Minicourse on *information, complexity and organisation in multidimensional symbolic dynamics*

On the limit between the computable and the uncomputable

Silvere Gangloff

April 15, 2021

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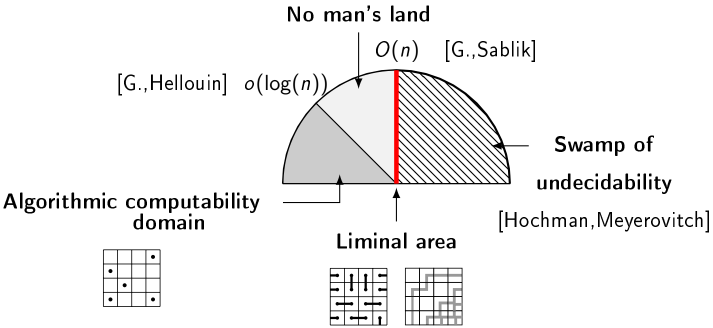
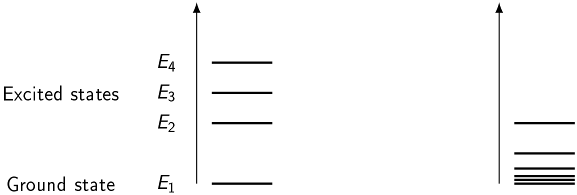
Multidimensional SFT: a computational 'transition':

Reminder (third lecture):



Uncomputability in quantum physics:

Energy states:



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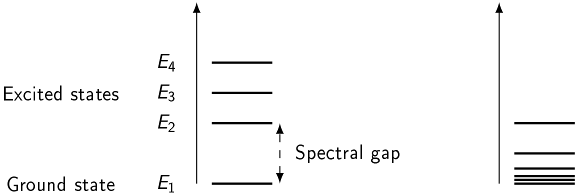

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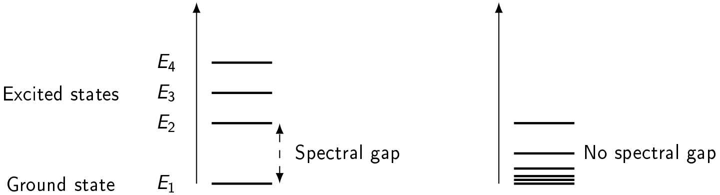

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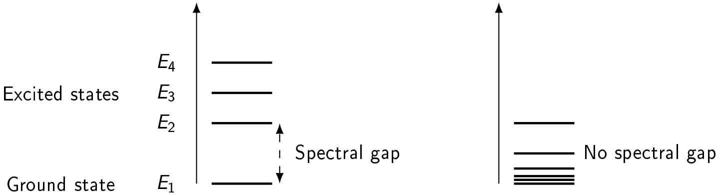

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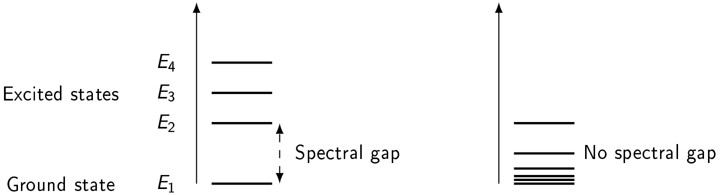

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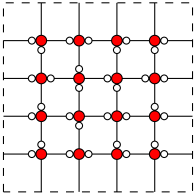
Kreinovich(2017): Why Some Physicists Are Excited About the Undecidability of the Spectral Gap Problem and Why Should We

Uncomputability in quantum physics: Exactly solvable models

Square ice model [Pauling(1935)]:



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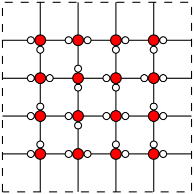

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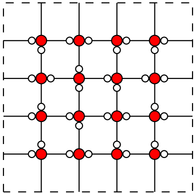

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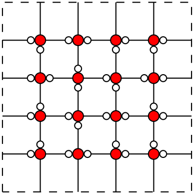

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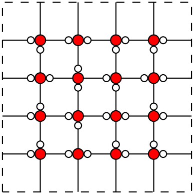

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Questions:

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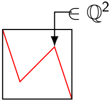
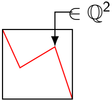
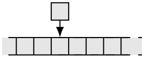
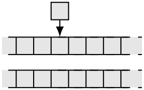
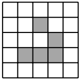
2. How does 'organisation' emerge from simple interactions between elements of matter ?
3. Are the models for which uncomputability occur physically significant ? Can we formulate a restriction which ensures computability ?

Computability of (topological) entropy:

Milnor (2002): is the *entropy* of a dynamical system effectively computable ?

Computable	Uncomputable
$[0,1]$ 	$[0,1]^4$  [Koiran 2001]
 [Jeandel 2014]	 [Delvenne Blondel 2004]
Expansive [D'amico Manzini Margara 2003]	Non contraint [Hurd Kari Culik 1992]



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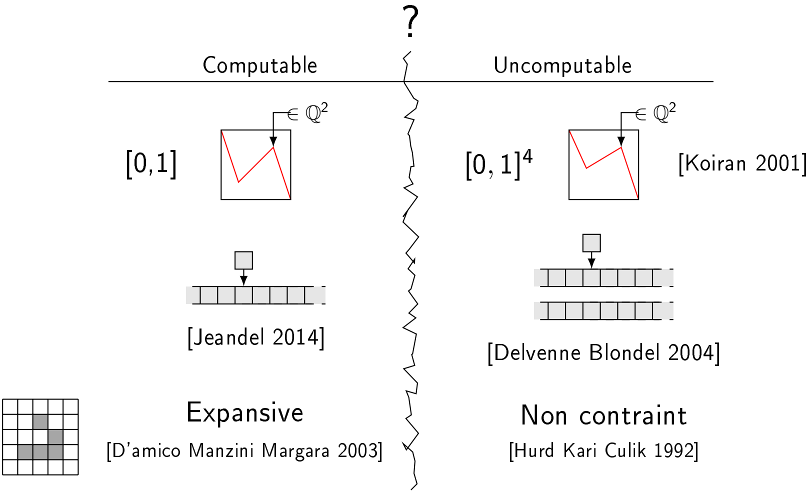


Reminders:

Alphabet $\mathcal{A}$ finite. **Patterns:**(d=1) elements of $\mathcal{A}^{\mathbb{U}}$, $\mathbb{U} \subset \mathbb{Z}$.

Subshifts(d=1): set of patterns $\mathcal{F}$.

$$X_{\mathcal{F}} = \{x \in \mathcal{A}^{\mathbb{Z}} \ : \ \forall \mathbb{U} \subset \mathbb{Z}, x_{|\mathbb{U}} \notin \mathcal{F}\}.$$

For every subshift X on alphabet $\mathcal{A}$ there exists $\mathcal{F}$ s.t. $X = X_{\mathcal{F}}$.

When $\mathcal{F}$ finite : of finite type; when $\mathcal{F}$ recursively enumerable (set of outputs of a computing machine): effective.

Reminders:

Language: $\mathcal{L}(X)$: set of patterns which appear in some $x \in X$.

Entropy(d=1): $N_n(X)$: number of words $w \in \mathcal{L}(X)$, $|w| = n$.

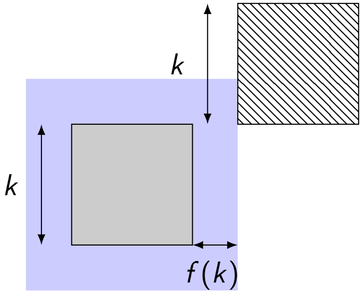
$$h(X) = \lim_{n \to +\infty} \frac{\log_2(N_n(X))}{n} \underset{\text{Th}}{=} \inf_{n \to +\infty} \frac{\log_2(N_n(X))}{n}$$

Π_1-computable: $x \in \mathbb{R}$: exists an algorithm $n \mapsto r_n$ with $r_n \downarrow x$.

Lemma: when X is effective, $h(X)$ is Π_1-computable.

Reminders:

f-block gluing:



When $d = 1$: square patterns $\rightarrow$ words.

Decidable subshifts: caracterization of a threshold

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$\boxed{\text{Algorithm}}$

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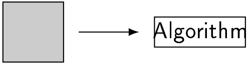

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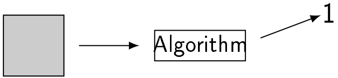

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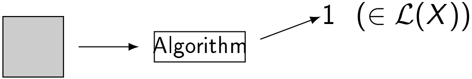

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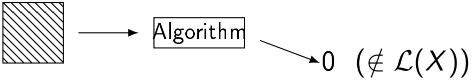

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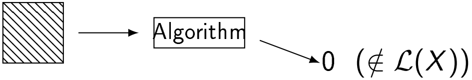

Assume $\Sigma(f) \underset{Def}{=} \sum_{n} \frac{f(k)}{k^2}$ computable.

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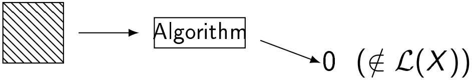

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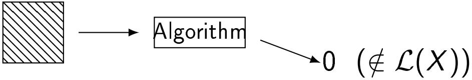

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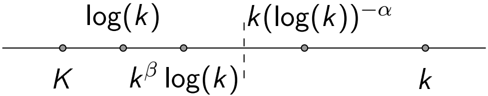

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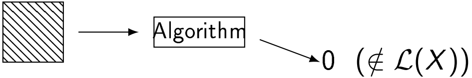

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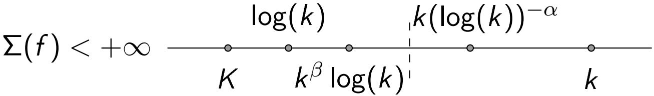

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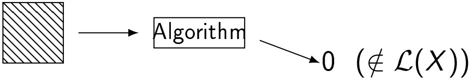

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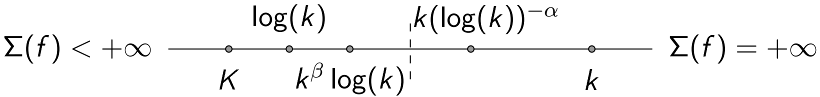

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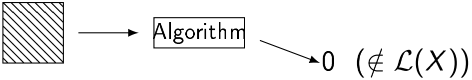

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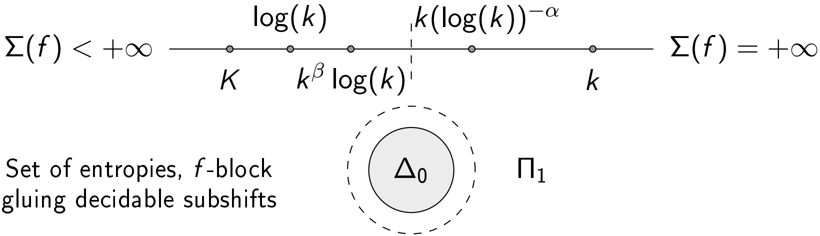

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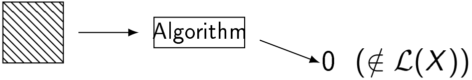

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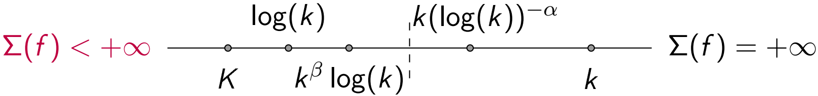

Set of entropies, f-block
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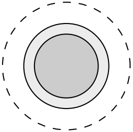

Π_1

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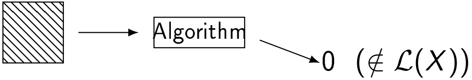

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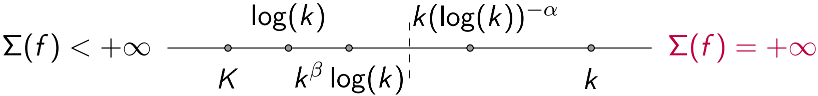

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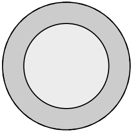

Π_1

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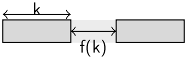

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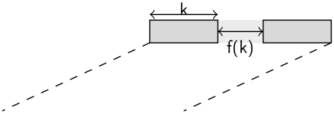

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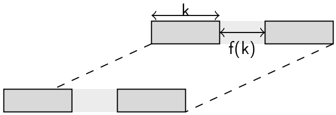

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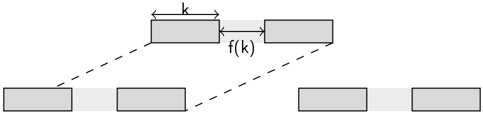

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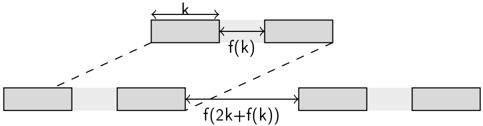

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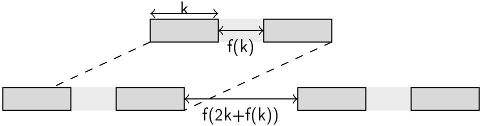

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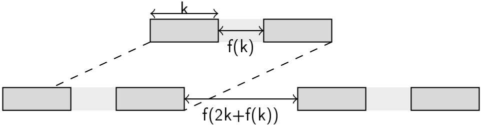

$$\vdots$$

$$\frac{\log(N_k(X))}{k} - |\mathcal{A}| \cdot \sum_{k}^{+\infty} \frac{f(2^l)}{2^l} \leq h \leq \frac{\log(N_k(X))}{k}$$

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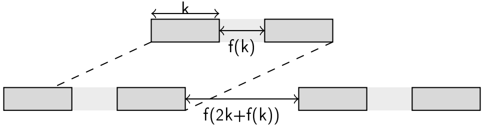

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Since X is decidable, $k \mapsto N_k(X)$ is computable, hence h is computable.

Above the threshold I. Objects: $\Sigma(f) = +\infty$

Known: obstruction $\rightarrow$ let us prove realization

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Bounded density shifts.

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$$\overset{k}{\longleftrightarrow}$$

$$\boxed{0}\boxed{1}\boxed{0}\boxed{1}\boxed{1}\boxed{0}\boxed{0}\boxed{1}$$

$$> p_k$$

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$$\overset{k=8}{\longleftrightarrow}$$

0	1	0	1	1	0	0	1

$$> p_8$$

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Since (p_k) is computable, $X_{\mathcal{F}}$ is decidable:

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Bounded density shifts. Consider $(p_k)_k \in \mathbb{N}^{\mathbb{N}}$ non-decreasing and computable. Define $\mathcal{F}$ as set of:

$$\overset{k=8}{\longleftrightarrow}$$

0	1	0	1	1	0	0	1

$> p_8 = 3$

Since (p_k) is computable, $X_{\mathcal{F}}$ is decidable:

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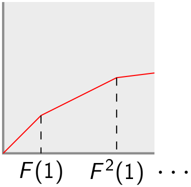

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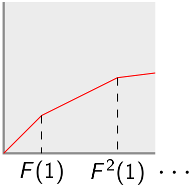

f-block gluing $\Leftrightarrow \forall n,\ p_{F(n)} \geq 2p_n + 4$

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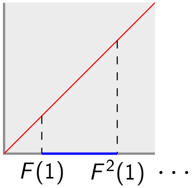

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$h > \alpha_1 + 2^{-1}$

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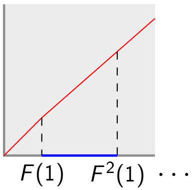

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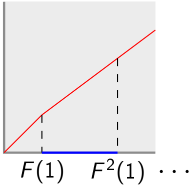

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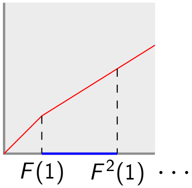

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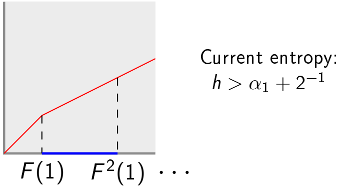

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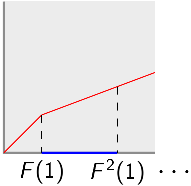

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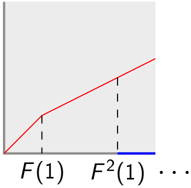

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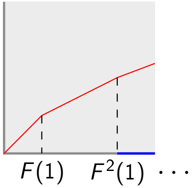

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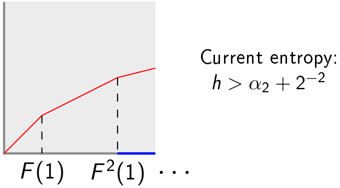

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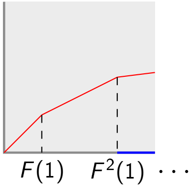

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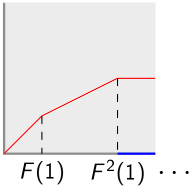

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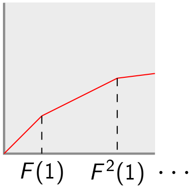

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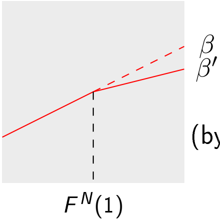
Above the threshold III. Tracking tool: $\Sigma(f) = +\infty$

Entropy change: $\beta = (\beta_1, \beta_2, ..)$ slopes:



$0 \geq \Delta h \geq -H(1/F^N(1))$

$H(\epsilon) = \epsilon \log(\epsilon) + (1-\epsilon)\log(1-\epsilon)$

(by bounding preimages of a transformation)

Above the threshold III. Tracking: $\Sigma(f) = +\infty$

Let us assume that $h_\infty < \alpha$.

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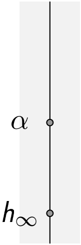

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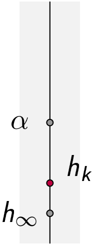

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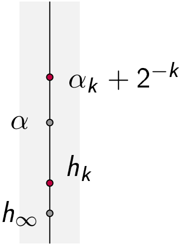

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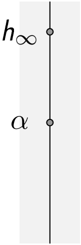

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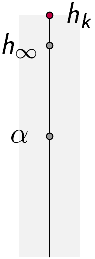

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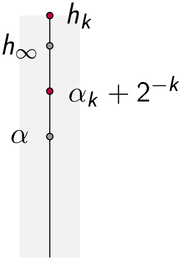

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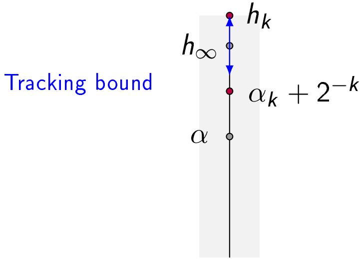

Block gluing condition is minimal:

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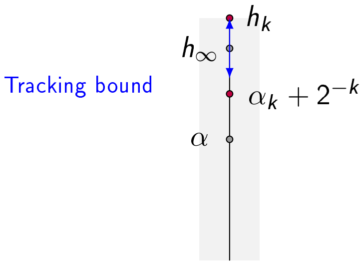

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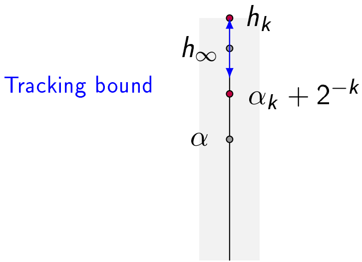

Block gluing condition is minimal: $\forall l \geq k$, $p_{F(l)} = 2p_l + 4$.

Repeat + limit:

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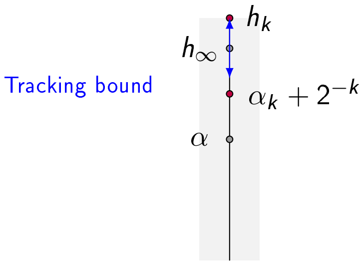

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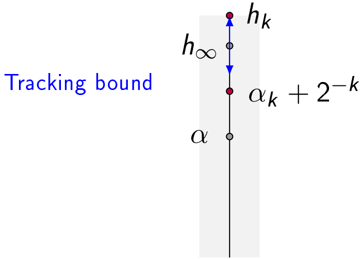

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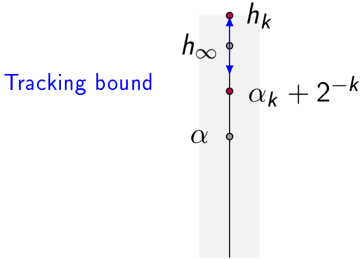

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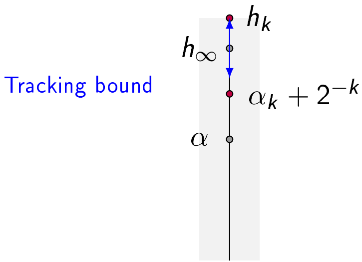

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$\Sigma(f) = +\infty$: $\inf \frac{p_l}{l} = h_\infty = 0 \rightarrow$ contradiction.

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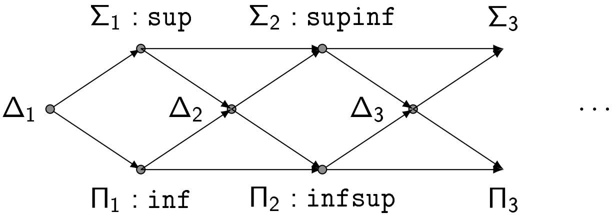
3. For other classes of dynamical systems ? [$\rightarrow$ better understanding of the threshold phenomenon]

Computability in general:

X.Zheng, K.Weihrauch: The arithmetical hierarchy of real numbers.



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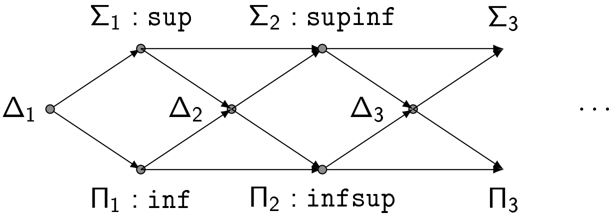

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Theorem: for all m, $\Sigma_m \subsetneq \Delta_{m+1}$, $\Pi_m \subsetneq \Delta_{m+1}$, $\Delta_m \subsetneq \Sigma_m$, $\Delta_m \subsetneq \Pi_m$.

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Question: Classification of classes of dynamical systems according to possible values of entropy ?

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Theorem[G.,Herrera,Rojas,Sablik(2019)]: the entropy of a topological **computable** dynamical system (X, f) is Σ_2-computable.

If in the arithmetical hierarchy, possible classes are: Δ_1, Σ_1, Π_1, Δ_2, Σ_2.

Definition of computable dynamical system

A topological dynamical system is some (X, f), where X is 'computable' and f is 'computable'.

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Definition: A function $f : X \to X$ is **computable** when there exists an algorithm which on input m enumerates $I_m \subset \mathbb{N}$ such that

$$f^{-1}(B_m) = \bigcup_{n \in I_m} B_n,$$

Characterization of computable functions on Cantor sets

Lemma: a function $\mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ is computable when there exists a non-decreasing computable function $\varphi : \mathbb{N} \to \mathbb{N}$ and an algorithm which provided as input the $\varphi(n)$ first elements of some $x \in \mathcal{A}^{\mathbb{N}}$ outputs the n first elements of $f(x)$.

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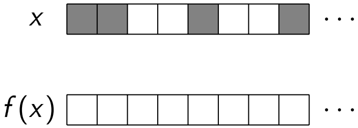

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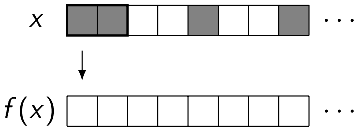

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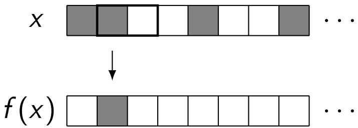

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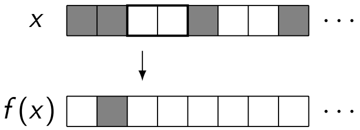

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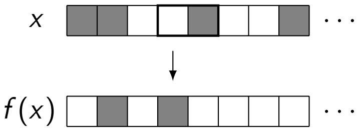

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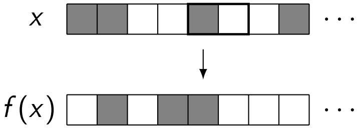

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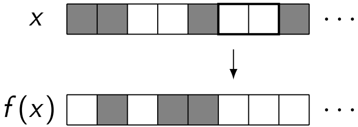

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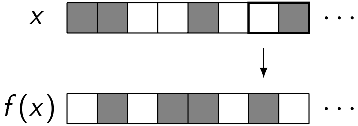

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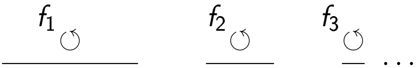

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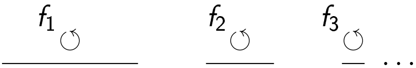

5. f surjective with entropy $\sup_n h_n$.

Interval maps: a case of jump

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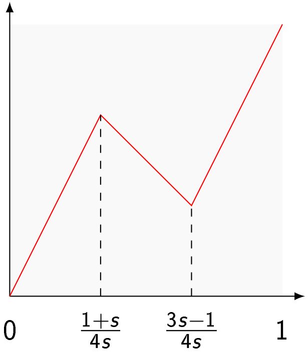

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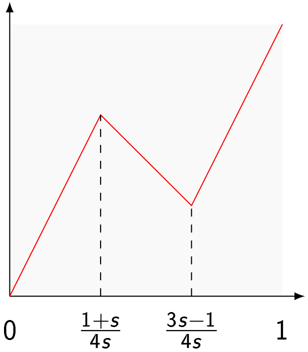

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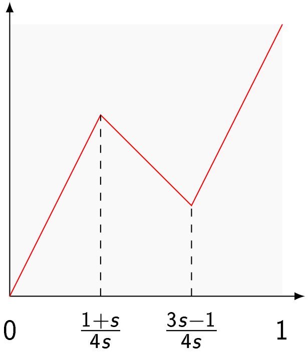

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Thus for all $s \in \mathbb{Q}$, the entropy of $([0,1], f_s)$ is s.

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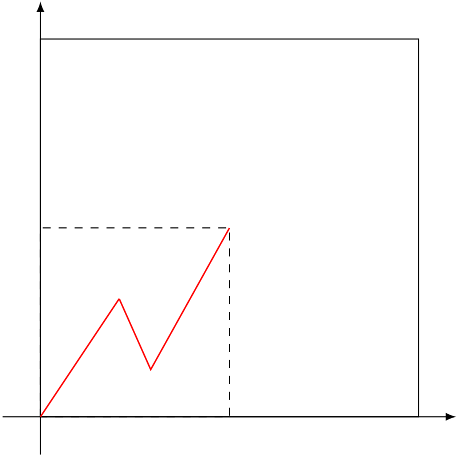

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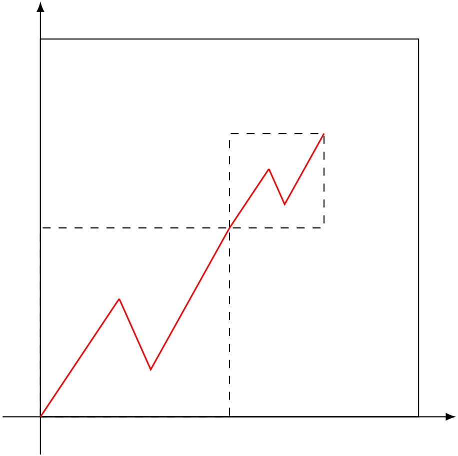

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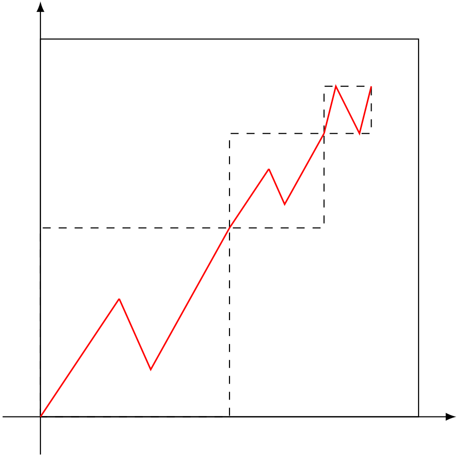

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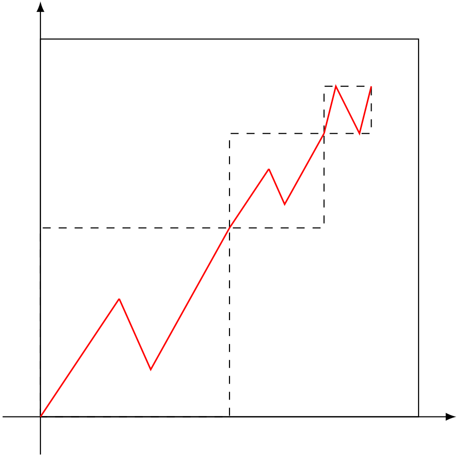

Computable map, entropy s.

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3. Do you have other ideas of classes of systems and dynamical constraints ?