A Cantor dynamical system is slow if and only if it has only attracting finite orbits.

S.Gangloff, joint work with **P.Oprocha**

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Problem

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Embedding in $\mathbb{R}$ with vanishing derivative : exists $\phi : X \to \mathbb{R}$ injective and $g : \mathbb{R} \to \mathbb{R}$ s.t. $\phi \circ f = g \circ \phi$, and $g'_{|\phi(X)|} \equiv 0$.

Problem :

What are the Cantor systems which can be embedded in $\mathbb{R}$ with vanishing derivative ?

Partial result

Result of P.Oprocha and J.Boroński :

Minimal system $\equiv$ every orbit $\{f^n(x), n \geq 0\}$ is dense.

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Theorem[Oprocha,Boroński] : Every minimal Cantor system can be embedded in $\mathbb{R}$ with vanishing derivative.

Jarník's theorem

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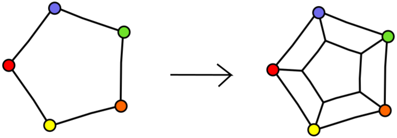
Thus we only need to characterize the Cantor systems (X, f) with a function $\phi : X \to \mathbb{R}$ injective whose derivative is zero.

Graph coverings :

Graph morphisms :



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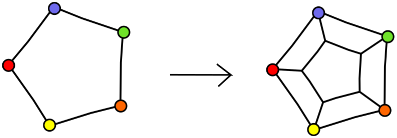

Graph coverings : $(G_n)_{n\geq 1} = (V_n, E_n)_{n\geq 1}$ finite simple oriented graphs and $\pi_n : G_{n+1} \to G_n$ *graph morphisms*.

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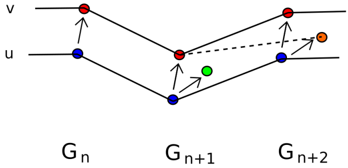

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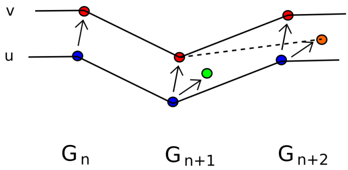

By setting $f(\mathbf{u}) = \mathbf{v}$, we define a continuous function $V \to V$, and thus a dynamical system (V, f).

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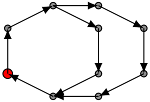
Conjugacy : $x \in X$ is associated with a unique sequence (u_n).

Gambeaudau-Martens representation of minimal systems :

If (X, f) is minimal, the graphs G_n can be taken under the following form :



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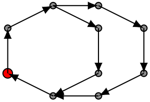

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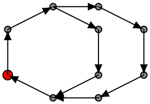

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We will expose this proof for particular systems : odometers.

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$$
\begin{array}{rl}
x: & 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ \ \cdots \\
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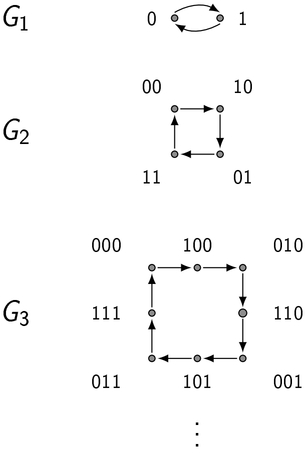

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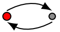
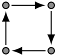
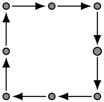


G_2



G_3



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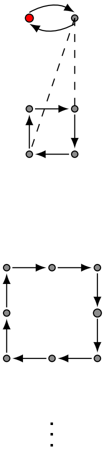

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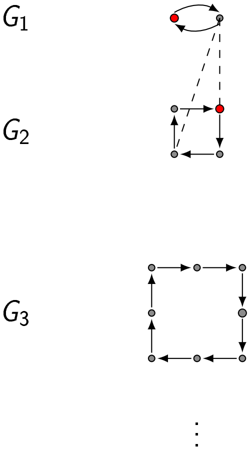

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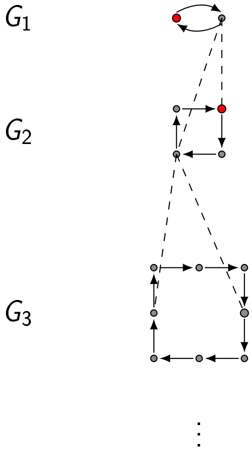

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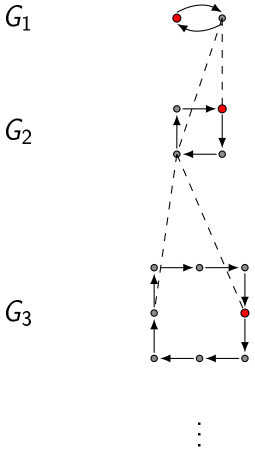

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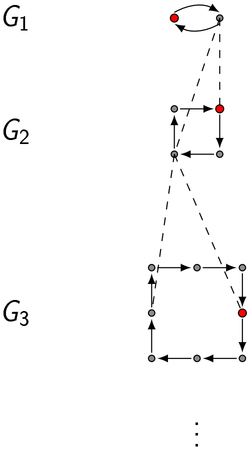

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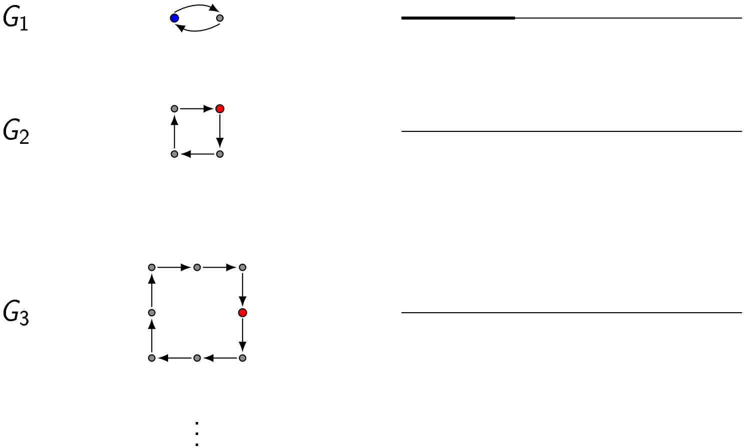

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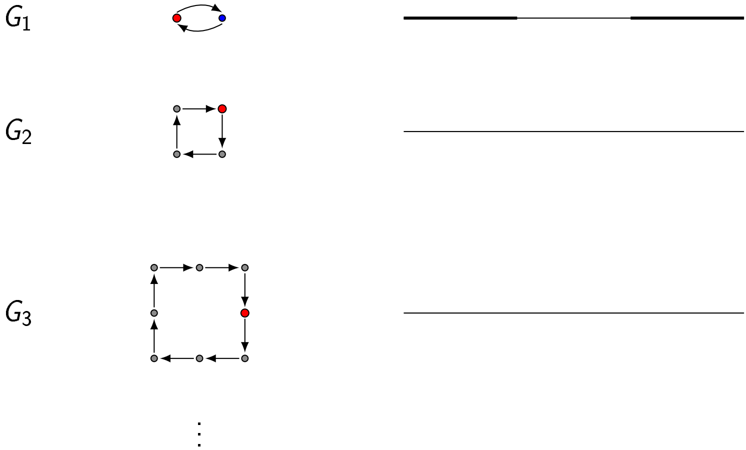

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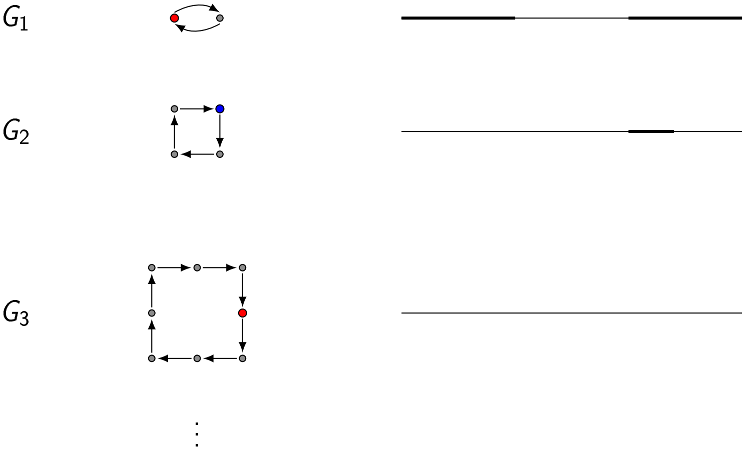

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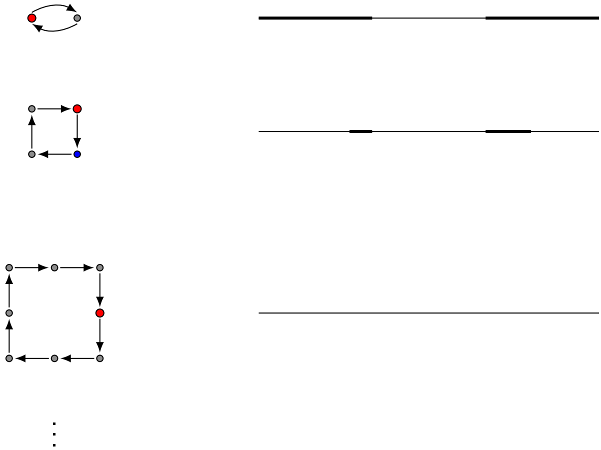

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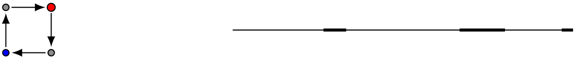

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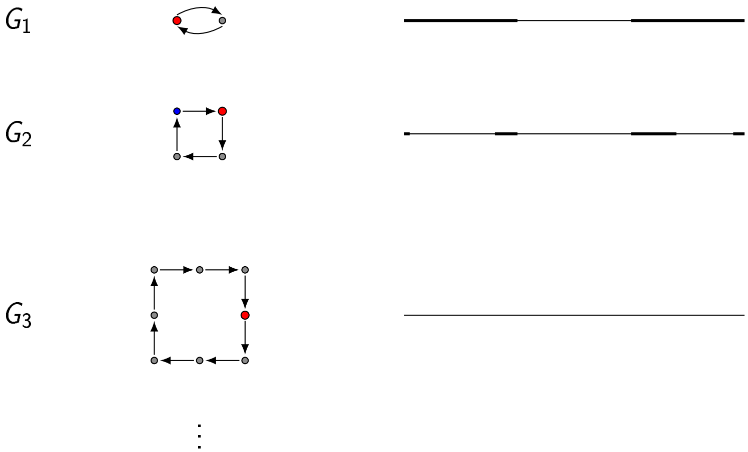

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Since $|\phi'(\mathbf{u})| \leq 2^{-k}$ for all k, it is zero.

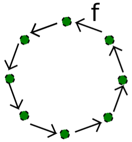
Complete characterization

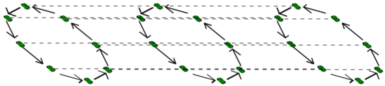
Characterization result :

Theorem[Gangloff,Oprocha] : A Cantor system can be embedded in $\mathbb{R}$ with vanishing derivative if and only if all its finite orbits are attractors.

Finite orbits and attractors in the graph coverings :

Finite orbit :





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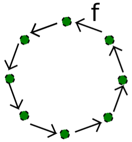

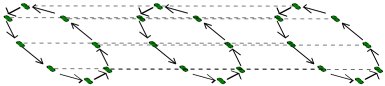

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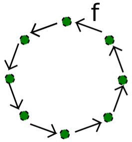

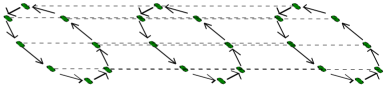

Attractor : closed set C such that for an open neighborhood U of C, $\bigcap_n f^n(U) = C$.

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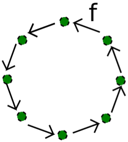

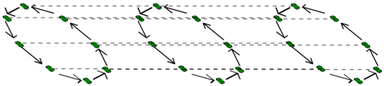

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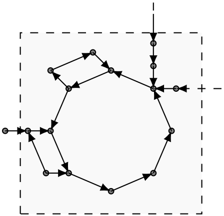

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3. This has to be true for embeddable Cantor systems.

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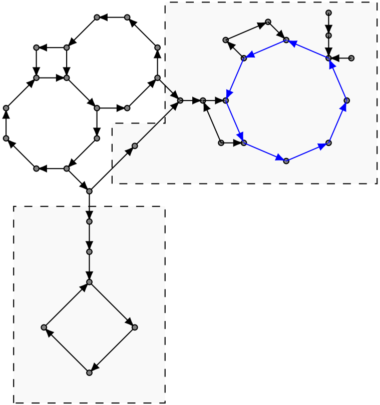
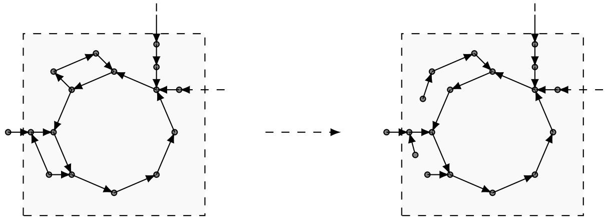
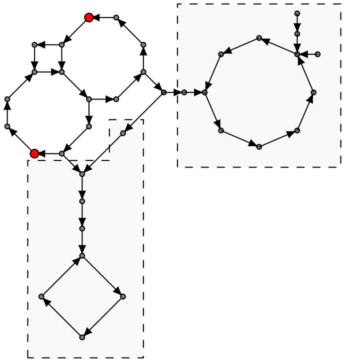


Rectification of finite attractors in supercyclical partitions :



Marking and shrinking processes :

1.



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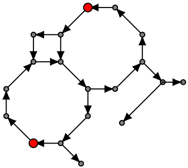

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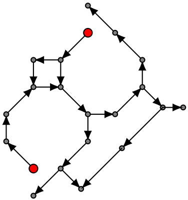
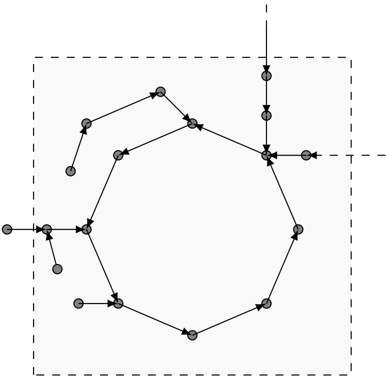


How to deal with finite attractors : distance to the orbit :



How to deal with finite attractors : choosing intervals :