Minicourse on *information, complexity and organisation in multidimensional symbolic dynamics*

From unidimensional to multidimensional symbolic dynamics; some questions about sofic subshifts

Silvere Gangloff

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sgangloff@agh.edu.pl ; silvere.gangloff@gmx.com

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▷ The space $\mathcal{A}^{\mathbb{Z}}$ is **compact**.

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Reciprocally for all $\mathcal{F}$, $X_{\mathcal{F}}$ is always a subshift.

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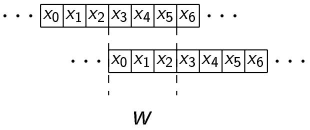

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where $N_n(X, f, \mathcal{U})$ is the minimal cardinality of a subcover of:

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Also $N_n(X, \sigma, \mathcal{U}_0)$, simplified $N_n(X)$, is the cardinality of

$$\mathcal{L}_n(X) \underset{\text{def}}{=} \{w \in \mathcal{A}^* \ : |w| = n \text{ and } X \cap [w]_0 \neq \emptyset\}.$$

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Examples of subshifts

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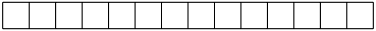

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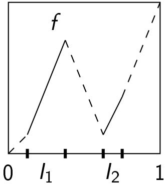

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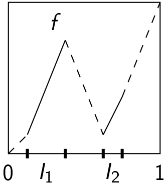

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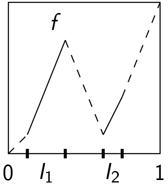

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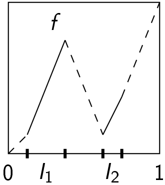

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Here $([0,1], f)$ has a subsystem isomorphic to $X_{\{11\}}$.

Higher block presentation

Take $X \subset \mathcal{A}^{\mathbb{Z}}$ subshift. For $x \in X$, define $\varphi(x)$ as follows:

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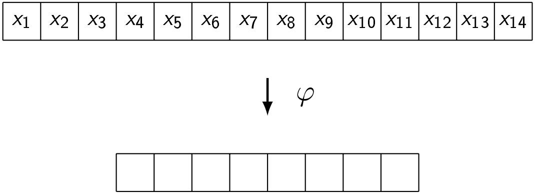

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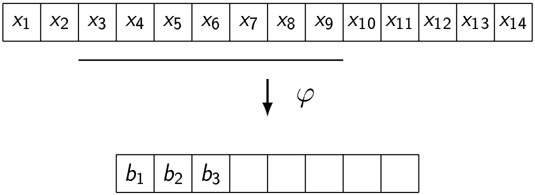

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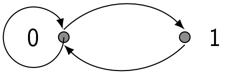

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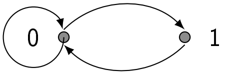

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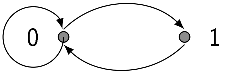

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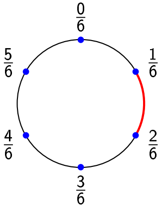

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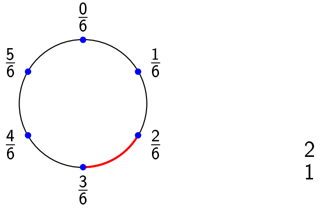

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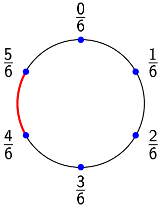

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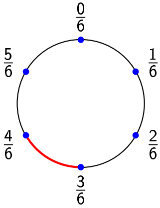

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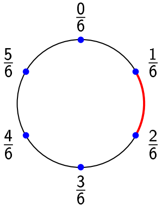

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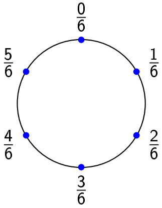

1	3
3	4
4	2
2	1
1	3

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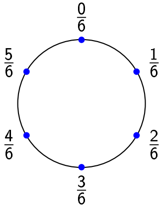

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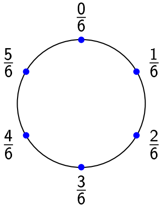

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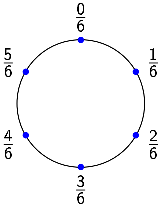

$$
\begin{matrix}
1 & 3 & 4 & 2 & 1 \\
3 & 4 & 2 & 1 & 3 \\
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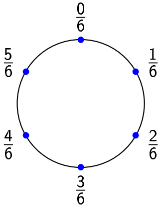

$$\begin{array}{ccccccc} & & & \vdots & & & \\ & 1 & 3 & 4 & 2 & 1 & \\ & 3 & 4 & 2 & 1 & 3 & \\ \cdots & 4 & 2 & 1 & 3 & 4 & \cdots \\ & 2 & 1 & 3 & 4 & 2 & \\ & 1 & 3 & 4 & 2 & 1 & \\ & & & \vdots & & & \end{array}$$

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$$|r_n - x| < 2^{-n}.$$

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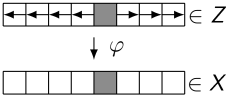

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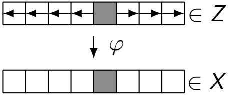

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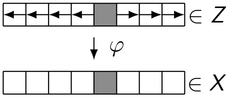

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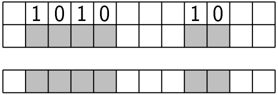
Interpretation in terms of **information transport**: arrows constitute a signal of gray symbol presence.

Other examples

Even subshift:



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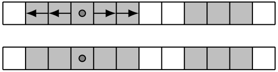

Characterization of unidimensional sofic subshifts

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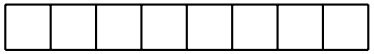
Follower sets: $F_1 = \{$$\}$; $F_2 = \{$$\}$.

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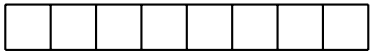
Follower sets: $F_1 = \{$$\}$; $F_2 = \{$$\}$.
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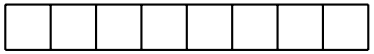
Follower sets: $F_1 = \{$$\}$; $F_2 = \{$$\}$.
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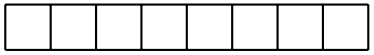
Follower sets: $F_1 = \{$$\}$; $F_2 = \{$$\}$.
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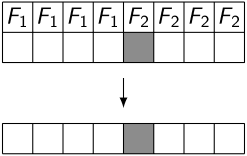

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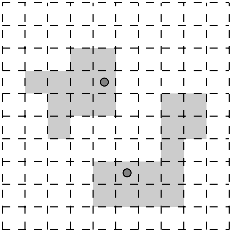
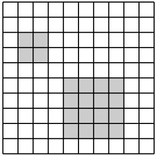


Other example: **L.Brown-Westrick:** seas of squares with computability condition:



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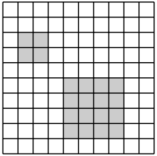

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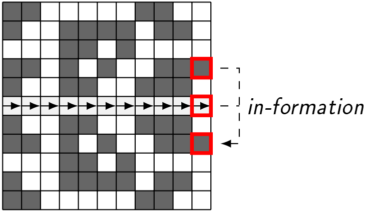

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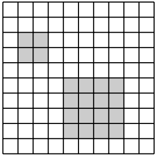

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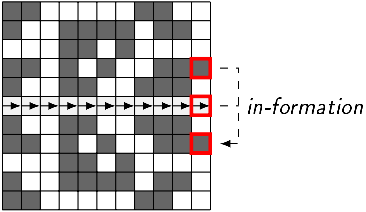

Some necessary conditions are known (cf. Romashchenko, Destombes 2018).

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Examples of sofic shifts might be candidate counterexamples, but we lack tools to understand the possible covers, and their entropy.