The gap between $O(\log(n))$-transitivity and $\Theta(n)$-transitivity classes is empty for Hom shifts

Benjamin Hellouin, Silvère Gangloff and Piotr Oprocha

January 2022

Abstract

Multidimensional subshifts of finite type (SFT) have been studied for decades through the spectrum of topological dynamical properties such as transitivity and mixing. It is common intuition that, in dimension two, the maximum, over pairs of patterns of same size n, of the distance which may separate them in a configuration of a considered shift: *(i)* is bounded; *(ii)* depends logarithmically on n; *(iii)* or depends linearly on n.

In this article, we provide a partial proof of this fact on a particular class of nearest neighbor shifts called *Hom shifts*, in which neighbor symbols on the grid $\mathbb{Z}^2$ are forced to be neighbors in a fixed finite non-oriented simple graph G. More precisely we prove, for these shifts and for a "vertical transitivity" property, that when the above rate does not depend linearly on n, it is dominated by $\log(n)$. We also prove, contradicting a conjecture by R.Pavlov and M.Schraudner, that there exists a Hom shift for which this rate is $\Theta(\log(n))$.

1 Introduction

Multidimensional subshifts of finite type (SFT) form a class of dynamical systems which appear across various areas of mathematics, including *ergodic theory* (symbolic encoding of multidimensional maps such as the $\times 2 \times 3$ dynamical system), *statistical and quantum physics* (exactly solvable models such as dimers or square ice), as well as *logics* (results about the undecidability of the tiling problem). Formally a shift of finite type is the pair formed by: the $\mathbb{Z}^d$-shift action σ on some $\mathcal{A}^{\mathbb{Z}^d}$,for $\mathcal{A}$ finite set and $d \geq 1$; a subset X of $\mathcal{A}^{\mathbb{Z}^d}$, which is closed for the infinite product of the discrete topology and stable by the shift action.

Dynamical properties such as transitivity, minimality or mixing have been used in the study of complexity of dynamical systems, and in particular multidimensional shifts of finite type. For these particular systems, one relatively recent motivation for their study has been the effect of some of these properties such as *strong irreducibility* or *block gluing* one the possibility to compute algorithmically the entropy of the shift.

Minimal 'gluing' distance. In this context, these properties are translated into the possibility to '*glue*' two patterns (which are elements of $\mathcal{A}^{\mathbb{U}}$, $\mathbb{U}$ finite subset of $\mathbb{Z}^d$), each of which appear in at least one element of the shift, in the same element. For two fixed patterns of same size n, we can define a distance between occurrences of these patterns in the same element of the shift, and consider the minimum over these distances. For a fixed n, consider the maximum over all the pairs of patterns of size n of this minimal distance, which as a function of n is called *gap function*. When dealing with dynamical properties of multidimensional shifts of finite type, it appears that this maximum: *(i)* is bounded; *(ii)* is $\Theta(\log(n))$ as a function of n; *(iii)* or is $\Theta(n)$. In particular it is not known whether or not this function can have other possible behaviors.

Hom shifts. In general, tools are lacking in order to attempt proving this kind of intuition. The strategy followed here is to restrict the scope to a subclass of systems rather than multidimensional shifts of finite type in general. We choose to focus on *Hom shifts*, meaning that the set on which the shift acts is determined by a finite simple and non-oriented graph G: the set $\mathcal{A}$ is the set of vertices of G, and X is the set of elements x of $\mathcal{A}^{\mathbb{Z}^d}$ such that for two elements $\mathbf{i}, \mathbf{j}$ of $\mathbb{Z}^d$ which are neighbors, $x_{\mathbf{i}}$ and $x_{\mathbf{j}}$ are forced to be neighbors in G. Furthermore, we restrict to $d = 2$ and study a specific form of the transitivity property for simplicity.

Motivations for focusing on Hom shifts. There are multiple reasons which motivate this restriction to the subclass of Hom shifts: first, formal results have been obtained by N.Chandgotia and B.Marcus [CM18] on dynamical properties - transitivity, topological mixing and block gluing - of Hom shifts, relating them to algebraic topological properties of the graph G. Furthermore Hom shifts have many attractive properties: **(1)** They are invariant under rotations and symmetry, fact which makes Hom shifts a natural framework for models of isotropic physical systems (for instance, hard square shift, square ice, etc). **(2)** From the computer science point of view, we do not know of any embedding of universal computation that is invariant under isometries such as rotation and symmetry, which means that undecidability phenomena, which are the norm in dimension two and higher, should not appear (in particular we can expect positive results to be proven). **(3)** Dynamical properties of Hom shifts often translate into combinatorial properties of the corresponding graph, which is a fruitful approach in order to obtain positive results. Indeed, [CM18] expresses some transitivity and mixing properties (block gluing and phased black gluing) for Hom shifts in terms of asymptotics of the maximum distance between walks of length n on the graph, as n tends to infinity. They proved in particular that, when the graph is square-free, this maximum distance is either $O(1)$ or $\Theta(n)$.

Main results. For Hom shifts in general the situation appears to be different: we prove in the present paper that there exists a Hom shift whose gap function is $\Theta(\log(n))$, infirming a conjecture by R.Pavlov and M.Schraudner (section 6.3 in [CM18]) stating that the gap function is also $O(1)$ or $\Theta(n)$ for Hom shifts in general (without square-free assumption). Furthermore, we prove that there is no intermediate behavior strictly between $\Theta(\log(n))$ and $\Theta(n)$, meaning that if the gap function is not $\Theta(n)$, it is $O(\log(n))$. This is done by characterizing $O(\log(n))$ and $\Theta(n)$ behaviors of the gap functions with respectively the finiteness and infiniteness of the *quaternary cover* of the graph G, which consists roughly in the quotient of its universal cover by squares of G.

Significance of the results: it lies in the following: **(i)** it indicates that is is possible in principle to prove formally some distinctions between transitivity classes of multidimensional shifts of finite type in a precise way; **(ii)** the results deepen the formal connection done by B.Marcus and N.Chandgotia between transitivity classes of Hom shifts and algebraic topology of the underlying graphs, a connection that we expect to develop further in upcoming papers. **(iii)** This connection is expected to shed some light on the situation for multidimensional shifts of finite type in general, for which the effect of quantified block gluing on the algorithmic computability of entropy has been studied in [GS18]. In this context, transitivity classes are not well understood, leaving some difficult problems open: in particular it is not known if there are block gluing classes between $\Theta(\log(n))$ and $\Theta(n)$.

Structure of the text: We formulate our results precisely in Section 3, after some notations and definitions in Section 2, and present there the structure of the remainder of the paper.

2 Definitions and notations

For any set S, we will denote by S^* the set of finite words on S. For a word u, we will denote the number of its letters by $|u|$. We will usually write u as $u_0 \dots u_{|u|-1}$. The empty word is denoted by ϵ.

2.1 Graphs

In the whole text $G = (V_G, E_G)$ is a *non-oriented*, and simple (without multiple edges between two vertices) graph, where V_G denotes the set of vertices of G and E_G its set of edges. Depending on the context, this graph may not necessarily be finite.

Definition 2.1. *A* ***path*** *on the graph G is a word p in V_G^* such that for all $k \leq |p|-2$, $(p_k, p_{k+1}) \in E$. We denote by $l(p)$ the number $|p|-1$, and call it the* ***length*** *of p (equivalently this is the number of edges that the path follows). A* ***cycle*** *on G is a path c such that $c_0 = c_{l(c)}$. It is said to be* ***simple*** *when $i < j$ and $c_i = c_j$ imply that $i = 0$ and $j = l(c)$. Similarly we call* ***simple path*** *a path p such that for $i \neq j$, $p_i \neq p_j$.*

Denomination 2.2. *A path is said to be* ***non-backtracking*** *if it does not contain a word of the form aba, with $a, b \in V_G$.*

Remark 2.3. *Any path p can be transformed into a non-backtracking one by iterating a track-erasing transformation, which consists in replacing every subword aba by a, until there is none left. It is straightforward to check that the result, that we will denote by $\varphi(p)$, is the same regardless of the choice of subword at each step. We say that p* ***backtracks to*** *$\varphi(p)$.*

Notation 2.4. *We will denote by $\mathcal{C}_G^0$ the sets of simple cycles of G.*

Notation 2.5. *For a path p of length $n \geq 1$ on G written $p_0 \dots p_n$, we will denote respectively by $\rho_r(u)$ (resp. $\rho_l(u)$) the set of paths of the form*

$$p_1 \dots p_{n-1} x \quad (resp. x p_0 \dots p_{n-1}).$$

An element of this set is called a ***right shift*** *(resp. a left shift) of u.*

Notation 2.6. *For two paths p and q such that $q_0 = p_{l(p)}$, we will denote $p \odot q$ the path $p_0 \dots p_{l(p)-1} q_1 \dots q_{l(q)}$. For all cycle c, we will denote by c^n, $n \geq 1$ the cycles such that for all $n \geq 2$, $c^n = c \odot c^{n-1}$ and $c^1 = c$.*

Notation 2.7. *For any pair of different vertices $a, b \in V_G$, we denote by $\delta(a,b)$ the shortest length of a path in G which begins at a and ends at b. The* ***diameter*** *of the graph is:*

$$diam(G) := \max_{a,b \in V_G} \delta(a,b).$$

2.2 Hom shifts

In this section, G may be infinite: we will need this case when dealing with the quaternary covers of the graph G (Definition 5.7).

Notation 2.8. *The* ***Hom shift*** *corresponding to the graph G is the subset X_G of $V_G^{\mathbb{Z}^2}$ whose elements are the $x \in V_G^{\mathbb{Z}^2}$ such that for all $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{Z}^2$ adjacent positions, $x_{\boldsymbol{i}}$ and $x_{\boldsymbol{j}}$ are neighbors in G.*

Notation 2.9. *For any integer $n \geq 0$, we will also denote $X_G^{(n)}$ the subset of $\left(V_G^{n+1}\right)^{\mathbb{Z}}$ whose elements x are such that there exists some $z \in X_G$ such that $x = z_{|[\![0,n]\!] \times \mathbb{Z}}$.*

2.3 Transitivity gap

2.3.1 Definition

Notation 2.10. *For all integer $n \geq 1$ and $u, v \in V_G^{n+1}$, let us denote*

$$d_G(u,v) = \min\left\{k \geq 0 : \exists x \in X_G^{(n)} \mid x_0 = u \text{ and } x_k = v\right\}.$$

Definition 2.11. *The* ***transitivity gap function*** *of the Hom shift X_G is the function $\gamma_G : \mathbb{N}^* \to \mathbb{N}$ such that for all $n \geq 0$:*

$$\gamma_G(n) = \max_{u,v \in V_G^{n+1}} d_G(u,v).$$

In general it is difficult to compute exactly or obtain results about the function γ_G for a graph G. We instead look at equivalence classes for an equivalence relation which is natural to consider for the quantification of transitivity-like properties:

Notation 2.12. *Let us consider two functions $f, g : \mathbb{N}^* \to \mathbb{N}$. We will write $g(n) = O(f(n))$ when there exist rational numbers $c > 0$ and $K > 0$ such that for all n,*

$$g(n) \leq cf(n) + K.$$

We will write $g \equiv f$ when $f(n) = O(g(n))$ and $g(n) = O(f(n))$. The relation $\equiv$ is an equivalence relation, and we will denote by $\Theta(g(n))$ the equivalence class of g.

Denomination 2.13. *A Hom shift X_G is said to be $\Theta(g)$-transitive (resp. $O(g)$-transitive) when $\gamma_G \in \Theta(g(n))$ (resp. $O(g(n))$).*

We will also use the following notations:

Notation 2.14. *For all $n \geq 1$, let us denote by Δ_G^n the graph whose vertices are the paths on G of length n, and such that there is an edge between two paths p and q of length n if and only if $d_G(p, q) = 1$.*

Remark 2.15. *For all path p and q a right or left shift of p, $(p, q) \in \Delta_G^{l(p)}$.*

2.3.2 Some technical results

The following lemma tells, under some technical condition, that every path on G is at bounded distance, for d_G, of a cycle on G. This means that in this case, in order to evaluate the asymptotics of the function γ_G, we only need to consider the values of d_G on pairs of cycles.

Lemma 2.16. *Let us assume that every cycle in $\mathcal{C}_G^0$ has even length. For all $n \geq (diam(G) + 1)$ even and for all path p of length n on G, there exists a sequence of paths $p^{(0)}, p^{(1)}, ..., p^{(m)}$ with $m \leq (diam(G) + 1)$ such that $p^{(0)} = p$, $p^{(m)}$ is a cycle and for all $i \leq m - 1$, $d_G(p^{(i)}, p^{(i+1)}) = 1$.*

Proof. Let us consider a path p of length $n \geq \mathrm{diam}(G) + 1$ even, and assume that it is not a cycle, meaning that $p_0 \neq p_n$ (if p is a cycle the statement is trivial).

1. **There exists $m \leq \mathbf{diam}(G) + 1$ such that there exists a path from p_n to p_m which is of length m or $m - 1$:** Let us set, for all $k \geq 0$ with $k \leq \mathrm{diam}(G)$, $l_k = \delta(p_n, p_k) - k$. We have that $l_0 = \delta(p_n, p_0) > 0$ because p is not a cycle, and $l_{\mathrm{diam}(G)+1} \leq -1$ by definition of the diameter. Furthermore, for all $k \leq \mathrm{diam}(G)$, we have:

$$-1 \leq l_{k+1} - l_k \leq 1.$$

   Indeed, for all such k there exists a path of length $\delta(p_n, p_k) + 1$ from p_n to p_{k+1}, which implies $\delta(p_n, p_{k+1}) \leq \delta(p_n, p_k) + 1$. Furthermore we also have for a similar reason $\delta(p_n, p_k) \leq \delta(p_n, p_{k+1}) + 1$. This implies the above inequalities.

As a consequence there exists an integer m such that $l_m \in \{0, -1\}$, which means that $\delta(p_n, p_m)$ is m or $m-1$.

2. **We have $\delta(p_n, p_m) = m$:** Let us assume ad absurdum that $\delta(p_n, p_m) = m-1$. This implies that there exists a cycle of size $n-1$ in G, which is not possible by hypothesis, since $n-1$ is odd: indeed, consider the cycle which is obtained from p by replacing its part $p_0, ..., p_m$ by a path from p_n to p_m of smallest possible length. As a consequence, $\delta(p_n, p_m) = m$.

3. **Conclusion:** For the m found, let us denote q a path of length m from p_n to p_m. We moreover define, for all $k \leq m$, the path $p^{(k)} = p_k \dots p_n \odot q_0 \dots q_k$. It is straightforward that for all $k \leq m-1$, $p^{(k)}$ and $p^{(k+1)}$ are neighbors in Δ_G^n, and that $p^{(m)}$ is a cycle.

□

The following lemma will enable us to evaluate the asymptotics of the sequence $(\gamma_G(n))_{n\geq 1}$ using the asymptotics of $(\gamma_G(2n))_{n\geq 1}$:

Lemma 2.17. *For all n, we have $\gamma_G(2n) \leq \gamma_G(2n+1) \leq \gamma_G(2n) + 2$.*

Proof. The inequality $\gamma_G(2n) \leq \gamma_G(2n+1)$ is trivial. Let us prove the second one. Let us consider two paths p and q of length $2n+1$. We have that there exists a path $p^{(0)}, ..., p^{(l)}$ such that for all $i < l$, $(p^{(i)}, p^{(i+1)})$ is an edge in Δ_G^{2n}, $p^{(0)} = \rho_l(p)$ and $p^{(l)} = \rho_l(q)$. For all $i \geq 1$, let us set $q^{(i)} = p^{(i)} p_{2n}^{(i-1)}$ and $q^{(0)} = p$. For all $i < m$ we have $(q^{(i)}, q^{(i+1)})$ is an edge in Δ_G^{2n+1}, $q^{(0)} = p$ and $\rho_l(q^{(l)}) = \rho_l(q)$. As a consequence $d_G(q^{(l)}, q) \leq 2$: indeed there is a path in Δ_G^{2n+1} from $q^{(l)}$ to any left shift of $q^{(l)}$ to q. As a consequence $d_G(p, q) \leq l + 2$, and thus $\gamma_G(2n+1) \leq \gamma_G(2n) + 2$. □

3 Statement of the problem and results

The problem addressed in this paper is the following:

Problem 3.1. *1. What are the equivalences classes, for the relation $\equiv$, of functions γ_G, for G finite and undirected simple graph? 2. Is it decidable, provided such a graph G, which equivalence class the function γ_G belongs to?*

Let us notice that as a consequence of Remark 2.15, for all G and n, we have $\gamma_G(n) < +\infty$. Furthermore $\gamma_G(n) = O(n)$. In this paper we prove the following theorem, which deals with the "*upper part*" of the spectrum of possible behaviors of γ_G:

Theorem 3.2. *Whenever $\gamma_G \notin \Theta(n)$, $\gamma_G \in O(\log(n))$. Furthermore, there exists a finite non-oriented simple graph K such that $\gamma_K \in \Theta(\log(n))$.*

The remainder of this text is devoted to the proof of Theorem 3.2. It is decomposed as follows. In Section 4, we define a representation of cycles on the graph G in terms of elements of $\mathcal{C}_G^0$ which will be used extensively throughout the paper. In Section 5 we define the notion of decomposability into squares for simple cycles of the graph G, which will also be used in the following. In particular we define the quaternary cover $\mathcal{U}_4(G)$ which is roughly obtained by quotienting the universal cover of G by the squares of G. We then divide the proof of Theorem 3.2 in three section: In Section 6 we prove that if the quaternary cover of G is finite, then X_G is $O(\log(n))$-transitive ; In Section 7, the existence of a graph K such that γ_K is $\Theta(\log(n))$; and in Section 8, that if the quaternary cover of G is infinite, then X_G is $\Theta(n)$-transitive. Finally in Section 9 we discuss briefly problems that are left open.

4 Writing cycles as trees of simple cycles

In the following, we will use a representation of cycles on G as a rooted finite tree whose vertices are attached with a simple cycle of G, a representation that we call *cactus*. We thus introduce notation for rooted trees in Section 4.1, before providing details on the cactus representation in Section 4.2.

4.1 Notations for rooted trees

Definition 4.1. *A **rooted tree** is some pair (T, r) where T is a tree (undirected simple graph without cycle), and r is a vertex of T, called the **root**.*

Notation 4.2. *For a finite rooted tree (T, r), we denote by $n(T, r)$ the following number:*

$$n(T, r) := \max_{a \in V_T} \delta(a, r).$$

*It is called **depth** of the rooted tree (T, r). For all $k \leq n(T, r)$, we denote by $\ell_k(T, r)$ the set of vertices a of T such that $\delta(a, r) = k$. This set is called the k**th level** of the rooted tree. In particular $\ell_0(T, r) = \{r\}$.*

Denomination 4.3. *For all $k \leq n(T, r)$ positive, and all element b of $\ell_k(T, r)$, there is a unique element a of $\ell_{k-1}(T, r)$ such that that (a, b) is an edge of T. It is called the **predecessor** of b, and denoted $\boldsymbol{p}(b)$.*

Definition 4.4. *An **ordered rooted tree** is a tuple $(T, r, \leq)$ where (T, r) is a finite rooted tree and $\leq$ is a total order on V_T such that for all $k \leq n(T, r)$ positive and $a \in \ell_{k-1}(T, r)$, $b \in \ell_k(T, r)$, $a \leq b$.*

4.2 The cactus representation of cycles

4.2.1 Definition

Definition 4.5. *A **cactus** C on G is a tuple $(T, r, \leq, \xi, \chi)$, where $(T, r, \leq)$ is an ordered rooted tree, $\xi : V_T \to \mathcal{C}_G$, $\chi : V_T \backslash \{r\} \to \mathbb{N}$ functions satisfying the following properties:*

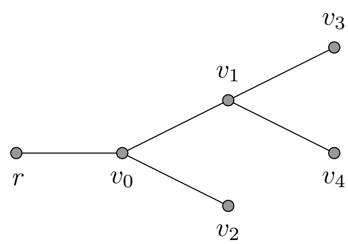
1. *The length of* $\xi(r)$ *is* 1.

2. *For all* $a \neq r$, $\chi(a) \leq l(\xi(\boldsymbol{p}(a)))$ *and* $\xi(\boldsymbol{p}(a))_{\chi(a)} = \xi(a)_0$.

3. *For all* $a, b \neq r$, *if* $a \leq b$ *and* $\boldsymbol{p}(a) = \boldsymbol{p}(b)$, *then* $\chi(a) \leq \chi(b)$.

Furthermore this cactus is said to be ***simple*** *when for all* $a \in V_T$, $\xi(a) \in \mathcal{C}_G^0$.

Exemple 4.6. *In order to illustrate this definition, let us consider* G *to be the Kenkatabami graph* K *represented on Figure 4. One example of cactus on this graph is* $(T, r, \leq, \xi, \chi)$ *such that:* T *is the tree*



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and $r \leq v_0 \leq v_1 \leq v_2 \leq v_3 \leq v_4$*; the function* ξ *defined by* $\xi(r) = \epsilon_2$, $\xi(v_0) = \epsilon_2\gamma_1\mu_3\omega\mu_4\gamma_2\epsilon_2$, $\xi(v_1) = \gamma_1\epsilon_1\delta_1\mu_2\gamma_1$, $\xi(v_2) = \gamma_2\mu_5\delta_3\epsilon_3\gamma_2$, $\xi(v_3) = \epsilon_1\gamma_3\mu_6\omega\mu_1\delta_1\epsilon_1$, $\xi(v_4) = \mu_2\omega\mu_1\delta_1\mu_2$*; the function* χ *is defined by* $\chi(v_0) = 0$, $\chi(v_1) = 1$, $\chi(v_2) = 5$, $\chi(v_3) = 1$ *and* $\chi(v_4) = 3$.

4.2.2 Definition of a cycle $\pi(C)$ *encoded* by a cactus C

The idea behind the definition of the previous section is to encode any cycle on G with a cactus, in the sense that for any cycle c there exists a cactus such that by 'gluing' the cycles $\xi(a)$, $a \in V_T \backslash \{r\}$, together we obtain the cycle c. The tree indicates which cycle is glued on which one (the predecessor), and the function χ indicates *where* it is glued. Furthermore, the order $\leq$ indicates in which order the cycles $\xi(a)$ are glued.

More formally, for every cactus C, we will construct a cycle $\pi(C)$. We will need the following notations:

Notation 4.7. *For two cycles* c *and* c' *such that there exists* $k \leq l(c)$, $c_k = c'_0$, *we will denote by* $c \oplus_k c'$ *the following cycle:*

$$c \oplus_k c' = c_0 \dots c_{k-1} c' c_{k+1} \dots c_{l(c)}.$$

For $C = (T, r, \leq, \xi, \chi)$, we define $\pi(C)$ inductively on the number $n(T, r)$. When $n(T, r) = 0$, we set $\pi(C) = \xi(r)$. Let us assume then that for some $k \geq 0$, $\pi(C)$ is defined whenever $n(T, r) \leq k$, and assume that $n(T, r) = k + 1$.

For every $a \in \ell_1(T, r)$ and $j \in \{\chi(b) : b \in V_T \mid a = \boldsymbol{p}(b)\}$, let us denote $C_{a,j} = (T_{a,j}, a, \leq_{a,j}, \xi_{a,j}, \chi_{a,j})$ the cactus such that: **(i)** $T_{a,j}$ is the maximal tree containing a in the graph obtained from T by disconnecting a from r and from the vertices b such that $\boldsymbol{p}(b) = a$ and $\chi(b) \neq j$; **(ii)** $\leq_{a,j}$ is obtained by restriction of $\leq$ on $T_{a,j}$; **(iii)** $\xi_{a,j}$ and $\chi_{a,j}$ are obtained by restriction of ξ and χ respectively on $T_{a,j}$, with the only difference that $\xi_{a,j}(a) = \xi(a)_j$ and for all b such that $\boldsymbol{p}(b) = a$ and $\chi(b) = j$, $\chi_{a,j}(b) = 0$.

For every $a \in \ell_1(T, r)$ and $j \in \{\chi(b) : b \in V_T \mid a = \boldsymbol{p}(b)\}$, we have straigthforwardly that $n(T_{a,j}, a) = k$, which means that $\pi(C_{a,j})$ is already defined. For every $a \in \ell_1(T, r)$ we set $\gamma_a = \gamma_a^{(l(\xi(a)))}$, where $\gamma_a^{(0)} = \xi(a)$ and for all $j < l(\xi(a))$, $\gamma_a^{(j+1)} = \gamma_a^{(j)} \oplus_j \pi(C_{a,j})$ whenever $j \in \{\chi(b) : b \in V_T, a = \boldsymbol{p}(b)\}$, and $\gamma_a^{(j+1)} = \gamma_a^{(j)}$ otherwise. In the end, we set $\pi(C)$ to be equal to:

$$\pi(C) = \gamma_{a_1} \oplus_{\xi(r)} \dots \oplus_{\xi(r)} \gamma_{a_s},$$

where $a_1 \leq \dots \leq a_s$ are the elements of $\ell_1(T, r)$.

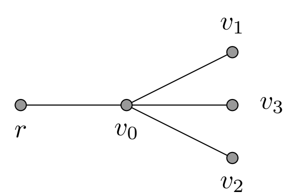
Exemple 4.8. *For C the cactus defined in Example 4.6, the cycle $\pi(C)$ is the following cycle:*

$$\pi(C) = \epsilon_2\gamma_1\epsilon_1\gamma_3\mu_6\omega\mu_1\delta_1\epsilon_1\delta_1\mu_2\omega\mu_1\delta_1\mu_2\gamma_1\mu_3\omega\mu_4\gamma_2\mu_5\delta_3\epsilon_3\gamma_2\epsilon_2.$$

4.2.3 Encoding any cycle with a simple cactus of bounded depth

We will say informally that any cactus C encodes the cycle $\pi(C)$. Such an encoding is not unique:

Exemple 4.9. *For C the cactus defined in Example 4.6, the cycle $\pi(C)$ can be also encoded by the cactus $C' \neq C$ (meaning that $\pi(C') = \pi(C)$), with $C' = (T', r', \leq', \xi', \chi')$, such that: T' is the tree*



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the function ξ' is given by $\xi'(r) = \epsilon_2$, $\xi'(v_0) = \epsilon_2\gamma_1\epsilon_1\gamma_3\mu_1\delta_1\mu_2\omega\mu_4\gamma_2\epsilon_2$, $\xi'(v_1) = \delta_1\epsilon_1\delta_1$, $\xi'(v_2) = \omega\mu_1\delta_1\mu_2\gamma_1\mu_3\omega$, $\xi'(v_3) = \mu_2\omega\mu_1\delta_1\mu_2$; the function χ is defined by $\chi'(v_0) = 0$, $\chi'(v_1) = 5$, $\chi'(v_2) = 7$, $\chi'(v_3) = 9$.

This means that it is possible in principle to encode any cycle with a cactus having useful properties:

Lemma 4.10. *For any cycle c on the graph G, there exists a simple cactus $C = (T, r, \leq, \xi, \chi)$ with $n(T, r) \leq |V_G|$ such that $\pi(C) = c$.*

Proof. Let us consider a cycle $c = c_0 \dots c_{l(c)}$. If $l(c) = 0$, the statement of the lemma is trivial. Let us assume that $l(c) > 0$.

1. **Preliminary notations:** Let us denote by $\leq$ the order on the set $\bigcup_{k\geq 1}\mathbb{N}^k$ such that for all $\mathbf{u},\mathbf{v}\in\mathbb{N}^k,\mathbb{N}^l$, $\mathbf{u}\leq\mathbf{v}$ if and only if $k<l$ or $k=l$ and $\mathbf{u}\leq_{\text{lex}}\mathbf{v}$. Let us also denote $r=(0)$.

   Let us construct recursively a finite sequence of cactuses (C_j) whose final term is a simple cactus C such that $\pi(C)=c$.

2. **First term:** Let us denote by $0=i_0<i_1\ldots<i_s=l(c)$ the consecutive integers i such that $c_i=c_0$. We set $C_1=(T_1,r,\leq,\xi_1,\chi_1)$ the cactus such that $V_{T_1}=\{(0)\}\cup\{(0,0),...,(0,s-1)\}$, $E_{T_1}=\{(0,(0,j)):0\leq j\leq s-1\}$, $\xi_1(0)=c_0$, $\xi_1((0,j))=c_{i_j}\ldots c_{i_{j+1}}$ for all $j\leq s-1$, and $\chi_1(j)=i_j$ for all $j\leq s-1$. Let us note that in every $\chi_1(j)$, the vertex c_0 appears only on the leftmost and rightmost positions.

3. **Recursion hypotheses:** Let us assume that we have constructed a finite sequence $(C_j=(T_j,r,\leq,\xi_j,\chi_j))_{j=1..l}$ of cactuses with $l\geq 1$ which satisfy the following properties:

   (a) For all $j\leq l$, $n(T_j,r)=j$.

   (b) For all $j\leq l$, $V_{T_j}\subset\bigcup_{k\geq 1}\mathbb{N}^k$.

   (c) For all $j\leq l$ and every leaf $\mathbf{v}$ of T_j, there is a set $V_{\mathbf{v}}\subset V_G$ with $|V_{\mathbf{v}}|=\delta(r,\mathbf{v})$ such that the vertices of $\xi_j(\mathbf{v})$, except for the leftmost and rightmost ones, do not belong to $V_{\mathbf{v}}$.

   (d) For all $\mathbf{v}\in\ell_i(T_j,r)$ with $i<j$, the cycle $\xi_j(\mathbf{v})$ is simple. Furthermore, for all $\mathbf{v}\neq r$ and $j\leq l$, $l(\xi_j(\mathbf{v}))\neq 0$.

   (e) For all $j\leq l$, $\pi(C_j)=c$.

4. **Recursion:** If for every vertex $a\in V_{T_l}$, $\xi(a)\in\mathcal{C}_G^0$, we stop the construction here. Otherwise we construct a cactus C_{l+1} that is appended to the sequence. Let us denote by $\mathbf{v}_1\leq ...\leq\mathbf{v}_s$ the elements of $\ell_l(T_l,r)$ whose value for ξ is not a simple cycle. Let us set $C^{(0)}=C_l$, and assume that we have constructed a finite sequence $C^{(i)}=(T^{(i)},(0),\leq,\xi^{(i)},\chi^{(i)})$, for i from 0 to $k<s$. Let us consider w the first letter, from left to right, in the cycle $\xi(\mathbf{v}_k)$ which is different from the endpoints of this cycle and appears in it at least twice. Let us write $\xi(\mathbf{v}_k)=w_0...w_m$. There exists a sequence $0<i_0<...<i_t<m$ such that for all j, $w_{i_j}=w$ and for any q different from all the i_j, $w_q\neq w$. We construct a cactus $C^{(k+1)}=(T^{(k+1)},(0),\leq,\xi^{(k+1)},\chi^{(k+1)})$ whose tree is $T^{(k+1)}$ obtained by adding t vertices to $T^{(k)}$ which are labeled by the $t-1$ different tuples obtained by concatenating $\mathbf{v}_{k+1}$ with $j<t$, and connecting them to the vertex $\mathbf{v}_{k+1}$. We then set $\xi^{(k+1)}(\mathbf{v}_{k+1})$ to be equal to $w_0...w_{i_0}w_{i_t+1}...w_k$ and for any other $\mathbf{v}$, $\xi^{(k+1)}(\mathbf{v})=\xi^{(k)}(\mathbf{v})$, and attribute to $\xi^{(k+1)}$ has the value $w_{i_j}...w_{i_{j+1}}$ on the $(j+1)$th of the above tuples. The value for $\chi^{(k+1)}$ is set to i_0 for each of them, and to $\chi^{(k+1)}(\mathbf{v})$ for any other vertex $\mathbf{v}$.

   Finally we set $C_{l+1}:=C^{(s)}$.

Verification of recursion hypotheses for C_{l+1}: It is straightforward that

$$n(T_{l+1}, r) = n(T_l, r) = l + 1,$$

and that for all $j < l + 1$ and $\mathbf{v} \in \ell_j(T_{l+1}, r)$, $\xi_{l+1}(\mathbf{v})$ is simple. The condition (b) is also straightforward. For $\mathbf{v}$ which is in V_{T_l}, the condition (c) is also satisfied. For $\mathbf{v} \in V_{T_{l+1}} \backslash V_{T_l}$, and w the first letter, from left to right, different from endpoints which lies at least twice in $\xi_{l+1}(\mathbf{p}(\mathbf{v}))$, the set $V_{\mathbf{p}(\mathbf{v})} \cup \{w\}$ satisfies condition (c). By construction $\pi(C_{l+1}) = c$.

5. **Conclusion:** The sequence (C_l) does not contain more than $|V_G|$ terms because of condition (c). Let us consider the last term C_m. Because of condition (d) and the fact that it is the last term, for all $\mathbf{v}$ vertex of T_m, the cycle $\xi_m(\mathbf{v})$ is simple. Since $\pi(C_m) = c$, the lemma is proved.

□

Remark 4.11. *Let us notice that in Lemma 4.10, the bound $|V_G|$ can be reached only if there is a vertex of G with a self-loop.*

5 Decomposability of simple cycles into squares

Denomination 5.1. *In the following, we will call **square** any simple cycle of length four.*

In the present section, we define the notion of decomposability into squares for simple cycles of G, and prove some properties related to this notion which will be useful in the remainder of the text.

5.1 Definition

Definition 5.2. *Let us consider two non-backtracking paths p, q. We say that p and q **differ by a square** when $\varphi(p \odot_k s) = q$ for some k and some square s. Figure 1 illustrates the types of pairs (p, q) of paths which differ by a square. Let us also denote $\sim_\square$ the equivalence relation between paths on G such that $p \sim_\square q$ if and only if there is a sequence $(p^{(k)})_{k \in [\![1,m]\!]}$ of paths such that for all $k < m$, $p^{(k+1)}$ and $p^{(k)}$ differ by a square, $p^{(0)} = p$ and $p^{(m)} = q$.*

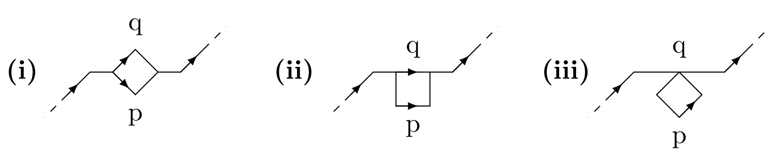

Figure 1: Partial representation of two paths p, q which differ by a square.

Definition 5.3. *A simple cycle c on G is said to be* ***decomposable into squares*** *when c is equivalent for $\sim_{\square}$ to a cycle of length 0.*

Lemma 5.4. *If a simple cycle is decomposable into squares, then it is of even length.*

Proof. It is sufficient to see that for p, q two paths which differ by a square, $l(p) - l(q)$ is even. $\square$

Definition 5.5. *A graph is said to be* ***square-dismantlable*** *when all of its simple cycles are decomposable into squares.*

As a consequence of Lemma 5.4 and Lemma 2.16 we have the following:

Corollary 5.6. *If the graph G is square-dismantlable, then for all n even and p a path of length n, there exists a cycle c of length n such that $d_G(p, c) \leq (diam(G) + 1)$.*

5.2 Quaternary cover

In this section, we define the quaternary cover of the graph G, which will be central in the dichotomy between $\Theta(n)$-transitive Hom shifts and $O(\log(n))$-transitive ones.

Definition 5.7. *The* ***quaternary cover*** *of G is the (common) isomorphic class of graphs, G_a, for $a \in V_G$, defined as follows:*

1. *its vertices are the equivalence classes for $\sim_{\square}$ of non-backtracking paths on G which begin at a;*
2. *there is an edge between two vertices $\overline{p}$ and $\overline{q}$ if and only if there exist two paths $p' \in \overline{p}$ and $q' \in \overline{q}$ and a vertex $b \in V_G$ such that $p' = q'b$ or $q' = p'b$.*

We denote this isomorphic class $\mathcal{U}_4(G)$.

In the following we will consider and represent $\mathcal{U}_4(G)$ as a graph whose vertices are not labelled. One can find some examples on Figure 2. One may notice that the third graph from the bottom is isomorphic to its quaternary cover. Proposition 5.8 tells that this is true for any square dismantlable graph. On the other hand a graph which is not square-dismantlable may have a finite or infinite quaternary cover (instance the second and fourth graphs from the bottom).

Proposition 5.8. *If the graph G is square-dismantlable, then $\mathcal{U}_4(G)$ is a finite graph. Furthermore there is an isomorphism between G and $\mathcal{U}_4(G)$.*

Proof. Let us fix any vertex a_0 of G. For all a in G, choose some non-backtracking path p_a from a_0 to a. The class of p_a for the equivalence relation $\sim_{\square}$ consists in paths which begin at a_0 and ends at a. As a consequence, all the classes $\overline{p_a}$ are different. Furthermore, because G is square-dismantlable, every path is equivalent for $\sim_{\square}$ to a simple path, meaning that every equivalence class is equal to some $\overline{p_a}$. Furthermore for all a, b there is an edge between a and b if and only if there is an edge between $\overline{p_a}$ and $\overline{p_b}$, which yields the statement. $\square$

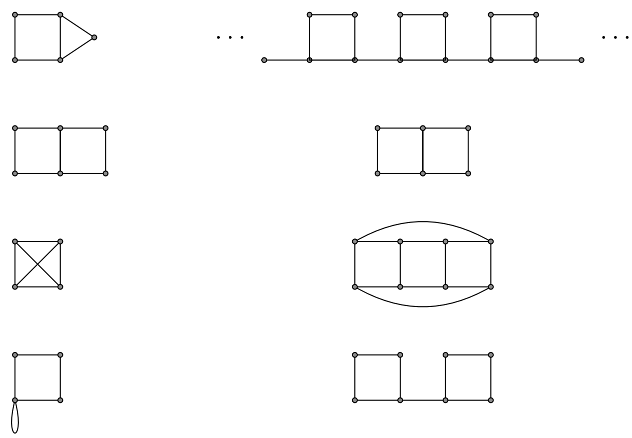

Figure 2: Some finite non-oriented simple graphs (on the left) and for each the corresponding quaternary cover (on the right).

In fact the function $G \mapsto \mathcal{U}_4(G)$ is a projection onto the set of square-dismantlable graphs:

Proposition 5.9. *The quaternary cover of a graph G is always square-dismantlable.*

Proof. Let us fix some vertex a and label the vertices of $\mathcal{U}_4(G)$ with classes of paths beginning with a. Let us also consider a simple cycle of $\mathcal{U}_4(G)$ and prove that it is decomposable into squares. The first vertex of this cycle is the class for $\sim_\square$ of a path $p = p^{(0)}$ on G which begins with a. The other vertices correspond to classes of paths $p^{(1)}, ..., p^{(k)}$ such that $p^{(k)} = p^{(0)}$ and for all $i < k$, $p^{(i+1)}$ is equivalent to $p^{(i)}a_i$ for some vertex a_i of G. As a consequence, the path $pa_1 \dots a_k$ on G is equivalent to p. This means that there exists a sequence of paths $q^{(0)}, ..., q^{(l)}$ such that for all $i \leq l-1$, $q^{(i)}$ and $q^{(i+1)}$ differ by a square, and $q^{(0)} = pa_1 \dots a_k$, $q^{(l)} = p$. For all i we can write $q^{(i)} = s^{(i)}d^{(i)}$, where $s^{(i)}$ has the same lengh as p. We also denote $r^{(i)}$ the path on $\mathcal{U}_4(G)$ which consists in the classes of paths $s^{(i)}$, $s^{(i)}d_0^{(i)}$, ... , $q^{(i)}$. We have that $q^{(l)} = s^{(l)} = p$ and as a consequence, $r^{(l)}$ is reduced to the class of the length 0 path a. Furthermore, for all i, if q_{i+1} and q_i differ only on their prefix of length $l(p)$, then $r^{(i+1)} = r^{(i)}$. Otherwise, $r^{(i+1)}$ and $r^{(i)}$ differ by a square. As a consequence one can extract from $r^{(i)}$ a subsequence such that consecutive elements differ by a square. Since $r^{(l)}$ is reduced to the class of a and $r^{(0)}$ is the path $p^{(0)}, ..., p^{(k)}$, this path is decomposable into squares. $\square$

The interest of the concept of quaternary cover comes from the following lemma, which allows us to relate dynamical transitivity-like properties of X_G to the ones of $X_{\mathcal{U}_4(G)}$.

Notation 5.10. *For any element of any labelling of the graph $\mathcal{U}_4(G)$, for all vertex of this graph, we denote $\theta(\overline{p})$ the last element of any of the paths in the class $\overline{p}$.*

Lemma 5.11. *Let us consider a finite graph G, and $\boldsymbol{i}_0 \in \mathbb{Z}^2$. For all $x \in X_G$, there exists a configuration z of $X_{\mathcal{U}_4(G)}$ such that $z_{\boldsymbol{i}_0} = x_{\boldsymbol{i}_0}$ and for all $\boldsymbol{i} \in \mathbb{Z}^2$, $\theta(z_{\boldsymbol{i}}) = x_{\boldsymbol{i}}$.*

Proof. Let us fix some configuration $x \in X_G$ and an element $\mathbf{i}_0$ of $\mathbb{Z}^2$. Let us use a labelling of $\mathcal{U}_4(G)$ with paths beginning with $x_{\boldsymbol{i}_0}$. For all $\mathbf{i} \in \mathbb{Z}^2$, all the paths $x_{\mathbf{i}_0} \dots x_{\mathbf{i}_m}$, for a sequence $\mathbf{i}_0, ..., \mathbf{i}_m$ of elements of $\mathbb{Z}^2$ such that for all $i < m$, $\mathbf{i}_k$ and $\mathbf{i}_{k+1}$ are neighbors in $\mathbb{Z}^2$, and such that $\mathbf{i}_m = \mathbf{i}$, are equivalent for the relation $\sim$. Then consider the configuration z of $X_{\mathcal{U}_4(G)}$ such that for all $\mathbf{i}$, $z_{\boldsymbol{i}}$ is the corresponding equivalence class. Straightforwardly we have that z is well defined and for all $\mathbf{i} \in \mathbb{Z}^2$, $\theta(z_{\boldsymbol{i}}) = x_{\boldsymbol{i}}$. □

The following is straightforward from Lemma 5.11.

Proposition 5.12. *For any function g such that $X_{\mathcal{U}_4(G)}$ is $O(g(n))$-transitive, X_G is also $O(g)$-transitive.*

6 When $|\mathcal{U}_4(G)| < +\infty$, X_G is $O(\log(n))$-transitive

In this section, we prove that if $\mathcal{U}_4(G)$ is finite, then X_G is $O(\log(n))$-transitive. Because of Proposition 5.12 and Proposition 5.9 it is sufficient to prove that if G is square-dismantlable, then X_G is $O(\log(n))$-transitive (Theorem 6.13). Before proving this theorem, we prove a series of lemmas.

We will need the following technical notations:

Notation 6.1. *For two cycles c, c' on G, we will denote by $c\mathcal{R}_0c'$ (resp. $c\mathcal{R}_1c'$) the following relation: c and c' have the same length, begin and end with the same letters and there exists a left (resp. right) shift of c' which is neighbor of c in $\Delta_G^{l(c)}$. Furthermore, for two paths c, c' of the same length, we denote $d_G^{\mathcal{R}}(c, c')$ the minimal number m such that there exists a sequence $(c^{(k)})_{0 \leq k \leq m}$ of cycles such that $c^{(0)} = c$, $c^{(m)} = c'$ and for all $k < m$, $c^{(k)}\mathcal{R}_0c^{(k+1)}$ or $c^{(k)}\mathcal{R}_1c^{(k+1)}$.*

Notation 6.2. *For all $c = c_0 \dots c_{l(c)}$ cycle on G, let us denote $\sigma_0(c)$ and $\sigma_1(c)$ the* ***circular left and right shifts*** *of c: $\sigma_1(c) = c_1 \dots c_{l(c)}c_1$ and $\sigma_0(c) = c_{l(c)-1}c_0 \dots c_{l(c)-1}$.*

Remark 6.3. *For all c cycle on G, $\sigma_0(c) \in \rho_l(c)$ and $\sigma_1(c) \in \rho_r(c)$. As a consequence, both $\sigma_0(c)$ and $\sigma_1(c)$ are neighbors of c in $\Delta_G^{l(c)}$. Furthermore, whenever $c\mathcal{R}_0c'$ (resp. $c\mathcal{R}_1c'$) then c and $\sigma_0(c')$ (resp. $\sigma_1(c')$) are neighbors in $\Delta_G^{l(c)}$.*

Lemma 6.4. *For all c, c' cycles having the same length, we have the inequality*

$$d_G(c, c') \leq 2d_G^{\mathcal{R}}(c, c').$$

Proof. Let us consider a sequence $(c^{(k)})_{0\leq k\leq m}$ of paths and $(\epsilon_k)_{0\leq k\leq m-1}$ a sequence of elements of $\{0,1\}$ such that $c^{(0)} = p$, $c^{(m)} = c'$ and for all $k < m$, $c^{(k)}\mathcal{R}_{\epsilon_k}c^{(k+1)}$. Let us consider the sequence $(d^{(k)})_{0\leq k\leq 2m}$ of cycles such that for all $k \leq m$,

$$d^{(k)} = \sigma_{\epsilon_k} \circ \ldots \circ \sigma_{\epsilon_1}(c^{(k)}),$$

and for $k > m$,

$$d^{(k)} = \sigma_{\epsilon_{k-m}} \circ \ldots \circ \sigma_{\epsilon_1}(d^{(m)}).$$

This sequence is a path from c to c' in $\Delta_G^{l(c)}$. This yields the statement of the lemma. □

Lemma 6.5. *For any cycles c, c', $r^{(1)}, ..., r^{(k)}$ which all begin and end at c_0, the distance for $d_G^{\mathcal{R}}$ between the two paths*

$$r^{(1)} \odot c \odot r^{(2)} \odot c \odot \ldots \odot r^{(k-1)} \odot c \odot r^{(k)}$$

$$r^{(1)} \odot c' \odot r^{(2)} \odot c' \odot \ldots \odot r^{(k-1)} \odot c' \odot r^{(k)}$$

is smaller than $d_G^{\mathcal{R}}(c,c')$.

Proof. It is sufficient to see that for c, c' paths such that $c\mathcal{R}_\epsilon c'$ that for

$$c^{(1)} := r^{(1)} \odot c \odot r^{(2)} \odot c \odot \ldots \odot r^{(k-1)} \odot c \odot r^{(k)},$$

and

$$c^{(2)} := r^{(1)} \odot q \odot r^{(2)} \odot q \odot \ldots \odot r^{(k-1)} \odot q \odot r^{(k)},$$

we have $c^{(1)}\mathcal{R}_\epsilon c^{(2)}$. □

Lemma 6.6. *For any two cycles c, c' which differ by a square and t a cycle such that $l(t) = 2$ beginning and ending at c_0. For all $k \geq \max(l(c') - l(c), 0)$,*

$$d_G^{\mathcal{R}}(t^k \odot c, t^{k+(l(c)-l(c'))/2} \odot c') \leq 2.$$

Proof. Let us notice that when two cycles c, c' differ by a square, then we have $(l(c') - l(c)) \in \{-4, -2, 0, 2, 4\}$. Let us consider each of these cases one by one:

(i) $\boldsymbol{(l(c') - l(c)) = 0}$: for all $k \geq 0$, we have $(t^k \odot c)\ \mathcal{R}_0\ (t^k \odot c')$.

(ii) $\boldsymbol{(l(c') - l(c)) = 2}$: for all $k \geq 1$, $(t^k \odot c)\ \mathcal{R}_0\ (t^{k-1} \odot c')$.

(iii) $\boldsymbol{(l(c') - l(c)) = 4}$: there is some j and some vertices x, y, z of G such that $c' = c_0 ... c_j xyz c_{j+1} ... c_{l(c)}$. Then for all $k \geq 0$, for $u^{(1)} := t^k \odot c$,

$$u^{(2)} := t^{k-1} \odot c_0 ... c_j x c_j c_{j+1} ... c_{l(c)},$$

$$u^{(3)} := t^{k-2} \odot c_0 ... c_j xyz c_j c_{j+1} ... c_{l(c)},$$

we have $u^{(1)}\mathcal{R}_0 u^{(2)}\mathcal{R}_0 u^{(3)}$.

(iv) $\boldsymbol{(l(c') - l(c)) = -2}$: for all $k \geq 1$, $(t^k \odot c)\ \mathcal{R}_0\ (t^{k+1} \odot c')$.

(v) $\boldsymbol{(l(c') - l(c)) = -4}$: there is some j and some vertices x, y, z of G such that $c = c'_0 ... c'_j xyz c'_{j+1} ... c'_{l(c')}$. Then for all $k \geq 0$, for $u^{(1)} := t^k \odot c$, and

$$u^{(2)} := t^{k+1} \odot c'_0 ... c'_j x c'_{j+1} ... c'_{l(c')}),$$

$$u^{(3)} := t^{k+2} \odot c'_0 ... c'_j c'_{j+2} z ... c'_{l(c')},$$

we have $u^{(1)} \mathcal{R}_0 u^{(2)} \mathcal{R}_0 u^{(3)}$.

These facts all imply that $d_G^{\mathcal{R}}(t^k \odot c, t^{k+(l(c)-l(c'))/2} \odot c') \leq 2$. □

Lemma 6.7. *Let c be a cycle decomposable into squares and let t be a cycle such that $l(t) = 2$ and beginning and ending at c_0. For $c^{(0)}, \ldots, c^{(m)}$ any sequence of cycles such that for all $i \leq m-1$, $c^{(i+1)}$ and $c^{(i)}$ differ by a square, and such that $c^{(0)} = c$ and $c^{(m)}$ is of length 0, then*

$$d_G^{\mathcal{R}}(t^{2m} \odot c, t^{2m+l(c)/2}) \leq 2m.$$

Proof. For all $j \leq m$, let us denote $\gamma^{(j)} := t^{2m+(l(c)-l(c^{(j)}))/2} \odot c^{(j)}$. This sequence is well-defined because for all j, $l(c^{(j+1)}) - l(c^{(j)}) \leq 4$, and as a consequence

$$2m + (l(c) - l(c^{(j)}))/2 \geq 0.$$

As a consequence of Lemma 6.6, for all $j \leq m-1$, $d_G^{\mathcal{R}}(\gamma^{(j)}, \gamma^{(j+1)}) \leq 2$. Thus by triangular inequality we get

$$d_G^{\mathcal{R}}(t^{2m} \odot c, t^{2m+l(c)/2}) \leq 2m.$$

□

Corollary 6.8. *Let c be a cycle decomposable into squares and let t be a cycle such that $l(t) = 2$ beginning and ending at c_0. For $c^{(0)}, \ldots, c^{(m)}$ any sequence of cycles such that for all $i \leq m-1$, $c^{(i+1)}$ and $c^{(i)}$ differ by a square, and such that $c^{(0)} = c$ and $c^{(m)}$ is of length 0, then*

$$d_G^{\mathcal{R}}(t^{2m} \odot c^m, t^{2m+ml(c)/2}) \leq 2m^2.$$

Proof. This is a straightforward consequence of Lemma 6.7 and Lemma 6.5. □

Lemma 6.9. *Let c be a non-trivial cycle decomposable into squares and let t be a trivial cycle beginning and ending at c_0. There exists an integer $m_c \geq 1$ such that for all $n \geq 0$, a sequence of positive integers $(k_j)_{1 \leq j \leq l}$ such that*

$$k_1 + \ldots + k_l = (2^n - 1) m_c$$

and $(r^{(j)})_{1 \leq j \leq l+1}$ cycles beginning and ending at c_0, the distance for d_G between the paths

$$d_t^{2m_c} \odot r^{(1)} \odot c^{k_1} \odot \ldots c^{k_l} \odot r^{(l+1)}$$

and

$$t^{2m_c} \odot r^{(1)} \odot t^{k_1 l(c)/2} \odot \ldots t^{k_l l(c)/2} \odot r^{(l+1)}$$

is smaller than $8m_c^2 n$.

Proof. Let us denote Γ the function which to any cycle d of length $(2^n+1)m_c$ such that for all j between 0 and m,

$$d_{4m_c+(k_1+\ldots+k_j)l(c)} = d_0$$

associates the cycle

$$d_{|\llbracket 0,4m_c\rrbracket} \odot r^{(1)} \odot d_{|\llbracket 4m_c,4m_c+k_1 l(c)\rrbracket} \odot \ldots \odot d_{|\llbracket 4m_c+(k_1+\ldots k_{l-1})l(c),4m_c+(k_1+\ldots k_l)l(c)\rrbracket} \odot r^{(l+1)}.$$

For all $n \geq 0$ and for all $k \leq n$, let us denote $\gamma_{n,k}$ the cycle

$$\gamma_{n,k} = t^{2m_c} \odot c^{2^{n-k}-1} \odot \left(t^{m_c l(c)/2} \odot c^{2^{n-k}-1}\right)^{2^k-1}$$

Let us prove that for all $k < n$,

$$d_G(\Gamma(\gamma_{n,k}), \Gamma(\gamma_{n,k+1})) \leq 8m_c^2.$$

Using the cycles

$$\gamma_{n,k+1,j} = t^{2m_c-j} \odot c^{2^{n-k}-1} \odot t^j \odot c^{2^{n-k}-1} \odot \left(t^{m_c l(c)/2-j} \odot c^{2^{n-k}-1} \odot t^j \odot c^{2^{n-k}-1}\right)^{2^k-1}$$

we get that the distance for $d_G^{\mathcal{R}}$ between $\Gamma(\gamma_{n,k+1})$ and

$$\Gamma\left(c^{2^{n-k}-1} \odot t^{2m_c} \odot c^{2^{n-k}-1} \odot \left(t^r \odot c^{2^{n-k}-1} \odot t^{2m_c} \odot c^{2^{n-k}-1}\right)^{2^k-1}\right)$$

is smaller than $2m_c$, where $r := m_c l(c)/2 - 2m_c$.
Similarly as in Corollary 6.8, we get that:

$$d_G^{\mathcal{R}}(\Gamma(\gamma_{n,k}), \Gamma(\gamma_{n,k+1})) \leq 4m_c^2.$$

By triangular inequality, we obtain the inequality

$$d_G^{\mathcal{R}}(\Gamma(t^{2m_c} \odot c^{(2^n-1)m_c}), \Gamma(t^{m_c(2+(2^n-1)l(c)/2)})) \leq 4m_c^2 n.$$

Finally,

$$d_G(\Gamma(t^{2m_c} \odot c^{(2^n-1)m_c}), \Gamma(t^{m_c(2+(2^n-1)l(c)/2)})) \leq 8m_c^2 n.$$

□

Lemma 6.10. *If G is square dismantlable then there is some $\alpha_G > 0$ such that for every simple cycle c on G, t cycle such that $l(t) = 2$ which begins and ends at c_0, integer n, any sequence of positive integers $(k_j)_{1\leq j\leq l}$ such that*

$$k_1 + \ldots + k_l = n,$$

and $(r^{(j)})_{1\leq j\leq l+1}$ cycles beginning and ending at c_0, the distance for d_G between the paths

$$t^{2m_c} \odot r^{(1)} \odot c^{k_1} \odot \ldots c^{k_l} \odot r^{(l+1)}$$

and

$$t^{2m_c} \odot r^{(1)} \odot t^{k_1 l(c)/2} \odot \ldots t^{k_l l(c)/2} \odot r^{(l+1)}$$

is smaller than $\alpha \cdot \log_2(n)$.

Proof. Let us set α the following number:

$$\alpha := \max_{c \in \mathcal{C}_G^0} 8m_c^2.$$

Since $|\mathcal{C}_G^0| < +\infty$, α_G is well defined. Let us consider a cycle c, assume first that $n \geq m_c$ and decompose n as $n = km_c - r$ with $k \geq 0$ and $r < m_c$. For t provided by Lemma 6.9,

$$t^{2m_c} \odot r^{(1)} \odot c^{k_1} \odot \ldots c^{k_l} \odot r^{(l+1)}$$

is a prefix of:

$$t^{2m_c} \odot r^{(1)} \odot c^{k_1} \odot \ldots c^{k_l} \odot r^{(l+1)} \odot c^{r+m_c\left(2^{\lceil \log_2(k) \rceil +1} - k\right)}.$$

By Lemma 6.9, the distance for d_G between this cycle and

$$t^{2m_c} \odot r^{(1)} \odot t^{k_1 l(c)/2} \odot \ldots t^{k_l l(c)/2} \odot r^{(l+1)} \odot t^{\left(r+m_c\left(2^{\lceil \log_2(k) \rceil +1} - k\right)\right) l(c)/2}$$

is smaller than $\alpha \log_2(k)$. By restriction, the distance for d_G between

$$t^{2m_c} \odot r^{(1)} \odot c^{k_1} \odot \ldots c^{k_l} \odot r^{(l+1)}$$

and

$$t^{2m_c} \odot r^{(1)} \odot t^{k_1 l(c)/2} \odot \ldots t^{k_l l(c)/2} \odot r^{(l+1)}$$

is also smaller than $\alpha \cdot \log_2(k) \leq 2\alpha \cdot \log_2(n)$. This inequality is still satisfied replacing α by any $\alpha_G > \alpha$. We thus take α_G sufficiently large so that it is also satisfied for $n < m_c$. This ends the proof. $\square$

Remark 6.11. *For any cycle c and a t a cycle such that $l(t) = 2$ and which begins and ends at c_0, and any $m \geq 0$, we have straightforwardly*

$$d_G^{\mathcal{R}}(t^m \odot c, c \odot t^m) \leq m$$

by "moving" the copies of t from the left to the right one by one. However by repetitive application of circular shift σ_0 we obtain another useful bound:

$$d_G^{\mathcal{R}}(t^m \odot c, c \odot t^m) \leq l(c).$$

We will use this remark with the following generalization of Lemma 6.5:

Lemma 6.12. *Let us consider three sequences of cycles $(c^{(i)})_{1 \leq i \leq k}, (d^{(i)})_{1 \leq i \leq k}$, $(r^{(i)})_{1 \leq i \leq k+1}$ which all begin and end at the same point c_0, and a double sequence*

$$(c^{(i,j)})_{\substack{0 \leq j \leq l_i \\ 1 \leq i \leq k}}$$

such that for all i and $j < l_i$, $c^{(i,j)} \mathcal{R}_0 c^{(i,j+1)}$, $c^{(i,0)} = c^{(i)}$ and $c^{(i,l_i)} = d^{(i)}$. The distance for $d_G^{\mathcal{R}}$ between the two paths

$$r^{(1)} \odot c^{(1)} \odot r^{(2)} \odot c^{(2)} \odot \ldots \odot r^{(k)} \odot c^{(k)} \odot r^{(k+1)}$$

$$r^{(1)} \odot d^{(1)} \odot r^{(2)} \odot d^{(2)} \odot \ldots \odot r^{(k)} \odot d^{(k)} \odot r^{(k+1)}$$

is smaller than $\max_i l_i$.

Proof. Let us denote $L := \max_i l_i$. It is sufficient to see that for all i, it is possible to complete the sequence $(c^{(i,j)})_{0 \leq j \leq l_i}$ into $(c^{(i,j)})_{0 \leq j \leq L}$ using iterates of σ_0 on $c^{(i,l_i)}$. The remainder is straightforward. □

Theorem 6.13. *If G is square dismantlable, X_G is $O(\log(n))$-transitive.*

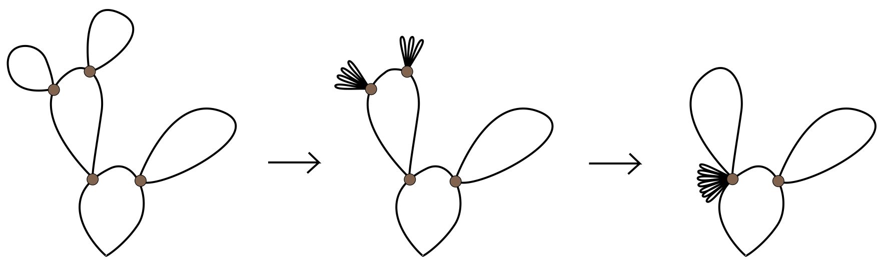

Figure 3: Illustration of the steps 3 (on the left),4 (on the right) in the proof of Theorem 6.13.

Proof.

1. **Compressing any cycle into an iteration of a length 2 cycle:** Let us consider a cycle c of length n on the graph G. Let us prove that

$$d_G(t^{2m_c} \odot c, t^{2m_c + l(c)/2}) \leq |V_G||\mathcal{C}_G^0|\alpha_G \log(n) + |V_G||\mathcal{C}_G^0| \cdot \max_{c \in \mathcal{C}_G^0} l(c).$$

This implies, together with Lemma 2.16 and Lemma 2.17, that $\gamma_G(n) = O(\log(n))$.

2. **Using the cactus representation to write c relatively to its leaves:** From Lemma 4.10 we know that there exists a simple cactus $C = (T, r, \leq, \xi, \chi)$ such that $\pi(C) = c$ with depth bounded by $|V_G|$. Let us set $k := n(T,r)$ and denote by $w^{(1)} \leq ... \leq w^{(m_c)}$ the elements of $\ell_k(T,r)$. The cycle c can thus be written as follows:

$$c = \gamma^{(1)} \odot \xi(w^{(1)}) \odot \gamma^{(2)} \odot \ldots \odot \gamma^{(m_c)} \odot \xi(w^{(m_c)}) \odot \gamma^{(m_c+1)},$$

and since the cactus C is simple, each of the cycles $\xi(w^{(i)})$ is simple.

3. **Compressing the cycles associated to leaves by ξ:** For each element of $\mathcal{C}_G^0$ successively, we use Lemma 6.10, and obtain ultimately that the distance d_G between $t^{2m_c} \odot c$ and the cycle $t^{2m_c} \odot u$, where

$$u := \gamma^{(1)} \odot \left(t^{(1)}\right)^{a_1} \odot \gamma^{(2)} \odot \left(t^{(2)}\right)^{a_2} \odot \gamma^{(3)} \odot \ldots \odot \gamma^{(m_c)} \odot \left(t^{(m_c)}\right)^{a_{m_c}} \odot \gamma^{(m_c+1)},$$

and each of the $t^{(i)}$ is of length 2, is at most $|\mathcal{C}_G^0|\alpha_G \log(n)$.

Let us consider the cactus C' which is obtained from C by replacing each leaf $w^{(i)}$ of C by a_i vertices $w^{(i,j)}$, $1 \leq j \leq a_i$ that are connected to $\boldsymbol{p}(w^{(i)})$ such that for all i, j, $\chi(w^{(i,j)}) = \chi(w_i)$ and $\xi(w^{(i,j)}) = t^{(i)}$. For each i the vertices $w^{(i,j)}$ are ordered in any way, and for $i \neq i'$, $w^{(i,j)} \leq w^{(i',j)}$ if and only if $w_i \leq w_{i'}$. Furthermore for every vertex a which was already in the tree of C, and any i, j, $a \leq w^{(i,j)}$. We have $\pi(C') = u$.

4. Let us consider then the cactus C'' obtained from C' by disconnecting the $w^{(i,j)}$ from $\boldsymbol{p}(w^{(i)})$ and reconnecting them to $\boldsymbol{p}(\boldsymbol{p}(w^{(i)}))$. Their value for ξ is not changed, but we change the value of $w^{(i,j)}$ for χ to $\chi(\boldsymbol{p}(w^{(i)}))$. We also set $w^{(i,j)} \leq \boldsymbol{p}(w^{(i)})$. Let us also denote $u'' := \pi(C'')$. Using multiple times Remark 6.11 and Lemma 6.12, we get that the distance for d_G between $t^{2m_c} \cdot u'$ and $t^{2m_c} \cdot u''$ is smaller than:

$$|\mathcal{C}_G^0| \cdot \max_{c \in \mathcal{C}_G^0} l(c)$$

5. As a consequence there exists a cycle c' represented by a cactus of depth $n(T, r) - 1$ such that

$$d_G(c, c') \leq |\mathcal{C}_G^0|\alpha_G \log(n) + \mathcal{C}_G^0| \cdot \max_{c \in \mathcal{C}_G^0} l(c).$$

By repeating this, we obtain that

$$d_G(t^{2m} \odot c, t^{m+l(c)}/2) \leq |V_G||\mathcal{C}_G^0|\alpha_G \log(n) + |V_G||\mathcal{C}_G^0| \cdot \max_{c \in \mathcal{C}_G^0} l(c).$$

This completes the proof the the first point, and thus the proof of the theorem.

□

7 There exists K with X_K $\Theta(\log(n))$-transitive

Let us prove here the following, which implies that the set of graphs whose Hom shift is $\Theta(\log(n))$-transitive is not empty, disproving the conjecture of R.Pavlov and M.Schraudner.

Theorem 7.1. *There exists a graph K such that X_K is $\Theta(\log(n))$-transitive.*

Proof. Let us denote K the graph shown on Figure 4 (we will use the notations of the figure for the vertices), that we also call Kenkatabami graph. Since the graph is straightforwardly square dismantable, Theorem 6.13 applies, and X_K is $O(\log(n))$-transitive. It is sufficient then to prove that $\log(n) = O(\gamma_K(n))$.

1. **Notations:** Let us denote by c the anti-clockwise exterior cycle of the graph K:

$$c = \epsilon_1\gamma_1\epsilon_2\gamma_2\epsilon_3\gamma_3\epsilon_1.$$

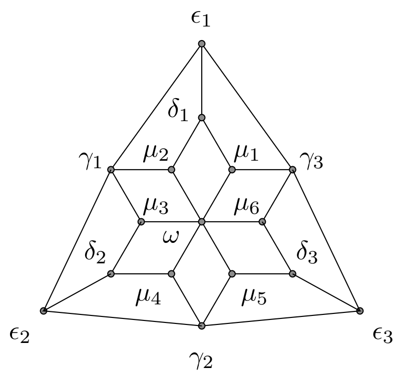

Figure 4: The Kenkatabami graph.

For any path p on the graph K, we will call c-blocks of p the maximal words of the form c^n which appear in p; the order of a block is the number of times c appears in it. We will also denote by $\mu_c(p)$ the maximal order of a c-block in a path p.

2. **Lower bound on $\mu_c(p_2)$ for p_2 neighbor of p_1:**

   (i) Claim: We claim that for n large enough and p, q vertices of $\Delta_K^{nl(c)}$ which are neighbors in this graph, then:

$$\mu_c(p) \geq \frac{1}{2}\mu_c(q) - 3.$$

   (ii) Characteritics of the neighbors of c^n in $\Delta_K^{nl(c)}$: For all $n \geq 4$, the neibhors of c^n in the graph $\Delta_K^{nl(c)}$ are of the form uwv, uw or wv, u or v, where the words u, v are respectively right shift of a prefix of c^n and left shift of a sufix of c^n, and $|w| \leq 2$ in the first case, $|w| = 2$ in the second and third cases.

   Let us denote by q the cycle c^n and consider p one neighbor of q in $\Delta_K^{nl(c)}$. We assume that it is not a shift of q.

   Since the only common neighbor of ϵ_1 and ϵ_2 (resp. ϵ_1 and ϵ_3, ϵ_2 and ϵ_3) is γ_1 (resp. γ_3, γ_2), and that the only common neighbor of γ_1 and γ_2 (resp. γ_1 and γ_3, γ_2 and γ_3) is ϵ_1 (resp. ϵ_3, ϵ_2), then if for some k we have $p_{[\![2k,2k+2]\!]} = q_{1+[\![2k,2k+2]\!]}$, then we have the same equality replacing k by j for all $j \leq k$. Similarly if for some k we have $p_{2+[\![2k,2k+2]\!]} = q_{1+[\![2k,2k+2]\!]}$, then we have the same equality replacing k by j for all $j \geq k$.

   Thus if for all k, $p_{[\![2k,2k+2]\!]} = q_{1+[\![2k,2k+2]\!]}$ or $p_{2+[\![2k,2k+2]\!]} = q_{1+[\![2k,2k+2]\!]}$, since p is not a shift of q, there is some k such that for all $j \leq k$ we have $p_{[\![2k,2k+2]\!]} = q_{1+[\![2k,2k+2]\!]}$ and for all $j > k$ we have $p_{2+[\![2k,2k+2]\!]} = q_{1+[\![2k,2k+2]\!]}$. As a consequence p is of the form uwv with $|w| = 1$.

Let us assume the contrary, and assume that there exists some j such that $p_{2+[\![2j,2j+2]\!]} \neq q_{1+[\![2j,2j+2]\!]}$ and $p_{[\![2j,2j+2]\!]} \neq q_{1+[\![2j,2j+2]\!]}$. We also assume that $q_{1+[\![2j,2j+2]\!]} = \gamma_1\epsilon_2\gamma_2$ (other cases are processed similarly). This implies that p_{2j+2} can not be equal to γ_1, because otherwise we would have $p_{2j+3} = \epsilon_2$ and then $p_{2j+4} = \gamma_2$, which is impossible by hypothesis ($p_{2+[\![2j,2j+2]\!]} \neq q_{1+[\![2j,2j+2]\!]}$). For the similar reasons, p_{2j+2} can not be equal to γ_2 (using the hypothesis $p_{[\![2j,2j+2]\!]} \neq q_{1+[\![2j,2j+2]\!]}$).

As a consequence $p_{2j+2} = \delta_2$. This forces $p_{2j+1} \in \{\epsilon_2, \mu_3\}$. Furthermore the only common neighbor of ϵ_1 and ϵ_2, as well as ϵ_1 and μ_3, is γ_1, which forces $p_j = \gamma_1$. Using repeatedly similar arguments, we obtain that $p_{[\![1,2j+1]\!]}$ is a shift of a prefix of q. We have also forced $p_{2j+3} \in \{\mu_4, \epsilon_2\}$. Since γ_2 is the only common neighbor of μ_4 and ϵ_3, as well as ϵ_2 and ϵ_3, this forces $p_{2j+4} = \gamma_2$. Similarly we have that $p_{[\![2j+4,n|c|]\!]}$ is a shift of a suffix of q. We thus have that p is of form uwv with $|w| = 2$.

(iii) Lower bound on $\mu_c(p)$ for p a neighbor of c^n in $\Delta_K^{nl(c)}$:

Let us consider u a prefix or suffix of c^n. It is clear that $\mu_c(u) \geq \lfloor \frac{|u|}{|c|} \rfloor$. Then if u is a shift of such a prefix or suffix, $\mu_c(u) \geq \frac{|u|}{|c|} - 2$. As a consequence of point **(ii)**, any neibhor p of c^n in $\Delta_K^{nl(c)}$ has a subword which is a shift of a prefix or suffix of c^n and whose length is at least $\frac{n|c|-2}{2}$. Thus we have that

$$\mu_c(p) \geq \frac{1}{|c|}\frac{n|c|-2}{2} - 2 \geq \frac{n}{2} - 3.$$

(iv) Proof of the claim: Let us consider p and q two words of length n large enough which are neighbors in $\Delta_K^{nl(c)}$. For $n = \mu_c(p)$, p has c^n as a subword. The value of μ_c on the corresponding subword of q is thus larger than $\frac{n}{2} - 3$, which means that $\mu_c(q) \geq \frac{n}{2} - 3 = \frac{\mu_c(p)}{2} - 3$.

3. **Lower bound on $\gamma_K(n)$:** Let us consider n large enough. In this paragraph, we denote by p_1 the cycle c^n. Thus $\mu_c(p_1) = n$. If we consider $p_1 ... p_m$ a sequence of paths such that p_{i+1} is neighbor of p_i in $\Delta_K^{nl(c)}$, then $\mu_c(p_{i+1}) \geq \frac{1}{2}\mu_c(p_i) - 3$, which indicates that:

$$\mu_c(p_{i+1}) \geq \frac{1}{2^i}\mu_c(p_1) - \sum_{k=0}^{i-1}\frac{3}{2^k} \geq \frac{n}{2^i} - 6.$$

As a consequence the distance d between the p_1 and any path which does not contain any c has to satisfy:

$$\frac{n}{2^d} - 6 \leq 0,$$

which implies that:

$$d \geq \log_2(n) - \log_2(6).$$

As a consequence for n large enough

$$\gamma_K(n) \geq \log_2(n) - \log_2(6).$$

This implies that $\log(n) = O(\gamma_K(n))$.

□

8 When $|\mathcal{U}_4(G)| = +\infty$, X_G is $\Theta(n)$-transitive

We have proven in Section 6 that when the quaternary cover of G is infinite, X_G is $\Theta(n)$-transitive, completing the dichotomy between the $\Theta(\log(n))$ and $\Theta(n)$ transitivity classes:

Theorem 8.1. *If the quaternary cover of G is infinite, then X_G is $\Theta(n)$-transitive.*

Proof. Let us label $\mathcal{U}_4(G)$ with the paths beginning at a vertex a of G. Since this graph is infinite and the degree of each of its vertices is bounded from above by $|V_G|$, for all n there exists a vertex $\overline{p_n}$ which is at distance $2n$ from $\overline{a}$. Let us denote by $a_1, ..., a_{2n}$ a path of length n between these two vertices. Let us prove that the distance between $u := aa_1...a_{2n}$ and $v := (aa_1a)^n$ in Δ_G^{2n} is at least n. Let us assume on the contrary that there exists some element x of X_G such that for some $k < n$, x coincide with u and v respectively on $[\![0, 2n]\!] \times \{0\}$ and $[\![0, 2n]\!] \times \{k\}$. We know (Lemma 5.11) that there exists a unique configuration z of $X_{\mathcal{U}_4(G)}$ such that $z_{0,k} = x_{0,k} = a$ and for all $\mathbf{i} \in \mathbb{Z}^2$, $\theta(z_{\mathbf{i}}) = x_{\mathbf{i}}$.

As a consequence, $z_{2n,k}$ is the class $\overline{p_n}$ and since $\overline{v} = \overline{v_0}$, $z_{2n,0}$ is the class of a path of length smaller or equal to k. By the triangular inequality, the distance between $\overline{p_n}$ and $\overline{a}$ is strictly smaller than $2n$ which is impossible.

We conclude that $\gamma_G(2n) \geq n$ for all $n \geq 1$, which implies that X_G is $\Theta(n)$-transitive. □

9 Open problems

We leave two main problems open. The *first problem* is the classification of transitivity classes for Hom shifts. We conjecture the following:

Conjecture 9.1. *The only transitivity classes of Hom shifts are $O(1)$, $\Theta(\log(n))$ and $\Theta(n)$.*

There are several examples of $O(1)$-transitive Hom shifts (one can find some examples on Figure 5). From these ones we can also construct other examples of $\Theta(\log(n))$-transitive Hom shifts by 'gluing' these graphs together (see for instance Figure 7 and Figure 6). What intuitively separates these two sets of graphs is the way squares are assembled to form the graph. However this has been difficult thus far to formalize this intuition.

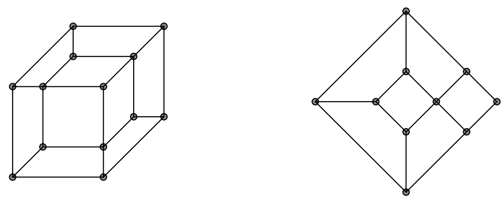

Figure 5: Example of a graph G whose associated Hom shift is $O(1)$-transitive

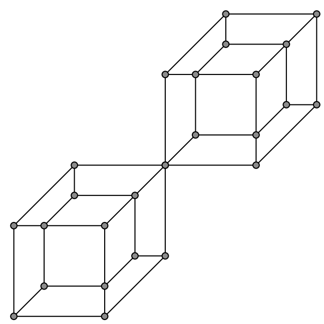

Figure 6: Example of $\Theta(\log(n))$-transitive Hom shift.

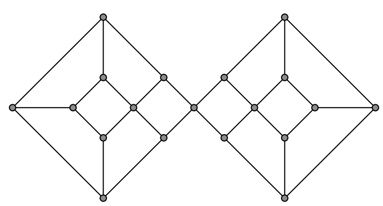

Figure 7: Example of $\Theta(\log(n))$-transitive Hom shift.

The *second problem* is the one of finding an algorithm which, provided a graph G, decides in which transitivity class the corresponding Hom shift belongs. In particular, since we know that there is no intermediate class between $O(\log(n))$ and $\Theta(n)$, and that wether the Hom shifts belong to one or the other depends on the finiteness of the quaternary cover, we would like to answer the following question:

Question 9.2. *Is there an algorithm which decides, provided a finite simple non-oriented graph G, if $\mathcal{U}_4(G)$ is finite or not ?*

Here the difficulty is the closeness to known undecidability results: while the quaternary cover is finite if and only if the quotient by squares of the *fundamental group* of G is finite, it is not possible to decide if a group defined by a finite number of generators and relations is finite or not.

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