

On exact computation of square ice entropy

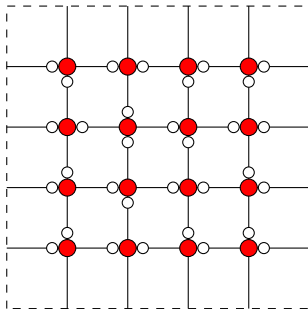
Silvère Gangloff

LIP, ENS Lyon

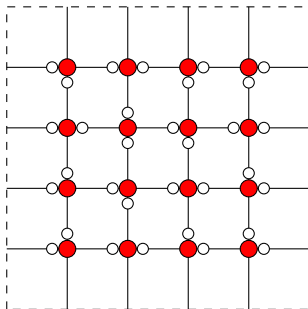
March 11, 2019

I. Representations of square ice

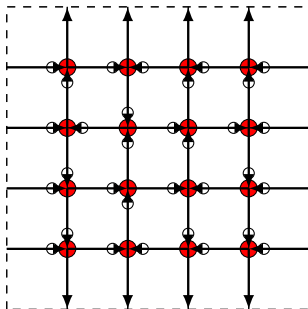
Square ice model [Pauling-Lieb]:



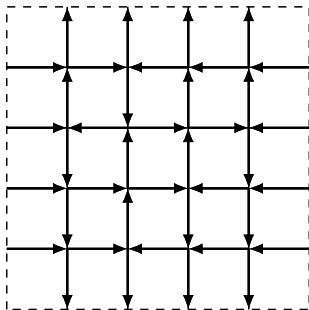
Wang tiles representation [Six-vertex model]



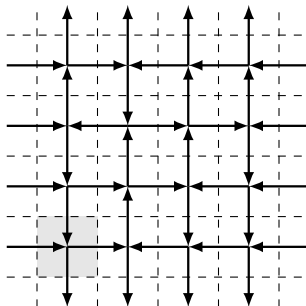
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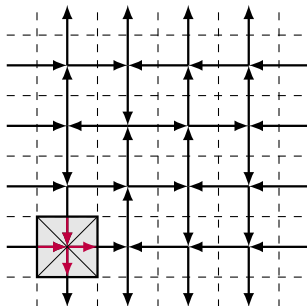
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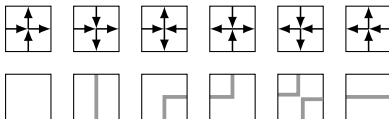
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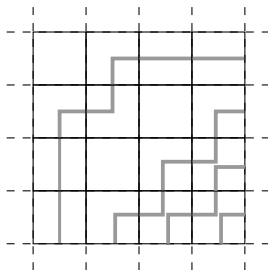
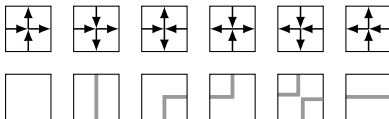
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Discrete curves subshift $[X^s]$:



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Example of apparition in symbolic dynamics:

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→ **Proof under some hypothesis**

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→ **Proof under some hypothesis**

S. Gangloff, *A proof that square ice entropy is $\frac{3}{2} \log_2(4/3)$* , 2019.

II. Subshifts of finite type and entropy

SFT (subshift of finite type): subset of $\mathcal{A}^{\mathbb{Z}^2}$, defined by a finite set of **forbidden patterns**.

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Forbidden patterns $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ et $\begin{bmatrix} 1 & 1 \end{bmatrix}$.

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Ex: **Hard square shift**, or hard core model.

Forbidden patterns $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ et $\begin{bmatrix} 1 & 1 \end{bmatrix}$.

0	0	0	0	0
0	0	0	1	0
0	0	0	0	1
0	0	0	1	0
1	0	0	0	0

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0	0	0	0	0	
0	0	0	1	1	oops
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0	0	0	0	0	

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0	0	0	0	0
0	0	0	1	0
0	0	0	0	1
1	0	1	0	0
0	0	0	0	0

Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles

0	0
0	0

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Hard core

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0	1

$$\mathcal{N}_2(X) = 7$$

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Free tiles

1	1
1	1

$$\mathcal{N}_2(X) = 2^{2^2}$$

$$\mathcal{N}_N(X) = 2^{N^2}$$

Hard core

1	0
0	1

$$\mathcal{N}_2(X) = 7$$

$$\mathcal{N}_N(X) = 2^{N^2(h(X)+o(1))}$$

Entropy of a SFT X :

$$h(X) = \inf_N \frac{\log_2(\mathcal{N}_N(X))}{N^2}$$

Free tiles

$$h = 1$$

Hard core

$$h \geq 1/2$$

Square ice [Lieb 67]

$$h = \frac{3}{2} \log_2(4/3)$$

Entropy of a SFT X :

$$h(X) = \inf_N \frac{\log_2(\mathcal{N}_N(X))}{N^2} = \inf_N \frac{\log_2(\mathcal{N}_N^{loc}(X))}{N^2}$$

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Free tiles

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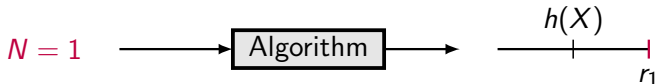
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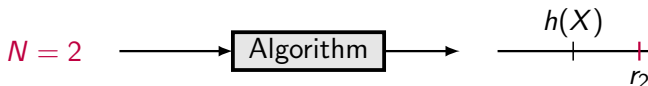
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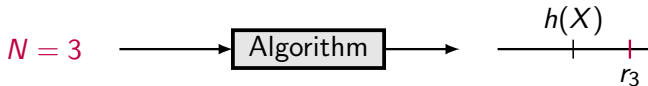
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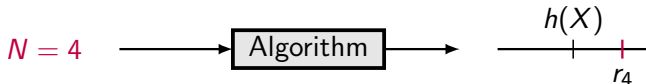
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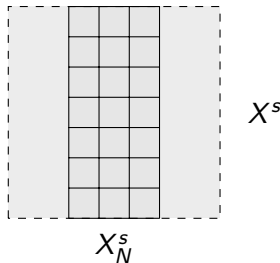
II. Lieb transfer matrices approach

Entropy of square ice:

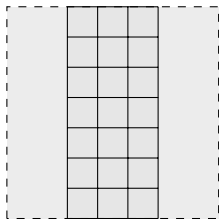
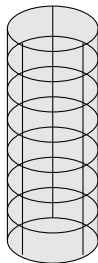
$$h(X^s) = \lim_{M,N} \frac{\log_2(\mathcal{N}_{M,N}(X^s))}{MN}.$$

Stripes subshifts:

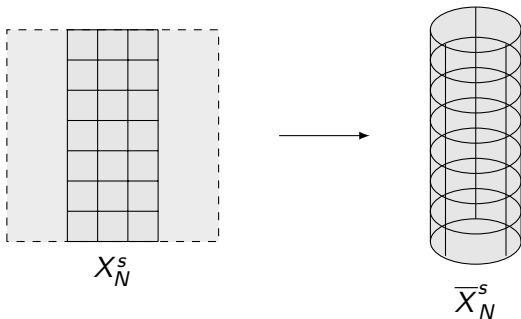
$$h(X^s) = \lim_N \frac{h(X_N^s)}{N}$$



Cylindric stripes subshifts:

 X_N^s  \overline{X}_N^s

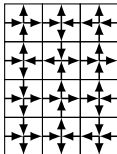
Cylindric stripes subshifts:



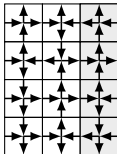
Symmetry properties of square ice imply:

$$h(X^s) = \lim_N \frac{h(\overline{X}_N^s)}{N}$$

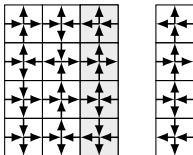
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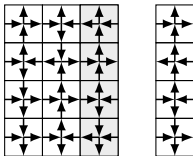
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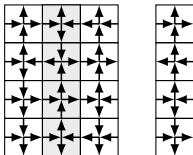
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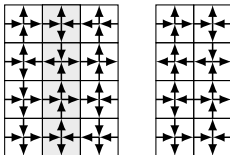
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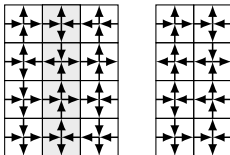
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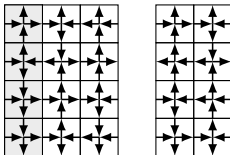
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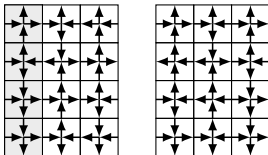
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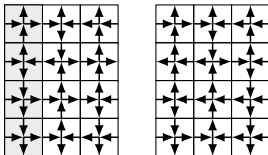
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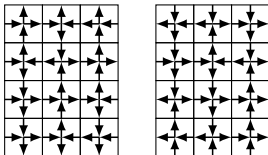
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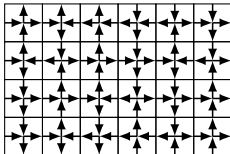
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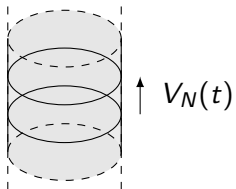
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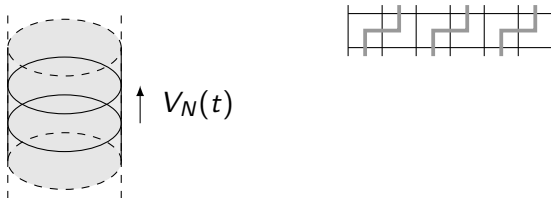
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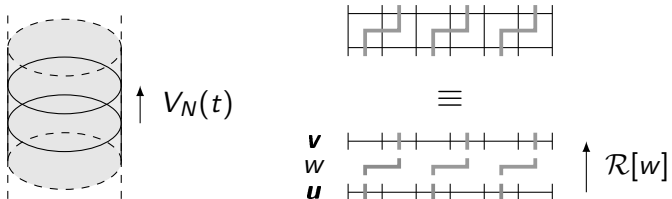
Lieb transfer matrices:



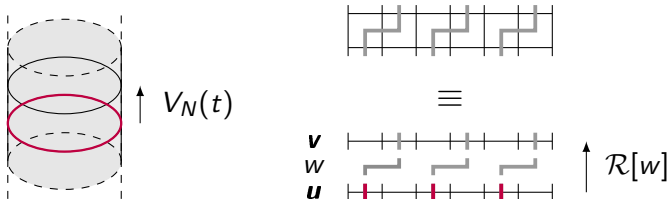
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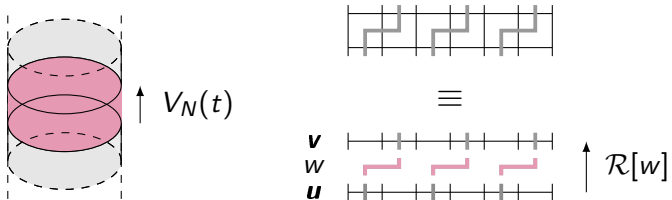
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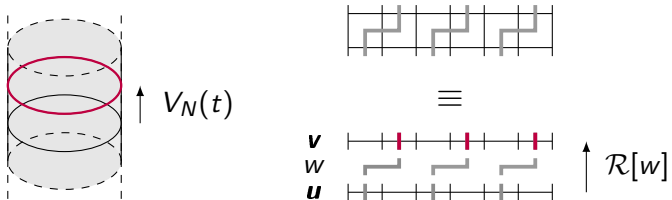
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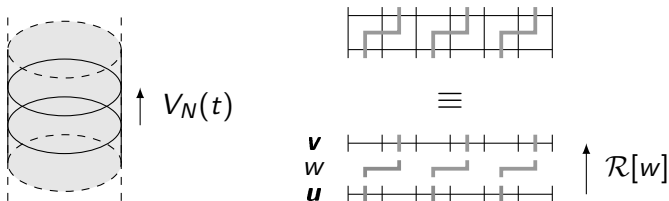
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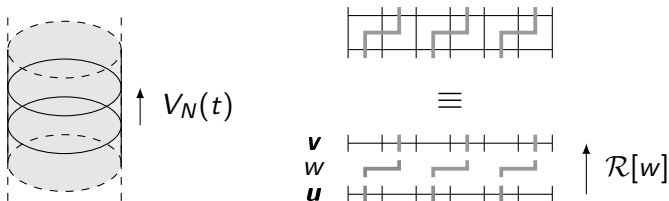
Lieb transfer matrices:



$$V_N(t)[\mathbf{u}, \mathbf{v}] = \sum_{\mathbf{u} \mathcal{R}[w] \mathbf{v}} t^{|w|}.$$

where $|w| = \#$ of $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$ and $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$

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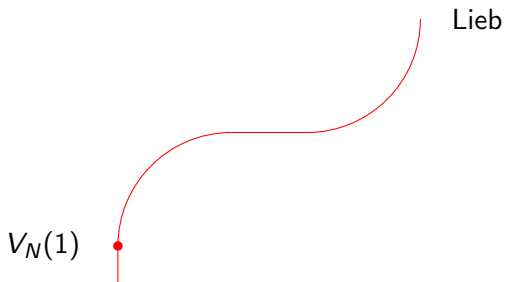
where $|w| = \#$ of $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$ and $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$

$$h(X^s) = \lim_N \frac{\log_2(\lambda_{\max}(V_N(1)))}{N}$$

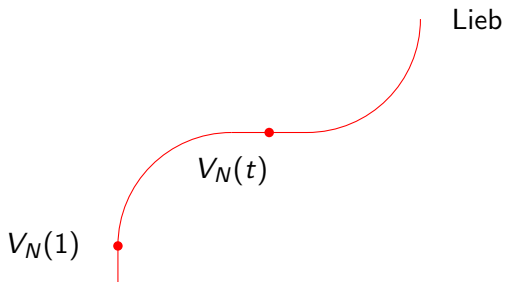
Computing maximal eigenvalue of $V_N(1)$, strategy:

$$V_N(1) \quad \bullet$$

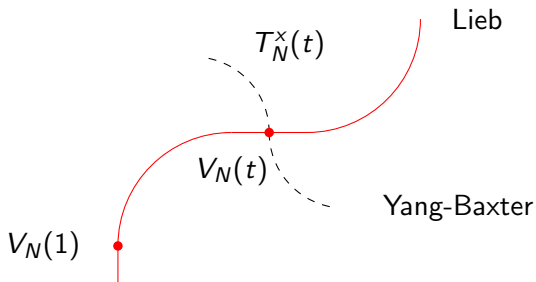
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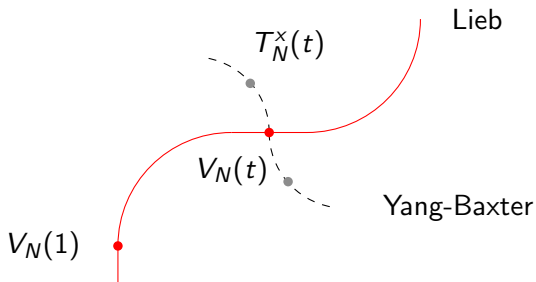
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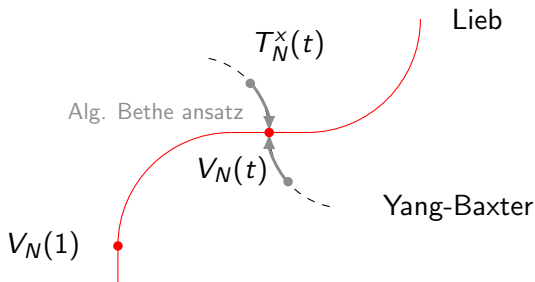
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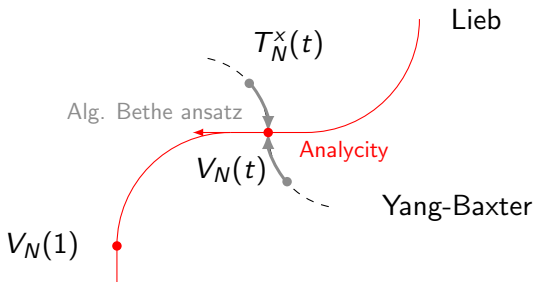
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III. Yang-Baxter transfer matrices

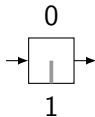
R-matrices and monodromy matrices:

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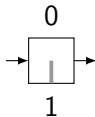


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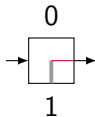
$$R(0,1) = \left(\begin{array}{c} \\ \end{array} \right)$$

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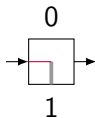
$$R(0,1) = \begin{pmatrix} 0 & \\ & \end{pmatrix}$$

R-matrices and monodromy matrices: \rightarrow : input/output:



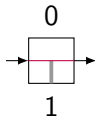
$$R(0,1) = \begin{pmatrix} 0 & \lambda \\ & \end{pmatrix}$$

R-matrices and monodromy matrices: \rightarrow : input/output:



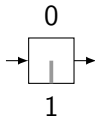
$$R(0,1) = \begin{pmatrix} 0 & \lambda \\ 0 & 1 \end{pmatrix}$$

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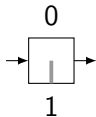
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R-matrices and monodromy matrices: \rightarrow : input/output:



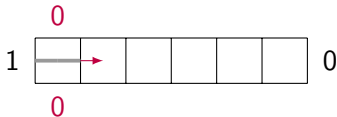
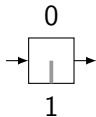
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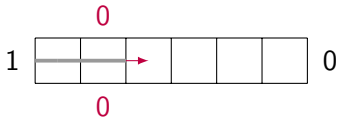
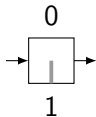
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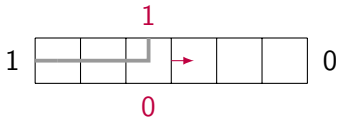
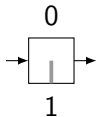
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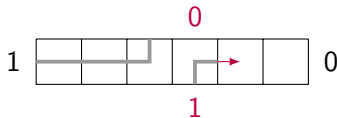
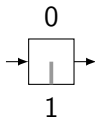
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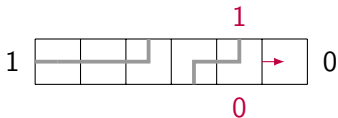
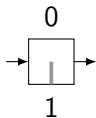
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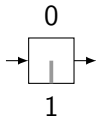
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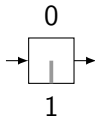
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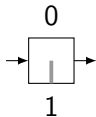
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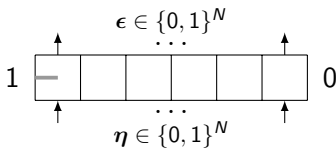


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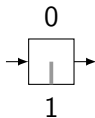
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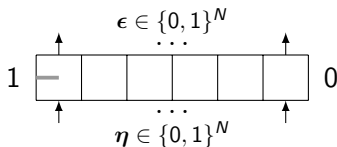
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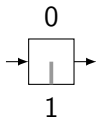


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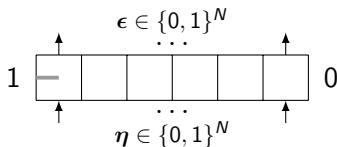


$$M_N(1,0)[\epsilon, \eta]$$

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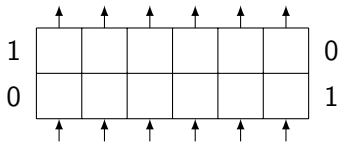
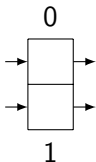


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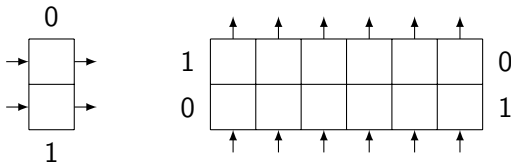
Yang-Baxter transfer matrices:

$$T_N[\epsilon, \eta] = \sum_{u \in \{0,1\}} M_N(u, u)[\epsilon, \eta].$$

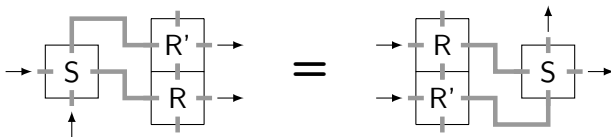
Composition of these matrices and condition for commutation:



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Yang-Baxter equation:



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$$Np_j = 2\pi j - (n+1)\pi - \sum_{k=1}^n \Theta_t(p_j, p_k)$$

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→ **Known:** existence and analyticity in t .

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IV. Asymptotics

Asymptotics of counting functions:

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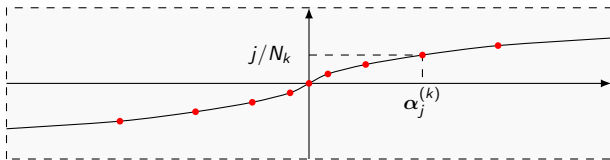
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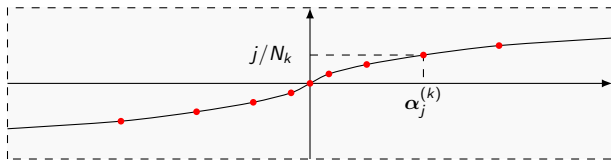
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$$\lim_N \frac{\log_2(\lambda_{\max}(V_N(1)))}{N} = \lim_k \frac{1}{N_k} \sum_{j=1}^{n_k} f(\boldsymbol{\alpha}_j^{(k)}) = \int_{\mathbb{R}} f(\alpha) \rho_t(\alpha) d\alpha.$$

Strategy:

- 1 Extend $\xi_t^{(k)}$ on a stripe including \mathbb{R} :

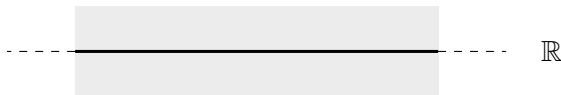
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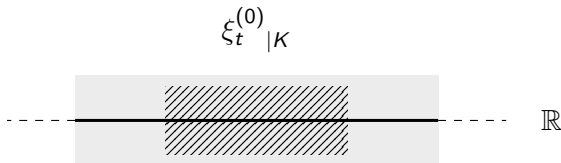
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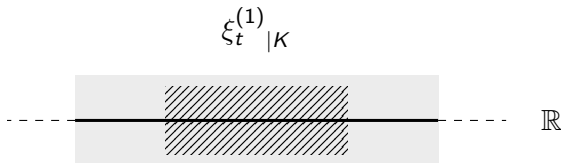
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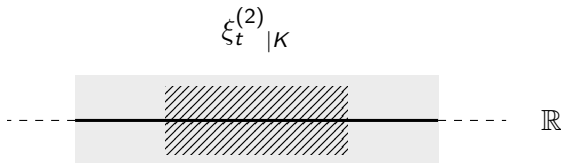
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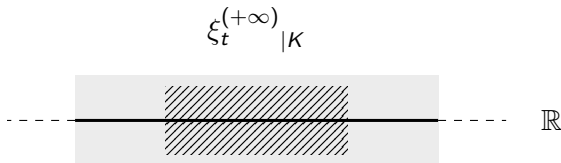
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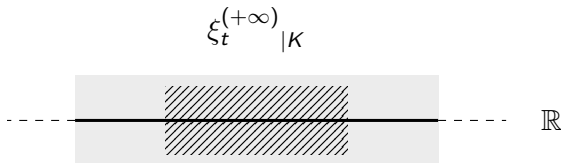
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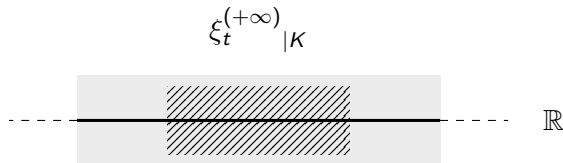
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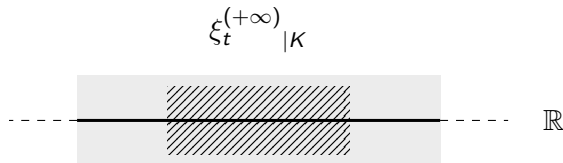
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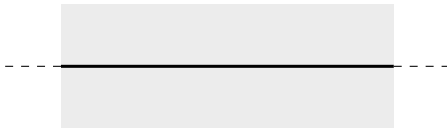
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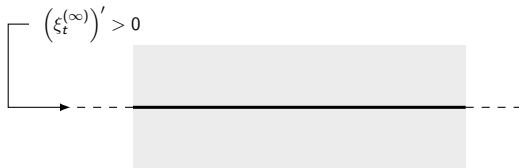


- 2 Assume $(\xi_t)^{\nu(k)} \rightarrow \xi_t^{(+\infty)}$ on any compact K .
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- 4 Thus, $\xi_t^{(k)} \rightarrow \xi_t^{(\infty)}$.

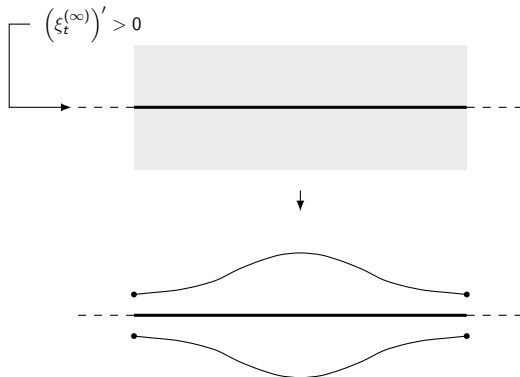
Rarefaction of the roots and $\xi_t^{(k)}$ biholomorphisms: $\epsilon > 0$:



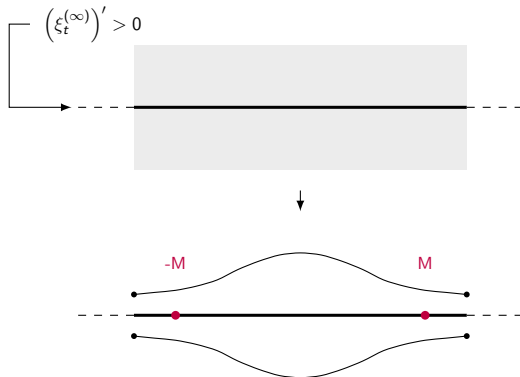
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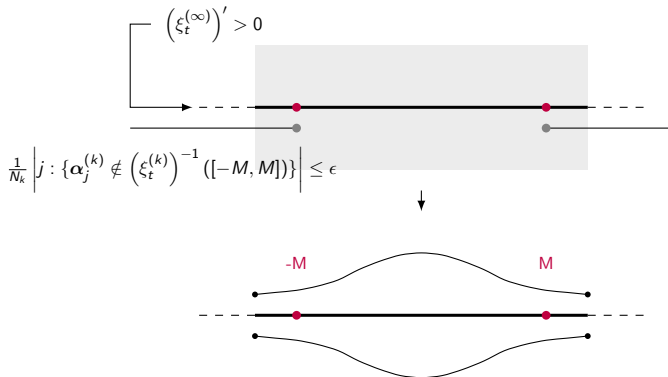
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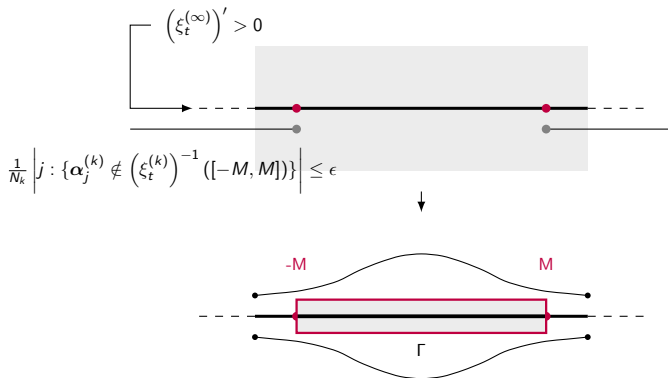
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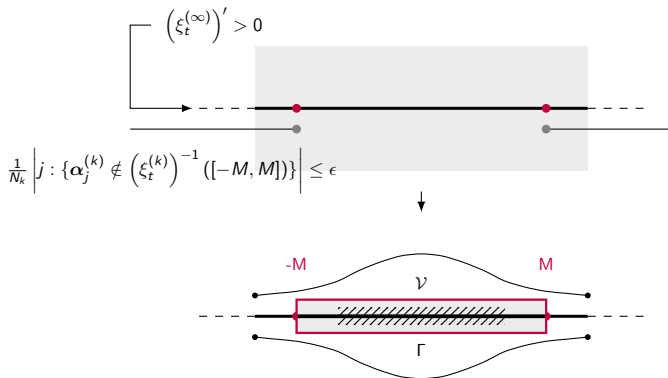
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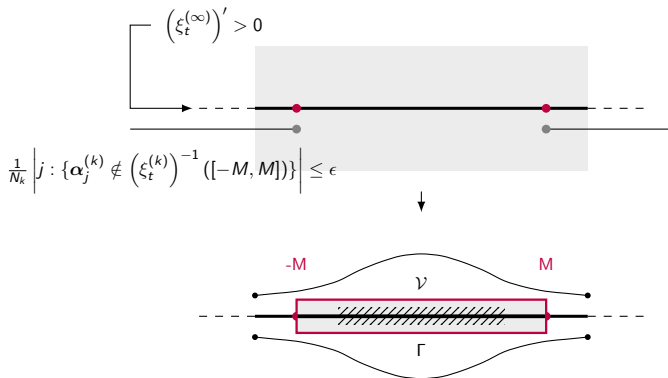
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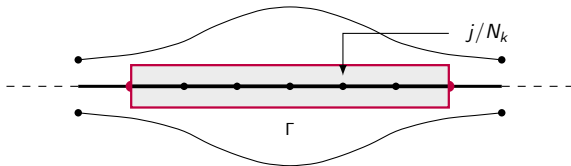


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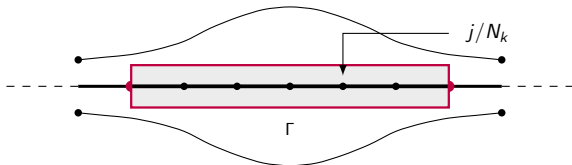


The functions have distinct values on \mathcal{V} and Γ . Thus they are biholomorphisms onto \mathcal{V} .

Lace integral expression of $\xi_t^{(k)}$:



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By residues theorem:

$$\xi_t^{(k)}(\alpha) = \frac{1}{2\pi} \kappa_t(\alpha) + \frac{n_k + 1}{2N_k} + \oint_{\Gamma} \theta_t \left(\left(\xi_t^{(k)} \right)^{-1}(\alpha) \right) \frac{e^{2i\pi s N_k}}{e^{2i\pi s N_k} - 1} ds + O(\epsilon).$$

Fredholm integral equation: Limit and change of variable:

$$\xi_t^{(\infty)}(\alpha) = \frac{1}{2\pi} \kappa_t(\alpha) + \frac{1}{4} + \int_0^{+\infty} \theta_t(\alpha) \left(\xi_t^{(\infty)} \right)'(\alpha) d\alpha.$$

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Solution by Fourier transforms.

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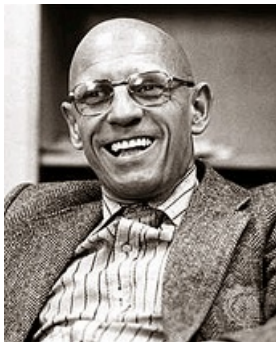
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Through an expression of $\rho_t = \left(\xi_t^{(\infty)}\right)'$ and lace integrals computations:

$$h(X^s) = \frac{3}{2} \log_2(4/3).$$

V. Comments

Why mathematical physics are hard to read for mathematicians?



Archaeology of Knowledge, 1969

Concept of *discursive formation*

Mathematics and mathematical physics are distinct discursive formations ;
different conceptions of units of meaning, etc.

Further research:

- 1 Extensions: eight-vertex model [Baxter], dimer model [Lieb]..



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- 1 Extensions: eight-vertex model [Baxter], dimer model [Lieb]..



- 2 **Hard core model** ? Tridimensional ice ? Kari-Culik tilings ?



- 3 Transformation of entropy by subshifts operators ?

