On Differentiation of Digital Duopoly With Heterogeneity in Values

Seyoung Won

Abstract

This paper presents a Hotelling-type product differentiation model originated from with an allowance for the heterogenity in the vertical quality values, horizontal position stances and double financing scheme that are common in the duopoly of digital economies. Firms with zero marginal cost - here this is referred as digital firms because it is common in digital economy setting - have an incentive to be polarized in their stances and to hurt consumer utility in the stance space for profit maximization.

This research contributes to the literature by emphasizing the importance of possible heterogeneity in the product differentiation model as such heterogeneity could lead to an underestimation of consumer disutility. Under symmetric pricing scheme, digital firms are likely to maximum-differentiate in horizontal stance whereas minimum-differentiate in vertical quality values; firms have an incentive to sabotage consumer utility in the stance horizon if the disutility cost is too small. Heterogeneity is noteworthy because typical homogeneity assumption may underestimate the importance of such disutility cost in the stance horizon. The zero marginal cost digital industries tend to be more polarized towards the extremes in the stance compared to the analog industries, whereas they have incentive to minimize the quality difference between them.

It is particularly interesting that this research could help explain the extreme divergence of political stances of current two-party system, where the parties are treated like firms competing for consumers on a line of political stance; which gives an alternative reasoning to the traditional median voter theorem which cannot explain current political extrema.

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1 Introduction

Do firms differentiate in digital industries? If yes, is the differentiation greater than that of analog industries? In both digital and analog industries the differentiation of a product can be largely divided into two categories: horizontal and vertical differentiation. The horizontal preference differentiation space is called '*stance*' and the vertical quality differentiation space is called '*value*' throughout the paper. Whilst those two types of differentiation are usually distinguished from one to the other and classified as separate classes, preferences on the horizontal spectrum and qualities in the vertical space interact. For example in a political economics context, a consumer can choose a newspaper by his political opinion on the horizontal preference space or by the quality of a newspaper in the vertical space, or possibly by both of them. Vertical quality may be related to the opinion of the newspaper, for example, left-wing newspapers may produce a better quality news on refugees, and right-wing newspapers may have a better economics section. In such cases the vertical quality may be dependent on the stance of newspapers on a spectrum of political opinions.

Perhaps this is not only applicable to the political economics context, but also to something more general in digital industries. Digital firms try to build their own environment and ecosystems for their users, and such attempts could be understood as horizontal differentiation. For instance, Apple Music has a vertical quality of offering streaming music; the quality would include the total size of the music pool and functions for users such as downloading. A user can listen to Apple Music in iOS or Android, but the quality offered by Apple Music would be better in iOS because it offers additional function of synchronization across all Apple devices. The consumer a priori has a preference in devices and operating systems like iOS or Android before he chooses Apple Music for streaming service. This example suggests a digital firm has two differentiation spaces, horizontal environment and vertical quality of a product, and the quality offered can be dependent on the horizontal preference.

The intention of this paper is to examine how changes in the advertisement receipts and the marginal costs impact the product differentiation under pricing symmetry. The advertisement receipts and the marginal costs are chosen as parameters because they are the key differences between digital and analog industries - digital industries have lower marginal costs and more advertisement receipts due to lower replication costs and greater global accessibility (Goldfarb and Tucker, 2019).

In order to cover both horizontal and vertical differentiation, this paper uses the model with an allowance for heterogeneity in values. The model framework uses two sources of financing for firms because in digital industries firms can typically finance via two sources: the price received from the users of digital products and advertisement receipts from other firms and advertising agencies.

The pricing symmetry is justified with observations from the real world. Shiller and Waldfogel (2011) demonstrate that the pricing in the digital music industry is surprisingly uniform, due to the bundling strategies firms adopt. The pricing of digital music in 2019 has been changed from the uniform pricing to the monthly payment for streaming services. However, the main question about the uniformness of pricing has not yet been explained, according to Goldfarb and Tucker (2019), because the monthly-paying streaming service is analogous to the bundling of music. In 2019, the pricing can be said to be still uniform and symmetric, for example Apple Music and its main competitor Spotify Premium both offer music streaming services for USD$9.99[1].

With symmetric pricing, the differentiation problem will be illustrated with the well-known Hotelling model, first proposed by Hotelling (1929). However, the Hotelling model with linear disutility costs has an issue with the discontinuity of best response functions in the pricing game as d'Aspremont, Gabszewicz, and Thisse (1979) highlight and therefore with the linear disutility costs, the existence of a Nash equilibrium is not guaranteed if the locations of two firms are too close. In order to cope with this problem, quadratic disutility costs[2] are used in this paper. Quadratic disutility costs lead firms to locate at the extremes of the Hotelling line whereas linear disutility costs lead firms to locate at the center of the line, because with quadratic disutility costs the market share effect is dominated by the strategic effects of firms' actions (Tirole, 1988).

To capture symmetry in the strategic interactions between firms, the model of this paper assumes all consumers have a quadratic disutility cost. This is because the Hotelling model with a mix of the quadratic and linear disutility cost may result in the non-existence of a Nash equilibrium again if the consumers with a linear disutility cost exceed the half

[1]Apple Music and Spotify have identical pricing schemes: $9.99 for 'Individual' plan, $4.99 for 'Student' plan and $14.99 for 'Family' plan. Please see the webpage of Apple and Spotify [Online; accessed 20-OCT-2019]

[2]Discontinuity of best response price function can be solved adopting quadratic disutility (transportation) cost, see d'Aspremont et al. (1979).

of the total population (Egli, 2007). This also guarantees the disutility cost to be convex enough to have maximal differentiation at the extremes of the Hotelling line (Economides, 1986). Within the Hotelling model framework, De Palma, Ginsburgh, Papageorgiou, and Thisse (1985) study heterogeneity problems on the Hotelling line and they have discovered that the principle of minimum differentiation holds if a sufficient level of heterogeneity is provided. This result, however, is dependent on the assumption that firms cannot determine a priori differences in consumer preferences. This assumption may not hold in digital industries because lowered tracking costs are a typical characteristic of digital economics and it enables one to see the differences in consumer preferences in a way that traditional industries could not (Goldfarb and Tucker, 2019). Gabszewicz and Thisse (1979) discuss vertical product differentiation within the Hotelling model framework, but it is not a good fit for examining digital economies as the price is the only financing source for firms.

Considering the characteristics of digital industries, I follow the setup of Gabszewicz, Laussel, and Sonnac (2001) and Gabszewicz, Laussel, and Sonnac (2002) for the model framework. The model proposed by them has a merit in providing a framework for two financing sources of the price and the advertisement receipts, although the original model proposed by them does not incorporate vertical product differentiation. With additional valuation terms in the vertical quality space, the model framework can cover both horizontal and vertical differentiation. Throughout the model analysis, I concentrate on symmetric pricing cases because the pricing symmetry is one of observed characteristics of digital industries as in the cases of Apple Music and Spotify.

The model framework of this paper consists of a four-stage sequential game due to addition of variables for the valuation terms, whereas the original model only has three stages. I show that allowing for heterogeneity gives insights for economic phenomena: 1) *When firms maximum-differentiate, firms will try to make a market to have a certain level of disutility from the distance in the stance*; 2) *homogeneity assumption has a caveat of underestimating the power of disutility cost of consumers*; and 3) *if advertisement receipt increases and/or marginal cost decreases, the firms' stances tend to diverge to the extremes.*

This paper contributes to the existing literature by proposing the Hotelling model with two financing sources that covers both horizontal and vertical differentiation. While the findings are not general assertions due to the specific functional form used in the model, this paper has contributions in supporting arguments like online newspapers are more politically extreme than traditional paperback newspapers or digital firms have an incentive to build their own digital ecosystem, not only because of the direct payment from consumers but also because imposing a certain level of disutility to consumers benefits the firms. The model also suggests the naive assumption of homogeneity can lead to the misunderstanding of the consumer disutility costs.

The rest of this paper is organized as follows. Section 2 introduces the model setup of the heterogeneous values in detail. Section 3 derives the subgame perfect Nash equilibria and implications associated with the model, focusing on symmetric price setting. Section 4 then introduces the comparison between the model with the allowance for heterogeneity and the model with the homogeneity assumption proposed by Gabszewicz et al. (2001) and Gabszewicz et al. (2002). Section 5 discusses the findings and the contributions from the model. Section 6 states the limitation of the model and possible future research. Section 7 finally concludes.

2 Model

Before I discuss the proper setup of the model, as an overview, consider a simple example of a market with uniformly distributed consumers on $[0, 1]$ and two firms can choose each of their locations in between. With a help of Figure 1, suppose one firm chooses the location A and the other firm chooses the location B. On this spectrum, A is located to the left of B and choices are made simultaneously (Case 1). Suppose another case, where the first firm chooses the location A' which is on the right side to the second firm's location B (Case 2). Because the locations are chosen simultaneously and each firm can only choose a single location, Case 2 is topologically indifferent to the case which one firm have the location A left to the location of the other firm B (Case 3).

Figure 1: Assumption on the locations

In terms of the market, Case 2 and Case 3 of Figure 1 is the same with the firms swapped in positions. Firm B in Case 2 plays a role of Firm A in Case 3 and vice versa. Therefore, without loss of generality, I assume that Firm A is on the left side of Firm B.

Another problem to consider is the coverage of the market when the value terms are entered. If a consumer is incentive compatible, then he would not buy a product if his utility is negative. In other words, he would not buy when the value of a product does not exceed the total cost he pays. If the value is not sufficiently large, the market may not be fully covered and consequently the market with two firms may face local monopolies of two firms. Figure 2 is a graphical illustration of this problem. Suppose the locations of Firm 1 and Firm 2 are a and $1 - b$ respectively. Firm 1 offers the value of v_1 and Firm 2 offers the value of v_2. If the values v_1 and v_2 are not sufficiently large, then there are two marginal consumers[3] for each firm whose utilities are zero and the market is not fully covered. In such case, there is no strategic interaction between the firms as they

[3]There are two marginal consumers on the left ($\underline{x}$) and the right ($\bar{x}$) of each firm

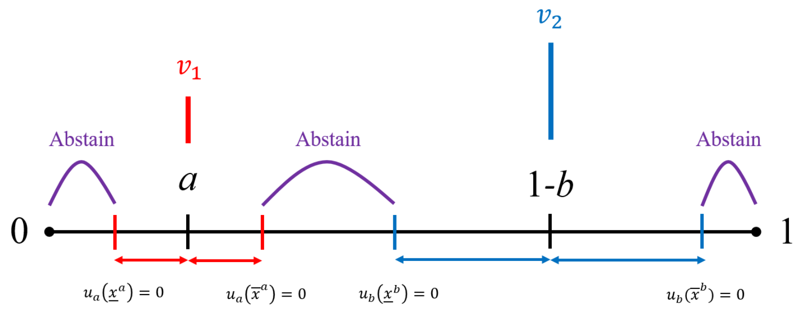

Figure 2: Local Monopolies of two firms

do not meet each other and therefore there is no competition going on in each of the isolated local monopoly spaces. Such existence of local monopolies of two firms in digital space is counter-intuitive.[4] Therefore I suppose that the market is fully covered, so there always exists a marginal consumer on the boundary of two firms' demand coverage who is indifferent between purchasing from any of the two firms. This is a stronger assumption than the incentive compatibility condition, but not unrealistic, especially in the political economics context where everyone has to vote[5] and therefore need information for voting.

The following setup is borrowed from Gabszewicz et al. (2001) and Gabszewicz et al. (2002), with a modification of the allowance for heterogeneity in values of each of firms' product. Each of two firms produces products at a constant marginal cost per product, $c \geq 0$. The product purchase choices of consumers are mutually exclusive. Those consumers are uniformly distributed in the market space, and they have variance in their stances from the left extreme to the right extreme[6]. This diversification of stances are normalized to a unit interval of $[0, 1]$ with a uniform distribution where 0 represents the far-left extreme and 1 represents the far-right extreme, and the hierarchy of intermediate stances between the far-left and the far-right can be set in order within an interval of $[0, 1]$. At any given point in the stance interval $[0, 1]$ there exists a consumer whose ideal stance is the given point. At the ideal point, the consumer has a disutility of zero and further away from the point, the consumer suffers disutility from not having an ideal

[4]Digital industries exhibit a great advantage in catching the niche markets, represented as the long tail effects, and 'local' monopoly rarely happens (Goldfarb and Tucker, 2019).

[5]For example, voting is compulsory in the Australian federal election

[6]This can be understood in terms of political opinion in choosing a newspaper or preference in the operating system of a product.

product. To be specific, this disutility is measured by the quadratic function

$$tx_i^2 + p_i \tag{1}$$

where x denotes the location of a consumer i in the stance interval $[0, 1]$ whose ideal stance point is $x \in [0, 1]$. The term p_i represents the price of product i where $i \in \{1, 2\}$ as there are two firms, Firm 1 and Firm 2. The term t denotes a measure of intensity for a certain stance in the interval, where this parameter is associated with the cost of disutility of a consumer for not having an ideal product. Such disutility changes as the square of the distance between the ideal product in stance space and the chosen firm's stance location. The locations of Firm 1 and Firm 2 in the stance interval are represented as a and $1 - b$ respectively, where the distance of Firm 1 from the far-left extreme 0 is a and the distance of Firm 2 from the far-right extreme 1 is b. The location of Firm 1 is assumed to be on the left side to Firm 2 without loss of generality, which means $a + b \leq 1 \iff 1 - a - b \geq 0$. Firm 1 offers products with the value of v_1 and Firm 2 offers products with the value of v_2. Inside the stance interval of $[0, 1]$, there exists a marginal consumer $\tilde{x}$ who is indifferent between choosing a product from Firm 1 and from Firm 2 as portrayed in Figure 3. This marginal consumer represents the capstone between the consumers buying from Firm 1 and from Firm 2, that is, demands for each of firms. Demand is not only subject to location choices but also dependent on the values and prices of products offered by each firm. Those locations, values and prices are determined by firms in an orderly manner, so firms first decide the location on the stance spectrum, then determine the values they offer and set pricing.

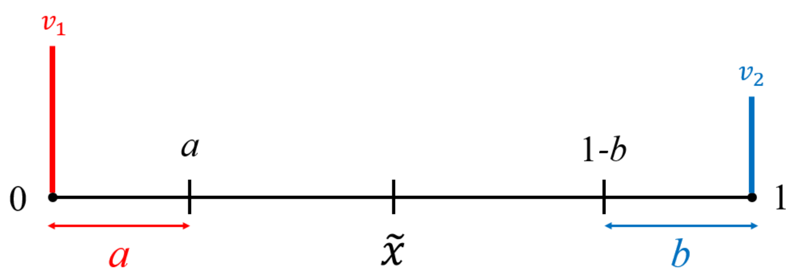

Figure 3: The Setup of the Hotelling model

Suppose once the demand for each firm is known, the advertisement slots in the product is purchased from outside advertisement agencies. This could be understood as putting advertising banners in online newspaper or on digital platforms where newspapers or platforms themselves are the products of firms. The fees of those slots in Firm 1 and Firm 2 purchased by advertisement agencies are denoted as s_1 and s_2 respectively. The advertisement agencies have a willingness to pay for advertisement in the slots of Firm 1 and Firm 2, represented by a constant parameter $4k$, $k \in [0, 0.25]$ so $4k \in [0, 1]$. This willingness to pay k represents the advertisement agencies are intending to purchase the advertising banner of a firm; it could be understood as a parameter for advertisement revenue: the larger k value is, the larger unit advertisement revenue is. I assume this willingness to pay increases as there are more potential consumers demanding a product of firm: the larger the population of consumers of Firm i, the higher the willingness to pay for advertisement where $i \in \{1, 2\}$. Therefore the willingness to pay $4k$ is multiplied by the number of consumer population of Firm i and together with the prices for advertisement s_i, the utilities of an advertisement agency with a willingness to pay of $4k$ is represented by

$$(i)\ n_1 \times 4k - s_1, \quad (ii)\ n_2 \times 4k - s_2, \quad (iii)\ n_1 \times 4k - s_1 + n_2 \times 4k - s_2 \tag{2}$$

where n_1 and n_2 represent the number of consumers buying from Firm 1 and Firm 2 respectively, and I suppose the advertising agency can purchase advertising slots (i) from Firm 1 only, (ii) from Firm 2 only or (iii) from both Firm 1 and Firm 2 and these three utility functions correspond to the utilities of the advertising agency in each of the three cases.

The following section covers the four-stage game played by firms. In the first stage, each of the firms chooses their locations in the stance interval, a for Firm 1 and $1 - b$ for Firm 2. In the second stage, firms select the values of their product v_1 and v_2 of Firm 1 and 2 respectively. In the third stage, they set prices p_1 and p_2 for Firm 1 and 2 respectively. Finally, in the fourth stage, they choose the advertisement fees s_1 and s_2 to be charged from outside advertisement agencies.

3 Subgame Perfect Nash Equilibria

3.1 The Advertisement and Price Stage Game

As mentioned previously, the strategies of Firm 1 and Firm 2 in this fourth stage game is to charge advertisement fees s_1 and s_2 respectively. The stance locations a and

$1-b$ have already been chosen in the first stage while values $v_1(a,b)$ and $v_2(a,b)$ have been chosen in the second stage. After second stage, prices of firms $p_1(a,b,v_1,v_2)$ and $p_2(a,b,v_1,v_2)$ have been chosen in the third stage. To those prices correspond demands $n_1(a,b,v_1,v_2,p_1,p_2)$ and $n_2(a,b,v_1,v_2,p_1,p_2)$ for Firm 1 and Firm 2 respectively.

With those corresponding demands $n_1(a,b,v_1,v_2,p_1,p_2)$ and $n_2(a,b,v_1,v_2,p_1,p_2)$, the procedure of the last stage game follows that of Gabszewicz et al. (2001) and Gabszewicz et al. (2002). It results in the equilibrium advertisement payout $R_i(s_i^*), i \in \{1,2\}$ for $k \in [0, 0.25]$:

$$R_i(s_1^*, s_2^*) = kn_i \tag{3}$$

The derivation is in **Appendix A.1**. This formula suggests that each firm gets advertisement receipts proportional to the number of consumers it serves. In the next game those equilibrium advertisement payouts are known to two firms and those payouts kn_i are taken into consideration of profit maximization of two firms.

Now consider the third-stage game, where firms choose the prices $p_1(a,b,v_1,v_2)$ and $p_2(a,b,v_1,v_2)$. Through the outcomes of the first and second stage games, consumers realize what stances and values the firms offer, and with such information they make their choices. Among those choices of consumers, there exists a marginal consumer $\tilde{x}$ whose choice is indifferent between choosing Firm 1 or Firm 2. This means for a marginal consumer, for $i \in \{1,2\}$, the value from Firm i subtracted by the sum of the price of a product from Firm i and the quadratic disutility cost associated with choosing Firm i's product from Equation (1) should be equivalent, which is represented by

$$v_1 - t(\tilde{x} - a)^2 - p_1 = v_2 - t(1 - b - \tilde{x})^2 - p_2$$

This equation can be rearranged into a function of the marginal consumer:

$$\tilde{x} = a + \frac{(p_2 - v_2) - (p_1 - v_1)}{2t(1 - a - b)} + \frac{1 - a - b}{2}$$

The proof is in **Appendix A.2**. From Figure 3 it is visible that $\tilde{x}$ represents the consumers buying from Firm 1 and $1 - \tilde{x}$ represents the consumers buying from Firm 2. These demands for Firm 1 and Firm 2 are now denoted as $n_1(a,b,p_1,p_2,v_1,v_2)$ and $n_2(a,b,p_1,p_2,v_1,v_2)$ respectively.

I construct the corresponding profit functions:

$$\Pi_1 = (p_1 - c) * n_1 + k * n_1 - \frac{v_1^2}{2} \quad \text{and} \quad \Pi_2 = (p_2 - c) * n_1 + k * n_1 - \frac{v_2^2}{2}$$

where p_1 and p_2 denote the prices of Firm 1 and Firm 2, c denotes the constant marginal cost and k is the advertisement payout received by outside advertisement agencies from Equation (3). There exist value production costs for each of two firms, assumed strictly decreasing and strictly convex, which are represented by the explicit functional form of $v_i^2/2, i \in \{1, 2\}$. Namely, if the value of a product increases there is an associated increase in the production cost for the value. This value production cost is assumed to be linearly dependent on the price, so $\partial v_i^2/\partial p_i = 2v_i$. Hence the profit maximization problem for Firm $i, i \in \{1, 2\}$ is

$$\max \ \Pi_i = (p_i - c + k) * n_i - \frac{v_i^2}{2} \tag{4}$$

By taking the first order conditions with respect to prices, two firms determine the best pricing strategies as a function of their own and the other's choices of the price, value and stance. The functional form of the first order conditions are:

$$\frac{\partial}{\partial p_1}\Pi_1 = 0 = a + \frac{v_1 - 2p_1 - v_2 + p_2 + c - k}{2t(1-a-b)} + \frac{1-a-b}{2} - v_1$$

$$\frac{\partial}{\partial p_2}\Pi_2 = 0 = b + \frac{v_2 - 2p_2 - v_1 + p_1 + c - k}{2t(1-a-b)} + \frac{1-a-b}{2} - v_2$$

Simultaneously solving these first order conditions give the best response functions of prices with respect to own and the other's price, value and stance:

$$p_1 = \max\{0, \frac{1}{2}[p_2 + c - k + v_1 - v_2 + t(1-a-b)(1+a-b-2v_1)]\} \tag{5}$$

$$p_2 = \max\{0, \frac{1}{2}[p_1 + c - k + v_2 - v_1 + t(1-a-b)(1+b-a-2v_2)]\} \tag{6}$$

Remark 1. *If $v_1 = v_2 = 0$ the best response price functions and profit maximization problem reduce to the original problem of Gabszewicz et al. (2001) and Gabszewicz et al. (2002).*

These best response functions show that the price cannot be negative. I focus on symmetric pricing cases of two firms, which are both firms choose either positive prices

(henceforth the *positive price case*) or prices of zero (henceforth the *zero price case*).
In the *zero price case* the prices of Firm 1 and Firm 2 are simply 0.
In the *positive price case* the prices of Firm 1 and Firm 2 can be solved simultaneously, and the functional forms of equilibrium prices are:

$$p_1^*(v_1, v_2, a, b) = c - k + \frac{1}{3}(v_1 - v_2) + t(1 - a - b)(1 + \frac{a - b - 4v_1 - 2v_2}{3}) \tag{7}$$

$$p_2^*(v_1, v_2, a, b) = c - k + \frac{1}{3}(v_2 - v_1) + t(1 - a - b)(1 + \frac{b - a - 4v_2 - 2v_1}{3}) \tag{8}$$

Equations (7) and (8) suggest (see **Appendix A.3** for the proof):

Lemma 1. *If prices are both positive, in order to have a situation where an increase in values can lead to an increase in prices, the firms require* $1/4 \geq t$ *when* $(a, b) = (0, 0)$.

3.2 The Value Stage Game

This section considers the *positive price case* first. In Section 3.1, the best response functions of prices are determined to be Equations (5) and (6). Therefore firms take this account for their choices of value, and consequently the value can be represented solely by the stances a and $1 - b$ of two firms, by solving the profit maximization problem of firms from Equation (4) with respect to their values. Simultaneously solving the first order conditions of the endogenized profit functions with respect to values results in the best response value functions $v_1(a, b)$ and $v_2(a, b)$. The full functional forms and derivations are in **Appendix B**. The derivations also imply:

Lemma 2. *If firms are located at each of the extremes of the stance spectrum, there exists a threshold* $\hat{t} = 1$ *which makes the best response choice of values to be zero for both firms.*

The proof of **Lemma 2** is in **Appendix B.1**. Figure 4 is a graphical illustration of **Lemma 2**. If firms are located at each of the extremes, i.e. $(a, b) = (0, 0)$, then the equilibrium value functions $v_1^*(a, b)$ and $v_2^*(a, b)$ have the same functional form of:

$$v_1^*(0, 0) = v_2^*(0, 0) = \frac{1 - 6t + 5t^3}{4 - 18t - 30t^2 - 10t^3} \tag{9}$$

which is a function decreasing in t. I assume that firms would rather produce values of zero than offer a product of negative value. Then the minimum of those value terms are

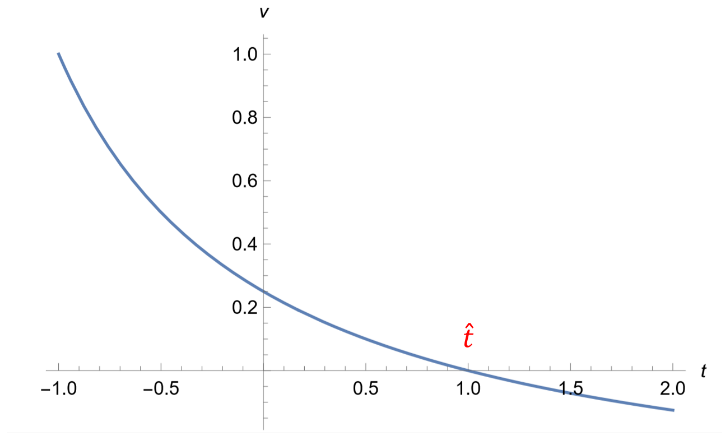

Figure 4: Change in values with respect to the change in disutility costs t

constrained at zero. Therefore as illustrated in Figure 4, there is a threshold consumer disutility cost $\hat{t} = 1$ where firms start to offer the zero value product.

Lemma 2 in the *positive price case* implies three messages: 1) if prices are both positive and if firms are located at the extremes, the values which two firms each offer become identical through strategic interactions; 2) the value offered to customers decreases as the consumer disutility cost of the market t increases; and 3) at a certain level of disutility cost $\hat{t}$, firms would rather choose to offer zero-value products.

Now consider the *zero price case*. If both prices are zero then the profit functions are:

$$\Pi_1(v_1, v_2, a, b) = (k - c) * n_1 - \frac{v_1^2}{2} \quad \text{and} \quad \Pi_2(v_1, v_2, a, b) = (k - c) * n_2 - \frac{v_2^2}{2}$$

The corresponding first order conditions are:

$$\frac{\partial}{\partial v_1}\Pi_1 = 0 = (k - c)\frac{1}{2t(1 - a - b)} - v_1 \quad \text{and} \quad \frac{\partial}{\partial v_2}\Pi_2 = 0 = (k - c)\frac{1}{2t(1 - a - b)} - v_2$$

The first order conditions give the optimal values that firms offer in the *zero price case*:

$$v_1^* = v_2^* = \frac{k - c}{2t(1 - a - b)}$$

which suggests that in the *zero price case*, the values offered are positively dependent on the advertisement receipts less marginal cost $(k - c)$. This suggests:

Remark 2. *If prices are both zero, the best response choices of values are zero for both firms if $k = c$ or t is infinitely high.*

3.3 The Stance Stage Game

This section begins with the *positive price case* first. In **Section 3.2** the value terms are endogenized to $v_1(a, b)$ and $v_2(a, b)$. Therefore, substituting the equilibrium value functions $v_1^*(a, b)$ and $v_2^*(a, b)$ into the endogenized profit functions $\Pi_1(v_1, v_2, a, b)$ and $\Pi_2(v_1, v_2, a, b)$ in **Appendix B** brings the equilibrium profit functions to be optimized in terms of the stance locations, $\Pi_1^*(a, b)$ and $\Pi_2^*(a, b)$. These profit functions are solely in terms of the stance parameters a and b and there is a unique Nash equilibrium $(a, b) = (0, 0)$ because $\partial\Pi_1^*/\partial a > 0$ and $\partial\Pi_2^*/\partial b > 0$. This provides a proposition:

Proposition 1. *If prices are positive, both firms locate at the extremes* $(a, b) = (0, 0)$ *and their profits are dependent on the disutility cost* t *where there exists a very small threshold* $\tilde{t}$ *which makes the profits of both firms transitions from negative to positive.*

The proof is in **Appendix C.1**. Figure 5 below is a visualization of **Proposition 1**. The graph shows that the profits are an increasing function of consumer disutility cost t

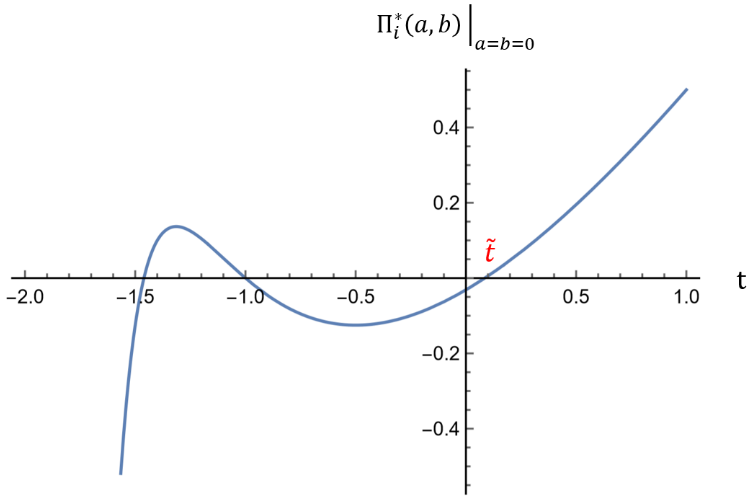

Figure 5: Unique positive threshold disutility cost t

when $(a, b) = (0, 0)$. The threshold $\tilde{t} = \frac{1}{16}(3\sqrt{17} - 11) \approx 0.08558$. Firms are located at the extremes of the stance horizon because the only Nash equilibrium is $(a, b) = (0, 0)$. In this equilibrium, firms have an identical best response value function represented in Equation (9) and consequently an identical equilibrium profit function via their strategic interactions. The equilibrium value function v_i^* is a decreasing function of the disutility cost t as stated in **Lemma 2**. The value function v_i is assumed to be non-negative for both firms, thus if the disutility cost $t \geq 1$ then both firms offer a value of zero. When the value offered is 0, the profit functions of firms are strictly decreasing in their choices in the stances a and b and strictly increasing functions of the consumer disutility cost t (see **Appendix C.2** for the proof).

Proposition 1 claims that if prices are positive then firms maximum-differentiate horizontally. Moreover, it suggests that there is an incentive for firms to harm consumer utility for profit maximization purposes because the profit is positively dependent on the consumer disutility cost t. If the consumer disutility cost for the horizontal stance is too small, firms can make a loss instead of profit. If t is too small then consumers are much likely to be indifferent between two firms, thus intensifying the competition between the firms. As a result of this intense competition, firms may end up getting losses. In contrast, if t is large enough then the competition becomes relatively mild and firms can enjoy more profits.

Now consider the *zero price case.* Plugging the endogenized values $v_1(a,b)$ and $v_2(a,b)$ into the firms' profit functions where prices are both zero gives the profit functions of the firms:

$$\Pi_1^* = (k-c)\left(\frac{1+a-b}{2}\right) - \frac{1}{2}\left(\frac{k-c}{2t(1-a-b)}\right)^2$$

$$\Pi_2^* = (k-c)\left(\frac{1+b-a}{2}\right) - \frac{1}{2}\left(\frac{k-c}{2t(1-a-b)}\right)^2$$

And their corresponding first order conditions with respect to their stances:

$$\frac{\partial}{\partial a}\Pi_1 = \frac{\partial}{\partial b}\Pi_2 = \frac{1}{2}(k-c) - \frac{(k-c)^2}{4(1-a-b)^3t^2} \tag{10}$$

This implies:

Lemma 3. *If prices are both zero, the principle of minimum differentiation does not hold. The higher $(k-c)$ is, the more firms horizontally differentiate. $(k-c) > 0$ is not a sufficient condition for strictly positive $\partial\Pi_1/\partial a$ and $\partial\Pi_2/\partial b$.*

The proof is in **Appendix C.3**. Lemma 3 states that if the advertisement receipts increases and/or the marginal cost decreases, then firms in the *zero price case* are more likely to differentiate horizontally.

4 Model Comparison

In this section I compare the original homogeneous value model of Gabszewicz et al. (2001) and Gabszewicz et al. (2002) with the heterogeneous value model I have presented. Even if the model allows for heterogeneity in values, as stated in **Remark 1**, the best response functions of prices (Equation (5) and (6)) reduce to:

$$p_1 = \max\{0, \frac{1}{2}[p_2 + c - k + t(1-a-b)(1+a-b)]\}$$

$$p_2 = \max\{0, \frac{1}{2}[p_1 + c - k + t(1-a-b)(1+b-a)]\}$$

if $v_1 = v_2 = 0$. If the values of the products of Firm 1 and Firm 2 are exogenous, that is, the values are not subject to the stances of firms, it simply means that firms are producing zero quality products. But, if the values are endogenous, the stances of firms determine $v_1(a,b)$ and $v_2(a,b)$. This then implies a condition of which the values need to be determined.

In the *positive price case*, the equilibrium values being zero means that $t \geq 1$ according to **Lemma 2**. However, the condition for the disutility cost t is different in the stance equilibria from the original homogeneous model of Gabszewicz et al. (2001).

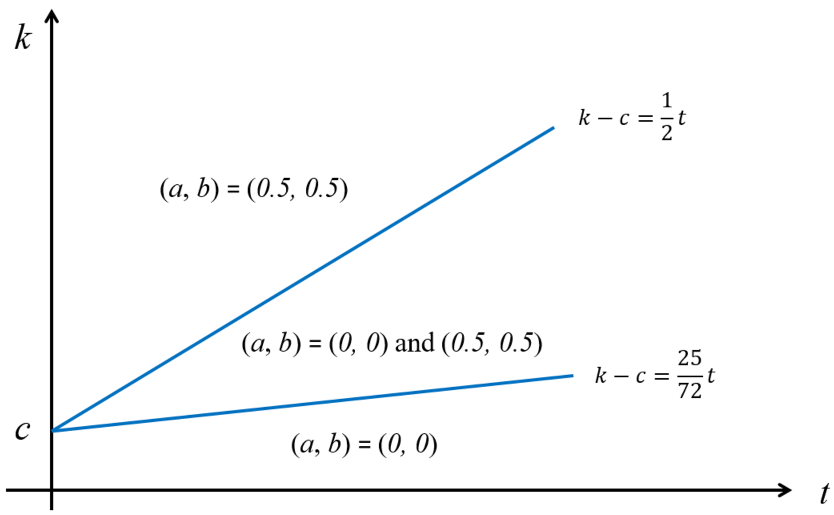

Figure 6: Stance Equilibria from the Homogeneous model

Figure 6 shows the stance equilibria from the homogeneous model of Gabszewicz et al. (2001) and Gabszewicz et al. (2002). In the *positive price case* of the homogeneous model, it is necessary to have $k - c < \frac{25}{72}t \iff \frac{72}{25}(k-c) < t$ because the Nash equilibrium of the *positive price case* is strictly $(a, b) = (0, 0)$ in the homogeneous model, as $\partial\Pi_1/\partial a$ and $\partial\Pi_2/\partial b$ are strictly decreasing.[7]

On the other hand, in the *positive price case* of the heterogeneous model, it requires $1 \leq t$ to have zero values for both firms' products according to **Lemma 2**. As represented in **Remark 1** the heterogeneous model reduces to the homogeneous model if $v_1 = v_2 = 0$. By construction, $k \in [0, 0.25]$ and $c \geq 0$ and therefore the maximum value of $(k - c)$ is 0.25. This limit in $k - c$ implies that the maximum sufficient condition for the divergence of firms suggested by the homogeneous condition is $\frac{72}{25} \times 0.25 < t \iff 0.72 < t$. This suggests:

Proposition 2. *The homogeneous model may underestimate the disutility cost of consumers and consequently the values need to be offered.*

The proof is in **Appendix D.1**. The difference between the conditions on t comes from allowing heterogeneity. Typically the homogeneous quality Hotelling model assumes that the values are sufficient enough to have a marginal consumer. However, if it forces the marginal consumer to exist, then firms would rationally choose to offer values as minimal as possible for profit maximization purposes, if the production of value is costly. Allowance for heterogeneity extends the model to quantify what the minimal possible value would be. Moreover, the homogeneous model differs from the heterogeneous model because the quality of products that firms offer is exogenously fixed to be the same. By contrast, in the heterogeneous model, even if the firms end up offering the same quality products, such result is due to the strategic interactions between two firms. **Proposition 2** states the homogeneous model may underestimate the power of consumer disutility costs t, and such underestimation may harm the profit maximization of a firm.

In the *zero price case*, the equilibrium values being zero means that $k = c$ according to **Remark 2** and consequently the profits of both firms are zero as well. This is intuitive because if the prices are zero, the only source of financing is from the advertisement receipts and therefore the values offered depend on the advertisement k and the marginal cost c. In the *positive price case*, having zero values is independent of $(k - c)$ according to **Lemma 2**. The conditions on the term $(k - c)$ suggest:

[7]For more details about the homogeneous model please see Gabszewicz et al. (2001).

Proposition 3. *Under pricing symmetry, the higher* $(k - c) \geq 0$ *is, the more firms horizontally differentiate.*

The proof is in **Appendix D.2**. **Proposition 3** suggests that higher advertisement receipts k and/or decreases in the marginal cost c induce firms to horizontally differentiate more.

One example of **Proposition 3** is the newspaper market. In the case where both online and physical newspapers are sold for free, online newspapers can gather more advertisement receipts than traditional physical newspapers because online newspapers can be seen across the world whereas physical copies are more difficult to disseminate. Moreover, online newspapers have lower marginal costs than their physical counterparts due to the characteristics of the digital industry - namely, the replication cost is nearly zero for digital copies. The horizontal stances represent the political opinions of newspapers in terms of duopoly and my model implies that online newspapers have more extreme political stances than traditional paperbacks.

There is an empirical support for the argument because it is consistent with the research done by Nie, Miller, Golde, Butler, and Winneg (2010). Since the 1980s the cheaper production costs enable cable news networks to rely on smaller audiences whose positions are further from the center, or whose political interests lie in more niche issues. Internet news providers with even lower marginal costs than cable network service providers occupy the market for non-centrist political news. This is largely compatible with the argument from **Proposition 3**, especially considering that the cable network or internet news Nie et al. (2010) examined are typically sold at a price of zero - either from bundling, or because of the nature of the digital market.

5 Discussion

This paper contributes to the existing literature through the introduction of heterogeneity along the Hotelling line and an interpretation of the product differentiation under the digital duopoly. The heterogeneous Hotelling model framework provides three propositions related to the digital industries.

Firstly, **Proposition 1** asserts the principle of maximum differentiation in the *positive price case*, and the incentive of the firms to harm consumer utility for profits. The firm profits are positively dependent on the consumer disutility cost t and if $t < \tilde{t}$, that is, if the consumer disutility cost is too low, then the firms have losses. Firms have a clear motivation to increase the disutility cost t if possible. Secondly, **Proposition 2** claims the homogeneity assumption may lead to the underestimation of the consumer disutility costs, which no previous literature on the Hotelling model has pointed out. Thirdly, **Proposition 3** provides a supporting argument for the polarization on the stance horizon with respect to the increase in the advertisement receipts and the decrease in the marginal costs. This is typically important in digital industries because the increase in the advertisement receipts and the decrease in the marginal cost are the major characteristics which distinguish the digital industries from the other traditional industries (Goldfarb and Tucker, 2019). This suggests, although not strictly true in the *zero price case*, the firms are likely to maximum-differentiate, especially in the digital industries.

The model also suggests the firms minimum-differentiate vertically. Through the strategic interactions of firms, in both of the *positive case* and the *zero price case*, firms end up having an identical value function as demonstrated in **Section 3.2**. This makes sense intuitively because as firms tend to maximum-differentiate horizontally, the firms want to soften the competition and consequently both firms provide the identical value to consumers.

6 Future Research

The model presented in this paper relies on three critical assumptions: consumer distribution, the fully covered market and the value-production cost function $v_i^2/2$. Firstly, this paper assumes uniformly distributed consumers and changes in the distribution may result in significantly different predictions. For example, the Subgame Perfect Nash Equilibrium this paper discusses in **Section 3** may be different from the maximal differentiation if the consumer distribution is a normal distribution, or a skewed distribution. If consumers are normally distributed for instance, the incentive to locate in the center increases as there is a greater population around the mean. Secondly, the assumption of the full market coverage enables the comparison between this model and the original homogeneous value model from Gabszewicz et al. (2002), but as previously noted in the overview section, this is a much stronger assumption than the incentive compatibility. Depending on the market and the context apart from political economics, the market may not be fully covered and consequently there may be no marginal consumer. Thirdly, the profit function is built upon the value production cost function $v_i^2/2$ but such costs may be different in functional form. The value production costs may be too high so the penalty for increasing values is too heavy, or perhaps too low. Lastly, the model framework of this paper is based on the Hotelling model, but one can consider a framework under differentiated Bertrand model to examine the problems this paper discusses.

The original paper of Gabszewicz et al. (2001) and Gabszewicz et al. (2002) provide a Subgame Perfect Nash Equilibria and the necessary conditions of k, c, and t to solve the general equilibrium problem. In this paper, only symmetric pricing cases are considered so it fails to obtain such a general equilibrium condition. Therefore possible future works would be to examine asymmetric pricing cases to find the general equilibrium solution. Empirical examination of **Proposition 2** is another possible future research, if the assumed homogeneity actually leads to an underestimation of consumers' disutility costs and consequently misleads the product quality determination in the real world.

7 Conclusion

I have introduced heterogeneity in values into the Hotelling model with a four-staged sequential game where there are two firms with two financing sources from the advertisement and price receipts. This allowance for heterogeneity under symmetric pricing situations provides visible economic phenomena and insights. Firstly, firms have an incentive to damage consumer utility in the stance horizon up to a certain level if the cost for disutility is too small. Secondly, allowing for heterogeneity reveals that the homogeneity assumption may underestimate the importance of such disutility cost in the stance line. Thirdly, digital industries tend to be more polarized towards the extremes in the stance compared to analog industries. The firms are likely to maximum-differentiate horizontally, whereas minimum-differentiate vertically.

These implications may fail if the assumptions are not matched to actual cases. For example, a non-uniform distribution of consumers may result in minimal differentiation of digital firms in the stance horizon if the center contains a sufficiently higher concentration of the population and therefore motivating firms to move inward. The market may not be fully covered and the cost function may be different to what the model is established upon. This paper does not serve as a general model for product differentiation in digital industries as the model framework has a specific functional form and is dependent on multiple assumptions, but it captures the gist of economic phenomena about the differentiation in digital industries.

The model suggests that even if the end-result is to provide symmetric, and therefore homogeneous value, the journey to the end still matters. The product characteristic of homogeneity or heterogeneity is a serious issue as it indirectly measures the significance of consumer disutility costs in horizontal differentiation. This paper contributes to the supporting of arguments for the opinion polarization in digital industries and firms' potential incentive to harm consumer utility in favor of greater profits.

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A Appendix A

A.1 Appendix A.1

I suppose $u(0)$, $u(1)$, $u(2)$, and $u(1,2)$ to be the utilities of advertisement agencies obtained by buying advertising slots from none of the firms, from only Firm 1, from only Firm 2, and from both Firm 1 and Firm 2 respectively. Without loss of generality, assume $n_2 \geq n_1$ and from Equation (2) those utilities give the following:

$$(i) \quad u(1) \geq u(0) \Rightarrow n_1 \times 4k - s_1 \geq 0 \iff n_1 \times 4k \geq s_1 \iff 4k \geq \frac{s_1}{n_1}$$

$$(ii) \quad u(2) \geq u(0) \Rightarrow n_2 \times 4k - s_2 \geq 0 \iff n_2 \times 4k \geq s_2 \iff 4k \geq \frac{s_2}{n_2}$$

$$(iii) \quad u(2) \geq u(1) \Rightarrow n_2 \times 4k - s_2 \geq n_1 \times 4k - s_1 \iff \frac{s_2 - s_1}{n_2 - n_1}$$

$$(iv) \quad u(1,2) \geq u(2) \Rightarrow n_1 \times 4k - s_1 + n_2 \times 4k - s_2 \geq n_1 \times 4k - s_1 \iff 4k \geq \frac{s_2}{n_2}$$

$$(v) \quad u(1,2) \geq u(1) \Rightarrow n_1 \times 4k - s_1 + n_2 \times 4k - s_2 \geq n_2 \times 4k - s_2 \iff 4k \geq \frac{s_1}{n_1}$$

The possible resulting cases[8] are either of

$$\frac{s_1}{n_1} \leq \frac{s_2}{n_2} \leq \frac{s_2 - s_1}{n_2 - n_1} \quad \text{or} \quad \frac{s_2 - s_1}{n_2 - n_1} \leq \frac{s_2}{n_2} \leq \frac{s_1}{n_1}$$

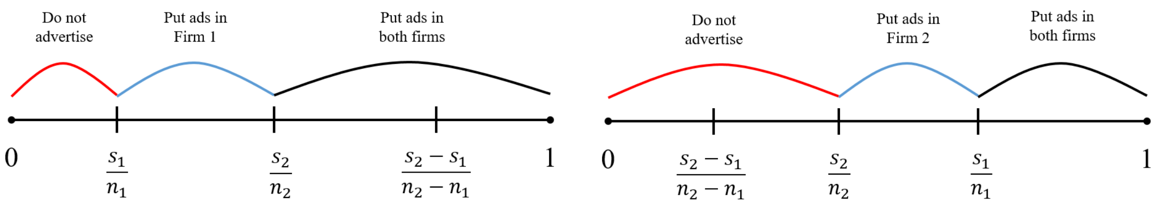

Figure 7: Advertisement Slots Purchase

Figure 7 represents the possible two cases. It shows that the demand for Firm 1 is in the interval of $[s_1/n_1, 1]$ and the demand for Firm 2 is in the interval of $[s_2/n_2, 1]$. Consequently the demands D for advertisement slots in Firm 1 and Firm 2, and the

[8] The possible cases are either $s_1/n_1 \geq s_2/n_2$ or $s_2/n_2 \geq s_1/n_1$ and the magnitude of $(s_2 - s_1)/(n_2 - n_1)$ depends on the relative sizes of those two terms

payouts R for Firm 1 and Firm 2 are:

$$D_1(s_1,s_2)=4k\cdot(1-\frac{s_1}{n_1}),\quad D_2(s_1,s_2)=4k\cdot(1-\frac{s_2}{n_2}),$$
$$R_1(s_1,s_2)=4k\cdot(1-\frac{s_1}{n_1})\cdot s_1,\quad R_2(s_1,s_2)=4k\cdot(1-\frac{s_2}{n_2})\cdot s_2$$

The first order condition of the payout functions R_1 and R_2 with respect to the corresponding advertisement fees s_1 and s_2 give the optimal advertisement fees $s_i^*, i\in\{1,2\}$ and the equilibrium payout $R_i(s_i^*), i\in\{1,2\}$:

$$\frac{\partial}{\partial s_i}(s_i-\frac{s_i^2}{n_i})=0 \implies 1-\frac{2}{n_i}s_i=0 \implies s_i^*=\frac{n_i}{2}$$

$$R_i(s_1^*,s_2^*)=4k\cdot(1-\frac{s_i}{n_i})\cdot s_i=4k\cdot(1-\frac{\frac{n_i}{2}}{n_i})\cdot\frac{n_i}{2}=kn_i$$

■

A.2 Appendix A.2

The demand for Firm 1 is $\tilde{x}$, which is calculated below.

$$v_1-t(\tilde{x}-a)^2-p_1=v_2-t(1-b-\tilde{x})^2-p_2 \iff$$
$$t(\tilde{x}-a)^2+p_1-v_1=t(1-b-\tilde{x})^2+p_2-v_2 \iff$$
$$(p_2-v_2)-(p_1-v_1)=t(\tilde{x}-a)^2-t(1-b-\tilde{x})^2 \iff$$
$$(p_2-v_2)-(p_1-v_1)=t(\tilde{x}-a-1+b+\tilde{x})(\tilde{x}-a+1-b-\tilde{x}) \iff$$
$$(p_2-v_2)-(p_1-v_1)=t(1-a-b)(2\tilde{x}-1-a+b) \iff$$
$$2\tilde{x}-(1+a-b)=\frac{(p_2-v_2)-(p_1-v_1)}{t(1-a-b)} \iff$$
$$\tilde{x}=\frac{(p_2-v_2)-(p_1-v_1)}{2t(1-a-b)}+\frac{1+a-b}{2}=a+\frac{(p_2-v_2)-(p_1-v_1)}{2t(1-a-b)}+\frac{1-a-b}{2}$$

Consequently, the demand for Firm 2 is $1-\tilde{x}$ and its formula is:

$$\begin{aligned}
1-\tilde{x} &= 1-\left[a+\frac{(p_2-v_2)-(p_1-v_1)}{2t(1-a-b)}+\frac{1-a-b}{2}\right] \\
&= \frac{1-a+b}{2}+\frac{(p_1-v_1)-(p_2-v_2)}{2t(1-a-b)} \\
&= b+\frac{(p_1-v_1)-(p_2-v_2)}{2t(1-a-b)}+\frac{1-a-b}{2}
\end{aligned}$$

Consequently the demand function for Firm 1 and Firm 2 are:

$$\text{Demand for Firm 1} = \begin{cases} \text{If} \quad \tilde{x}<0, & \text{then} \quad n_1 = 0 \\ \text{If} \quad 0 \leq \tilde{x} \leq 1, & \text{then} \quad n_1 = \tilde{x} \\ \text{If} \quad \tilde{x} > 1, & \text{then} \quad n_1 = 1 \end{cases}$$

$$\text{Demand for Firm 2} = \begin{cases} \text{If} \quad 1-\tilde{x}<0, & \text{then} \quad n_2 = 0 \\ \text{If} \quad 0 \leq 1-\tilde{x} \leq 1, & \text{then} \quad n_2 = 1-\tilde{x} \\ \text{If} \quad 1-\tilde{x} > 1, & \text{then} \quad n_2 = 1 \end{cases}$$

■

A.3 Appendix A.3 - Proof of Lemma 1

Lemma 1. *If prices are both positive, in order to have a situation where an increase in values can lead to an increase in prices, the firms require* $1/4 \geq t$ *when* $(a,b)=(0,0)$.

Proof of **Lemma 1**

Equations (7) and (8) suggest that for the given equilibrium price p_1^*,

$$\begin{aligned}
\frac{1}{3}v_1 - t(1-a-b)\times\frac{4}{3}v_1 \geq 0 &\iff \text{An increase in } v_1 \text{ leads to an increase in } p_1^* > 0 \\
&\iff \frac{1}{4} \geq t(1-a-b)
\end{aligned}$$

In other words, $\frac{1}{4} \geq t(1-a-b)$ is a necessary condition to have an increase in the value v_1 leading to an increase in the equilibrium price p_1^* as determined by strategic interactions. By symmetry, an analogous argument applies to Firm 2 as well. This condition is always satisfied regardless of the value of disutility cost t if $a = b = 0.5$, i.e. if both firms locate at the center. If firms are at the extremes ($a = b = 0$), then it requires $1/4 \geq t$. ■

B Appendix B

In **Section 3.1** the profit maximization problem is supposed by Equation (4) and this profit function can be endogenized in terms of values v_1, v_2 and stances a, b with the best response price functions (Equation (5) and (6)). Plugging in those best response functions to profit maximisation problem gives the profit functions to be optimised in **The Value Stage Game**:

$$\Pi_1(v_1, v_2, a, b) = \left[\frac{1}{3}(v_1 - v_2) + t(1-a-b)(1 + \frac{a - b - 4v_1 - 2v_2}{3})\right] \times$$
$$\left[a + \frac{1-a-b}{2} + \frac{1}{2t(1-a-b)}\left(\frac{1}{3}(v_1 - v_2) + \frac{2}{3}t(1-a-b)(b - a + v_1 - v_2)\right)\right] - \frac{v_1^2}{2}$$

and

$$\Pi_2(v_1, v_2, a, b) = \left[\frac{1}{3}(v_2 - v_1) + t(1-a-b)(1 + \frac{b - a - 4v_2 - 2v_1}{3})\right] \times$$
$$\left[b + \frac{1-a-b}{2} + \frac{1}{2t(1-a-b)}\left(\frac{1}{3}(v_2 - v_1) + \frac{2}{3}t(1-a-b)(a - b + v_2 - v_1)\right)\right] - \frac{v_2^2}{2}$$

Simultaneously solving the first-order conditions $\frac{\partial}{\partial v_1}\Pi_1 = 0$ and $\frac{\partial}{\partial v_2}\Pi_2 = 0$ gives:

$$v_1^*(a, b) = \frac{1 + 2(-3 + a^2 + 2a + 4b - b^2)t + (a - b)(1-a-b)^2t^2 + (5 + a - b)(1-a-b)^3t^3}{2(2 - 9(1-a-b)t - 15(1-a-b)^2t^2 - 5(1-a-b)^3t^3)}$$

$$v_2^*(a, b) = \frac{1 + 2(-3 + b^2 + 2b + 4a - a^2)t + (b - a)(1-a-b)^2t^2 + (5 + b - a)(1-a-b)^3t^3}{2(2 - 9(1-a-b)t - 15(1-a-b)^2t^2 - 5(1-a-b)^3t^3)}$$

These equations are the equilibrium value functions of Firm 1 and Firm 2. The values offered by firms are assumed to be non-negative. Therefore the best response value functions are:

$$\begin{cases} v_1(a, b) = \max\{0, \frac{1+2(-3+a^2+2a+4b-b^2)t+(a-b)(1-a-b)^2t^2+(5+a-b)(1-a-b)^3t^3}{2(2-9(1-a-b)t-15(1-a-b)^2t^2-5(1-a-b)^3t^3)}\} \\ v_2(a, b) = \max\{0, \frac{1+2(-3+b^2+2b+4a-a^2)t+(b-a)(1-a-b)^2t^2+(5+b-a)(1-a-b)^3t^3}{2(2-9(1-a-b)t-15(1-a-b)^2t^2-5(1-a-b)^3t^3)}\} \end{cases}$$ ■

B.1 Appendix B.1 - Proof of Lemma 2

Lemma 2. *If firms are located at the extremes of the stance spectrum, there exists a threshold* $\hat{t} = 1$ *which makes the best response choice of values to be zero for both firms.*

Proof of **Lemma 2**

The derivations in **Appendix B** imply that if both firms are located at each of the extremes, i.e. $(a, b) = (0, 0)$, then the value functions become identical in the form of:

$$v_1(0,0) = v_2(0,0) = \frac{1 - 6t + 5t^3}{4 - 18t - 30t^2 - 10t^3}$$

The unique real solution for the equation $v_1(0, 0) = v_2(0, 0) = 0$ is $t = 1$. The graphical illustration is presented in Figure 4. ■

C Appendix C

C.1 Appendix C.1 - Proof of Proposition 1

Proposition 1. *If prices are positive, both firms locate at the extremes* $(a, b) = (0, 0)$ *and their profits are dependent on the disutility cost* t *where there exists a very small threshold* $\tilde{t}$ *which makes the profits of both firms transitions from negative to positive.*

Proof of **Proposition 1**

If the prices are both positive, the corresponding profit functions of Firm 1 and Firm 2 are:

$$\Pi_1^*(a,b) = -\frac{(-1 + (5 - 3a - 2a^2 - 7b + 2b^2)t + (5 + a - b)(1 - a - b)^2t^2)^2}{8(-2 + (-1 + a + b)t)^2(1 + 5(-1 + a + b)t - 5(-1 + a + b)^2t^2)^2} \times \frac{(1 + 10(-1 + a + b)t - 19(1 - a - b)^2t^2 + 8(-1 + a - b)^3t^3)}{8(-2 + (-1 + a + b)t)^2(1 + 5(-1 + a + b)t - 5(-1 + a + b)^2t^2)^2}$$

and

$$\Pi_2^*(a,b) = -\frac{(-1 + (5 - 3b - 2b^2 - 7a + 2a^2)t + (5 + b - a)(1 - a - b)^2t^2)^2}{8(-2 + (-1 + a + b)t)^2(1 + 5(-1 + a + b)t - 5(-1 + a + b)^2t^2)^2} \times \frac{(1 + 10(-1 + a + b)t - 19(1 - a - b)^2t^2 + 8(-1 + b - a)^3t^3)}{8(-2 + (-1 + a + b)t)^2(1 + 5(-1 + a + b)t - 5(-1 + a + b)^2t^2)^2}$$

The sign of $\Pi_1^*(a,b)$ is dependent on the term $-(1+10(-1+a+b)t-19(1-a-b)^2t^2+8(-1+a-b)^3t^3)$ in the numerator because all other terms are always positive as the square of real number is always positive. Rearrange the term without the negative sign and call it $f(a)$ such that:
$f(a) = (1-10(1-a-b)t-19(1-a-b)^2t^2+8(-1+a-b)^3t^3)$.
The first derivative $f'(a)$ is:

$$\begin{aligned} f'(a) &= -10(-1) + (-19) \times 2(1-a-b)(-1)t^2 + 24(-1+a-b)^2t^3 \\ &= 10 + 38(1-a-b)t^2 + 24(1-a+b)^2t^3. \end{aligned}$$

If $1-a-b>0$ then $1-a+b>0$ as well because $b \geq 0$. Therefore if $1-a-b>0$ then $f'(a)>0$. The sign of $\Pi_1^*(a,b)$ is the opposite of the sign of $f(a)$ by construction. Therefore $\Pi_1^*(a,b)$ is decreasing in a if $1-a-b>0$. By symmetry $\Pi_2^*(a,b)$ is decreasing in b if $1-a-b>0$. The condition $1-a-b>0$ has been assumed to be true from the very beginning of the model setup. I have shown that:

$$\frac{\partial}{\partial a}\Pi_1^*(a,b) < 0 \quad \text{and} \quad \frac{\partial}{\partial b}\Pi_2^*(a,b) < 0 \quad \text{if } 1-a-b>0$$

Firm 1 can only choose its own location a. It chooses the minimum possible value $a=0$ because its profit is decreasing in a. For the same reason Firm 2 chooses $b=0$. This proves $(a,b)=(0,0)$ is the unique Nash Equilibrium for profit maximization in the *positive price case.*

At $(a,b)=(0,0)$ the profit functions $\Pi_1^*(a,b)$ and $\Pi_2^*(a,b)$ are identical and reduce to:

$$\begin{aligned} \Pi_i^*(a,b)|_{a=b=0}, i \in \{1,2\} &= -\frac{(-1+(5)t+(5)(1)^2t^2)^2(1+10(-1)t-19(1)^2t^2+8(-1)^3t^3)}{8(-2+(-1)t)^2(1+5(-1)t-5(-1)^2t^2)^2} \\ &= -\frac{(-1+5t+5t^2)^2(1-10t-19t^2-8t^3)}{8(2+t)^2(1-5t-5t^2)^2} \end{aligned}$$

This is a cubic function of t. The equation $\Pi_i^*(a,b)|_{a=b=0}=0$ has three solutions: $t=-1$, $t=\frac{1}{16}(-11-3\sqrt{17})$, and $t=\frac{1}{16}(-11+3\sqrt{17})$. The disutility cost t cannot be negative. Therefore the only possible t satisfying $\Pi_i^*(a,b)|_{a=b=0}=0$ is $\hat{t}=\frac{1}{16}(3\sqrt{17}-11) \approx 0.08558$. The graphical illustration is presented in Figure 5 in Section 3.3. ■

C.2 Appendix C.2

As stated in **Remark 1**, if both firms offer zero value then the problem reduces to the original question of Gabszewicz et al. (2001) and Gabszewicz et al. (2002). Then the profit function given in Equation (4) reduces to:

$$\max\ \Pi_i = (p_i - c + k) * n_i$$

The profit function has no value production cost term because the value is zero. The best response price functions are:

$$p_1 = \max\{0, \frac{1}{2}[p_2 + c - k + t(1-a-b)(1+a-b)]\}$$

$$p_2 = \max\{0, \frac{1}{2}[p_1 + c - k + t(1-a-b)(1+b-a)]\}$$

Solving those best response functions in the *positive case* simultaneously gives the equilibrium prices:

$$p_1^*(a,b) = c - k + t(1-a-b)(1 + \frac{a-b}{3})$$
$$p_2^*(a,b) = c - k + t(1-a-b)(1 + \frac{b-a}{3})$$

These equations can be derived by plugging $v_1 = v_2 = 0$ in Equation (7) and (8) as well. The corresponding equilibrium profit functions are:

$$\Pi_1^*(a,b) = \frac{t}{18}(1-a-b)(3-a+b)^2 \quad \text{and} \quad \Pi_2^*(a,b) = \frac{t}{18}(1-a-b)(3-b+a)^2$$

Note $\Pi_1^*(a,b)$ (respectively $\Pi_2^*(a,b)$) is a strictly decreasing function of a (respectively b) for $a, b \in [0,1]$. Additionally both $\Pi_1^*(a,b)$ and $\Pi_2^*(a,b)$ are strictly increasing functions of the disutility cost t because $1-a-b \geq 0$. ■

C.3 Appendix C.2 - Proof of Lemma 3

Lemma 3. *If prices are both zero, the principle of minimum differentiation does not hold. The higher $(k-c)$ is, the more firms horizontally differentiate. $(k-c) > 0$ is not a sufficient condition for strictly positive $\partial\Pi_1/\partial a$ and $\partial\Pi_2/\partial b$.*

Proof of **Lemma 3**

Note that $\frac{\partial}{\partial a}\Pi_1 > 0 \iff \Pi_1$ is an increasing function of a, and $\frac{\partial}{\partial b}\Pi_2 > 0 \iff \Pi_2$ is an increasing function of b. In other words, from Equation (10), Π_1 and Π_2 are decreasing with respect to a and b respectively if

$$\frac{1}{2}(k-c) < \frac{(k-c)^2}{4(1-a-b)^3t^2} \iff 2(1-a-b)^3 < \frac{(k-c)}{t^2}$$

where $1-a-b > 0$ and $k-c > 0$. If $(a,b) = (0.5, 0.5)$, meaning that if firms are located at the center of stance interval, this inequality *always* holds so Π_1 and Π_2 are decreasing at the center. This means $(a,b) = (0.5, 0.5)$ is not a Nash Equilibrium, therefore the principle of minimal differentiation does not hold if prices are both zero. Greater $(k-c)$ value drives this equation more likely to hold. Which means $(k-c) > 0$ is not a sufficient condition for strictly positive $\partial\Pi_1/\partial a$ and $\partial\Pi_2/\partial b$. ■

D Appendix D

D.1 Appendix D.1 - Proof of Proposition 2

Proposition 2. *The homogeneous model may underestimate the disutility cost of consumers and consequently the values need to be offered.*

Proof of **Proposition 2**

According to **Lemma 1**, in the *positive price case* the firms require $1/4 \geq t$ because $(a,b) = (0,0)$ is the unique Nash equilibrium in the *positive price case* according to **Proposition 1**. If the firms produce zero-value products, i.e. the heterogeneous model reduces to the homogeneous model, then it requires $1 \leq t$ according to **Lemma 2**. The condition for the maximum differentiation under the homogeneous model is $0.72 < t$ as shown before. If $0.72 < t \leq 1$ then the firms produce positive value in order to grasp demand in the heterogeneous model, but offer zero value in the homogeneous model. This means the homogeneous model underestimates the power of disutility costs of consumers.

For example if $t = 0.72$ and a firm decides to offer zero value, the symmetry in value offerings breaks down and consumer demands will be skewed toward the firm offering positive value, which means that for a given utility the firm offering zero value underestimates the disutility cost of consumers and does not make enough compensation for consumers with enough value offering. ■

D.2 Appendix D.2 - Proof of Proposition 3

Proposition 3. *Under pricing symmetry, the higher* $(k - c) \geq 0$ *is, the more firms horizontally differentiate.*

Proof of **Proposition 3**

According to **Proposition 1** the principle of maximum differentiation holds in the *positive price case.* According to **Lemma 3**, $(k - c)$ plays a crucial role in determining the sign of the partial derivatives of the profit functions. This suggests the higher $(k - c)$ is, the more firms horizontally differentiate in the *zero price case.* Therefore under pricing symmetry, in both *positive price case* and *zero price case*, an increase in $(k - c)$ leads the firms to polarize to each of the extremes on the stance horizon. ■