Entropy of subhifts: a sharp computational threshold phenomenon.

Silvere Gangloff, Benjamin Hellouin

10 décembre 2018

Effect of quantified irreducibility on the computability of subshifts entropy, Gangloff, Hellouin, Discr. Cont. Dyn. Sys. (2018)

Milnor (2002) : is the entropy of a dynamical system computable ?

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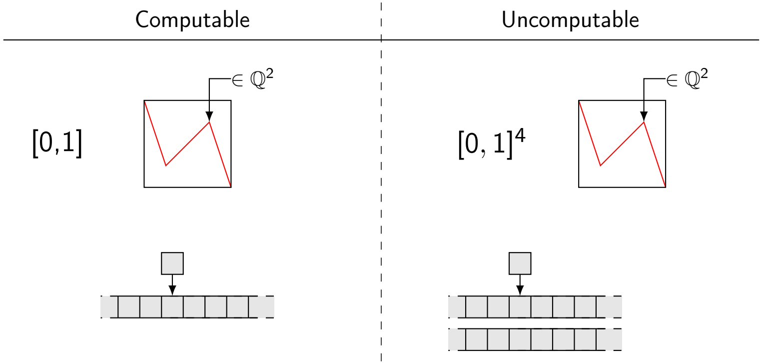
Examples of results : Koiran 2001, Jeandel 2014, Delvenne and Blondel 2004.



Entropy of subshifts :

Subshift : set of $\cdots \boxed{1}\boxed{0}\boxed{1}\boxed{0}\boxed{0}\boxed{1}\boxed{0}\boxed{1} \cdots$; forbidden : $\boxed{1}\boxed{1}$

Possible : $\boxed{0}\boxed{0}\boxed{0}$ $\boxed{1}\boxed{0}\boxed{0}$ $\boxed{0}\boxed{1}\boxed{0}$ $\boxed{0}\boxed{0}\boxed{1}$ $\boxed{1}\boxed{0}\boxed{1}$; $N_3 = 5$, $N_n = 2^{n(h+o(1))}$.

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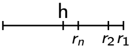

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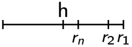
$1, 2, \ldots, n \longrightarrow \boxed{\text{Algorithm}} \longrightarrow$ 

Computable $\equiv$ comp. speed

About multidimensional SFT :

Definition : set of

$$\begin{array}{c} \vdots \\ \cdots \begin{array}{|c|c|c|c|} \hline 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline \end{array} \cdots \\ \vdots \end{array}$$

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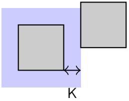

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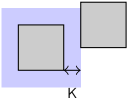
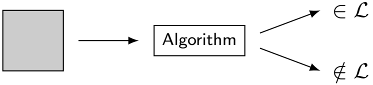


2017 : Gap function $K \rightarrow f(n)$ (G., Sablik) : f-block gluing

Decidable subshifts : a computational threshold for entropy

Decidable :



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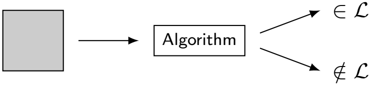

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Theorem[G.,Hellouin] : define $\Sigma(f) = \sum_n f(n)/n^2$, assumed computable.

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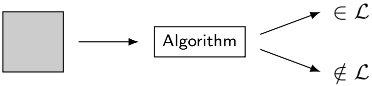

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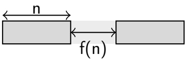
Question : $\Sigma(f)$ not computable ?

Under the threshold : $\Sigma(f) < +\infty \Rightarrow h$ computable



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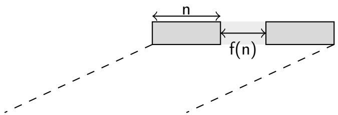

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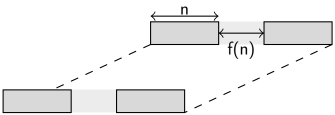

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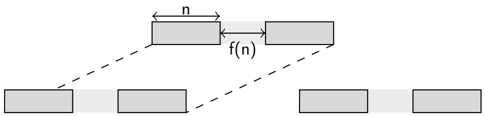

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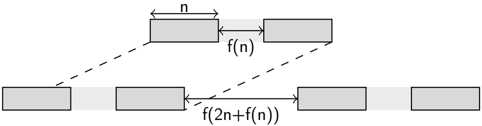

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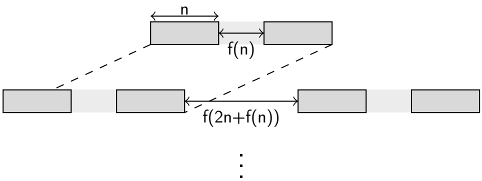

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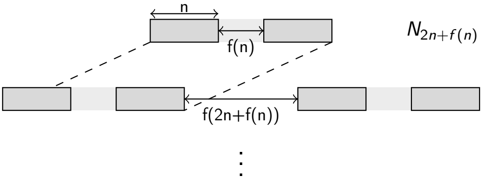

$$N_{2n+f(n)} \geq N_n^2$$

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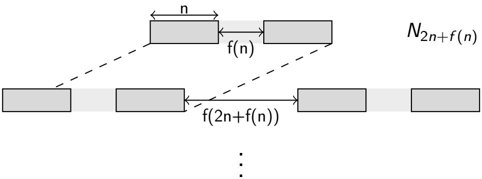

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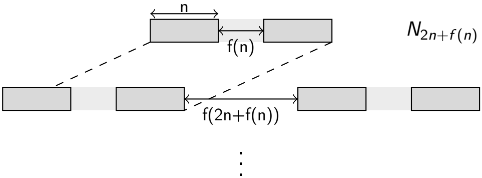

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Repetition+log :

$$\frac{\log(N_n)}{n} - |\mathcal{A}|.\sum_{n}^{+\infty} \frac{f(2^k)}{2^k} \leq h \leq \frac{\log(N_n)}{n}$$

Above the threshold I. Objects : $\Sigma(f) = +\infty$

Objective : realization of any Π_1-comp. number.

Bounded density shifts : $(p_n)_n \in \mathbb{N}^{\mathbb{N}}$ growing, forbidden :

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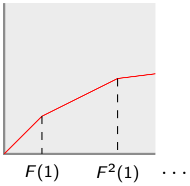
$F(n) \equiv 2n + f(n)$; (p_n) discretised of :



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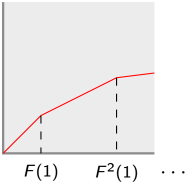

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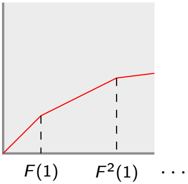

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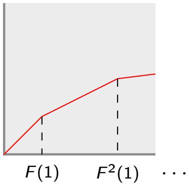

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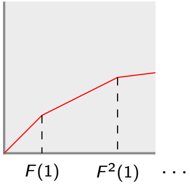

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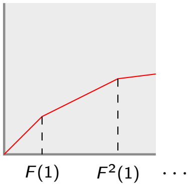

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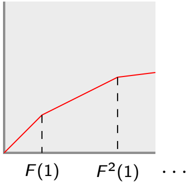

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Set $\alpha_n \to \alpha$, Π_1-comp.

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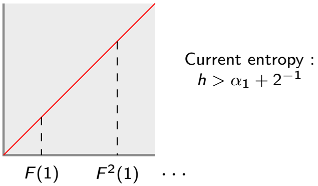

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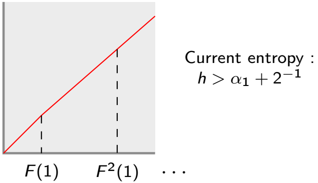

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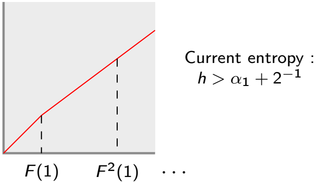

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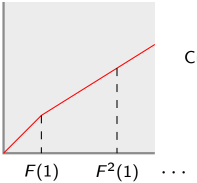

Current entropy :
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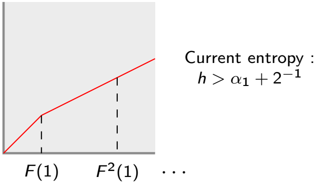

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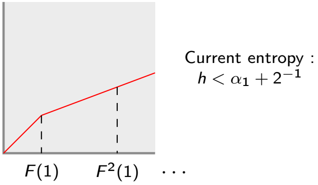

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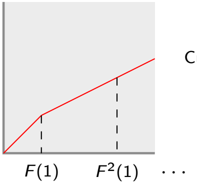

Current entropy :
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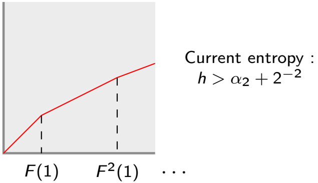

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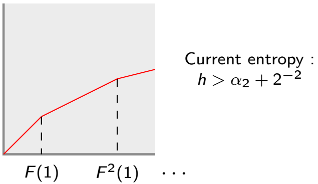

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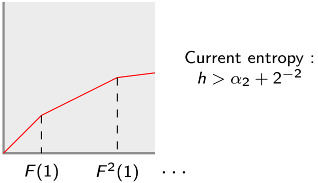

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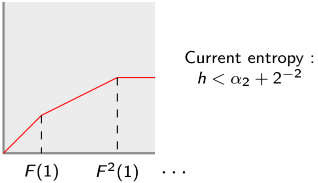

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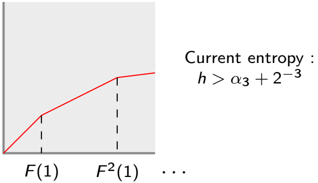

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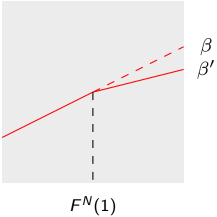
Entropy change : $\beta = (\beta_1, \beta_2, ..)$ slopes :



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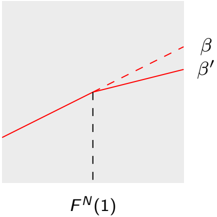

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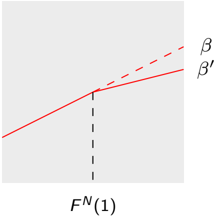

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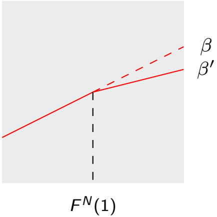

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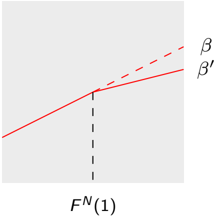

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