Report on past research

Silvère Gangloff

January 2, 2020

After a master in fundamental mathematics and a one year research internship, I started formally research during my doctoral thesis preparation under supervision of Mathieu Sablik. At this time, I was interested in how computation models interact with dynamical systems, motivated by numerous general questions such as how to account for the generality of the Church-Turing thesis (human computation capability is captured by the one of Turing machines) in our understanding of the actual computation done by human mind (one can see how the only existence of a living being is related to its particular dynamics), on which I had given thoughts during my research internship. After a couple of months, Mathieu proposed me to work on more concrete and precisely stated mathematical questions of this flavor.

My work in this direction led to the solution [GS17a] of an open problem that appeared in a seminal paper by M. Hochman and T. Meyerovitch, *A characterization of entropies of multidimensional subshifts of finite type* [HM10]. This text established computability theory concepts as useful tools in order to analyse dynamical systems, in particular through a method of construction of dynamical systems embedded with Turing machine computations that allows to characterize some dynamical characteristics (such as possible entropy values) of systems in the class of multidimensional subshifts of finite type. The open problem that we solved was about the possibility to characterize the values of entropy for transitive subshifts of finite type. Further refinements in this direction led us to other very interesting results, including: some highly technical constructions which provided an answer to the difficult folklore question of the possibility of embedding computations in minimal subshifts of finite type [GS17b] [GS17c]; together with B. Hellouin, a precise characterization computational threshold phenomenon - meaning a limit between some computability regime and some uncomputability one [GH18]. We also proposed, in a joint paper with C. Rojas and A. Herrera [GHRS], to extend the approach of entropy values characterization to other classes of systems, such as interval maps and onto maps of the Cantor set.

After my doctoral thesis, I worked as a post-doctoral researcher with N. Aubrun and M. Rao, who proposed me another approach to entropy of multidimensional subshifts of finite type, through exactly solved models in statistical and quantum mechanics. For some of these models (which can be seen as subshifts of finite type), physicists have computed exactly the value of entropy [K61][L67][Baxter], and the aim was to understand these computations and understand the properties that make the entropy of these models computable. At this time, I got interested in the square ice model, for which the computation method [L67] for entropy seemed to be the most general. After some intensive work, I provided a complete and fully rigorous mathematical proof of this computation [G19a], and a complement on a rigorous version of a more general method for some part of this proof [G19b]. During this post-doctoral year, in a common work with A. Talon, we used some results of my PhD thesis area of research in order to prove the computability of the growth rate the number of dominating sets on grid graphs [GT].

In the following, I shall give some definitions of my main objects of study (subshifts of finite type) and some background in order to provide then a more precise description of these results.

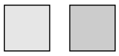
1 Turing computations embedding in multidimensional SFT

Subshifts of finite type appear in various fields of mathematics: in the work of C. Shannon [Shannon] as models for discrete communication channels, as discretisation of continuous dynamical systems (Rudolph's contribution to the $\times 2 \times 3$ conjecture [R90]), as statistical physics models [Baxter], or tilings of the plane [Rob71]. They are defined as follows:

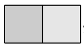
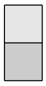
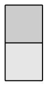
Definition 1. *A subshift of finite type (**SFT** for short) on alphabet (or finite set) $\mathcal{A}$ is a subset of $\mathcal{A}^{\mathbb{Z}^k}$, for some $k \geq 1$, defined as the set of elements of this set in which no pattern (formally, a pattern is an element of $\mathcal{A}^{\mathbb{U}}$, where $\mathbb{U}$ is a finite subset of $\mathbb{Z}^k$) in a fixed finite set appear.*

In the following, we consider that $k = 2$. In this case, these SFT are called bidimensional. The elements of the alphabet are often represented as square tiles written with some symbols and the SFT itself is represented through large square patterns which exhibit typical behavior of its elements.

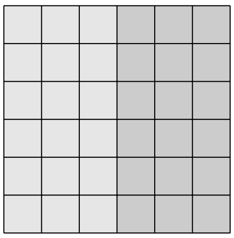
Example 1. *For instance, if the alphabet is given by the symbols*



and by the forbidden patterns

,  *and* 

then elements of the SFT exhibit the following behavior:



meaning that elements of the SFT are the ones that are colored light gray on a left half plane and dark gray on a right half plane that together cover $\mathbb{Z}^2$.

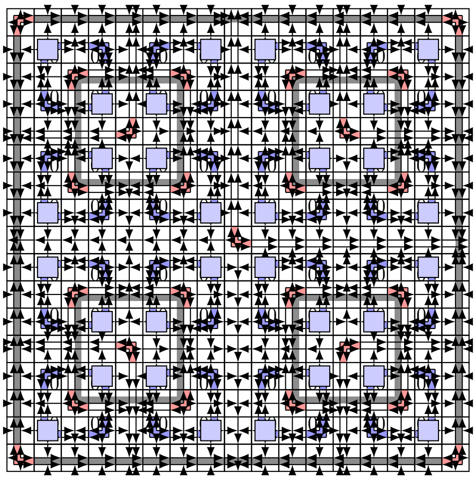
A subshift of finite type X is a compact space, and together with the shift action, they form a (bidimensional) dynamical system. This action, denoted usually σ, makes an element $\mathbf{u}$ of $\mathbb{Z}^2$ act on an element of X by shifting all the colors by the vector $\mathbf{u}$.

Let us denote that the definition of SFT can be relaxed: for instance, one can consider subshifts of finite type on the Cayley graph of a finitely generated group, or an infinite set of forbidden patterns (subshift, but not of finite type). These classes of systems are more flexible, and allow other types of mathematical statements.

1.1 Embedding Turing computations in SFT

The work of M. Hochman, in particular with T. Meyerovitch [HM10], has shown that one can use Turing machines in order to understand the dynamics of multidimensional subshifts of finite type. The technique used in their characterization theorem is often referred to as *embedding Turing computations* in the SFT. This means constructing a SFT whose elements are forced to exhibit in them arbitrary larges finite space-time diagrams of the Turing machine. This kind of construction has been initially developped by R. Berger [B66] and then R. Robinson [Rob71] in order to construct some class of SFT for a proof that it is not possible to find an algorithm which given a set of local rules decides if the corresponding subshift is empty or not (this result is commonly called the *undecidability of the tiling problem* for the plane): the idea is basically to attribute to any Turing machine a SFT in which are embedded computations of this Turing machine, adding the local rule that any halt state is forbidden. The result follows from the fact that it is not possible to decide if a Turing machine eventually halts or not (A. Turing).

The technique of R. Robinson relies on some SFT whose elements look typically like the following (the symbols are not fully represented, in order to simplify the exposition):



thus exhibiting some hierarchical structures (the gray squares). The set of light blue squares included in one of the gray squares, and not any other, form a grid on which is superimposed some space-time diagram of a Turing machine (describing its dynamics over a fixed input). This technique has been applied by M.Hochman and T.Meyerovitch to prove the following result:

Theorem 1 (Hochman, Meyerovitch)**.** *The possible values of entropy of subshifts of finite type are the Π_1-computable non-negative real numbers.*

The term entropy in the theorem refers to topological entropy, defined for any dynamical system on a compact space. For a SFT X, it is expressed as:

$$h(X) = \lim_n \frac{\log_2(N_n(X)))}{n^2},$$

where $N_n(X)$ denotes the number of size n square patterns that appear in an element of X. Moreover, a Π_1-computable real number x is a real number such that there exists an algorithm which on input n (integer) outputs a rational number r_n such that the sequence (r_n) is non-increasing and converges towards x.

In the proof of this theorem, the computations of the machines are used in order to control the density of some *random bits* displayed over the grid, which generate entropy.

2 Personal results

In their article [HM10], M. Hochman and T. Meyerovitch asked if their result would still be true under the dynamical constraint of **transitivity**. This question has been extended to the study of the effect of dynamical constraints on the possibility to embed Turing computations in bidimensional SFT. It was already known for instance that a SFT which satisfies the block gluing property has computable entropy [PS15]. This means that there exists an algorithm which on input n outputs a rational number r_n such that $|x - r_n| \leq 2^{-n}$. Moreover, a SFT is said to be block gluing when any two square patterns having the same size can be glued together and then completed in a coloration of the grid in the SFT, provided that the distance between the two patterns is larger than a constant (independent from the patterns).

In a first work with M. Sablik, we generalized the definition of block gluing by changing the minimal gluing distance between the pattern to a function of their size. We then studied how the function affects the computability of entropy of subshifts of finite type with this property. This study was motivated by the fact that this is a particular instance of problem of characterizing the limit between the computable and the non-computable. This problem, which has been considered for classical problems (such as PCP, the halting problem, or Hilbert's tenth problem) is generally considered as very difficult, and precise characterizations of the limit are very rare. Understanding precisely the limit for some problems could lead to, in the long run, to a better understanding of the phenomenon of undecidability.

2.1 Block gluing with gap function

2.1.1 Bidimensional SFT: a computational transition

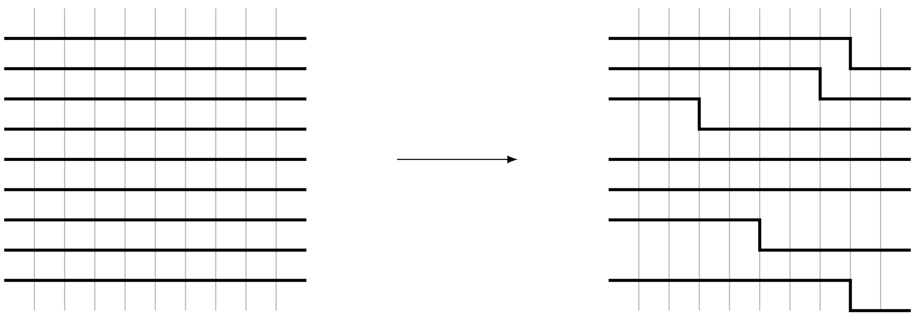
Roughly, for a function $f : \mathbb{N} \to \mathbb{N}$, a subshift is f-block gluing when two cuboid patterns of size n in that appear in elements of this subshift can be 'glued' whenever they are separated by a distance at least $f(n)$. A subshift is linearly block gluing when it is f-block gluing for a function f such that there exists some $C > 0$ such that for all $n \geq 1$, $n/C \leq f(n) \leq Cn$.

When f is constant, the f-block gluing property is the same as R. Pavlov and M. Shraudner's block gluing [PS15]. This property implies that the entropy is computable.

The main theorems that we proved are the following ones. The first theorem provides the first and believed necessary ingredient towards computation embedding in subshifts of finite type - aperiodicity, through hierarchical structures - under the constraint of linear block gluing. With the second one, we go further and state a characterization of the possible entropies under this constraint.

Theorem 2 ([GS17a]). *There exists an aperiodic linearly block gluing SFT.*

For the proof of this theorem, we introduced the notion of net gluing with gap function. This property means that two patterns can be glued one relatively to the other on a sub-lattice of $\mathbb{Z}^2$. We also introduced an operator on subshifts on some fixed alphabet. This operator transforms linearly net gluing subshifts into linearly block gluing ones. The principle of this transformation is to distort the graph of $\mathbb{Z}^2$, as illustrated on the following figure, multiple times and in different directions.



We then apply this transformation on the subshift of R. Robinson which is known to be aperiodic and that we prove to be linearly net gluing.

Theorem 3 ([GS17a]). *The numbers that are the entropy of a linearly block gluing bidimensional SFT are the Π_1-computable numbers. Moreover, a bidimensional SFT which is f-block gluing with $f(n) = o(\log(n))$ has computable entropy.*

As linearly block gluing implies transitivity, this answers the question asked in [HM10], about the possibility to realize every Π_1-computable number as the entropy of a transitive SFT. This was also proved recently by B. Durand and A. Romaschenko [DR17], using the fixed-point paradigm. This construction consists in coloring a part of every computing unit, with control on the frequency of the colored part in $\mathbb{Z}^2$. Random bits are superimposed to these colored parts, generating entropy. This subshift is also net gluing. However, its gap function f is superlinear: $f(n) = n^\alpha$, with $\alpha > 1$. The condition of linearity imposes restriction on the growth of the computing units. Since the fixed-point construction relies on this growth so that the computing machines have enough time to end their computations, the linearly net gluing property can not be derived directly from this paradigm.

The difficulty in the proof of this theorem comes from the fact that the construction by M. Hochman and T. Meyerovitch is very rigid:

1. the identification mechanism for the symbols on which the machines act is done over areas (infinite columns) unrelated to the structures which prevents the block gluing property: the obstruction is that a column constituted with a symbol can not appear over a column of another one.

2. the machines implemented in the SFT can have degenerated (non-typical) behaviors, and this also prevents the gluing property.

In order to adapt the construction we needed to change its general scheme. We needed to adapt the identification mechanism to more local areas, and simulate the degenerated behaviors of the machines as well as normal (intended) behaviors. We allocated areas for normal machines and areas for simulating ones. A machine itself can not 'know' it has a simulation function, so we had to use error signals so that the machine exchanges information with another place in $\mathbb{Z}^d$ where this information is located. With these modifications, we have constructed some linear net gluing SFT. In order to transform them into linearly block gluing ones, we adapted the transformation used to prove Theorem 2, so that the entropy of the image by this transformation can be expressed with the entropy of the initial subshift.

A notable aspect of this construction is that in order to keep the dynamical properties, we have to introduce parasitic entropy (which is not generated by random bits). However, this parasitic entropy can be imposed to be as small as we want, and the random bits serve to supplement this entropy into the aimed one.

The same phenomenon is observable while dealing with the dynamical constraint of minimality.

2.1.2 Decidable subshifts: characterization of a threshold

For multidimensional SFT, we thus proved that there is a transition area for the quantitifed block gluing property, for a gap function between linear and logarithmic, distinguishing a computability and an uncomputability regimes for topological entropy.

In order to refine our understanding of this type of transition, we defined, with B. Hellouin, a notion of irreduciblity with gap function. A subshift is said to be f-irreducible, for a function $f : \mathbb{N} \to \mathbb{N}$, when any two patterns - not necessarily cuboid ones - can be glued whenever the distance between them is at least the maximum of their diameters. We study the effect of the gap function for the irreducibility property on the computability of the entropy for decidable subshifts - meaning subshifts whose language (the set of patterns that appear in at least one element of the subshift) is decidable. Subshifts of finite type are not necessarily in this class. However, $o(n)$-irreducible ones are. For the class of decidable subshifts, we characterize the threshold with a summability condition:

Theorem 4 ([GH18]). *Let $f : \mathbb{N} \to \mathbb{N}$ be a computable function such that $f(0) = 0$. If $\sum_n \frac{f(n)}{n^2}$ converges to a computable number, there exists an algorithm that (uniformly) computes the entropy of f-irreducible decidable $\mathbb{Z}^d$-subshifts. If $\sum_n \frac{f(n)}{n^2} = +\infty$, the set of numbers that are entropy of a decidable f-irreducible $\mathbb{Z}^d$-subshift are exactly the Π_1-computable numbers.*

The main tools used in the proof of this result are the bounded density subshifts, introduced by B. Stanley [Stanley], and defined on alphabet $\{0,1\}$ by rules which consist in forbidding for all $n \geq 1$ the number of 1 symbols in a length n word to be greater than p_n, where $(p_n)_n$ is a sequence of positive integers. We program bounded density subshifts in order to prove the second point of the theorem, and the condition on the function f serves to ensure that the entropy of the constructed subshift is the intended one. A notable aspect of the construction is that we use sequences $(p_n)_n$ coming from the discretisation of concave piecewise linear maps: this allows programming these sequences by programming only the sparse sub-sequence $(p_{F^n(1)})_n$, where $F(k) = 2k + f(k)$ for all $k \geq 1$. The block gluing condition is translated into a lower bound on the possible values that $p_{F^{n+1}(1)}$ can take, according to $p_{F^n(1)}$.

The main obstacle to having a similar result for f-block gluing SFT seems to be the way hierarchical structures -used to embed computations - are constructed, preventing sub-linear block gluing.

2.2 Minimality

In a second work with M.Sablik [GS17b], we studied the properties of another topological invariant, the entropy dimension, of minimal $\mathbb{Z}^d$-SFT. A $\mathbb{Z}^d$-subshift is minimal when all the elements of this subshift share the same patterns.

It is a known fact that the entropy of a minimal SFT is zero. However, the study of the entropy dimension reveals that this class of systems is also rich, in the sense that this is possible to construct SFT in this class that embed Turing computations. The entropy dimension of a subshift X is defined as

$$D_h(X) = \lim_n \frac{\log(\log(N_n(X)))}{\log(n)},$$

when this limit exists, where $(N_n(X))_n$ is the complexity sequence of the subshift. This entropy dimension was introduced in [C97] in order to measure the complexity of zero entropy systems. In [M11], T. Meyerovitch characterized the numbers that are entropy dimension of a multidimensional SFT.

The construction of T. Meyerovitch uses similar ideas as in [HM10], except that the bits on which the machines act are chosen using a selection process through the hierarchical structures of the subshift of [Rob71], thought as tree structures, which is similar to the construction process of a Cantor set. Random bits then generate the entropy dimension.

We used the ideas we developed in our work on the block gluing property (the simulation of degenerated behavior of the machines and the error signals) in order to adapt T. Meyerovitch's characterization to minimality. We proved the following characterization:

Theorem 5 ([GS17b])**.** *The numbers that are entropy dimension of a minimal $\mathbb{Z}^3$-SFT are the Δ_2-computable numbers in* $[0,2]$.

Here a real number is said to be Δ_2-computable when there exists an algorithm which on input n outputs a rational number r_n such that the sequence $(r_n)_n$ converges to x.

The difficulty of the proof comes from the fact that the simulating machines start on an initial tape which is written with a word which has no constraint: one configuration can have all its simulating machines starting on tapes written with some words, and another configuration with other words.

The idea is to use counters that alternate all the possible initial tapes in any configuration. We use the same idea in order to alternate all the possible local layouts of random bits, although for this we need a specific mechanism to increment the counters. We code these counters so that their periods at different scales are different Fermat numbers $2^{2^n}+1$, which are coprime (this is Golbach's theorem). We need this in order to ensure the minimality property.

In the constructions that we made for the block gluing and minimality, dynamical constraints were translated into a heuristic principle of **separation of information**. This means that information has to be localized in different parts of the structures in order to ensure the dynamical constraints, so that only coherent information can appear in the degenerated versions of these structures. For instance, in the construction for the block gluing, the information that tells if the machine is simulating or computing can not be transported by the machine head. If this was the case, one would not be able to simulate a degenerated behavior of a machine head which transports the information that the machine is not simulating. This forces the use of signals, and in particular **error signals**. This mechanism allows a machine which ended its computation in an error state to communicate this information to its initial location where its initialization is verified. This allows the inclusion of degenerated behaviors of the machines in degenerated structures among the possible behaviors of the machines in typical structures.

2.3 Computable dynamical systems on computable compact metric spaces

During my doctoral thesis, I also collaborated with C. Rojas, expert in computable analysis. We worked on extending the questions that we considered with M. Sablik on subshifts of finite type to more general dynamical systems [GHRS]. Naturally, in order to consider computability problems, one should consider computable objects. In particular, we needed to define the right notion of computable dynamical system. In order to build this definition, we relied on computable analysis.

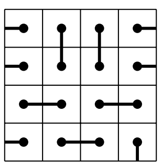
After that, we proved that the entropy of any computable dynamical system is always a Σ_2-computable number, where a real number x is Σ_2-computable when there exists an algorithm which on input (m, n) outputs a rational number $r_{m,n}$ such that $x = \sup_m \inf_n r_{m,n}$.

We could then study the effect of dynamical constraints on this statement, such as surjectivity for systems defined by a function of the Cantor set. We used the result that we obtained with B. Hellouin in order to realize any Σ_2-computable number as entropy of such a dynamical system hence obtaining a complete characterization.

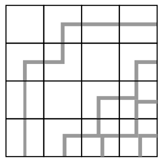
Another important direction in this research has been to consider continuous dynamical systems, which is more original. Using extensively the notion of horseshoe for dynamical systems on the compact interval and a formula of entropy, we were able to prove a characterization of values of entropy for these systems as the Σ_1-computable numbers, where a real number x is Σ_1-computable when there exists an algorithm which on input n outputs a rational number r_n such that $x = \sup_n r_n$. We also proved that some constraint on the total variation implies computability of entropy, and that remarkably this is a characterization.

3 Rigorous exact computation of entropy

This part corresponds to the main project of my first postdoctoral year. It is known that for some models of statistical and quantum mechanics, known as *exaclty solvable models*, that can be seen as bidimensional subshifts of finite type, there exists a formula for the entropy, which has not been necessarily proven. The simplest of these models, from this point of view, is the dimer model, whose elements typically have the following behavior:

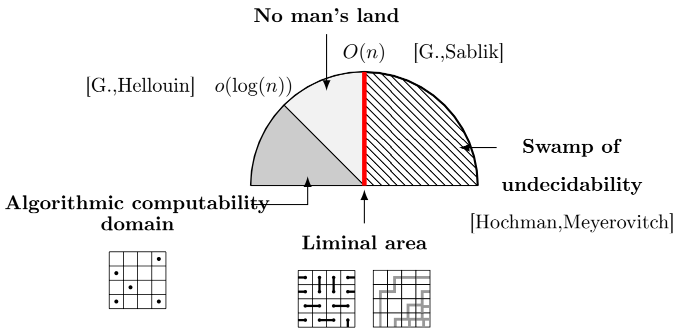


Its entropy has been computed rigorously by P.W.Kasteleyn [K61] using a method that is very specific to this model, seeing the number of size n square patterns as the Pfaffian of a certain matrix, which can be computed using only linear algebra. The second model of interest is the square ice model, which is equivalent to a SFT whose elements typically look like:



Its entropy has been computed by E.H.Lieb [L67] with a method that relied on some unproven hypotheses.

It is interesting to see that these two models, for which there exists a method to compute exactly the entropy, satisfy the linear block gluing property and in fact lie in a "liminal area", shown in the following diagram, which we suspect separates the computability domain from uncomputability one. On can also see on this diagram a representation of the hard core model, whose entropy can be computed algorithmically, while no formula is known for its entropy.



Naturally, I got interested in the computing methods for the models in the liminal area, and particularly square ice, since the method seemed to be partly applicable in principle to other subshifts of finite type, and also by other and recent rigorous results on square ice by H. Duminil-Copin and al [D15].

It appeared that the literature on the subject is particularly obscure, illustrating the culture barrier between physics and mathematics. Major obstacles were of various nature, including not only the fact that the argumentation of E.H.Lieb contained non-rigorous parts, but also a non-differenciation between rigorous arguments and non-rigorous ones, the fact that arguments helping to complete some parts of the proof in the literature (without references) were spread in the literature (without knowing which ones could be already treated somewhere), the non-triviality of the relation between the references of E.H.Lieb and their application to the problem, the presence of non-rigorous arguments in these references, as well the unstability of the notations across articles and the unstability of the language used to describe the results making extremely difficult any bibliographic research. My work has been primarily to formulate and prove rigorously and systematically each part of the method. It appeared later that we could complete the proof using some ideas of a stream of research, which ended with the work of K. Kozlowski [K15], for proving the unproven hypothesis of E.H. Lieb. The end result of this work is the following theorem:

Theorem 6. *The entropy of the square ice model is equal to* $\frac{3}{2}\log_2(4/3)$.

The computation method relies on a relation between entropy and the sequence of maximal eigenvalue of adjacency matrices of the model, where the nth adjacency matrix of the model describes which length n finite rows can be adjacent in an element of the subshift by attributing coefficient 1 to these pairs of rows and 0 to the others. These matrices are generalised into transfer matrices, that can have any coefficient in $\mathbb{C}$ for the pairs of rows that can be adjacent.

In order to have access to the maximal eigenvalue of each of the adjacency matrices, the natural method is to find an eigenvector, and then prove that the corresponding eigenvalue is the maximal one. This is done by E.H.Lieb for an analytic path of transfer matrices that crosses the adjacency matrix for each order n. The proof of coincidence with the maximal eigenvalue uses the analyticity by first proving the coincidence for a subpath and then extending this coincidence to the whole path, in particular for the adjacency matrix. The iarticle of E.H. Lieb relied on an intuitive guess of the form of the eigenvector, well known as the coordinate Bethe ansatz. This eigenvector has been derived non rigorously later by a method that relies on another method, called algebraic Bethe ansatz. The difference between this method and the coordinate one is that it provides a more systematic method to propose an eigenvalue for a transfer matrix whose coefficients for pairs of rows are obtained as a product of coefficients related to local combinations of symbols. This method constructs, for any transfer matrix in the path constructed by E.H.Lieb, another transversal path of transfer matrices such that all the matrices in this path commute. The proposed candidate eigenvector, and then the corresponding eigenvalue, is obtained by applying a product of these matrices to a vector. The commutation property ensures that this vector is an eigenvector.

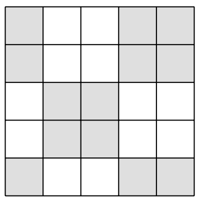
Motivated by the fact that the elements of the algebraic Bethe ansatz can be in principle generalized to other subshifts of finite type, I proved rigorously the following, in a second paper on square ice:

Theorem 7. *The eigenvector given by the algebraic Bethe ansatz is equal to the one given by coordinate Bethe ansatz.*

4 Computability of the growth rate of dominating sets

During my post-doctoral research year, A. Talon, PhD student under supervision of Michael Rao and on advice of him, came to me with some question about the possibility to prove the existence of a growth rate of the number of dominating sets on square grids, when the size of the grid tends towards infinity.

The notion of dominating set comes from graph theory and can be defined for any graph. The definition is the following: for a graph G, a dominating set is a subset of vertices S such that any vertex has a neighbor in S. For exemple, on the following figure we have represented a dominating set of some grid (which is the dual graph of the actual "grid" on the picture), as the set of vertices colored gray:



There exist some variations that are of interest, such as minimal (for the inclusion) dominating sets. For any of these variations, we denote N_n the number of dominating sets of the size n square grid. The question of A. Talon was the existence of the limit

$$\lim \frac{\log_2(N_n)}{n^2}.$$

It appeared that it is possible to prove the existence of this growth rate by comparing the number of dominating sets of the grid with the number of square patterns of a bidimensional subshift of finite type. This comparison is not trivial since the difference between the two notions differ by the presence of border effects. The existence derives then from the existence of entropy:

Theorem 8 ([GT])**.** *The growth rate of the number of dominating sets on size n square grids exists.*

Moreover, we proved that the subshift of finite is block gluing (which demanded a.careful non trivial proof), which yields an algorithm for computing the growth rate of dominating sets:

Theorem 9 ([GT])**.** *This growth rate is a computable real number.*

This work could lead in principle to other applications of symbolic dynamics results on the computability of entropy to other combinatorial problems in graph theory. On the side of symbolic dynamics, this provides some examples of block gluing subshifts of finite type for which this property is not completely trivial to prove. On could exploit this direction in order to construct examples of block gluing subshifts of finite type with a gap function in between linear and logarithmic.

References

[Baxter] R.Baxter, Exxactly solved models in statistical mechanics *Academic Press*, 1982.

[B66] R. Berger Undecidability of the domino problem. *Memoirs of the American Mathematical Society*, 1966, Numb. 66.

[BZ12] J.Buzzi and L.Zambotti, Approximate maximizers of intricacy functionals. *Propability and related fields*, 2012, Vol. 153, Iss. 3-4.

[C97] M.Carvalho, Entropy dimension of dynamical systems. *Portualiae Mathematica*, 1997, Vol.54, Iss. 1.

[D15] H.Duminil-Copin and M. Gagnebin and M. Harel and I. Manolescu and V. Tassion, Discontinuity of the phase transition for the planar random-cluster and Potts models with $q > 4$. Arxiv 2016

[DR17] B.Durand and A.Romashchenko, On the expressive power of quasiperiodic tilings. Arxiv 2017

[GHRS] S. Gangloff and A. Herrera and C. Rojas and M. Sablik, On the computability properties of topological entropy: a general approach. Arxiv 2019, Submitted

[GH18] S.Gangloff and B.Hellouin, Effect of quantified irreducibility on the computability of subshift entropy. *Discrete and continuous dynamical systems*, 2019, Vol. 39, Iss. 4.

[G19a] S.Gangloff, A proof that square ice entropy is $\frac{3}{2}\log_2(4/3)$. Arxiv, 2019. Submitted

[G19b] S.Gangloff, From coordinate to algebraic Bethe Ansatz for square ice. Arxiv, 2019. Submitted

[GT] S.Gangloff and A.Talon, Asymptotic growth rate of square grids dominating sets: a symbolic dynamics approach. Arxiv, 2019. Submitted

[GS17b] S. Gangloff and M. Sablik, A characterization of the entropy dimensions of minimal z3-sfts. arxiv, 2017. Submitted

[GS17c] S. Gangloff and M. Sablik. Simulation of minimal effective dynamical systems on the Cantor set by minimal tridimensional SFT. Arxiv, 2017. Submitted

[GS18] S. Gangloff and M. Sablik, Simulation of minimal effective dynamical systems on the Cantor set by minimal tridimensional SFT. Arxiv, 2018. Submitted

[GS17a] S. Gangloff and M. Sablik, Block gluing intensity of bidimensional sft: computability of the entropy and periodic points. *Journal d'Analyse math/'ematique*, to appear.

[HM10] M. Hochman and T. Meyerovitch, A characterization of the entropies of multidimensional shifts of finite type. *Annals of Mathematics*, Vol. 171, Iss. 3.

[K61] P.W.Kasteleyn, The statistics of dimers on a lattice : I. The number of dimer arrangements on a quadratic lattice. *Physica*, 1961, Vol. 27, Iss. 12.

[K15] K.K.Kozlowski, On condensation properties of Bethe roots associated with the XXZ chain. *Communications in Mathematical Physics*, 2018, Vol. 357.

[L67] E.H.Lieb Residual entropy of square ice *Physical review*, 1967, Vol. 162, Iss. 1.

[M11] T.Meyerovitch Growth-type invariants for Zd subshifts of finite type and classes arithmetical of real numbers *Inventiones Mathematicae*, 2011, Vol. 184, Iss. 3.

[PS15] R. Pavlov and M. Schraudner, Entropies realizable by block gluing shifts of finite type. *Journal d'Analyse Mathématique*, Vol. 126, Iss. 1.

[PW18] K.Petersen and B.Wilson Dynamical intricacy and average sample complexity. Arxiv 2018.

[Rob71] R. Robinson, Undecidability and nonperiodicity for tilings of the plane. *Inventiones Mathematicae*, 1971, Vol 12.

[R90] D.J. Rudolph, $\times 2 \times 3$ invariant measures and entropy. *Ergodic theory and dynamical systems*, 1990, Vol. 10, Iss. 2.

[Shannon] C. Shannon, A mathematical theory of communication. *Bell System Technical Journal*, 1948, Vol. 27, Iss. 3.

[Stanley] B.Stanley, Bounded density shifts. *Ergodic theory and dynamical systems*, 2013, Vol. 33, Iss. 6.