Heaps

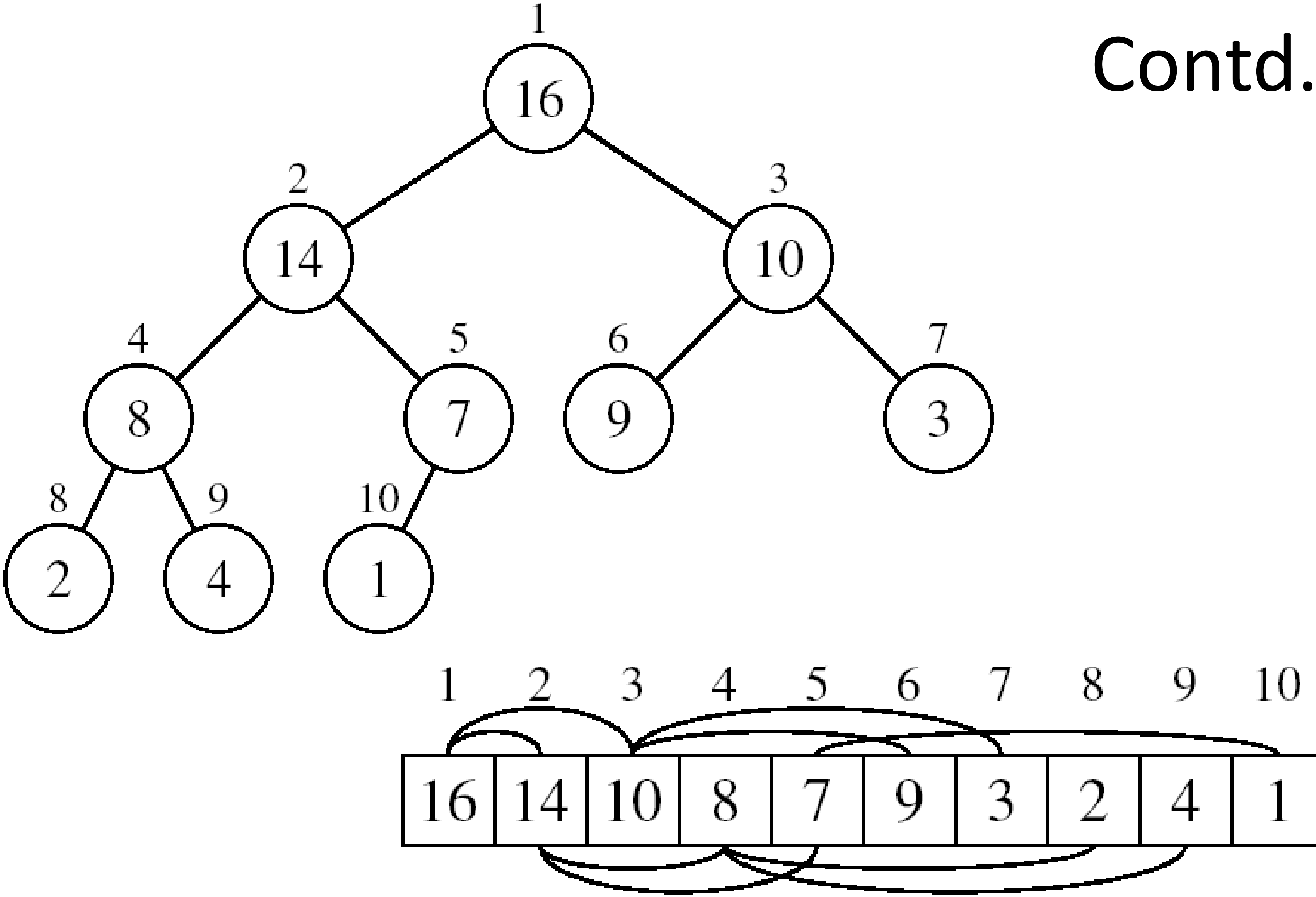
Heap Data Structure

- It is a nearly complete binary tree.
  - All levels are full, except possibly the last one, which is filled from left to right.
  - Due to this, they are commonly stored using arrays as there is no memory wastage.
- Values in the nodes satisfy a heap property, the specifics of which depend on the kind of heap.

Array Representation of Heaps

- A heap can be stored as an array *A*.
  - Root of tree is $A[1]$
  - Left child of $A[i] = A[2i]$
  - Right child of $A[i] = A[2i + 1]$
  - Parent of $A[i] = A[\lfloor i/2 \rfloor]$
  - Number of elements in the array = $A.length$
  - Number of elements in the heap which are stored within array A = $A.heap\text{-}size$
  - $0 \leq A.heap\text{-}size \leq A.length$
- The elements in the subarray $A[(\lfloor n/2 \rfloor+1) \,..\, n]$ are leaves

Contd…



Types of Heaps

- **Max-heaps** (largest element at root), satisfy the *max-heap property:*
  – for every node i other than the root,

$$A[\mathrm{PARENT}(i)] \geq A[i]$$

- **Min-heaps** (smallest element at root), satisfy the *min-heap property:*
  – for every node i other than the root,

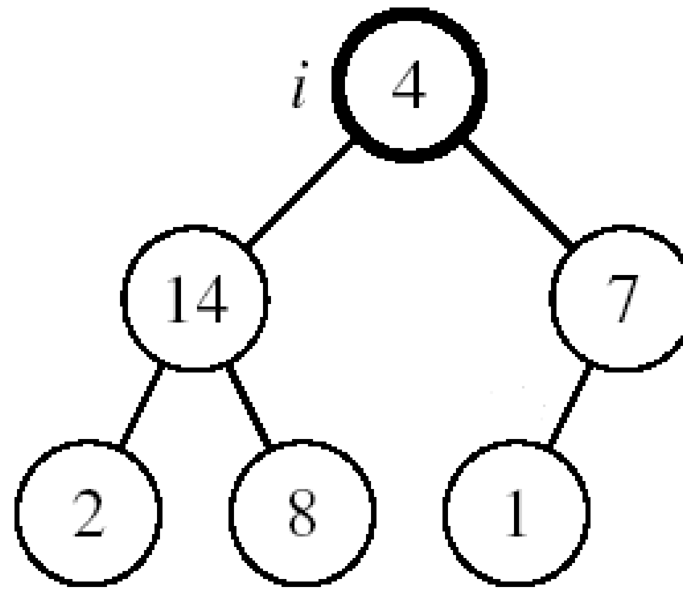
$$A[\mathrm{PARENT}(i)] \leq A[i]$$

Operations on Heaps

- Maintain/Restore the max-heap property
  - MAX-HEAPIFY
- Create a max-heap from an unordered array
  - BUILD-MAX-HEAP
- Sort an array in place
  - HEAPSORT
- Priority queues
  - MAX-HEAP-INSERT,
  - HEAP-EXTRACT-MAX,
  - HEAP-INCREASE-KEY, and
  - HEAP-MAXIMUM

Maintaining the Heap Property

- Suppose a node is smaller than a child
  - Left and Right subtrees of *i* are max-heaps
- To eliminate the violation:
  - Exchange with larger child
  - Move down the tree
  - Continue until node is not smaller than children



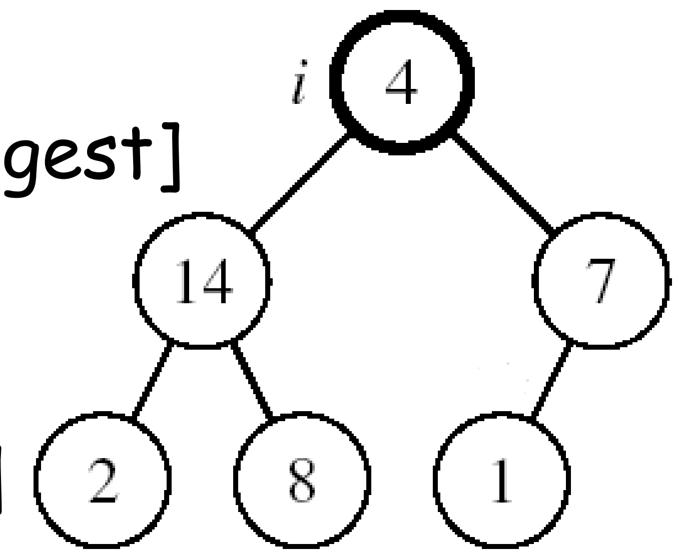
Maintaining the Heap Property

MAX-HEAPIFY(*A*, i)

```
l = LEFT(i)
r = RIGHT(i)
if l ≤ A.heap-size and A[l] > A[i]
    then largest = l
    else largest = i
if r ≤ A.heap-size and A[r] > A[largest]
    then largest = r
if largest ≠ i
    then exchange A[i] ↔ A[largest]
MAX-HEAPIFY(A, largest)
```

Assumptions:

- Left and Right subtrees of **i** are max-heaps
- *A*[i] may be smaller than its children

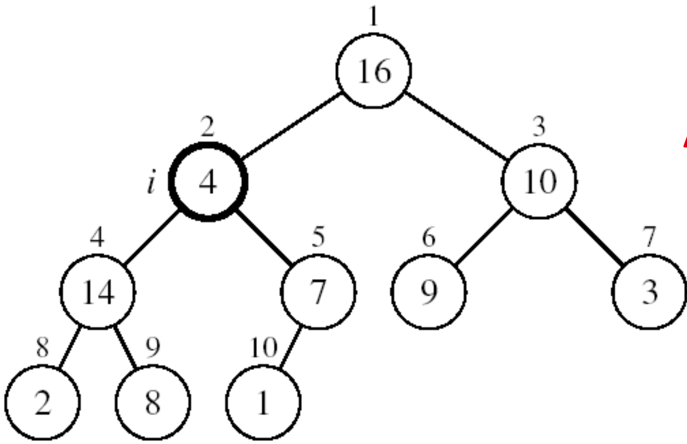


Example

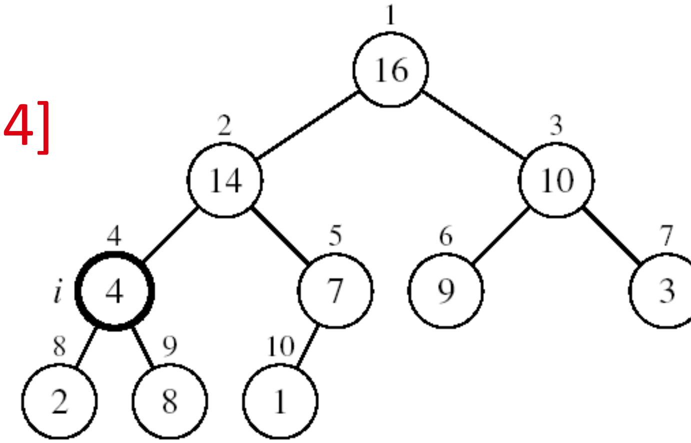
$A.heap\text{-}size = 10$

$A.length = 10$

<u>MAX-HEAPIFY(A, 2)</u>



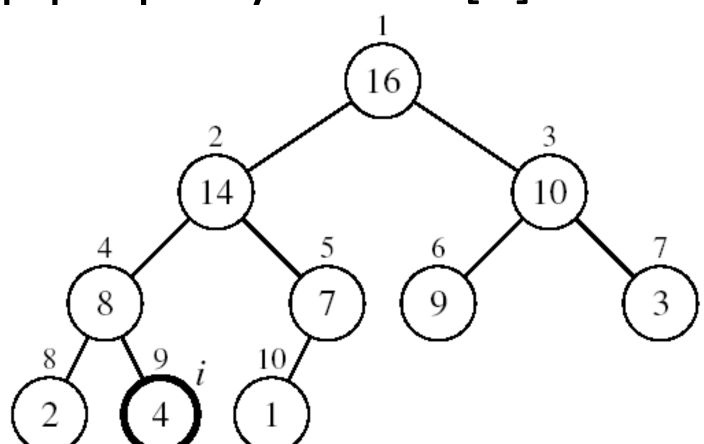
A[2] ↔ A[4]



A[2] violates the heap property

A[4] violates the heap property

A[4] ↔ A[9]



Heap property restored

MAX-HEAPIFY Running Time

- It traces path from root to a leaf.
- In worst case length of length path is h.
- Running time of MAX-HEAPIFY is $O(h)$ or $O(\lg n)$
  - Since the height of the heap is $\lg n$.

OR

- In the worst case (last level is exactly half-filled), the children's subtrees each have size at most 2n/3.
- Thus, running time of MAX-HEAPIFY

$$T(n) \leq T(2n/3) + \Theta(1)$$

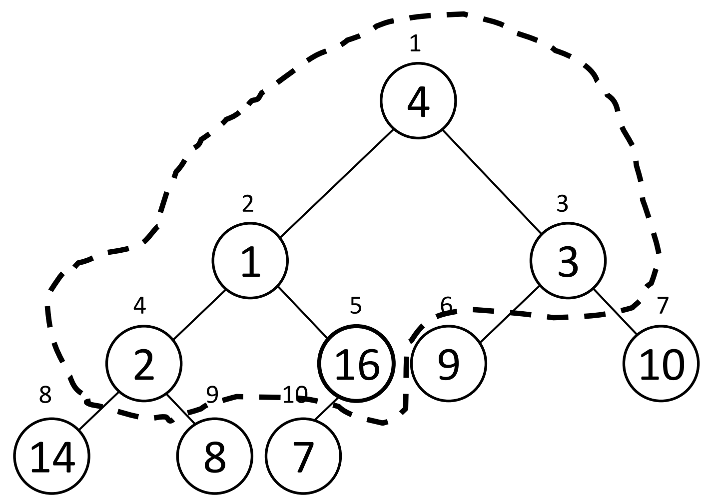
- Case 2 of master theorem. $T(n) = O(\lg n)$

Building a Heap

- Convert an array $A[1 \ldots n]$, where $n = A.length$ into a max-heap.
- The elements in the subarray $A[(\lfloor n/2 \rfloor+1) \ldots n]$ are leaves.
- Apply MAX-HEAPIFY on elements between 1 and $\lfloor n/2 \rfloor$.

<u>BUILD-MAX-HEAP(A)</u>

1. $A.heap\text{-}size = A.length$
2. **for** $i = \lfloor A.length/2 \rfloor$ **downto** 1
3. MAX-HEAPIFY(A, i)

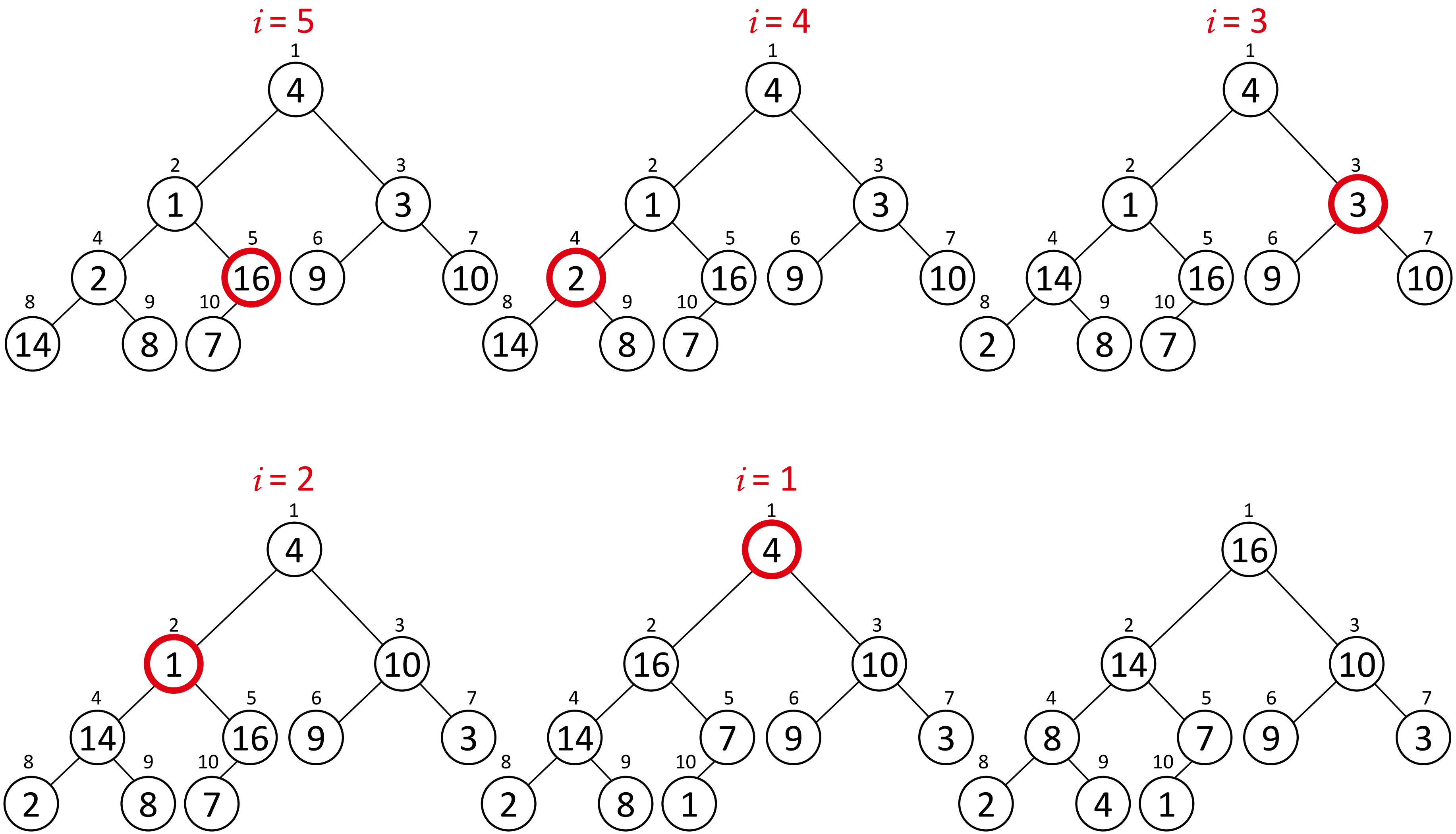


A:

4	1	3	2	16	9	10	14	8	7

Example:

A										
	4	1	3	2	16	9	10	14	8	7



Running Time of BUILD MAX HEAP

<u>BUILD-MAX-HEAP(A)</u>

1. A.heap-size = A.length
2. **for** i = $\lfloor A.length/2 \rfloor$ **downto** 1 $O(n)$
3. MAX-HEAPIFY(A, i) $O(\lg n)$

$\Rightarrow$ Running time: $O(n \lg n)$

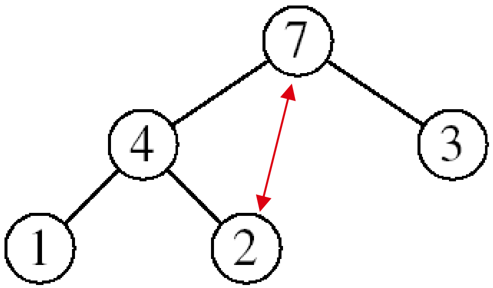
- This is not an asymptotically tight upper bound

Running time of BUILD-MAX-HEAP is: $T(n) = O(n)$

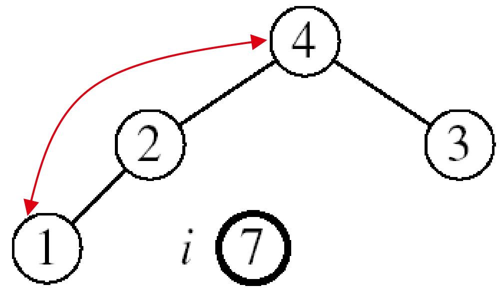
Heapsort

- Goal:
  – Sort an array using heaps.
- Idea:
  – Build a **max-heap** from the array
  – Swap the root (the maximum element) with the last element in the array
  – "Discard" this last node by decreasing the heap size
  – Call MAX-HEAPIFY on the new root
  – Repeat this process until only one node remains

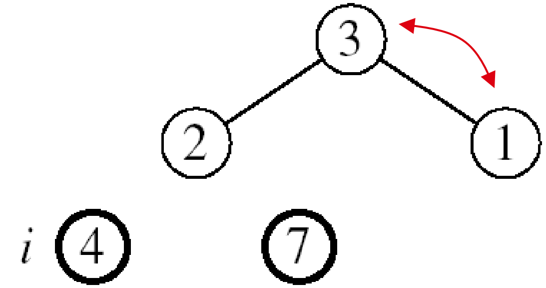
Sort: 4, 7, 3, 1, 2   A=[7, 4, 3, 1, 2]



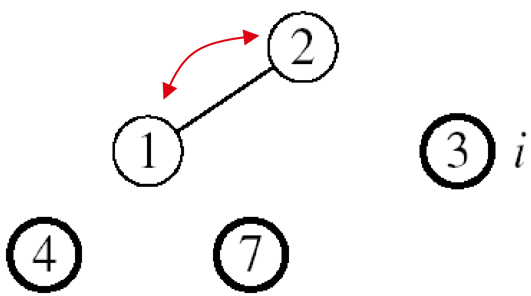
MAX-HEAPIFY(A, 1)



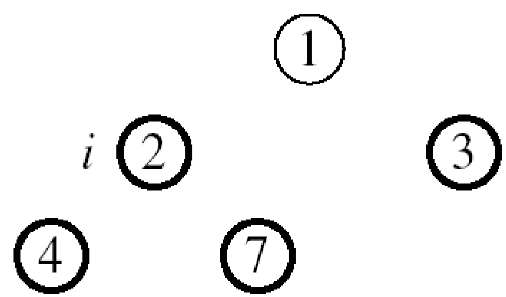
MAX-HEAPIFY(A, 1)



MAX-HEAPIFY(A, 1)



MAX-HEAPIFY(A, 1)



A	1	2	3	4	7

HEAPSORT(*A*)

1.	BUILD-MAX-HEAP(*A*)	$O(n)$
2.	**for** i = *A.length* **downto** 2	n – 1 times
3.	**do** exchange *A*[1] with *A*[i]	constant
4.	*A.heap-size* = *A.heap-size* - 1	constant
5.	MAX-HEAPIFY(*A*, 1)	$O(\lg n)$

- Running time: $O(n \lg n)$

Applications

- Heap sort
- Priority queues: Query for minimum or maximum value in a dynamic collection of values.
- Dijkstra's algorithm for finding the shortest path between a pair of nodes uses heap to pick the closest unexplored node at each iteration to continue the search from it.
  Example: routing of network packets between two nodes.
- Prim's algorithm for finding the Minimum Spanning Tree uses heap to select a new minimum-cost edge that expands your current minimum spanning tree.
  Example: wire layout for a service network, such as electricity or cable. Aim is to provide service coverage to an entire area with the minimum wiring cost possible.
- Huffman encoding (data compression).
- Used by an operating system for dynamic memory allocation.