Use of Holography to study Heavy Ion Collision

Seyf Kaddachi, 11.14.2024

Based on the paper by Paul M. Chesler and Laurence G. Yaffe :

- “Numerical solution of gravitational dynamics in asymptotically anti-de Sitter spacetimes”. **arXiv:** 1309.1439

Heavy Ion Collision

The different stages:
Collision of highly Lorentz contracted nuclei → liberation of a very large phase space density of partons.

The initial distribution of partons is highly anisotropic.

Origins of anisotropy:

- Typical transverse momenta much larger than longitudinal momenta.
- A magnetic field is created by the moving charged nuclei. The magnetic field decays fast after the collision.

These partons subsequently interact and scatter.
After a “thermalization time”/ “isotropization time”, the gas of interacting partons may be modeled as a relativistic fluid (a quark-gluon plasma) whose stress tensor is nearly isotropic.

- Isotropization time of the dense parton gas is remarkably short, less than 1 fm/c.
- Using hydrodynamic as EFT requires that the stress-energy tensor be nearly isotropic.

This QGP expands, cools, and eventually reaches a temperature where hadrons reform, fly outward, and ultimately reach the detector

The Need for Holography

Studying real time quantum dynamics in QCD at strong coupling

- from the creation of the initial out-of-equilibrium state
- through local thermalization, hydrodynamic evolution,
- hadronization, and eventual freeze-out

is not currently possible.

Holography provides unique opportunities to study strongly-coupled out-of-equilibrium dynamics, by mapping the dynamics to dual gravitational dynamics in 1 spatial dimension higher .

For large N_c SYM, the dual theory is classical gravity.

Holography

Holography is a gauge/gravity duality that relates certain quantum field theories in 4 spacetime dimensions to gravitational physics in 5 spacetime dimensions with asymptotically AdS geometry.

The boundary geometry corresponds to the spacetime geometry of the SYM field theory.

Time-dependent deformation of the 4d boundary geometry produces gravitational radiation which propagates into the fifth dimension. This radiation will produce a black brane and a corresponding horizon inside the bulk.

Holography

The 5d geometry will relax onto a smooth and slowly varying form. This relaxation is dual to the relaxation of non-hydrodynamic degrees of freedom in the quantum field theory

By solving the corresponding gravitational problem numerically, and using the gauge/gravity dictionary, one can compute the boundary field theory stress-energy tensor at all times, from the first excitation of the initial state to the late-time onset of hydrodynamics.

Therefore, by studying the evolution of the 5d black hole geometry, one can gain insight into the creation and relaxation of the SYM plasma

Warm-up example:
Homogeneous Isotropization

We mentioned that the QGP is initially highly anisotropic.
We study now the holographic model of the istropization.

r is the “bulk” radial coordinate. The AdS boundary is at $r \to \infty$.

We assume some symmetries of the QGP 4d spacetime.
Symmetries:

- Translation (i.e. 2-dim spatial homogeneity) and O(2) rotation invariance in the transverse xy plane.
- Diffeomorphism

Homogeneous Isotropization

Metric: $ds^2 = 2\, dr\, dt\ - 2\, A(t,r)\, dt^2 + \Sigma(t,r)^2\, (e^{B(t,r)}\, dx^2 + e^{B(t,r)} dy^2 + e^{-2B(t,r)} dz^2\,)$

Fields: $A(t,r)$, $B(t,r)$ and $\Sigma(\mathrm{t,r})$.
$B(t,r)$ is the "anisotropy" function.

Residual gauge symmetry:
The form of the metric is invariant under the residual diffeomorphism r → r + λ(t), where λ(t) is an arbitrary function.
Residual gauge fixing by e.g. fixing the apparent horizon at fixed $\bar{r_h}$.

With these symmetries, the late time behavior of the 5d geometry (and the corresponding late-time stress-energy tensor) are known analytically. We will therefore be able to compare directly our numerical results, supposed to be valid at all times, to the known late-time behavior.

Homogeneous Isotropization

Relation between boundary stress-energy tensor and the bulk fields:

- **Boundary energy density:** $T^{00} = -\frac{3}{2}\kappa\, a^{(4)}$ with $\kappa = \frac{L^3}{4\,\Pi\, G_N}$.

L as the spacetime curvature scale.

$a^{(4)}$ is the coefficient of the $1/r^2$ term in the Laurent expansion in powers of r of the field A near the AdS boundary. $A = \frac{1}{2}(r+\lambda)^2 - \partial_t\lambda + a^{(D)}\, r^{2-D} + O(r^{1-D})$

In principle $a^{(4)}$ $(t, \vec{x}\,)$, but for the symmetries present in our setup, $a^{(4)} = const$.

Inserting the holographic relation $G_N = \frac{\pi}{2} L^3/N_c^2$ (appropriate for N = 4 SYM), we get

$$T^{00} = -\frac{3}{4\pi^2} \mathrm{N}_\mathrm{c}^2\ a^{(4)}$$

- **Boundary equilibrium temperature:**

The energy density of an equilibrium, strongly coupled N = 4 SYM plasma at temperature T is given by

$$T_{eq}^{00} = \frac{3}{8} N_c^2\, \pi^2 T^4$$

Hence equilibrium temperature $T_{eq} = \frac{2^{1/4}}{\pi}\left(-a^{(4)}\right)^{1/4}$

Homogeneous Isotropization

- **stress-energy tensor:**

$T^{0i} = 0$ (because of symmetries of our setup.)

$T^{ij} = \kappa\ diag(\, b^{(4)}(t) - \frac{1}{2}\, a^{(4)}\,,\ b^{(4)}(t) - \frac{1}{2}\, a^{(4)}\ , -2\, b^{(4)}(t)\ - \frac{1}{2}\, a^{(4)}\,)$.
where $B(t,r) \sim\ b^{(4)}\,(t)\, r^{-4} + O(r^{-5})$.

- **pressure anisotropy:**

$$\delta p \equiv T_{zz} - \frac{1}{2}\left(T_{xx} + T_{yy}\right) \propto b^4(t)$$

Homogeneous Isotropization

EOM :

The 5D Einstein equations reduce (due of the present symmetries) to 1+1 dimensional partial differential equations.

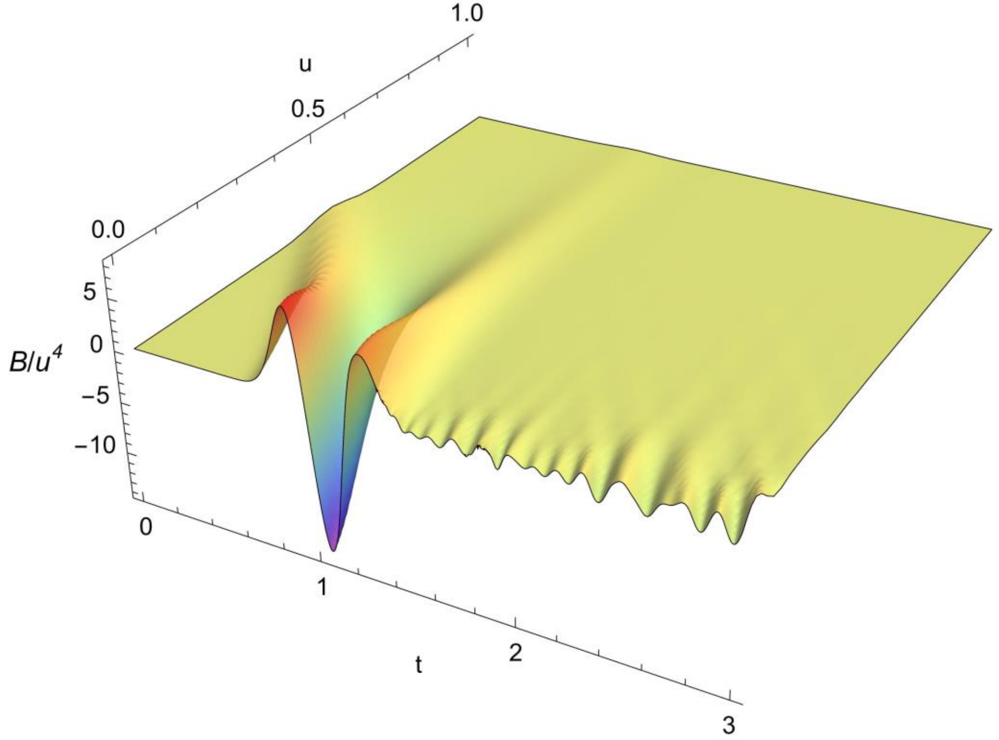
$$
\begin{aligned}
2\quad 0 &= \Sigma\,(\dot\Sigma)' + 2\Sigma'\,\dot\Sigma - 2\Sigma^2\,, &&\longrightarrow \dot\Sigma \\
3\quad 0 &= \Sigma\,(\dot B)' + \tfrac{3}{2}\left(\Sigma'\dot B + B'\,\dot\Sigma\right), &&\longrightarrow \dot B \\
4\quad 0 &= A'' + 3B'\dot B - 12\Sigma'\,\dot\Sigma/\Sigma^2 + 4\,, &&\longrightarrow A \\
0 &= \ddot\Sigma + \tfrac{1}{2}\left(\dot B^2\,\Sigma - A'\,\dot\Sigma\right), && \\
1\quad 0 &= \Sigma'' + \tfrac{1}{2}B'^2\,\Sigma\,, &&\longrightarrow \Sigma
\end{aligned}
$$

where, for any function $h(r,\tau)$,

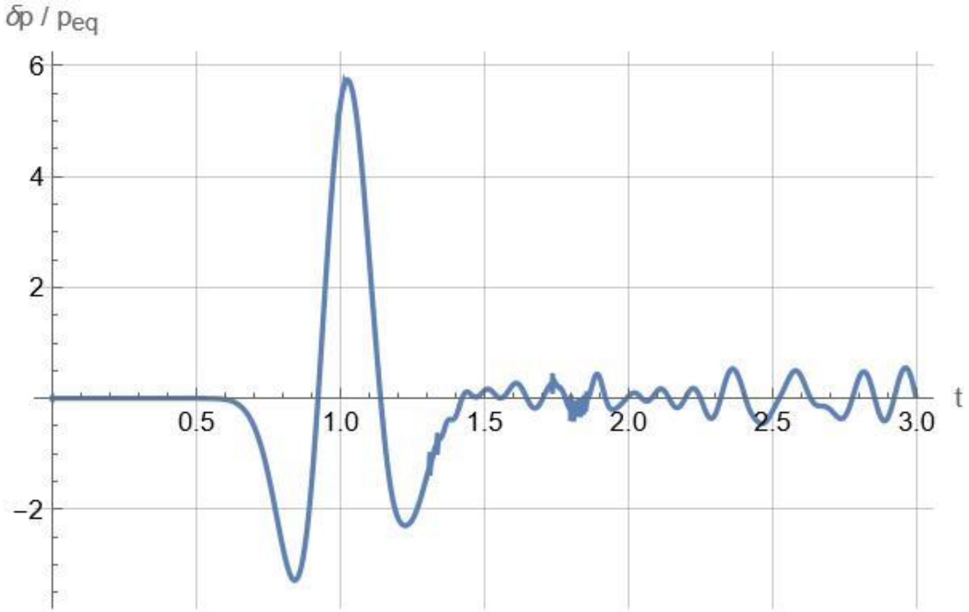
$$
h' \equiv \partial_r h, \qquad \dot h \equiv \partial_\tau h + \tfrac{1}{2}A\,\partial_r h\,.
$$

Initial conditions at $t = 0$:

- $B(t=0,u) = \beta\, u^4 \exp(-\frac{(u-u_0)^2}{2\,\omega^2})$ with $\begin{cases} \beta = 0.9 \\ u_0 = 0.5 \\ \omega = 1/20 \end{cases}$
- Initial energy density: $T^{00} = -\frac{3}{2}\kappa\, a^{(4)}$ with $a^{(4)}(t=0) = -1.5$



Normalized anisotropy function



Anitsotropy pressure δp over p_{eq}

Homogeneous Isotropization

$$p_{\mathrm{eq}} = \tfrac{1}{8} N_c^2 (\pi T)^4,$$

The bulk geometry evolves toward an isotropic equilibrium geometry, which is just the static Schwarzschild black-brane solution.

The approach to equilibrium shows exponentially damped oscillations.

The characteristic relaxation time is comparable or shorter than $1/T_{eq}$, even when the system is initially quite far from equilibrium with $\delta p/p_{eq}$ initially of O(10).

This is a basic test of the numerics; no problems with numerical instabilities, potentially preventing evolution to arbitrarily late times, are seen.

Other Examples

- In the dual gravitational description, collisions of **shock waves in SYM** turn into a problem of colliding gravitational shock waves in five dimensions.

  The corresponding gravitational problem is reduced (due to symmetries) to a 2 + 1 dimensional problem. (r-coordinate and longitudinal coordinate z .)

- **Turbulent flows in D spatial dimensions** are dual, via holography, to dynamical black hole solutions in asymptotically AdS_D+2 spacetime.

Tools to solve radial ODEs numerically

- **Discretization:** To integrate the radial ODEs numerically, one must discretize the radial and spatial coordinates. Functions are then represented as finite arrays of function values on some specified set of grid points. Estimate derivatives with suitable finite difference approximations.

- **Spectral methods:** solve the radial differential equations by converting the differential equation into a straightforward linear algebra problem.

Some techniques to increase numerical precision/performance

- Make a **change of variable** which maps the unbounded radial coordinate r to a finite interval. We just invert, and define $u \equiv 1/r$.

- **Subtractions**: it is very helpful to define subtracted functions in which the (known) leading pieces which diverge as $u \to 0$ are removed, and/or to rescale the subtracted functions by appropriate powers of u so that the resulting functions vanish linearly, or approach a constant, as $u \to 0$.

  E.g. for homogeneous isotropization:

$$a(t,u) \equiv A(t,u) - \frac{1}{2}\,\Sigma(t,u)^2$$
$$b(t,u) \equiv u^{-3}\,B(t,u)$$
$$\sigma(t,u) \equiv \Sigma(t,u) - \frac{1}{u}$$

- **Parallelization:** radial ODEs can be solved independently at each spatial point. The radial ODEs can be integrated, in parallel, using independent CPUs for each point in space.