

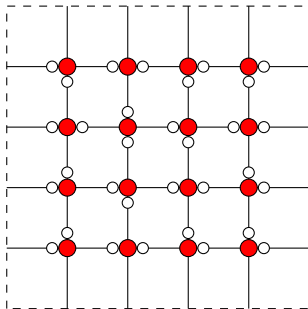
Calcul de l'entropie résiduelle de la glace carrée

Silvère Gangloff

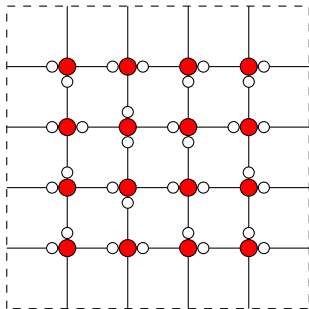
LIP, ENS Lyon

October 11, 2018

États stables de la glace carrée [Pauling-Lieb]:

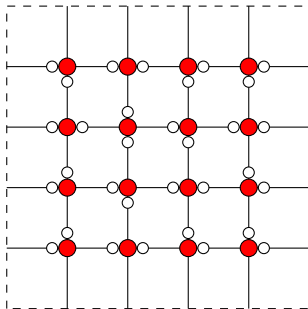


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Valeur de l'entropie?

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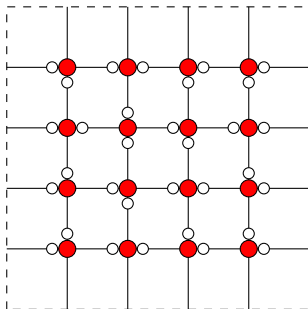
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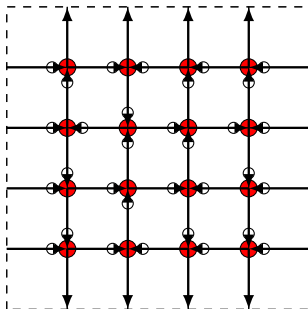
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But de l'exposé: 'calcul' de la l'entropie de la glace.

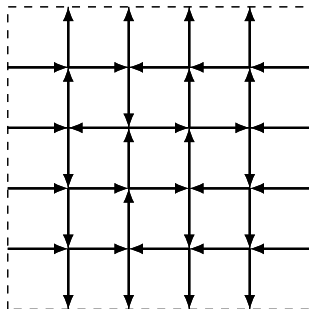
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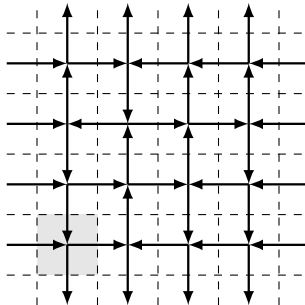
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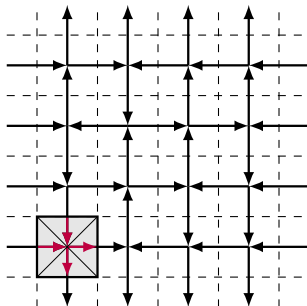
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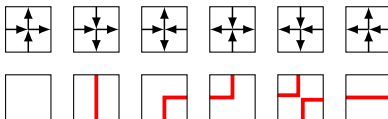
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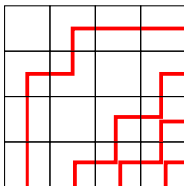
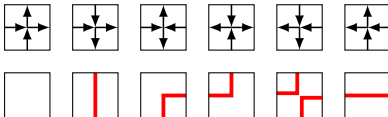
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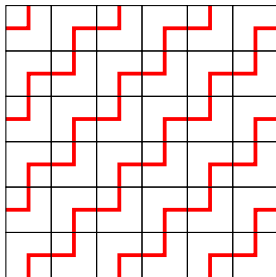
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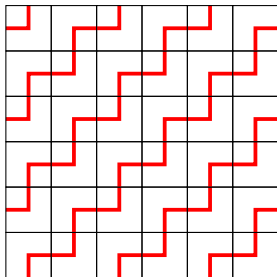
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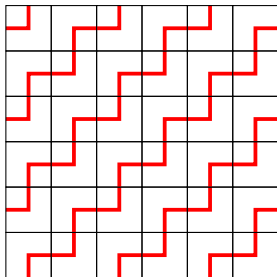


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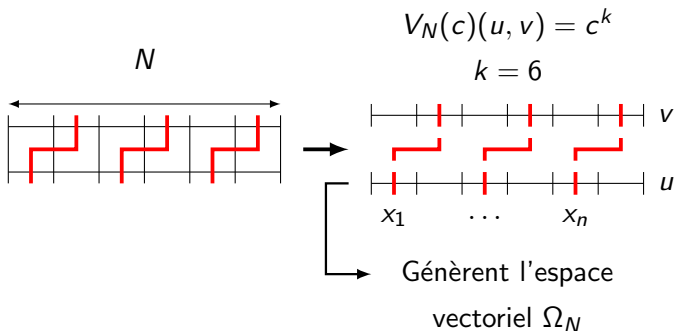


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Suffisant: compter les motifs valides sur un tore

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$$\Omega_N = \bigoplus_{k=0}^N \Omega_N^{(k)}, \quad k : \text{nombre de courbes}$$

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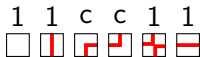
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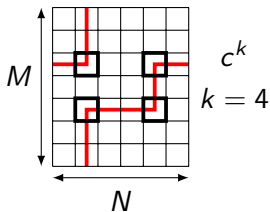
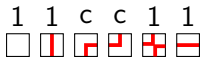
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$$P_{N,M}(c) = \sum_{\text{motifs}} c^{k(\text{motif})}$$

$$P_{N,M}(c) = \text{Tr}(V_N(c)^M)$$

$$h_1 = h_{top}$$

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La fonction $c > 0 \mapsto \lambda_{\max}(V_N(c))$ est un **analytique**. [Yang²]

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où $\Theta(0,0)$,

$$e^{-i\Theta(x,y)} = e^{i(x-y)} \frac{e^{ix} + e^{-iy} - 2\Delta}{e^{-ix} + e^{-iy} - 2\Delta},$$

et $\Delta = (2 - c^2)/2$,

$$\varphi_n(c)(x_1, \dots, x_n) = \sum_{\sigma \in \Sigma_n} A_\sigma \prod_{k=1}^n \exp(ip_{\sigma(k)} x_k)$$

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Remarque: $\Lambda_n(c) \neq 0?$, $\varphi_n(c) \neq 0?$

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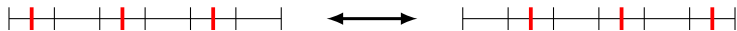
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Donc $\lambda_{\max}(V_N(\infty)) = 1$

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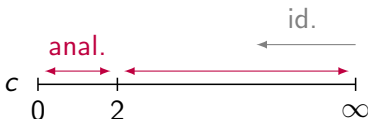
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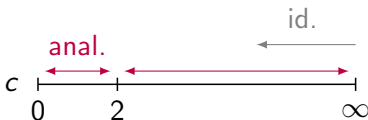


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Problème: identification $c > 2$, on veut $c = 1$.

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[1] E.H. Lieb, T. Shultz, D. Mattis, *Two soluble models of an antiferromagnetic chain*, Annals of Physics, 1961.

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Intégrales contours dans \mathbb{C} :

$$h_1 = (4/3)^{3/2}$$

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- ③ Entropie du Kari-Culik ?
- ④ Transformation d'entropie par opérateurs ?

