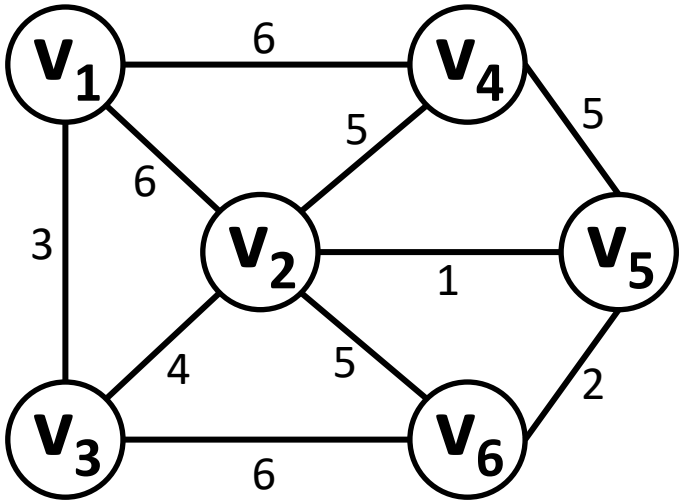
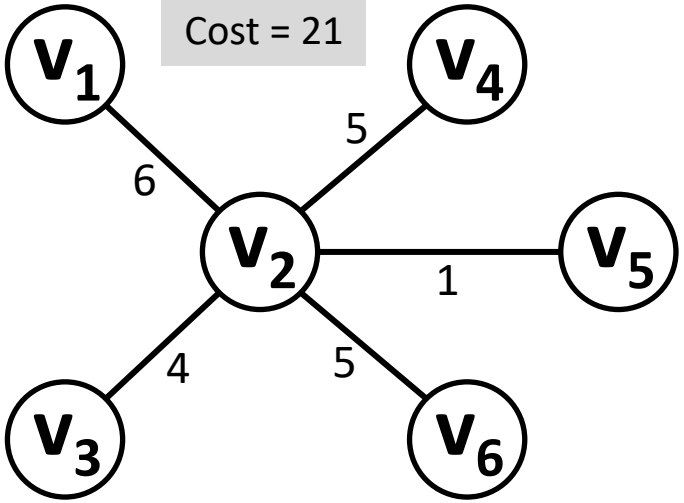
Graphs

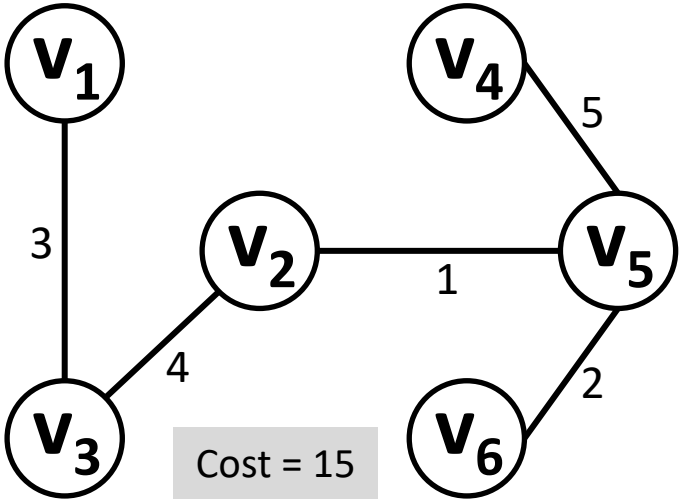
Minimum Spanning Tree



Connected Graph

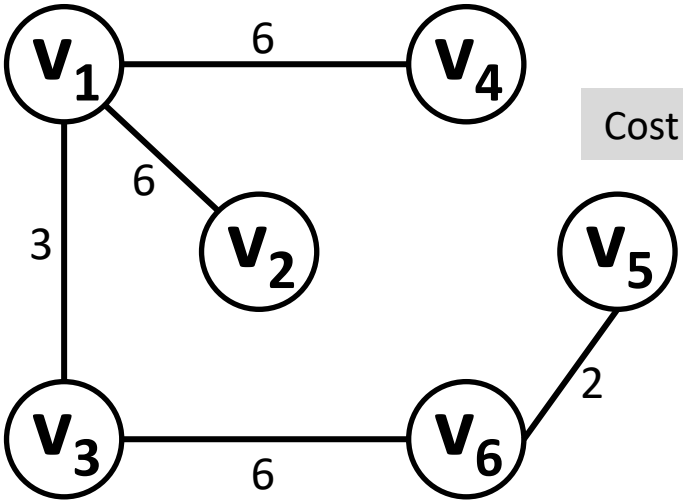
Cost = 21





Cost = 15

Cost = 23



Introduction

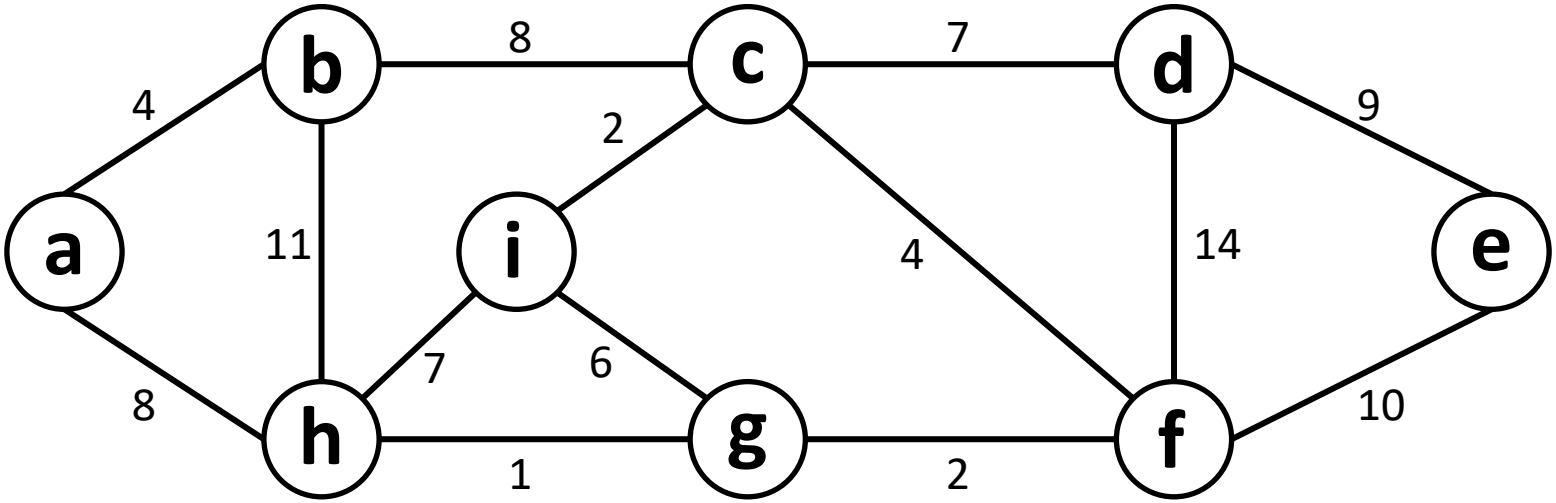
- Let, $G = (V,E)$ be a connected, undirected graph and $w(u,v)$ be a weight/cost of an edge $(u,v) \in E$.
- Then an acyclic subset $T \subseteq E$, that connects all of the vertices in V, is called a "spanning tree".
- The problem termed as "minimum spanning tree" or "minimum-weight spanning tree" aims to minimize total weight given as

$$w(T) = \sum_{(u,v) \in T} w(u,v)$$

Contd…

- Algorithms to solve the minimum spanning tree problem:
  - Kruskal's algorithm and
  - Prim's algorithm.

Kruskal's Algorithm



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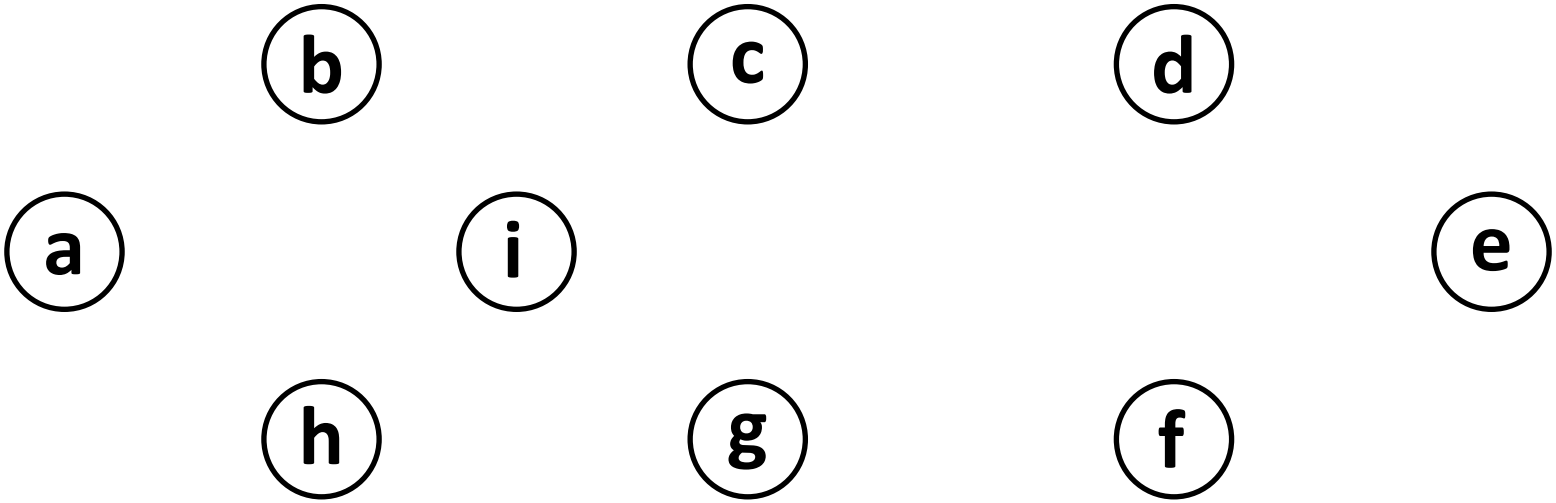
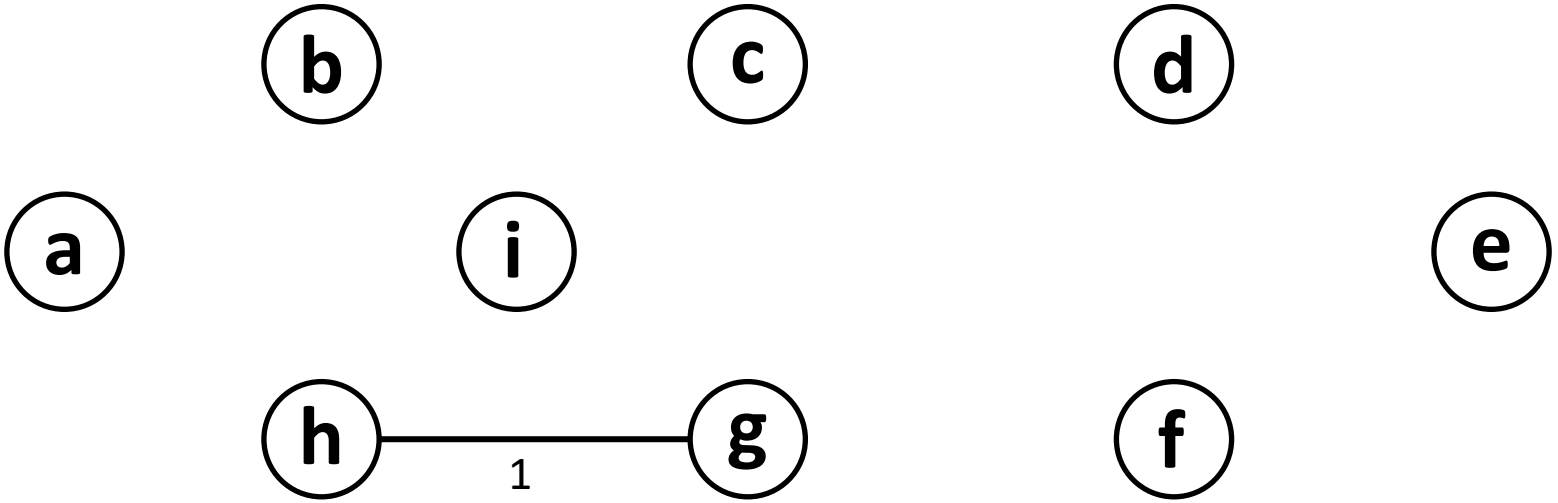


Edge	Weight
(g,h)	1
(f,g)	2
(c,i)	2
(a,b)	4
(c,f)	4

Edge	Weight
(g,i)	6
(c,d)	7
(h,i)	7
(a,h)	8
(b,c)	8

Edge	Weight
(d,e)	9
(e,f)	10
(b,h)	11
(d,f)	14

Contd…



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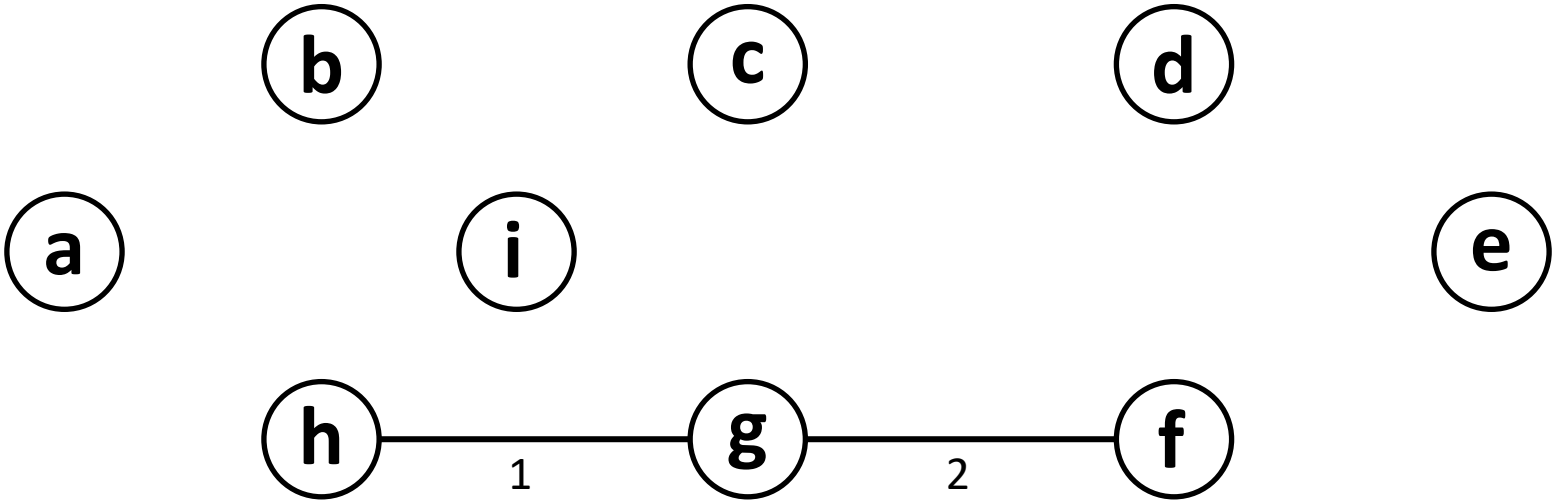
(b,h)	11
(d,f)	14

Contd…



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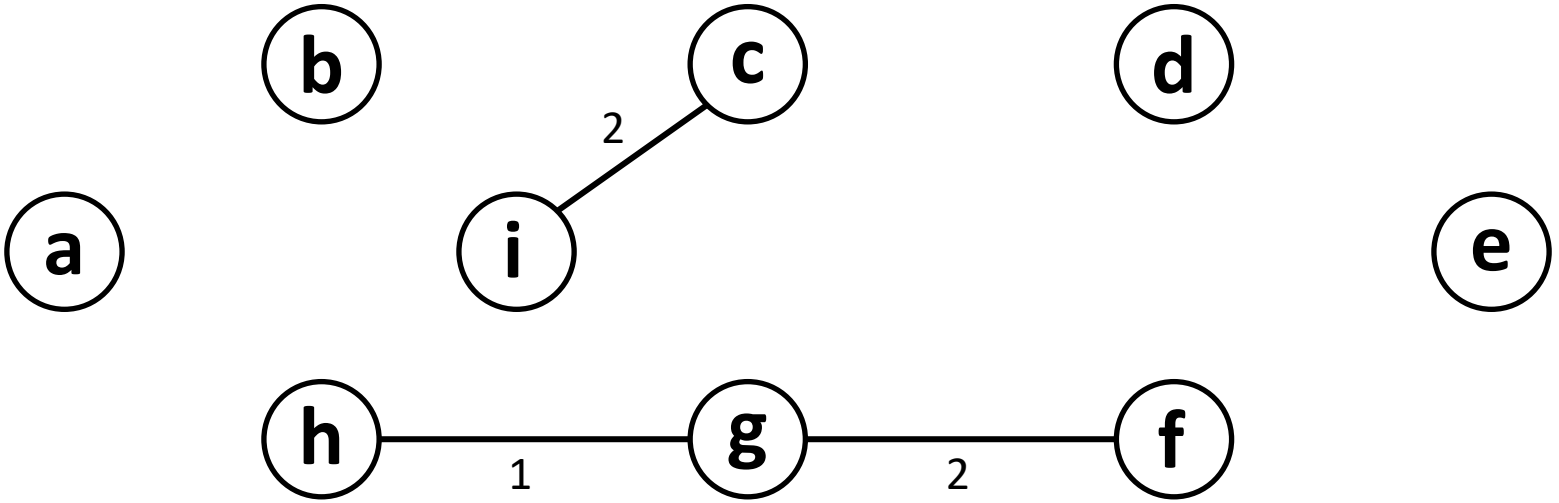
Contd…



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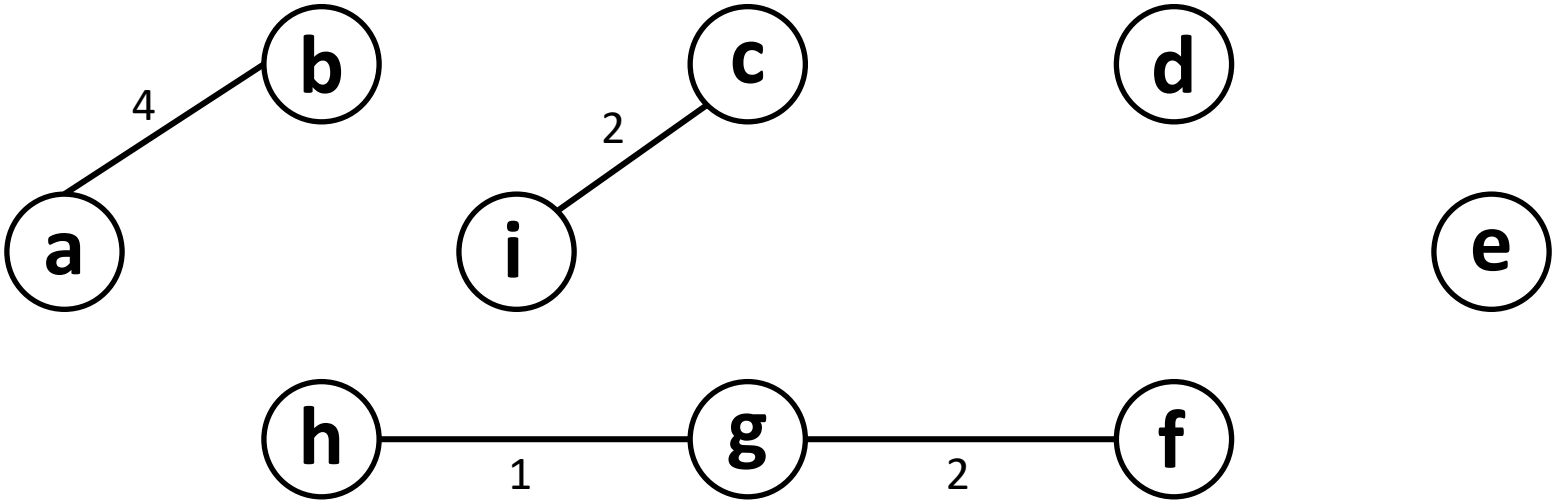
Contd…



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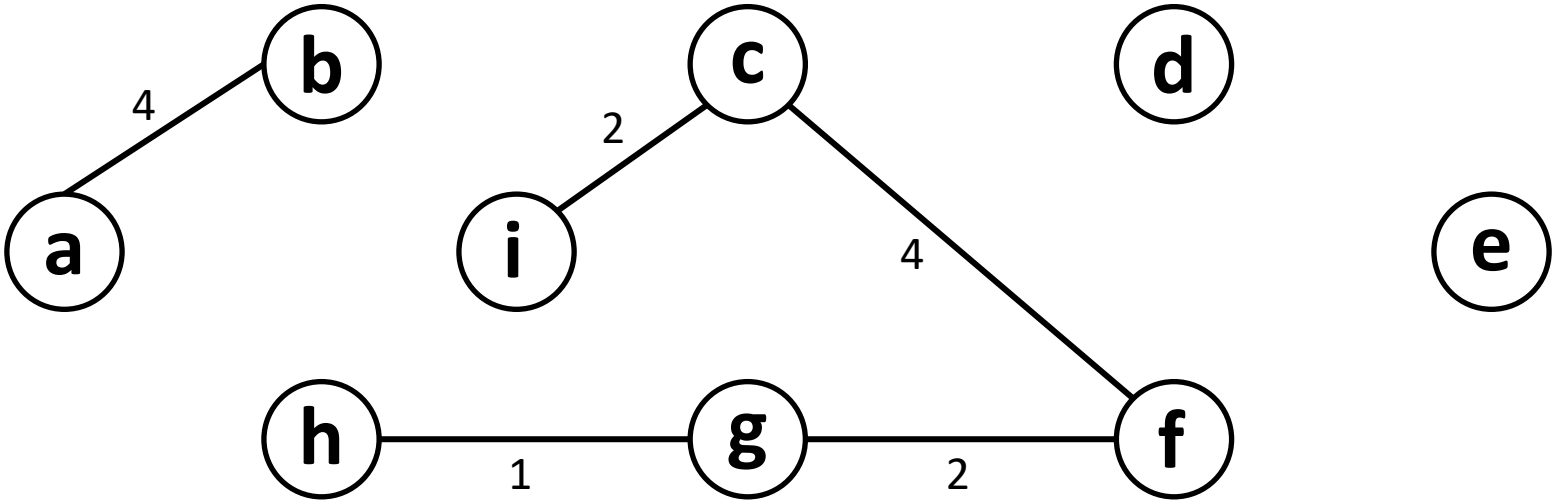
Contd…



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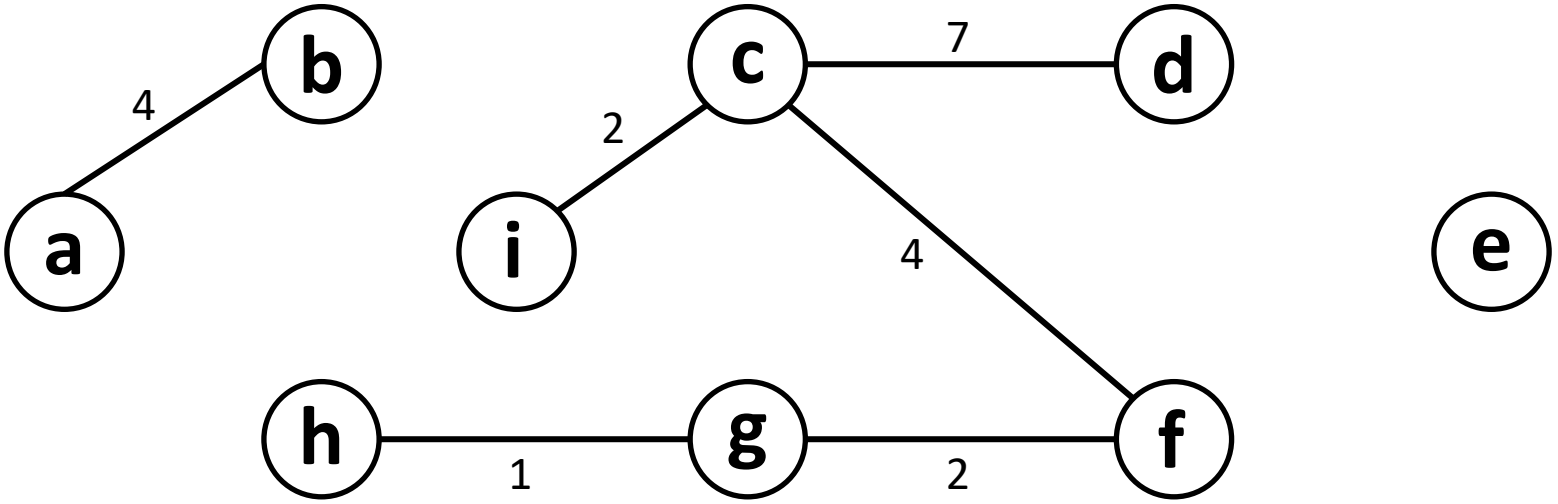
Contd…



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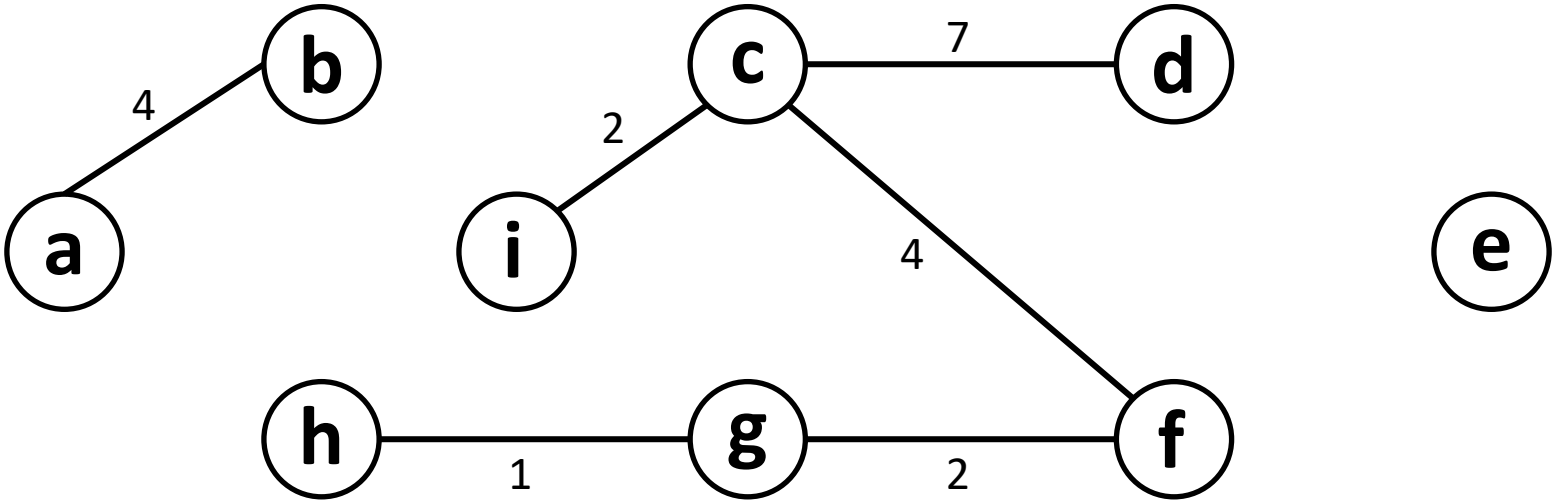
Contd…



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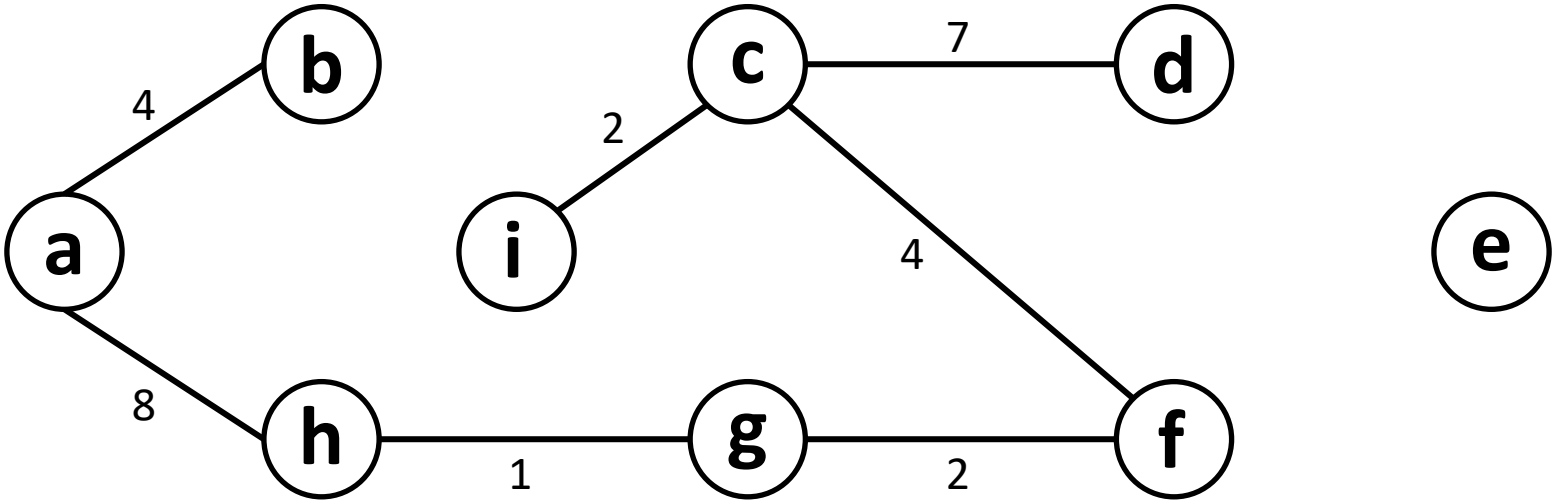
Contd…



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(a,h)	8
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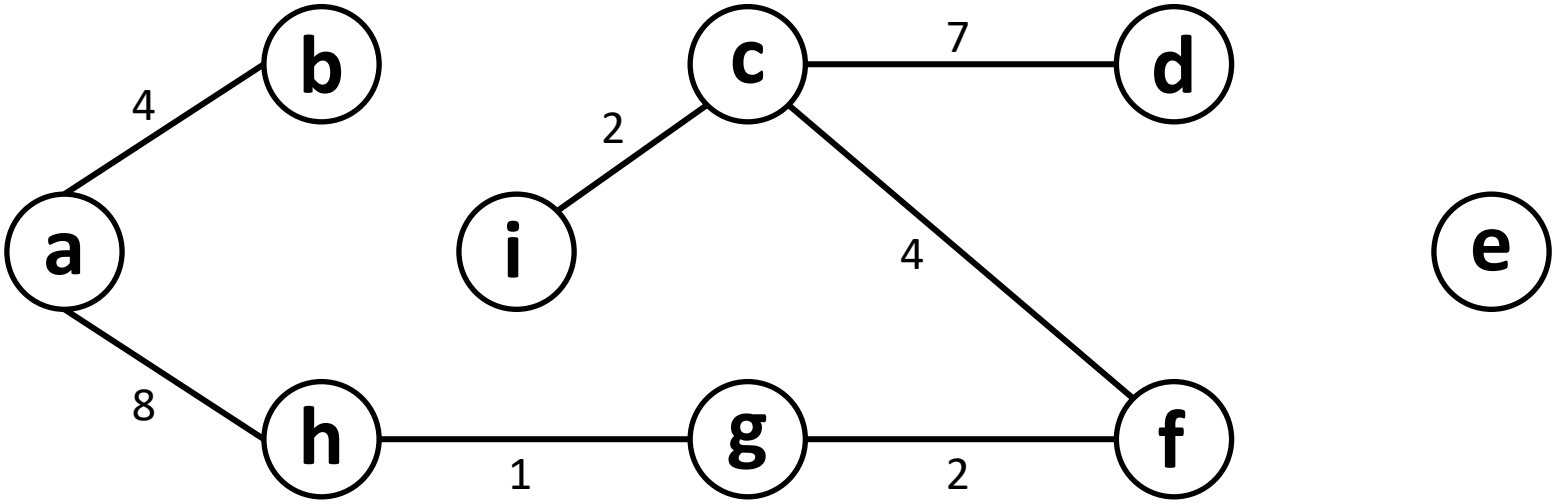
Contd…



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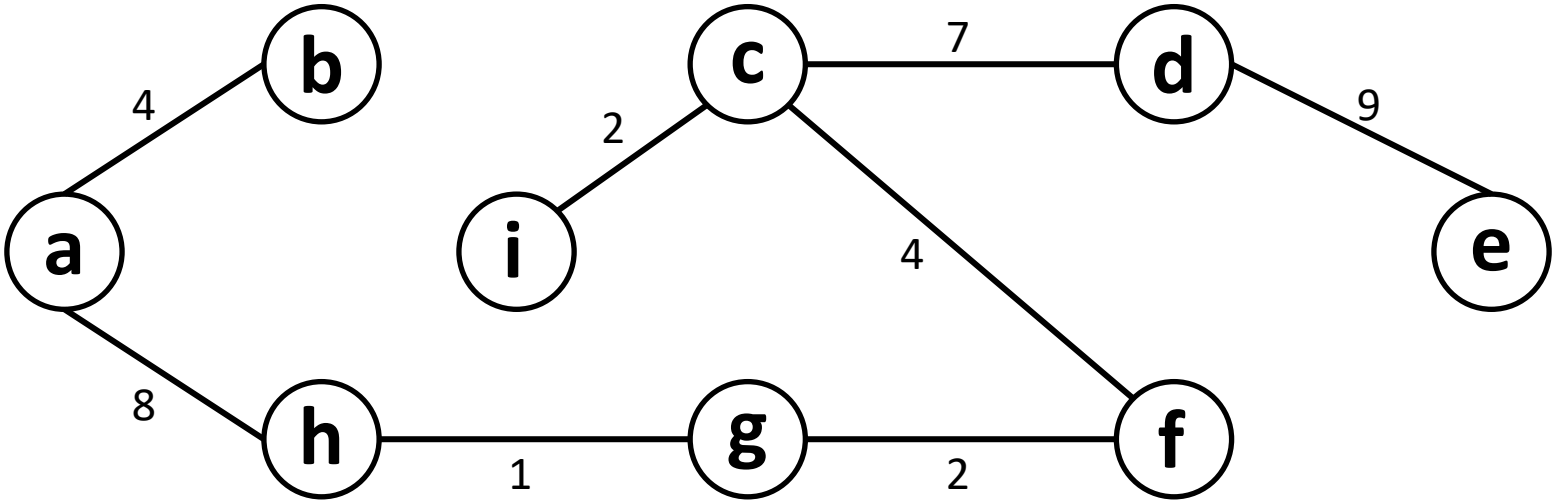

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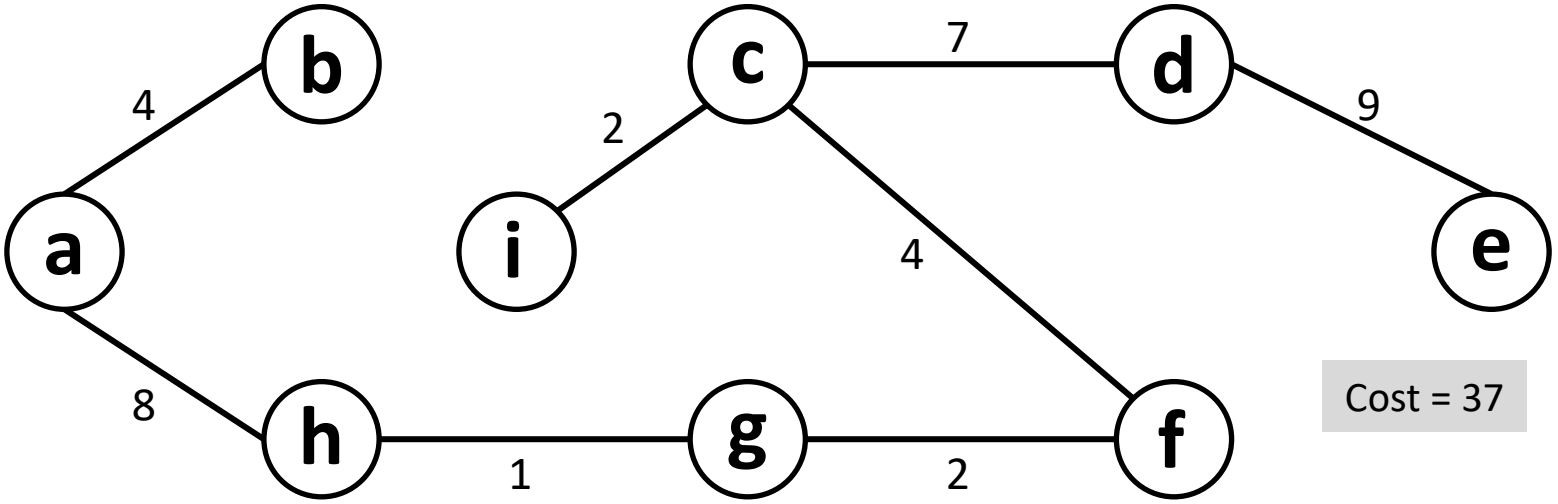

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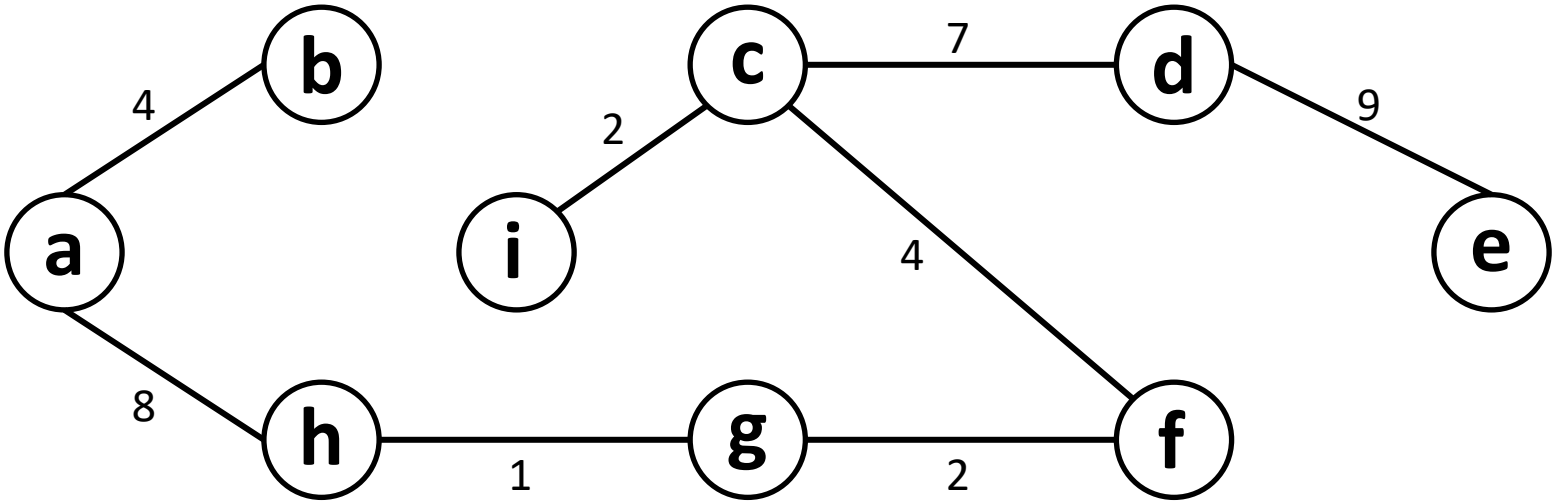
(a,h)	8
(b,c)	8

Edge	Weight
(d,e)	9
(e,f)	10
(b,h)	11
(d,f)	14

Contd…

A = {(g,h), (f,g), (c,i), (a,b), (c,f), (c,d), (a,h), (d,e)}



Cost = 37

Implementation

```
MST-KRUSKAL(G, w)
1  A = ∅
2  for each vertex v ∈ G.V
3       MAKE-SET(v)
4  sort the edges of G.E into nondecreasing order by weight w
5  for each edge (u, v) ∈ G.E, taken in nondecreasing order by weight
6       if FIND-SET(u) ≠ FIND-SET(v)
7            A = A ∪ {(u, v)}
8            UNION(u, v)
9  return A
```

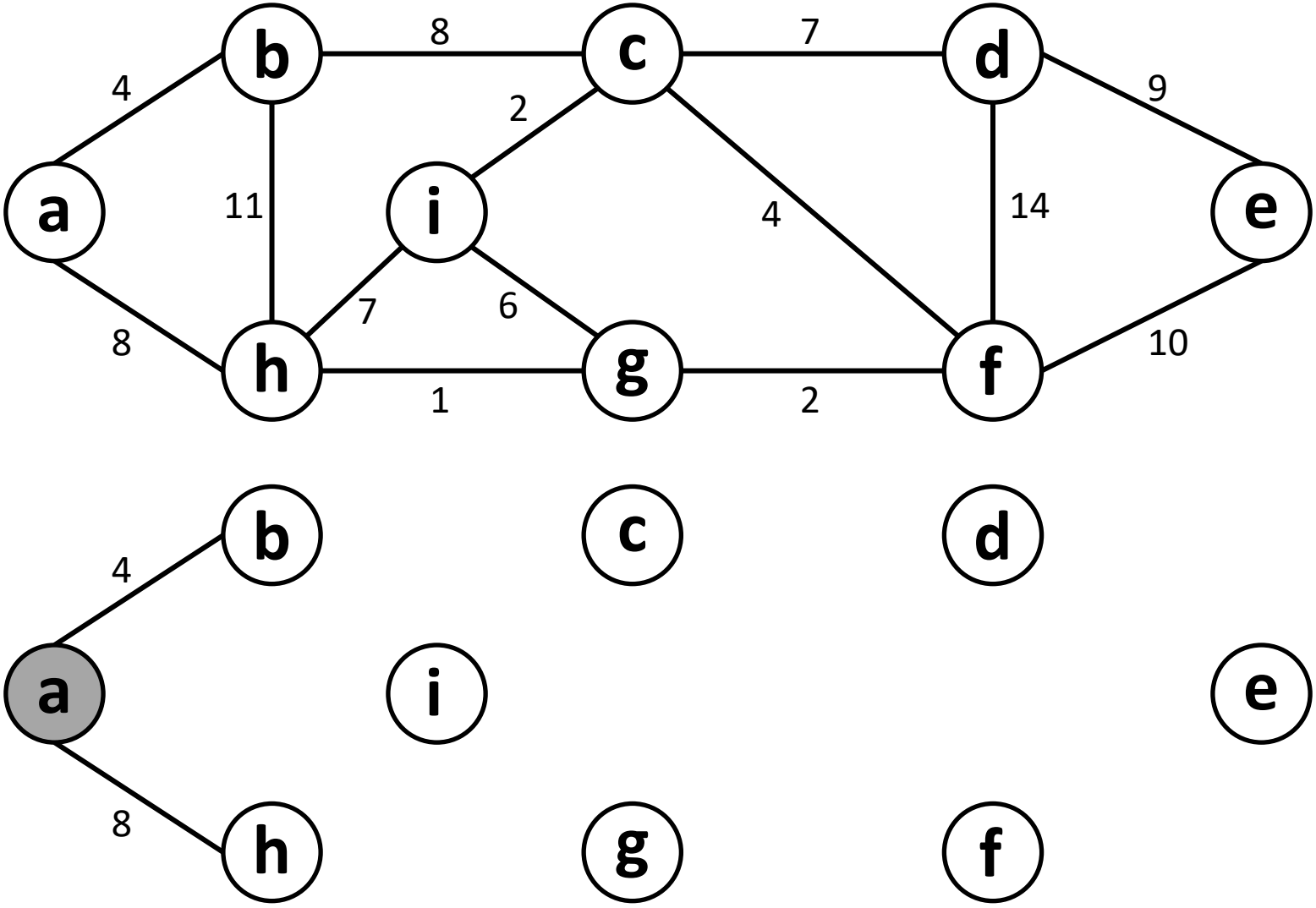
```
MAKE-SET(x)
1  x.p = x
2  x.rank = 0
```

```
UNION(x, y)
1  LINK(FIND-SET(x), FIND-SET(y))
```

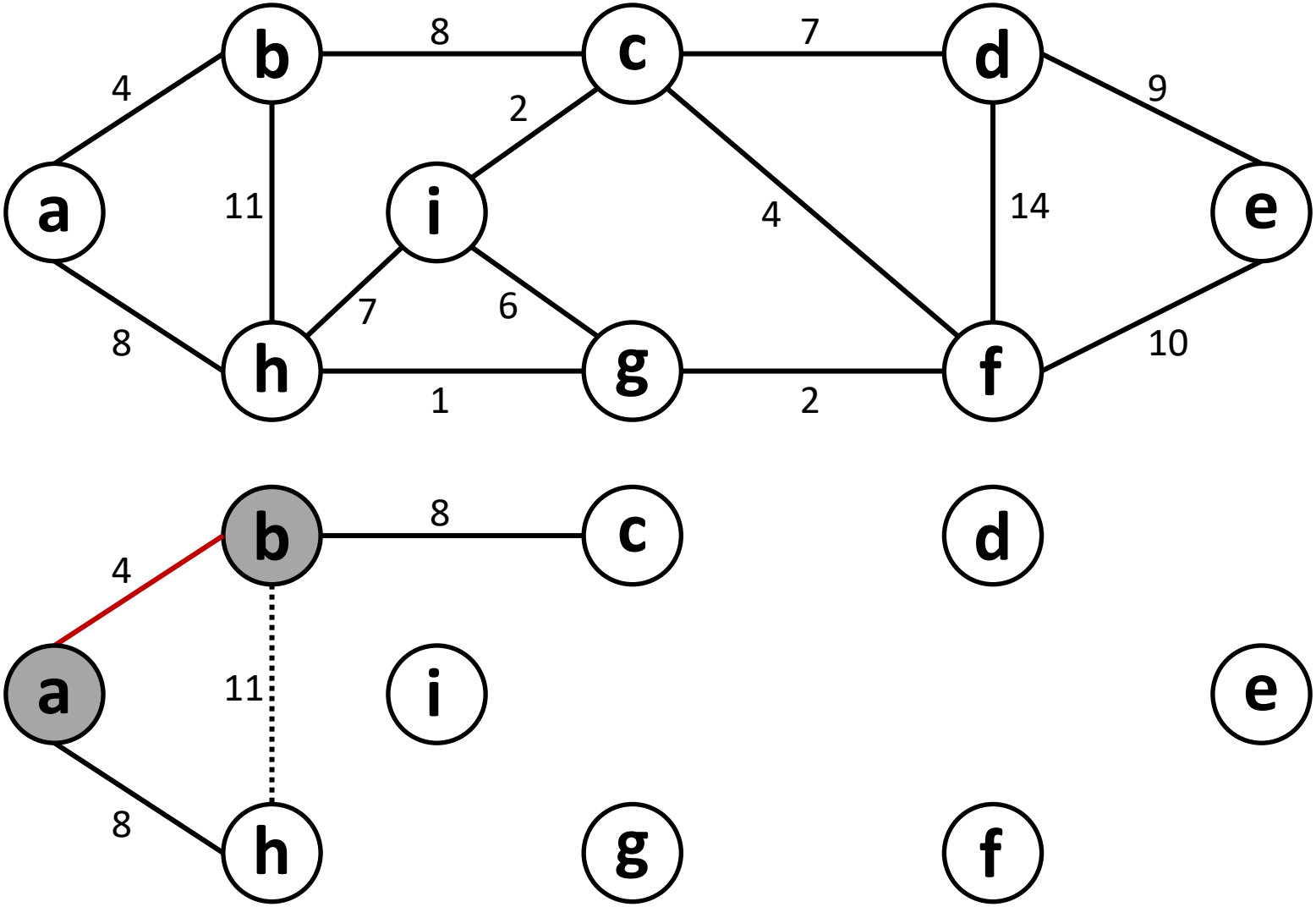
```
FIND-SET(x)
1  if x ≠ x.p
2       x.p = FIND-SET(x.p)
3  return x.p
```

```
LINK(x, y)
1  if x.rank > y.rank
2       y.p = x
3  else x.p = y
4       if x.rank == y.rank
5            y.rank = y.rank + 1
```

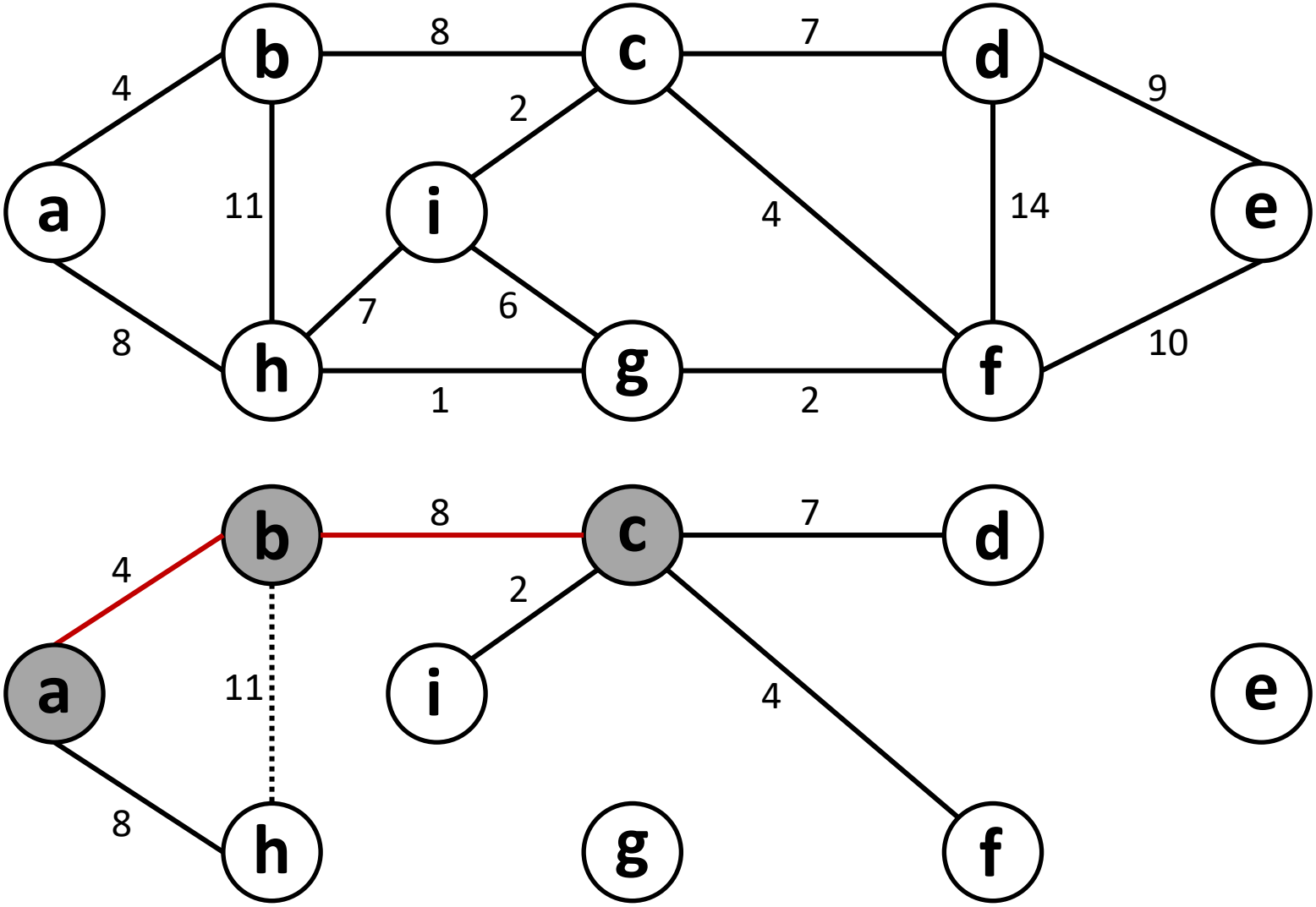
Prim's Algorithm



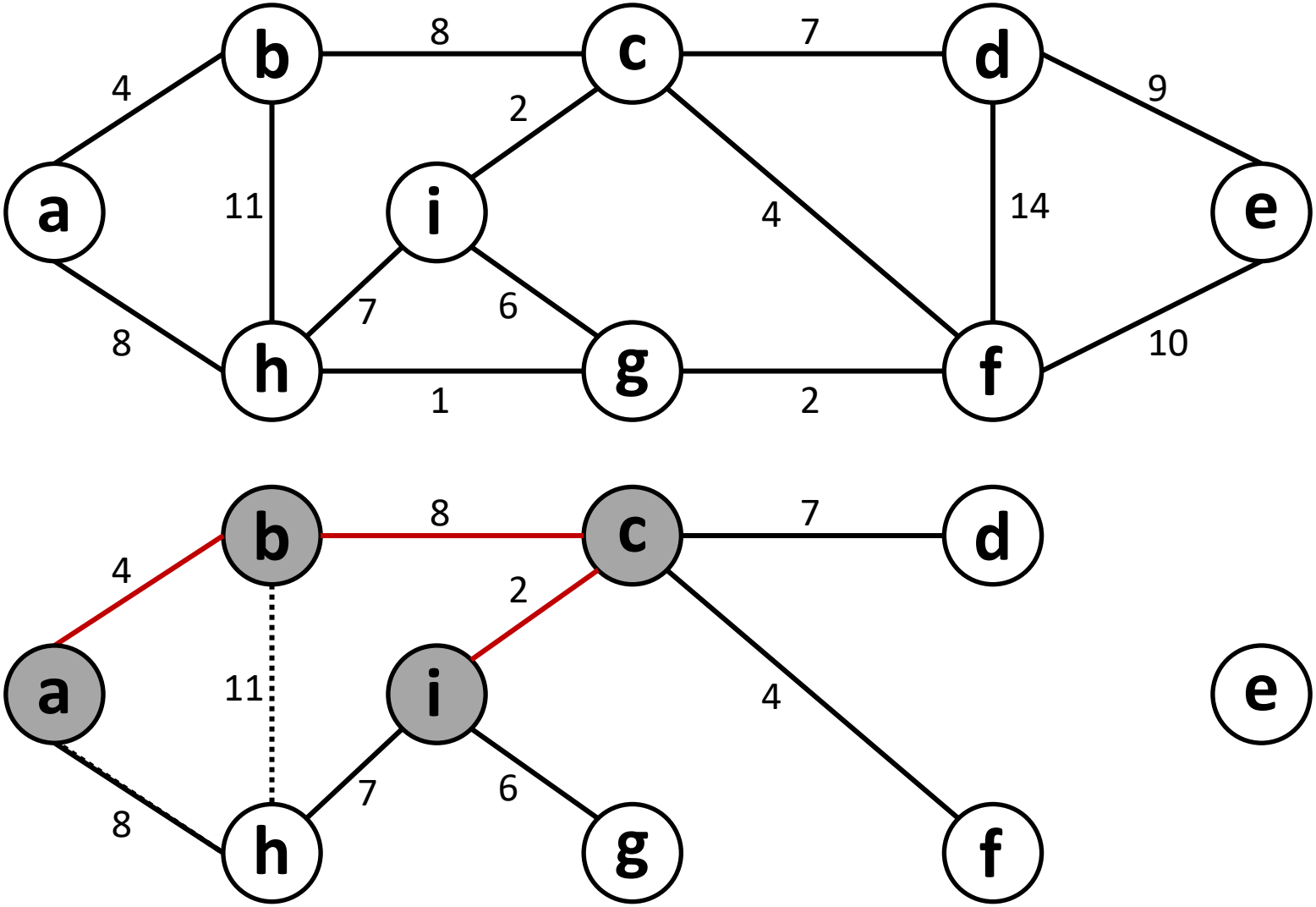
Prim's Algorithm



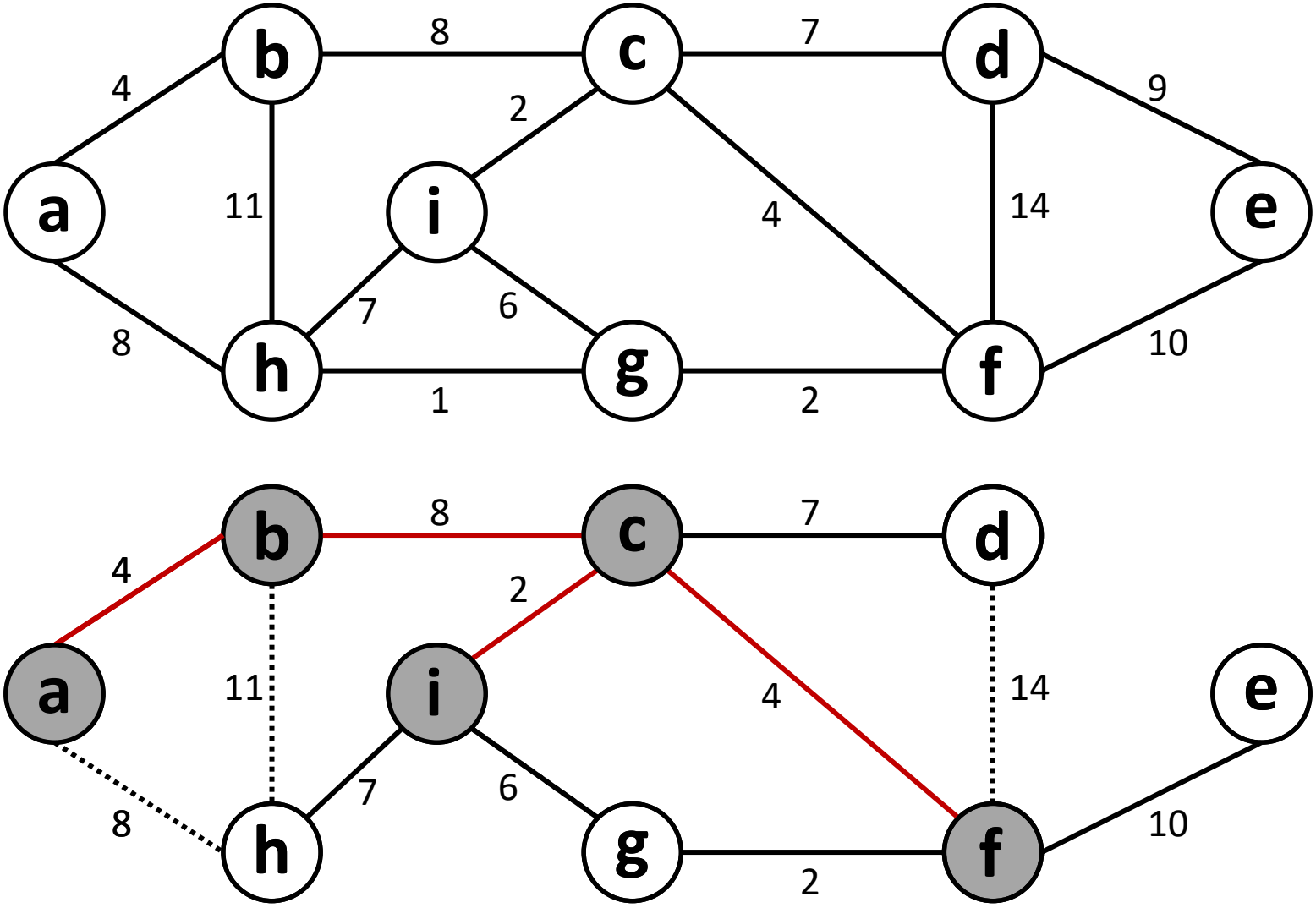
Prim's Algorithm



Prim's Algorithm



Prim's Algorithm



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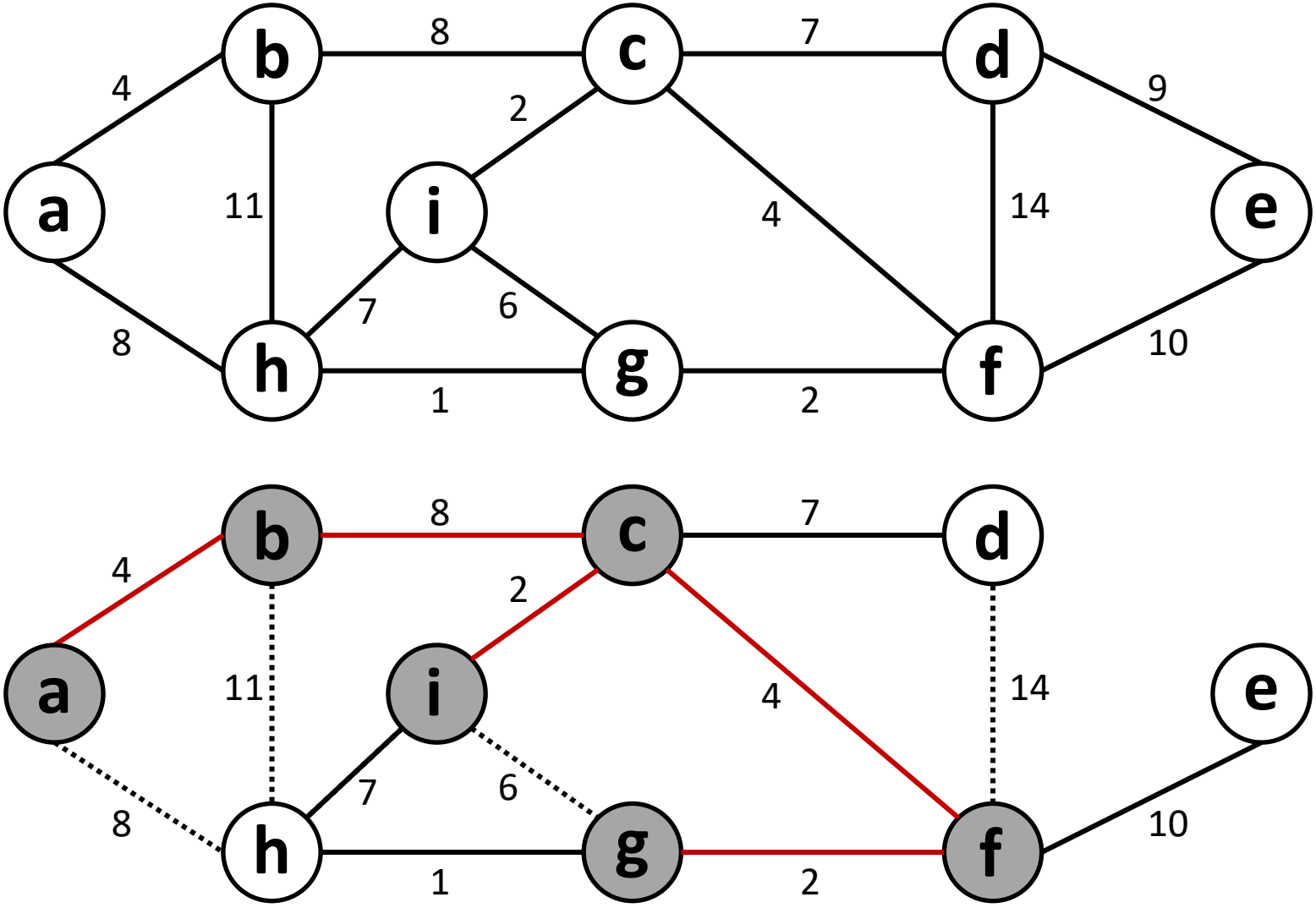

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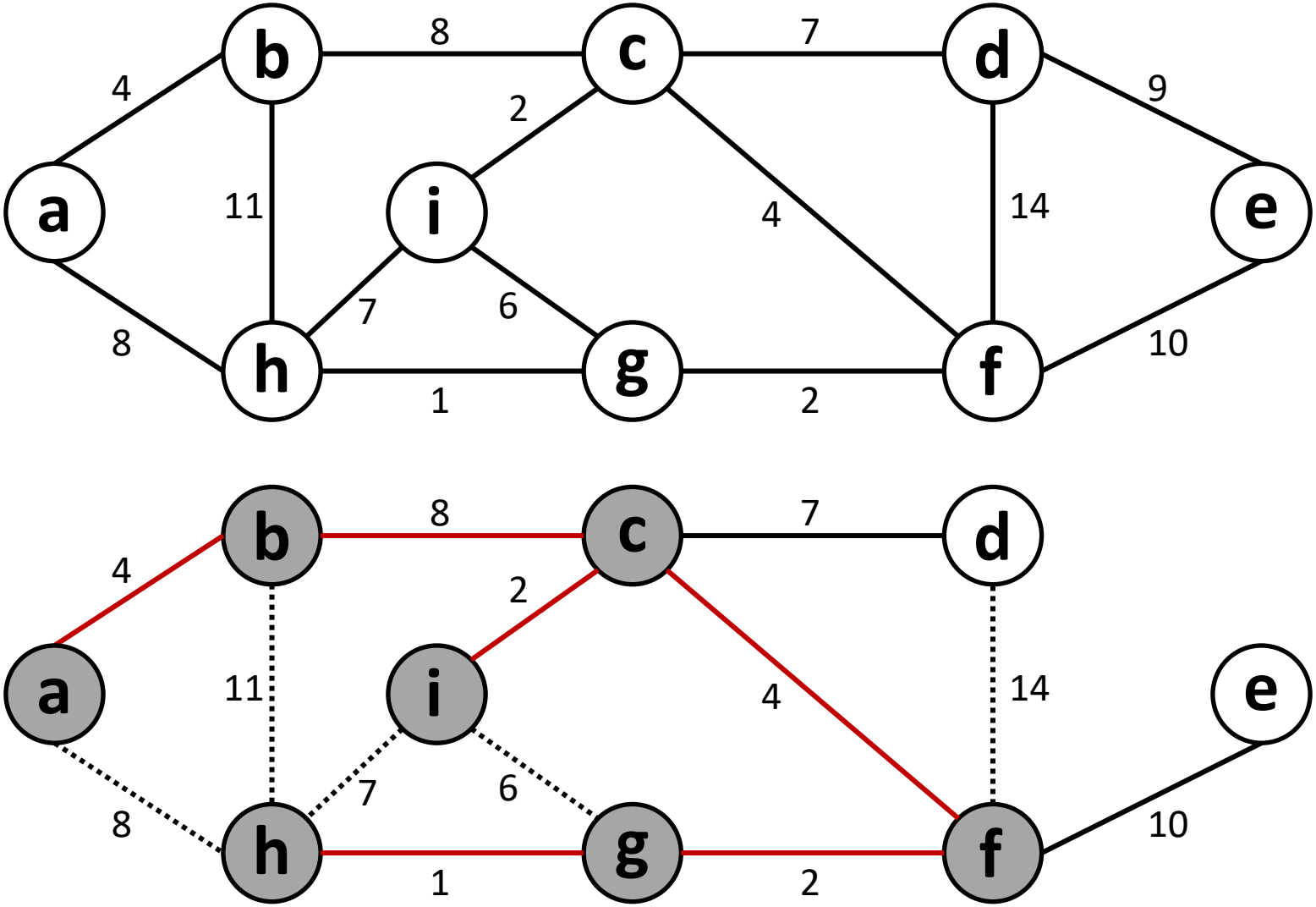

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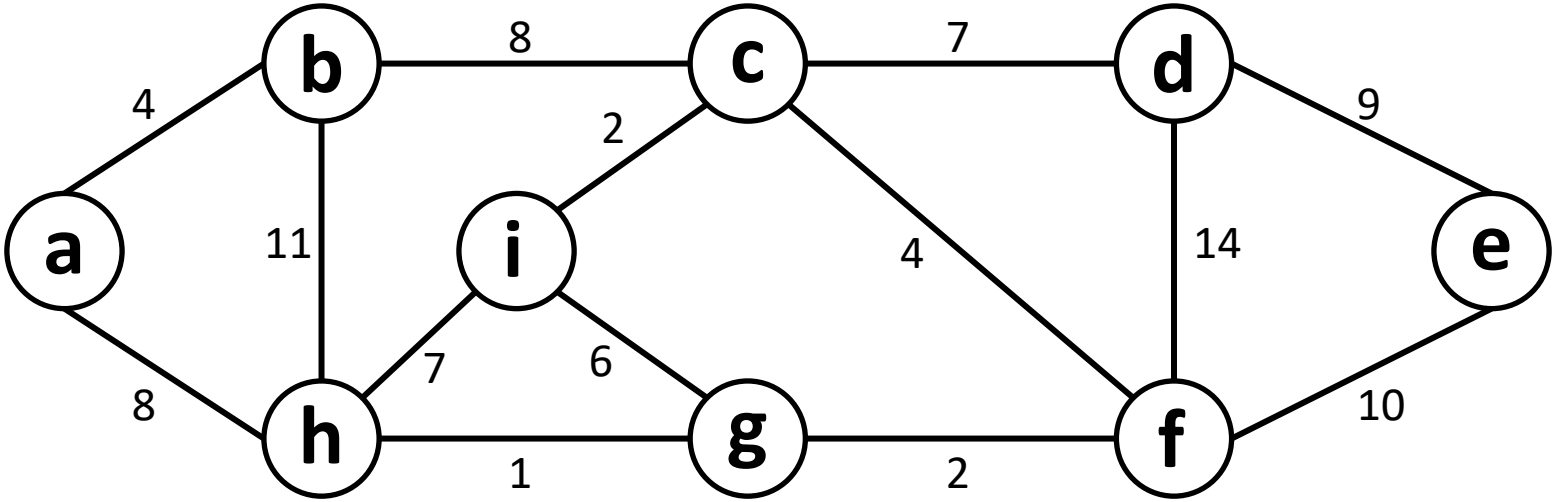

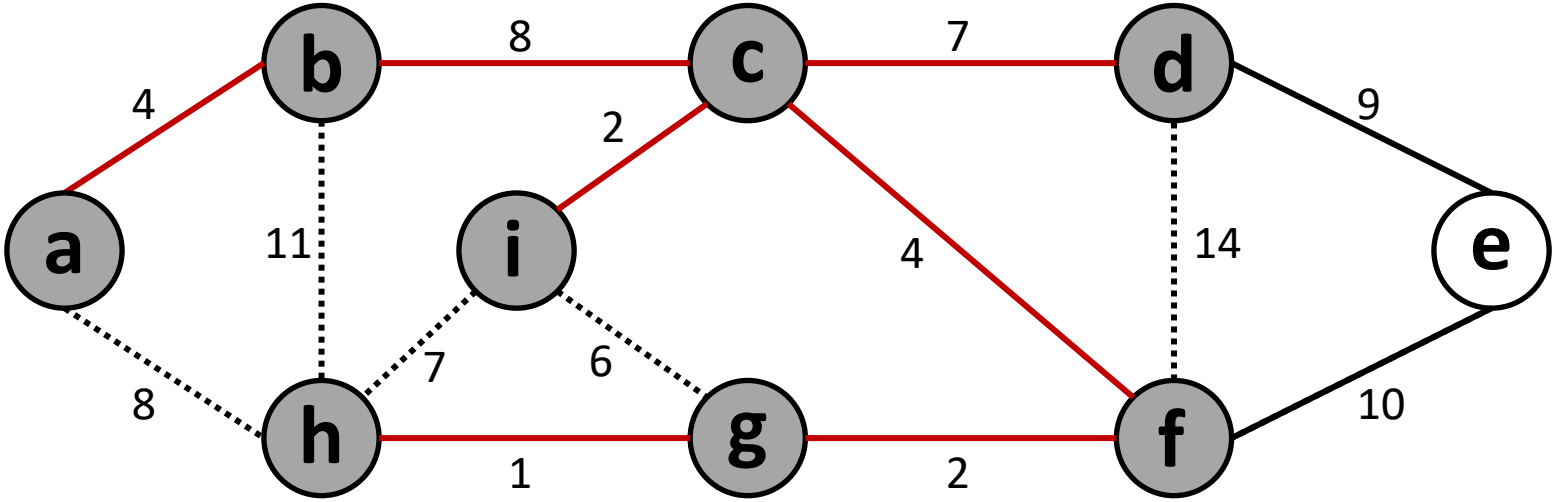

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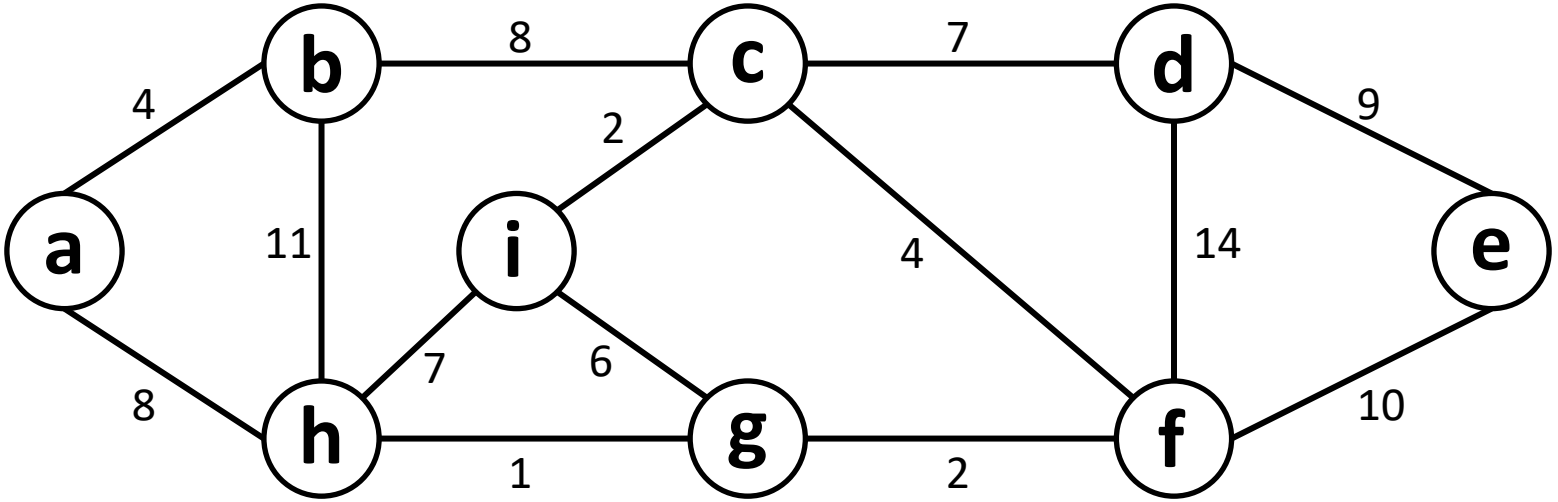
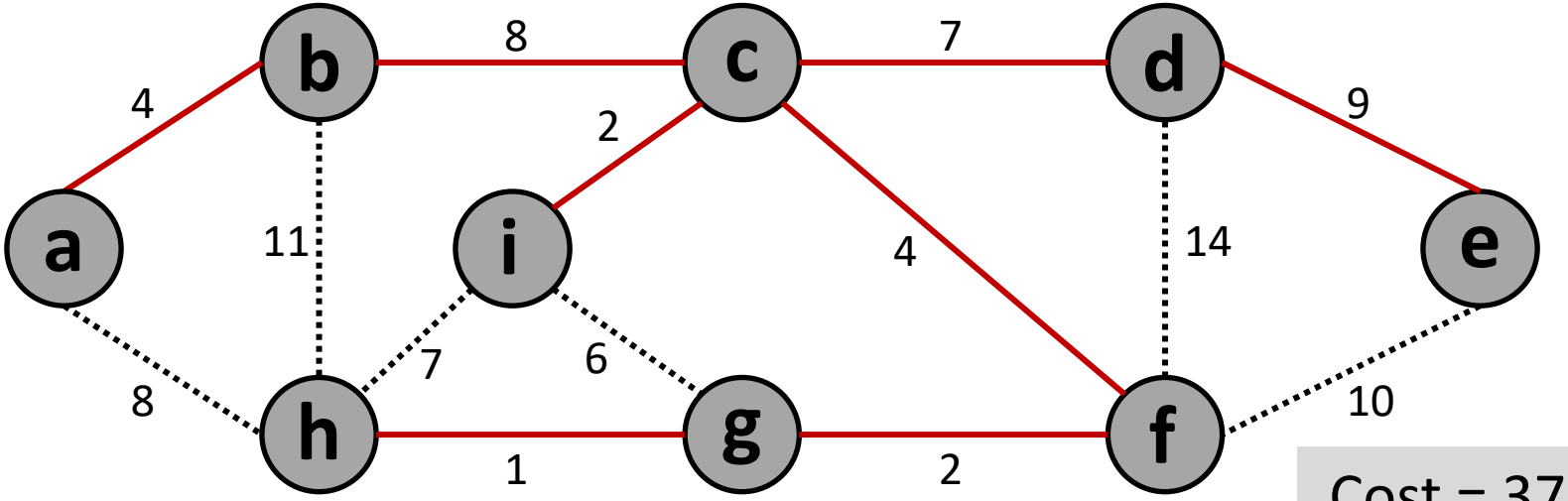




Cost = 37

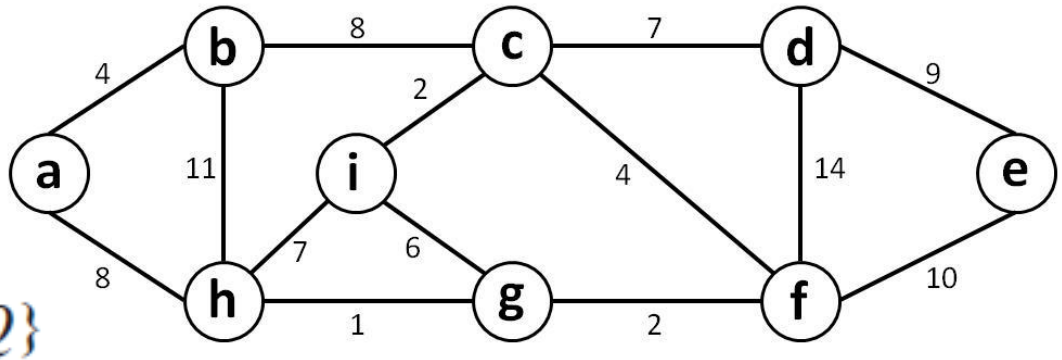
Implementation

a	→	b	4	→	h	8					
b	→	a	4	→	c	8	→	h	11		
c	→	b	8	→	d	7	→	f	4	→	i	2
d	→	c	7	→	e	9	→	f	14		
e	→	d	9	→	f	10					
f	→	c	4	→	d	14	→	e	10	→	g	2
g	→	f	2	→	h	1	→	i	6		
h	→	a	8	→	b	11	→	g	1	→	i	7
i	→	c	2	→	g	6	→	h	7		

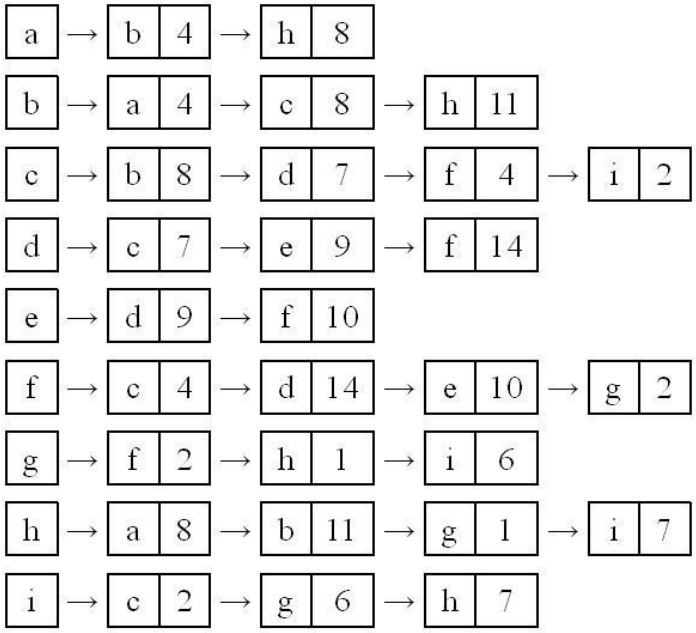
MST-PRIM(G, w, r)

```
 1  for each u ∈ G.V
 2      u.key = ∞
 3      u.π = NIL
 4  r.key = 0
 5  Q = G.V
 6  while Q ≠ ∅
 7      u = EXTRACT-MIN(Q)
 8      for each v ∈ G.Adj[u]
 9          if v ∈ Q and w(u, v) < v.key
10              v.π = u
11              v.key = w(u, v)
```

$$A = \{(v, v.\pi) : v \in V - \{r\} - Q\}$$



Example - Execution



Vertex	π	key
a	NIL	0
b	NIL	∞
c	NIL	∞
d	NIL	∞
e	NIL	∞
f	NIL	∞
g	NIL	∞
h	NIL	∞
i	NIL	∞

Q

ab-4
ah-8

```
MST-PRIM(G, w, r)
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```
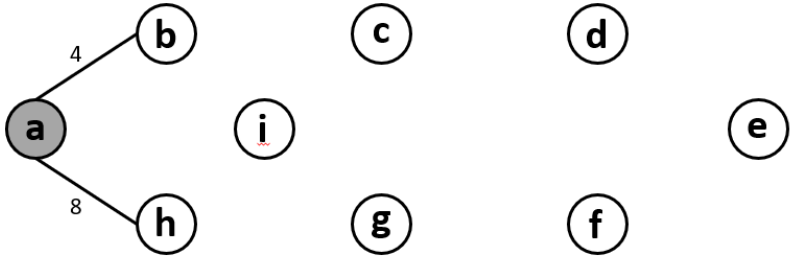
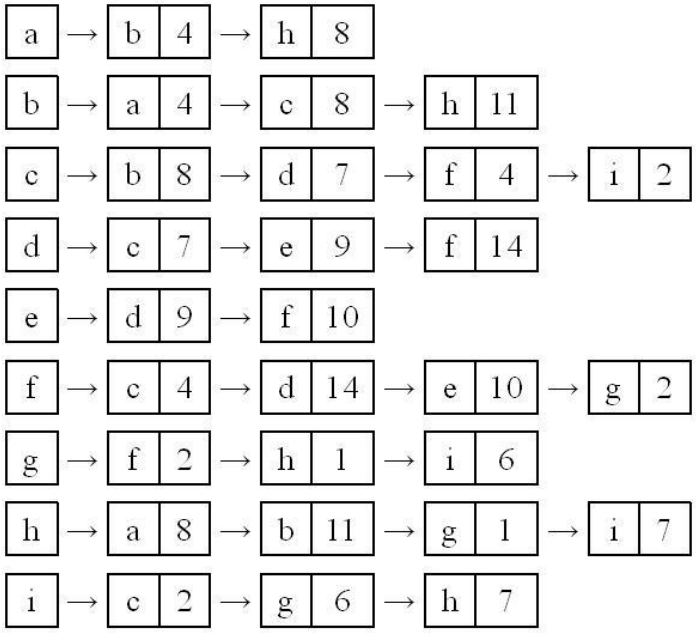



Contd…

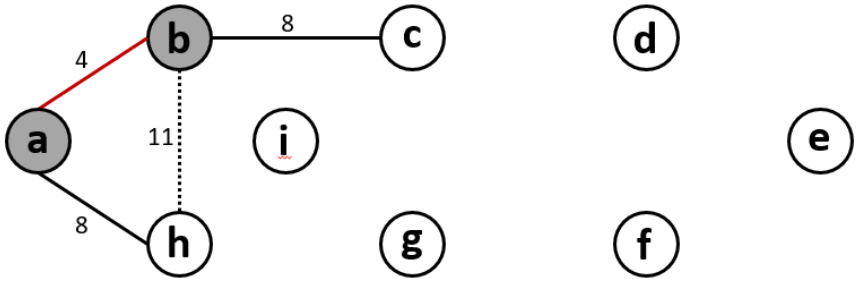


Vertex	π	key
a	NIL	0
b	a	4
c	NIL	∞
d	NIL	∞
e	NIL	∞
f	NIL	∞
g	NIL	∞
h	NIL	∞
i	NIL	∞

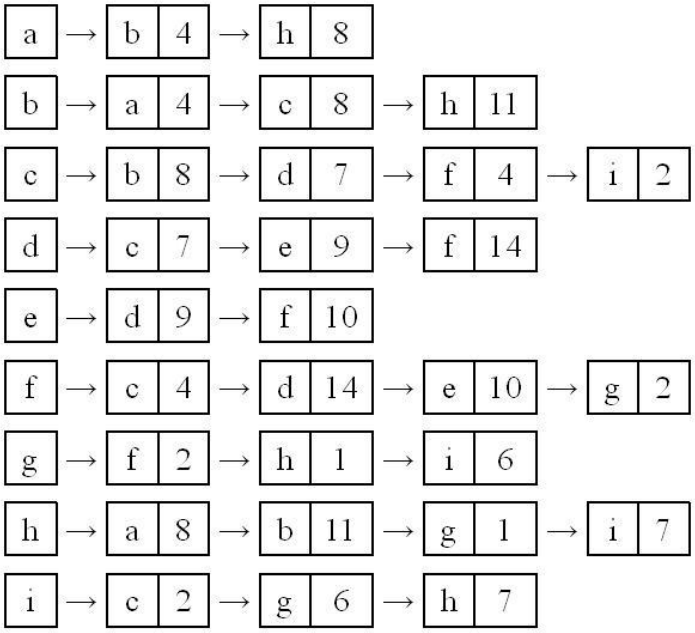
```
MST-PRIM(G, w, r)
 1  for each u ∈ G.V
 2      u.key = ∞
 3      u.π = NIL
 4  r.key = 0
 5  Q = G.V
 6  while Q ≠ ∅
 7      u = EXTRACT-MIN(Q)
 8      for each v ∈ G.Adj[u]
 9          if v ∈ Q and w(u, v) < v.key
10              v.π = u
11              v.key = w(u, v)
```

Q

bc-8
ah-8
bh-11



Contd…

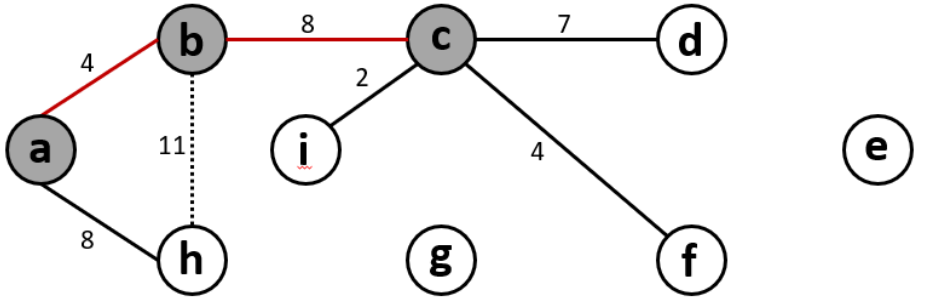


Vertex	π	key
a	NIL	0
b	a	4
c	b	8
d	NIL	∞
e	NIL	∞
f	NIL	∞
g	NIL	∞
h	NIL	∞
i	NIL	∞

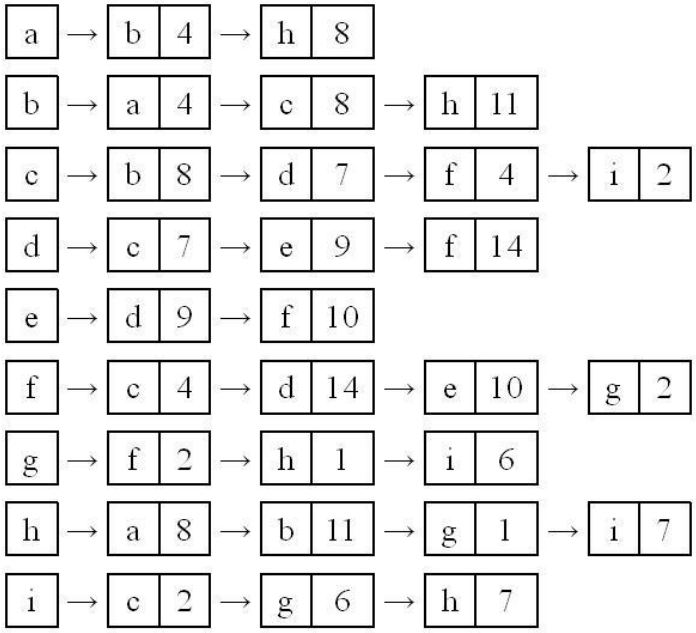
```
MST-PRIM(G, w, r)
 1  for each u ∈ G.V
 2      u.key = ∞
 3      u.π = NIL
 4  r.key = 0
 5  Q = G.V
 6  while Q ≠ ∅
 7      u = EXTRACT-MIN(Q)
 8      for each v ∈ G.Adj[u]
 9          if v ∈ Q and w(u, v) < v.key
10              v.π = u
11              v.key = w(u, v)
```

Q

ci-2
cf-4
cd-7
ha-8
bh-11



Contd…

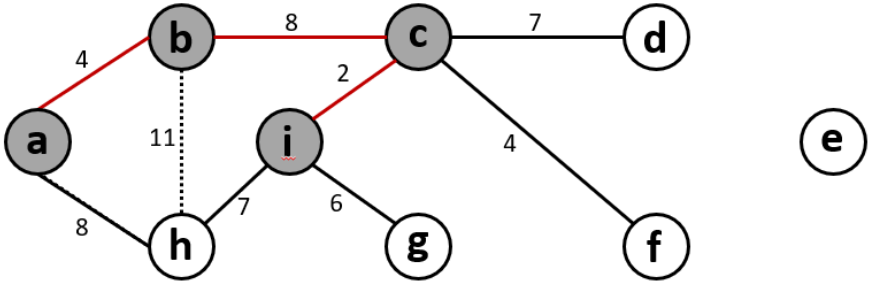


Vertex	π	key
a	NIL	0
b	a	4
c	b	8
d	NIL	∞
e	NIL	∞
f	NIL	∞
g	NIL	∞
h	NIL	∞
i	c	2

```
MST-PRIM(G, w, r)
 1  for each u ∈ G.V
 2      u.key = ∞
 3      u.π = NIL
 4  r.key = 0
 5  Q = G.V
 6  while Q ≠ ∅
 7      u = EXTRACT-MIN(Q)
 8      for each v ∈ G.Adj[u]
 9          if v ∈ Q and w(u, v) < v.key
10              v.π = u
11              v.key = w(u, v)
```

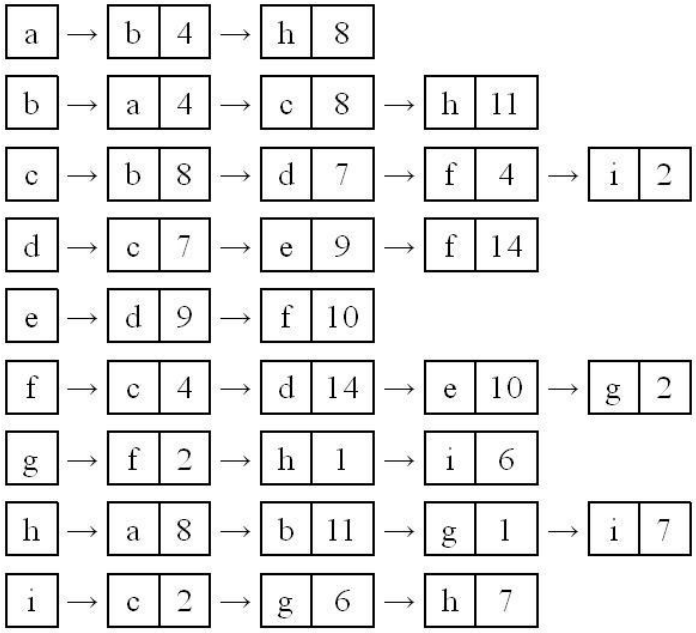
Q

cf-4
ig-6
cd-7
ih-7
ha-8
bh-11



Contd…

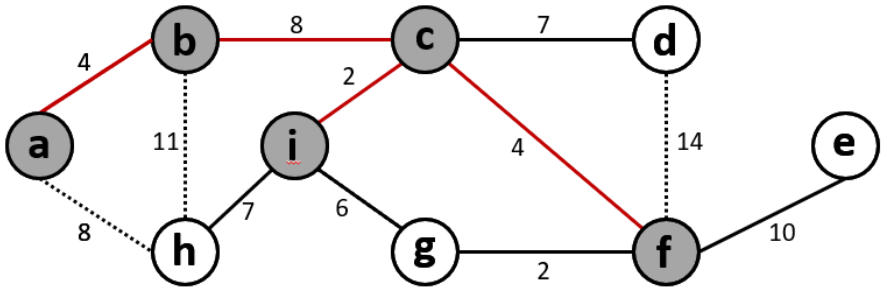
Vertex	π	key
a	NIL	0
b	a	4
c	b	8
d	NIL	∞
e	NIL	∞
f	c	4
g	NIL	∞
h	NIL	∞
i	c	2




```
MST-PRIM(G, w, r)
 1  for each u ∈ G.V
 2      u.key = ∞
 3      u.π = NIL
 4  r.key = 0
 5  Q = G.V
 6  while Q ≠ ∅
 7      u = EXTRACT-MIN(Q)
 8      for each v ∈ G.Adj[u]
 9          if v ∈ Q and w(u, v) < v.key
10              v.π = u
11              v.key = w(u, v)
```

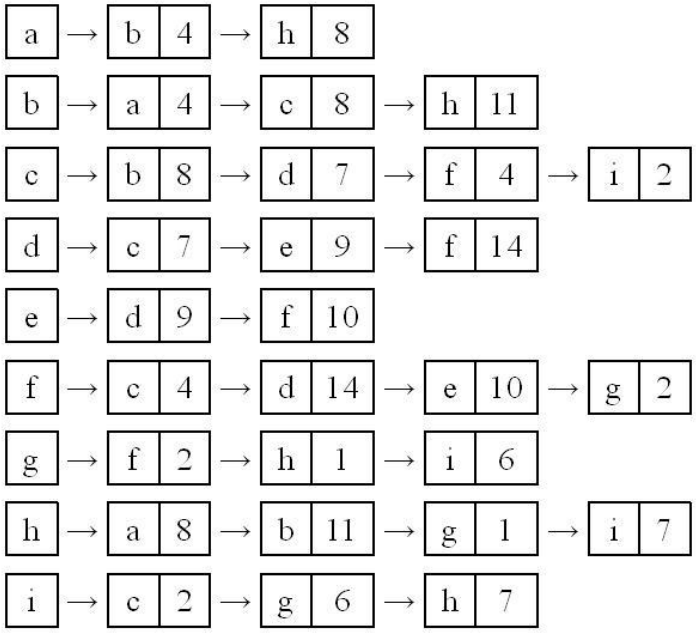
Q

fg-2
ig-6
cd-7
ih-7
ha-8
fe-10
bh-11
fd-14



Contd…

Vertex	π	key
a	NIL	0
b	a	4
c	b	8
d	NIL	∞
e	NIL	∞
f	c	4
g	f	2
h	NIL	∞
i	c	2

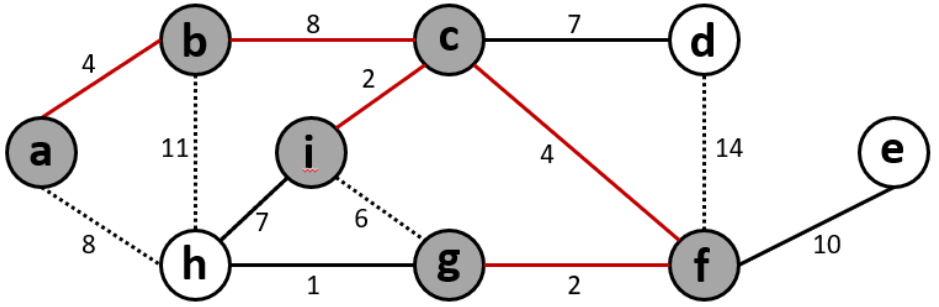



```
MST-PRIM(G, w, r)
 1  for each u ∈ G.V
 2      u.key = ∞
 3      u.π = NIL
 4  r.key = 0
 5  Q = G.V
 6  while Q ≠ ∅
 7      u = EXTRACT-MIN(Q)
 8      for each v ∈ G.Adj[u]
 9          if v ∈ Q and w(u, v) < v.key
10              v.π = u
11              v.key = w(u, v)
```

Q

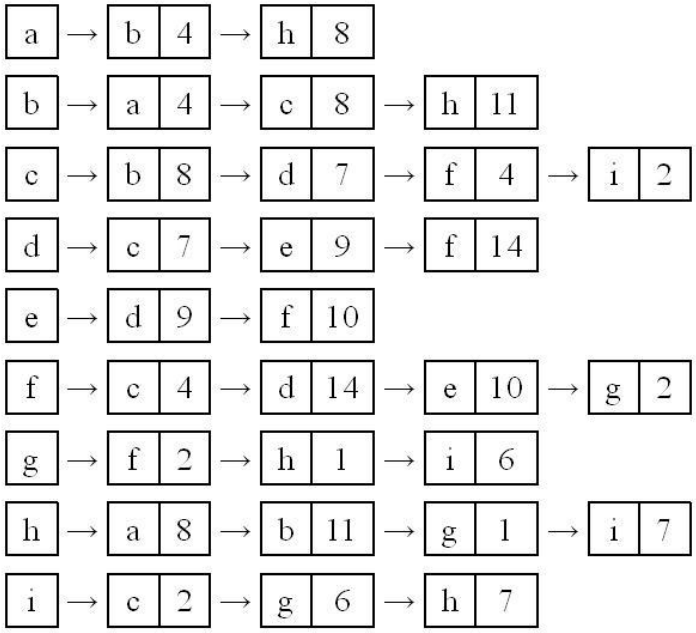
gh-1
ig-6
cd-7
ih-7
ha-8
fe-10

bh-11
fd-14



Contd…

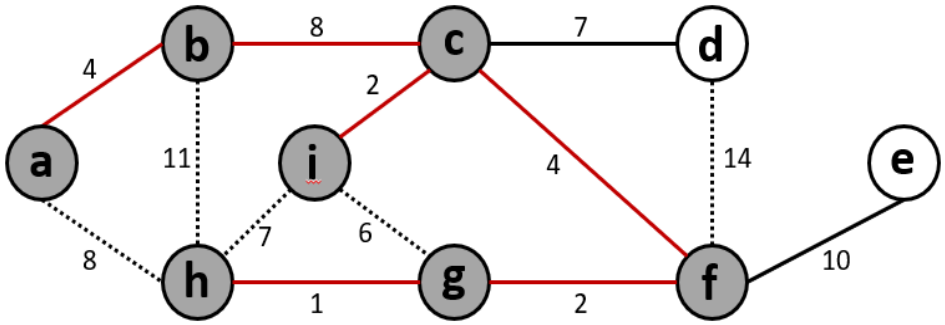
Vertex	π	key
a	NIL	0
b	a	4
c	b	8
d	NIL	∞
e	NIL	∞
f	c	4
g	f	2
h	g	1
i	c	2




```
MST-PRIM(G, w, r)
 1  for each u ∈ G.V
 2      u.key = ∞
 3      u.π = NIL
 4  r.key = 0
 5  Q = G.V
 6  while Q ≠ ∅
 7      u = EXTRACT-MIN(Q)
 8      for each v ∈ G.Adj[u]
 9          if v ∈ Q and w(u, v) < v.key
10              v.π = u
11              v.key = w(u, v)
```

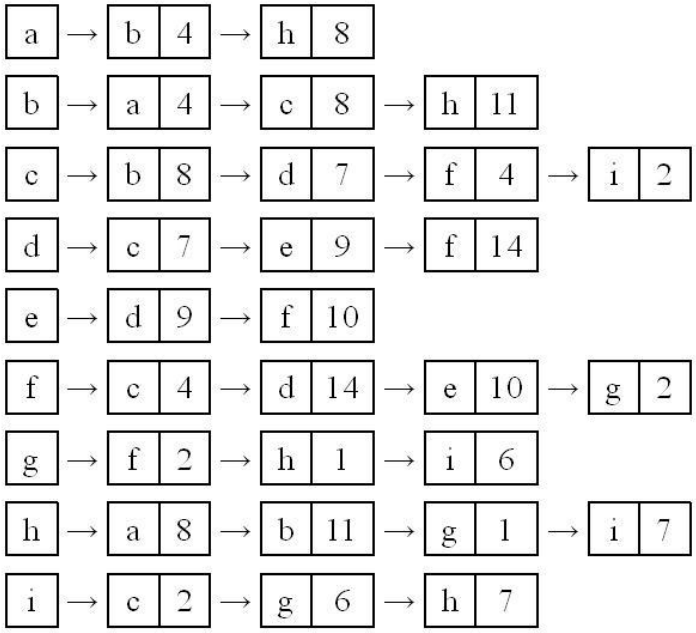
Q

ig-6
cd-7
ih-7
ha-8
fe-10
bh-11
fd-14



Contd…

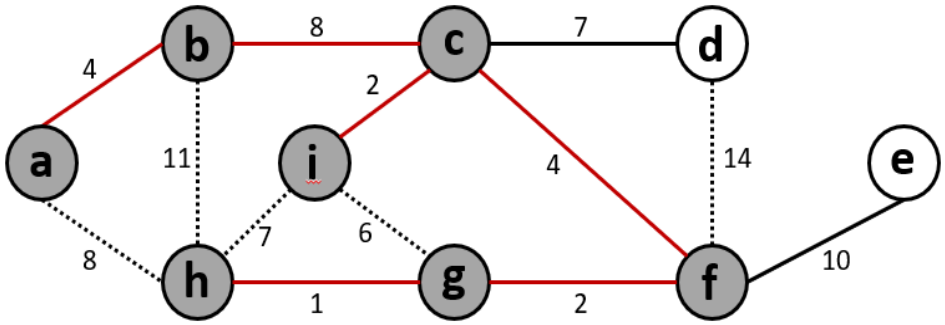
Vertex	π	key
a	NIL	0
b	a	4
c	b	8
d	NIL	∞
e	NIL	∞
f	c	4
g	f	2
h	g	1
i	c	2



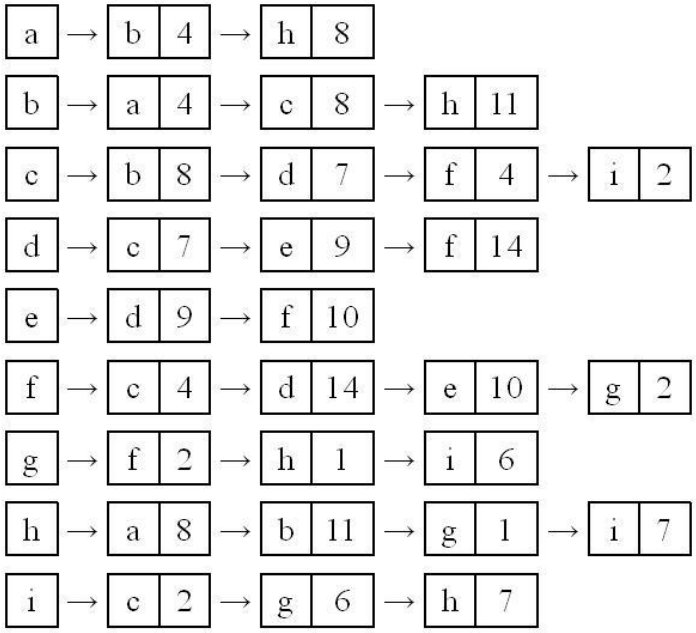
Q

cd-7
ih-7
ha-8
fe-10
bh-11
fd-14

```
MST-PRIM(G, w, r)
 1  for each u ∈ G.V
 2      u.key = ∞
 3      u.π = NIL
 4  r.key = 0
 5  Q = G.V
 6  while Q ≠ ∅
 7      u = EXTRACT-MIN(Q)
 8      for each v ∈ G.Adj[u]
 9          if v ∈ Q and w(u, v) < v.key
10              v.π = u
11              v.key = w(u, v)
```



Contd…

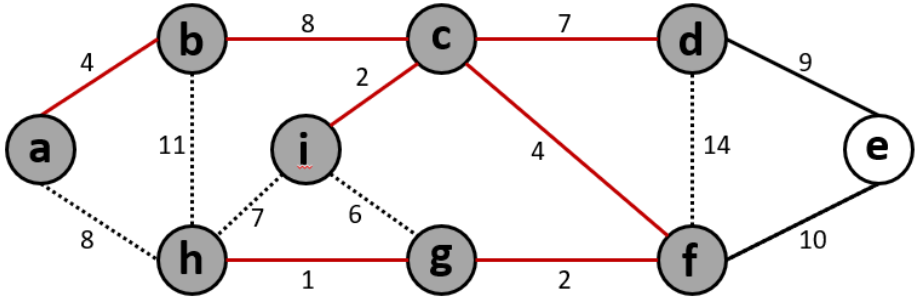


Vertex	π	key
a	NIL	0
b	a	4
c	b	8
d	c	7
e	NIL	∞
f	c	4
g	f	2
h	g	1
i	c	2

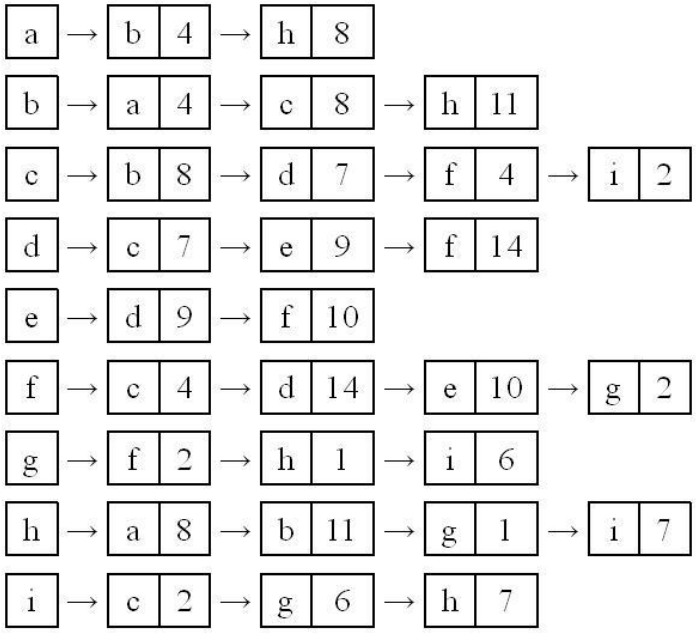
```
MST-PRIM(G, w, r)
 1  for each u ∈ G.V
 2      u.key = ∞
 3      u.π = NIL
 4  r.key = 0
 5  Q = G.V
 6  while Q ≠ ∅
 7      u = EXTRACT-MIN(Q)
 8      for each v ∈ G.Adj[u]
 9          if v ∈ Q and w(u, v) < v.key
10              v.π = u
11              v.key = w(u, v)
```

Q

ih-7
ha-8
de-9
fe-10
bh-11
fd-14



Contd…

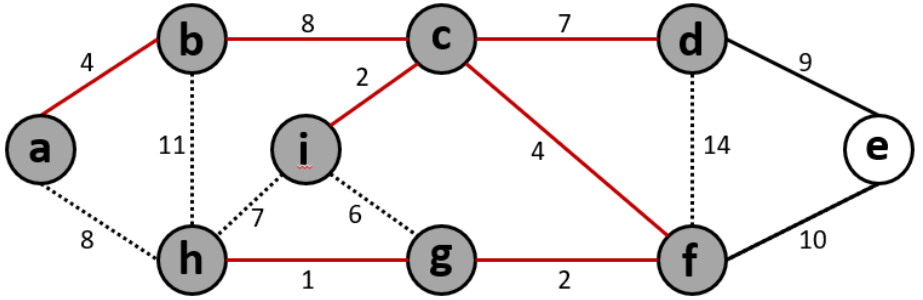


Vertex	π	key
a	NIL	0
b	a	4
c	b	8
d	c	7
e	NIL	∞
f	c	4
g	f	2
h	g	1
i	c	2

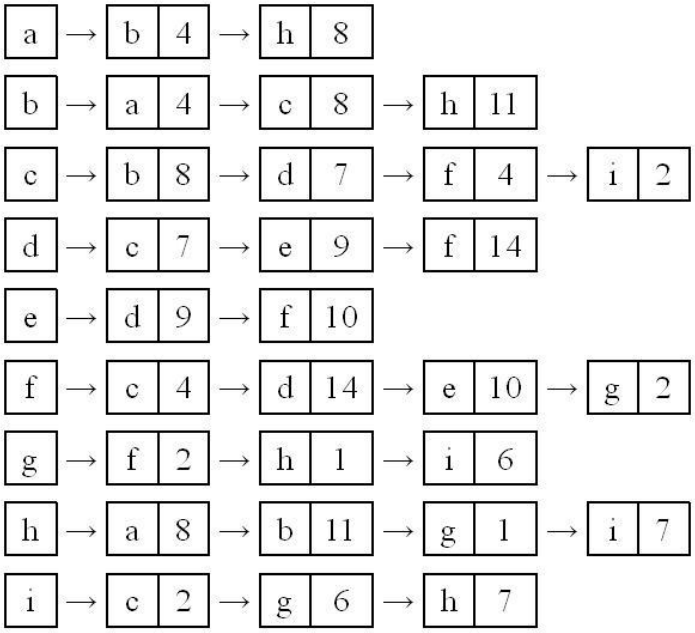
```
MST-PRIM(G, w, r)
 1  for each u ∈ G.V
 2      u.key = ∞
 3      u.π = NIL
 4  r.key = 0
 5  Q = G.V
 6  while Q ≠ ∅
 7      u = EXTRACT-MIN(Q)
 8      for each v ∈ G.Adj[u]
 9          if v ∈ Q and w(u, v) < v.key
10              v.π = u
11              v.key = w(u, v)
```

Q

ha-8
de-9
fe-10
bh-11
fd-14



Contd…

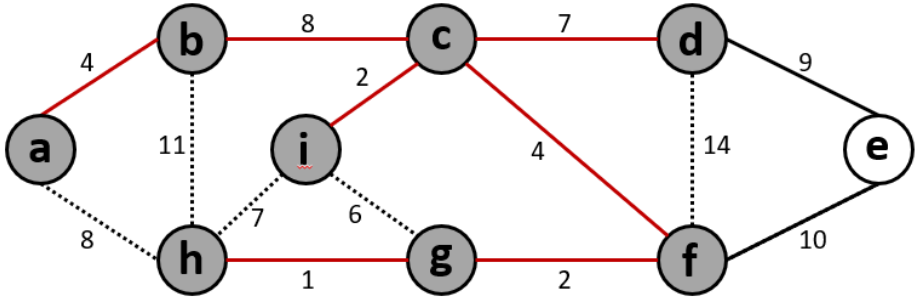


Vertex	π	key
a	NIL	0
b	a	4
c	b	8
d	c	7
e	NIL	∞
f	c	4
g	f	2
h	g	1
i	c	2

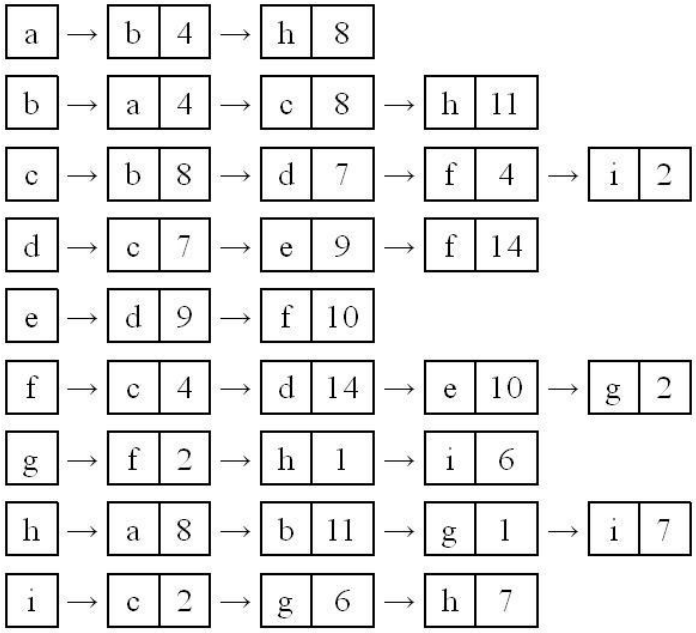
```
MST-PRIM(G, w, r)
 1  for each u ∈ G.V
 2      u.key = ∞
 3      u.π = NIL
 4  r.key = 0
 5  Q = G.V
 6  while Q ≠ ∅
 7      u = EXTRACT-MIN(Q)
 8      for each v ∈ G.Adj[u]
 9          if v ∈ Q and w(u, v) < v.key
10              v.π = u
11              v.key = w(u, v)
```

Q

de-9
fe-10
bh-11
fd-14



Contd…

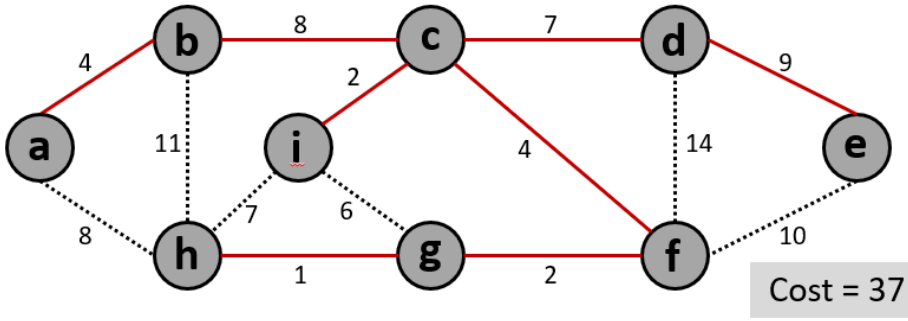


Vertex	π	key
a	NIL	0
b	a	4
c	b	8
d	c	7
e	d	9
f	c	4
g	f	2
h	g	1
i	c	2

```
MST-PRIM(G, w, r)
 1  for each u ∈ G.V
 2      u.key = ∞
 3      u.π = NIL
 4  r.key = 0
 5  Q = G.V
 6  while Q ≠ ∅
 7      u = EXTRACT-MIN(Q)
 8      for each v ∈ G.Adj[u]
 9          if v ∈ Q and w(u, v) < v.key
10              v.π = u
11              v.key = w(u, v)
```

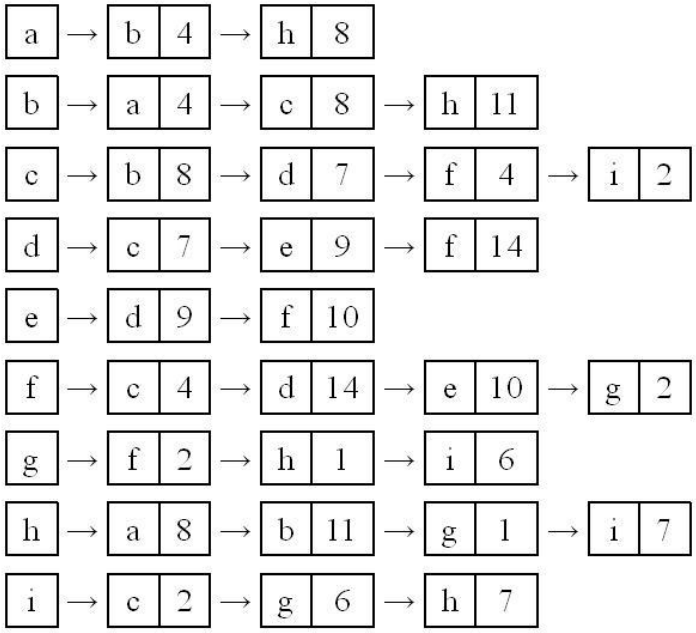
Q

fe-10
bh-11
fd-14



Contd…

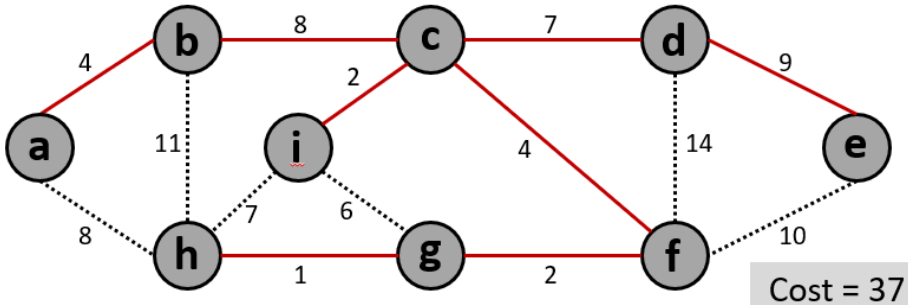
Vertex	π	key
a	NIL	0
b	a	4
c	b	8
d	c	7
e	d	9
f	c	4
g	f	2
h	g	1
i	c	2



Q

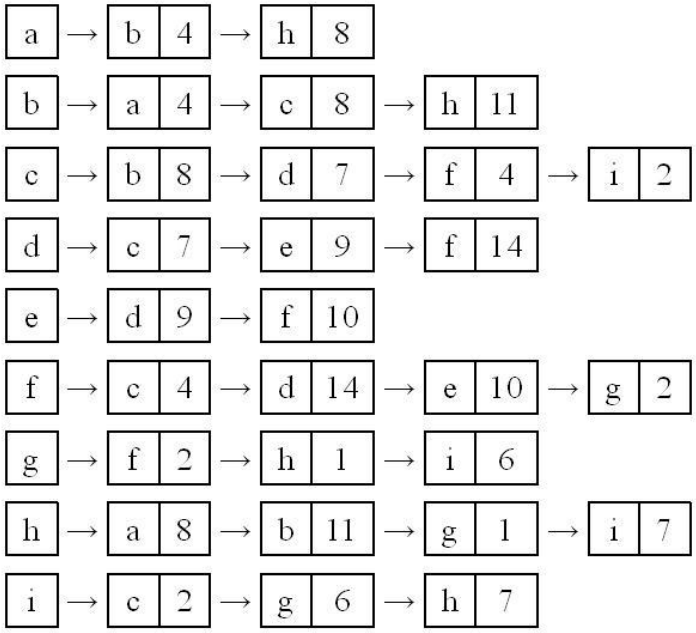
bh-11
fd-14

```
MST-PRIM(G, w, r)
 1  for each u ∈ G.V
 2      u.key = ∞
 3      u.π = NIL
 4  r.key = 0
 5  Q = G.V
 6  while Q ≠ ∅
 7      u = EXTRACT-MIN(Q)
 8      for each v ∈ G.Adj[u]
 9          if v ∈ Q and w(u, v) < v.key
10              v.π = u
11              v.key = w(u, v)
```



Cost = 37

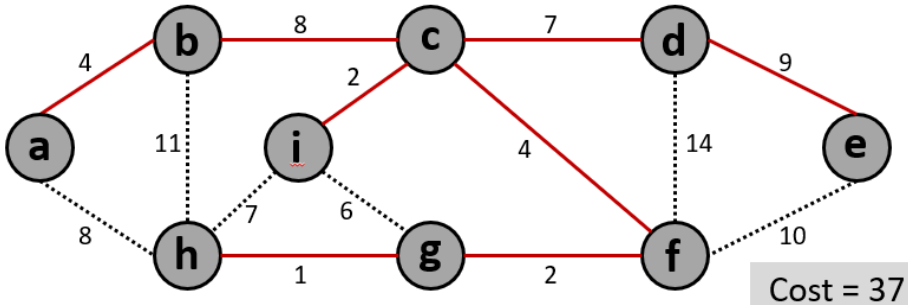
Contd…



Vertex	π	key
a	NIL	0
b	a	4
c	b	8
d	c	7
e	d	9
f	c	4
g	f	2
h	g	1
i	c	2

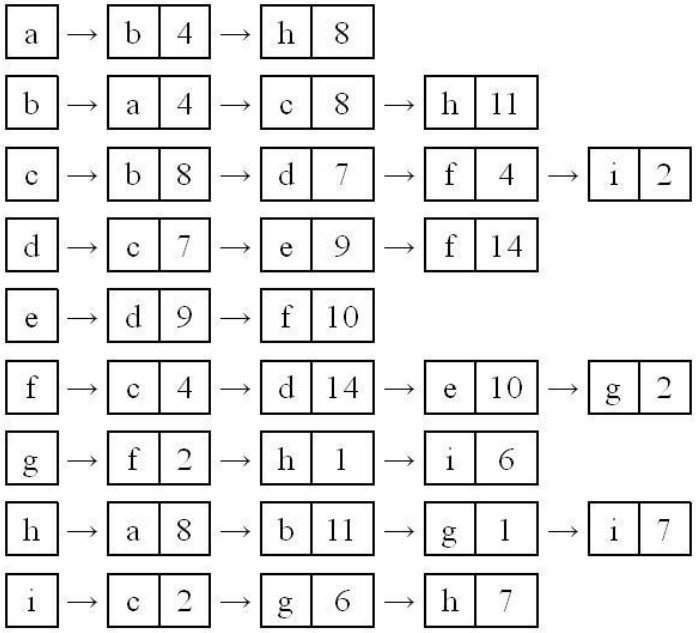
Q fd-14

```
MST-PRIM(G, w, r)
 1  for each u ∈ G.V
 2      u.key = ∞
 3      u.π = NIL
 4  r.key = 0
 5  Q = G.V
 6  while Q ≠ ∅
 7      u = EXTRACT-MIN(Q)
 8      for each v ∈ G.Adj[u]
 9          if v ∈ Q and w(u, v) < v.key
10              v.π = u
11              v.key = w(u, v)
```



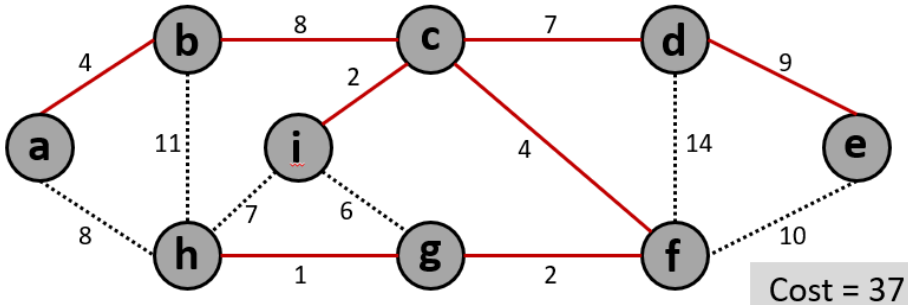
Contd…

Vertex	π	key
a	NIL	0
b	a	4
c	b	8
d	c	7
e	d	9
f	c	4
g	f	2
h	g	1
i	c	2



Q

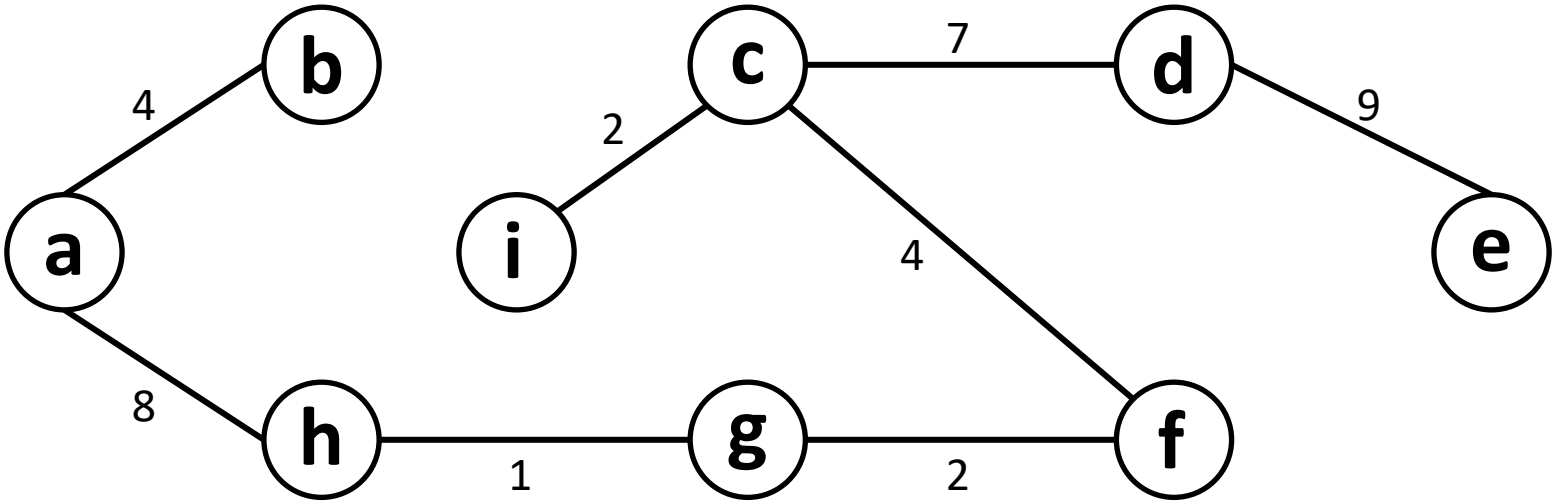
```
MST-PRIM(G, w, r)
 1  for each u ∈ G.V
 2      u.key = ∞
 3      u.π = NIL
 4  r.key = 0
 5  Q = G.V
 6  while Q ≠ ∅
 7      u = EXTRACT-MIN(Q)
 8      for each v ∈ G.Adj[u]
 9          if v ∈ Q and w(u, v) < v.key
10              v.π = u
11              v.key = w(u, v)
```



Cost = 37

Contd…

A = {(b,a), (c,f), (d,c), (e,d), (f,g), (g,h), (h,a), (i,c)}



Cost = 37