On exact computation of square ice entropy

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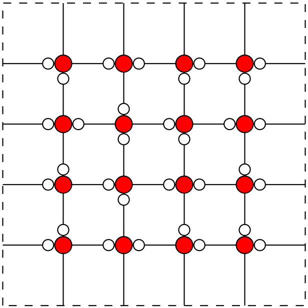
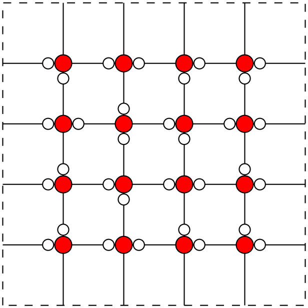
I. Representations of square ice

Square ice model [Pauling-Lieb]:



Wang tiles representation [Six-vertex model]



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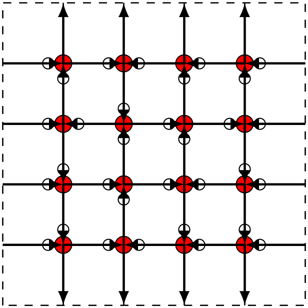

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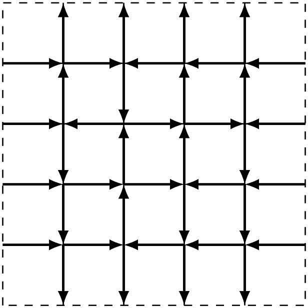

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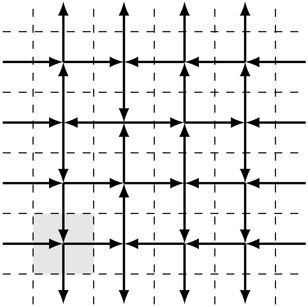

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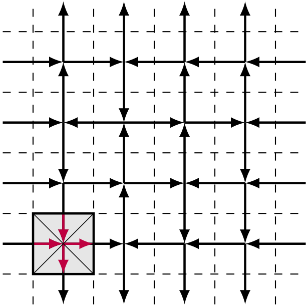
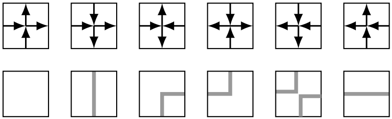


Discrete curves subshift $[X^s]$:



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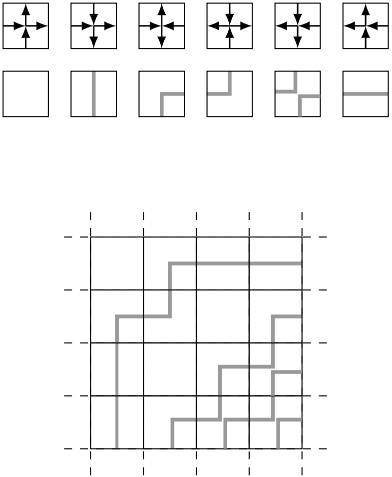

Example of apparition in symbolic dynamics:

S.Gangloff, M. Sablik, *Quantified block gluing, aperiodicity and entropy of multidimensional SFT* , 2017.

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S. Gangloff, *A proof that square ice entropy is* $\frac{3}{2}\log_2(4/3)$, 2019.

II. Subshifts of finite type and entropy

SFT (subshift of finite type): subset of $\mathcal{A}^{\mathbb{Z}^2}$, defined by a finite set of **forbidden patterns**.

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Forbidden patterns $\boxed{\begin{matrix}1\\1\end{matrix}}$ et $\boxed{1\,\vert\,1}$.

SFT (subshift of finite type): subset of $\mathcal{A}^{\mathbb{Z}^2}$, defined by a finite set of **forbidden patterns**.

Ex: **Hard square shift**, or hard core model.

Forbidden patterns $\begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline \end{array}$ et $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}$.

0	0	0	0	0
0	0	0	1	0
0	0	0	0	1
0	0	0	1	0
1	0	0	0	0

SFT (subshift of finite type): subset of $\mathcal{A}^{\mathbb{Z}^2}$, defined by a finite set of **forbidden patterns**.

Ex: **Hard square shift**, or hard core model.

Forbidden patterns $\boxed{\begin{array}{c}1\\ \hdashline 1\end{array}}$ et $\boxed{1 \,\vdots\, 1}$.

$$\begin{array}{c:c:c:c:c}
0 & 0 & 0 & 0 & 0 \\ \hdashline
0 & 0 & 0 & 1 & 0 \\ \hdashline
0 & 0 & 0 & 0 & 1 \\ \hdashline
1 & 0 & 0 & 1 & 0 \\ \hdashline
0 & 0 & 0 & 0 & 0
\end{array}$$

SFT (subshift of finite type): subset of $\mathcal{A}^{\mathbb{Z}^2}$, defined by a finite set of **forbidden patterns**.

Ex: **Hard square shift**, or hard core model.

Forbidden patterns $\boxed{\begin{array}{c}1\\ \hline 1\end{array}}$ et $\boxed{\begin{array}{c|c}1&1\end{array}}$.

$$
\begin{array}{c|c|c|c|c}
0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 1 & 0 \\ \hline
0 & 0 & 0 & 0 & 1 \\ \hline
1 & 0 & 1 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0
\end{array}
$$

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Forbidden patterns $\boxed{\begin{array}{c} 1 \\ \hline 1 \end{array}}$ et $\boxed{\begin{array}{c|c} 1 & 1 \end{array}}$.

0	0	0	0	0
0	0	0	0	0
0	0	0	1	1
1	0	1	0	0
0	0	0	0	0

oops

SFT (subshift of finite type): subset of $\mathcal{A}^{\mathbb{Z}^2}$, defined by a finite set of **forbidden patterns**.

Ex: **Hard square shift**, or hard core model.

Forbidden patterns $\boxed{\begin{array}{c}1\\ \hline 1\end{array}}$ et $\boxed{\begin{array}{c|c}1&1\end{array}}$.

$$\begin{array}{c|c|c|c|c}
0&0&0&0&0\\ \hline
0&0&0&1&0\\ \hline
0&0&0&0&1\\ \hline
1&0&1&0&0\\ \hline
0&0&0&0&0
\end{array}$$

Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles

0	0
0	0

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$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles

0	0
1	0

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0	0
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1	0
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0	1

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Free tiles

1	1
1	1

$$\mathcal{N}_2(X) = 2^{2^2}$$

Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles

1	1
1	1

$$\mathcal{N}_2(X) = 2^{2^2}$$
$$\mathcal{N}_N(X) = 2^{N^2}$$

Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles	**Hard core**
$\begin{array}{\|c\|c\|}\hline 1 & 1 \\ \hline 1 & 1 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 0 & 0 \\ \hline 0 & 0 \\ \hline\end{array}$
$\mathcal{N}_2(X) = 2^{2^2}$ $\mathcal{N}_N(X) = 2^{N^2}$	

Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles

1	1
1	1

$$\mathcal{N}_2(X) = 2^{2^2}$$
$$\mathcal{N}_N(X) = 2^{N^2}$$

Hard core

0	0
1	0

Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles

1	1
1	1

$$\mathcal{N}_2(X) = 2^{2^2}$$
$$\mathcal{N}_N(X) = 2^{N^2}$$

Hard core

0	0
0	1

Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles	**Hard core**
$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array}$	$\begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 0 \\ \hline \end{array}$

$$\mathcal{N}_2(X) = 2^{2^2}$$
$$\mathcal{N}_N(X) = 2^{N^2}$$

Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles

1	1
1	1

$$\mathcal{N}_2(X) = 2^{2^2}$$
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Hard core

0	1
0	0

Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array}$$

$$\mathcal{N}_2(X) = 2^{2^2}$$
$$\mathcal{N}_N(X) = 2^{N^2}$$

Hard core

$$\begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array}$$

Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles	**Hard core**
$\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}$	$\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}$

$$\mathcal{N}_2(X) = 2^{2^2}$$
$$\mathcal{N}_N(X) = 2^{N^2}$$

Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles

1	1
1	1

$$\mathcal{N}_2(X) = 2^{2^2}$$
$$\mathcal{N}_N(X) = 2^{N^2}$$

Hard core

1	0
0	1

$$\mathcal{N}_2(X) = 7$$

Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles	**Hard core**
$\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}$	$\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}$
$\mathcal{N}_2(X) = 2^{2^2}$	$\mathcal{N}_2(X) = 7$
$\mathcal{N}_N(X) = 2^{N^2}$	$\mathcal{N}_N(X) = 2^{N^2(h(X)+o(1))}$

Entropy of a SFT X:

$$h(X) = \inf_N \frac{\log_2(\mathcal{N}_N(X))}{N^2}$$

Free tiles	**Hard core**	**Square ice [Lieb 67]**
$h = 1$	$h \geq 1/2$	$h = \frac{3}{2}\log_2(4/3)$

Entropy of a SFT X:

$$h(X) = \inf_N \frac{\log_2(\mathcal{N}_N(X))}{N^2} = \inf_N \frac{\log_2(\mathcal{N}_N^{loc}(X))}{N^2}$$

Free tiles	**Hard core**	**Square ice [Lieb 67]**
$h = 1$	$h \geq 1/2$	$h = \frac{3}{2}\log_2(4/3)$

Entropy of a SFT X:

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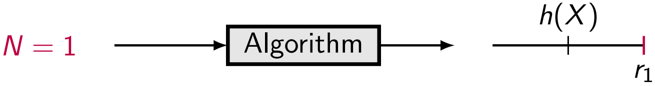
Free tiles	**Hard core**	**Square ice [Lieb 67]**
$h = 1$	$h \geq 1/2$	$h = \frac{3}{2}\log_2(4/3)$

Computable from above:



Entropy of a SFT X:

$$h(X) = \inf_N \frac{\log_2(\mathcal{N}_N(X))}{N^2} = \inf_N \frac{\log_2(\mathcal{N}_N^{loc}(X))}{N^2}$$

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$h = 1$	$h \geq 1/2$	$h = \frac{3}{2}\log_2(4/3)$

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Free tiles	**Hard core**	**Square ice [Lieb 67]**
$h = 1$	$h \geq 1/2$	$h = \frac{3}{2}\log_2(4/3)$

Computable from above:

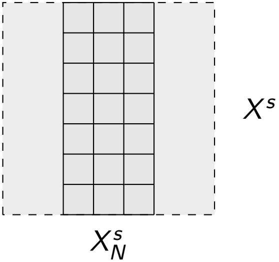
II. Lieb transfer matrices approach

Entropy of square ice:

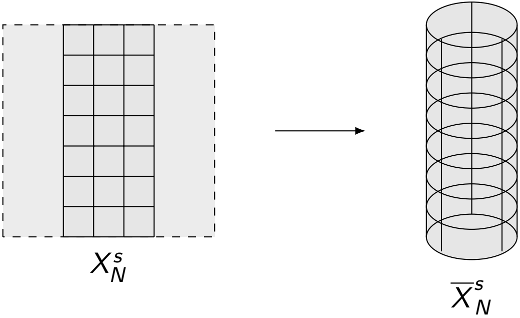
$$\boxed{h(X^s) = \lim_{M,N} \frac{\log_2(\mathcal{N}_{M,N}(X^s))}{MN}}.$$

Stripes subshifts:

$$\boxed{h(X^s) = \lim_{N} \frac{h(X^s_N)}{N}}$$



Cylindric stripes subshifts:



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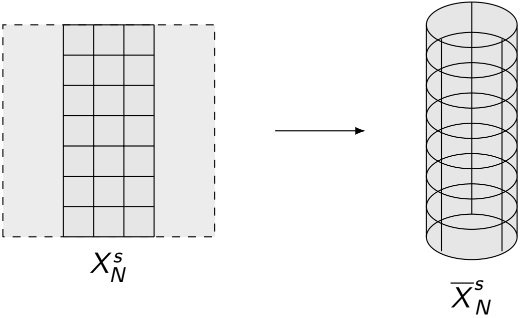
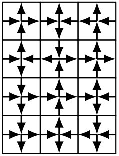


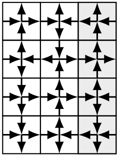
Symmetry properties of square ice imply:

$$\boxed{h(X^s) = \lim_N \frac{h(\overline{X}_N^s)}{N}}$$

Symmetry properties of square ice:



Symmetry properties of square ice:



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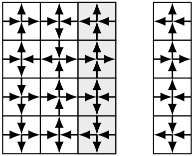
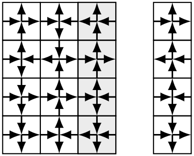


Symmetry properties of square ice:



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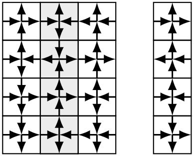

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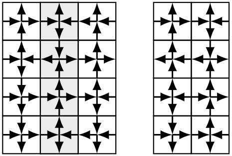

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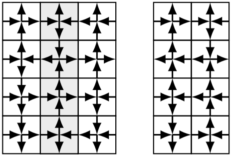
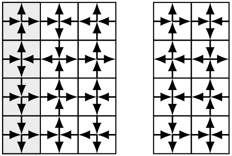


Symmetry properties of square ice:



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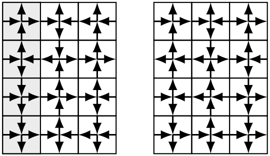

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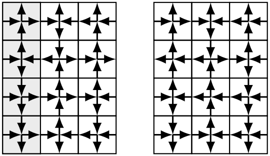

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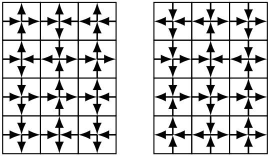

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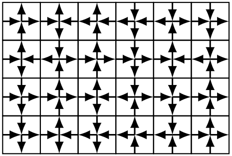
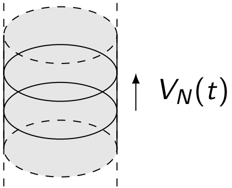


Lieb transfer matrices:



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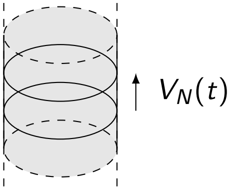

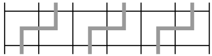

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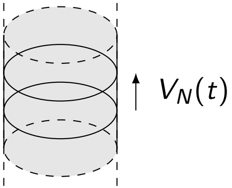

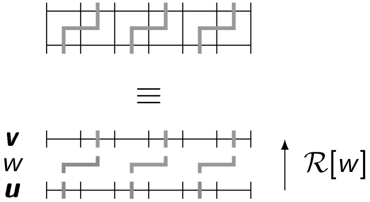

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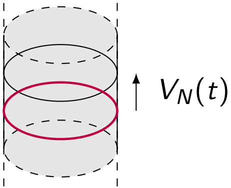

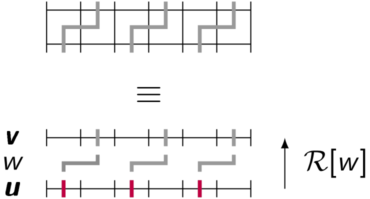

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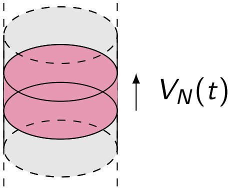
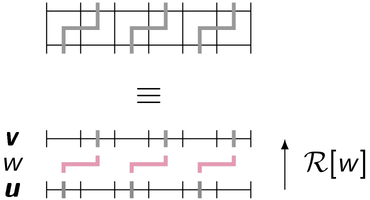
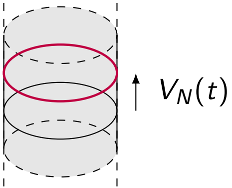
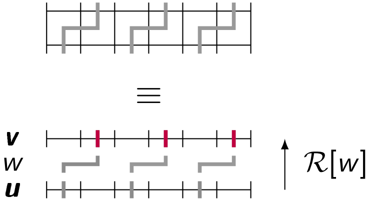




Lieb transfer matrices:





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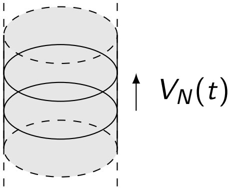

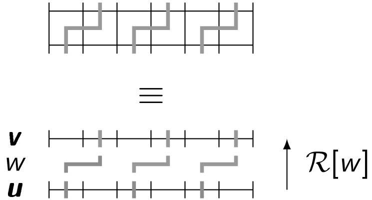

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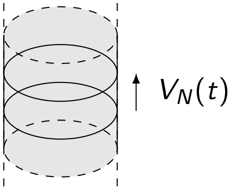

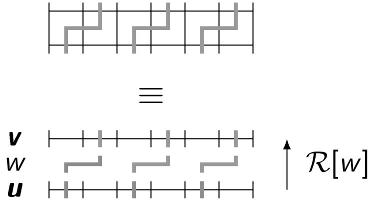

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Computing maximal eigenvalue of $V_N(t)$, strategy:

$V_N(1)$ •

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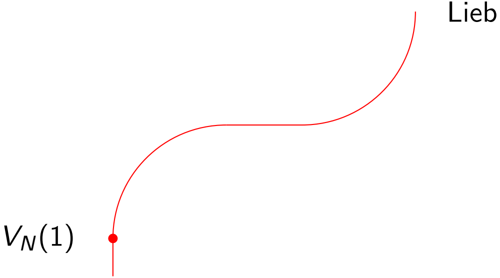

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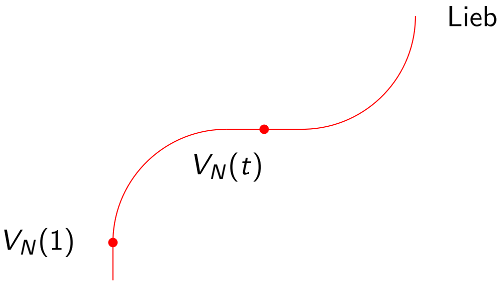

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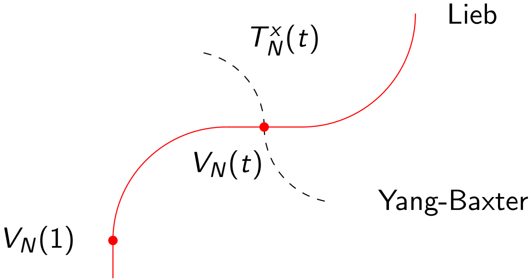

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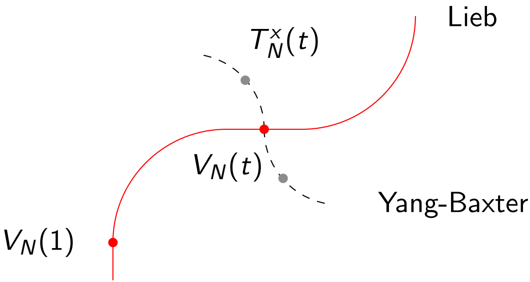

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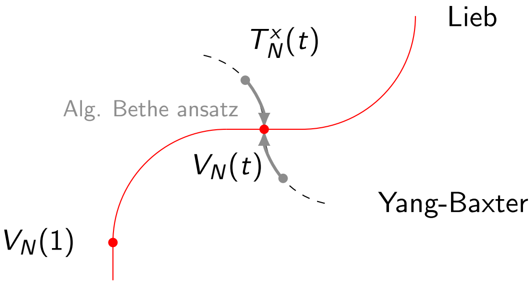

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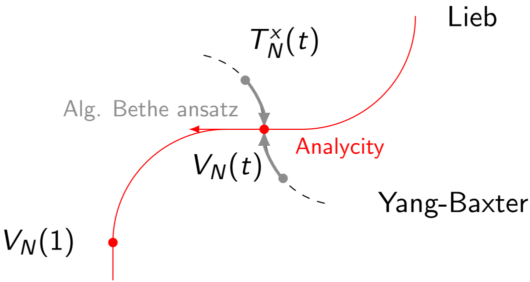

III. Yang-Baxter transfer matrices

R-matrices and monodromy matrices:

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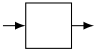

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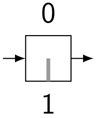

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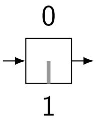

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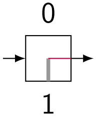

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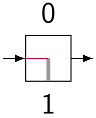

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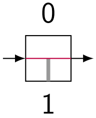

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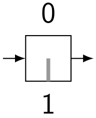

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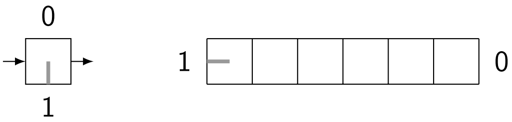

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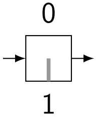

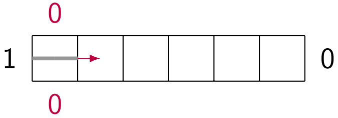

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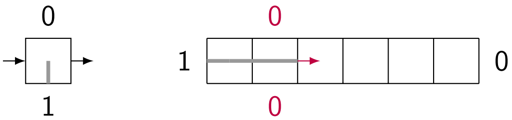

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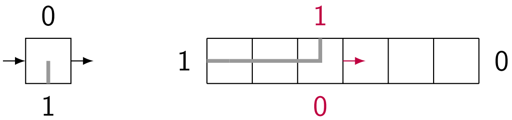

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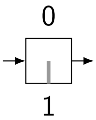

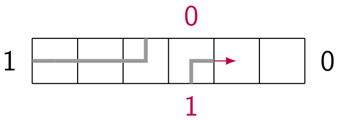

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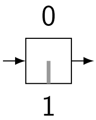

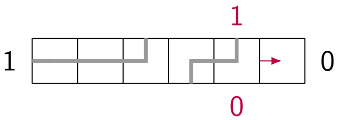

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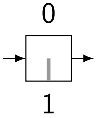

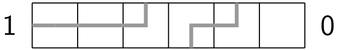

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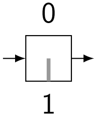

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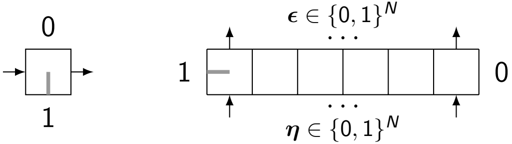

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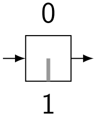

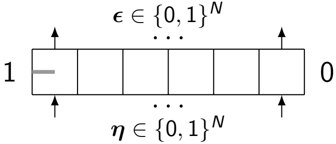

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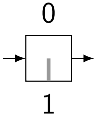

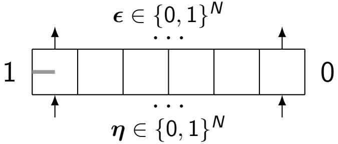

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Yang-Baxter transfer matrices:

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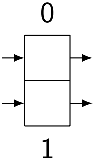

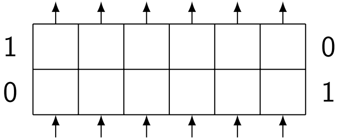

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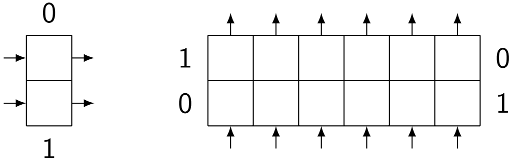
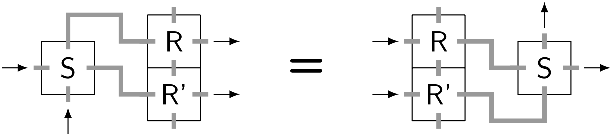


Yang-Baxter equation:



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IV. Asymptotics

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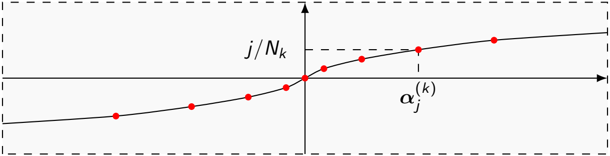

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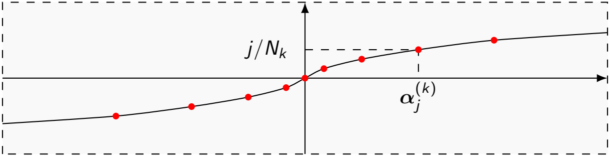

$$\lim_N \frac{\log_2(\lambda_{\max}(V_N(1)))}{N} = \lim_k \frac{1}{N_k}\sum_{j=1}^{n_k} f(\boldsymbol{\alpha}_j^{(k)}) = \int_{\mathbb{R}} f(\alpha)\rho_t(\alpha)d\alpha.$$

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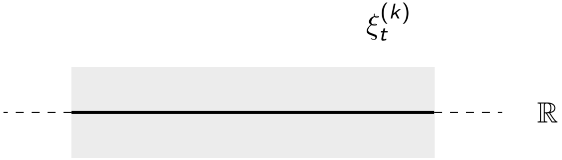

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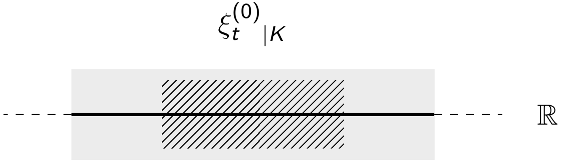

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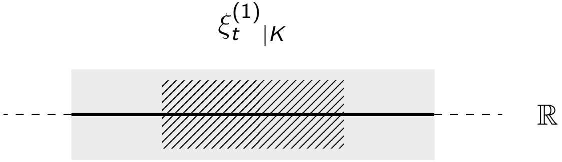

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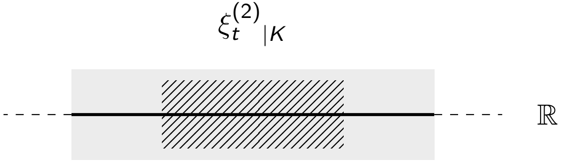

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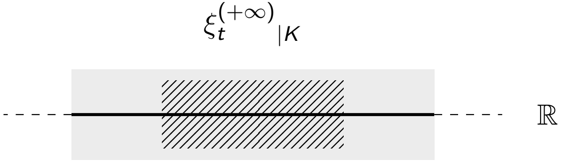

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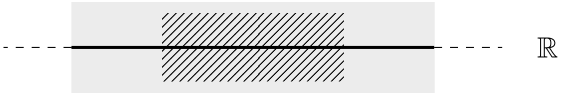

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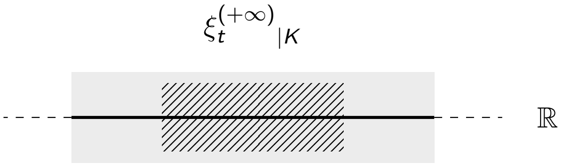

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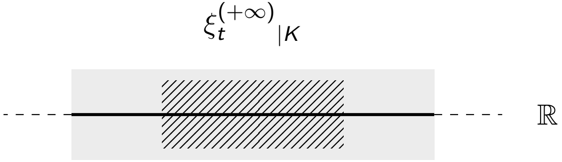

2. Assume $(\xi_t)^{\nu(k)} \to \xi_t^{(+\infty)}$ on any compact K.
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4. Thus, $\xi_t^{(k)} \to \xi_t^{(\infty)}$.

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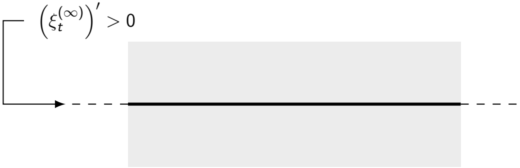

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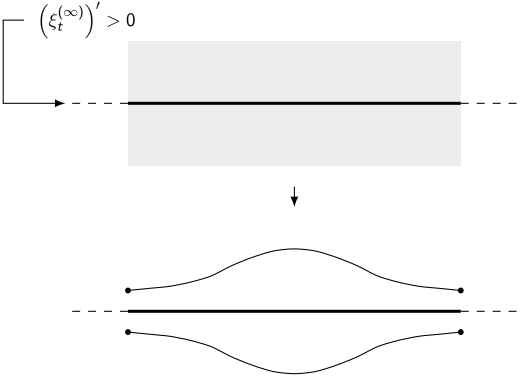

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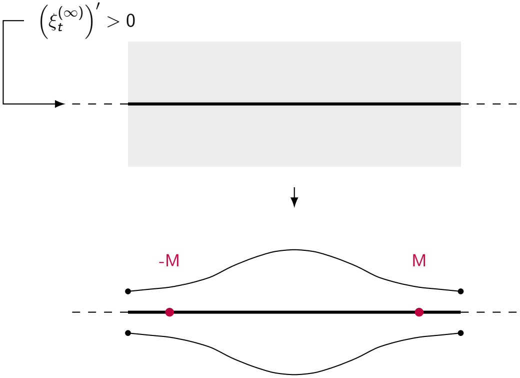

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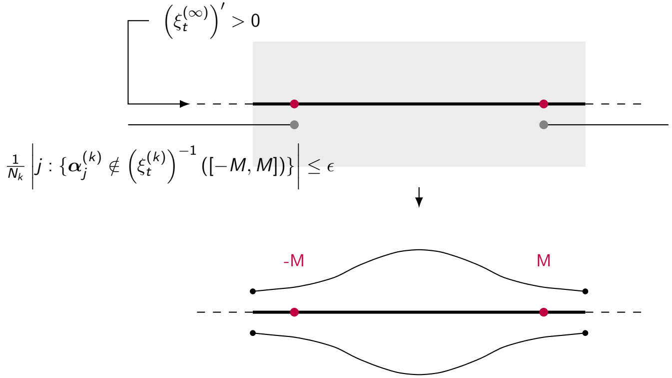

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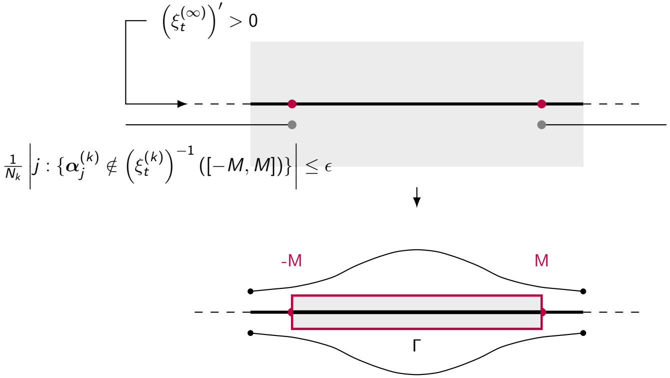

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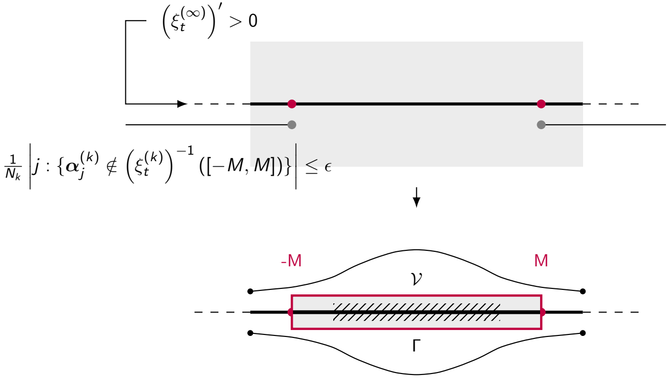

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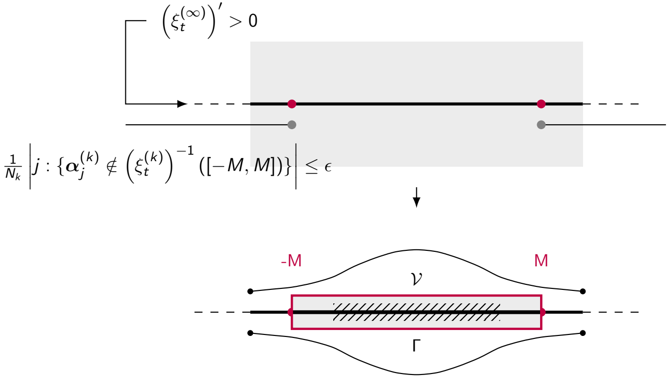
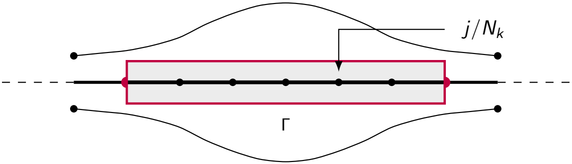


The functions have distinct values on $\mathcal{V}$ and Γ. Thus they are bihilomorphisms onto $\mathcal{V}$.

Lace integral expression of $\xi_t^{(k)}$:



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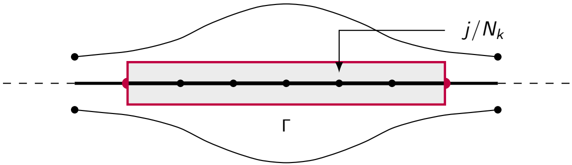

By residues theorem:

$$\xi_t^{(k)}(\alpha) = \frac{1}{2\pi}\kappa_t(\alpha) + \frac{n_k+1}{2N_k} + \oint_\Gamma \theta_t\left(\left(\xi_t^{(k)}\right)^{-1}(\alpha)\right)\frac{e^{2i\pi s N_k}}{e^{2i\pi s N_k}-1}ds + O(\epsilon).$$

Fredholm integral equation: Limit and change of variable:

$$\xi_t^{(\infty)}(\alpha) = \frac{1}{2\pi}\kappa_t(\alpha) + \frac{1}{4} + \int_0^{+\infty} = \theta_t(\alpha)\left(\xi_t^{(\infty)}\right)'(\alpha)d\alpha.$$

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Solution by Fourier transforms.

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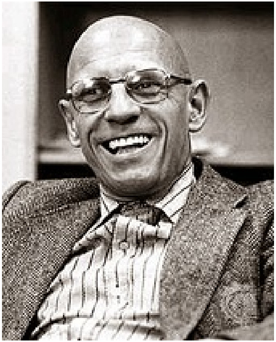
Through an expression of $\rho_t = \left(\xi_t^{(\infty)}\right)'$ and lace integrals computations:

$$\boxed{h(X^s) = \frac{3}{2}\log_2(4/3)}.$$

V. Comments

Why mathematical physics are hard to read for mathematicians?



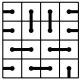
Archaeology of Knowledge,1969

Concept of *discursive formation*

Mathematics and mathematical physics are distinct discursive formations ; different conceptions of units of meaning, etc.

Further research:

1. Extensions: eight-vertex model [Baxter], dimer model [Lieb]..



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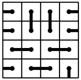
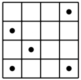


2. **Hard core model** ? Tridimensional ice ? Kari-Culik tilings ?



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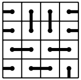

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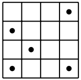

3. Transformation of entropy by subshifts operators ?