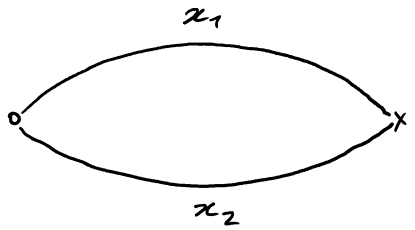
a) Parallel Network User Equilibrium (UE) Analysis

Single Origin-Destination Pair



$$c_1(x_1) = 1 + \frac{1}{x_1}$$

$$c_2(x_2) = 3 + x_2$$

There exists 3 complementary cases

1. all traffic on route 1
2. all traffic on route 2
3. split traffic

i) Find validity constraints for case 1

$$\therefore \; x_1 = d$$

$$x_2 = 0$$

$$c_1 = 1 + \frac{1}{d}$$

$$c_2 = 3$$

UE valid when $c_1 \leq c_2$

$$\therefore \; 1 + \frac{1}{d} \leq 3$$

$$\frac{1}{2} \leq d$$

Find validity constraints for case 2

$\therefore \ x_1 = 0$

$x_2 = d$

$$c_1 = 1 + \frac{1}{0}$$

$$= \infty$$

$$c_2 = 3 + d$$

$\therefore$ as $c_2 \leq c_1 \ \forall d$

There always exists a UE solution $\forall d$ in case 2

ii) Find validity constraints for case 3

$\therefore \ x_1 = x_1$

$x_2 = d - x_1$

$$\therefore \ c_1 = 1 + \frac{1}{x_1}$$

$$c_2 = 3 + d - x_1$$

For UE solution $c_1 = c_2$

$$\therefore \ 1 + \frac{1}{x_1} = 3 + d - x_1$$

$$x_1 + 1 = 3x_1 + dx_1 - x_1^2$$

$$0 = x_1^2 + x_1(-2-d) + 1$$

$$\therefore \ x_1 = \left(1 + \frac{d}{2}\right) \pm \left(\sqrt{d} + \frac{d}{2}\right)$$

As $0 < x_1 < d$

when

$$x_1 = 1 + \frac{d}{2} + \sqrt{d} + \frac{d}{2}$$
$$= d + 1 + \sqrt{d}$$
$$\therefore x_1 > d$$

$\therefore$ solution is invalid

when

$$x_1 = 1 + \cancel{\frac{d}{2}} - \sqrt{d} - \cancel{\frac{d}{2}}$$
$$= 1 - \sqrt{d}$$
$$x_1 > 0$$

$\therefore$ valid when $d < 1$

iii) Investigate system costs

$$f = c_1(x_1)x_1 + c_2(x_2)x_2$$

Consider cases

① $f_1 = 1 + d$ $\qquad d \geq \frac{1}{2}$

② $f_2 = d^2 + 3d$ $\qquad d \in \mathbb{R}$

③ $f_3 = d^2 + 2d + 2d\sqrt{d}$ $\qquad d < 1$

Assuming rational actors will always find the lowest system cost

consider $d < \frac{1}{2}$, find min system cost

investigate cases ②, ③

$$\cancel{d^2} + 3\cancel{d} > \cancel{d^2} + 2\cancel{d} + 2\cancel{d}\sqrt{d}$$
$$1 > 2\sqrt{d}$$
$$\frac{1}{4} > d$$

$\therefore$ case ② > case ③ : $d < \frac{1}{4}$

Consider $d > \frac{1}{4}$, find min system cost

investigate cases ①, ②

$$1 + d > d^2 + 3d$$
$$0 > d^2 + 2d - 1$$
$$\therefore -1 - \sqrt{2} < d < -1 + \sqrt{2}$$
$$d < 0.414$$

but d only valid from $d \geq \frac{1}{2}$ for case ①

Therefore

$$f_{UE} = \begin{cases} 1 + d & d \geq \frac{1}{2} \\ d^2 + 3d & \frac{1}{4} < d < \frac{1}{2} \\ d^2 + 2d + 2d\sqrt{d} & d < \ldots \end{cases}$$

$$f_{UE} = \begin{cases} 1+d & d \ge \frac{1}{2} \\ d^2+3d & \frac{1}{4} < d < \frac{1}{2} \\ d^2+2d+2d\sqrt{d} & d \le \frac{1}{4} \end{cases}$$

For the region $-1+\sqrt{2} < d < \frac{1}{2}$ the min system cost is not a UE solution $\therefore$ the 'price of anarchy' > 1 in this region.