Calcul de l’entropie résiduelle de la glace carrée

Silvère Gangloff

LIP, ENS Lyon

October 4, 2018

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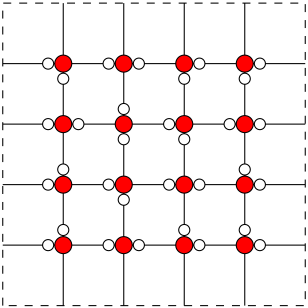

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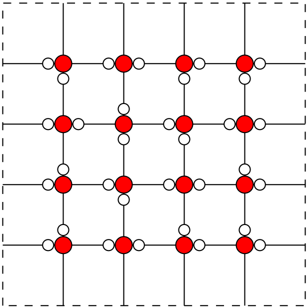

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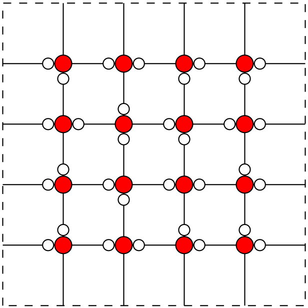

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Valeur de l'entropie?

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But de l'exposé: 'calcul' de la l'entropie de la glace.

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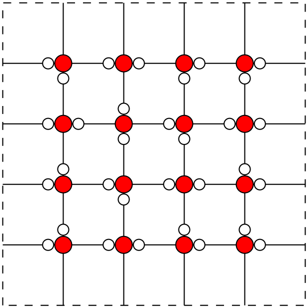

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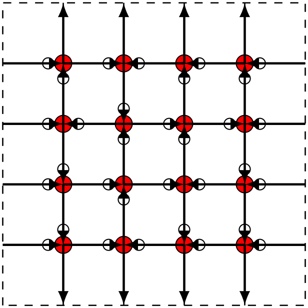

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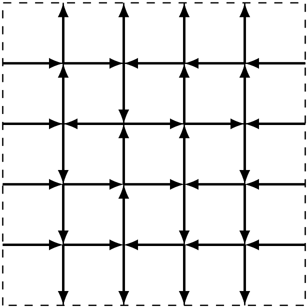

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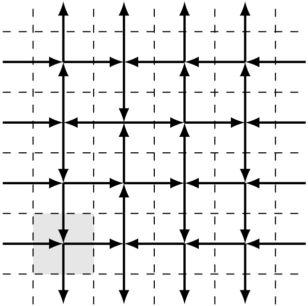

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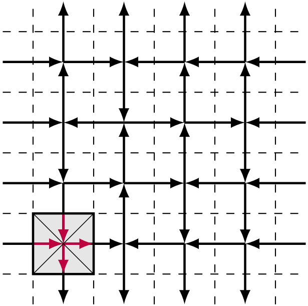

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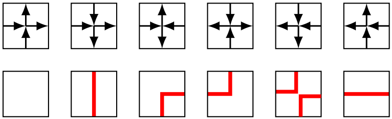

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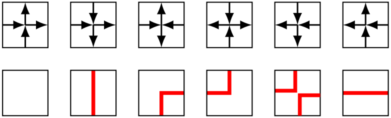

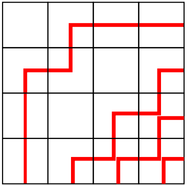

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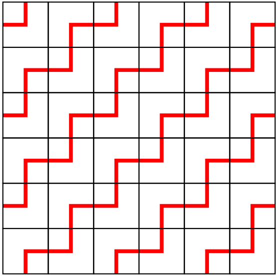

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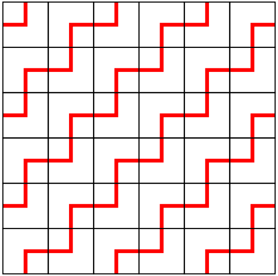

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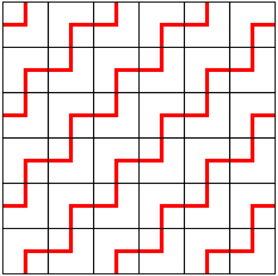

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Suffisant: compter les motifs valides sur un tore

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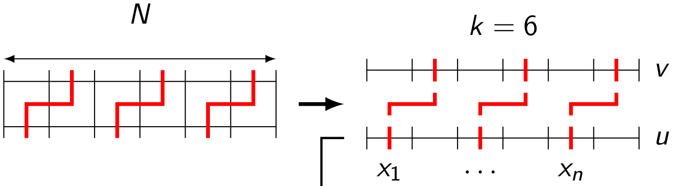
$$V_N(c)(u, v) = c^k$$

$$k = 6$$



Génèrent l'espace vectoriel Ω_N

$$\Omega_N = \bigoplus_{k=0}^{N} \Omega_N^{(k)}, \quad k : \text{nombre de courbes}$$

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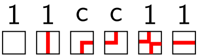

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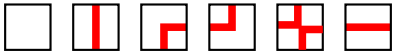

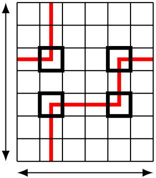

M

N

c^k

$k = 4$

$$P_{N,M}(c) = \sum_{\text{motifs}} c^{k(\text{motif})}$$

$$P_{N,M}(c) = Tr(V_N(c)^M)$$

$$\boxed{h_1 = h_{top}}$$

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Donc $\exists P$ inversible t.q. $\forall c > 0$,

$$V_N(c) = P \begin{pmatrix} \lambda_1(c) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_{2^N}(c) \end{pmatrix} P^{-1}$$

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où $\Theta(0,0)$,

$$e^{-i\Theta(x,y)} = e^{i(x-y)} \frac{e^{ix} + e^{-iy} - 2\Delta}{e^{-ix} + e^{-iy} - 2\Delta},$$

et $\Delta = (2 - c^2)/2$,

$$\varphi_n(c)(x_1, ..., x_n) = \sum_{\sigma \in \Sigma_n} A_\sigma \prod_{k=1}^{n} \exp(ip_{\sigma(k)} x_k)$$

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Remarque: $\Lambda_n(c) \neq 0$?, $\varphi_n(c) \neq 0$?

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Donc $\lambda_{max}(V_N(\infty)) = 1$

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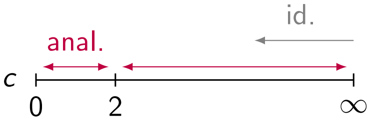

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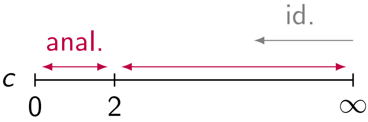

Problème: identification $c > 2$, on veut $c = 1$.

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[1] E.H. Lieb, T. Shultz, D. Mattis, *Two soluble models of an antiferromagnetic chain*, Annals of Physics, 1961.

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Intégrales contours dans $\mathbb{C}$:

$$\boxed{h_1 = (4/3)^{3/2}}$$

Commentaires:

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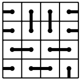

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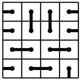
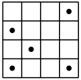
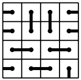


2. Hard core model ? La glace cubique ?



Commentaires:

1. Extensions: eight-vertex model [Baxter], dimer model [Lieb]..



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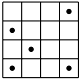

3. Perron-Frobenius: simplification de l'ansatz ?

   "If all eignevectors are given by the Bethe ansatz and span the 2^N dimensional vectorial space (which is the case) [...]", Baxter.

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4. Transformation d'entropie par opérateurs ?