TOTAL ORDERING BASED ON SPACE FILLING CURVES FOR MULTIVALUED MORPHOLOGY.

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Abstract. This paper addresses the problem of the extension of morphological operators to the case of multicomponent images. It basically comes to the definition of a well suited vectorial ordering relation. We briefly discuss the possible extensions found in the literature and propose an ordering scheme based on the bit mixing paradigm. This choice is justified using the space filling curve equivalent representation of some classical total ordering relations. Results concerning the extension of alternating sequential filters by reconstruction to the case of colour images are presented.

Key words: vectorial ordering, space filling curves, multivalued morphology

1. Introduction

Mathematical morphology theory has now become a widely used non linear technique for image processing. Initially designed as a set theory, it was then generalized to the set of gray-level images with the help of umbras, and, more recently, to any complete lattice, which is the appropriate mathematical framework for morphology [1] [12] [13].
This minimum necessary algebraic structure is a set T such that:
- T is induced by a (partial) ordering relation " ≤ " ;
- for any family (finite or not) of elements in T, there exists a smallest majorant called the sup (for supremum) and a greatest minorant called the inf (for infimum).

Concurrently, multicomponent image processing has become an important field of research. If the image has N different components, each pixel is now represented by an N-dimensional vector.
To extend morphological operators to multicomponent images, it is of course possible to process each component separately and to recombine the results afterwards (Marginal processing). This approach is not fully satisfying for different reasons :
- since the output vector will rarely be one of the input vectors, the appearance of, for instance, false colours (in the case of colour images) is quite inevitable,
- if the image has N different components, N processings are required, leading to time consuming operators,
- finally, this approach does not take into account the inter-component correlation.

For these reasons, it is preferable to use a purely vectorial approach that process the image all at once. But we then have to define a well suited vectorial lattice structure to work in and,

in particular, a vectorial ordering relation is required to define the sup and the inf of any family of N-dimensional vectors.

Section 2 presents the unifying approach to multivariate data ordering formalized in [2]. Section 3 presents a specific total vectorial ordering relation proposed in [3] based on the bit mixing paradigm. Section 4 presents the equivalence existing between any total vectorial order and a space filling curve. Before the conclusion, section 5 presents the application of the bit mixing paradigm to the colour extension of ASF by reconstruction.

2. Vectorial ordering relations : a formal unifying approach

The key point to extend morphological operators to the N-dimensional case is the definition of a vectorial ordering relation that induces a lattice structure on the data. Goutsias [2] presents a unifying formalism based on the used of a vector transform h from the working space $\Re^N$ into $\Re^Q$ (with $Q \leq N$ *a priori*) followed by a marginal ordering on $\Re^Q$:

$$\begin{aligned} h : \Re^N &\rightarrow \Re^Q \\ X &\rightarrow h(X) \end{aligned}$$

Assuming an ordering relation has previously be defined on $\Re^Q$, an order can be defined on $\Re^N$ in a straightforward way using the following relation :

$$\forall (X,Y) \in \Re^N \times \Re^N ,\ X \leq Y \Leftrightarrow h(X) \leq h(Y)$$

Depending on the used transform and on the value of Q, different situations are possible. The two next sections respectively discuss the case Q>1 and Q=1.

2.1. Q > 1 : partial ordering relation

The only existing order on $\Re^Q$, with Q>1, is the canonic order (the well known lexicographic order corresponds to the case Q=1 with the appropriate transform). In this structure, two vectors X and Y from the product space $\Re^Q$ are compared in the following way :

$$X \leq Y \Leftrightarrow X(i) \leq Y(i) ,\ \forall\, i \in \{1,...,Q\}$$

This order is a partial order. For instance vectors (1,2,3) and (3,2,1) can not be compared. Nevertheless, this canonic lattice structure is often used for multivalued morphology [4] [5]. This ordering relation can be used directly on the input data (case h = Identity application). In this case, basic operations such as the supremum or the infimum, and all their possible compositions, are strictly equivalent to a marginal processing. It is also possible to use other transforms h such as the Karhunen Loeve transform or the principal components analysis [6] and to use the canonic lattice structure on the transformed data after this change of base. But, in every case, a partial order is induced, leading unavoidably to the appearance of false colours : the maximum of a set of vectors is not defined and the supremum is rarely one of the input vectors.

Furthermore, using a partial ordering relation poses an algorithmic problem. All existing fast programs have been meant to deal with gray level images, i.e. with totally ordered complete lattices, and most of them cannot be directly adapted to partially ordered lattices.

2.2. Q = 1 : total (pre-)ordering relation

To avoid the problem of false colour appearance, it is necessary to use a transform h with Q=1. In this case, the maximum of a set $\{X_j\}$ of vectors is always defined.

• If h is not injective, this maximum may not be uniquely defined. The non-antisymmetrical relation induced by such a transform is a **pre-ordering** relation, leading to what are called "morphological-like" filters (the theoretical background is not strictly respected) [6]. Goutsias [2] explains how to solve this theoretical problem by defining a new equivalence relation between the vectors to restore the antisymmetrical property of the induced relation, but the problem of uniquely defining the supremum (or equivalently here, the maximum) still remains. Nevertheless, some good results may be provided using pre-ordering relations. For instance, at each pixel is associated the value of its i^{th} component, with i fixed *a priori* or adaptatively determined :

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With the general formalism, if the image has N different components coded each on p bits, this ordering relation corresponds to the following transform h :

$$\begin{aligned} h: T^{N} &\rightarrow Im_h(T^{N}) \\ X &\rightarrow \sum_{i=1}^{N} X(N+1-i) \cdot 2^{p \cdot (i-1)} \end{aligned}$$

where T is the set $\{0,1,...,2^p-1\}$. This order is not fully satisfying either since it induces an important dissymmetry between the different components of the image.

3. The bit mixing paradigm

Following the same formalism, we propose here a transform h that bijectively maps N-dimensional vectors into $\mathfrak{R}$. This transform is based on the binary representation of each component of the considered vector X. If the N components of X are coded with p bits each, the N.p available bits are mixed up together to build the N.p bits long scalar value h(X). Of course, there are many different ways to mix up these N.p bits, but we choose the one that minimizes the unavoidable dissymmetry between the different components. So, we first take the first bit of X(1), then the first bit of X(2), until the first bit of X(N), and we repeat the process with the second bit, and so on until the p^{th} bit. Mathematically, this can be written as follows: $X \in \mathfrak{R}^N$ has N components X(i), each one coded on p bits $X(i)_j \in \{0,1\}$ with $j \in \{1,...,p\}$. The considered mapping h can then be written as follows:

$$h(X) = \sum_{j=1}^{p} \left\{ 2^{N \cdot (p-j)} \cdot \sum_{i=1}^{N} 2^{N-i} \cdot X(i)_j \right\}$$

In the same way , we can define the reciprocal mapping h^{-1}. Let S be an integer scalar coded with N.p bits $S_j \in \{0,1\}$ with $j \in \{1,...,N.p\}$. Then, each component X(i) of $h^{-1}(S) = X \in \mathfrak{R}^N$

is defined by:

$$X(i) = \sum_{j=1}^{p} 2^{p-j} \cdot s_{N \cdot (p-j) + N - i + 1}$$

In the following, for the sake of clarity and simplicity, we will restrict this paper, without any loss of generality, to the case of the colour images, represented in the classical 3*8 bits Red Green Blue base. Figure 1 presents the coding and the decoding of such a colour pixel.

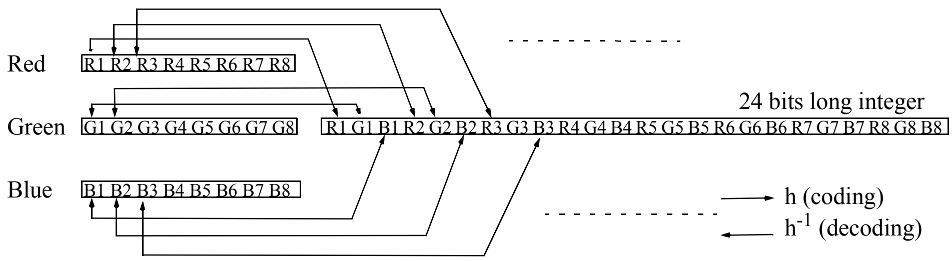


Figure 1 : Coding (h) and decoding (h^{-1}) of one colour pixel.

As previously said, we chose the coding that minimizes the unavoidable dissymmetry inherent to this kind of coding : approximately the same weight is attached to each component of the image (all the components are processed quite symmetrically). Furthermore, no significative changes have been observed in the application presented §5 by inverting the order of the components (the remaining small dissymmetry is of no consequence). Nevertheless, if some *a priori* information concerning the relative importance of the different components is available, it can be taken into account by attaching a higher weight to the most significative component (for instance by putting its 2 or 3 first bits at the head of the coding).

On the contrary, to increase the symmetry between the relative importance of each component, the following variant can be used : during the coding, the order of the mixed bits is randomly changed at each range, giving for instance the following result, with the same notations as in figure 2 : R1.G1.B1.G2.B2.R2.B3.R3.G3...

Of course, the same order is kept for all the pixels of the image. We will see in section 4 that this variant theoretically improves the quality of the coding.

The order induced by the canonic lattice structure is probably the most natural and intuitive vectorial order, even if it is just a partial one. A key-point is that the order induced by the bit mixing paradigm or its variant can be regarded as a simple extension of the canonic order (if two vectors are comparable in the canonic structure, they will keep the same order with our relation), but this partial order has been completed into a total one : two vectors are now always comparable.

4. Total vectorial ordering & space filling curves

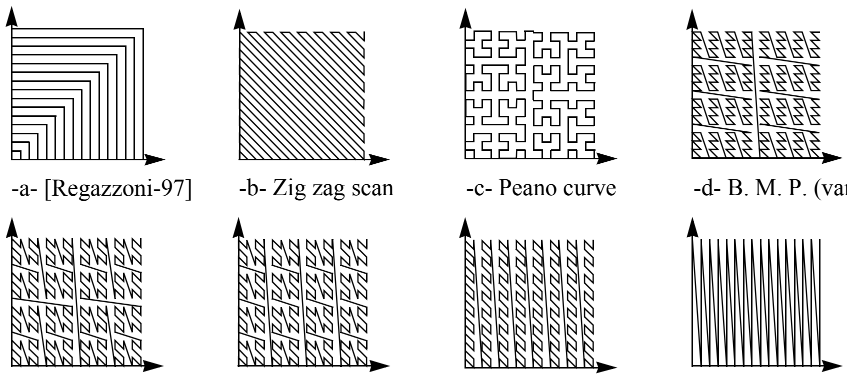
4.1. Presentation

For a given N-dimensional space, a space filling curve is a curve that goes through each point of the space one single time [15]. It gives a monodimensional representation of a multidimensional space. It thus induces a total order on the space : to compare two vectors, one just has to compare their curvilinear abscissas along the curve. Reciprocally, for each total

ordering relation, it is possible to build a corresponding space filling curve by joining the smallest vector of the space to the greatest, passing through all the other points increasingly (according to the chosen ordering relation). Therefore, there is a double equivalence :

(total order on $\Re^N$) $\Leftrightarrow$ (bijective application $h:\Re^N \rightarrow \Re$) $\Leftrightarrow$ (space filling curve in $\Re^N$)

Bidimensional space filling curves have already been used in image processing as scanning methods. Such curves are presented on figure 2 in the case of a bidimensional space where each component is coded on 4 bits (from 0 to 15).
For instance, the Peano scan (figure 2-c), that allows to fully exploit the spatial correlation between adjacent pixels, has been used in [7] to improve the performances of recursive median filter. [11] used a tridimensional Peano curve to improve the compression quality of colour images. The “zig-zag” scan (figure 2-b) has been used in the J-PEG norm for image coding (see for instance [8]). Regazzoni [9] used the curve shown on the figure 2-a to propose a vector median filter with an approach that is quite similar to the one presented in this paper. The layered structure of that curve shows that the corresponding order is essentially based on the euclidean norm of the vectors. Figure 2-h shows the curve corresponding to the lexicographic order in a bidimensional space. The high anisotropy of the curve shows the dissymmetry induced by this order between the different components.



-a- [Regazzoni-97] -b- Zig zag scan -c- Peano curve -d- B. M. P. (variant)

-e- B.M.P. (1) -f- B.M.P. (2) -g- B.M.P. (3) -h- Lexicographic

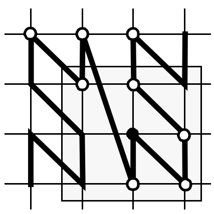
Figure 2 : Examples of bidimensional space filling curves.

4.2. Bit mixing paradigm & space filling curve

The curves corresponding to the order induced by the bit mixing paradigm and its variant are presented respectively on the figures 2-e and -d. We can make some comments that remain true in the general case (N components coded on p bits) :
- The curve is self-similar (fractal structure). That leads to a “hierarchical” comparison of the vectors. On the curve (figure 2-e), this can be seen as a division of the space into four main blocks. If the two vectors we want to compare are in two different blocks (i.e. if the first bits of their coded form are different), it suffices for the comparison. Otherwise, if they are in the same block, this block is divided into four smaller blocks, and so on, taking more and more details of the curve into account, until the comparison is made possible (each scale of the curve corresponds to one bit of the coding).
- Figures 2-f and -g show the curves obtained by mixing the bits respectively two by two and three by three. Mixing them four by four gives, in this case, the lexicographic order.

The curve gradually evolves from an almost isotropic curve (figure 2-e) to a highly anisotropic one (lexicographic order), giving more and more importance to one component.

• All the curves, at different degrees, do not respect the space topology : some points that are very close in the original space are moved away on the curve. This can be interpreted in term of noise enhancement in the image. Similarly, the curve can bring closer together points that were far away from each other in the original space. This can mask some transitions in the image. These perturbations are inevitable since no bijective transform from $\Re^N$ into $\Re$ (the equivalent representation of a space filling curve) can be linear. So, we tried to choose the curve that minimizes this topological distortion. For this, we built a quantitative criterion that measures the perturbation undergone by the neighbourhood notion during the transformation. For each point of the original colour space, we calculated the number of his neighbours that still belong to his neighbourhood along the curve (figure 3) and we averaged the results for the whole space. Table 1 gives the results obtained with some presented curves. Apart from the Peano curve, the topology is preserved at best when the variant of the bit mixing paradigm is used. We nevertheless did not use the Peano curve since the order it induces does not respect the canonic lattice structure. For instance, it decides that vector (0,8) is greater than vector (15,15). As a consequence, we chose the bit mixing paradigm to induce a complete lattice structure on the data set. Next section presents some results obtained with this approach.



In the original colour space, using 8-connectivity, the considered point (black dot) has 8 neighbours (included in the gray square). After the coding, i.e. along the curve, his 8 nearest neighbours are the 8 white dots. The entire neighbourhood has not been preserved. The criterion we used to quantify this phenomenon is the average intersection of these two neigbourhoods. For instance, for the presented point, five white dots are included in the gray square : 5/8 = 62,5 %.

Figure 3 : Quantitative evaluation of the topological distortion

TABLE 1
Topology preservation with different space filling curves

Curve	Neighbourhood preservation ratio.
Peano	58 %
Bit mixing paradigm (variant)	50 %
Bit mixing paradigm	47 %
Zig-zag scan	27 %
Regazzoni [9]	26 %
Lexicographic	25 %

5. Application

Using the bit mixing paradigm approach, we tested the colour extension of the alternating sequential filters (ASF) by reconstruction [10]. These operators are connected filters. Connected filters play a major role in mathematical morphology: they are the filters that do not introduce any discontinuity in the image: the edge information is either totally removed or exactly preserved, but the point is that it is never moved or blurred. An ASF by reconstruc-

tion is the composition of several morphological openings and closings by reconstruction that are applied alternatively and successively with a growing structuring element size. Those filters have a good ability to reduce noise and to simplify an image and are, as a consequence, very useful in many situations (preprocessing for segmentation, coding ...).
The implementation of this vectorial extension of morphological operators to the colour case is very easy and fast. Since only sup- and inf- operations are involved, the processing scheme presented in figure 4 can be used. We can see that the optimal algorithms that are available for the monodimensional case can be used directly on the coded data (in a scalar form).

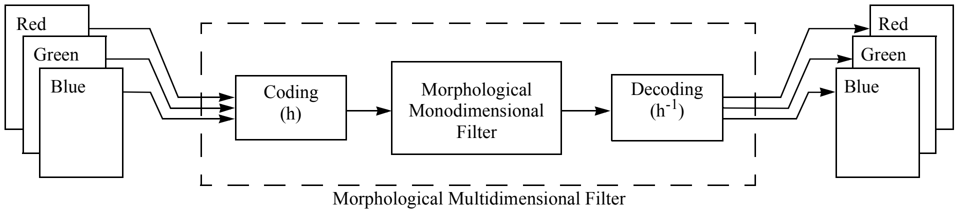
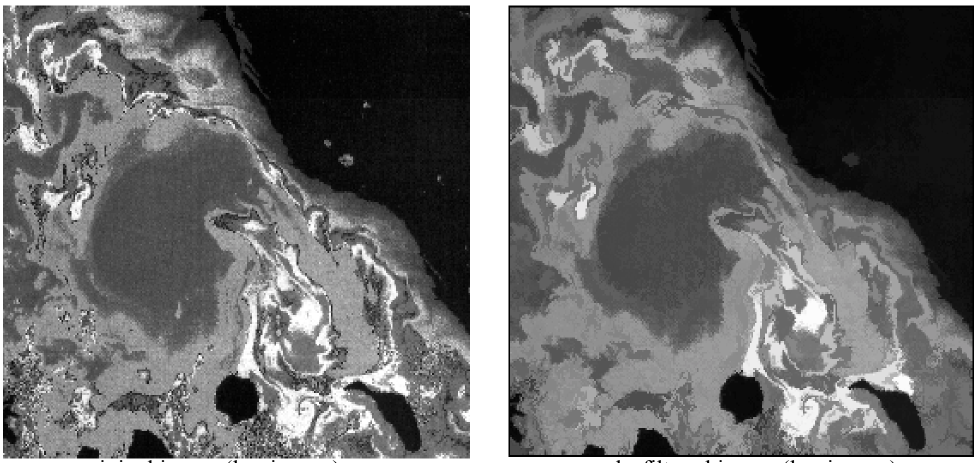


Figure 4 : Processing synopsis to filter a colour image with a “gray-level” filter.

To illustrate the good behaviour of the filter with the proposed structure, we can look at pictures 5(a) & (b). They respectively show the luminance of an original colour image (aerial view from the Canaries islands) and the same image filtered by the following sequence :

$$\text{(filtered image)} = \text{decoding } \mathbf{o}\ \overline{\gamma}_5\ \mathbf{o}\ \overline{\varphi}_5\ \mathbf{o}\ \overline{\gamma}_3\ \mathbf{o}\ \overline{\varphi}_3\ \mathbf{o} \text{ coding (initial image)}$$

where $\overline{\gamma}_{\mathbf{n}}$ and $\overline{\varphi}_{\mathbf{n}}$ respectively represent the opening and the closing by reconstruction with a square flat structuring element of size n.



-a- original image (luminance)
(aerial view of the Canarie Islands)

-b- filtered image (luminance)
($\overline{\gamma}_5\ \mathbf{o}\ \overline{\varphi}_5\ \mathbf{o}\ \overline{\gamma}_3\ \mathbf{o}\ \overline{\varphi}_3$)

Figure 5 -a- & -b-

The filtered image is visibly much simpler than the original one ; some details have been lost, but preserved edges remain absolutely unchanged, which is the typical intended behaviour of the used filter. We also can notice the loss of contrast due to the properties of the openings and the closings (respectively anti-extensivity and extensivity) leading to a kind of “tarnished” result ; the dynamics of the image gradually decreases. This is also typical for this kind of filters.

6. Conclusion

Using the bit mixing paradigm, we proposed a bijective transform that encodes multidimensional data in a scalar form. This transform induces a total ordering relation on the vectors and thus can be used to extend morphological operators to multicomponent images. We justified this choice by studying its equivalent representation as a space filling curve, and we tested the method on the colour extension of alternating sequential filters by reconstruction. Of course, we could have used the very same approach on colour images in the Hue Saturation Intensity base, taking the hue more or less into account adaptively to the saturation for instance [14], or on any other multicomponent images (multispectral, multitemporal and s.o.). The method seems to be very promising. It actually presents several advantages:

- it prevents the appearance of false colours since any output vector is necessarily one of the input vectors,
- since the transform is bijective, the definition of, for instance, the supremum of any set of vectors is unambiguous,
- it is very simple to implement (coding and decoding are very easy and fast),
- it is well suited to all already existing optimized algorithms that deal with the one dimensional case. The existing programs are used directly without any change (except for the coding and decoding steps that have to be added). We just need to check that the program can deal with integers coded on the right number of bits (for instance, 24 and not only 8).
- the properties of the operators we wanted to extend are well preserved.

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where T is the set $\{0,1,...,2^p\text{-}1\}$. This order is not fully satisfying either since it induces an important dissymmetry between the different components of the image.

3. The bit mixing paradigm

Following the same formalism, we propose here a transform h that bijectively maps N-dimensional vectors into $\Re$. This transform is based on the binary representation of each component of the considered vector X. If the N components of X are coded with p bits each, the N.p available bits are mixed up together to build the N.p bits long scalar value h(X). Of course, there are many different ways to mix up these N.p bits, but we choose the one that minimizes the unavoidable dissymmetry between the different components. So, we first take the first bit of X(1), then the first bit of X(2), until the first bit of X(N), and we repeat the process with the second bit, and so on until the p^{th} bit. Mathematically, this can be written as follows: $X \in \Re^N$ has N components X(i), each one coded on p bits $X(i)_j \in \{0,1\}$ with $j \in \{1,...,p\}$. The considered mapping h can then be written as follows:

$$h(X) = \sum_{j=1}^{p} \left\{ 2^{N \cdot (p-j)} \cdot \sum_{i=1}^{N} 2^{N-i} \cdot X(i)_j \right\}$$

In the same way , we can define the reciprocal mapping h^{-1}. Let S be an integer scalar coded with N.p bits $S_j \in \{0,1\}$ with $j \in \{1,...,N.p\}$. Then, each component X(i) of $h^{-1}(S) = X \in \Re^N$

is defined by:

$$X(i) = \sum_{j=1}^{p} 2^{p-j} \cdot s_{N \cdot (p-j) + N - i + 1}$$

In the following, for the sake of clarity and simplicity, we will restrict this paper, without any loss of generality, to the case of the colour images, represented in the classical 3*8 bits Red Green Blue base. Figure 1 presents the coding and the decoding of such a colour pixel.

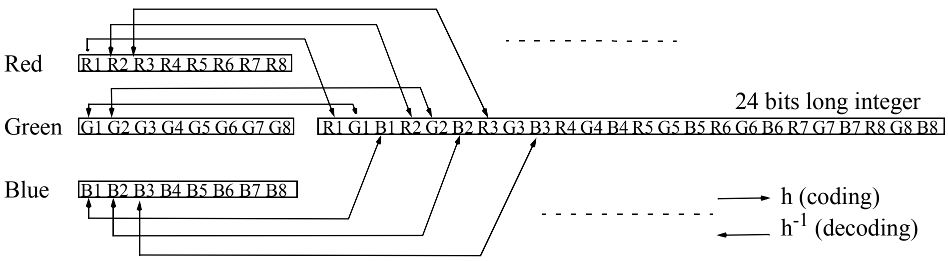

Figure 1 : Coding (h) and decoding (h^{-1}) of one colour pixel.

As previously said, we chose the coding that minimizes the unavoidable dissymmetry inherent to this kind of coding : approximately the same weight is attached to each component of the image (all the components are processed quite symmetrically). Furthermore, no significative changes have been observed in the application presented §5 by inverting the order of the components (the remaining small dissymmetry is of no consequence). Nevertheless, if some *a priori* information concerning the relative importance of the different components is available, it can be taken into account by attaching a higher weight to the most significative component (for instance by putting its 2 or 3 first bits at the head of the coding).
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Of course, the same order is kept for all the pixels of the image. We will see in section 4 that this variant theoretically improves the quality of the coding.
The order induced by the canonic lattice structure is probably the most natural and intuitive vectorial order, even if it is just a partial one. A key-point is that the order induced by the bit mixing paradigm or its variant can be regarded as a simple extension of the canonic order (if two vectors are comparable in the canonic structure, they will keep the same order with our relation), but this partial order has been completed into a total one : two vectors are now always comparable.

4. Total vectorial ordering & space filling curves

4.1. Presentation

For a given N-dimensional space, a space filling curve is a curve that goes through each point of the space one single time [15]. It gives a monodimensional representation of a multidimensional space. It thus induces a total order on the space : to compare two vectors, one just has to compare their curvilinear abscissas along the curve. Reciprocally, for each total

ordering relation, it is possible to build a corresponding space filling curve by joining the smallest vector of the space to the greatest, passing through all the other points increasingly (according to the chosen ordering relation). Therefore, there is a double equivalence :

(total order on $\Re^N$) $\Leftrightarrow$ (bijective application $h:\Re^N \rightarrow \Re$) $\Leftrightarrow$ (space filling curve in $\Re^N$)

Bidimensional space filling curves have already been used in image processing as scanning methods. Such curves are presented on figure 2 in the case of a bidimensional space where each component is coded on 4 bits (from 0 to 15).
For instance, the Peano scan (figure 2-c), that allows to fully exploit the spatial correlation between adjacent pixels, has been used in [7] to improve the performances of recursive median filter. [11] used a tridimensional Peano curve to improve the compression quality of colour images. The “zig-zag” scan (figure 2-b) has been used in the J-PEG norm for image coding (see for instance [8]). Regazzoni [9] used the curve shown on the figure 2-a to propose a vector median filter with an approach that is quite similar to the one presented in this paper. The layered structure of that curve shows that the corresponding order is essentially based on the euclidean norm of the vectors. Figure 2-h shows the curve corresponding to the lexicographic order in a bidimensional space. The high anisotropy of the curve shows the dissymmetry induced by this order between the different components.

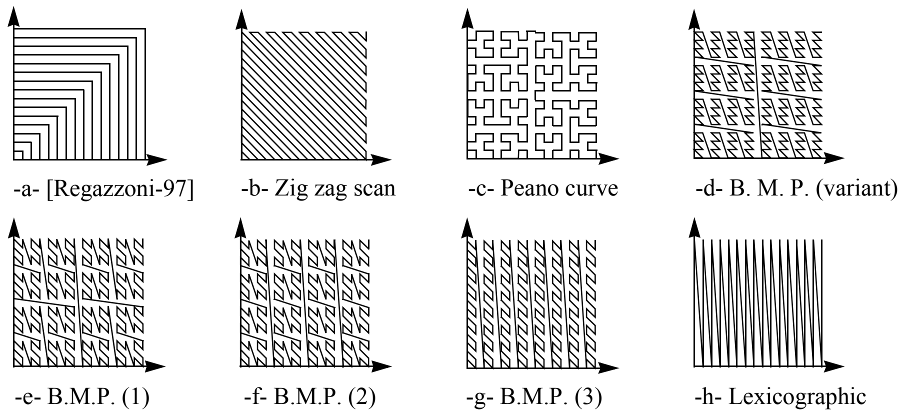


Figure 2 : Examples of bidimensional space filling curves.

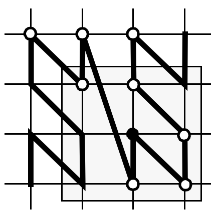
4.2. Bit mixing paradigm & space filling curve

The curves corresponding to the order induced by the bit mixing paradigm and its variant are presented respectively on the figures 2-e and -d. We can make some comments that remain true in the general case (N components coded on p bits) :

- The curve is self-similar (fractal structure). That leads to a “hierarchical” comparison of the vectors. On the curve (figure 2-e), this can be seen as a division of the space into four main blocks. If the two vectors we want to compare are in two different blocks (i.e. if the first bits of their coded form are different), it suffices for the comparison. Otherwise, if they are in the same block, this block is divided into four smaller blocks, and so on, taking more and more details of the curve into account, until the comparison is made possible (each scale of the curve corresponds to one bit of the coding).
- Figures 2-f and -g show the curves obtained by mixing the bits respectively two by two and three by three. Mixing them four by four gives, in this case, the lexicographic order.

The curve gradually evolves from an almost isotropic curve (figure 2-e) to a highly anisotropic one (lexicographic order), giving more and more importance to one component.

• All the curves, at different degrees, do not respect the space topology : some points that are very close in the original space are moved away on the curve. This can be interpreted in term of noise enhancement in the image. Similarly, the curve can bring closer together points that were far away from each other in the original space. This can mask some transitions in the image. These perturbations are inevitable since no bijective transform from $\Re^N$ into $\Re$ (the equivalent representation of a space filling curve) can be linear. So, we tried to choose the curve that minimizes this topological distortion. For this, we built a quantitative criterion that measures the perturbation undergone by the neighbourhood notion during the transformation. For each point of the original colour space, we calculated the number of his neighbours that still belong to his neighbourhood along the curve (figure 3) and we averaged the results for the whole space. Table 1 gives the results obtained with some presented curves. Apart from the Peano curve, the topology is preserved at best when the variant of the bit mixing paradigm is used. We nevertheless did not use the Peano curve since the order it induces does not respect the canonic lattice structure. For instance, it decides that vector (0,8) is greater than vector (15,15). As a consequence, we chose the bit mixing paradigm to induce a complete lattice structure on the data set. Next section presents some results obtained with this approach.



In the original colour space, using 8-connectivity, the considered point (black dot) has 8 neighbours (included in the gray square). After the coding, i.e. along the curve, his 8 nearest neighbours are the 8 white dots. The entire neighbourhood has not been preserved. The criterion we used to quantify this phenomenon is the average intersection of these two neigbourhoods. For instance, for the presented point, five white dots are included in the gray square : 5/8 = 62,5 %.

Figure 3 : Quantitative evaluation of the topological distortion

TABLE 1
Topology preservation with different space filling curves

Curve	Neighbourhood preservation ratio.
Peano	58 %
Bit mixing paradigm (variant)	50 %
Bit mixing paradigm	47 %
Zig-zag scan	27 %
Regazzoni [9]	26 %
Lexicographic	25 %

5. Application

Using the bit mixing paradigm approach, we tested the colour extension of the alternating sequential filters (ASF) by reconstruction [10]. These operators are connected filters. Connected filters play a major role in mathematical morphology: they are the filters that do not introduce any discontinuity in the image: the edge information is either totally removed or exactly preserved, but the point is that it is never moved or blurred. An ASF by reconstruc-

tion is the composition of several morphological openings and closings by reconstruction that are applied alternatively and successively with a growing structuring element size. Those filters have a good ability to reduce noise and to simplify an image and are, as a consequence, very useful in many situations (preprocessing for segmentation, coding ...).
The implementation of this vectorial extension of morphological operators to the colour case is very easy and fast. Since only sup- and inf- operations are involved, the processing scheme presented in figure 4 can be used. We can see that the optimal algorithms that are available for the monodimensional case can be used directly on the coded data (in a scalar form).

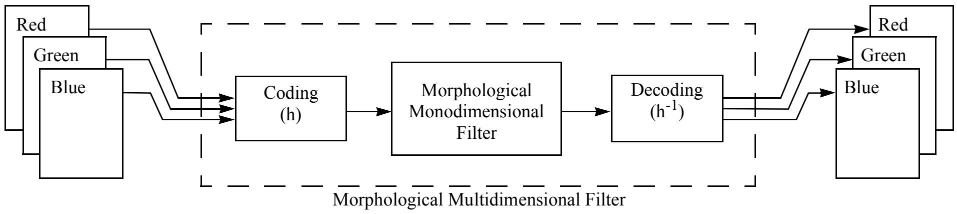
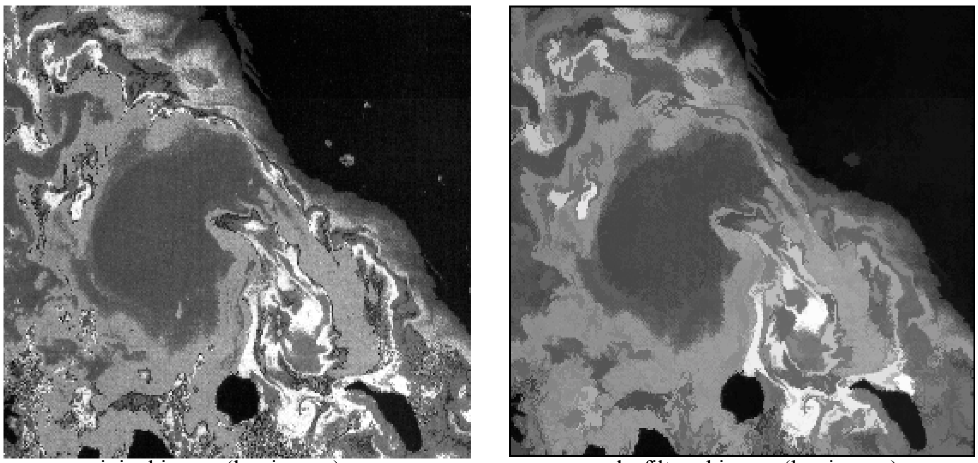


Figure 4 : Processing synopsis to filter a colour image with a “gray-level” filter.

To illustrate the good behaviour of the filter with the proposed structure, we can look at pictures 5(a) & (b). They respectively show the luminance of an original colour image (aerial view from the Canaries islands) and the same image filtered by the following sequence :

$$\text{(filtered image)} = \text{decoding } \mathbf{o}\ \overline{\gamma}_5\ \mathbf{o}\ \overline{\varphi}_5\ \mathbf{o}\ \overline{\gamma}_3\ \mathbf{o}\ \overline{\varphi}_3\ \mathbf{o} \text{ coding (initial image)}$$

where $\overline{\gamma}_{\mathbf{n}}$ and $\overline{\varphi}_{\mathbf{n}}$ respectively represent the opening and the closing by reconstruction with a square flat structuring element of size n.



-a- original image (luminance)
(aerial view of the Canarie Islands)

-b- filtered image (luminance)
($\overline{\gamma}_5\ \mathbf{o}\ \overline{\varphi}_5\ \mathbf{o}\ \overline{\gamma}_3\ \mathbf{o}\ \overline{\varphi}_3$)

Figure 5 -a- & -b-

The filtered image is visibly much simpler than the original one ; some details have been lost, but preserved edges remain absolutely unchanged, which is the typical intended behaviour of the used filter. We also can notice the loss of contrast due to the properties of the openings and the closings (respectively anti-extensivity and extensivity) leading to a kind of “tarnished” result ; the dynamics of the image gradually decreases. This is also typical for this kind of filters.

6. Conclusion

Using the bit mixing paradigm, we proposed a bijective transform that encodes multidimensional data in a scalar form. This transform induces a total ordering relation on the vectors and thus can be used to extend morphological operators to multicomponent images. We justified this choice by studying its equivalent representation as a space filling curve, and we tested the method on the colour extension of alternating sequential filters by reconstruction. Of course, we could have used the very same approach on colour images in the Hue Saturation Intensity base, taking the hue more or less into account adaptively to the saturation for instance [14], or on any other multicomponent images (multispectral, multitemporal and s.o.). The method seems to be very promising. It actually presents several advantages:

• it prevents the appearance of false colours since any output vector is necessarily one of the input vectors,

• since the transform is bijective, the definition of, for instance, the supremum of any set of vectors is unambiguous,

• it is very simple to implement (coding and decoding are very easy and fast),

• it is well suited to all already existing optimized algorithms that deal with the one dimensional case. The existing programs are used directly without any change (except for the coding and decoding steps that have to be added). We just need to check that the program can deal with integers coded on the right number of bits (for instance, 24 and not only 8).

• the properties of the operators we wanted to extend are well preserved.

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TOTAL ORDERING BASED ON SPACE FILLING CURVES FOR MULTIVALUED MORPHOLOGY.

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Abstract. This paper addresses the problem of the extension of morphological operators to the case of multicomponent images. It basically comes to the definition of a well suited vectorial ordering relation. We briefly discuss the possible extensions found in the literature and propose an ordering scheme based on the bit mixing paradigm. This choice is justified using the space filling curve equivalent representation of some classical total ordering relations. Results concerning the extension of alternating sequential filters by reconstruction to the case of colour images are presented.

Key words: vectorial ordering, space filling curves, multivalued morphology

1. Introduction

Mathematical morphology theory has now become a widely used non linear technique for image processing. Initially designed as a set theory, it was then generalized to the set of gray-level images with the help of umbras, and, more recently, to any complete lattice, which is the appropriate mathematical framework for morphology [1] [12] [13].
This minimum necessary algebraic structure is a set T such that:
- T is induced by a (partial) ordering relation " $\leq$ " ;
- for any family (finite or not) of elements in T, there exists a smallest majorant called the sup (for supremum) and a greatest minorant called the inf (for infimum).

Concurrently, multicomponent image processing has become an important field of research. If the image has N different components, each pixel is now represented by an N-dimensional vector.
To extend morphological operators to multicomponent images, it is of course possible to process each component separately and to recombine the results afterwards (Marginal processing). This approach is not fully satisfying for different reasons :
- since the output vector will rarely be one of the input vectors, the appearance of, for instance, false colours (in the case of colour images) is quite inevitable,
- if the image has N different components, N processings are required, leading to time consuming operators,
- finally, this approach does not take into account the inter-component correlation.

For these reasons, it is preferable to use a purely vectorial approach that process the image all at once. But we then have to define a well suited vectorial lattice structure to work in and,

in particular, a vectorial ordering relation is required to define the sup and the inf of any family of N-dimensional vectors.

Section 2 presents the unifying approach to multivariate data ordering formalized in [2]. Section 3 presents a specific total vectorial ordering relation proposed in [3] based on the bit mixing paradigm. Section 4 presents the equivalence existing between any total vectorial order and a space filling curve. Before the conclusion, section 5 presents the application of the bit mixing paradigm to the colour extension of ASF by reconstruction.

2. Vectorial ordering relations : a formal unifying approach

The key point to extend morphological operators to the N-dimensional case is the definition of a vectorial ordering relation that induces a lattice structure on the data. Goutsias [2] presents a unifying formalism based on the used of a vector transform h from the working space $\Re^N$ into $\Re^Q$ (with Q ≤ N *a priori*) followed by a marginal ordering on $\Re^Q$:

$$\begin{aligned} h : \Re^N &\rightarrow \Re^Q \\ X &\rightarrow h(X) \end{aligned}$$

Assuming an ordering relation has previously be defined on $\Re^Q$, an order can be defined on $\Re^N$ in a straightforward way using the following relation :

$$\forall (X,Y) \in \Re^N \times \Re^N ,\ X \le Y \Leftrightarrow h(X) \le h(Y)$$

Depending on the used transform and on the value of Q, different situations are possible. The two next sections respectively discuss the case Q>1 and Q=1.

2.1. Q > 1 : partial ordering relation

The only existing order on $\Re^Q$, with Q>1, is the canonic order (the well known lexicographic order corresponds to the case Q=1 with the appropriate transform). In this structure, two vectors X and Y from the product space $\Re^Q$ are compared in the following way :

$$X \le Y \Leftrightarrow X(i) \le Y(i) ,\ \forall\, i \in \{1,...,Q\}$$

This order is a partial order. For instance vectors (1,2,3) and (3,2,1) can not be compared. Nevertheless, this canonic lattice structure is often used for multivalued morphology [4] [5]. This ordering relation can be used directly on the input data (case h = Identity application). In this case, basic operations such as the supremum or the infimum, and all their possible compositions, are strictly equivalent to a marginal processing. It is also possible to use other transforms h such as the Karhunen Loeve transform or the principal components analysis [6] and to use the canonic lattice structure on the transformed data after this change of base. But, in every case, a partial order is induced, leading unavoidably to the appearance of false colours : the maximum of a set of vectors is not defined and the supremum is rarely one of the input vectors.

Furthermore, using a partial ordering relation poses an algorithmic problem. All existing fast programs have been meant to deal with gray level images, i.e. with totally ordered complete lattices, and most of them cannot be directly adapted to partially ordered lattices.

2.2. Q = 1 : total (pre-)ordering relation

To avoid the problem of false colour appearance, it is necessary to use a transform h with Q=1. In this case, the maximum of a set $\{X_j\}$ of vectors is always defined.

• If h is not injective, this maximum may not be uniquely defined. The non-antisymmetrical relation induced by such a transform is a **pre-ordering** relation, leading to what are called “morphological-like” filters (the theoretical background is not strictly respected) [6]. Goutsias [2] explains how to solve this theoretical problem by defining a new equivalence relation between the vectors to restore the antisymmetrical property of the induced relation, but the problem of uniquely defining the supremum (or equivalently here, the maximum) still remains. Nevertheless, some good results may be provided using pre-ordering relations. For instance, at each pixel is associated the value of its i^{th} component, with i fixed *a priori* or adaptatively determined :

$$\begin{aligned} h : \Re^N &\to \Re \\ X &\to X(i) \end{aligned}$$

Other techniques based on **reduced ordering** using different generalized distances (Mahalanobis and s.o.) can also be used [6].

• The only true **total ordering** relation that is commonly presented in the literature is the lexicographic order. It is defined as follows :

$$\forall (X,Y) \in \Re^N \times \Re^N ,\ X < Y \Leftrightarrow \exists\, k \in \{1,\dots,N\} \,/\, \begin{array}{l} X(i) = Y(i)\ \forall i \in \{1,\dots,k\text{-}1\} \\ X(k) < Y(k) \end{array}$$

With the general formalism, if the image has N different components coded each on p bits, this ordering relation corresponds to the following transform h :

$$\begin{aligned} h :\ T^N &\to Im_h(T^N) \\ X &\to \sum_{i=1}^{N} X(N+1-i) \cdot 2^{p \cdot (i-1)} \end{aligned}$$

where T is the set $\{0,1,\dots,2^p\text{-}1\}$. This order is not fully satisfying either since it induces an important dissymmetry between the different components of the image.

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Following the same formalism, we propose here a transform h that bijectively maps N-dimensional vectors into $\Re$. This transform is based on the binary representation of each component of the considered vector X. If the N components of X are coded with p bits each, the N.p available bits are mixed up together to build the N.p bits long scalar value h(X). Of course, there are many different ways to mix up these N.p bits, but we choose the one that minimizes the unavoidable dissymmetry between the different components. So, we first take the first bit of X(1), then the first bit of X(2), until the first bit of X(N), and we repeat the process with the second bit, and so on until the p^{th} bit. Mathematically, this can be written as follows: $X \in \Re^N$ has N components X(i), each one coded on p bits $X(i)_j \in \{0,1\}$ with $j \in \{1,\dots,p\}$. The considered mapping h can then be written as follows:

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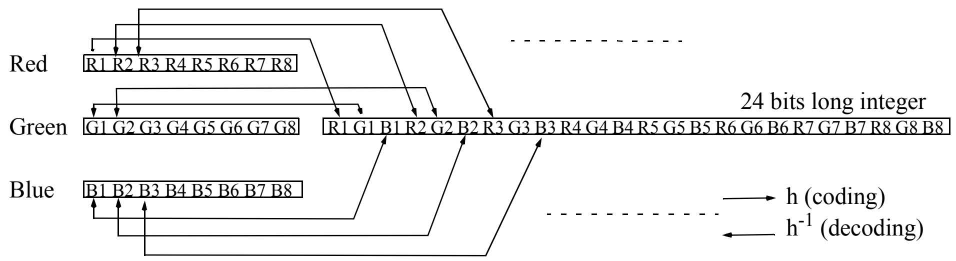

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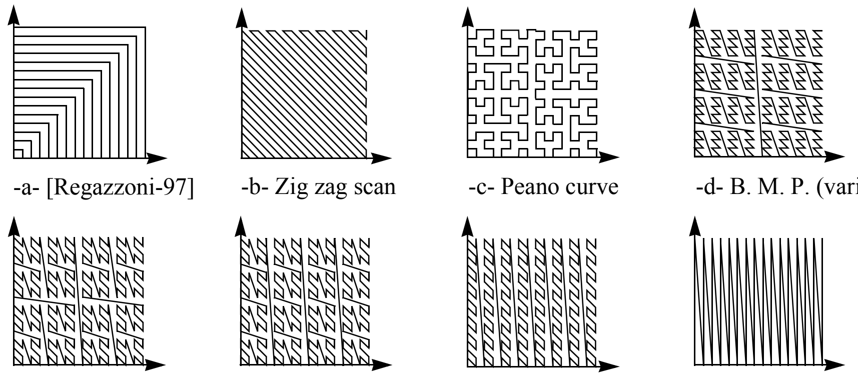

-a- [Regazzoni-97] -b- Zig zag scan -c- Peano curve -d- B. M. P. (variant)

-e- B.M.P. (1) -f- B.M.P. (2) -g- B.M.P. (3) -h- Lexicographic

Figure 2 : Examples of bidimensional space filling curves.

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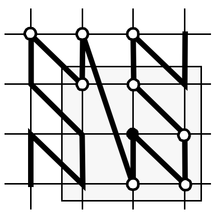
• Figures 2-f and -g show the curves obtained by mixing the bits respectively two by two and three by three. Mixing them four by four gives, in this case, the lexicographic order.

The curve gradually evolves from an almost isotropic curve (figure 2-e) to a highly anisotropic one (lexicographic order), giving more and more importance to one component.

- All the curves, at different degrees, do not respect the space topology : some points that are very close in the original space are moved away on the curve. This can be interpreted in term of noise enhancement in the image. Similarly, the curve can bring closer together points that were far away from each other in the original space. This can mask some transitions in the image. These perturbations are inevitable since no bijective transform from $\Re^N$ into $\Re$ (the equivalent representation of a space filling curve) can be linear. So, we tried to choose the curve that minimizes this topological distortion. For this, we built a quantitative criterion that measures the perturbation undergone by the neighbourhood notion during the transformation. For each point of the original colour space, we calculated the number of his neighbours that still belong to his neighbourhood along the curve (figure 3) and we averaged the results for the whole space. Table 1 gives the results obtained with some presented curves. Apart from the Peano curve, the topology is preserved at best when the variant of the bit mixing paradigm is used. We nevertheless did not use the Peano curve since the order it induces does not respect the canonic lattice structure. For instance, it decides that vector (0,8) is greater than vector (15,15). As a consequence, we chose the bit mixing paradigm to induce a complete lattice structure on the data set. Next section presents some results obtained with this approach.



In the original colour space, using 8-connectivity, the considered point (black dot) has 8 neighbours (included in the gray square). After the coding, i.e. along the curve, his 8 nearest neighbours are the 8 white dots. The entire neighbourhood has not been preserved. The criterion we used to quantify this phenomenon is the average intersection of these two neigbourhoods. For instance, for the presented point, five white dots are included in the gray square : 5/8 = 62,5 %.

Figure 3 : Quantitative evaluation of the topological distortion

TABLE 1
Topology preservation with different space filling curves

Curve	Neighbourhood preservation ratio.
Peano	58 %
Bit mixing paradigm (variant)	50 %
Bit mixing paradigm	47 %
Zig-zag scan	27 %
Regazzoni [9]	26 %
Lexicographic	25 %

5. Application

Using the bit mixing paradigm approach, we tested the colour extension of the alternating sequential filters (ASF) by reconstruction [10]. These operators are connected filters. Connected filters play a major role in mathematical morphology: they are the filters that do not introduce any discontinuity in the image: the edge information is either totally removed or exactly preserved, but the point is that it is never moved or blurred. An ASF by reconstruc-

tion is the composition of several morphological openings and closings by reconstruction that are applied alternatively and successively with a growing structuring element size. Those filters have a good ability to reduce noise and to simplify an image and are, as a consequence, very useful in many situations (preprocessing for segmentation, coding ...).
The implementation of this vectorial extension of morphological operators to the colour case is very easy and fast. Since only sup- and inf- operations are involved, the processing scheme presented in figure 4 can be used. We can see that the optimal algorithms that are available for the monodimensional case can be used directly on the coded data (in a scalar form).

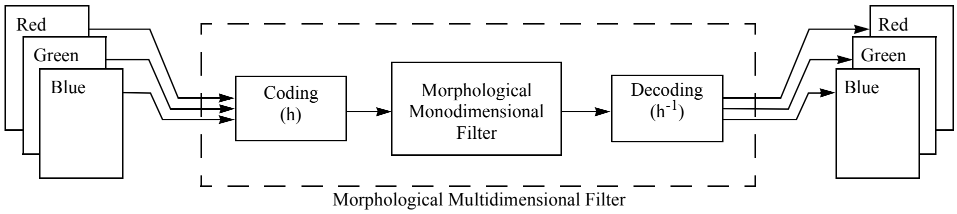
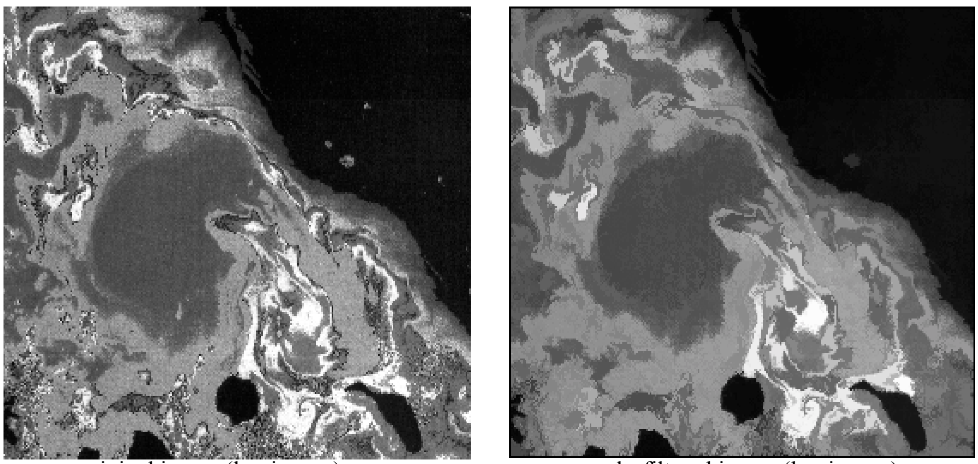


Figure 4 : Processing synopsis to filter a colour image with a “gray-level” filter.

To illustrate the good behaviour of the filter with the proposed structure, we can look at pictures 5(a) & (b). They respectively show the luminance of an original colour image (aerial view from the Canaries islands) and the same image filtered by the following sequence :

$$\text{(filtered image)} = \text{decoding} \circ \bar{\gamma}_5 \circ \bar{\varphi}_5 \circ \bar{\gamma}_3 \circ \bar{\varphi}_3 \circ \text{coding (initial image)}$$

where $\bar{\gamma}_n$ and $\bar{\varphi}_n$ respectively represent the opening and the closing by reconstruction with a square flat structuring element of size n.



-a- original image (luminance)
(aerial view of the Canarie Islands)

-b- filtered image (luminance)
($\gamma_5 \circ \varphi_5 \circ \gamma_3 \circ \varphi_3$)

Figure 5 -a- & -b-

The filtered image is visibly much simpler than the original one ; some details have been lost, but preserved edges remain absolutely unchanged, which is the typical intended behaviour of the used filter. We also can notice the loss of contrast due to the properties of the openings and the closings (respectively anti-extensivity and extensivity) leading to a kind of “tarnished” result ; the dynamics of the image gradually decreases. This is also typical for this kind of filters.

6. Conclusion

Using the bit mixing paradigm, we proposed a bijective transform that encodes multidimensional data in a scalar form. This transform induces a total ordering relation on the vectors and thus can be used to extend morphological operators to multicomponent images. We justified this choice by studying its equivalent representation as a space filling curve, and we tested the method on the colour extension of alternating sequential filters by reconstruction. Of course, we could have used the very same approach on colour images in the Hue Saturation Intensity base, taking the hue more or less into account adaptively to the saturation for instance [14], or on any other multicomponent images (multispectral, multitemporal and s.o.). The method seems to be very promising. It actually presents several advantages:

- it prevents the appearance of false colours since any output vector is necessarily one of the input vectors,
- since the transform is bijective, the definition of, for instance, the supremum of any set of vectors is unambiguous,
- it is very simple to implement (coding and decoding are very easy and fast),
- it is well suited to all already existing optimized algorithms that deal with the one dimensional case. The existing programs are used directly without any change (except for the coding and decoding steps that have to be added). We just need to check that the program can deal with integers coded on the right number of bits (for instance, 24 and not only 8).
- the properties of the operators we wanted to extend are well preserved.

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