Research Project: Frontiers of uncomputability of entropy for multidimensional subshifts of finite type

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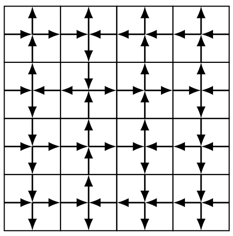
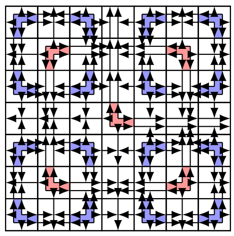
I. Computability and uncomputability of entropy, known results:

Multidimensional subshifts of finite type are sets of d-dimensional infinite words (elements of $\mathcal{A}^{\mathbb{Z}^d}$, where $d \geq 1$ and $\mathcal{A}$ is a finite set). Elements of these sets are often represented through typical patterns, as below. The one on the right corresponds to the six vertex model, a major object of statistical mechanics, and the one on the left corresponds to the Robinson tiling, another major object of logics and decidability. Multidimensional subshifts of finite type are studied not only as topological dynamical systems (in symbolic dynamics) but also in statistical physics, as lattice models, and as computation models. Computing and understanding the entropy of these models is an important problem for all these points of view. In dynamical systems theory (and in particular symbolic dynamics), it is a topological invariant, and as such is a tool in the project of classification of dynamical systems. It is also widely present in statistical physics, as a thermodynamic quantity, and a quantity of information for computation theories.





Uncomputability in symbolic dynamics: a renewal of interest in computing entropy of multidimensional subshifts of finite type occurred recently in symbolic dynamics with the recent work of M. Hochman and T. Meyerovitch [HM11], published in *Annals of Mathematics* (2011). While possible values of entropy for unidimensional subshifts of finite type were characterized algebraically, they proved a characterization of these values for multidimensional ones with a recursion-theoretic criterion, thus answering a long standing open problem. This was already known that the entropy of multidimensional subshifts may be uncomputable, meaning that there exist no algorithm which on input n outputs a rational number which approximates the entropy of the subshift up to 2^{-n}. The work of M. Hochman and T. Meyerovitch goes further and shows that the study of multidimensional subshifts as a class of dynamical systems is intertwined with computability theory.

Uncomputability in statistical physics: In parallel, and more recently, an independent work of T. Cubit, D. Perez-Garcia and M. Wolf, internal to the community of statistical physics, provided a proof of the undecidability of the spectral gap, another quantity closely related to entropy. This is an important result for physicists, since the spectral gap is a tool to decide various other properties of the models; the uncomputability result has (and the first one to have) thus a direct influence on the practice of physics [K17]. A a short presentation of this work was published in the journal *Nature* (2015) [BCLP15].

Existing computation methods: on the other hand, entropy has been computed,although non rigorously, for very particular bidimensional subshifts of finite type called exactly solvable models, such as the dimer model and the six vertex model [L67]. More recently, new results were proved that enumerate exactly the number of patterns of the six vertex model under border conditions [K2]. The exact value of entropy of the hard cores model (which consists in a subshift on the alphabet $\{0, 1\}$ defined by forbidding two 1 symbols to be neighbours) is still not known, but can be computed algorithmically. This can be derived from strong irreducibility [HM11] of the model but a more efficient algorithm was constructed, based on percolation techniques [P12]. The properties of these models which ensure the algorithmic computability of entropy have been extracted, such as symmetry of the local rules [F97], and irreducibility properties. Following the work on entropy, one could wonder if some dynamical restrictions make the spectral gap decidable.

The **main aim of this project** is to understand more precisely the nature of the frontiers between uncomputability and computability of entropy in the class of multidimensional subshifts of finite type (and more precisely bidimensional ones), thus providing tools to localise computable problems in statistical physics. In the next section, I will describe some results in this direction that I proved with my co-authors during my doctoral thesis.

II. A computational threshold phenomenon:

There is no definite way to approach these frontiers. A particular one is through quantified versions of the irreducibility properties. In this restricted context, approaching the frontier means characterizing the threshold on the rate of irreducibility.

Muldimensional subshifts of finite type: drawing a transition domain. In a first long article written with M. Sablik [GS17], we studied this for bidimensional subshifts of finite type, approaching the threshold from below and generalizing the computability of entropy to $o(\log(n))$-irreducibility. Approaching from above, we proved that the possible values of entropy are not changed by the constraint of the linear form of irreducibility. As a corollary, we answered the problem 9.1 of M. Hochman and T. Meyerovitch [HM11]. This text is also interesting for other reasons, since it provides an operator point of view on constructions of multidimensional subshifts of finite type. Moreover, it provides solutions for embedding universal computation under minimality restriction, under which most of known constructions, such as the one of M. Hochman and T. Meyerovitch, break down. For instance, we adapted our techniques in two other long articles in order to provide a characterization of the possible values of another topological invariant, the entropy dimension, under minimality restriction [GS171], and adapt another classical result about simulation of effective systems by subshifts of finite type under this restriction [GS172].

Characterization of a threshold for decidable subshifts: The obstacle in fulfilling the gap between $o(\log(n))$ and linear ($O(n)$) from above for subshifts of finite type comes from the lack of tools for embedding Turing computations, since known techniques rely on the classical structure of R. Robinson or the fixed-point principle [DRS12], and both prevent $o(n)$-block gluing. From below, it comes from combinatorial reasons. In order to understand better this transition, we considered with B. Hellouin [GH18] the more flexible class of decidable subshifts. For a subshift in this class, there exists an algorithm which decides if a pattern can be observed in a configuration of the system. For this class, we were able to characterize a sharp threshold for the quantified irreducibility property: above the threshold the values of entropy are not affected, and below it is computable. The threshold is formulated as a summability condition on the rate function. The tools involved are combinatorial, and an important difficulty was to understand how certain operators acts on subshifts entropy. An important question left open is the characterization of the entropies of irreducible subshifts of finite type below the threshold.

III. Robustness of existing computation methods:

Extending existing methods: Another strategy to approach the frontier is to consider the exactly solvable lattice models of statistical physics in order to understand the properties of these models that make entropy algorithmically computable, and extend the algorithms to broader classes. The interest for these computations methods in the symbolic dynamics community comes also from a growing interest in the possibility of another computation embedding technique, due to K. Culik and J. Kari [CK96], involved in undecidability results, to imply the characterization of M. Hochman and T. Meyerovitch. The difficulty is that it is much more difficult to understand values of entropy of the subshifts obtained by this method.

Communication between physics and mathematics communities: However, the literature on exactly solvable models is left unstructured and dispersed. Moreover, some proofs seem to be lacking: a first recent work by H. Duminil-Copin et al. [DGHMT16] in this direction has provided a rigorous proof of the computation of weighted entropies for the ferromagnetic range of parameters for the six vertex model. As a consequence, an important preliminary step for this project, which constitutes my current work, is to understand and transmit these techniques in more structured documents, which separate rigorous and non-rigorous statements, and complete the proofs when possible.

IV. Forms of computation under dynamical constraints:

On a more informal tone, an important problem originating in cybernetics and the works of A. Turing and J. Von Neumann has been to uncover computational universality in biological systems, by comparing Turing machines with computing units in the living. As a reference to this problem, one important aspect of the constructions that we made in our work with M. Sablik [GS17] is that the universal computation embedding becomes more complex as we approach the frontier; in particular through the addition of natural information processing mechanisms preventing from breaking the dynamical restrictions. Some analogies came up naturally between these mechanisms and ones that can be observed in the living, which were developed in my doctoral thesis. One could go further and wonder if, and how, dynamical constraints exerted on the living are sculpting computation in biological systems. A side aspect of the project is to provide answers to this problem through further developments of these analogies, for instance with models involved in predictive coding theory in neuroscience [F12].

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