{K}_\varnothing\chi \in hd(w)\}$ and $\{\psi \mid \mathsf{H}_D\psi \in hd(w) \wedge D \subseteq C \wedge \forall a \in D((\mathbf{s}')_a = \psi)\}$. Suppose the opposite. In other words, assume there are

$$\text{formulae} \quad \mathsf{K}_\varnothing\chi_1, \ldots, \mathsf{K}_\varnothing\chi_n \in hd(w) \quad (7)$$
$$\text{and formulae} \quad \mathsf{H}_{D_1}\psi_1, \ldots, \mathsf{H}_{D_m}\psi_m \in hd(w) \quad (8)$$
$$\text{such that} \quad D_1, \ldots, D_m \subseteq C, \quad (9)$$
$$\forall i \leq m \, \forall a \in D_i \, ((\mathbf{s}')_a = \psi_i), \quad (10)$$

and $\chi_1, \ldots, \chi_n, \psi_1, \ldots, \psi_m \vdash \varphi$. By Lemma 9,

$$\mathsf{H}_\varnothing\chi_1, \ldots, \mathsf{H}_\varnothing\chi_n, \mathsf{H}_{D_1}\psi_1, \ldots, \mathsf{H}_{D_m}\psi_m \vdash \mathsf{H}_{\cup_{i=1}^m D_i}\varphi.$$

Then, by the Empty Coalition axiom,

$$\mathsf{K}_\varnothing\chi_1, \ldots, \mathsf{K}_\varnothing\chi_n, \mathsf{H}_{D_1}\psi_1, \ldots, \mathsf{H}_{D_m}\psi_m \vdash \mathsf{H}_{\cup_{i=1}^m D_i}\varphi.$$

Thus, by Lemma 8 and assumption (9),

$$\mathsf{K}_\varnothing\chi_1, \ldots, \mathsf{K}_\varnothing\chi_n, \mathsf{H}_{D_1}\psi_1, \ldots, \mathsf{H}_{D_m}\psi_m \vdash \mathsf{H}_C\varphi.$$

Hence, $hd(w) \vdash \mathsf{H}_C\varphi$ by assumption (7) and assumption (8). Then, $\neg\mathsf{H}_C\varphi \notin hd(w)$ due to the consistency of the set $hd(w)$, which contradicts the assumption $\neg\mathsf{H}_C\varphi \in hd(w)$ of the lemma. Therefore, set X is consistent.

Let set $\hat{X}$ be a maximal consistent extension of set X and w' be the sequence $w :: \varnothing :: \hat{X}$. Note that $w' \in W$ by Definition 10 and because $\{\chi \mid \mathsf{K}_\varnothing\chi \in hd(w)\} \subseteq X \subseteq \hat{X} = hd(w')$ by the choice of set X.

Also note that $(w, \mathbf{s}', w') \in M$ by Definition 12 and because $\{\psi \mid \mathsf{H}_D\psi \in hd(w) \wedge \forall a \in D((\mathbf{s}')_a = \psi)\} \subseteq X \subseteq \hat{X} = hd(w')$ by the choice of set X. Thus, $h :: \mathbf{s}' :: w'$ is a history by Definition 5.

Finally, $\neg\varphi \in hd(w')$ because $\neg\varphi \in X \subseteq \hat{X} = hd(w')$ by the choice of set X. ⊠

Lemma 18 The system $ETS(X_0)$ is regular.

Proof. Let $w \in W$ and $\mathbf{s} \in V^{\mathcal{A}}$. By Definition 2, it suffices to show that there is an epistemic state w' such that $(w, \mathbf{s}, w') \in M$. Indeed, let history h be a single-element sequence w. Note that $\neg\mathsf{H}_{\mathcal{A}}\bot \in hd(w) = hd(hd(h))$ by the Unachievability of Falsehood axiom and due to the maximality of set $hd(w)$. Thus, by Lemma 17, there is a history $h :: \mathbf{s}' :: w'$ such that $\mathbf{s} =_{\mathcal{A}} \mathbf{s}'$. Hence, $(hd(h), \mathbf{s}', w') \in M$ by Definition 5. At the same time, $\mathbf{s} =_{\mathcal{A}} \mathbf{s}'$ implies that $\mathbf{s} = \mathbf{s}'$ by Definition 4. Thus, $(hd(h), \mathbf{s}, w') \in M$. Therefore, $(w, \mathbf{s}, w') \in M$ because $hd(h) = w$. ⊠

Completeness: Final Steps

Lemma 19 $h \Vdash \varphi$ iff $\varphi \in hd(hd(h))$ for each history h and each formula $\varphi \in \Phi$.

Proof. We prove the statement by induction on the structural complexity of formula φ. If φ is an atomic proposition p, then $h \Vdash p$ iff $hd(h) \in \pi(p)$, by Definition 9. Hence, $h \Vdash p$ iff $p \in hd(hd(h))$ by Definition 13.

The cases when formula φ is a negation or an implication follow from Definition 9 and the maximality and the consistency of the set $hd(hd(h))$ in the standard way.

Next, suppose that formula φ has the form $\mathsf{K}_C\psi$.
$(\Rightarrow)$: Let $\mathsf{K}_C\psi \notin hd(hd(h))$. Then, either by Lemma 14 (when set C is empty) or by Lemma 15 (when set C is nonempty), there is a history h' such that $h \approx_C h'$ and $\neg\psi \in hd(hd(h'))$. Hence, $h' \nVdash \psi$ by the induction hypothesis. Therefore, $h \nVdash \mathsf{K}_C\psi$ by Definition 9.
$(\Leftarrow)$: Let $h \nVdash \mathsf{K}_C\psi$. By Definition 9, there is a history h' such that $h \approx_C h'$ and $h' \nVdash \psi$. Thus, $\psi \notin hd(hd(h'))$ by the induction hypothesis. Then, $K_C\varphi \notin hd(hd(h))$ by Lemma 12.

Finally, let formula φ be of the form $\mathsf{H}_C\psi$.
$(\Rightarrow)$: Assume $h \Vdash \mathsf{H}_C\psi$. Then, by Definition 9, there is a strategy profile $\mathbf{s} \in \Phi^C$ such that for any history $h' :: \mathbf{s}' :: w'$, if $h \approx_C h'$ and $\mathbf{s} =_C \mathbf{s}'$, then $h' :: \mathbf{s}' :: w' \Vdash \psi$. Thus, by Lemma 3,

$$\text{for any history } h :: \mathbf{s}' :: w', \text{ if } \mathbf{s} =_C \mathbf{s}', \text{ then} \quad (11)$$
$$h :: \mathbf{s}' :: w' \Vdash \psi.$$

Suppose that $\mathsf{H}_C\psi \notin hd(hd(h))$. Then, $\neg\mathsf{H}_C\psi \in hd(hd(h))$ due to the maximality of the set $hd(hd(h))$. Hence, by Lemma 17, there is a history $h :: \mathbf{s}' :: w'$ such that $\mathbf{s} =_C \mathbf{s}'$ and $\neg\psi \in hd(w')$. Thus, $\psi \notin hd(w')$ due to the consistency of set $hd(w')$. Hence, by the induction hypothesis, $h :: \mathbf{s}' :: w' \nVdash \psi$, which contradicts statement (11).
$(\Leftarrow)$: Assume that $\mathsf{H}_C\psi \in hd(hd(h))$. By Lemma 16, there is a strategy profile $\mathbf{s} \in \Phi^C$ such that for any history $h' :: \mathbf{s}' :: w'$ if $h \approx_C h'$ and $\mathbf{s} =_C \mathbf{s}'$, then $\psi \in hd(w')$. Hence, by the induction hypothesis, for any history $h' :: \mathbf{s}' :: w'$ if $h \approx_C h'$ and $\mathbf{s} =_C \mathbf{s}'$, then $h' :: \mathbf{s}' :: w' \Vdash \psi$. Therefore, $h \Vdash \mathsf{H}_C\psi$ by Definition 9. ⊠

Theorem 2 If $h \Vdash \varphi$ for each history h of each regular epistemic transition system, then $\vdash \varphi$.

Proof. Suppose that $\nvdash \varphi$. Consider any maximal consistent set X_0 such that $\neg\varphi \in X_0$. Let h_0 be a single-element sequence consisting of just set X_0. By Definition 5, sequence h_0 is a history of the canonical epistemic transition system $ETS(X_0)$. Then, $h_0 \Vdash \neg\varphi$ by Lemma 19. Therefore, $h_0 \nVdash \varphi$ by Definition 9. ⊠

Theorem 2 can be generalized to the strong completeness theorem in the standard way. We also believe that the number of states and the domain of choices in the canonical model can be made finite using filtration on subformulae.

Conclusion

We have extended the recent study of the interplay between knowledge and strategic coalition power (Ågotnes

and Alechina 2016; Fervari et al. 2017; Naumov and Tao 2017c; 2017a) to the case of perfect recall. Our main results are the soundness and the completeness theorems.

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