Square ice topological entropy: (i) archeology

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Abstract

In this text, we provide a complete, almost self-contained computation of the entropy of the square ice. We also formulate questions related to the dynamical properties of this model allowing this computation.

1 Introduction

The square ice model (or six vertex model) is a classical object of statistical mechanics. It is one example of lattice model for which a method is known to compute exactly some physical characteristics, such as the residual entropy, computed by E.H. Lieb [L67] through the so-called transfer matrix method. His work relies on the diagonalisation by C.N. Yang and C.P. Yang [YY66] of some Heisenberg Hamiltonian depending on a parameter. This work itself relies on a previous work by E.H. Lieb, T. Shultz and D. Mattis [LSM61], who diagonalised the Hamiltonian which corresponds to a particular value.

In the terms of symbolic dynamics, the square ice model is a bidimensional subshift of finite type (SFT). This means that it is a set of infinite bidimensional words on some alphabet, whose letters are possible local configurations of molecules of water. These words represent possible stable states of square ice. The residual entropy is then the topological entropy of this SFT.

It is of current interest in the field of symbolic dynamics to study the topological entropy on multidimensional SFT as a class. It was already known that, contrary to unidimensional ones, the entropy of multidimensional SFT may be uncomputable [CHK92]. Recently M. Hochman and T. Meyerovitch [HM10] proved moreover that the possible values of entropy for d-dimensional SFT are the numbers that can be approximated from above by the ouputs of an algorithm, exhibiting that dynamics of multidimensional SFT are intertwined with computability theory. Even more recently was discovered that this uncomputability phenomenon breaks down under strong dynamical constraints [PS15]. Current research attempts to understand the frontier between the uncomputability and the computability of entropy for multidimensional SFT. For instance, approaching the frontier from the uncomputable, the author, together with M. Sablik [GS17a] proved that the characterization of M. Hochman and T. Meyerovitch stands under a relaxed form of the restriction studied in [PS15], which includes the lattice models known to be exactly solvable. Another strategy is to extend the existing methods to compute exactly the entropy, approaching the frontier from the computable. This text provide a rigorous, and complete computation. It is of interest in itself, for it may take some consequent time for someone not used to statistical physics reasonings to find and understand the arguments in multiple documents that are used with changes of notations. It will also serve as a ground for our approach of the frontier. For this purpose, in the description of this computation, we point out crucial properties of the SFT allowing the computation of its entropy, and formulate question out of these observations. We are also careful to recall basic analysis statements for people who are not used to them.

This text is organized as follows: in Section 2, we recall definitions of symbolic dynamics relative to bidimensional subshifts of finite type and introduce representations of square ice. We describe in Section 3.2 the transfer matrix method for the square ice. The principle of this method is to consider the bidimensional SFT as a limit of unidimensional ones. Each of these subhsifts has entropy equal to the maximal eigenvalue of a matrix describing it. We provide an original and rigorous proof that in computing the entropy, one can restrict to a particular subset of patterns which can be wrapped on a torus without breaking the rules of the SFT. This simplifies the matrices, and allows the search of vectors that satisfy the eigenequations under a particular form (Bethe ansatz).

We prove in Section 4 that the Bethe ansatz provides the maximal eigenvalues and compute the entropy, gathering and completing arguments coming from multiple references.

2 Background

2.1 Bidimensional subshifts of finite type

2.1.1 Definition

Let $\mathcal{A}$ be some finite set, called **alphabet**. For all $d \geq 1$ (in this text we consider only $d = 1, 2$), the set $\mathcal{A}^{\mathbb{Z}^d}$ is a topological space with the infinite power of the discrete topology on $\mathcal{A}$. Its elements are called **configurations**. Let us denote σ the action of $\mathbb{Z}^d$ on this space defined by the following equality for all $\mathbf{u} \in \mathbb{Z}^d$ and x element of the space:

$$(\sigma^{\mathbf{u}}.x)_{\mathbf{v}} = x_{\mathbf{v}+\mathbf{u}}.$$

A compact subset X of this space is called a **subshift** when this subset is stable under the action of the shift. This means that for all $\mathbf{u} \in \mathbb{Z}^d$:

$$\sigma^{\mathbf{u}}.X \subset X.$$

Consider some finite subset $\mathbb{U}$ of $\mathbb{Z}^d$. An element p of $\mathcal{A}^{\mathbb{U}}$ is called a **pattern** on **support** $\mathbb{U}$. This pattern **appears** in a configuration x when there exists a translate $\mathbb{V}$ of $\mathbb{U}$ such that $x_{\mathbb{V}} = p$. We say that it appears in another pattern q on support containing $\mathbb{U}$ such that the restriction of q on $\mathbb{U}$ is p. It appears in a subshift X when it appears in a configuration of X. Such a pattern is also called **globally admissible** for X. The set of patterns of X that appear in it is called the **language** of X. When $d = 2$, the number of patterns on support $\mathbb{U}_{m,n} \equiv [\![0, m-1]\!] \times [\![0, n-1]\!]$ that appear in X (these patterns are called (m, n)-blocks) is denoted $N_{m,n}(X)$, and the number of patterns on support $\mathbb{U}_n \equiv [\![0, n-1]\!]^2$ that appear in X (called n-blocks) is denoted $N_n(X)$.

A subshift X defined by forbidding patterns in some finite set $\mathcal{F}$ to appear in the configurations, formally:

$$X = \left\{x \in \mathcal{A}^{\mathbb{Z}^d} : \forall \mathbb{U} \subset \mathbb{Z}^d, x_{\mathbb{U}} \notin \mathcal{F}\right\}$$

is called a subshift of **finite type** (SFT). In a context where the set of forbidden patterns defining the SFT is fixed, a pattern is called **locally admissible** for this SFT when no forbidden pattern appears in it.

A **morphism** between two $\mathbb{Z}^d$-subshifts X, Z is a continuous map $\varphi : X \to Z$ such that $\varphi \circ \sigma^{\mathbf{v}} = \sigma^{\mathbf{v}} \circ \varphi$ for all $\mathbf{v} \in \mathbb{Z}^d$ (the map commutes with the shift action). An **isomorphism** is an invertible morphism.

2.1.2 Topological entropy

Let X be a bidimensional subshift ($d = 2$). The **topological entropy** of X is defined as:

$$h(X) \equiv \inf_{n \geq 1} \frac{\log_2(N_n(X))}{n^2}.$$

It is a well known fact in topological dynamics that this infimum is a limit:

$$h(X) = \lim_{n \geq 1} \frac{\log_2(N_n(X))}{n^2}.$$

It is a topological invariant, meaning that when there is an isomorphism between two subshifts, these two subshifts have the same entropy [LM95].

Let us then consider, for all $n \geq 1$, the subshift X_n obtained from X by restricting to the width n infinite strip $\{0, ..., n-1\} \times \mathbb{Z}$. Formally, this subshift is defined on alphabet $\mathcal{A}^n$ and by $z \in X_n$ if and only if there exists $x \in X$ such that for all $k \in \mathbb{Z}$, $z_k = (x_{1,k}, ..., x_{n,k})$.

Proposition 1 (Folklore)**.** *The entropy of X can be computed through the sequence $(h(X_n))_n$:*

$$h(X) = \lim_n \frac{h(X_n)}{n}.$$

We include a proof of this statement, for completeness:

Proof.

- **$\leq$ by cutting squares in rectangles:** From the definition of X_n, this equality means

$$h(X) = \lim_n \lim_m \frac{\log(N_{m,n})}{nm}.$$

Since for any m, n, k, the set $\mathbb{U}_{km,kn}$ is the union of mn translates of $\mathbb{U}_k$, a pattern on support $\mathbb{U}_{km,kn}$ can be seen as an array of patterns on $\mathbb{U}_k$. As a consequence,

$$N_{km,kn}(X) \leq (N_{k,k}(X))^{m,n},$$

and

$$\lim_m \frac{\log(N_{m,n}(X))}{nm} = \lim_m \frac{\log(N_{km,kn}(X))}{k^2 nm} \leq \lim_m \frac{\log(N_{k,k}(X))}{k^2} = \frac{N_{k,k}}{k^2}.$$

As a consequence, for all k,

$$\lim_n \lim_m \frac{\log(N_{m,n}(X))}{nm} \leq \frac{N_{k,k}(X)}{k^2},$$

and this implies

$$\lim_n \lim_m \frac{\log(N_{m,n}(X))}{nm} \leq h(X).$$

- **$\geq$ by cutting rectangles in squares:** For all m, n, by considering a pattern on $\mathbb{U}_{mn,nm}$ as a union of translates of $\mathbb{U}_{m,n}$, we get that:

$$N_{mn,nm}(X) \leq (N_{m,n}(X))^{mn}.$$

Thus,

$$h(X) = \lim_m \frac{N_{mn,nm}(X)}{m^2 n^2} \leq \lim_m \frac{N_{m,n}(X)}{mn}.$$

As a consequence,

$$h(X) \leq \lim_n \lim_m \frac{\log(N_{m,n}(X))}{nm} \leq h(X)$$

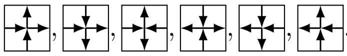
□

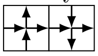
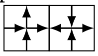
2.2 Representations of square ice

The square ice can be defined as an isomorphic class of subshifts of finite type, whose elements can be thought as its representations. The most widely used is the six vertex model (whose name derives from that the elements of the alphabet represent vertices of a regular grid) and is presented in Section 2.2.1. In this text, we will use another representation, presented in Section 2.2.2, whose configurations consist of drifting discrete curves, representing possible particle trajectories.

2.2.1 The six vertex model

The **six vertex model** is the subshift of finite type described as follows:

Symbols: .

Local rules: Considering two adjacent positions in $\mathbb{Z}^2$, the arrows corresponding to the common edge of the symbols on the two positions have to be directed the same way. For instance, the pattern  is allowed, while  is not.

Global behavior: The symbols draw a lattice whose edges are oriented in such a way that all the vertices have two incoming arrows and two outgoing ones. This is called an Eulerian orientation of the square lattice. See an example of admissible pattern on Figure 1.

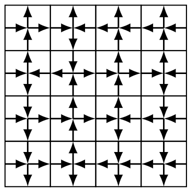
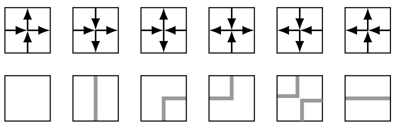


Figure 1: An example of locally and thus globally admissible pattern of the six vertex model.

Remark 1. *One can see that locally admissible patterns of this SFT are always globally admissible.*

2.2.2 Drifting discrete curves

From the six vertex model, we derive another representation of square ice through an isomorphism, which consist in transforming the letters in the following way:



The pattern on Figure 1 can be represented as on Figure 2. In this SFT, the local rules are that any outgoing segment of curve in a non-blank symbol is extended in its direction on the next position.

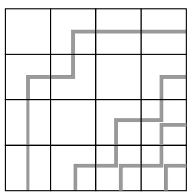


Figure 2: Representation of pattern on Figure 1.

In the following, we denote X this SFT.

3 Transfer matrix method

In this section, we present the transfer matrix method used by E.H. Lieb [L67] in order to compute the entropy of square ice (since all the representations of square ice are isomorphic, their entropy is the same, hence this notion is not ambiguous). In Subsection 3.1, we provide a proof that in order to compute the entropy, it is sufficient to evaluate the asymptotic growth of patterns that can be wrapped on a torus without breaking the local rules. In Subsection 3.2, we present the transfer matrices, which describe the dynamics of the restriction of the SFT to infinite stripes. In Subsection 3.3, we introduce a notion of weighted entropies, corresponding to the notion of partition function in statistical mechanics. Roughly, it consists in viewing the entropy as a particular value of a function, whose analycity is helpful to deduce its values on its whole interval of definition from this knowledge on a particular domain. We relate its values to the maximal eigenvalue of a matrix function. In Subsection 3.4, we describe the Bethe ansatz, providing vectors which verify the eigenequations for the transfer matrices. In Section 4, we will prove that it provides the maximal eigenvalue on a particular domain, extending this property to the whole interval, and in particular the parameter corresponding to the topological entropy.

3.1 Toroidal hypothesis

3.1.1 Toroidal patterns

Consider some alphabet $\mathcal{A}$. For all $m, n \geq 1$, we denote by $\Pi_{m,n}$ the torus $\mathbb{Z}^2/(n\mathbb{Z} \times m\mathbb{Z})$. Let us also denote $\pi_n : \mathbb{U}_{m,n} \to \Pi_{m,n}$ the canonical projection, and $\phi_{m,n} : \mathcal{A}^{\mathbb{U}_{m,n}} \to \mathcal{A}^{\Pi_{m,n}}$ the application such that for all $\mathbf{u} \in \mathbb{U}_{m,n}$ and $p \in \mathcal{A}^{\mathbb{U}_{m,n}}$,

$$(\phi_{m,n}(p))_{\pi_{m,n}(\mathbf{u})} = p_{\mathbf{u}}.$$

Informally, this application wraps patterns over $\mathbb{U}_{m,n}$ onto the torus $\Pi_{m,n}$.

We say that a pattern p on support $\mathbb{U} \subset \mathbb{U}_{m,n}$ appears in a pattern q on $\Pi_{m,n}$ when there exists a pattern p' on $\mathbb{U}_{m,n}$ whose image by $\pi_{m,n}$ is q and there exists an element $\mathbf{u} \in \Pi_{m,n}$ such that for all $\mathbf{v} \in \mathbb{U}$,

$$q_{\mathbf{u}+\pi_{m,n}(\mathbf{v})} = p_{\mathbf{v}}.$$

Let X be a subshift of finite type on $\mathbb{Z}^2$. A (m,n)-**toroidal pattern** of X is an element of $\mathcal{A}^{\mathbb{U}_{m,n}}$ whose image by $\phi_{m,n}$ does not contain any forbidden pattern for X, meaning that this pattern can be wrapped on a torus without breaking the rules defining X. We define in a similar way **cylindric patterns** as patterns that can be wrapped on a vertical cylinder.

3.1.2 Toroidal hypothesis

Let us denote by $N^{(t)}_{m,n}(X)$ the number of (m,n)-toroidal patterns of X, and $N^{(c)}_{m,n}(X)$ the number of (m,n)-cylindric patterns of X.

Definition 1. *A bidimensional subshift of finite type X satisfies the **toroidal hypothesis** when*

$$h(X) = \lim_{m} \lim_{n} \frac{\log(N^{(t)}_{m,n}(X))}{mn}.$$

Remark 2. *The toroidal hypothesis implies in particular that*

$$h(X) = \lim_{m} \lim_{n} \frac{\log(N^{(c)}_{m,n}(X))}{mn},$$

since for all m, n,

$$N_{m,n}(X) \geq N^{(c)}_{m,n}(X) \geq N^{(c)}_{m,n}(X).$$

3.1.3 Verification of the hypothesis for lattice models

Lemma 1. *The discrete curves SFT (Section 2.2.2) satisfies the toroidal hypothesis.*

Idea of the proof: *We define a familly of operations on blocks. Each of them allows a curve that is cut on the right side of the block to be continued starting from the left side (as if on a torus). Through application of a sequence of these operations, one can transform any block into a cylindric one. We then define a familly of erasing operations, which act reversely. The set of images of a fixed pattern through these operations contains the preimages of this pattern through the continuing operations. Bounding the numbers of possible images of a block through a sequence of erasing operations, we obtain an upper bound on the number of blocks. This bound involves the number of cylindric blocks.*

Remark 3. *The statement of Lemma 1 was assumed by E.H. Lieb [L67] without proof. An argument was proposed recently in [DC16], comparing blocks having linearly related sizes, which seems not sufficient to derive the equality of entropies. The proof we propose is however very specific to the square ice.*

Proof. 1. **Continuing operations:** Let us define, for all $i \in [\![0, N-1]\!]$, an operation on the N-blocks in the language of the discrete curves. This operation continues the trajectory of a discrete curve which leaves the block on the right at height $\bar{i}$ (this denotes the projection of i onto $\mathbb{Z}/N\mathbb{Z}$.

- **Bidimensional reading and writing machine:** This operation consists in the action of a reading and writing head that moves on the torus $\Pi_{m,n}$, initialized on position $\overline{(n-1, i)}$, where the curve leaves the block. The head has two possible states $\uparrow, \rightarrow$, which designate the direction of the movement of the head. It starts in state $\rightarrow$. Iteratively, the head moves in the direction of the arrow, and then reads and write on the position reached, and changes its state. These changes are made according to the transition rule is described as follows:
  - $\uparrow$, are transformed into $\uparrow$, ,
  - $\uparrow$, into $\uparrow$, ,
  - $\uparrow$, into $\rightarrow$, ,
  - $\rightarrow$, into $\rightarrow$, ,
  - $\rightarrow$, into $\rightarrow$, ,
  - $\rightarrow$, into $\uparrow$, ,

  When the transition is undefined, the writing step is not executed, and the process stops. When on the top row and in state $\uparrow$ at the moving step, the process stops.

- **Undefined transitions:**
  The transitions that are left undefined are never reached. Assume this is possible. Let us consider that the head arrives in state $\rightarrow$ (the other case is similar). There are three possibilities for the symbol met by the head: , or .
  In all these cases, the symbol on the left of this one, the position from where the head comes from, before the last writing step is either , or . In the first case and third cases, the transition was already undefined, which is impossible (since the head moved). In the second case, the head would have gone upwards instead, which is a contradiction.

- **Global behavior:** This process ends when the head reaches the top row and stops. Indeed, if at some point the head enters a row which is not crossed by any curve, then it goes straightly to the top. If this is not the case, then each time the head does not cross the row straightly, it means that it encounters the symbol , traces and starts rotating on the row. Since it has traced the symbol on this row, it cannot cross this position again (since the transition is undefined). As a consequence, it has to deviate.
  One can see a simple example of trajectory under this process on Figure 3.

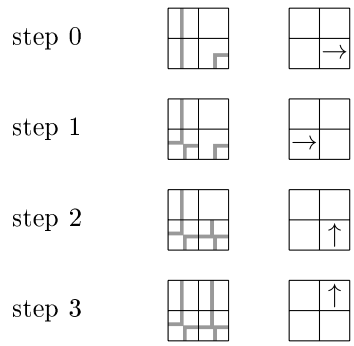


Figure 3: Illustration of the continuing curves operations on a simple pattern, for $m = n = 2$ and $i = 0$. Step j shows the pattern, the position and the state of the head after jth writing.

2. **Erasing operations:**

   Let us define for all i some operation on (m, n)-blocks having an outgoing curve on the ith position of the top row. This operation erases a part of this curve between two times at which the curve crosses the right side of the block. As well as the continuing operations, the erasing ones are defined by a moving and writing head, whose transition rule is described as follows:

   - $\downarrow$, are transformed into $\downarrow$, ,
   - $\downarrow$, into $\downarrow$, or $\leftarrow$, ,
   - $\leftarrow$, into $\leftarrow$, ,
   - $\leftarrow$, into $\downarrow$, or $\leftarrow$, .

   This transition rule defines $t(i)$ different erasing operations for this position i, where $t(i)$ is the number of times this curve crosses the rightmost column. Each is defined by stoping the process on a position where the curve crosses this column, before writing.

3. **Transformation of (m, n)-blocks into cylindric ones:**

   Let us consider the application $\mathcal{T}_{m,n}$ which transforms (m, n)-blocks in the language of the discrete curves into cylindric ones by considering successively for each i if a curve is cut on the right side at height i. If this is the case, $\mathcal{T}$ executes the continuing operation at height i. For all i, after considering heights $1, ..., i$, any curve crossing the right side of the block is continued until the top row. As a consequence, for all block, its image by $\mathcal{T}_{m,n}$ is cylindric.

4. **Some cylindrical hypothesis:**

   Let us observe that the transitions of the erasing operations are reversing the transitions of the continuing ones. As a consequence, any preimage of a cylindric block is obtained by the application of a sequence of erasing operations.

   Since in any (m, n)-block, there are at most $n + m$ curves (each curve crosses else the right side or the down side) and that each of these curves intersects the rightmost column at most n times, the number of patterns that can be sent to a fixed cylindric pattern through a sequence of continuing operations is at most n^{n+m}. As a consequence, for all m, n,

   $$N^{(c)}_{m,n}(X) \leq N_{m,n}(X) \leq n^{m+n}.N^{(c)}_{m,n}(X),$$

   thus

   $$h(X_m) = \lim_n \frac{\log_2(N^{(c)}_{m,n}(X))}{n}.$$

5. **Toroidal hypothesis:**

   Since each (m, n)-cylindric block can be extended into a $(m, n+m)$-toroidal one (this is formulated and proved as the irreducibility of the transfer matrix, in Lemma 2),

$$\lim_n \frac{\log_2(N^{(c)}_{m,n}(X))}{n} = \lim_n \frac{\log_2(N^{(t)}_{m,n}(X))}{n},$$

   As as a consequence,

$$h(X_m) = \lim_n \frac{\log_2(N^{(t)}_{m,n}(X))}{n}.$$

   From Proposition 1, we get the toroidal hypothesis.

□

Remark 4. *The toroidal hypothesis seems hard to prove for other models. For instance we don't know if it is verified by the dimer model (Question 1). It is however for the hard core model, since block gluing implies toroidal hypothesis. It is clear that it cannot be verified for any bidimensional SFT, since positive entropy and toroidal hypothesis imply the existence of periodic points (any toroidal pattern can be assembled with itself to create a periodic configuration). As a consequence, it breaks any known construction of SFT with uncomputable entropy. We thus ask Question 2.*

Question 1. *Does the dimer model [K61] verify the toroidal hypothesis?*

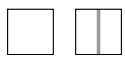
This question seems non trivial, since natural application of the method used for block gluing SFT or the square ice do not work.

Question 2. *Is there a bidimensional SFT which verifies the toroidal hypothesis and whose entropy is uncomputable ?*

3.2 The transfer matrix

3.2.1 Interlacing relation

In order to simplify the exposition we assimilate the following symbols to 0 and 1 respectively:



Consider u, v two words in $\{0,1\}^N$, and w a $(N,1)$-cylindric pattern of the subshift X. We say that the pattern w **connects** u to v (we denote this $u\mathcal{R}[w]v$), when for all $k \in [\![1, N]\!]$, $u_k = 1$ (resp. $v_k = 1$) if and only if w has an incoming (resp. outgoing) curve on the bottom (resp. top) at position $k-1$. This definition is illustrated on Figure 4.

Let us denote $\mathcal{R} \subset \{0,1\}^N \times \{0,1\}^N$ the relation defined by $u\mathcal{R}v$ if and only if there exists a $(N,1)$-cylindric pattern w of the discrete curves shift such that $u\mathcal{R}[w]v$.

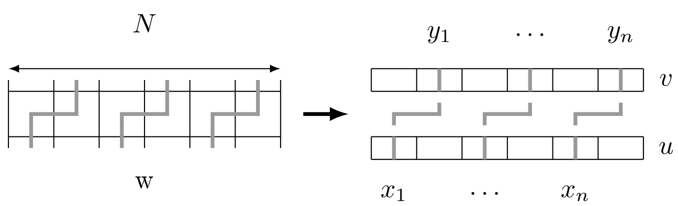


Figure 4: Illustration for the definition of the relation $\mathcal{R}$.

Then there is a natural invertible map from the set of (M, N)-cylindric patterns to the sequences of words $u_1, ..., u_{M+1}$ such that for all $i \leq M$, $u_i\mathcal{R}u_{i+1}$. It is thus sufficient to compute the asymptotics of the number of such sequences in order to compute the entropy of square ice.

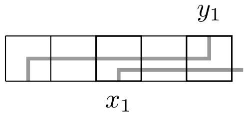


Figure 5: Illustration of (impossible) crossing situation, which would imply non-authorized symbols.

Let us also note that $u\mathcal{R}v$ implies that the number of 1 symbols in u is equal to the number of 1 symbols in v.

We say that two words u, v having same length are **interlaced** when there exist two increasing sequences $(x_i)_{1\leq i\leq n}$ and $(y_i)_{1\leq i\leq n}$ such that $u_k = 1$ if and only if $k = x_i$ for some i, and $v_k = 1$ if and only if $k = y_i$ for some i, and $x_1 \leq y_1 \leq x_2 ... \leq x_n \leq y_n$ or $y_1 \leq x_1 \leq y_2 ... \leq y_n \leq x_n$.

For a word u in $\{0,1\}^N$, we denote $|u|_1$ the number of $k \leq N$ such that $u_k = 1$.

Proposition 2. *For two length N words u, v, we have $u\mathcal{R}v$ if and only if $|u|_1 = |v|_1 \equiv n$ and u, v are interlaced.*

Proof.

- ($\Rightarrow$): assume that $u\mathcal{R}[w]v$ for some w. Let us denote (x_i) (resp. (y_i)) the increasing sequence of positions where u (resp. v) has symbol 1. The curve crossing u at position x_1 has to cross v at position y_1. Indeed, if this was not the case, another curve would join another position of u to y_1, thus crossing the first curve (impossible). This is illustrated on Figure 5.

  Since the curve crossing x_1 has to not cross $x_2 + 1$ (since it would cross the curve crossing position x_2), then $y_1 < x_2 + 1$, i.e. $y_1 \leq x_2$. Repeating this argument, one has $x_1 \leq y_1 \leq x_2 ... \leq x_n \leq y_n$. The case $x_1 \geq y_1$ is similar.

- ($\Leftarrow$): if we have the interlacing relation between u and v, one can easily see that connecting the curve from x_i to y_i, one gets $u\mathcal{R}[w]v$ for some w.

$\square$

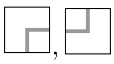
Proposition 3. *When $u\mathcal{R}v$ and $u \neq v$, there exists a unique w such that $u\mathcal{R}[w]v$. When $u = v$ there are exactly two possibilities, either the word $w = u = v$ or the one connecting x_i to x_{i+1} for all i.*

Proof. Consider two words $u \neq v$ such that $u\mathcal{R}v$. There exists at least one i such that x_i (notation of Proposition 2) is not equal to any y_j. This forces any w such that $u\mathcal{R}[w]v$ to connect the position x_i to the smallest $y_j \geq x_i$. Since they are interlaced (Proposition 2), $x_{i+1} \geq y_j$. Thus it has to be connected to y_{j+1}. Repeating this argument, we get the unicity of w. $\square$

3.2.2 Transfer matrix

Let $N \geq 1$ be an integer, and $c > 0$. Let us define Ω_N the space $\mathbb{C}^2 \bigotimes ... \bigotimes \mathbb{C}^2$, tensor product of N copies of $\mathbb{C}^2$. This vector space is generated by the basis $\{0,1\}^N$. The notation of vectors in the basis and vectors in Ω_N might seem ambiguous. We will point out the meaning of the notations when needed.

For all N and $(N,1)$-cylindric pattern w, let $|w|$ denote the number of symbols



in this pattern. For instance, for the word w on Figure 4, $|w| = 6$. For all $c \geq 0$, let us define $V_N(c) \in \mathcal{M}_{2^N}(\mathbb{C})$ the matrix such that for all $u, v \in \{0,1\}^N$,

$$V_N(c)_{u,v} = \sum_{u\mathcal{R}[w]v} c^{|w|}.$$

Let us recall that a non-negative matrix A is called **irreducible** when there exists some $k \geq 1$ such that all the coefficients of A^k are positive. Let us also recall the Perron-Frobenius theorem for symmetric, non-negative and irreducible matrices.

Theorem 1 (Perron-Frobenius)**.** *Let A be a symmetric, non-negative and irreducible matrix. Then A has a positive eigenvalue λ such that any other eigenvalue μ of A satisfies $|\mu| \leq \lambda$. Moreover, there exists some eigenvector u for the eigenvalue λ with positive coordinates such that if v is another eigenvector (not necessarily for λ) with positive coordinates, then $v = \alpha.u$ for some $\alpha > 0$.*

Let us prove the unicity of the positive eigenvector up to a multiplicative constant:

Proof. Let us denote u the Perron-Frobenius eigenvector and v another vector whose coordinates are all positive, associated to the eigenvalue μ. Then

$$\mu u^t.v = (Au)^t.v = u^t A v = \lambda u^t.v$$

Thus, since $u^t.v > 0$, then $\mu = \lambda$, and by Perron-Frobenius, there exists some $\alpha > 0$ such that $v = \alpha.u$. □

Lemma 2. *The matrix $V_N(c)$ is symmetric, non-negative and irreducible.*

Proof.

- **Symmetry:** since the interlacing relation is symmetric, $V_N(c)(u, v) > 0$ if and only if $V_N(c)(v, u) > 0$. When this is the case, and $u \neq v$ (the case $u = v$ is trivial), there exists a unique w connecting u to v. The coefficient of this word is exactly $c^{n-\#\{k:u_k=v_k=1\}}$, where $n = \#\{k : u_k = 1\} = \#\{k : v_k = 1\}$, and this coefficient is indifferent to the exchange of u and v.

- **Irreducibility:** Let us consider some word u and denote $x_1 < ... < x_n$ the positions of its curves, and consider another word v. If they are interlacing, $V_N(c)(u, v) > 0$. If they are not, let us denote $\omega(u, v)$ the maximal number of y_j that lie in some $[\![x_i, x_{i+1}[\![$. Let us see that there exists some u' such that $u\mathcal{R}u'$ and $\omega(u', v) < \omega(u, v)$. For all i such that $[\![x_i, x_{i+1}[\![\cap \{y_j\}$ is empty and $[\![x_{i+1}, x_{i+2}[\![\cap \{y_j\}$ has more than one element, let us consider the word w that displaces the curve crossing x_{i+1} to the maximal $y_j \in [\![x_{i+1}, x_{i+2}[\![$. The other elements of x are fixed by w. As a consequence, since $\omega(u, v) \leq N$ for all u, v, $V_N(c)^N > 0$, and thus $V_N(c)$ is irreducible.

□

3.3 Weighted entropies

Let us define, for all $c \geq 0$, the **weighted entropy** for weight c the number

$$h_c(X) = \inf_N \inf_M \frac{\log_2(||V_N(c)^M)||_1)}{NM},$$

where $||.||_1$ is the matricial norm obtained by summing all the modules of the coefficients. Since this is a norm, the sequence $(||V_N(c)^M||)_M$ is sub-multiplicative, and this implies that the infima in the definitions are limits.

In particular, since $V_N(1)(u, v)$ is the number of ways to relate u to v by a $(N, 1)$-cylindric pattern, $Tr(V_N(1)^M)$ is exactly the number of (M, N)-cylindric patterns of the discrete curves shift. As a consequence, because of the toroidal hypothesis, $h_1(X)$ is the topological entropy of this subhift.

Since the matrix $V_N(c)$ is symmetric, one can diagonalize this matrix in $\mathcal{M}_{2^N}(\mathbb{R})$ and get

$$h_c(X) = \lim_N \frac{\log_2(\lambda(V_N(c))}{N},$$

where $\lambda(V_N(c))$ denotes the Perron-Frobenius eigenvalue of $V_N(c)$.

Remark 5 (Random symbols). *When c is an integer, one can think of the weighted entropy corresponding to c as the entropy of the SFT obtained by attributing a "random" symbol in $\{1, ..., c\}$ on positions where a curve drifts. When c tends towards infinity, this focuses on configurations whose curves are compacted and perpetually drifting, hence simplifying the combinatorics. This technique is very similar to the one used in [GS17a] to compute the entropy of distortion subshifts, except that in this context the combinatorial simplicity is obtained by focusing on straight curves adding random bits in $\{0, 1\}$. The configurations with maximal contribution to the partition function are often called* ***ground states*** *in statistical mechanics.*

3.4 Bethe ansatz

We introduce in this section a well known method in statistical mechanics, in order to provide a solution of the eigenequation for the transfer matrix.

3.4.1 The auxiliary function Θ

Let us denote $\mu : (-1, 1) \to (0, \pi)$ the unique function such that for all $\Delta \in (-1, 1)$,

$$\cos(\mu(\Delta)) = -\Delta,$$

and Θ the function on the set of (Δ, x, y) such that $\Delta \in (-1, 1)$ and $-(\pi-\mu(\Delta)) \leq x, y \leq (\pi-\mu(\Delta))$ defined by $\Theta(0, 0, 0) = 0$ and for all Δ, x, y,

$$\exp(-i\Theta(\Delta, x, y)) = \exp(i(x - y)).\frac{e^{-ix} + e^{iy} - 2\Delta}{e^{-iy} + e^{ix} - 2\Delta}.$$

This function exists and is analytic. In a context in which Δ is fixed, we will simplify the notation: $\Theta(\Delta, x, y) = \Theta(x, y)$. By a unicity argument, one can see that for all x, y, $\Theta(x, y) = -\Theta(y, x)$. For the same reason, $\Theta(x, -y) = -\Theta(-x, y)$.

3.4.2 The ansatz

In all the following N and n are arbitrary integers such that $N = 2n$. For $z \neq 1$, we denote $L(z) = 1 + \frac{c^2 z}{1-z}$ and $M(z) = 1 - \frac{c^2}{1-z}$. For a sequence $(x_j)_{j \leq n}$, which we assimilate to a word on $\{0, 1\}^N$ (the element of x denote the position of the 1 symbols in this word), and $(p_1, ..., p_n) \in (-(\pi - \mu), (\pi - \mu))^n$, we define a candidate eigenvector $\psi(p, \Delta)$ in $\Omega_N = \mathbb{C}^2 \otimes ... \otimes \mathbb{C}^2$ by setting its coordinate relative to x as:

$$\psi(p, \Delta)(\mathbf{x}) = \sum_{\sigma \in \mathfrak{S}_n} A_\sigma(p, \Delta) \prod_{k=1}^{n} e^{ip_{\sigma(k)} x_k},$$

where

$$A_\sigma(p, \Delta) = \epsilon(\sigma) \prod_{1 \leq k < l \leq n} e^{ip_{\sigma(k)}}. \left(e^{-ip_{\sigma(k)}} + e^{ip_{\sigma(l)}} - 2\Delta\right).$$

The interest of searching an eigenvalue of this form is that attributing exponential factors to each word, we expect that most of the coefficients on the orbits will cancel under the action of the matrix, and that only one orbit is not canceled out.

Remark 6. *Searching the eigenvector under this form seems, however, artificial, since we know that the state corresponding to the maximal eigenvalue is unique, up to a multiplicative constant, and positive. It is possible that understanding better the structure of the action of the matrix would lead directly to a positive solution.*

Theorem 2. *Consider some $(p_1, ..., p_n)$ such that for all j, $p_j \neq 0$ and which verifies* ***Bethe equation:***

$$Np_j = 2\pi \left(j - \frac{n+1}{2}\right) - \sum_{k=1}^{n} \Theta(p_j, p_k).$$

Then ψ satisfies the eigenequation $V\psi(p,\Delta) = \Lambda(\Delta,p)\psi(p,\Delta)$ for

$$\Lambda(\Delta,p) = \prod_{k=1}^{n} L(e^{ip_k}) + \prod_{k=1}^{n} M(e^{ip_k}).$$

A clear and elementary proof of this theorem can be found in [DC16].

4 Computation of the square ice topological entropy

4.1 Outline

The remainder of this text is devoted to provide a rigorous and complete computation of the entropy of square ice, by first proving that the Bethe ansatz provides the maximal eigenvalue of the transfer matrix, for any value of $c \geq 0$. The proof of this statement consists essentially in gathering arguments of various articles, and filling some gaps in this collection of arguments. As much as possible, we point out the origin of these arguments along the proof. For rigourous argumentation, we rely mainly on a recent article [DC16], in which the authors, motivated by percolation problems, provide a rigorous computation of the weighted entropies for $c > 2$.

Theorem 3 ([DC16])**.** *For all $c > 2$, and $\lambda > 0$ such that $\cosh(\lambda) = \frac{1}{2}c^2 - 1$,*

$$h_c(X) = \frac{\lambda}{2} + \sum_{k=1}^{\infty} \frac{1}{k} e^{-k\lambda} \tanh(k\lambda).$$

Their proof relies mainly on the existence of solution to the Bethe equation when $c > 2$, using arguments adapted from [YY66]. Then they use the asymptotics of the vector and value obtained by the Bethe ansatz when c tends to infinity to show that the largest eigenvalue and the value given by Bethe ansatz coincide. This relies on an identification with the largest (simple) eigenvalue of the matrix when c tends to infinity.

In order to compute the topological entropy, we need to identify the largest eigenvalue with the Bethe ansatz for $c < 2$. This is what we do in the following, complementing the work of [DC16].

For the existence of solutions to the Bethe equation, we rely quite straightforwardly on [DC16]. For the identification with the Perron-Frobenius eigenvalue, we make more precise an argument from [YY66II], identifying these values around $c = \sqrt{2}$.

This part is organized as follows: in Section 4.2, we study a continuous version of the Bethe equation. The interest of this equation comes from the fact that with a minor transformation, involving a variable change, this equation can be transformed into a convolution equation. This equation can be solved exactly using the continuous Fourier transform. In the perspective of possible adaptations variations of the model, we give all the details of this computation. In Section 4.3, we prove the existence of solutions to the discrete Bethe equation for $c < 2$, adapting arguments from [DC16]. In Section 4.4 we give details on the diagonalisation of some Hamiltonian which commutes with the transfer matrix for $c = \sqrt{2}$, allowing the identification of Bethe value to the Perron-Frobenius eigenvalue. In Section 4.5 we detail the computation of entropy, relying on asymptotics of the solutions of the Bethe equation.

In the following, we fix Δ_1 such that

$$\frac{1}{2} < \Delta_1 < \frac{\sqrt{2}}{2}.$$

4.2 Continuous version of Bethe equation

4.2.1 Some map to the unit circle

In this section, we provide some details about the properties of a map having values in the unit circle defined and used in [YY66] in order to compute the solution of the Continuous Bethe equation.

4.2.1.1 Definition Let us denote $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$. The function $(-1,1) \times \mathbb{R} \to \mathbb{U}$:

$$(\Delta, \alpha) \mapsto \frac{e^{i\mu(\Delta)} - e^{\alpha}}{e^{i\mu(\Delta)+\alpha} - 1}$$

is continuously derivable, hence there exists a unique continuously derivable function k mapping $(-1,1) \times \mathbb{R}$ to itself such that $k(0,0) = 0$ and for all α,

$$e^{ik(\Delta,\alpha)} = \frac{e^{i\mu(\Delta)} - e^{\alpha}}{e^{i\mu(\Delta)+\alpha} - 1}.$$

In the following sections, for the sake of notation, we denote $\mu = \mu(\Delta)$ and $k(\alpha) = k(\Delta, \alpha)$ in a context when Δ is fixed.

4.2.1.2 Computation of the derivative

Computation 1. *Let us fix Δ. For all $\alpha \in \mathbb{R}$,*

$$\boxed{k'(\alpha) = \frac{\sin(\mu)}{\cosh(\alpha) - \cos(\mu)}.}$$

Proof.

- **Computation of $\cos(k(\alpha))$ and $sin(k(\alpha))$:**

$$e^{ik(\alpha)} = \frac{\left(e^{-i\mu+\alpha} - 1\right)\left(e^{i\mu} - e^{\alpha}\right)}{\left|e^{i\mu+\alpha} - 1\right|^2} = \frac{e^{\alpha} + e^{2\alpha}e^{-i\mu} - e^{i\mu} + e^{\alpha}}{(\cos(\mu)e^{\alpha} - 1)^2 + (\sin(\mu)e^{\alpha})^2}.$$

Thus by taking the real part,

$$\cos(k(\alpha)) = \frac{2e^{\alpha} + (e^{2\alpha} - 1)\cos(\mu)}{\cos^2(\mu)e^{2\alpha} - 2\cos(\mu)e^{\alpha} + 1 + (1 - \cos^2(\mu))e^{2\alpha}}$$

$$\cos(k(\alpha)) = \frac{2e^{\alpha} + (e^{2\alpha} - 1)\cos(\mu)}{e^{2\alpha} - 2\cos(\mu)e^{\alpha} + 1} = \frac{1 - \cos(\mu)\cosh(\alpha)}{\cosh(\alpha) - \cos(\mu)},$$

where we factorized by $2e^{\alpha}$ for the second equality. As a consequence:

$$\cos(k(\alpha)) = \frac{\sin^2(\mu) + \cos^2(\mu) - \cos(\mu)\cosh(\alpha)}{\cosh(\alpha) - \cos(\mu)} = \frac{\sin^2(\mu)}{\cosh(\alpha) - \cos(\mu)} - \cos(\mu).$$

A similar computation gives

$$\sin(k(\alpha)) = \frac{\sin(\mu)\sinh(\alpha)}{\cosh(\alpha) - \cos(\mu)}$$

- **Deriving the expression $\cos(k(\alpha))$:**

As a consequence, for all α:

$$-k'(\alpha)\sin(k(\alpha)) = -\frac{\sin^2(\mu)\sinh(\alpha)}{(\cosh(\alpha) - \cos(\mu))^2} = -\frac{\sin(k(\alpha))^2}{\sinh(\alpha)}.$$

Thus, for all α but in a discrete subset of $\mathbb{R}$,

$$k'(\alpha) = \frac{\sin(\mu)}{\cosh(\alpha) - \cos(\mu)}.$$

This identity is thus verified on all $\mathbb{R}$, by continuity.

□

4.2.1.3 Domain and invertibility

Proposition 4. *The function k has images in $]-\pi+\mu, \pi-\mu[$ and is invertible from $\mathbb{R}$ to this interval.*

Proof.

- **Injectivity:**

  Since $\mu \in (0, \pi)$, then $\sin(\mu) > 0$ and we have the inequality $\cosh(\alpha) \geq 1 > \cos(\mu)$. As a consequence, k is strictly increasing, and thus injective.

- **The equality $k(\alpha) = n\pi$ imples $\alpha = 0$:**

  Assume that for some α, $k(\alpha) = n\pi$ for some integer n. If n is odd, then:

  $$e^{\alpha} - e^{i\mu} = e^{i\mu+\alpha} - 1.$$

  $$e^{\alpha} + 1 = e^{i\mu}.(e^{\alpha} + 1),$$

  and thus $e^{i\mu} = 0$, which is impossible, since $\mu \in (0, \pi)$. If n is even, then

  $$-e^{\alpha} + e^{i\mu} = e^{i\mu+\alpha} - 1.$$

  As a consequence, since $e^{i\mu} \neq -1$, we have $e^{\alpha} = 1$, and thus $\alpha = 0$.

- **Extension of the images:**

  Since when α tends towards $+\infty$ (resp. $-\infty$), the function tends towards $-e^{i\mu}$ (resp. $e^{i\mu}$), $k(\alpha)$ tends towards some $n\pi - \mu$ (resp. $m\pi + \mu$). and from the above property, $n = 1$ (reps. $m = -1$). Thus the image of k is the set $]-(\pi-\mu), \pi-\mu[$.

  Thus k is an invertible map from $\mathbb{R}$ to $(\mu - \pi, \pi - \mu)$.

□

4.2.2 Properties of the function Θ relative to k

The following equality originates in [YY66]. We provide some details of a relatively simple way to compute it.

Computation 2. *For any numbers α, β:*

$$\boxed{\frac{d}{d\alpha}\left(\Theta(k(\alpha), k(\beta))\right) = -\frac{d}{d\beta}\left(\Theta(k(\alpha), k(\beta))\right) = -\frac{\sin(2\mu)}{\cosh(\alpha-\beta) - \cos(2\mu)}}$$

Proof. Le us fix Δ.

- **Deriving the equation that defines Θ:**

  Let us denote, for all x, y:

  $$G(x, y) = \frac{x(1-2\Delta y) + y}{x + y - 2\Delta}.$$

  Then we have that for all x, y

  $$\partial_1 G(x, y) = \frac{(1-2\Delta y).(x+y-2\Delta) - (x(1-2\Delta y)+y)}{(x+y-2\Delta)^2}$$

  $$\partial_1 G(x, y) = -2\Delta \frac{1+y^2-2\Delta y}{(x+y-2\Delta)^2}$$

  By definition of Θ, for all α, β,

  $$\exp(-i\Theta(k(\alpha), k(\beta))) = G(e^{ik(\alpha)}, e^{-ik(\beta)}).$$

Thus we have, by deriving this equality:

$$-i\frac{d}{d\alpha}(\Theta(k(\alpha),k(\beta)))\exp(-i\Theta(k(\alpha),k(\beta))) = ik'(\alpha)e^{ik(\alpha)}\partial_1 G(e^{ik(\alpha)},e^{-ik(\beta)}).$$

$$\frac{d}{d\alpha}(\Theta(k(\alpha),k(\beta))) = -k'(\alpha)e^{ik(\alpha)}\frac{\partial_1 G(e^{ik(\alpha)},e^{-ik(\beta)})}{G(e^{ik(\alpha)},e^{-ik(\beta)})}$$

$$\frac{d}{d\alpha}\Theta(k(\alpha),k(\beta)) = \frac{(k'(\alpha)2\Delta e^{ik(\alpha)})(1+e^{-2ik(\beta)}-2\Delta e^{-ik(\beta)})}{(e^{ik(\alpha)}+e^{-ik(\beta)}-2\Delta)(e^{ik(\alpha)}+e^{-ik(\beta)}-2\Delta.e^{i(k(\alpha)-k(\beta))})}$$

Factoring by $e^{i(k(\alpha)-k(\beta))}$:

$$\frac{d}{d\alpha}\Theta(k(\alpha),k(\beta)) = 2\Delta k'(\alpha).\frac{e^{ik(\beta)}+e^{-ik(\beta)}-2\Delta}{(e^{ik(\alpha)}+e^{-ik(\beta)}-2\Delta)(e^{ik(\beta)}+e^{-ik(\alpha)}-2\Delta)}.$$

- **Simplification of a term $e^{ik(\alpha)}+e^{-ik(\beta)}-2\Delta$:**
  Let us denote the function F defined by

$$F(\alpha,\beta) = e^{ik(\alpha)}+e^{-ik(\beta)}-2\Delta.$$

  By definition of k and $-2\Delta = e^{-i\mu}+e^{i\mu}$ we have:

$$F(\alpha,\beta) = \frac{e^{i\mu}-e^{\alpha}}{e^{i\mu+\alpha}-1}+\frac{e^{i\mu+\beta}-1}{e^{i\mu}-e^{\beta}}+e^{i\mu}+e^{-i\mu}.$$

$$F(\alpha,\beta) = \frac{(e^{i\mu}-e^{\alpha})(e^{i\mu}-e^{\beta})+(e^{i\mu+\alpha}-1)(e^{i\mu+\beta}-1)+(e^{i\mu}+e^{-i\mu}).(e^{i\mu+\alpha}-1)(e^{i\mu}-e^{\beta})}{(e^{i\mu+\alpha}-1)(e^{i\mu}-e^{\beta})}.$$

$$F(\alpha,\beta) = \frac{e^{3i\mu+\alpha}+e^{\beta-i\mu}-e^{i\mu}.(e^{\alpha}+e^{\beta})}{(e^{i\mu+\alpha}-1)(e^{i\mu}-e^{\beta})}.$$

- **Simplification of Θ's derivative:**
  For all α,β, we have

$$\frac{1}{k'(\alpha)}\frac{d}{d\alpha}\Theta(\alpha,\beta) = 2\Delta\frac{F(\beta,\beta)}{F(\alpha,\beta).F(\beta,\alpha)}.$$

  As a consequence of last point,

$$\frac{1}{k'(\alpha)}\frac{d}{d\alpha}\Theta(k(\alpha),k(\beta)) = \frac{-(e^{i\mu+\alpha}-1)(e^{i\mu}-e^{\alpha})(e^{-i\mu}+e^{i\mu})(e^{3i\mu+\beta}+e^{\beta-i\mu}-2e^{i\mu}.e^{\beta})}{(e^{3i\mu+\alpha}+e^{\beta-i\mu}-e^{i\mu}.(e^{\alpha}+e^{\beta}))(e^{3i\mu+\beta}+e^{\alpha-i\mu}-e^{i\mu}.(e^{\beta}+e^{\alpha}))}.$$

$$\frac{1}{k'(\alpha)}\frac{d}{d\alpha}\Theta(k(\alpha),k(\beta)) = -\frac{(e^{i\mu+\alpha}-1)(e^{i\mu}-e^{\alpha})e^{\beta}.(e^{2i\mu}-1).(e^{2i\mu}-e^{-2i\mu})}{e^{2i\mu}(e^{2i\mu+\alpha}+e^{\beta-2i\mu}-(e^{\alpha}+e^{\beta}))(e^{2i\mu+\beta}+e^{\alpha-2i\mu}-(e^{\beta}+e^{\alpha}))}.$$

  Since in the denominator of the fraction in square of the modulus of some number, we rewrite it.

$$\frac{1}{k'(\alpha)}\frac{d}{d\alpha}\Theta(k(\alpha),k(\beta)) = -\frac{(e^{i\mu+\alpha}-1)(e^{i\mu}-e^{\alpha})e^{\beta}.(e^{2i\mu}-1).(e^{2i\mu}-e^{-2i\mu})}{e^{2i\mu}\left((e^{\alpha}+e^{\beta})^2(\cos(2\mu)-1)^2+(e^{\alpha}-e^{\beta})^2\sin^2(2\mu)\right)}.$$

We rewrite also the other terms, by splitting the $e^{2i\mu}$ in the denominator in two parts, one makes appear $\sin(\mu)$, and the other one, the square modulus:

$$\frac{1}{k'(\alpha)}\frac{d}{d\alpha}\Theta(\alpha,\beta) = -4|e^{i\mu+\alpha}-1|^2\frac{e^{\beta}.\sin(\mu).\sin(2\mu)}{(e^{\alpha}+e^{\beta})^2(\cos(2\mu)-1)^2+(e^{\alpha}-e^{\beta})^2\sin^2(2\mu)}.$$

By writing $\sin^2(2\mu) = 1-\cos^2(2\mu)$ and then factoring by $1-\cos(2\mu)$:

$$\frac{1}{k'(\alpha)}\frac{d}{d\alpha}\Theta(\alpha,\beta) = -4\frac{|e^{i\mu+\alpha}-1|^2}{1-\cos(2\mu)}\frac{e^{\beta}.\sin(\mu).\sin(2\mu)}{(e^{\alpha}+e^{\beta})^2(1-\cos(2\mu))+(e^{\alpha}-e^{\beta})^2(1+\cos(2\mu))}.$$

Developping the denominator and factoring it by $4e^{\alpha+\beta}$, we obtain:

$$\frac{1}{k'(\alpha)}\frac{d}{d\alpha}\Theta(\alpha,\beta) = -\frac{|e^{i\mu+\alpha}-1|^2}{e^{\alpha}(1-\cos(2\mu))}.\frac{\sin(\mu).\sin(2\mu)}{\cosh(\alpha-\beta)-\cos(2\mu)}.$$

We have left to see that

$$\frac{\sin(\mu)k'(\alpha).|e^{i\mu+\alpha}-1|^2}{e^{\alpha}(1-\cos(2\mu))} = 1.$$

This derives directly from $1-\cos(2\mu) = 2\sin^2(\mu)$ and the value of $k'(\alpha)$ given by Computation 1.

- **The other equality:**

  We obtain the value of $\frac{d}{d\beta}(\Theta(k(\alpha),k(\beta)))$ through the equality $\Theta(x,y) = -\Theta(y,x)$ for all x,y.

$\square$

4.2.3 Bounds on Θ

For a fixed Δ and for all x, let us denote Ψ_x the function:

$$\Psi_x : y \mapsto \Theta(x,y) + \Theta(x,-y)$$

Proposition 5. *Let us fix $\Delta_0 \in (0,1/2)$. For all x and Δ, the function Ψ_x is increasing. Moreover, there exists some positive $L < 2\pi$ such that for all x and $\Delta \in [\Delta_0,\Delta_1]$*

$$\Psi_x(x,0) - \Psi_x(x,-(\pi-\mu)) \leq L.$$

Proof.

- Let us observe that for all x,y, $G(x,y) = G(y,x)$. As a consequence,

  $$\partial_1 G(x,y) = \partial_2 G(x,y).$$

  Since for all x,y,

  $$\Theta(x,-y) = -\Theta(-x,y),$$

  then for all β,

  $$\frac{d}{d\beta}\Theta(k(\alpha),-k(\beta)) = -\frac{d}{d\beta}\Theta(-k(\alpha),k(\beta)) = -k'(\beta)e^{ik(\beta)}\frac{\partial_2 G(e^{-ik(\alpha)},e^{ik(\beta)})}{G(e^{-ik(\alpha)},e^{ik(\beta)})}.$$

  As a consequence,

  $$\frac{d}{d\beta}(\Psi_{k(\alpha)}(k(\beta))) = 2\frac{\sin(2\mu)}{\cosh(\alpha-\beta)-\cos(2\mu)} \geq 0,$$

  since $\Delta \in [0,\Delta_1]$. As a consequence Ψ_x is increasing.

- As a consequence, for all x,

$$\Psi_x(x,0) - \Psi_x(x,-\pi) \leq 2\int_{-\infty}^{0} \frac{\sin(2\mu)}{\cosh(\alpha-\beta)-\cos(2\mu)} d\beta \leq \int_{-\infty}^{+\infty} \frac{\sin(2\mu)}{\cosh(\alpha-\beta)-\cos(2\mu)} d\beta,$$

where $\alpha = k^{-1}(x)$. As a consequence of the formula given in the part "Computation of the Fourier transform of Ξ_μ", in the proof of following Proposition 6 for γ tending towards 0, this last integral is smaller than $4\pi\frac{\pi-\mu}{\pi}$. For $\Delta \in [\Delta_0, \Delta_1]$, this number is smaller than

$$L = 4\pi\frac{\pi - \mu(\Delta_0)}{\pi} < 4\pi\frac{\pi-\mu(0)}{\pi} = 4\pi\frac{\pi - \pi/2}{\pi} = 2\pi.$$

□

4.2.4 Computation of the solution to the continuous Bethe equation

This part is a detailed and well ordered version of a computation of [YY66II].

Proposition 6. *Let ρ be a continuous function on the set of (Δ, x) with $\Delta \in (-1,1)$ and $x \in [-(\pi-\mu(\Delta)), \pi-\mu(\Delta)]$, whose integral is $1/2$ and for all Δ, x, we have (**continuous Bethe equation**):*

$$2\pi\rho(\Delta,x) = 1 + \int_{-(\pi-\mu(\Delta))}^{\pi-\mu(\Delta)} \partial_1\Theta(x,y)\rho(y)dy.$$

Then for all x,

$$\boxed{\rho(\Delta,x) = \frac{1}{2k'(k^{-1}(x))}\frac{\pi}{\mu(\Delta)}\frac{1}{\cosh(\pi k^{-1}(x)/2\mu(\Delta))}}.$$

In a context when Δ is fixed, we will denote $\rho(x) = \rho(\Delta, x)$.

Proof. Let us fix $\Delta \in (-1,1)$.

- **Change of variable:**

  We first consider the change of variable $x = k(\alpha)$ and $y = k(\beta)$ [YY66II].

  The equation becomes

$$2\pi\rho(k(\alpha)) = 1 + \int_{-\infty}^{+\infty} \partial_1\Theta(k(\alpha),k(\beta))\rho(k(\beta))k'(\beta)d\beta.$$

  As a consequence, by multiplying by $k'(\alpha)$,

$$2\pi k'(\alpha)\rho(k(\alpha)) = k'(\alpha) + \int_{-\infty}^{+\infty} \partial_1\Theta(k(\alpha),k(\beta))\rho(k(\beta))k'(\alpha)k'(\beta)d\beta.$$

  Since

$$\frac{d}{d\alpha}\Theta(k(\alpha),k(\beta)) = k'(\alpha)\partial_1\Theta(k(\alpha),k(\beta)),$$

  and from Computation 2, the equation becomes:

$$R(\alpha) = \Xi_\mu(\alpha) - \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Xi_{2\mu}(\alpha-\beta)R(\beta)d\beta,$$

  where

$$\Xi_\mu(\alpha) = \frac{\sin(\mu)}{\cosh(\alpha)-\cos(\mu)}$$

  and $R(\alpha) = 2\pi k'(\alpha)\rho(k(\alpha))$.

- **Fourier transform:**

We consider the Fourier transform of R:

$$\hat{R}(\omega) = \int_{-\infty}^{+\infty} R(\alpha) e^{i\omega\alpha} d\alpha,$$

which is possible since R is $L^1(\mathbb{R})$.

Thus, since the convolution product is transformed in a product by Fourier transform:

$$\hat{R}(\omega) = \hat{\Xi}_\mu(\omega) - \frac{1}{2\pi} \hat{\Xi}_{2\mu}(\omega) \hat{R}(\omega).$$

As well, denote $\hat{\Xi}_\mu$ the Fourier transform of Ξ_μ.

Thus

$$\hat{R}(\omega) = \frac{\hat{\Xi}_\mu(\omega)}{1 + \frac{1}{2\pi} \hat{\Xi}_{2\mu}(\omega)}$$

- **Singularities of Ξ_μ:**

The singularities of the function Ξ_μ in this domain are exactly the numbers $i(\mu + 2k\pi)$ for $k \geq 0$ and $i(-\mu + 2k\pi)$ for $k \geq 1$, since for $\alpha \in \mathbb{C}$, $\cosh(\alpha) = \cos(\mu)$ if and only if

$$\cos(i\alpha) = \cos(\mu).$$

This implies that $\alpha = i(\pm\mu + 2k\pi)$ for some k.

- **Computation of the Fourier transform of Ξ_μ:**

We have, for all γ:

$$\int_{-\infty}^{+\infty} \Xi_\mu(\alpha) e^{i\alpha\gamma} d\alpha = 2\pi \frac{\sinh[(\pi - \mu)\gamma]}{\sinh(\pi\gamma)}$$

Indeed, one can consider the integrals

$$\int_{\Gamma_N} \Xi_\mu(\alpha) e^{i\alpha\gamma} d\alpha,$$

where Γ_N is the border of $\mathcal{D}_N = [-N, N] + i[0, N]$.

For all lace Γ,

$$\int_\Gamma \Xi_\mu(\alpha) e^{i\alpha\gamma} d\alpha = \int_\Gamma \frac{\sinh(i\mu)}{i(\cosh(\alpha) - \cosh(i\mu))} e^{i\alpha\gamma} d\alpha$$

As a consequence, the residue of Ξ_μ in $i(\mu + 2k\pi)$ is

$$Res(\Xi_\mu, i\mu + 2k\pi) = \frac{e^{i\gamma . i(\mu+2k\pi)}}{i} = \frac{1}{i} e^{-\gamma(\mu+2k\pi)}.$$

As well,

$$Res(\Xi_\mu, -i\mu + 2k\pi) = \frac{e^{i\gamma . i(-\mu+2k\pi)}}{i} = -\frac{1}{i} e^{-\gamma(-\mu+2k\pi)}.$$

By the residue theorem,

$$\int_{\Gamma_N} \Xi_\mu(\alpha) e^{i\alpha\gamma} d\alpha = 2\pi i \left(\sum Res(\Xi_\mu, i(\mu + 2k\pi)) - \sum Res(\Xi_\mu, i(-\mu + 2k\pi)) \right),$$

where the sums are over the k such that $i(\mu + 2k\pi)$ (resp. $i(-\mu + 2k\pi)$) is in $\mathcal{D}_N$.

Since only the contribution on $[-N,N]$ of the integral is non zero asymptotically, and by convergence of the integral and the sums,

$$\int_{-\infty}^{+\infty} \Xi_\mu(\alpha)e^{i\alpha\gamma}d\alpha = 2\pi e^{-\gamma\mu} + 2\pi\sum_{k=1}^{+\infty}(-e^{\gamma\mu}+e^{-\gamma\mu})e^{-2\gamma k\pi}$$

$$\int_{-\infty}^{+\infty} \Xi_\mu(\alpha)e^{i\alpha\gamma}d\alpha = 2\pi e^{-\gamma\mu} + 2\pi(-e^{\gamma\mu}+e^{-\gamma\mu})\left(\frac{1}{1-e^{-2\gamma\pi}}-1\right)$$

$$\int_{-\infty}^{+\infty} \Xi_\mu(\alpha)e^{i\alpha\gamma}d\alpha = 2\pi e^{-\gamma\mu} + 2\pi(-e^{\gamma\mu}+e^{-\gamma\mu})\frac{e^{-\gamma\pi}}{e^{\gamma\pi}-e^{-\gamma\pi}}$$

$$\int_{-\infty}^{+\infty} \Xi_\mu(\alpha)e^{i\alpha\gamma}d\alpha = 2\pi\frac{e^{-\gamma(-\pi+\mu)}-e^{\gamma(-\pi-\mu)}-e^{\gamma(\mu-\pi)}+e^{\gamma(-\pi-\mu)}}{e^{\gamma\pi}-e^{-\gamma\pi}} = 2\pi\frac{\sinh(\gamma(\pi-\mu))}{\sinh(\gamma\pi)}.$$

- **Getting back to $\hat{R}$:**

  We obtain:

$$\hat{R}(\omega) = \frac{2\pi\sinh(\omega(\pi-\mu))}{\sinh(\pi\omega)+\sinh(\omega(\pi-2\mu))} = \frac{4\pi\sinh(\omega(\pi-\mu))}{e^{\omega\pi}.(1+e^{-2\mu\omega})-e^{-\omega\pi}.(1+e^{2\mu\omega})}.$$

  Thus

$$\hat{R}(\omega) = \frac{4\pi\sinh(\omega(\pi-\mu))}{e^{\omega(\pi-\mu)}.(e^{\mu\omega}+e^{-\mu\omega})-e^{-\omega(\pi-\mu)}.(e^{-\mu\omega}+e^{\mu\omega})} = \frac{\pi}{\cosh(\mu\omega)}.$$

- **Inverse transform:**

$$R(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\pi}{\cosh(\mu\omega)}e^{-i\omega\alpha}d\omega = \frac{1}{\mu}\int_{-\infty}^{\infty}\frac{1}{2\cosh(u)}e^{-i\frac{u}{\mu}\alpha}du,$$

  where we used the variable change $u=\mu\omega$.

  Using the computation of the Fourier transform of Ξ_μ for $\mu=\pi/2$,

$$\int_{-\infty}^{+\infty}\frac{1}{\cosh(\alpha)}e^{i\alpha\gamma}d\alpha = 2\pi\frac{\sinh(\pi\gamma/2)}{\sinh(\pi\gamma)} = \frac{\pi}{\cosh(\pi\gamma/2)}.$$

  Thus we have

$$R(\alpha) = \frac{1}{2\mu}\frac{\pi}{\cosh(\pi\alpha/2\mu)} = \frac{\pi}{2\mu}\frac{1}{\cosh(\pi\alpha/2\mu)}$$

□

4.3 Existence of solutions to the Bethe equation

In this part, we prove the existence of solutions to Bethe equation (introduced in Theorem 2) for N sufficiently large. The proof of this fact originates from [YY66], but the proof we propose here is basically the proof presented in [DC16], with minor modifications, in particular in order to adapt it to the case $c<2$.

4.3.1 Some operators and their properties

4.3.1.1 Definition Consider some $\Delta \in (-1, 1)$. Let us denote

$$\mathcal{S}_n(\Delta) = \{\mathbf{p} = (p_1, ..., p_n) : -\pi + \mu < p_1 < ... < p_n < \pi - \mu \text{ and } p_{n-j+1} = -p_j, \forall j\}$$

For the sake of notation, we denote $\mathcal{S}_n = \mathcal{S}_n(\Delta)$ when Δ is fixed. We see any $\mathbf{p} \in \mathcal{S}_n$ as a function by associating it with $\rho_{\mathbf{p}} : [-(\pi - \mu), \pi - \mu] \mapsto \mathbb{R}$ defined by

$$\rho_{\mathbf{p}}(t) = \frac{1}{N(p_{j+1} - p_j)}$$

for all j between 0 and n and $t \in [p_j, p_{j+1})$, where we defined $p_0 \equiv -(\pi - \mu)$ and $p_{n+1} \equiv (\pi - \mu)$. For f a function $[-(\pi - \mu), \pi - \mu] \to \mathbb{R}$, we denote

$$||f|| = ||k'.f \circ k||_\infty.$$

Let us denote $\mathbb{I}_n$ the identity operator on the set of $(\Delta, \mathbf{p})$ such that $\Delta \in (-1, 1)$ and $\mathbf{p} \in \mathcal{S}_n$ and $\mathbb{T}_n$ which to some $(\Delta, \mathbf{p})$ associates the sequence

$$\left(\frac{1}{N}\left(2\pi\left(j - \frac{n+1}{N}\right) - \sum_{k=1}^{n} \Theta(p_j, p_k)\right)\right)_j.$$

A sequence $\mathbf{p}$ is solution of the Bethe equation if and only if $\mathbb{T}_n(\mathbf{p}) = \mathbf{p}$. Let us also denote $\mathbb{T}_\infty$ the operator on functions $f : [-(\pi - \mu), \pi - \mu] \to \mathbb{R}$ associate the function

$$x \mapsto \frac{1}{2\pi}\left(1 + \int_{-(\pi-\mu)}^{\pi-\mu} \partial_1\Theta(x, y) f(y) dy\right).$$

The fixed points of $\mathbb{T}_\infty$ are the solution of the continuous Bethe equation.

4.3.1.2 Contraction properties of $\mathbb{T}_\infty$ and $\mathbb{T}_n$

Lemma 3. *There exists some positive number $\epsilon_0 < 1$ such that for all $\Delta \in [0, \Delta_1]$, f, g,*

$$||\mathbb{T}_\infty(f) - \mathbb{T}_\infty(g)|| \leq \epsilon_0.||f - g||.$$

Proof.

- **Triangular inequalities:** By definition, for all x,

  $$\mathbb{T}_\infty(f)(x) - \mathbb{T}_\infty(g)(x) = \frac{1}{2\pi}\int_{-(\pi-\mu)}^{\pi-\mu} \partial_1\Theta(x, y)(f(y) - g(y))dy.$$

  After the changes of variables $y = k(\beta)$ and $x = k(\alpha)$, we get that

  $$k' \circ k^{-1}(x)|\mathbb{T}_\infty(f)(x) - \mathbb{T}_\infty(g)(x)| \leq \frac{1}{2\pi}\left|\int_{-\infty}^{+\infty} \Xi_{2\mu}(\alpha - \beta)k'(\beta)(f(k(\beta)) - g(k(\beta)))d\beta\right|$$

  As a consequence,

  $$||\mathbb{T}_\infty(f) - \mathbb{T}_\infty(g)|| \leq m.||f - g||,$$

  where ϵ_0 is the maximum of $|\Xi_{2\mu}(\alpha)|$ for $\Delta \in [0, \Delta_1]$ and $\alpha \in \mathbb{R}$.

- **Analyzing and enframing of ϵ_0:**
  Recall that for all α and Δ,

  $$\Xi_{2\mu}(\alpha) = \frac{\sin(2\mu)}{\cosh(\alpha) - \cos(2\mu)}.$$

When $\Delta \in [0, \Delta_1]$, $\mu \in [\pi/2, 3\pi/4]$, since $\Delta_1 < \sqrt{2}/2$. This implies that $\sin(2\mu) \leq 0$ and

$$\max_{\alpha} \Xi_{2\mu}(\alpha) = \frac{-\sin(2\mu)}{1 - \cos(2\mu)}.$$

This function is an increasing function of μ:

$$\frac{d}{d\mu}\left(\frac{-\sin(2\mu)}{1-\cos(2\mu)}\right) = 2\frac{-\cos(2\mu).(1-\cos(2\mu)) + \sin^2(2\mu)}{(1-\cos(2\mu))^2} = \frac{2}{(1-\cos(\mu))} > 0.$$

As a consequence,

$$0 \leq \epsilon_0 \leq -\frac{\sin(2\mu(\Delta_1))}{1-\cos(2\mu(\Delta_1))} < -\frac{\sin(2 \times 3\pi/4)}{1-\cos(2\times 3\pi/4)} = 1.$$

□

Lemma 4. *There exist some $\epsilon_1 \in (0,1)$ and some $K_1, K_2, K_3 > 0$ and N_0 such that if $N \geq N_0$ and $\mathbf{p}$ is a fixed point of $\mathbb{T}_n$, then*

$$||\rho_{\mathbf{p}} - \rho|| \leq \epsilon_1.||\rho_{\mathbf{p}} - \rho|| + \frac{K_1}{N} + \frac{K_2}{N}.||\rho_{\mathbf{p}} - \rho|| + K_3.||\rho_{\mathbf{p}} - \rho||^2.$$

Proof. Consider a solution of the Bethe equation $\mathbb{T}_n(\mathbf{p}) = \mathbf{p}$, and ρ a solution of the continuous Bethe equation $\mathbb{T}_\infty(\rho) = \rho$. Thus ρ is given by Proposition 6.

- **Direct consequences of the definition of $||.||$:** Since $(\Delta, x) \mapsto \rho(\Delta, x)$ is positive (cosh is positive and k' too (Proof of Proposition 4) and continuous on $[\Delta_0, 1]$, we can write, by definition of the norm $||.||$, that for all j,

$$\left|\rho(p_j) - \frac{1}{N(p_{j+1} - p_j)}\right| \leq \frac{||\rho_{\mathbf{p}} - \rho||}{\min(k' \circ k^{-1})},$$

where the minimum is taken over the (Δ, x) such that $\Delta \in [\Delta_0, 1]$ and $x \in [-(\pi - \mu(\Delta)), \pi - \mu(\Delta)]$. As a consequence:

$$\left|p_{j+1} - p_j - \frac{1}{\rho(p_j).N}\right| \leq \frac{||\rho_{\mathbf{p}} - \rho||}{\min(k' \circ k^{-1})} \frac{(p_{j+1} - p_j)}{\rho(p_j)}.$$

Thus,

$$p_{j+1} - p_j \leq \frac{1}{\rho(p_j).N} + \frac{||\rho_{\mathbf{p}} - \rho||}{\min(k' \circ k^{-1})} \frac{(p_{j+1} - p_j)}{\rho(p_j)}.$$

$$p_{j+1} - p_j \leq \frac{1}{N \min(\rho)} + \frac{2\pi.||\rho_{\mathbf{p}} - \rho||}{\min(\rho).\min(k' \circ k^{-1})}.$$

Injecting this in the second equation of this paragraph, we get that

$$\left|p_{j+1} - p_j - \frac{1}{\rho(p_j).N}\right| \leq \frac{||\rho_{\mathbf{p}} - \rho||}{N \min^2(\rho).\min(k' \circ k^{-1})} + \frac{2\pi||\rho_{\mathbf{p}} - \rho||^2}{\min^2(\rho).\min^2(k' \circ k^{-1})}.$$

- **Relating the Bethe equation to its continuous version:**

  1. **Auxiliary function:**
     Consider the function $f_{\mathbf{p}} : \mathbb{R} \to \mathbb{R}$ defined by

$$f_{\mathbf{p}}(x) = \frac{1}{2\pi}\left(x + \frac{1}{N}\sum_{k=1}^{n} \Theta(x, p_k)\right).$$

Since $\mathbf{p}$ is a fixed point of $\mathbb{T}_n$, for any $t \in [p_j, p_{j+1})$,

$$\rho_p(t) = \frac{(f_p(p_{j+1}) - f_p(p_j))}{(p_{j+1} - p_j)},$$

and as a consequence, there exists some $\xi_j \in (p_j, p_{j+1})$ such that $\rho_p(t) = f'_p(\xi_j)$. Then for all j and $t \in [p_j, p_{j+1})$, by triangular inequalities,

$$\begin{array}{rcl} k'(k^{-1}(t))|\rho_p(t) - \rho(t)| & \leq & k'(k^{-1}(t)).\left(|f'_p(\xi_j) - \rho(\xi_j)| + |\rho(\xi_j) - \rho(t)|\right) \\ & \leq & k'(k^{-1}(\xi_j)).\left(|f'_p(\xi_j) - \rho(\xi_j)| + |\rho(\xi_j) - \rho(t)|\right) \\ & + & |k'(k^{-1}(t)) - k'(k^{-1}(\xi_j))|.\left(|f'_p(\xi_j) - \rho(\xi_j)| + |\rho(\xi_j) - \rho(t)|\right) \end{array}$$

Thus,

$$||\rho_p - \rho|| \leq \left(1 + \frac{\eta(k' \circ k^{-1}, \delta(\mathbf{p}))}{\min(k' \circ k^{-1})}\right).||f'_p - \rho|| + 2\max(k' \circ k^{-1})\eta(\rho, \delta(\mathbf{p})),$$

where for f a function and $\epsilon > 0$, $\eta(f, \epsilon)$ is the equicontinuity modulus, and $\delta(\mathbf{p})$ is the maximum of the $p_{j+1} - p_j$. As a consequence,

$$||\rho_p - \rho|| \leq \left(1 + \frac{\eta(k' \circ k^{-1}, \delta(\mathbf{p}))}{\min(k' \circ k^{-1})}\right).||f'_p - \rho|| + 2\max(k' \circ k^{-1})\max(\rho')\delta(\mathbf{p}).$$

2. **Relation of this auxiliary function to the continuous Bethe equation:**
Let us denote, for all $\alpha \in \mathbb{R}$, $R_{\mathbf{p}}(\alpha) = \rho_{\mathbf{p}}(\alpha)k'(\alpha)$.
By definition of $f_{\mathbf{p}}$, for all x,

$$2\pi f'_{\mathbf{p}}(x) = 1 + \frac{1}{N}\sum_{k=1}^{n} \partial_1\Theta(x, p_k).$$

Using Taylor inequality we get that

$$\left|2\pi f'_{\mathbf{p}}(x) - 1 - \int_{-(\pi-\mu)}^{\pi-\mu} \partial_1\Theta(x, y)\rho_p(y)dy\right|$$

is smaller than

$$\frac{2}{N}\max(\partial_1\Theta) + \sum_{k=0}^{n}\int_{p_k}^{p_{k+1}} \frac{1}{N(p_{k+1} - p_k)}\left|\partial_1\Theta(x, p_k) - \partial_1\Theta(x, y)\right| dy,$$

which is smaller than

$$\frac{2}{N}\max(\partial_1\Theta) + \sum_{k=0}^{n}\int_{p_k}^{p_{k+1}} \frac{1}{N(y - p_k)}\left|\partial_1\Theta(x, p_k) - \partial_1\Theta(x, y)\right| dy,$$

smaller than, by Taylor inequalities:

$$\frac{2}{N}\max(|\partial_1\Theta|) + \frac{1}{N}\max(|\partial_2\partial_1\Theta|).$$

We used here that Θ is analytic on the set of (Δ, x) such that $\Delta \in [\Delta_0, 1]$ and $x \in [-(\pi - \mu(\Delta), \pi - \mu(\Delta)]$. As a consequence:

$$|f'_{\mathbf{p}}(x) - \rho(x)| \leq |\mathbb{T}_\infty(\rho_{\mathbf{p}})(x) - \mathbb{T}_\infty(\rho)(x)| + \frac{2}{N}\max(|\partial_1\Theta|) + \frac{1}{N}\max(|\partial_2\partial_1\Theta|)$$

$$||f'_{\mathbf{p}} - \rho|| \leq (1m).||\rho_{\mathbf{p}} - \rho|| + 2\frac{\max(k' \circ k^{-1})}{N}(\max(\partial_1\Theta) + \max(|\partial_2\partial_1\Theta)|).$$

- **Combination of these two points:**
Let us consider ϵ_1 such that
$$\epsilon_0 < \epsilon_1 < 1,$$
where ϵ_0 is given by Lemma 3. As a consequence of the first and the the second point, there exist some $K_1, K_2, K_3 > 0$ such that for N great enougth
$$||\rho_{\mathbf{p}} - \rho|| \leq \epsilon_1.||\rho_{\mathbf{p}} - \rho|| + \frac{K_1}{N} + \frac{K_2}{N}.||\rho_{\mathbf{p}} - \rho|| + K_3.||\rho_{\mathbf{p}} - \rho||^2.$$

□

4.3.2 Existence of solution to the Bethe equation

The following theorem is an adaptation of Theorem 2.3 in [DC16] for the case $\Delta \in (-1, 1)$. Main ideas dates back to [YY66]. The first statement of the enumeration will help to identify the value obtained by Bethe ansatz to the Perron-Frobenius eigenvalue, and the second one provides asymptotic behavior of the solutions of the Bethe equation. In this section, we will need the implicit functions theorem, that we recall here:

Theorem 4 (Implicit functions). *Let us consider $k \geq 1$ and $m \geq 1$ integers, and $U \subset \mathbb{R} \times \mathbb{R}^k$ an open set. Let $\mathbb{T} : U \to \mathbb{R}^m$ an analytic function. Assume that there exists some $v \in U$ such that the differential $d\mathbb{T}(v)$ is invertible and $\mathbb{T}(v) = 0$. Then there exists some open set $V \subset U$ which contains v, a real interval I and an analytic function $w : I \to \mathbb{R}$ such that $\mathbb{T}(x, s) = 0$ for $(x, s) \in V$, $x \in \mathbb{R}$ and $s \in \mathbb{R}^k$,*
$$\mathbb{T}(x, s) = 0 \Leftrightarrow s = w(x).$$

We will use this theorem on
$$U = \{(\Delta, \mathbf{p}) : \Delta \in (-1, 1), \mathbf{p} \in \mathcal{S}_n(\Delta)\} \subset \mathbb{R}^{n+1},$$
and on the operator $\mathbb{T} = \mathbb{I}_n - \mathbb{T}_n$, in the proof of the following theorem:

Theorem 5. *There exists some $K > 0$ and some N_1 such that for any $N \geq N_1$ such that $N \equiv 0[4]$, there exists a family of solutions $\boldsymbol{p}_\Delta$, $\Delta \in [0, \Delta_1]$ to the Bethe equations such that*

1. *$\Delta \mapsto \boldsymbol{p}_\Delta$ is analytic.*
2. *$||\rho_{p_\Delta} - \rho|| \leq 1/N$ for all $\Delta \in [0, \Delta_1]$.*

Remark 7. *As a consequence, the functions $\Delta \mapsto \psi(p_\Delta, \Delta)$ and $(\Delta, p) \mapsto \Lambda(\Delta, p)$ are analytic. Moreover, when n is even, since we construct solutions in $\mathcal{S}_n$, no element of the sequence p_Δ can be equal to 0. Hence one can apply the Bethe ansatz formulated above.*

Ideas of the proof: *we would like to apply the implicit function theorem to $\mathbb{I}_n - \mathbb{T}_n$, starting on a neighborhood of $(0, \boldsymbol{p}_0)$, where $\boldsymbol{p}_0$ verifies the Bethe equation at $\Delta = 0$, then extending progressively. In order to apply this theorem, we need the Jacobian matrix (of the partial derivatives of $\mathbb{I}_n - \mathbb{T}_n$) to be invertible. This is the argument used in [DC16]. This is known by a simple analysis at $\Delta = 0$ (point 1 of the proof). In order to prove that, we rely on asymptotic simplicity of this matrix whenever the distribution of some solution to the Bethe equation is sufficienly close to ρ, the solution of the continuous Bethe equation (point 2). The strategy is then to extend progressively the solution from $\Delta = 0$, where the Bethe equation is simplified. In this process, we need to ensure that the proximity of the solution of the Bethe equation to ρ is not broken (for the Jacobian matrix to be invertible). For this, we rely on the fact that the operator underlying the continuous Bethe equation are uniformly contracting on a sub-interval of $(-1, 1)$ (point 3, Lemma 4).*

Proof. 1. **Analysis of the case $\Delta = 0$:**

For $\Delta = 0$, $\Theta = 0$ and the Bethe equation has a unique solution $\mathbf{p}_0$ given by:

$$p_j = \frac{2\pi}{N}\left(j - \frac{n+1}{2}\right).$$

Since $\Delta = 0$, $\mu = \pi/2$. Because $N = 2n$, for all j we have

$$p_j < \frac{2\pi(n-1)/2}{N} < \frac{\pi}{2} = \pi - \mu.$$

As well, $p_j > -\pi/2 = -(\pi - \mu)$. Since the sequence $\mathbf{p}$ is also symmetrical, it is in $\mathcal{S}_n(0)$. Moreover the inequality $||\rho_{\mathbf{p}} - \rho|| \leq K/N$ is verified for any $K > 0$. Indeed, $\rho(x) = \frac{1}{2\pi}$ for all x, and for all i and $x \in [p_j, p_{j+1})$,

$$\rho_{\mathbf{p}} = \frac{1}{N(p_{j+1} - p_j)} = \frac{1}{2\pi} = \rho(x).$$

2. **When a solution $\mathbf{p}$ of the Bethe equation for Δ is sufficiently close to ρ, the Jacobian matrix is invertible at $(\Delta, \mathbf{p})$.**

   - **Statement:**
     Formally: for any $K > 0$ and $\Delta_0 \in (0, 1/2)$, there exists some N_1 such that for any $\Delta \in [\Delta_0, \Delta_1]$ and $N \geq N_1$, the matrix $d(\mathbb{I} - T)(\Delta, p)$ is invertible whenever p is a solution of the Bethe equation such that $||\rho_p - \rho|| \leq K/N$.
   - **Proof:**

     (a) **Expression of the Jacobian matrix:**
     The tangeant vectorial space at any point is

     $$\mathbb{R}^n_{\text{sym}} \equiv \{(x_1, ..., x_n) \in \mathbb{R}^n : \forall j, x_{n+1-j} = -x_j\}.$$

     Let us denote the matrix A of the Jacobian matrix of $\mathbb{I} - \mathbb{T}_n$ relative to the basis $(e_j)_{j \leq n/2}$ of this space, where $(e_j)_{j \leq n}$ is the canonical basis of $\mathbb{R}^n$. The coefficient (j, k) of this matrix is

     $$A_{j,k} = \frac{\partial}{\partial p_k}((\mathbb{I} - \mathbb{T}_n)(\mathbf{p})_j).$$

     When $j \neq k$, this coeffient is equal to

     $$A_{j,k} = \frac{1}{N}(\partial_2 \Theta(p_j, p_k) - \partial_2 \Theta(p_j, -p_k)).$$

     When $j = k$,

     $$A_{j,k} = 1 + \frac{1}{N}\sum_{l \neq j} \partial_1 \Theta(p_j, p_l) - \frac{1}{N}\partial_2 \Theta(p_j, -p_j).$$

     (b) **A sufficient condition to verify:**
     Let us denote B the diagonal matrix with entries $N(p_{j+1} - p_j)$. Since B is invertible, in order to prove that A is invertible, it is sufficient to prove that $C \equiv AB$ is invertible. For this purpose, we show that the diagonal terms of this matrix are dominant, meaning that for all j,

     $$|C_{j,j}| > \sum_{k \neq j} |C_{j,k}|.$$

(c) **Lower bound for the coefficients of C:**
We have directly that for all j,

$$C_{j,j} \geq \frac{1}{\rho_{\mathbf{p}}(p_j)} \left(1 + \frac{1}{N} \sum_{k \neq j} \partial_1 \Theta(p_j, p_k) \right) - \frac{1}{N} ||\partial_2 \Theta||_{\infty}.$$

As a consequence:

$$C_{jj} \geq \frac{1}{\rho_{\mathbf{p}}(p_j)} \left(1 + \frac{1}{N} \int_{-(\pi-\mu)}^{\pi-\mu} \partial_1 \Theta(p_j, y) \rho(y) dy dy \right) + O\left(||\rho_{\mathbf{p}} - \rho|| \right)$$

Since ρ is a fixed point of $\mathbb{T}_{\infty}$:

$$C_{jj} \geq \frac{2\pi\rho(p_j)}{\rho_{\mathbf{p}}(p_j)} + O\left(||\rho_{\mathbf{p}} - \rho|| \right) = 2\pi + O\left(||\rho_{\mathbf{p}} - \rho|| \right),$$

where the $O(.)$ is independant from the parameters.

(d) **Off-diagonal terms are non-negative:**
For $j \neq k \leq n/2$,

$$C_{j,k} = (p_{k+1} - p_k).(\partial_2 \Theta(p_j, p_k) - \partial_2 \Theta(p_j, -p_k)).$$

Since the functions

$$\Psi_x : y \mapsto \Theta(x, y) + \Theta(x, -y)$$

is increasing on $[-\pi, 0]$ for all $x \in [-\pi, 0]$, $C_{j,k} \geq 0$.

(e) **Upper bound for the off-diagonal terms:**
As a direct consequence, for all j,

$$\begin{array}{rcl} \sum_{k \neq j \leq n/2} C_{j,k} & \leq & \sum_{k \leq n/2} (p_{k+1} - p_k)(\partial_2 \Psi_x(p_j, p_k)) + O(1/N) + O(||\rho_{\mathbf{p}} - \rho||) \\ & \leq & \Psi_x(p_j, 0) - \Psi_x(p_j, -(\pi - \mu)) + O(1/N) + O(||\rho_{\mathbf{p}} - \rho||) \\ & \leq & L + O(1/N) + O(||\rho_{\mathbf{p}} - \rho||) \end{array}.$$

(f) **Combining the bounds:**
By combination of the upper and lower bounds, and positivity of the $C_{j,k}$ when $j \neq k$:

$$|C_{j,j}| - \sum_{k \neq j} |C_{j,k}| \geq C_{j,j} - \sum_{k \neq j} C_{j,k} \geq 2\pi - L + O(1/N) + O(||\rho_{\mathbf{p}} - \rho||).$$

As a consequence, for N large enough, if $||\rho_{\mathbf{p}} - \rho|| \leq K/N$, then A is invertible.

3. **The second hypothesis of the theorem is not broken during extension:** Let us consider some ϵ_2 such that

$$1 > \epsilon_2 > \epsilon_1.$$

There exists some $K > 0$ such that for N_2 large enough, for any $N \geq N_2$, there exists no solution $p \in \mathcal{S}_n$ of the Bethe equation such that

$$\epsilon_2.\frac{K}{N} \leq ||\rho_p - \rho|| \leq \frac{K}{N}.$$

Indeed, by applying Lemma 4, if this inequality was verified, we would have

$$||\rho_p - \rho|| \leq \epsilon_1 \frac{K}{N} + \frac{K_1}{N} + \frac{KK_2}{N^2} + \frac{K^2 K_3}{N^2}.$$

We choose K such that

$$\epsilon_1 K + K_1 < \epsilon_2 . K$$

For this choice, for N great enough, we obtain that

$$\epsilon_2 \frac{K}{N} \leq ||\rho_p - \rho|| < \epsilon_2 \frac{K}{N},$$

which is impossible.

4. **Repetition:** One can see that the matrix A is the identity when $\Delta = 0$. As a consequence we can build a solution around $(0, \mathbf{p}_0)$. Then we choose a Δ_0 included in this neighborhood and > 0. We consider $K > 0$ given by the last point. By the last point, the solutions constructed around $(0, \mathbf{p}_0)$ can not break the second condition of the theorem. As a consequence, one can extend again, using the second point, as long as $\Delta < \Delta_1$.

□

4.4 Analysis of Heseinberg Hamiltonian and identification at $c = \sqrt{2}$

4.4.1 Analysis of the Heisenberg Hamiltonian

In this section, following the technique introduced by Lieb, Schultz and Mattis [LSM61], we analyse some particular Hamiltonian, which is proved to commute with the transfer matrix $V_N(\sqrt{2})$ [DC16], and compute its eigenvalues. It will serve, in Section 4.4.2, to identify the Perron-Frobenius eigenvalue to the value provided by the Bethe ansatz.

4.4.1.1 Lowering and Raising operators Let us recall that $\Omega_N = \mathbb{C}^2 \otimes ... \otimes \mathbb{C}^2$. In this section, for the sake of notation, we identify $\{1, ..., N\}$ with $\mathbb{Z}/N\mathbb{Z}$. Let us denote a the matrix in $\mathcal{M}_2(\mathbb{C})$ equal to

$$a \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and a^* the matrix

$$a^* \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

For all $j \in \mathbb{Z}/N\mathbb{Z}$, we denote a_j the matrix in $\mathcal{M}_{2^N}(\mathbb{C})$ equal to

$$a_j \equiv id \otimes ... \otimes a \otimes ... \otimes id,$$

where id denotes the identity, and a acts on the jth copy of $\mathbb{C}^2$. We denote a_j^* the matrix

$$a_j^* \equiv id \otimes ... \otimes a^* \otimes ... \otimes id.$$

In other terms, a_j (resp. a_j^*) acts on a vector x of the basis of Ω_N by sending it to the null vector if $x_j = 0$ (resp. $x_j = 1$) and to the vector z such that $z_j = 0$ (resp. $z_j = 1$) and $z_k = x_k$ when $k \neq j$ if $x_j = 1$ (resp. $x_j = 0$).

Lemma 5. *The matrices a_j and a_j^* verify the following properties, for all j and $k \neq j$:*

- $a_j a_j^* + a_j^* a_j = id.$
- $a_j^2 = a_j^{*2} = 0.$
- a_j, a_j^* *commute both with* a_k *and* a_k^*.

Proof. • By straightforward computation, we get

$$aa^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$a^*a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Thus $aa^* + a^*a$ is the indentity of $\mathbb{C}^2$. As a consequence, for all j,

$$a_j a_j^* + a_j^* a_j = id \otimes ... \otimes id,$$

which is the identity of Ω_N.

- The second set of equalities come from

$$a^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$a^{*2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

- The last set come from the fact that any operator on $\mathbb{C}^2$ commutes with the identity.

□

4.4.1.2 Heisenber hamiltonian The **Hamiltonian** is the matrix in $\mathcal{M}_{2^N}(\mathbb{C})$ defined as:

$$H = - \sum_{j \in \mathbb{Z}/N\mathbb{Z}} \left(a_j^* a_{j+1} + a_j a_{j+1}^* \right)$$

This matrix is non-positive and irreducible and symmetric, so we can apply Perron-Frobenius theorem for these matrices on the hamiltonian.

Remark 8. *The action of $H_i \equiv a_j^* a_{j+1} + a_j a_{j+1}^*$ on an element of the basis of Ω_N is to inverse the symbols of positions j and $j+1$ whenever they are different. When they are equal, the image is the null vector. Thus one can think to H as 'linking' two vectors of the basis x, y (meaning $H(x,y) \neq 0$) whenever they differ by moving a 1 symbol to a neighbor position.*

Remark 9. *The problem of non-coherence of this proof in litterature appears clearly for this point, on which the authors don't agree on notations and precise definition of the Hamiltonian. Our definition corresponds exactly to taking $\gamma = 0$ in the definition of [LSM61], to $\Delta = 0$ for [YY66]. It corresponds, up to a factor -1, to the Hamiltonian at $\Delta = 0$ for [DC16].*

The following lemma corresponds to Lemma 5.1 in [DC16], for $\Delta = 0$:

Lemma 6. *The matrix $V_N(\sqrt{2})$ and the hamiltonian H commute:*

$$H.V_N(\sqrt{2}) = V_N(\sqrt{2}).H$$

Remark 10. *As a consequence, a vector is an eigenvector of $V_N(\sqrt{2})$ if and only if it is an eigenvector of H (however the eigenvalues are not necessarily the same).*

4.4.1.3 Anihilation and creation operators

Let us denote σ the matrix of $\mathcal{M}_2(\mathbb{C})$ defined as:

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let us denote, for all $j \in \mathbb{Z}/N\mathbb{Z}$, c_j the matrix

$$c_j = \sigma \otimes ... \sigma \otimes a \otimes id \otimes ... \otimes id,$$

and

$$c_j^* = \sigma \otimes ... \sigma \otimes a^* \otimes id \otimes ... \otimes id.$$

Let us recall that two matrices P, Q anticommute when $PQ = -QP$.

Lemma 7. *These operators verify the following properties for all j and $k \neq j$:*

- $c_j c_j^* + c_j^* c_j = id$.
- c_j^* *and* c_j *anticommute with both* c_k^* *and* c_k.
- $a_{j+1}^* a_j = -c_{j+1}^* c_j$ *and* $a_j^* a_{j+1} = -c_j^* c_{j+1}$.

Proof.

- Since $\sigma^2 = id$, for all j,

$$c_j c_j^* + c_j^* c_j = a_j a_j^* + a_j^* a_j.$$

From Lemma 5, we now that this operator is equal to identity.

- We can assume without loss of generality that $j < k$. Let us prove that c_j anticommutes with c_k (the other cases are similar):

$$c_j c_k = id \otimes ... \otimes a\sigma \otimes \sigma \otimes ... \otimes \sigma \otimes \sigma a \otimes id \otimes ... \otimes id.$$

$$c_j c_k = id \otimes ... \otimes \sigma a \otimes \sigma \otimes ... \otimes \sigma \otimes a\sigma \otimes id \otimes ... \otimes id.$$

Hence it is sufficient to see:

$$\sigma a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

$$a\sigma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -\sigma a$$

- Let us prove the first equality (the other one is similar):

$$c_{j+1}^* c_j = id \otimes ... \otimes id \otimes \sigma a \otimes a^* \otimes id \otimes ... \otimes id.$$

We have just seen in the last point that $\sigma a = -a$. As a consequence $c_{j+1}^* c_j = -a_{j+1}^* a_j$. □

4.4.1.4 Action of a symmetric orthogonal matrix

Let us denote c^* is the vector $(c_1^*, ..., c_N^*)$ and c^t is the transpose of the vector $(c_1, ..., c_N)$. Let us consider a symmetric and orthogonal matrix $U = (u_{i,j})_{i,j}$ in $\mathcal{M}_N(\mathbb{R})$ and denote

$$b = U.c^t = (b_1, ..., b_N)$$

and

$$b^* = c^*.U^t = (b_1^*, ..., b_N^*).$$

Lemma 8. *For all j and $k \neq j$:*

- *b_j and b_j^* anticommute with both b_k and b_k^* and $b_j b_j^* + b_j^* b_j = id$.*
- *Let us denote $\nu = (0, ..., 0)$. All the vectors $\psi_\alpha = (b_1^*)^{\alpha_1} ... (b_N^*)^{\alpha_N} \nu$ are not equal to zero.*
- *For all j and α, we have:*
  - $b_j^* b_j \psi_\alpha = \boldsymbol{0}$ *if* $\alpha_j = 0$,
  - $b_j^* b_j \psi_\alpha = \psi_\alpha$ *if* $\alpha_j = 1$.

Proof.

- Let us see this first point on b_j and b_k^*. We can write

$$b_j = \sum_i u_{i,j} c_i$$

$$b_k^* = \sum_i u_{k,i} c_i^* = \sum_i u_{i,k} c_i^*.$$

Thus

$$b_j b_k^* = \sum_i \sum_{l \neq i} u_{i,j} u_{l,k} c_i c_l^* + \sum_i u_{i,j} u_{i,k} c_i c_i^*.$$

From Lemma 7,

$$b_j b_k^* = -\sum_l \sum_{i \neq l} u_{i,j} u_{l,k} c_l^* c_i + \sum_i u_{i,j} u_{i,k} (\mathbf{1} - c_i^* c_i).$$

Since the matrix U is orthogonal,

$$b_j b_k^* = -\sum_l \sum_{i \neq l} u_{i,j} u_{l,k} c_l^* c_i - \sum_i u_{i,j} u_{i,k} c_i^* c_i = -b_k^* b_j.$$

This step is the reason why we use the operators c_i instead of the operators a_i.

- For all k,

$$b_k^* = \sum_l u_{k,l} a_l^*.$$

As a consequence, for a sequence $k_1, ..., k_s$,

$$b_{k_1}^* ... b_{k_s}^* = \sum_{l_1} ... \sum_{l_s} u_{k_1, l_1} ... u_{k_s, l_s} a_{l_1}^* ... a_{l_s}^*.$$

Since $(a^*)^2 = 0$, the sum can be considered on the integers $l_1, ..., l_s$ such that they are two by two distinct. The operator $a_{l_1}^* ... a_{l_s}^*$ acts on ν by changing the 0 on positions $l_1, ..., l_s$ into 1. The contribution of this vector in the sum is the sum

$$\sum_{\sigma \in \Sigma_s} u_{k_1, l_{\sigma 1}} ... u_{k_1, l_{\sigma s}}.$$

These sums cover all the determinants of size s sub-matrices of U. If these sums were all zero, this would mean that the determinant of U is zero. This is impossible since U is orthogonal, thus it is invertible. As a consequence all the vectors ψ_α are not equal to zero.

- When $\alpha_j = 0$, from the fact that when $j \neq k$, b_j and b_k^* commute, we get that

$$b_j \psi_\alpha = (b_1^*)^{\alpha_1} ... (b_N^*)^{\alpha_N} b_j \nu,$$

and $b_j \nu = 0$, since for all j, $a_j \nu = 0$. As a consequence $b_j^* b_j \nu = 0$. When $\alpha_j = 1$, then

$$b_j^* b_j \psi_\alpha = (b_1^*)^{\alpha_1} ... (b_{j-1}^*)^{\alpha_{j-1}} b_j^* b_j b_j^* (b_{j+1}^*)^{\alpha_{j+1}} ... (b_N^*)^{\alpha_N} \nu.$$

For the first point:

$$b_j^* b_j \psi_\alpha = (b_1^*)^{\alpha_1} ... (b_{j-1}^*)^{\alpha_{j-1}} b_j^* (id - b_j^* b_j)(b_{j+1}^*)^{\alpha_{j+1}} ... (b_N^*)^{\alpha_N}.$$

$$b_j^* b_j \psi_\alpha = (b_1^*)^{\alpha_1} ... (b_{j-1}^*)^{\alpha_{j-1}} b_j^* (id - b_j^* b_j)(b_{j+1}^*)^{\alpha_{j+1}} ... (b_N^*)^{\alpha_N} = \psi_\alpha - (b_1^*)^{\alpha_1} ... (b_N^*)^{\alpha_N} b_j \nu = \psi_\alpha.$$

□

4.4.1.5 Diagonalisation of the Hamiltonian

Theorem 6. *The eigenvalues of H are exactly the numbers:*

$$2 \sum_{\alpha_j = 1} \cos\left(\frac{2\pi j}{N}\right),$$

for $\alpha \in \{0,1\}^N$.

Proof. 1. **Rewriting H:**

From Lemma 7, we can write H as:

$$H = -\sum_j c_j^* c_{j+1} + c_{j+1}^* c_j.$$

The Hamiltonian H can be then rewritten as $H = c^* M c^t$, where M is the matrix defined by blocks

$$M = \frac{1}{2} \begin{pmatrix} \mathbf{0} & \mathbf{1} & & \mathbf{1} \\ \mathbf{1} & \ddots & \ddots & \\ & \ddots & \ddots & \mathbf{1} \\ \mathbf{1} & & \mathbf{1} & \mathbf{0} \end{pmatrix},$$

where $\mathbf{1}$ denotes the identity matrix on $\mathbb{C}^2$, and $\mathbf{0}$ denotes the null matrix. Let us denote M' the matrix of $\mathcal{M}_{2^N}(\mathbb{R})$ obtained from M by replacing $\mathbf{0}, \mathbf{1}$ by $0, 1$.

2. **Diagonalisation of M:** The matrix M' is symmetric and thus can be diagonalised in $\mathcal{M}_{2^N}(\mathbb{R})$ in an orthogonal basis. It is rather straightforwards to see that the vectors ψ^k, for any $k \in \{0, ..., N-1\}$ are an orthonormal family of eigenvectors of M' for the eigenvalue $\lambda_k = \cos(\frac{2\pi k}{N})$, where for all $j \in \{1, ..., N\}$,

$$\psi_j^k = \sqrt{\frac{2}{N}} \left(\sin(\frac{2\pi k j}{N}), \cos(\frac{2\pi k j}{N}) \right).$$

This comes from the equalities

$$\cos(x - y) + \cos(x + y) = 2\cos(x)\cos(y),$$

$$\sin(x - y) + \sin(x + y) = 2\cos(x)\sin(y),$$

applied to $x = k(j-1)$ and $y = k(j+1)$. This family of vectors is free, since the Vandermonde matrix with coefficients $e^{2\pi k j/N}$ is invertible. As a consequence, one can write

$$U' M' U'^t = D',$$

where D' is the diagonal matrix whose diagonal coefficients are the numbers λ_k, and U' is the orthogonal matrix given by the vectors ψ^k. Replacing any coefficient of these matrices by the product of this coefficient with the identity, one gets an orthogonal matrix U and a diagonal one D such that:

$$U M U^t = D.$$

3. **Construction of eigenvectors for H:** Let us consider the vectors ψ_α constructed in Lemma 8 for the matrix U of the last point, which is symmetric and orthogonal. From the expression of H, we get that

$$H\psi_\alpha = 2 \sum_{j:\alpha_j=1} \cos\left(\frac{2\pi j}{N}\right).$$

Since ψ_α is non zero, this is an eigenvector of H.

4. **The family (ψ_α) is a basis of Ω_N:**

From cardinality of this family (the number of possible α, equal to 2^N), this is sufficient to prove that this family is free. Let us assume that there are some scalars x_α such that

$$\sum_\alpha x_\alpha . \psi_\alpha$$

We apply first $b_1^* b_1 .... b_N^* b_N$ and get that

$$x_{(1,...,1)} \psi_{(1,...,1)} = 0,$$

and thus $x_{(1,...,1)} = 0$. Then we apply successively the operators $\prod_{j \neq k} b_j^* b_j$ for all k, and obtain that for all $\alpha \in \{(0,1,...,1),(1,0,1,...,1),...\}$, $x_\alpha = 0$. By repeating this argument, we obtain that all the coefficient x_α are null. As a consequence $(\psi_\alpha)_\alpha$ is a base of eigenvectors for H. As a consequence, the eigenvalues obtained at the last point cover all the eigenvalues of H.

$\square$

Remark 11. *The analysis of the transfer matrix of the discrete curves shift relies alot on the analysis of the Hamiltonian H. Can we explain the reason that makes these two operators commute ? Is there a canonical way to associate a Hamiltonian to the transfer matrix, possibly under some restrictions on the SFT ?*

In the analysis of the Hamiltonian presented above, the definition of the operators a_j is not specific to the square ice model or a model under toroidal hypothesis and can be formulated for any subshift of finite type. It seems that some condition allowing the analysis is that the Hamiltonian (and possibly the transfer matrix) is expressed in a particular way as a function of these operators.

4.4.2 Identification

In this section, we prove the following:

Theorem 7. *For all $N = 2n$ such that n is even, and for $\Delta \in [0, \Delta_1)$,*

$$\lambda(V_N(c)) = \Lambda(p_\Delta).$$

Proof. 1. **From the Hamiltonian to the transfer matrix:**

Let us recall that $V_N(\sqrt{2})$ commutes with H (Lemma 6). Assume that Bethe vector is not equal to zero in the neighborhood of $c = \sqrt{2}$. This means that this is an eigenvector of the matrix, and then of the Hamiltonian. The Theorem 2.3 in [DC16] states that this vector is associated to the eigenvalue $2\sum_{k=1}^n \cos(\frac{2\pi k}{N})$ (just take $\Delta = 0$ in the theorem). From Theorem 6, we know that this is the largest eigenvalue of H on the subspace of Ω_N generated by the elements of $\{0,1\}^N$ having exactly n times the symbol 1. Indeed, the sum in the statement of the theorem is maximal when $\alpha_1 = ... = \alpha_n = 1$ and $\alpha_{n+1} = ... = \alpha_N = 0$ for these sequences α.

As a consequence, from Perron-Frobenius theorem, the Bethe vector is positive around $c = \sqrt{2}$. From the same theorem, it is associated to the maximal eigenvalue of $V_N(\sqrt{2})$. As a consequence, the Bethe value is equal to the largest eigenvalue for $c = \sqrt{2}$. By continuity of the Bethe value as a function of c, this is also true on a neighborhood of this parameter. Since the Bethe value is analytic (Remark 7) and the largest eigenvalue of $V_N(c)$ is also analytic in c (by the Implicit functions theorem on the characteristic polynomial, using the fact that the largest eigenvalue is simple), one can identify these two functions on the interval $[0, \Delta_1)$.

We have left to prove that the bethe vector is non zero at $c = \sqrt{2}$:

2. **The Bethe vector is $\neq 0$ at $\Delta = 0$**

   When $\Delta = 0$, the vector given by the Bethe ansatz is such that for all x,

$$\psi(x) = \sum_{\sigma \in \Sigma_n} A_\sigma(0). \prod_{k=1}^{n} e^{ip_{\sigma(k)}x_k},$$

   where

$$A_\sigma(0) = \epsilon(\sigma). \prod_{1 \leq k < l \leq n} (1 + e^{i(p_{\sigma(k)} + p_{\sigma(l)})}).$$

   Let us prove that $A_\sigma(0)$ is independant from σ. Indeed,

$$\prod_{1 \leq k < l \leq n} (1 + e^{i(p_{\sigma(k)} + p_{\sigma(l)})}) = \prod_{1 \leq \sigma^{-1}(k) < \sigma^{-1}(l) \leq n} (1 + e^{i(p_k + p_l)}) = \prod_{1 \leq c < d \leq n} (1 + e^{i(p_c + p_d)})$$

   Since for all $l \neq k$ the condition $\sigma^{-1}(k) < \sigma^{-1}(l)$ is verified or $\sigma^{-1}(l) < \sigma^{-1}(k)$ is verified, exclusively. This means that there is exactly one time $e^{i(p_k+p_l)}$ in the product for each l, k such that $l \neq k$.

   Moreover,

$$A_\sigma(0) \neq 0$$

   since no $p_k + p_l$ can be equal to $\pm\pi$. Indeed, let us recall that in this case, the Bethe equation is simple to solve and for all j,

$$p_j = \frac{2\pi}{N} \left(j - \frac{n+1}{2} \right).$$

   Since n is even, the p_j are in $] - \pi/2, \pi/2[$.

   To see that $\psi(x) \neq 0$ for all x, it is sufficient to see that this coefficient is the determinant of a Vandermonde matrix.

□

4.5 Computation of square ice entropy

In this last section, we compute the entropy of the square ice model (Theorem 8). Before proving this, we need prove that the elements of the sequence solution to the Bethe ansatz are asymptotically distributed according to the density function ρ:

Proposition 7. *Consider $\Delta \in [0, \Delta_1)$. For all N, let $\mathbf{p}(N) \in \mathcal{S}_n$ be a solution of Bethe equation. The sequence of measures $(\mu_N)_N$ defined by:*

$$\mu_N \equiv \frac{1}{N} \sum_{k=1}^{n} \delta_{p_k(N)}$$

converges weakly to $\rho(x)dx$.

Proof. Consider any continuous function g on $[-(\pi-\mu),(\pi-\mu)]$, and define another function $g_{\mathbf{p}(N)}$ on the same interval by $g_{\mathbf{p}(N)}(t) = g(p_j(N))$ whenever $t \in [p_j, p_{j+1})$. Then

$$\int g(x)d\mu_N(x) = \frac{1}{N}\sum_{k=1}^{n} g(p_k(N)) = \sum_{k=0}^{n}\int_{p_k(N)}^{p_{k+1}(N)} \frac{g(p_k(N))}{N(p_{k+1}(N)-p_k(N))}dx + O\left(\frac{1}{N}\right)$$

$$\int g(x)d\mu_N(x) = \int g_{\mathbf{p}(N)}(x)\rho_{\mathbf{p}(N)}(x)dx + O\left(\frac{1}{N}\right)$$

($p_0(N)$ and $p_{n+1}(N)$ denote the bounds of the interval $[-(\pi-\mu),(\pi-\mu)]$). From the decomposition:

$$\int g(x)[\rho(x)dx - d\mu_N(x)] = \int g(x)[\rho(x)dx - \rho_{p(N)}(x)dx] + \int [g(x) - g_{p(N)}(x)]\rho_{p(N)}(x)dx,$$

and since $\rho_{\mathbf{p}(N)}$ converges uniformly towards ρ (Theorem 5), this integral converges towards 0. □

Theorem 8. *The topological entropy of square ice is equal to* $\frac{3}{2}\ln\left(\frac{4}{3}\right)$.

Remark 12. *This value corresponds to* $\ln(W)$ *in the initial computation of E.H.Lieb.*

Proof. Let us set $\Delta = \frac{1}{2}$ (equivalently $c = 1$). As a consequence, $\mu = 2\pi/3$.

- **Asymptotics of the Bethe eigenvalue:**

  The entropy of X is given by

  $$h(X) = \lim_N \frac{\log(\Lambda(\mathbf{p}(N)))}{N},$$

  where the limit is on integers N such that $N \equiv 0[4]$ and

  $$\Lambda(\mathbf{p}(N)) = \prod_{k=1}^{n} L(e^{ip_k(N)}) + \prod_{k=1}^{n} M(e^{ip_k(N)}).$$

  Since $c = 1$, for all z such that $|z| = 1$,

  $$M(z) = L(\overline{z}) = \frac{z}{z-1}.$$

  By symmetry of the sequences $\mathbf{p}(N)$,

  $$\prod_{k=1}^{n} e^{ip_k(N)} = \prod_{k=1}^{n} e^{ip_k(N)/2} = 1,$$

  and thus, since n is even,

  $$\Lambda(\mathbf{p}(N)) = 2\prod_{k=1}^{n}\frac{1}{e^{ip_k(N)}-1} = 2\prod_{k=1}^{n}\frac{e^{-ip_k(N)/2}}{e^{ip_k(N)/2}-e^{-ip_k(N)/2}}$$

  Since this eigenvalue is positive,

  $$\Lambda(\mathbf{p}(N)) = |\Lambda(\mathbf{p}(N))| = 2\prod_{k=1}^{n}\frac{1}{2|\sin(p_k(N)/2)|}.$$

  As a consequence,

  $$h(X) = -\lim_N \frac{1}{N}\sum_{k=1}^{N/2}\ln(2|\sin(p_k(N)/2)|) = -\int_{-(\pi-\mu)}^{\pi-\mu}\ln(2|\sin(x/2)|)\rho(x)dx,$$

  by Proposition 7.

- **Rewritings:**

Let us write

$$|\sin(x/2)| = \sqrt{\frac{1-\cos(x)}{2}}.$$

As a consequence,

$$h(X) = -\frac{\ln(2)}{2}\int_{-(\pi-\mu)}^{\pi-\mu}\rho(x)dx - \frac{1}{2}\int_{-(\pi-\mu)}^{\pi-\mu}\ln(1-\cos(x)).\rho(x)dx$$

Thus,

$$h(X) = -\frac{1}{2}\int_{-(\pi-\mu)}^{\pi-\mu}\ln(2-2\cos(x)).\rho(x)dx.$$

With the change of variable $x = k(\alpha)$:

$$h(X) = -\frac{1}{2}\int_{-\infty}^{+\infty}\ln\left(2(1-\cos(k(\alpha)))\right)\rho(k(\alpha))k'(\alpha)d\alpha.$$

Since

$$R(\alpha) = 2\pi\rho(k(\alpha))k'(\alpha) = \frac{\pi}{2\mu\cosh(\pi\alpha/2\mu)} = \frac{3}{4\cosh(3\alpha/4)}$$

and

$$\cos(k(\alpha)) = \frac{\sin^2(\mu)}{\cosh(\alpha)-\cos(\mu)} - \cos(\mu) = \frac{3}{4(\cosh(\alpha)+1/2)} + \frac{1}{2},$$

We have that:

$$h(X) = -\frac{3}{16\pi}\int_{-\infty}^{+\infty}\ln\left(1-\frac{3}{2\cosh(\alpha)+1}\right)\frac{1}{\cosh(3\alpha/4)}d\alpha.$$

Using the variable change $e^\alpha = x^4$, $d\alpha.x = 4dx$,

$$h(X) = -\frac{3}{16\pi}\int_{0}^{+\infty}\ln\left(1-\frac{3}{x^4+1/x^4+1}\right)\frac{2}{(x^3+1/x^3)}\frac{4}{x}dx.$$

By symmetry of the integrand:

$$h(X) = -\frac{3}{4\pi}\int_{-\infty}^{+\infty}\frac{x^2dx}{x^6+1}\ln\left(\frac{(2x^4-1-x^8)}{1+x^4+x^8}\right)dx$$

$$h(X) = -\frac{3}{4\pi}\int_{-\infty}^{+\infty}\frac{x^2dx}{x^6+1}\ln\left(\frac{(x^2-1)^2(x^2+1)^2}{1+x^4+x^8}\right)dx$$

- **Application of the residues theorem:**

We apply the residue theorem to obtain (the poles of the integrand are $e^{i\pi/6}, e^{i\pi/2}, e^{i5\pi/6}$):

$$\int_{-\infty}^{+\infty}\frac{x^2\ln(x+i)}{x^6+1}dx = 2\pi i\left(\sum_{k=1,3,5}\frac{e^{ik\pi/3}\ln(e^{ik\pi/6}+i)}{6e^{i5k\pi/6}}\right).$$

$$\int_{-\infty}^{+\infty}\frac{x^2\ln(x-i)}{x^6+1}dx = -2\pi i\left(\sum_{k=7,9,11}\frac{e^{ik\pi/3}\ln(e^{ik\pi/6}-i)}{6e^{i5k\pi/6}}\right)$$

By summing these two equations, we obtain that $\int_{-\infty}^{+\infty} \frac{x^2 \ln(x^2+1)}{x^6+1} dx$ is equal to:

$$\begin{gathered}\tfrac{\pi}{3} \left[\ln(e^{i\pi/6}+i) - \ln(e^{i\pi/2}+i) + \ln(e^{i5\pi/6}+i)\right. \\ \left. + \ln(e^{i7\pi/6}-i) - \ln(e^{i9\pi/6}-i) + \ln(e^{i11\pi/6}-i)\right]\end{gathered}$$

This is equal to

$$\frac{\pi}{3}(\ln(|e^{i\pi/6}+i)|^2) - \ln(|e^{i\pi/2}+i|) + \ln(|e^{i5\pi/6}+i|^2) = \frac{2\pi}{3} \ln\left(\frac{3}{2}\right).$$

- **Other computations:**

  We do not include the following computation, since it is very similar to the previous one:

$$\int_{-\infty}^{+\infty} \frac{x^2}{x^6+1} \ln(1+x^4x^8) dx = \frac{2\pi}{3} \ln\left(\frac{8}{3}\right).$$

  For the last integral, we write $\ln((x^2-1)^2) = 2\text{Re}(\ln(x-1)+\ln(x+1))$ and obtain:

$$\int_{-\infty}^{+\infty} \frac{x^2}{x^6+1} \ln((x^2-1)^2) = \text{Re}\left(\int_{-\infty}^{+\infty} \frac{x^2}{x^6+1} \ln(x-1) + \int_{-\infty}^{+\infty} \frac{x^2}{x^6+1} \ln(x+1)\right) = \frac{2\pi}{3} \ln\left(\frac{1}{2}\right)$$

- **Summing these integrals:**

  As a consequence

$$h(X) = -\frac{3}{4\pi}\frac{2\pi}{3}\left(\ln\left(\frac{1}{2}\right) + 2\ln\left(\frac{3}{2}\right) - \ln\left(\frac{8}{3}\right)\right) = \frac{1}{2}\ln\left(\frac{4^3}{3^3}\right) = \frac{3}{2}\ln\left(\frac{4}{3}\right).$$

□

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