Dynamics of holomorphic maps

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Preface

This text is written for the students in the Master program at the University of Paris 6. Only a knowledge in complex analysis in one variable and in measure theory is required. We begin with the theory of iteration of holomorphic polynomials and of rational fractions, but our aim is to introduce the readers to the current research in complex dynamics of several variables. We introduce the main dynamical objects and their properties in a way so that they can be easily extended to the case of higher dimension after introducing the necessary tools in complex analysis of several variables. Since the number of lectures is small, exercices are given; they contain important results which are used in the text. This version may contain mistakes. Your remarks and comments are welcome.

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Chapter 1

Fatou-Julia theory for rational fractions

1.1 Dynamics of polynomials

1.1.1 Critical and periodic points of polynomials

Let f be a holomorphic polynomial of degree $d \geq 2$ (we will see that the dynamics of polynomials of degree 1 is not interesting). Then f defines a continuous surjective map from $\mathbb{C}$ onto $\mathbb{C}$. For every $z \in \mathbb{C}$ the equation $f(w) = z$ admits exactly d solutions counted with multiplicities. In other words, *the fibers* $f^{-1}(z)$ of f contain exactly d points counted with multiplicities. More precisely, $f : \mathbb{C} \to \mathbb{C}$ is a *ramified covering of degree* d.

Definition 1.1.1.1. A point c is *critical of multiplicity* m if it is a solution of multiplicity m of the equation $f'(z) = 0$. The set of all critical points is *the critical set* of f. If c is a critical point, $f(c)$ is a *critical value* of f.

Since $\deg f' = d-1$, f admits exactly $d-1$ critical points in $\mathbb{C}$, counted with multiplicities. The polynomial $f^n := f \circ \cdots \circ f$ is *the iterate of order* n of f; it is of degree d^n.

Definition 1.1.1.2. A point p is *fixed of multiplicity* m if p is a solution of multiplicity m of the equation $f(z) = z$. A point p is *periodic of period* n *and of multiplicity* m if p is a fixed point of multiplicity m of f^n, that is, p is a solution of multiplicity m of the equation $f^n(z) = z$. We say that p is *pre-periodic* if there is $N \geq 0$ such that $f^N(p)$ is a periodic point.

There are exactly d fixed points in $\mathbb{C}$ counted with multiplicities (we will see that the infinity can be considered as a fixed point of multiplicity 1). Since f^n is a polynomial of degree d^n, f admits exactly d^n periodic points of period n in $\mathbb{C}$. If p is periodic of period n then it is also periodic of period kn for every $k \geq 1$.

Note that f^0 is the identity, that is, $f^0(z) = z$. One should distinguish $f^n(z)$ from the power $[f(z)]^n$.

Definition 1.1.1.3. The sequence of points $\mathrm{O}^+(p) := \{p, f(p), f^2(p), \ldots\}$ is *the orbit* of the point p. Any sequence $\{p_{-n}, \ldots, p_{-2}, p_{-1}, p_0\}$ such that $p_0 = p$ and $f(p_{-i-1}) = f(p_{-i})$ for $0 \leq i \leq n-1$, is *an inverse branch of order* n of p.

The orbit of a periodic point is called *a (periodic) cycle*. In general, a point may have several inverse branches. Pre-periodic points are the points whose orbits take only a finite number of values.

Exercise 1.1.1.4. *Describe the fibers, the critical set, the set of critical values, the periodic and pre-periodic points of the polynomial $f(z) = z^d$. Study the orbits a general point p. Find the number of inverse branches of order n of p. Same questions for a polynomial f of degree $d = 1$.*

Exercise 1.1.1.5. *Describe the critical set and the set of critical values of f^n in term of the critical set of f. Let d_n be the number (counted with multiplicities) of periodic points whose minimal periods are equal to n. Show that*

$$\lim_{n\to\infty} d_n d^{-n} = 1.$$

Definition 1.1.1.6. Let p be a periodic point of period n of f. We say that p is

1a. *repelling* if $|(f^n)'(p)| > 1$.

1b. *attracting* if $|(f^n)'(p)| < 1$.

2. *super-attracting* if $(f^n)'(p) = 0$.

3. *rationally indifferent* if $(f^n)'(p)$ is a root of unity.

4. *irrationally indifferent* if $|(f^n)'(p)| = 1$ and $(f^n)'(p)$ is not a root of unity.

The orbit of p is called respectively *a repelling cycle, an attracting cycle, a super-attracting cycle, a rationally indifferent cycle or an irrationally indifferent cycle.*

Definition 1.1.1.7. A subset K of $\mathbb{C}$ is *invariant* if $f(K) = K$ and *totally invariant* if $f^{-1}(K) = K$.

Exercise 1.1.1.8. *Let p be a periodic point of period n of f. Compute $(f^{kn})'(p)$ for $k \geq 1$. If $(f^n)'(p)$ satisfies one of the properties in Definition 1.1.1.6, show that $(f^{kn})'(p)$ satisfies the same property.*

Exercise 1.1.1.9. *Determine the type of the periodic points of the polynomial z^d. Determine all the finite totally invariant sets of z^d. Hint: $f^{-n}(z)$ contains d^n distinct points if $z \neq 0$.*

Exercise 1.1.1.10. *Show that completely invariant sets are invariant. Show that the complement of a totally invariant set is totally invariant. Describe the mimimal totally invariant set containing a given point* p*. Determine all the polynomials* f *admitting a totally invariant point.*

Exercise 1.1.1.11. *Let* h *be a polynomial of degree* 1*. Define* $g := h \circ f \circ h^{-1}$*. We say that* f *and* g *are analytically conjugate. Show that* $g^n = h \circ f^n \circ h^{-1}$*. Determine the relations between the fibers of* f *and the fibers of* g*. Same questions for the critical sets, the sets of critical values, the periodic and pre-periodic points, the type of periodic points, the orbits and inverse branches, the invariant and totally invariant sets. Show that any polynomial of degree* $d \geq 2$ *is conjugate to a polynomial of the form* $z^d + a_2 z^{d-2} + \cdots + a_d$*.*

Exercise 1.1.1.12. *Let* f *be a polynomial of degree* $d \geq 2$*. Let* Ω_∞ *be the set of points* z *such that* $|f^n(z)| \to +\infty$*. Show that* Ω_∞ *is a connected open neighbourhood of infinity which is totally invariant. Show also that the complement and the boundary of* Ω_∞ *are totally invariant. We call* Ω_∞ *the basin of* ∞*. We will see that the boundary* J *of* Ω_∞ *is the Julia set of* f *and its complement* F *is the Fatou set. The compact set* $\mathbb{C} \setminus \Omega_\infty$ *is called the filled Julia set. Show that* $\mathbb{C} \setminus \Omega_\infty$ *is the largest compact set invariant by* f*.*

Exercise 1.1.1.13. *Let* $p_0, \ldots, p_{m-1}$ *be a repelling periodic cycle of period* m *of* f *with* $p_{i+1} = f(p_i)$ *and* $p_m = p_0$*. Let* D_i *denote the open disc of center* p_i *and of radius* r*. Show that if* r *is small enough* $\overline{D}_i$ *is contained in* $f^m(D_i)$*.*

Exercise 1.1.1.14. *Let* $p_0, \ldots, p_{m-1}$ *be an attracting periodic cycle of period* m *of* f *with* $p_{i+1} = f(p_i)$ *and* $p_m = p_0$*. Let* D_i *denote the open disc of center* p_i *and of radius* r*. Show that if* r *is small enough,* $f^m(\overline{D}_i)$ *is contained in* D_i*. Let* $\widetilde{\Omega}$ *denote the set of points* z *such that* $f^{nm}(z)$ *converge to one of the points* p_i*. Show that* $\widetilde{\Omega}$ *is an invariant open set. Let* Ω *be the union of the components of* $\widetilde{\Omega}$ *which contain at least one point of the cycle. Show that* Ω *is also invariant. We call* $\widetilde{\Omega}$ *the basin of the cycle and* Ω *the immedidate basin. Hint: show that* $\widetilde{\Omega} = \cup_{n \geq 0} f^{-nm}(D_0 \cup \ldots \cup D_{m-1})$*.*

Exercise 1.1.1.15. *Let* D *be an open disc in* $\mathbb{C}$ *and* f *be a holomorphic function defined on* D *such that* $\overline{f(D)} \subset D$*. Show that* f *admits a unique fixed point* $p \in D$*. Hint: use Rouchés theorem. Moreover,* p *is attracting and* f^n *converge uniformly on* D *to* p*. Same problem for a simply connected domaine* D *in* $\mathbb{C}$*. Hint: use Schwarz lemma for* $\tau \circ f \circ \tau^{-1}$ *which fixes* 0 *where* $\tau(z) := (1 - z/p)/(z - 1/\overline{p})$*.*

Exercise 1.1.1.16. *Let* $f(z) = z^d + a_1 z^{d-1} + \cdots + a_d$*. If* $f(w) = 0$ *show that* $|w| \leq 2 \max_j |a_j|^{1/j}$*. Hint: if not, there is* $w \neq 0$ *such that* $|a_j|/|w^j| < 1/2^j$ *and* $1 + a_1/w + \cdots + a_d/w^d = 0$*.*

1.1.2 Dynamics near a fixed point

Let $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ denote a holomorphic function defined in a neighbourhood of 0 which fixes the origin, that is, $f(0) = 0$. We say that f is a *germ of holomorphic function* or simply a *holomorphic germ*. The iterates $f^n := f \circ \cdots \circ f$ are also defined in a neighbourhood of 0 but the domain of definition of f^n may be smaller than the domain of definition of f. The function f admits a Taylor expansion

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \cdots = \lambda z + F(z),$$

where $F(z) := a_2 z^2 + a_3 z^3 + \cdots$. If f is defined on a disc of center 0 and of radius strictly larger than r then the Cauchy formula implies that $|\lambda| \leq cr^{-1}$ and $|a_n| \leq cr^{-n}$ where $c := \max_{|z|=r} |f(z)|$.

Observe that if $\lambda \neq 0$ then f is *invertible*, that is, there exists a holomorphic germ $f^{-1} : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ satisfying $f \circ f^{-1}(z) = f^{-1} \circ f(z) = z$.

We assume that f is not constant. Then either $\lambda \neq 0$ or there is at least one coefficient a_n which does not vanish. According to the value of λ, the fixed point 0 is called *repelling, attracting, super-attracting, rationally indifferent or irrationally indifferent* exactly as in Definition 1.1.1.6.

Definition 1.1.2.1. We say that f is *linearizable* if there is an invertible holomorphic germ $h : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $h \circ f \circ h^{-1}(z) = \lambda z$.

Exercise 1.1.2.2. *Let A and t be strictly positive constants. Consider the function*

$$g(z) := z - A(t^2 z^2 + t^3 z^3 + \cdots) = z - A \sum_{n \geq 2} (tz)^n.$$

Determine the domain of definition of g. Let $g^{-1} : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ denote the inverse of g. Write

$$g^{-1}(z) = \alpha_1 z + \alpha_2 z^2 + \cdots .$$

Show that there are positive constants B and s such that $|\alpha_n| \leq Bs^n$ for $n \geq 1$. Compute α_1 and α_2. Prove that there are polynomials P_n of $n-2$ variables with positive coefficients depending on A and t such that

$$\alpha_n = P_n(\alpha_2, \ldots, \alpha_{n-1}) \quad \textit{for } n > 2.$$

Theorem 1.1.2.3 (Koenigs)**.** *Assume that $|\lambda| \neq 0, 1$. Then f is linealizable. More precisely, there is a unique holomorphic germ $h : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $h'(0) = 1$ and*

$$h \circ f \circ h^{-1}(z) = \lambda z.$$

Proof. We will find a germ $l : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $l^{-1} \circ f \circ l = \lambda z$ and

$$l(z) = z + b_2 z^2 + b_3 z^3 + \cdots = z + L(z),$$

where $L(z) := b_2z^2 + b_3z^3 + \cdots$. The germ $h := l^{-1}$ satisfies the theorem.

We deduce from the equation $l^{-1}\circ f\circ l(z) = \lambda z$ that $f\circ l(z) = l(\lambda z)$. Therefore,

$$\lambda l(z) + F(l(z)) = \lambda z + L(\lambda z)$$

which is equivalent to

$$\lambda z + L(\lambda z) - \lambda l(z) = F(l(z)).$$

Using the Taylor's expansion, we obtain

$$\sum_{n\geq 2} b_n(\lambda^n - \lambda)z^n = \sum_{n\geq 2} a_n(z + b_2z^2 + b_3z^3 + \cdots)^n.$$

Hence, there are polynomials Q_n of $n-2$ variables depending on the a_n with positive coefficients such that the previous equation is formally equivalent to

$$b_2 = a_2 \quad \text{and} \quad b_n = Q_n(b_2, \dots, b_{n-1}), \quad \text{for } n \geq 2.$$

This implies the uniqueness of l. We only have to show that the serie $l(z)$ is convergent.

Exercise 1.1.2.4. *Show that there are positive constants A and t such that $|\lambda^n - \lambda| \geq A^{-1/2}$ and $|a_n| \leq A^{1/2}t^n$ for every $n \geq 2$. If A and t satisfy these inequalities and β_n are as in Exercise 1.1.2.2, prove by induction that $|b_n| \leq \beta_n$. Hint: compare the coefficients of P_n and Q_n.*

We deduce that $|b_n| \leq Bs^n$. Hence, the serie $l(z)$ is convergent and defines a holomorphic germ. This completes the proof. □

We now consider the case of a super-attracting point. Let $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a holomorphic germ with the following Taylor expansion

$$f(z) = a_0z^p + a_1z^{p+1} + \cdots = z^p[a_0 + a_1z + \cdots] = z^p[a_0 + F(z)]$$

where $p \geq 2$, $a_0 \neq 0$ and $F(z) := a_1z + \cdots$.

Theorem 1.1.2.5. *There is an invertible holomorphic germ $h : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $h \circ f \circ h^{-1}(z) = z^p$.*

Proof. First, replace f by $\varphi \circ f \circ \varphi^{-1}$, where $\varphi(z) := \lambda z$ and $\lambda^{p-1} = a_0$. So, we can assume that $a_0 = 1$. Then

$$f(z) = z^p[1 + F(z)].$$

We describe the main idea of the proof. We first define locally invertible holomorphic functions g_n such that

$$g_n'(0) = 1 \quad \text{and} \quad [g_n(z)]^{p^{n+1}} = 1 + F(f^n(z)).$$

Then, we show that the infinite product $z\prod_{n\geq 0} g_n$ defines a holomorphic function h satisfying the desired identity $h \circ f \circ h^{-1}(z) = z^p$.

Exercise 1.1.2.6. *Let W be the disc of center 1 and of radius $1/2$. Let $z^{1/N}$ denote the first branch of the N-th root of $z \in W$. Show that*

$$|z^{1/N} - 1| < 2|z-1|/N < 1/N.$$

Hint: show that $|(z^{1/N})'| < 2/N$.

Let M and r be positive constants such that $|F(z)| \leq M|z|$ when $|z| \leq r$. Fix a positive constant ρ smaller than r, $1/(2M)$ and $1/4$. Let D denote the disc of center 0 and of radius ρ. We define the functions g_n on D. Observe that for $z \in D$ we have

$$|f(z)| \leq |z|^p(1 + M|z|) \leq |z|\rho^{p-1}(1 + M\rho) < |z|/2.$$

So, f maps D into D. We obtain by induction that

$$|f^n(z)| < |z|/2^n \text{ on } D.$$

It follows that

$$|F(f^n(z))| \leq M|z|/2^n < 1/2 \text{ on } D.$$

Using Exercise 1.1.2.6, we can define a holomorphic function g_n on D such that

$$g_n(z)^{p^{n+1}} = 1 + F(f^n(z)) \quad \text{and} \quad |g_n(z) - 1| \leq 2/p^{n+1}.$$

This estimate implies that the products

$$h_n(z) := zg_0(z)\ldots g_n(z) = z + \mathrm{o}(z)$$

converge uniformly on D to a holomorphic function h. Since $h_n(0) = 0$ and $h_n'(0) = 1$, we also have $h(0) = 0$ and $h'(0) = 1$. Moreover,

$$[g_n(f(z))]^{p^{n+1}} = 1 + F(f^{n+1}(z)) = [g_{n+1}(z)]^{p^{n+2}} = [g_{n+1}(z)^p]^{p^{n+1}}.$$

Hence, $g_n(f(z)) = g_{n+1}(z)^p$ and using the definition of $g_0(z)$ we obtain

$$h_n(f(z)) = f(z)g_0(f(z))\ldots g_n(f(z)) = f(z)[g_1(z)\ldots g_{n+1}(z)]^p = h_{n+1}(z)^p.$$

Letting $n \to \infty$ gives

$$h(f(z)) = h(z)^p.$$

This implies the result. □

Exercise 1.1.2.7. *Let $f(z) := z^d$. Find all invertible germs $h : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $h \circ f \circ h^{-1} = f$. Deduce the number of functions h satisfying Theorem 1.1.2.5.*

Exercise 1.1.2.8. *Using Theorem 1.1.2.5 show that there is a neighbourhood V of 0 such that for w small enough and $w \neq 0$, the equation $f(z) = w$ admits exactly p distinct solutions in V and these solutions are simple.*

Consider now the case of rational indifferent fixed point. This situation was studied by Leau. Replacing f by an iterate f^n, one can assume that

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots .$$

We also assume that $f(z) \neq z$, i.e. at least one coefficient a_n does not vanish.

Exercise 1.1.2.9. *Show that there is an integer $p \geq 1$ and an invertible holomorphic germ $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that*

$$\varphi \circ f \circ \varphi^{-1} = z - z^{p+1} + \mathrm{O}(z^{2p+1}).$$

Hint: use Taylor expansion. Study the orbits of $g(z) = z - z^{p+1}$ on the real half-lines $\{re^{2k\pi i/p},\ r > 0\}$ and $\{re^{(2k+1)\pi i/p},\ r > 0\}$ near 0 with $0 \leq k \leq p-1$.

From now on, assume that

$$f(z) = z - z^{p+1} + \mathrm{O}(z^{2p+1}).$$

Exercise 1.1.2.10. *Consider the function $\sigma(z) := z^{-p}$. Show that, for $t > 0$, the preimage of*

$$\widetilde{\Pi}(t) := \{z = x + iy,\ 2tx + t^2 y^2 > 1\}$$

by σ for $t > 0$ is the union of the following domains $\Pi_k(t)$ with $k = 0, \ldots, p-1$

$$\Pi_k(t) := \{re^{i\theta},\ r^p < t(1 + \cos(p\theta)),\ |2k\pi/p - \theta| < \pi/p\}.$$

Draw the domains $\Pi_k(t)$ for $p = 1, 2, 3, 4, 5$.

Definition 1.1.2.11. We say that $\Pi_k(t)$ is *a petal* and that the real half-line $\{re^{2k\pi i/p},\ r > 0\}$ is *the axis* of $\Pi_k(t)$.

Exercise 1.1.2.12. *Show that there is a holomorphic germ f such that*

$$f(z) = z - z^{p+1} + \mathrm{O}(z^{2p+1})$$

f is semi-conjugate by σ to the translation $z \mapsto z + p$, that is, f satisfies $\sigma \circ f(z) = \sigma(z) + p$. Deduce that f sends $\Pi_k(t)$ into $\Pi_k(t)$ if t is small enough. Show that f^n converge to 0 uniformly on $\Pi_k(t)$.

We have the following theorem.

Theorem 1.1.2.13 (Petal theorem)**.** *Assume that*

$$f(z) = z - z^{p+1} + \mathrm{O}(z^{2p+1}).$$

Then for t small enough

1. *f maps each petal $\Pi_k(t)$ into itself.*

2. *f^n converge to 0 uniformly on $\Pi_k(t)$.*

3. *$\arg(f^n)$ converge to $2k\pi/p$ uniformly on $\Pi_k(t)$.*

4. *$|f(z)| < |z|$ on a neighbourhood of the axis of $\Pi_k(t)$.*

5. *$f : \Pi_k(t) \to \Pi_k(t)$ is conjugate to the translation $z \mapsto z + p$.*

Proof. We consider only the case $k = 0$. The other cases are proved in the same way. We always assume that t is small enough. Let σ^{-1} the inverse branch of σ which defines a holomorphic bijection between $\mathbb{C} \setminus \mathbb{R}_-$ and the angular sector $S := \{re^{i\theta},\ |\theta| < \pi/p\}$. Define the function g by

$$g(w) := \sigma \circ f \circ \sigma^{-1}(w).$$

Exercise 1.1.2.14. *Show that g is defined on $\widetilde{\Pi}(t)$ and there are positive constants A, B and a holomorphic function $\gamma(w)$ on $\widetilde{\Pi}(t)$ such that*

$$g(w) = w + p + A/w + \gamma(w) \quad \text{and} \quad |\gamma(w)| \leq B/|w|^{1+1/p}.$$

1. *Deduce that g maps $\widetilde{\Pi}(t)$ into itself, $|g(w)| > |w|$ in a neighbourhood of $\widetilde{\Pi}(t) \cap \mathbb{R}$ and $\mathrm{Re}(g^n) \to +\infty$, $\mathrm{Im}(g^n)/\mathrm{Re}(g^n) \to 0$ on $\widetilde{\Pi}(t)$.*

2. *Deduce (1), (2), (3), (4) in Theorem 1.1.2.13.*

Exercise 1.1.2.15. *Show that there is a positive constant C such that*

1. *$|g(w) - w - p| \leq C/|w|$.*

If K is a compact subset of $\widetilde{\Pi}(t)$, prove the existence of constants $C_i > 0$ such that for $w \in K$ and $n \geq 2$

1. *$|g^n(w)| \geq C_1 n$ and $|\gamma(g^n(w))| \leq C_2 n^{-1-1/p}$.*

2. *$|g^n(w) - np| \leq C_3 \log n$.*

3. *$|1/g^n(w) - 1/(np)| \leq C_4 \log n/n^2$.*

Exercise 1.1.2.16. *Define*

$$u_n(w) := g^n(w) - np - (A/p)\log n.$$

Show that

$$u_{n+1}(w) + p = u_n(g(w)) - (A/p)\log(1 + 1/n)$$

and

$$u_{n+1}(w) - u_n(w) = A[1/g^n(w) - 1/(np)] + \gamma(g^n(w)) + (A/p)[1/n - \log(1 + 1/n)].$$

Deduce from the previous exercise that u_n converge to a holomorphic function uniformly on compact subsets of $\widetilde{\Pi}(t)$.

Let u denote the limit of the sequence (u_n). We deduce from the first part of Exercise 1.1.2.16 that

$$u(w) + p = u(g(w)).$$

This completes the proof of Theorem 1.1.2.13. □

Exercise 1.1.2.17. *Consider a holomorphic germ*

$$f(z) = \lambda z + \cdots ,$$

where $\lambda = e^{2k\pi i/q}$ *with* k, q *positive integers without common factor. Assume that* $f^q(z) \neq z$. *Show that there is an invertible holomorphic germ* φ *and a multiple* p *of* q *such that*

$$\varphi \circ f \circ \varphi^{-1}(z) = \lambda z - (\lambda/p) z^{p+1} + \mathrm{O}(z^{2p+1}).$$

Hint: use Taylor expansion. Show that

$$f^p(z) = z - z^p + \mathrm{O}(z^{2p+1}).$$

Deduce that f *sends each petal* $\Pi_k(t)$ *into a petal* $\Pi_l(t)$ *which is not necessarily the same.*

The situation in the case of irrational indifferent point is more delicate. We will not give here the proof of the following result.

Theorem 1.1.2.18 (Siegel-Yoccoz)**.** *Assume that* $|\lambda| = 1$ *and* λ *is not a root of unity.*

1. *If there are positive constants* c *and* k *such that* $|\lambda^n - 1| \geq cn^{-k}$ *for every* n, *then* f *is linearizable.*

2. *If* $\lambda z + z^2$ *is linearizable, then* f *is linearizable.*

Exercise 1.1.2.19. *For every* $k > 1$ *the set of* λ *in the unit circle* $\mathbb{S}^1$ *satisfying*

$$\liminf_{n\to\infty} n^k |\lambda^n - 1| = 0$$

has Lebesgue measure 0, *but it contains a* G_δ *dense subset (i.e. an intersection of a countable many dense open sets) of* $\mathbb{S}^1$.

Exercise 1.1.2.20. *Show that there is a* λ *such that*

$$\liminf_{k\to\infty} |\lambda^k - 1|^{1/2^k} = 0.$$

Let $f(z) = \lambda z + z^2$. *Show that* f *is not linearizable at* 0. *Hint: the product of periodic points of period* k *different of* 0, *is equal to* $|\lambda^k - 1|$, *hence there is a sequence of periodic points converging to* 0.

1.1.3 Fatou and Julia sets for polynomials

Let D be an open set in $\mathbb{C}$. Consider a family $\mathscr{F}$ of holomorphic functions on D.

Definition 1.1.3.1. The family $\mathscr{F}$ is *normal* if it is relatively compact in the following sense : for every sequence $(f_n) \subset \mathscr{F}$ one can extract a subsequence (f_{n_i}) which converge locally uniformly on D either to ∞ or to a holomorphic function.

Exercise 1.1.3.2. *Let Ω be a bounded domain in $\mathbb{C}$. Let $\mathscr{F}$ be the family of holomorphic functions on D with values in Ω. Let (f_n) be a sequence in $\mathscr{F}$.*

1. *Show that for every compact subset K of D there is a constant $c > 0$ satisfying $|f'| \leq c$ on K for every $f \in \mathscr{F}$. Hint: use Cauchy formula.*

2. *Let (a_m) be a sequence of points which is dense in D. Show that there is a sequence $(f_{n_i})_{i\geq 1}$ such that $(f_{n_i}(a_m))_{i\geq 1}$ is convergent for every m.*

3. *Prove that (f_{n_i}) converges locally uniformly on D to a function f. Deduce that f is continuous and holomorphic.*

Definition 1.1.3.3. Let f be a polynomial of degree $d \geq 2$. The *Fatou set* for f is the set F of all points a such that (f^n) is normal in a neighbourhood of a. *The Julia set* is the complement $J = \mathbb{C} \setminus F$ of the Fatou set.

So, F is the largest open set where (f^n) is normal. By definition, somehow the behavior of (f^n) is tame on F and chaotic on J. Note that J is compact in $\mathbb{C}$.

Exercise 1.1.3.4. *Let p be a positive integer. Prove that the Fatou and the Julia sets associated to f and to f^p are the same.*

Exercise 1.1.3.5. *Show that the Fatou set is equal to the union of Ω_∞ and $\mathbb{C} \setminus \overline{\Omega}_\infty$. Hint: use Exercise 1.1.3.2. Deduce that F, J are totally invariant, that J is the boundary of Ω_∞, and that the image of a component of F is also a component of F. Show that $\Omega_\infty \neq \mathbb{C}$ and deduce that J is not empty. Hint: use periodic points.*

Exercise 1.1.3.6. *Let $g : D \to D'$ be a holomorphic map between domains in $\mathbb{C}$. Assume that g is proper, that is, the inverse image of a compact subset of D' is compact in D. Show that g defines a finite ramified covering. If Ω is a Fatou component for a polynomial f, show that $f : \Omega \to f(\Omega)$ is proper.*

Exercise 1.1.3.7. *Show that J is perfect, that is, J does not contain isolated points. Hint: use the maximum modulus principle. Deduce that J is uncountable. Hint: use Baire's theorem.*

Exercise 1.1.3.8. *Let Ω be a bounded component of F. Show that Ω is simply connected or equivalently, the boundary of Ω is connected. Hint: use the maximum modulus principle.*

Exercise 1.1.3.9. *Let a be a periodic point of f. If a is repelling or rational indifferent, show that a is a point of J. Hint: compute $(f^n)'(a)$ or use the Taylor expansion of f^n. If a is attractive or linealizable irrational indifferent, show that a is a point of F. Show that the basin of an attracting cycle is a union of Fatou components and that it contains a critical point.*

Exercise 1.1.3.10. *Using Exercise 1.1.3.4 show that the petals associated to a rationally indifferent cycle, constructed as in Section 1.1.2 for an iterate of f, are contained in the Fatou set.*

1.1.4 Periodic and wandering Fatou components

One of the most important theorem in the Fatou-Julia theory is Sullivan's theorem which says that there are no wandering Fatou components. Here is the precise statement of this result.

Theorem 1.1.4.1 (Sullivan). *Let Ω be a Fatou component. Then Ω is preperiodic, that is, there are positive integers $p < q$ such that $f^p(\Omega) = f^q(\Omega)$. In particular, $f^p(\Omega)$ is periodic of period $q - p$.*

We will present here the main idea of the proof. Sullivan's theorem holds for rational fractions. We assume for simplicity that f is a polynomial. First assume there exists a wandering Fatou component Ω and we seek a contradiction. Observe that the Fatou components

$$\Omega, f(\Omega), f^2(\Omega), \ldots$$

are pairwise distinct since the critical set is finite. Replacing Ω by $f^p(\Omega)$ for p large enough allows us to assume that $f^n(\Omega)$ does not contain critical points for every n. Hence, f defines a biholomorphic map between $f^n(\Omega)$ and $f^{n+1}(\Omega)$.

Consider a smooth function μ with on Ω and extend μ to $\mathrm{O}(\Omega) := \cup_{n\in\mathbb{Z}} f^n(\Omega)$ by

$$\mu(f(z)) := [f'(z)/\overline{f'(z)}]\mu(z).$$

Define also $\mu = 0$ outside $\mathrm{O}(\Omega)$. We choose μ small enough in order to apply the following theorem on the Beltrami equation, see [4].

Theorem 1.1.4.2 (Ahlfors-Bers). *Assume that $|\mu| < 1$. There is a unique Lipschitz homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ such that $\varphi(z) - z \to 0$ when $|z| \to +\infty$ and*

$$\frac{\partial\varphi}{\partial\overline{z}} = \mu\frac{\partial\varphi}{\partial z}$$

almost everywhere on $\mathbb{C}$.

Of course, when $\mu = 0$ we obtain $\varphi(z) = z$. In general φ is holomorphic only outside the support of μ. Nevertheless, since μ satisfies

$$\mu(f(z)) = [f'(z)/\overline{f'(z)}]\mu(z)$$

one can prove that $f_\mu := \varphi \circ f \circ \varphi^{-1}$ is holomorphic. So, f_μ is conjugate to f by φ. Hence, its generic fiber contains exactly d points. We deduce that f_μ is a polynomial of degree d. We have constructed a map $\mu \mapsto f_\mu$. The contradiction comes from the fact that the space of μ is of infinite dimension but the space of all the polynomials of degree d is of finite dimension. The detailed proof contains some technical difficulties.

Definition 1.1.4.3. An invariant Fatou component Ω is

1a. *an attracting component* if it is the immediate basin of an attracting point.

1b. *a super-attracting component* if it is the immediate basin of a super-attracting point.

2. *a parabolic component or a Leau component* if it contains an invariant petal associated to a rationally indifferent fixed point.

3. *a Siegel disc* if the restriction of f to Ω is conjugate to a rotation on a disc.

Exercise 1.1.4.4. *Let $p_0, \ldots, p_{m-1}$ be an attracting cycle of period m. Show that they belong to different Fatou components.*

Exercise 1.1.4.5. *Let p be a rationally indifferent fixed point. Show that the petals associated to this point are contained in different Fatou components. Hint: apply the petal theorem to f^{-1} in a neighbourhood of p.*

Proposition 1.1.4.6 (see also Exercise 1.1.4.10)**.** *Let Ω be an invariant Fatou component. Assume there is at least one subsequence of (f^n) which converges locally uniformly on Ω to a non-constant function. Then*

1. *$f : \Omega \to \Omega$ is an automorphism of Ω, i.e. a bijective holomorphic function.*

2. *If a sequence (f^{n_i}) converges locally uniformly on Ω to a non-constant function φ then φ is also an automorphism of Ω.*

3. *There is a sequence (f^{n_i}) which converges locally uniformly on Ω to the identity.*

Proof. Let (k_i) be an increasing sequence of integers such that (f^{k_i}) converges locally uniformly on Ω to a non-constant function ψ.

Exercise 1.1.4.7. *Show that ψ has values in Ω.*

Replacing (k_i) by a subsequence, we can assume that the sequence of $l_i := k_{i+1} - k_i$ is strictly increasing. We have $f^{k_{i+1}} = f^{l_i} \circ f^{k_i}$. We deduce from this identity that if h is a limit of (f^{l_i}) on Ω then $\psi = h \circ \psi$. So, h is identity on $\psi(\Omega)$ which is an open set since ψ is not constant. It follows that h is identity on Ω. We have proved that f^{l_i} converge locally uniformly on Ω to the identity.

Let (m_i) be a subsequence of (l_i) such that $n_i < m_i$. If g is a limit value of $(f^{m_i-n_i})$, as above, using $f^{m_i-n_i}\circ f^{n_i} = f^{m_i}$ we deduce that $g\circ\varphi$ is identity on Ω. So, φ is invertible. Hence, it is an automorphism of Ω. This also holds for $n_i = 1$. We then deduce that $f:\Omega\to\Omega$ is an automorphism. □

Theorem 1.1.4.8. *Let p be an irrationally indifferent fixed point. Then the following properties are equivalent*

1. *p is in the Fatou set.*

2. *f is linealizable near p.*

3. *p is in a Siegel disc.*

Proof. Using a change of coordinate, we can assume $p = 0$. Define $\lambda := f'(0)$. It is clear that (3) implies (1). We show that (2) implies (1). Since f is linealizable near 0, for a suitable local coordinate w of 0 we have $f(w) = \lambda w$. Hence, $f^n(w) = \lambda^n w$. Since $|\lambda| = 1$, the family (f^n) is normal near 0 and 0 belongs to the Fatou set.

Now, we prove that (1) implies (3). Since $|(f^n)'(0)| = |\lambda^n| = 1$, the limit values of (f^n) on Ω are not constant. Let Ω denote the Fatou component containing 0. Proposition 1.1.4.6 implies that $f:\Omega\to\Omega$ is an automorphism. We have seen that Ω is simply connected. So by Riemann theorem, there is a biholomorphic map σ between Ω and the unit disc D of $\mathbb{C}$ such that $\sigma(0) = 0$. Define $g := \sigma\circ f\circ\sigma^{-1}$. Then, g is an automorphism of D such that $g(0) = 0$ and $g'(0) = \lambda$. Hence, $g(z) = \lambda z$, see Exercise 1.1.4.9 below. This completes the proof of the theorem. □

Exercise 1.1.4.9. *Find all the automorphisms g of the unit disc such that $g(0) = 0$. Hint: use the maximum modulus principle for $g(z)/z$. Find all the holomorphic automorphisms of the unit disc. Hint: use the automorphisms $(z-a)/(1-\overline{a}z)$ to reduce the problem to the case $g(0) = 0$.*

Exercise 1.1.4.10. *Let Ω be an invariant Fatou component. Assume there is a sequence (f^{n_i}) converging locally uniformly on Ω to a constant p. Show that $f(p) = p$. Hint: consider $f^{n_i}(z)$ and $f^{n_i}(f(z))$. Deduce that (f^n) converges locally uniformly on Ω to p. Hint: let $p_1,\ldots,p_m$ be the distinct fixed points of f and V_i be neighbourhoods of p_i which are disjoint; if K is a compact connected set in Ω show that $f^n(K)$ is in $\cup V_i$ for n large enough. Use also Proposition 1.1.4.6.*

The following result is due to Fatou-Cremer.

Theorem 1.1.4.11 (Fatou-Cremer)**.** *An invariant Fatou component is of one of the types in Definition 1.1.4.3.*

Proof. The unbounded component is the basin of infinity. Let Ω be a bounded invariant Fatou component. Recall that Ω is simply connected. If (f^n) admits a non-constant limit value on Ω, using Proposition 1.1.4.6 and Exercise 1.1.4.12 below, one proves that f admits fixed point in Ω.

Exercise 1.1.4.12. *Let g be an automorphism of the unit disc Δ. Assume g has no fixed point in Δ. Show that g^n converge locally uniformly to a constant function.*

The fixed point of f is in the Fatou set. Since the fact that (f^n) admits a non-constant limit value on Ω, the considered fixed point is irrational indifferent. Theorem 1.1.4.8 implies that Ω is a Siegel disc. If the limit values of (f^n) on Ω are constant, Exercise 1.1.4.10 implies that f^n converge locally uniformly to a fixed point p.

Exercise 1.1.4.13. *Let $g : \Delta \to \Delta$ be a holomorphic map. Assume that g admits a fixed point a in Δ which is not attractive. Show that g is an automorphism. Hint: show that we can assume $a = 0$ and consider $g(z)/z$.*

If p is in Ω, the previous exercise implies that p is attractive; so Ω is a the immediate basin of p. If not we want to prove that p is a rational indifferent fixed point and then Ω is a Leau domain. We can assume for simplicity that $p = 0$. So, Ω does not contain 0 and 0 cannot be attractive because it belongs to the boundary $b\Omega$ of Ω which is contained in the Julia set. It follows that f is injective in a neighbourhood of 0.

Exercise 1.1.4.14. *Show that 0 is not repelling. Deduce that $\lambda := f'(0)$ is of module 1.*

Consider a point a in Ω and a small disc of center 0. Since f^n converge to 0 on Ω, replacing a by $f^n(a)$, with n large enough we can assume that $f^n(a)$ is close to 0 and f^n is injective in a neighbourhood D of a. We can choose D connected and containing $f(a)$. Define $D' := D \cup f(D) \cup f^2(D) \cup \ldots$. Hence, D' is connected and $f(D') \subset D'$. Since $f^n(a) \neq 0$ for every n we can define on D' holomorphic functions

$$g_n(z) := \frac{f^n(z)}{f^n(a)}.$$

The fact that $f^n : D' \to \Omega$ is injective implies that $g_n \neq 0, 1, \infty$ on $D' \setminus \{a\}$. We will prove in Theorem 1.2.1.14 below that (g_n) is normal. Consider a subsequence (g_{n_i}) converging to a limit function φ.

Exercise 1.1.4.15. *Show that $\varphi \neq 0$. Hint: if $\varphi = 0$ using maximum principle prove that (g_{n_i}) converge to 0 on D'. Show that $\varphi \neq \infty$. Hint: consider $1/g_{n_i}$. Deduce that (g_n) is normal on D'. Show that φ is either constant or injective.*

Exercise 1.1.4.16. *Show that g_{n+1}/g_n converge to λ on Ω. Deduce that $\varphi(f(z)) = \lambda\varphi(z)$ and then $\varphi(f^n(a)) = \lambda^n$ on D'. Show that φ is not injective. Hint: use the fact that f is open at a.*

Hence φ is constant. We deduce from the previous exercise that $\lambda = 1$. □

1.1.5 Topological entropy and invariant measures

The entropy of a measure introduced by Kolmogorov and the topological entropy are fundamental notions in dynamics. We follow Bowen's approach for the topological entropy. Let n be a positive integer and fix a positive number ϵ.

Definition 1.1.5.1. Two points z and w in $\mathbb{C}$ are (n, ϵ)-*separated* if there is an integer i such that $0 \leq i \leq n-1$ and $|f^i(z) - f^i(w)| \geq \epsilon$.

The last inequality means that the distance between $f^i(z)$ and $f^i(w)$ is larger or equal to ϵ, before time n.

Consider a compact subset K of $\mathbb{C}$. Let $N(K, n, \epsilon)$ denote the maximal number of points in K which are pairwise (n, ϵ)-separated. This number increases when ϵ decreases to 0. Observe that we do note assume that K is invariant.

Definition 1.1.5.2. The *topological entropy* of f on K is

$$h_t(f, K) := \sup_{\epsilon>0} \limsup_{n\to\infty} \frac{1}{n} \log N(K, n, \epsilon).$$

Exercise 1.1.5.3. *Compute the topological entropy of z^d on the unit circle.*

For $n \geq 0$ and z, w in $\mathbb{C}$, define $\operatorname{dist}_n(z, w)$ by

$$\operatorname{dist}_n(z, w) := \max_{0\leq i\leq n-1} |f^i(z) - f^i(w)|.$$

Exercise 1.1.5.4. *Show that dist_n is a distance on $\mathbb{C}$ for every n and dist_0 is the Euclidean distance on $\mathbb{C}$.*

Definition 1.1.5.5. We call *Bowen (n, ϵ)-ball of center a* the set of points z such that $\operatorname{dist}_n(z, a) < \epsilon$.

Exercise 1.1.5.6. *Show that $N(K, n, \epsilon)$ is the maximal number of Bowen disjoint $(n, \epsilon/2)$-balls with centers in K. Let $M(K, n, \epsilon)$ be the minimal number of Bowen (n, ϵ)-balls needed to cover K. Show that*

$$M(K, n, \epsilon) \leq N(K, n, \epsilon) \leq M(K, n, \epsilon/2).$$

Hint: if F is maximal (n, ϵ)-separated then the (n, ϵ)-balls with center in F covers K; if G is a covering of K by $(n, \epsilon/2)$-balls, each ball contains 0 or 1 point of F. Deduce that

$$h_t(f, K) := \sup_{\epsilon>0} \limsup_{n\to\infty} \frac{1}{n} \log M(K, n, \epsilon).$$

Exercise 1.1.5.7. *Let $h(z) = az+b$ with $a \neq 0$ be an automorphism of $\mathbb{C}$. Define $g := h \circ f \circ h^{-1}$ and $H = h(K)$. Show that $h_t(g, H) = h_t(f, K)$.*

Exercise 1.1.5.8. *Let K and K' be compact subsets of $\mathbb{C}$. Show that*

$$h_t(f, K \cup K') = \max\{h_t(f, K), h_t(f, K')\}.$$

If K is a compact subset of a bounded Fatou component, show that $h_t(f, K) = 0$.

Exercise 1.1.5.9. *Show that*

$$h_t(f, K) = h_t(f, f(K)) = h_t(f, f^{-1}(K)).$$

Hint: use a covering of K by balls of radius ϵ.

Exercise 1.1.5.10. *Show that $h_t(f^p, K) = p h_t(f, K)$ for every $p \geq 1$. Hint: use in particular Exercise 1.1.5.9.*

Theorem 1.1.5.11 (Misiurewicz, Przytycki, Gromov)**.** *If K is the filled Julia set of f then*

$$h_t(f, K) = h_t(f, J) = \log d.$$

We will prove this result latter, see also Theorems 1.2.3.1 and 2.2.5.4.

Definition 1.1.5.12. Let ν be a positive (Borel) measure of finite mass with compact support in $\mathbb{C}$. Define the *push-forward* $f_*(\nu)$ by

$$f_*(\nu)(B) := \nu(f^{-1}(B)) \text{ for every Borel set } B$$

or equivalently

$$\int \varphi df_*(\nu) := \int (\varphi \circ f) d\nu \text{ for every continuous function } \varphi \text{ on } \mathbb{C}.$$

Exercise 1.1.5.13. *If δ_a denotes the Dirac mass at a, find $f_*(\delta_a)$. Show that $f_*(\nu)$ is a positive measure, that its mass is equal to the mass of ν and its support is $f(\mathrm{supp}(\nu))$. Show that $(f^n)_*(\nu) = (f_*)^n(\nu)$, and that the operator f_* is linear and continuous on positive measures.*

Exercise 1.1.5.14. *Let $c_1, \ldots, c_{d-1}$ denote the critical points of f where a point of multiplicity m is repeated m times. Let $b_i := f(c_i)$ be the corresponding critical values. Define the postcritical measure of f by*

$$\frac{1}{d} \sum_{i=1}^{d-1} \delta_{b_i}.$$

Let m_n denote the postcritical measure of f^n. Show that the sequence (m_n) is increasing and converges to a probability measure m_∞.

Definition 1.1.5.15. The measure ν is *invariant* if $f_*(\nu) = \nu$.

Exercise 1.1.5.16. *If ν is invariant show that ν is supported in the filled Julia set. Find all the invariant measures whose supports are finite sets.*

Theorem 1.1.5.17. *If K is a non-empty invariant compact set then the set $\mathscr{M}(K)$ of invariant probability measures of supported on K is a non-empty compact convex set.*

Proof. Recall that the set of all the probability measures with support in K is a compact convex set. Consider now the invariant measures. If ν_1 and ν_2 are invariant probability measures then $(\nu_1 + \nu_2)/2$ is also an invariant probability measure. Hence, $\mathscr{M}(K)$ is convex. If ν_n converge to ν, since f_* is continuous, $f_*(\nu_n)$ converge to $f_*(\nu)$. If ν_n are invariant then $f_*(\nu_n) = \nu_n$ converge also to ν; hence, $\nu = f_*(\nu)$. It follows that $\mathscr{M}(K)$ is compact.

Now we show that $\mathscr{M}(K)$ is not empty. Let ν be a probability measure with support in K. Since K is invariant, $(f^n)_*(\nu)$ are supported in K. Consider the Cesàro means

$$\nu_N := \frac{1}{N}\sum_{n=0}^{N-1}(f^n)_*(\nu).$$

They are probability measures with support in K. For every subsequence (ν_{N_i}) converging to a measure ν', we have

$$f_*(\nu_{N_i}) - \nu_{N_i} = \frac{1}{N_i}\big[(f^{N_i})_*(\nu) - \nu\big]$$

which converge to 0 since $(f^{N_i})_*(\nu)$ and ν have mass 1. Therefore, $f_*(\nu') = \nu'$ and ν' is an element of $\mathscr{M}(K)$. $\square$

Let $\mathscr{M}$ denote the set of invariant probability measures. Since they are supported in the filled Julia set, $\mathscr{M}$ is a compact convex set.

Definition 1.1.5.18. A measure in $\mathscr{M}$ is *ergodic* if it is extremal in $\mathscr{M}$, that is, if $\nu = (\nu_1 + \nu_2)/2$ with ν_1 and ν_2 in $\mathscr{M}$, then $\nu_1 = \nu_2 = \nu$.

Every compact convex set admits extremal points. So, ergodic measures exist.

Exercise 1.1.5.19. *Let $\{p_0, \ldots, p_{m-1}\}$ be a periodic cycle of minimal period m. Show that*

$$\nu := \frac{1}{m}\sum_{i=0}^{m-1}\delta_{p_i}$$

is an ergodic invariant measure.

Exercise 1.1.5.20. *Let ν be an invariant probability measure. Show that the following properties are equivalent*

1. *ν is ergodic.*

2. *$\nu(B) = 0$ or $\nu(B) = 1$ for any totally invariant Borel set B. Hint: consider the restriction of ν to B.*

3. *For every Borel sets A and B, we have*
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \nu(f^{-n}(A) \cap B) = \nu(A)\nu(B).$$
Hint: consider the characteristic function $\mathbf{1}_A$ of A and the limit values of the measures $\nu_N := \frac{1}{N}\sum_{n=0}^{N-1}(\mathbf{1}_A \circ f^n)\nu$.

4. *For every function φ in $L^1(\nu)$ (resp. $L^2(\nu)$ or $L^\infty(\nu)$) such that $\varphi \circ f = \varphi$ ν-almost everywhere, then φ is equal to a constant ν-almost everywhere. Hint: consider $B' := \{\alpha \leq \varphi \leq \beta\}$.*

5. *For every functions φ and ψ in $L^2(\nu)$ (resp. bounded, continuous or smooth functions) we have*
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int (\varphi \circ f^n)\psi d\nu = \Big(\int \varphi d\nu\Big)\Big(\int \psi d\nu\Big).$$
Hint: use the density of simple functions and the fact that the operators $\varphi \mapsto \frac{1}{N}\sum_{n=0}^{N-1} \varphi \circ f^n$ have bounded norms in $L^2(\mu)$.

Exercise 1.1.5.21. *Let $a \in \mathbb{C}$ with $|a| = 1$. Study the ergodicity of the map $z \mapsto az$ on the unit circle with respect to the Lebesgue measure. Hint: consider two cases: a is a root of unity and a determines an irational rotation.*

Theorem 1.1.5.22 (Birkhoff's ergodic theorem)**.** *Let ν be an ergodic invariant probability measure. Let φ be a function integrable with respect to ν. Then for ν-almost every $z \in \mathbb{C}$ we have*
$$\frac{1}{N} \sum_{n=0}^{N-1} \varphi(f^n(z)) \to \int \varphi d\nu.$$

Proof. Since φ can be written as a convergent serie of bounded functions, we can assume that φ is bounded. Consider the functions
$$\Phi_0 := 0 \quad \text{and} \quad \Phi_N := \sum_{n=0}^{N-1} \varphi(f^n(z))$$
and
$$\psi^* := \limsup_{N\to\infty} \frac{\Phi_N}{N} \quad \text{and} \quad \psi_* := \liminf_{N\to\infty} \frac{\Phi_N}{N}.$$
It is easy to show that ψ_* and ψ^* are bounded and $\psi_* \leq \psi^*$. Moreover, there is a constant $C > 0$ such that
$$|\psi^* \circ f - \psi^*| \leq C/N \quad \text{and} \quad |\psi_* \circ f - \psi_*| \leq C/N.$$

So $\psi^* \circ f = \psi^*$ and $\psi_* \circ f = \psi_*$. We deduce from Exercise 1.1.5.20 that ψ^* and ψ_* are constant ν-almost everywhere. We want to show that $\psi^* = \psi_* = \int \varphi d\nu$ ν-almost everywhere.

Using Fatou's lemma and the invariance of ν, we obtain

$$\psi^* = \int \psi^* d\nu \geq \limsup_{N\to\infty} \int \frac{1}{N}\Phi_N d\nu = \int \varphi d\nu.$$

Assume that $\psi^* > \int \varphi d\nu$. We seek a contradiction. Subtracting a constant from φ allows us to assume that $\psi^* = 0$ ν-almost everywhere and $\int \varphi d\nu < 0$. Define $\Psi_N := \max\{\Phi_0, \ldots, \Phi_N\}$ and $A_N := \{\Psi_N > 0\}$. Since $\psi^* > 0$ ν-almost everywhere, A_N increase to a set of ν-measure 1. On the other hand, since $\Phi_0 = 0$, we have on A_N

$$\Psi_N \circ f(x) + \varphi(x) = \max\{\Phi_0 \circ f(x), \ldots, \Phi_N \circ f(x)\} + \varphi(x) \geq \max_{1\leq n\leq N} \Phi_n(x) = \Psi_N(x).$$

Hence $\varphi(x) \geq \Psi_N(x) - \Psi_N \circ f(x)$ on A_N. This and the fact that $\Psi_N \geq 0$ with equality outside A_N imply

$$\int_{A_N} \varphi d\nu \geq \int_{A_N} \Psi_N d\nu - \int_{A_N} (\Psi_N \circ f) d\nu \geq \int \Psi_N d\nu - \int (\Psi_N \circ f) d\nu = 0$$

since ν is invariant. This contradicts the facts that $\int \varphi d\nu < 0$ and A_N increase to a set of total ν-measure. □

Exercise 1.1.5.23. *Let ν be an ergodic measure and A be a Borel set. For every $z \in \mathbb{C}$ and $N \geq 0$ denote by $T_N(z, A)$ the number of points in $\{z, \ldots, f^{N-1}(z)\} \cap A$. Show that for ν-almost every z we have*

$$\lim_{N\to\infty} \frac{T_N(z, A)}{N} = \nu(A).$$

Hint: apply Birkhoff's theorem to the characteristic function of A.

Definition 1.1.5.24. An invariant probability measure ν is *mixing* if for every Borel sets A and B we have

$$\lim_{n\to\infty} \nu(f^{-n}(A) \cap B) = \nu(A)\nu(B).$$

We deduce from Exercise 1.1.5.20 that if ν is mixing then it is ergodic.

Exercise 1.1.5.25. *Show that ν is mixing if and only if for every functions φ and ψ in $L^2(\mu)$ (resp. continuous or smooth) we have*

$$\lim_{n\to\infty} \int (\varphi \circ f^n)\psi d\nu = \Big(\int \varphi d\nu\Big)\Big(\int \psi d\nu\Big).$$

Show that the Lebesgue measure on the unit circle is not mixing for rotations.

Corollary 1.1.5.26. *Let ν be an ergodic invariant measure such that* $\log|f'|$ *is integrable with respect to ν. Then for ν-almost every $z \in \mathbb{C}$*

$$\lim_{n\to\infty} |(f^n)'(z)|^{1/n} = \exp\Big(\int \log|f'|d\nu\Big).$$

Proof. We have

$$\log|(f^n)'(z)| = \sum_{i=0}^{n-1} \log|f'(f^i(z))|.$$

So it is enough to apply the Birkhoff theorem to $\log|f'|$. □

Somehow, $\lim|(f^n)'|^{1/n}$ measure the rate of dilatation or contraction of the dynamical system. The previous corollary says that relatively to an ergodic measure, this quantity exists ν-almost everywhere and is equal to the constant

$$\chi_\nu := \int \log|f'|d\nu.$$

This is *the Lyapounov exponent of f with respect to ν.* A precise definition of Lyapounov exponents in a more general setting will be given latter.

Remark 1.1.5.27. For holomorphic maps of several variables, the dilatation or contraction depends on directions. In this case, there are several Lyapounov exponents.

We now introduce the entropy of measure. Let ν be a probability measure with compact support. Let $\xi := \{A_1, \ldots, A_m\}$ be a finite partition where A_i are pairwise disjoint Borel sets such that $\nu(A_1 \cup \ldots \cup A_m) = 1$.

Definition 1.1.5.28. The *entropy of the partition ξ with respect to ν* is the following non-negative number

$$H(\nu,\xi) := \sum_{i=1}^{m} -\nu(A_i)\log\nu(A_i).$$

This notion is independent of f.

Exercise 1.1.5.29. *Let $\eta = (B_1, \ldots, B_k)$ be another partition finer than ξ, that is, each A_i is a union of B_j. Show that $H(\nu,\eta) \geq H(\nu,\xi)$. Hint: the function $x \mapsto -x\log x$ is concave on $[0,1]$. Show that $H(\nu,\eta) \leq \log k$.*

Definition 1.1.5.30. Let $\xi := \{A_1, \ldots, A_m\}$ and $\eta = (B_1, \ldots, B_k)$ be two partitions. Define *the relative entropy* $H(\nu,\xi|\eta)$ by

$$\begin{aligned} H(\nu,\eta|\xi) &:= -\sum_i \nu(A_i)\sum_j \frac{\nu(B_j\cap A_i)}{\nu(A_i)}\log\frac{\nu(B_j\cap A_i)}{\nu(A_i)} \\ &:= -\sum_{i,j} \nu(B_j\cap A_i)\log\frac{\nu(B_j\cap A_i)}{\nu(A_i)}. \end{aligned}$$

Exercise 1.1.5.31. *If $\xi := \{A_1, \dots, A_m\}$ and $\eta = (B_1, \dots, B_k)$, let $\xi \vee \eta$ denote the partition formed by $A_i \cap B_j$. Show that*

1. *If ξ' is finer than ξ then $H(\nu, \eta|\xi') \leq H(\nu, \eta|\xi)$. In particular, we have $H(\nu, \eta|\xi) \leq H(\nu, \eta)$. Hint: use the concavity of $-x \log x$.*

2. *$H(\nu, \xi \vee \eta) = H(\nu, \xi) + H(\nu, \eta|\xi) \leq H(\nu, \xi) + H(\nu, \eta)$.*

3. *For all partitions ξ, η and η', we have $H(\nu, \eta \vee \eta'|\xi) \leq H(\nu, \eta|\xi) + H(\nu, \eta'|\xi)$. Hint: use 1. and 2.*

Now, assume that ν is invariant.

Exercise 1.1.5.32. *Show that $f^{-1}(\xi) := \{f^{-1}(A_1), \dots, f^{-1}(A_m)\}$ is a partition. Show that $H(\nu, f^{-1}(\xi)) = H(\nu, \xi)$.*

Let ξ_n^f denote the finite partition $\xi \vee f^{-1}(\xi) \vee \dots \vee f^{-n+1}(\xi)$, i.e. the partition formed by the disjoint Borel sets of type

$$A_{i_0} \cap f^{-1}(A_{i_1}) \cap \dots \cap f^{-n+1}(A_{i_{n-1}})$$

for $1 \leq i_0, \dots, i_{n-1} \leq m$. Observe that ξ_{n+1}^f is finer than ξ_n^f. Hence, the sequence of $H(\nu, \xi_n^f)$ is increasing.

Exercise 1.1.5.33. *Show that $H(\nu, \xi_n^f)$ is sub-additive, that is,*

$$H(\nu, \xi_{m+n}^f) \leq H(\nu, \xi_m^f) + H(\nu, \xi_n^f)$$

for $m, n \geq 0$. Hint: use previous exercises. Deduce that $\lim \frac{1}{n} H(\nu, \xi_n^f)$ exists and is equal to $\inf \frac{1}{n} H(\nu, \xi_n^f)$.

Define

$$h_f(\nu, \xi) := \lim_{n \to \infty} \frac{1}{n} H(\nu, \xi_n^f) = \inf_{n \geq 0} \frac{1}{n} H(\nu, \xi_n^f).$$

Exercise 1.1.5.34. *Show that $h_f(\nu, \eta) \leq h_f(\nu, \xi) + H(\nu, \eta|\xi)$. Hint: use Exercise 1.1.5.31.*

Definition 1.1.5.35. *The entropy* of the invariant probability measure ν is

$$h_f(\nu) := \sup_\xi h_f(\nu, \xi).$$

Exercise 1.1.5.36. *Show that $h_{f^n}(\nu) = n h_f(\nu)$.*

Theorem 1.1.5.37 (Variational principle)**.** *If K is a compact set containing the support of an invariant measure ν, then*

$$h_f(\nu) \leq h_t(f, K).$$

Proof. Let $\xi = \{A_1, \ldots, A_m\}$ be a finite partition of K. Choose compact subsets B_i of A_i such that $\nu(A_i \setminus B_i)$ are small enough. Let B_0 the complement of $\cup B_i$ and $\eta := \{B_0, B_1, \ldots, B_m\}$.

Exercise 1.1.5.38. *Show that $h_f(\nu, \xi) \leq h_f(\nu, \eta) + 1$. Hint: use Exercise 1.1.5.34.*

We cover $\mathbb{P}^1$ by open sets $\mathscr{C} := \{B_0 \cup B_1, \ldots, B_0 \cup B_m\}$.

Exercise 1.1.5.39. *Show that for $\epsilon > 0$ small enough, every disc of radius ϵ, is contained in one of the open sets $B_0 \cup B_i$.*

Define $f^{-n}\mathscr{C} := \{f^{-n}(B_0 \cup B_1), \ldots, f^{-n}(B_0 \cup B_m)\}$ and denote by $\mathscr{C}_n^f$ the covering $\mathscr{C} \vee f^{-1}\mathscr{C} \vee \ldots \vee f^{-n+1}\mathscr{C}$.

Exercise 1.1.5.40. *Any Bowen (n, ϵ)-ball is contained in one of the elements of $\mathscr{C}_n^f$. Deduce that the minimal number $L(n)$ of elements in $\mathscr{C}_n^f$ needed to cover K satisfies $L(n) \leq M(n, \epsilon/2)$ where $M(n, \epsilon/2) := M(K, n, \epsilon/2)$. Show that each element of $\mathscr{C}_n^f$ is formed by at most 2^n elements of η_n^f. Deduce that $\#\eta_n^f \leq 2^n L(n)$.*

It follows from the concavity of $-x \log x$ that

$$H(\nu, \eta_n^f) \leq \log \#\eta_n^f \leq \log[2^n L(n)] \leq \log[2^n M(n, \epsilon/2)].$$

Hence

$$h_f(\nu, \eta) \leq h_t(f, K) + \log 2$$

and

$$h_f(\nu, \xi) \leq h_f(\nu, \eta) + 1 \leq h_t(f, K) + \log 2 + 1.$$

Therefore,

$$h_f(\nu) \leq h_t(f, K) + \log 2 + 1.$$

This applied to f^n and Exercise 1.1.5.36 imply the result. $\square$

We deduce from Theorem 1.1.5.11 the following result.

Corollary 1.1.5.41. *For every invariant probability measure ν we have*

$$h_f(\nu) \leq \log d.$$

Exercise 1.1.5.42. *Let $f(z) = z^d$ and μ the Lebesgue probability measure on the unit circle. Compute $h_f(\nu)$.*

Let φ be a function on $\mathbb{C}$. Define $f^*(\varphi) := \varphi \circ f$ and

$$f_*(\varphi)(z) := \sum_{w \in f^{-1}(z)} \varphi(w)$$

where the points in $f^{-1}(z)$ are counted with multiplicities. It is clear that the operators f^* and f_* are linear.

Exercise 1.1.5.43. *Show that* $\frac{1}{d}f_*(f^*(\varphi)) = \varphi$*. Prove that* $f_*(\varphi)$ *is continuous if* φ *is continuous.*

Definition 1.1.5.44. Let ν be a positive measure with compact support in $\mathbb{C}$. Define the measure $f^*(\nu)$ by

$$\int \varphi df^*(\nu) := \int f_*(\varphi)d\nu.$$

Exercise 1.1.5.45. *Show that* $f^*(\nu)$ *is a positive measure and that its mass is equal to* d *times the mass of* ν*. Prove that* $f^*(\nu)$ *depends continuously on* ν*.*

Definition 1.1.5.46. A positive measure ν with compact support in $\mathbb{C}$ is *totally invariant* if $\frac{1}{d}f^*(\nu) = \nu$.

Exercise 1.1.5.47. *Show that if* ν *is totally invariant, it is invariant. Determine* $f^*(\delta_a)$ *where* δ_a *is the Dirac mass at* a*. If* δ_a *is totally invariant show that* $\{a\}$ *is totally invariant. Prove that there is a probability measure* μ *such that* $\frac{1}{d}f^*(\mu) = \mu$*. Hint: use Cesàro sum* $\frac{1}{N}\sum_{n=0}^{N-1} d^{-n}(f^n)^*(\nu)$ *where* ν *is a probability measure. Let* ν *be a totally invariant measure. Suppose* A *is a Borel set and that* f *is injective on* A*. Compare* $\nu(f(A))$ *and* $\nu(A)$*. Hint: compute* $f_*(1_A)$*.*

Exercise 1.1.5.48. *Determine all the totally invariant measures for* $f(z) := z^d$*.*

1.2 Dynamics of rational fractions

1.2.1 Riemann sphere and holomorphic maps

The *Riemann sphere* $\mathbb{P}^1$ is the compactification of the complex plane $\mathbb{C}$ by adding a point at infinity that we denote by ∞, that is, $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. We can identify the Riemann sphere with the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3 = \mathrm{O}x_1x_2x_3$, i.e. the sphere of center 0 and of radius 1, via the stereographic map. More precisely, the north pole $(0, 0, 1)$ is identified to ∞ and a point $z^* \neq (0, 0, 1)$ of $\mathbb{S}^2$ is identified to the point z in the plane $\mathrm{O}x_1x_2 \simeq \mathbb{C}$ which is the intersection of $\mathrm{O}x_1x_2$ with the line passing through $(0, 0, 1)$ and z^*.

For the moment, ∞ is somehow a special point. But this is a wrong impression. Consider the map $\tau_a : \mathbb{P}^1 \to \mathbb{P}^1$

$$\tau_a(z) := \frac{1}{z - a}$$

which is a bijection. It sends a to ∞ and ∞ to 0. So, the points a and ∞ are in some sense equivalent.

The map τ_a defines also a bijection from $\mathbb{P}^1 \setminus \{a\}$ onto $\mathbb{C}$. We say that $(\mathbb{P}^1\setminus\{a\}, \tau_a, \mathbb{C})$ defines a *chart* of $\mathbb{P}^1$. Observe that any couple of such charts covers all the Riemann sphere. One checks easily that $\tau_a \circ \tau_b^{-1}$ defines a holomorphic function on its domain of definition. Roughly speaking, one pass from a chart to another by a holomorphic map. One can imagine that $\mathbb{P}^1$ is obtained by sticking two copies $\mathbb{C}_a$ and $\mathbb{C}_b$ of the complex plane $\mathbb{C}$, one sticks the point $z \neq 1/(a-b)$ in $\mathbb{C}_b$ with the point $\tau_a(\tau_b^{-1}(z)) \neq 1/(b-a)$ in $\mathbb{C}_a$. The sticking respects the complex structures on $\mathbb{C}_a$ and $\mathbb{C}_b$.

The following definition is not intrinsic; it is given specially for $\mathbb{P}^1$, but it is easy to use.

Definition 1.2.1.1. Let D be an open subset of $\mathbb{P}^1$. We call *holomorphic function* any function $f : D \to \mathbb{C}$ such that

1. f is holomorphic on $D \setminus \{\infty\}$.
2. If D contains ∞ then $f(1/z)$ is holomorphic in a neighbourhood of 0.

It is clear that holomorphic functions on D are continuous.

Exercise 1.2.1.2. *Show that $f : D \to \mathbb{C}$ is holomorphic if and only if $f \circ \tau_a^{-1}$ is holomorphic on $\tau_a(D \setminus \{a\})$ for at least two different values of a.*

Exercise 1.2.1.3. *Assume that $a \notin D$. Show that $f : D \to \mathbb{C}$ is holomorphic if and only if $f \circ \tau_a^{-1}$ is holomorphic on $\tau_a(D)$.*

Exercise 1.2.1.4. *Let $f : \mathbb{P}^1 \to \mathbb{C}$ be a holomorphic function. Show that f is constant. Hint: use the maximum modulus principle.*

Definition 1.2.1.5. Let D be an open subset of $\mathbb{P}^1$. A continuous **map** $f : D \to \mathbb{P}^1$ is said to be *holomorphic* if near each point of D either f or $1/f$ is a holomorphic function[1].

It is clear that holomorphic maps are continuous.

Exercise 1.2.1.6. *Let $f : D \to \mathbb{P}^1$ be a holomorphic map. Show that $\tau_a \circ f$ is a holomorphic function on $D \setminus f^{-1}(a)$.*

Exercise 1.2.1.7. *Let P and Q be polynomials without common factor. Show that $f := P/Q$ defines a holomorphic maps from $\mathbb{P}^1$ to $\mathbb{P}^1$. Assume that $d :=$ $\max(\deg P, \deg Q) \geq 1$. Show that $f^{-1}(z)$ contains exactly d points counted with multiplicities for every $z \in \mathbb{P}^1$. We call d the degree of f.*

Exercise 1.2.1.8. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a holomorphic map. Show that f is a rational fraction. Hint: let $a_1, \ldots, a_d$ be the points in $f^{-1}(\infty)$ counted with multiplicity; show that $g(z) := f(z)(z-a_1)\ldots(z-a_d)$ is a polynomial.*

The distance on the unit sphere $\mathbb{S}^2$ induces a distance on $\mathbb{P}^1$ that we call *the spherical distance on $\mathbb{P}^1$:*

$$\mathrm{dist}_{\mathbb{P}^1}(z,w) := \frac{2|z-w|}{(1+|z|^2)^{1/2}(1+|w|^2)^{1/2}} \quad \text{and} \quad \mathrm{dist}_{\mathbb{P}^1}(z,\infty) := \frac{2}{(1+|z|^2)^{1/2}}.$$

[1] It is useful to compare this notion with the notion of meromorphic function.

Exercise 1.2.1.9. *Show that* $\mathrm{dist}_{\mathbb{P}^1}(z,w)$ *is equal to the Euclidean distance between* z^* *and* w^**. Deduce that* $\mathrm{dist}_{\mathbb{P}^1}$ *is a distance.*

With respect to this distance, the previous exercise shows that $\mathbb{P}^1$ has finite diameter. This is a difference with the Euclidean distance on $\mathbb{C}$. It is more difficult to compute with the spherical distance on $\mathbb{P}^1$ (see also Exercise 1.2.1.12 below) but the fact that $\mathbb{P}^1$ is compact is very useful.

Exercise 1.2.1.10. *Determine the holomorphic automorphisms of* $\mathbb{P}^1$ *which preserve the spherical distance. Such automorphisms are called isomorphisms. Show that the family of the isomorphisms is a group acting transitively on* $\mathbb{P}^1$*, that is, for every* a*,* b *in* $\mathbb{P}^1$ *there is an isomorphism* τ *such that* $\tau(a)=b$*.*

Let $f : D \to \mathbb{P}^1$ be a holomorphic map. For $z \in D$, define the *spherical derivative* of f at z by

$$|Df(z)|_{\mathbb{P}^1} := \lim_{w\to z} \frac{\mathrm{dist}_{\mathbb{P}^1}(f(z),f(w))}{\mathrm{dist}_{\mathbb{P}^1}(z,w)}.$$

Exercise 1.2.1.11. *Assume* $0 \in D$ *and* $f(0)=0$*. Compute* $|Df(0)|_{\mathbb{P}^1}$ *in term of* $f'(0)$*. Deduce that* $|Df(z)|_{\mathbb{P}^1}$ *exists for every* $z \in D$*. Hint: use Exercise 1.2.1.10.*

Exercise 1.2.1.12. *Assume that* D *and* $f(D)$ *are relatively compact in* $\mathbb{C}$*. Show that there is a constant* $c>0$ *such that*

$$c^{-1}|f'(z)| \le |Df(z)|_{\mathbb{P}^1} \le c|f'(z)|.$$

Exercise 1.2.1.13 (Brody renormalization lemma)**.** *For a holomorphic map* g *from a domain of* $\mathbb{C}$ *to* $\mathbb{P}^1$ *define*

$$\|g'(a)\| := \lim_{z\to a} \frac{\mathrm{dist}_{\mathbb{P}^1}(g(z),g(a))}{|z-a|}.$$

Let $g_n : D \to \mathbb{P}^1$ *be holomorphic maps from the unit disc* D *to* $\mathbb{P}^1$*.*

1. *Assume that for every compact* K *in* D *there is a constant* $c>0$ *such that* $\|g_n'\| \le c$ *on* K*. Show that* (g_n) *is equicontinuous and hence normal.*

2. *From now on, assume that* (g_n) *is not normal. Show that there are automorphisms* σ_n *of* D *such that, up to extracting a subsequence,* $h_n := g_n \circ \sigma_n$ *satisfy* $\lim_{n\to\infty} \|h_n'(0)\| = +\infty$*.*

3. *We are going to construct a non-constant map from* $\mathbb{C}$ *to* $\mathbb{P}^1$*. Reducing* D *allows us to assume that* h_n *are defined in a neighbourhood of* $\overline{D}$*. Show that there are* z_n *in* D *such that* $(1-|z|^2)\|h_n'(z)\|$ *is maximal at* z_n*.*

4. *Define*
$$\Phi_n(z) := h_n(\tau_n(z)) \quad \textit{where} \quad \tau_n(z) := \Big(\frac{z+z_n}{1+\overline{z}_n z}\Big).$$
Prove that
$$\|\Phi_n'(z)\|(1-|z|^2) \leq \|\Phi_n'(0)\| \quad \textit{and} \quad \lim_{n\to\infty} \|\Phi_n'(0)\| = +\infty.$$
Hint: show first that
$$\frac{|\tau_n'(z)|}{1-|\tau_n(z)|^2} \leq \frac{1}{1-|z|^2}.$$

5. *Define $R_n := \|\Phi_n'(0)\|$ and $l_n(z) := \Phi_n(z/R_n)$. Show that for every compact set K in $\mathbb{C}$, the sequence (l_n), with n large enough, is defined and equicontinuous on K. Deduce that there is a subsequence of (l_n) converging to a holomorphic map $l : \mathbb{C} \to \mathbb{P}^1$ with $l'(0) = 1$ and $|l'(z)| \leq 1$ for $z \in \mathbb{C}$.*

Theorem 1.2.1.14 (Montel)**.** *Let a, b, c be three distinct points in $\mathbb{P}^1$. Then the family $\mathscr{F}$ of holomorphic maps from an open set D of $\mathbb{C}$ to $\mathbb{P}^1 \setminus \{a, b, c\}$ is normal.*

Proof. We can assume that D is the unit disc of $\mathbb{C}$. Assume there is a non-normal sequence of holomorphic maps $f_n : D \to \mathbb{P}^1 \setminus \{a, b, c\}$. Using a change of coordinate, we can assume $a = 0$, $b = 1$ and $c = \infty$. Then f_n are holomorphic functions with values in $\mathbb{C} \setminus \{0, 1\}$.

Exercise 1.2.1.15. *Show that there are functions $g_{n,k}$ such that $(g_{n,k}(z))^{2^k} = f_n(z)$. Prove that $g_{n,k}$ omits the values in $R_k := \{e^{2i\pi p/2^k},\ 1 \leq p \leq 2^k\}$. Show that for each k, the family $(g_{n,k})$ is not normal.*

As in Exercise 1.2.1.13, we construct a non-constant holomorphic map $l_k : \mathbb{C} \to \mathbb{P}^1$ such that $l_k'(0) = 1$ using the functions $g_{n,k}$.

Exercise 1.2.1.16. *Using l_k construct a non-constant function $l : \mathbb{C} \to \mathbb{C}$ which omits the values in $\cup_{k\geq 0} R_k$. Hint: consider the two cases where (l_k) is normal or not normal; use Exercise 1.2.1.13.*

Since $\cup_{n\geq 0} R_n$ is dense in $\mathbb{S}^1$ and since l is open, l omits the values in $\mathbb{S}^1$. It follows that either $|l|$ or $|1/l|$ is bounded by 1. By Liouville's theorem, l is constant. This is a contradiction. □

There is another construction of the Riemann sphere which can be extended to define projective spaces in higher dimension. Consider in $\mathbb{C}^2 \setminus \{0\}$ the following equivalence relation : $(z_0, z_1) \sim (z_0', z_1')$ if and only if there is $\lambda \in \mathbb{C}^*$ such that $z_0' = \lambda z_0$ and $z_1' = \lambda z_1$. So, two points are equivalent if they belong to the same complex line passing through 0. Then the quotient $\mathbb{C}^2 \setminus \{0\}/\sim$ is identified to $\mathbb{P}^1$. Indeed, each complex line passing through 0, except $\{z_0 = 0\}$, intersects the vertical line $\{z_0 = 1\}$ at a unique point which determines the line. So, $\mathbb{C}^2 \setminus \{0\}/\sim$

can be identified with the union of $\{z_0 = 1\}$ with one point. Hence, it can be identified to $\mathbb{P}^1$.

There is a canonical map $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$. The space $\mathbb{P}^1$ has the quotient topology, that is, U is an open set in $\mathbb{P}^1$ if and only if $\pi^{-1}(U)$ is open in $\mathbb{C}^2 \setminus \{0\}$. The homogeneous coordinates of a point p in $\mathbb{P}^1$ is $[z_0 : z_1]$ provided that $\pi(z_0, z_1) = p$. So, we identify $[z_0 : z_1]$ with $[\lambda z_0 : \lambda z_1]$ for $\lambda \in \mathbb{C}^*$.

If P and Q are homogeneous polynomials of degree $d \geq 1$ without common factor, there is a unique rational function $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f \circ \pi = \pi \circ F$ where $F := (P, Q)$ is the polynomial map from $\mathbb{C}^2$ to $\mathbb{C}^2$. In the coordinate $[1 : z_1]$, we have $f(z_1) = P(1, z_1)/Q(1, z_1)$.

Exercise 1.2.1.17. *Given a rational function f on $\mathbb{P}^1$, construct F satisfying the above condition. Is F unique? What is the map F corresponding to a polynomial map in $\mathbb{C}$? What is F if f is an automorphism of $\mathbb{P}^1$?*

1.2.2 Fatou and Julia sets for rational fractions

Let D be an open set in $\mathbb{P}^1$. Consider a family $\mathscr{F}$ of holomorphic maps from D with values in $\mathbb{P}^1$.

Definition 1.2.2.1. The family $\mathscr{F}$ is *normal* if for any sequence $(f_n) \subset \mathscr{F}$ one can extract a subsequence (f_{n_i}) converging locally uniformly to a holomorphic map from D to $\mathbb{P}^1$.

Exercise 1.2.2.2. *Let D be an open set in $\mathbb{C}$ and $\mathscr{F}$ be a family of holomorphic functions on D. Show that $\mathscr{F}$ is normal in the sense of Definition 1.2.2.1 if and only if it is normal in the sense of Definition 1.1.3.1.*

Exercise 1.2.2.3. *Show that $\mathscr{F}$ is normal if and only if it is locally equicontinuous. Hint: use Exercises 1.1.3.2 and 1.2.1.12.*

Now, consider a rational fraction f of degree $d \geq 2$. A point c is *critical* for f if f is not injective on any neighbourhood of c. Using the charts of $\mathbb{P}^1$ introduced in the last section, we can consider f restricted to a neighbourhood of c as a usual holomorphic function. If c is a solution of multiplicity m of $f'(z) = 0$ then m is called *the multiplicity of* c. The image $f(c)$ of c is *a critical value* of f.

Exercise 1.2.2.4. *Show that f admits exactly $2d-2$ critical points counted with multiplicities. If f is a polynomial of degree $d \geq 2$ show that ∞ is a critical point of multiplicity $d-1$.*

Periodic, pre-periodic points and their multiplicities are defined as in the case of polynomials. The notions of (super-)attracting, repelling, rationally and irrationally indifferent fixed points and periodic cycles can be extended easily to the case of rational fractions.

Exercise 1.2.2.5. *Show that f^n admits d^n+1 periodic points of period n. If f is a polynomial of degree $d \geq 2$ show that ∞ is a super-attracting fixed point. What is the multiplicity of ∞ as a fixed point?*

We define the Fatou and Julia sets as in the case of polynomials. The Fatou set is the maximal open set F where (f^n) is normal. The Julia set J is the complement of F. We show as in the polynomial case that F and J are also the Fatou and Julia sets of f^p for every $p \geq 1$.

Exercise 1.2.2.6. *Show that J is non-empty. Hint: if (f^{n_i}) converges to h, Rouché's theorem contradicts the fact that h has finite degree. Show that if the interior of J is non-empty then $J = \mathbb{P}^1$. Hint: if $a \notin J$, we can assume $a = \infty$, hence J is compact in $\mathbb{C}$; show that (f^n) is normal in the interior of J.*

Exercise 1.2.2.7. *Let U be an open set which intersects J. Show that the complement of $\cup_{n\geq 0} f^n(U)$ contains at most 2 points. Hint: use Montel's theorem. Show that this set is totally invariant. Deduce that there is a set $\mathcal{E}$ which is maximal among totally invariant finite sets, that $\mathcal{E}$ contains at most 2 points and they are periodic super-attracting. Deduce that $\mathcal{E} \cap J = \varnothing$ and that J is the smallest totally invariant compact set which is infinite. Deduce also that J is the smallest totally invariant compact set which does not intersect $\mathcal{E}$.*

Exercise 1.2.2.8. *Let a be a non-exceptional point. Show that the closure of $\cup_{n\geq 0} f^{-n}(a)$ contains J. Hint: use 1.2.2.7.*

We will prove latter that J is perfect, in particular, it is uncountable. It is also true that repelling cycles are in the Julia set. The Sullivan theorem on the non-existance of wandering Fatou component is valid for rational fractions. The (super-)attracting invariant component, the Leau and Siegel domains are defined as in the polynomial case. We have seen in the polynomial case that all the Fatou components, except the basin of ∞, are simply connected. The situation for rational fractions is different.

Definition 1.2.2.9. An invariant Fatou component is called *a Herman ring* if the restriction of f to this component is conjugate to a rotation on a ring.

We have the following result of Fatou-Cremer.

Theorem 1.2.2.10 (Fatou-Cremer)**.** *If Ω is an invariant Fatou component, then Ω is either a (super-)attracting component, a Leau domain, a Siegel domain or a Herman ring.*

There exist rational fractions admitting Herman rings.

Exercise 1.2.2.11. *Assume that f admits a totally invariant Fatou component. Show that the other Fatou components are simply connected. In particular f admits no Herman ring. If f admits two totally invariant components, show that they are the only Fatou components. Hint: see Exercise 1.1.3.8.*

Exercise 1.2.2.12. *Assume there is an orbit* $\mathrm{O}^+(a) := \{a, f(a), f^2(a), \ldots\}$ *which is dense in* $\mathbb{P}^1$. *Show that the Fatou set is empty. Hint: use the Fatou-Cremer theorem.*

A rational fraction with empty Fatou set is called *chaotic*. For such a map, the sequence (f^n) is nowhere equicontinuous. Lattès proved that chaotic rational fractions do exist.

The following result is due to Fatou-Shishikura.

Theorem 1.2.2.13 (Fatou-Shishikura)**.** *The number of non-repelling cycles of* f *is at most equal to* $2d - 2$ *and this bound is sharp.*

1.2.3 Topological entropy and invariant measures

The notion of (n, ϵ)-separated set, the distances dist_n, the notion of Bowen balls and then the notion of topological entropy can be extended easily to the case of rational fractions or even to a continuous map on a compact metric space. We use here the spherical distance of $\mathbb{P}^1$ instead of the Euclidean distance in $\mathbb{C}$. Since $\mathbb{P}^1$ is compact, one can define the entropy of f on $K = \mathbb{P}^1$. This entropy is denoted by $h_t(f)$ and is called simply *the topological entropy* of f.

Theorem 1.2.3.1 (Misiurewicz, Przytycki, Gromov)**.** *If* f *is a rational fraction of degree* d *then*

$$h_t(f) = \log d.$$

Note that the inequality $h_t(f) \geq \log d$ is a consequence of Theorems 1.1.5.37 and 2.2.5.4.

The notion of invariant measure, ergodicity, mixing, Birkhoff's ergodic theorem, the Lyapounov exponent and the entropy of measure are valid for general differentiable self-maps on compact real manifolds, hence in particular for rational fractions. We only have to replace $|f'|$ by $|Df|_{\mathbb{P}^1}$. The variational principle is also valid here. It implies that if ν is an invariant probability measure then $h_f(\nu) \leq \log d$.

In the next section, we will construct an invariant measure μ with maximal entropy $\log d$.

Notes for Chapter 1. The iteration theory of rational maps was developed around 1920 by Fatou, Julia, Cremer, Leau, ... There are several monographs on that theory: Beardon [4], Carleson-Gamelin [15], Berteloot-Mayer [8] for example. The theorem on the non-wandering domain is due to Sullivan who introduced the use of the measurable Riemann mapping theorem in this area. Brody's lemma is due to Brody when the map is with values in an arbitrary compact complex manifolds. The proof of Montel's theorem given here is due to Ros. There are many sources for ergodic theory and dynamical system, see Walters [76], Katok-Hasselblatt [58].

Chapter 2

Equilibrium measure and properties

2.1 Potential and quasi-potential of a measure

2.1.1 Subharmonic and quasi-subharmonic functions

Let D be an open set in $\mathbb{C}$. Recall that a differential 1-form or a 2-form on D is an expression $\alpha dx + \beta dy$ or $\gamma dx \wedge dy$ respectively where α, β, γ are functions on D (a 0-form is simply a function). Recall also that dx and dy are $\mathbb{R}$-linear forms on the tangent space and $dy \wedge dx = -dx \wedge dy$. We deduce from the identity $dz = dx + idy$ and $d\overline{z} = dx - idy$ that $dx = \frac{1}{2}(dz + d\overline{z})$ and $dy = \frac{1}{2i}(dz - d\overline{z})$. Therefore, differential forms can be written as $\alpha' dz + \beta' d\overline{z}$ or $\gamma' dz \wedge d\overline{z}$. A 1-form $\alpha' dz + \beta' d\overline{z}$ is of bidegree $(1,0)$ if $\beta' = 0$ and of bidegree $(0,1)$ if $\alpha' = 0$. In the case of complex dimension 1, every 2-form is of bidegree $(1,1)$. We will come back to the notion of bidegree of forms in the higher dimension case. Recall that the integral of a 2-form $\gamma dx \wedge dy$ on D is defined by

$$\int_D \gamma dx \wedge dy := \int_D \gamma dx dy$$

when the last integral makes sense. So, we can identify $dx \wedge dy$ with the Lebesgue measure on $\mathbb{C}$. Note that $\frac{i}{2} dz \wedge d\overline{z} = dx \wedge dy$.

Let u be a function on an open set $D \subset \mathbb{C}$. Recall that

$$\frac{\partial}{\partial z} := \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

The operators d, d^c, ∂ and $\overline{\partial}$ are defined by

$$du := \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

and
$$\partial u := \frac{\partial u}{\partial z} dz, \quad \overline{\partial} u := \frac{\partial u}{\partial \overline{z}} d\overline{z}.$$
These operators act also on forms. We have

$$d(\alpha dx + \beta dy) := d\alpha \wedge dx + d\beta \wedge dy, \quad d^c := \frac{i}{2\pi}(\overline{\partial} - \partial)$$

and
$$\partial(\alpha' dz + \beta' d\overline{z}) := \partial\alpha' \wedge dz + \partial\beta' \wedge d\overline{z} = \partial\beta' \wedge d\overline{z}$$

and
$$\overline{\partial}(\alpha' dz + \beta' d\overline{z}) := \overline{\partial}\alpha' \wedge dz + \overline{\partial}\beta' \wedge d\overline{z} = \overline{\partial}\alpha' \wedge dz.$$
Recall also that
$$\Delta u := \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) u = 4 \frac{\partial^2 u}{\partial z \partial \overline{z}}.$$

Exercise 2.1.1.1. *Show that* $d = \partial + \overline{\partial}$, $d \circ d = 0$, $\partial \circ \partial = 0$, $\overline{\partial} \circ \overline{\partial} = 0$, $dd^c = \frac{i}{2\pi} \partial\overline{\partial}$, $d\overline{\partial} = \partial\overline{\partial}$, $d\partial = -\partial\overline{\partial}$ *and* $\frac{i}{2}\partial\overline{\partial} u = \frac{1}{4}\Delta u dx \wedge dy = \frac{1}{4}\Delta u(\frac{i}{2} dz \wedge d\overline{z})$.

Exercise 2.1.1.2. *Let* φ *be a* 1*-form of class* $\mathscr{C}^1$ *and* u *be a function of class* $\mathscr{C}^1$ *on* D. *Assume that* $\mathrm{supp}(\varphi) \cap \mathrm{supp}(u)$ *is compact in* D. *Show that*

$$\int \varphi \wedge \partial u = \int u \partial \varphi \qquad \text{and} \qquad \int \varphi \wedge \overline{\partial} u = \int u \overline{\partial} \varphi.$$

Hint: use the classical Stokes' formula.

Exercise 2.1.1.3. *Let* u *and* v *be* $\mathscr{C}^2$ *functions on* D *such that* $\mathrm{supp}(u) \cap \mathrm{supp}(v)$ *is compact in* D. *Show that*

$$\int u dd^c v = \int v dd^c u.$$

Hint: use Stokes' formula and the previous exercise.

The operators d, d^c, ∂, $\overline{\partial}$, $\partial\overline{\partial}$ and dd^c can be extended to forms or functions which are not regular. We use here the formulas obtained in the previous exercise, in order to define the action on distributions. More precisely, we define dd^c on functions as follows. Let $\mathscr{D}(D)$ denote the space of smooth functions with compact support on D.

Definition 2.1.1.4. Let u be a locally integrable function on D. Define $dd^c u$ as the continuous linear operator from $\mathscr{D}(D)$ to $\mathbb{C}$ (i.e. a current of bidegree $(1,1)$ in term of currents) given by the formula

$$\int v dd^c u := \int u dd^c v, \quad \text{for } v \in \mathscr{D}(D).$$

Recall that if a continuous linear operator from $\mathscr{D}(D)$ to $\mathbb{C}$ is positive, i.e. it takes positive values on positive functions, then the operator is a positive measure.

Exercise 2.1.1.5. *Let $u(z) = \log|z|$. Show that $dd^c u = \delta_0$. If f is holomorphic on D with isolated zeros a_i of multiplicity m_i. Show that $dd^c \log|f| = \sum m_i \delta_{a_i}$.*

Definition 2.1.1.6. A function $u : D \to \mathbb{R} \cup \{-\infty\}$ is *subharmonic* if

1. u is not identically equal to $-\infty$ on any component of D.
2. u is upper semi-continuous (u.s.c. for short), that is,
$$u(a) \geq \limsup_{z\to a, z\neq a} u(z), \qquad \text{for every } a \in D.$$
3. u satisfies the *submean inequality*, that is, if D contains the closed disc of center a and of radius $r > 0$ then
$$u(a) \leq \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta.$$

Note that the latter integral is the mean value of u on the circle of center a and of radius r.

Definition 2.1.1.7. A subset E of D is *polar* if there is a subharmonic function u on D such that $u = -\infty$ on E.

Exercise 2.1.1.8. *Let u be a subharmonic function on D. Show that*
$$\lim_{r\to 0}\frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta = u(a), \qquad \text{for } a \in D.$$
Show that
$$u(a) \leq \frac{1}{\pi r^2}\int_{D(a,r)} u(x,y)dxdy$$
where $D(a,r)$ is the disc of center a and of radius r. Prove that u is strongly u.s.c., that is, if E is a set of Lebesgue measure 0 then for every $a \in D$ we have $u(a) = \limsup u(z)$ if $z \to a$ and $z \notin E$.

Exercise 2.1.1.9. *Let u be a $\mathscr{C}^2$ function. Show that u is subharmonic if and only if Δu is positive or equivalently $dd^c u$ is a positive measure.*

Exercise 2.1.1.10. *Let (u_n) be a decreasing sequence of subharmonic functions on D. Show that either (u_n) decreases to $-\infty$ locally uniformly on at least one component of D or (u_n) decreases to a subharmonic function.*

Exercise 2.1.1.11. *Let $f_1, \ldots, f_n$ be holomorphic functions on D not all identically 0 on any component of D. Prove that $\log(|f_1|^2 + \cdots + |f_n|^2)$ is subharmonic. Deduce that finite sets in D are polar. Hint: consider smooth functions $\log(\epsilon + |f_1|^2 + \cdots + |f_n|^2)$ with $\epsilon > 0$.*

Exercise 2.1.1.12 (maximum principle)**.** *Let u be a subharmonic function on a connected open subset D of $\mathbb{C}$. Assume that u admits a local maximum at a point in D. Show that u is constant. Assume D is bounded. Define for $\xi \in bD$, $u^*(\xi) := \limsup_{z\to\xi, z\in D} u(z)$. Then for $z \in D$, $u(z) \le \max_{bD} u^*(z)$. Give a version of the maximum principle when U is unbounded.*

Exercise 2.1.1.13. *If u and v are subharmonic functions on D, show that* $\max(u, v)$ *is also subharmonic on D. Deduce that* $\log^+|z| := \max(0, \log|z|)$ *is subharmonic on $\mathbb{C}$.*

Definition 2.1.1.14. A function u on D is *harmonic* if u and $-u$ are subharmonic.

So, harmonic functions are continuous and have values in $\mathbb{R}$.

Exercise 2.1.1.15. *Let u be a harmonic function on a bounded domain D. Assume that u is continuous up to the boundary. If $u = 0$ on bD, show that $u = 0$.*

Exercise 2.1.1.16. *Let D be the unit disc. The Poisson kernel is defined as*

$$P(z,t) := \frac{1}{2\pi}\mathrm{Re}\Big(\frac{e^{it}+z}{e^{it}-z}\Big) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}$$

where $z = re^{i\theta}$. Let v be a continuous function on bD. We extend v to a function on $\overline{D}$ by

$$v(z) := \int_0^{2\pi} P(z,t)v(e^{it})dt, \qquad \textit{for } z \in D.$$

Show that v is continuous on $\overline{D}$ and is harmonic on D.

Exercise 2.1.1.17. *Let v be a harmonic function on a domain D containing the closed unit disc. Show that*

$$v(z) := \int_0^{2\pi} P(z,t)v(e^{it})dt, \qquad \textit{for } |z| < 1.$$

Hint: use Exercise 2.1.1.15. Deduce that harmonic functions are smooth and that bounded harmonic functions on $\mathbb{C}$ are constant (Liouville's theorem).

Exercise 2.1.1.18. *Let (u_n) be a sequence of harmonic functions on D converging locally in L^1 to a function u. Show that u is harmonic and u_n converge to u in $\mathscr{C}^\infty$. Hint: use Exercise 2.1.1.16.*

Exercise 2.1.1.19. *Let v be as in Exercise 2.1.1.16 and u be a subharmonic function on a neighbourhood of $\overline{D}$. Assume $u \le v$ on bD. Show that $u \le v$ on D. Hint: use the maximum principle for $u - v$. Deduce for $0 \le r' \le r$ that*

$$\int_0^{2\pi} u(r'e^{i\theta})d\theta \le \int_0^{2\pi} u(re^{i\theta})d\theta.$$

Theorem 2.1.1.20 (regularization)**.** *Let u be a subharmonic function on an open set $D \subset \mathbb{C}$. Let D' be an open set relatively compact in D. Then there is a sequence of smooth subharmonic functions (u_n) on D' such that u_n decrease to u on D'.*

Proof. Let $\delta > 0$ such that $\operatorname{dist}(D', bD) > \delta$. Let χ be a smooth positive decreasing function with compact support in $[0, \delta[$ and is constant near 0. Choose χ so that $\int \chi(r) r dr = 1/(2\pi)$. Define

$$\rho_n(z) := n^2 \chi(n|z|).$$

Exercise 2.1.1.21. *Show that ρ_n is radial, smooth and $\rho_n(z)(\frac{i}{2} dz \wedge d\bar{z})$ is a probability measure. Moreover, $\rho_n(z)(\frac{i}{2} dz \wedge d\bar{z})$ converge to δ_0 when $n \to \infty$.*

Define $u_n := u * \rho_n$, that is,

$$u_n(z) := \frac{i}{2} \int u(z + \xi) \rho_n(\xi) d\xi \wedge d\bar{\xi}.$$

Exercise 2.1.1.22. *Show that u_n is well-defined on D' and the sequence (u_n) is decreasing to u. Hint: use Exercise 2.1.1.19.*

By definition, it is clear that u_n satisfies the submean inequality. So u_n is subharmonic. □

Corollary 2.1.1.23. *If u is subharmonic on D, then $dd^c u$ is a positive measure on D.*

Proof. Since u_n is smooth subharmonic, by Exercise 2.1.1.9, $dd^c u_n$ is a positive measure on D'. Consider a smooth positive function φ with compact support in D'. We have

$$\langle dd^c u_n, \varphi \rangle = \langle u_n, dd^c \varphi \rangle \to \langle u, dd^c \varphi \rangle = \langle dd^c u, \varphi \rangle.$$

We deduce that $\langle dd^c u, \varphi \rangle$ is positive. So, $dd^c u$ restricted to D' is a positive measure. This holds for every D'. Hence, $dd^c u$ is a positive measure on D. □

Exercise 2.1.1.24. *Let u be a subharmonic function on D. Assume that D contains the closed unit disc. Show that*

$$u(z) \leq \int_0^{2\pi} P(z, t) u(e^{it}) dt$$

for $|z| < 1$. Hint: use Exercise 2.1.1.16 and Theorem 2.1.1.20.

Exercise 2.1.1.25. *Let $u : D \to \mathbb{R} \cup \{-\infty\}$ be a locally integrable function. Show that u is subharmonic if and only if u is strongly u.s.c. and $dd^c u$ is a positive measure. Hint: use Exercise 2.1.1.9 and a convolution as in Theorem 2.1.1.20.*

Exercise 2.1.1.26. *Let f be a holomorphic function on a domain D with values in a domaine D' of $\mathbb{C}$. Let u be a subharmonic or harmonic function on D'. Assume that f is not constant. Show that $u \circ f$ is subharmonic or harmonic on D. Hint: use Exercise 2.1.1.25.*

Remark 2.1.1.27. This exercise implies that the notions of subharmonicity and harmonicity are invariant under the holomorphic coordinate change. So, we can define subharmonic and harmonic functions on any open subset of $\mathbb{P}^1$.

Exercise 2.1.1.28. *Let $f : D \to D'$ be a proper holomorphic map between open sets in $\mathbb{P}^1$. If u is a function on D define the function $f_*(u)$ by*

$$f_*(u)(z) := \sum_{w \in f^{-1}(z)} u(w)$$

where the points in $f^{-1}(z)$ are counted with multiplicities. If u is subharmonic, show that $f_*(u)$ is also subharmonic and $dd^c f_*(u) = f_*(dd^c u)$. Hint: use Exercise 2.1.1.25.

We will not give the proof of the following result. The reader can prove it using logarithmic potentials.

Theorem 2.1.1.29 (compactness)**.** *Let (u_n) be a sequence of subharmonic functions on a domain D, locally uniformly bounded from above. Then either (u_n) converges locally uniformly to $-\infty$ on D or there is a subsequence (u_{n_i}) converging to a subharmonic function on D in L^p_{loc} for every $1 \le p < +\infty$.*

Exercise 2.1.1.30. *Show that a countable union of polar sets in D is polar. Hint: if u_n are subharmonic on D and c_n are positive numbers small enough, then $\sum c_n u_n$ is subharmonic and its pole set contains the pole sets of u_n.*

Theorem 2.1.1.31 (Hartogs' lemma)**.** *Let (u_n) be a sequence of subharmonic functions on D converging in L^1_{loc} to a subharmonic function u. Then*

$$\limsup_{n\to\infty} u_n \le u.$$

Moreover, if v is a continuous function on a compact subset K of D such that $u < v$ on K then $u_n < v$ on K for n large enough.

Proof. The second assertion applied to K reduced to a point implies the first assertion. Now, assume that the second assertion does not hold. Extracting a subsequence, we can assume there is a sequence of points $(a_n) \subset K$ convering to a point $a \in K$ such that $u_n(a_n) \ge v(a_n)$. Suppose for simplicity that $a = 0$. Since v is continuous, we can assume that $v(a_n) > v(0) - \epsilon > u(0) + 2\epsilon$ for some constant $\epsilon > 0$.

Since u is u.s.c., there is a constant $\delta > 0$ such that $u \leq u(0)+\epsilon$ on $\{|z| \leq 2\delta\}$. By Exercise 2.1.1.24, for $\delta < \rho < 2\delta$ and for a_n close enough to 0 we have

$$\int_0^{2\pi} P(a_n/\rho, t)u_n(\rho e^{it})dt \geq u_n(a_n) \geq v(a_n) \geq u(0) + 2\epsilon.$$

Since $P(a_n, \theta)$ converge uniformly to $1/(2\pi)$ and u_n converge to u in L^1_{loc}, we obtain that the mean of u on $\{\delta < |z| < 2\delta\}$ is larger than $u(0) + 2\epsilon$. This contradicts the choice of δ, which gives $u \leq u(0) + \epsilon$ on that set. □

Definition 2.1.1.32. A function u on D is *quasi-subharmonic* (qsh for short) if there is a smooth function v on D such that $u + v$ is subharmonic.

Exercise 2.1.1.33. *Show that for every compact set K in D, there is a constant $c > 0$ such that $dd^c u \geq -c(idz \wedge d\overline{z})$ on K.*

Exercise 2.1.1.34. *Let u be a subharmonic function on $\mathbb{P}^1$. Show that u is constant. Hint: use the maximum principle.*

2.1.2 Potential and quasi-potential of a measure

Definition 2.1.2.1. Let ν be a positive measure on D. *A potential* of ν is a subharmonic function u on D such that

$$dd^c u = \nu.$$

Exercise 2.1.2.2. *Let u and u' be two potentials of ν. Show that there is a harmonic function v such that $u' = u + v$. Hint: reduce to the smooth case using a convolution.*

Exercise 2.1.2.3. *Let ν be a positive measure having locally bounded potentials. Show that* supp(ν) *is perfect. Hint: if not $\nu = \nu' + c\delta_a$, $c > 0$; consider the sum of potentials of ν' and of $c\delta_a$.*

Exercise 2.1.2.4. *Let u and u' be subharmonic functions on D. Assume $dd^c u' \leq dd^c u$. Show that if u is locally bounded, u' is locally bounded. Hint: consider a potential of $dd^c u - dd^c u'$. Show that if u is continuous then u' is continuous.*

Definition 2.1.2.5. Let ν be a probability measure with compact support on $\mathbb{C}$. A *logarithmic potential* of ν is a potential u of ν on $\mathbb{C}$ such that

$$u - \log|z| = \mathrm{O}(1) \qquad \text{when } z \to \infty.$$

Exercise 2.1.2.6. *If u and u' be logarithmic potentials of a same measure ν, show that $u' = u + c$ where c is a constant. Hint: Use Exercises 2.1.2.2 and 2.1.1.17.*

Exercise 2.1.2.7. *Let u be a subharmonic function on $\mathbb{C}$. Show that u is a logarithmic potential of a probability measure with compact support if and only if $u - \log|z| = \mathrm{O}(1)$ when $z \to \infty$ and u is harmonic near ∞.*

Exercise 2.1.2.8 (exponential estimate)**.** *Define for a probability measure ν with compact support in $\mathbb{C}$:*

$$u(z) := \int \log|z-a| d\nu(a).$$

Show that u is a logarithmic potential of ν. Show that $e^{-\alpha u}$ is locally integrable for $0 \leq \alpha < 2$. Hint: use the convexity of $x \mapsto e^{-\alpha x}$. Deduce that if K is a compact set in D and v is subharmonic on D then there is $\alpha > 0$ such that $e^{-\alpha v}$ is integrable on K. Deduce that there is a constant $c > 0$ such that

$$\mathrm{area}\{z \in K,\ v(z) < -M\} \leq ce^{-\alpha M} \quad \textit{for } M \geq 0.$$

Hint: consider the restriction of ν to a neighbourhood of K and compare its logarithmic potentials with v; use Exercise 2.1.2.2.

Exercise 2.1.2.9. *Let E be a polar set in an open set D of $\mathbb{C}$. Show that E is polar in $\mathbb{C}$. Deduce that locally polar sets in $\mathbb{C}$ are polar. Hint: if u is subharmonic and $u = -\infty$ on E consider the logarithmic potential of $dd^c u$ restricted to a compact subset of D.*

Exercise 2.1.2.10 (Liouville's theorem)**.** *Let u be a subharmonic function on $\mathbb{C}$. Assume that u is bounded above. Show that u is constant. Hint: apply Exercise 2.1.1.24 to $u - v$ where v is a logarithmic potential of $dd^c u$ restricted to a compact set, see Exercise 2.1.2.8.*

Exercise 2.1.2.11. *Let ν be a positive smooth $(1,1)$-form on a domain D. Show that any potential of ν is smooth. Hint: use Exercise 2.1.2.8.*

Exercise 2.1.2.12 (regularization)**.** *Let u be a logarithmic potential of a probability measure ν with compact support. Show that there are smooth probability measures ν_n with logarithmic potentials u_n such that $\nu_n \to \nu$, $\mathrm{supp}(\nu_n) \to \mathrm{supp}(\nu)$ and u_n decreases to u. Hint: use a convolution.*

On the Riemann sphere $\mathbb{P}^1$ define

$$\omega := dd^c \log(1+|z|^2)^{1/2} + \delta_\infty.$$

Exercise 2.1.2.13. *Let τ be an automorphism of $\mathbb{P}^1$ which preserves ω, that is, $\tau_*(\omega) = \omega$ or equivalently $\tau^*(\omega) = \omega$ since τ is invertible. Prove that τ is an isomorphism. Show that the converse is also true.*

Exercise 2.1.2.14. *Show that ω is a smooth $(1,1)$-form on $\mathbb{P}^1$ nowhere vanishing and that ω defines a positive probability measure on $\mathbb{P}^1$. We call it the Lebesgue measure. Find u_a such that $dd^c u_a = \delta_a - \omega$.*

Exercise 2.1.2.15. *If u is a qsh function on $\mathbb{P}^1$, show that there is a constant $c > 0$ such that $dd^c u + c\omega$ is a positive measure.*

Exercise 2.1.2.16. *Let u be a logarithmic potential of a probability measure with compact support in $\mathbb{C}$. Show that $u - \log(1+|z|^2)^{1/2}$ is qsh. Hint: use the coordinate $w = 1/z$ in order to study this function near ∞.*

Definition 2.1.2.17. Let ν be a probability measure on $\mathbb{P}^1$. We call *quasi-potential* of ν every qsh function u such that

$$dd^c u + \omega = \nu.$$

Exercise 2.1.2.18. *If u is qsh such that $dd^c u \geq -\omega$, prove that $\nu := \omega + dd^c u$ is a probability measure. Deduce that u is a quasi-potential of ν.*

Exercise 2.1.2.19. *If u and u' be quasi-potentials of a same measure. Show that there is a constant c such that $u' = u + c$. Hint: using Exercise 2.1.2.2, show that there is a harmonic function c such that $u' = u + c$; show that c is constant using the maximum principle.*

Exercise 2.1.2.20. *Given a probability measure ν, find a quasi-potential u_ν of ν. Hint: use the idea of Exercise 2.1.2.9. Give an L^1 estimate on u_ν.*

Exercise 2.1.2.21. *Let u_1 and u_2 be quasi-potentials of ν_1 and ν_2. Show that $(u_1 + u_2)/2$ is a quasi-potential of $(\nu_1 + \nu_2)/2$.*

Exercise 2.1.2.22. *Suppose u is a qsh function on $\mathbb{P}^1$ such that $dd^c u \geq -\omega$. Show that $e^{-\alpha u}$ is integrable with respect to the Lebesgue measure on $\mathbb{P}^1$ for $0 \leq \alpha < 2$. Hint: Exercise 2.1.2.21 allows us to assume that $\nu := dd^c u + \omega$ has compact support in $\mathbb{C}$; show that $u + \log(1+|z|^2)^{1/2}$ is a logarithmic potential of ν and use Exercise 2.1.2.8. If v is an arbitrary qsh function on $\mathbb{P}^1$, show that $e^{-\alpha v}$ is integrable for some constant $\alpha > 0$. Deduce that there is a positive constant c such that*

$$\operatorname{area}\{v < -M\} \leq ce^{-\alpha M} \quad \textit{for } M \geq 0.$$

Exercise 2.1.2.23 (regularization)**.** *Let u be a quasi-potential of a probability measure ν in $\mathbb{P}^1$. Show that there are smooth probability measures ν_n with logarithmic potentials u_n such that $\nu_n \to \nu$, $\operatorname{supp}(\nu_n) \to \operatorname{supp}(\nu)$ and u_n decreases to u. Hint: write ν as a combination of a measure in $\mathbb{C}$ and a measure in $\mathbb{P}^1 \setminus \{0\}$ and use Exercise 2.1.2.12. Deduce that if v is a qsh function on $\mathbb{P}^1$ with $dd^c v \geq -c\omega$ there are qsh functions v_n decreasing to v and such that $dd^c v_n \geq -c\omega$.*

Definition 2.1.2.24. A subset of $\mathbb{P}^1$ is *polar* if it is contained in $\{u = -\infty\}$ where u is a qsh function on $\mathbb{P}^1$.

Exercise 2.1.2.25. *Show that a subset of $\mathbb{P}^1$ is polar if and only if it is locally polar.*

Exercise 2.1.2.26. *Let (a_n) be a dense sequence in the unit disc with $a_n \neq 0$. Fix $\lambda_n > 0$ such that $\sum \lambda_n \log|a_n/2| > -\infty$. Define $\varphi := \sum \lambda_n \log|(z-a_n)/2|$ and $\psi := \exp(\varphi)$. Show that φ and ψ are subharmonic, $0 \leq \psi < 1$ and ψ is not continuous in any open set.*

2.1.3 Space of dsh functions and Sobolev space

We first give some useful properties of qsh functions on $\mathbb{P}^1$.

Theorem 2.1.3.1 (compactness)**.** *Let (u_n) be a sequence of qsh functions on $\mathbb{P}^1$ such that $dd^c u_n \geq -\omega$. Assume that the sequence (u_n) is bounded from above. Then either (u_n) converges uniformly to $-\infty$ or there is a subsequence (u_{n_i}) converging in L^p, $1 \leq p < +\infty$, to a qsh function u with $dd^c u \geq -\omega$.*

Proof. Assume (u_n) does not converge uniformly to $-\infty$. So, there is a point a such that (u_n) does not converge to $-\infty$ uniformly in a neighbourhood of a. We can assume $a = 0$. So, $u_n + \log(1+|z|^2)^{1/2}$ are subharmonic and locally bounded from above uniformly on n. Extracting a subsequence, we can assume that $u_n + \log(1+|z|^2)^{1/2}$ converge in L^p_{loc} to a subharmonic function on $\mathbb{C}$, see Theorem 2.1.1.29. So, u_n converge to a qsh function u' on $\mathbb{C}$ with $dd^c u' \geq -\omega$. We deduce that u_n do not converge uniformly in a neighbourhood of $b := \infty$ to $-\infty$.

In the same way, using a coordinate so that b is the origin and a is infinite, we show that (u_n) converges in L^p to a qsh function u'' in $\mathbb{P}^1 \setminus \{a\}$ and $dd^c u'' \geq -\omega$. Of course, $u' = u''$ on $\mathbb{P}^1 \setminus \{a, b\}$. Define $u := u'$ on $\mathbb{P}^1 \setminus \{a\}$ and $u := u''$ on $\mathbb{P}^1 \setminus \{b\}$. Then u is qsh on $\mathbb{P}^1$ with $dd^c u \geq -\omega$ and (u_n) converges to u in L^p. □

Exercise 2.1.3.2 (Hartogs' lemma)**.** *Let u_n be qsh functions with $dd^c u_n \geq -\omega$. Assume that u_n converge in L^1 to a qsh function u. If v is a continuous function such that $u < v$, show that $u_n < v$ for n large enough. Hint: use Theorem 2.1.1.31.*

Corollary 2.1.3.3. *Let*

$$\mathscr{P}_1 := \Big\{u \text{ qsh on } \mathbb{P}^1,\ dd^c u \geq -\omega,\ \max_{\mathbb{P}^1} u = 0\Big\}$$

and

$$\mathscr{P}_2 := \Big\{u \text{ qsh on } \mathbb{P}^1,\ dd^c u \geq -\omega,\ \int_{\mathbb{P}^1} u\omega = 0\Big\}.$$

Then $\mathscr{P}_1$ and $\mathscr{P}_2$ are compact in L^p for $1 \leq p < +\infty$.

Proof. Let $(u_n) \subset \mathscr{P}_1$. It is clear that (u_n) does not converge uniformly to $-\infty$. Extracting a subsequence, we can assume (u_n) converges in L^p to a qsh function u with $dd^c u \geq -\omega$. We have to show that $\max u = 0$. If not, there is a constant $c > 0$ such that $u < -c$. By Exercise 2.1.3.2, $u_n < -c$ for n large enough. This contradicts the fact that $(u_n) \subset \mathscr{P}_1$. Therefore, $\mathscr{P}_1$ is compact in L^p.

Consider now a sequence $(u_n) \subset \mathscr{P}_2$. Define $m_n := \max u_n$ and $v_n := u_n - m_n$. So, $(v_n) \subset \mathscr{P}_1$. Extracting a subsequence, we can assume that v_n converge in L^p to a function $v \in \mathscr{P}_1$. On the other hand, since $\int u_n\omega = 0$, we have

$$m_n = -\int v_n\omega \to -\int v\omega.$$

Define $m := -\int v\omega$ and $u := v + m$. So, $u \in \mathscr{P}_2$ and u_n converge in L^p to u. Hence, $\mathscr{P}_2$ is compact in L^p. □

Definition 2.1.3.4. A function with values in $\mathbb{R} \cup \{\pm\infty\}$ defined on $\mathbb{P}^1$ outside a polar set is *dsh* if it is equal outside this polar set to a difference of two qsh functions. Two dsh functions are considered as *equal* if they are equal outside a polar set, that is, we identify two dsh functions if they satisfy this property.

It is clear that the set of dsh functions is a vector space that we denote by $\mathrm{DSH}(\mathbb{P}^1)$. Now we introduce a norm on $\mathrm{DSH}(\mathbb{P}^1)$.

Proposition 2.1.3.5. *Let u be a dsh function on $\mathbb{P}^1$. Then there are positive measures ν^+ and ν^- with the same mass and such that $dd^c u = \nu^+ - \nu^-$.*

Proof. Write $u = u^+ - u^-$ with $u^\pm$ qsh. Let $c > 0$ be a constant such that $dd^c u^\pm \geq -c\omega$. Define $\nu^\pm := dd^c u^\pm + c\omega$. We have $dd^c u = \nu^+ - \nu^-$. The fact that ν^+ and ν^- have the same mass is deduced from Exercise 2.1.2.18. □

Exercise 2.1.3.6. *Let ν^+ and ν^- be positive measures with the same mass. Show that there is a dsh function u such that $dd^c u = \nu^+ - \nu^-$. Moreover, the function u is unique up to a constant. Hint: use Exercise 2.1.2.18.*

Exercise 2.1.3.7. *Let $u : \mathbb{P}^1 \to \mathbb{R} \cup \{-\infty\}$ be an integrable function such that $dd^c u$ is a measure. Assume that u is strongly u.s.c. Show that u is dsh, see also Exercise 2.1.1.8. Hint: write $dd^c u = \nu^+ - \nu^-$, consider the sum of u with a quasi-potential of ν^- and use Exercise 2.1.1.25.*

For every dsh function u on $\mathbb{P}^1$ define

$$\|u\|_{\mathrm{DSH}} := \left|\int u\omega\right| + \min \|\nu^\pm\|$$

where ν^+ and ν^- are positive measures such that $dd^c u = \nu^+ - \nu^-$ and $\|\nu^\pm\|$ are their masses. We have seen that $\|\nu^+\| = \|\nu^-\|$.

Exercise 2.1.3.8. *Show that $\|\cdot\|_{\mathrm{DSH}}$ is a norm on $\mathrm{DSH}(\mathbb{P}^1)$. Show that there are qsh functions $u^\pm$ such that $\max u^\pm = 0$, $u = u^+ - u^- + m$, $dd^c u^\pm \geq -\|u\|_{\mathrm{DSH}}\omega$, $|m| \leq c\|u\|_{\mathrm{DSH}}$ and $\|u^\pm\|_{\mathrm{DSH}} \leq c\|u\|_{\mathrm{DSH}}$ where $c > 0$ is a constant independent of u.*

This norm induces a topology on $\mathrm{DSH}(\mathbb{P}^1)$, but we often use the following weak topology.

Definition 2.1.3.9. *Let u_n and u be dsh functions. We say that $u_n \to u$ in $\mathrm{DSH}(\mathbb{P}^1)$ if $u_n \to u$ in L^1 and $\|u_n\|_{\mathrm{DSH}}$ is bounded uniformly on n.*

Exercise 2.1.3.10. *Let $\mathrm{DSH}_0(\mathbb{P}^1)$ denote the space of dsh functions u such that $\int u\omega = 0$. Show that $(u_n) \subset \mathrm{DSH}_0(\mathbb{P}^1)$ converges to 0 if $(\|u_n\|_{\mathrm{DSH}})$ converges to 0. Is the converse true ? Show that bounded sets in $\mathrm{DSH}(\mathbb{P}^1)$ are relatively compact for the weak topology. Hint: use Exercise 2.1.3.8.*

Exercise 2.1.3.11. *Show that smooth functions are dense in $\mathrm{DSH}(\mathbb{P}^1)$. Hint: use regularization Theorem for qsh functions and Exercise 2.1.3.8.*

Definition 2.1.3.12. A positive measure ν in $\mathbb{P}^1$ is *PB* if qsh functions are ν-integrable.

Exercise 2.1.3.13. *Show that if ν is PB, it has no mass on polar sets. Show that the following properties are equivalent*

1. *ν is PB.*
2. *the quasi-potentials of ν are bounded.*
3. *ν admits locally bounded potentials.*

Exercise 2.1.3.14. *If ν is PB, show that dsh functions are ν-integrable and*

$$\left|\int |u| d\nu\right| \leq c\|u\|_{\mathrm{DSH}}$$

where $c > 0$ is a constant independent of u.

Definition 2.1.3.15. A probability measure ν on $\mathbb{P}^1$ is *PC* if it is PB and if the linear form $\varphi \mapsto \langle \nu, \varphi \rangle$ is continuous on $\mathrm{DSH}(\mathbb{P}^1)$ with respect to the weak topology on $\mathrm{DSH}(\mathbb{P}^1)$.

Exercise 2.1.3.16. *Show that the following properties are equivalent :*

1. *ν is PC.*
2. *The quasi-potentials of ν are continuous.*
3. *ν admits locally continuous potentials.*
4. *ν, which is defined on continuous functions, can be extended to a linear continuous form on $\mathrm{DSH}^1(\mathbb{P}^1)$.*

Show the the extension in the last item is unique and is given by $\varphi \mapsto \langle \nu, \varphi \rangle$ for φ dsh. Hint: use Exercise 2.1.3.11.

Exercise 2.1.3.17. *Let ν and ν' be probability measures on $\mathbb{P}^1$. Assume that $\nu' \leq c\nu$ for some constant $c > 0$. Show that if ν is PB then ν' is PB and if ν is PC then ν' is PC. Hint: use Exercise 2.1.2.4.*

Now, we introduce the classical Sobolev space that we will use latter. As in Definition 2.1.1.4, we can extend the operators d, ∂ and $\overline{\partial}$ to locally integrable function, see also Exercise 2.1.1.2. Let $\mathscr{D}^1(D)$ (resp. $\mathscr{D}^{1,0}(D)$ and $\mathscr{D}^{0,1}(D)$) denote the space of smooth 1-forms (resp. $(1,0)$-forms and $(0,1)$-forms) on D.

Definition 2.1.3.18. Let u be a locally integrable function on a domain D of $\mathbb{P}^1$. Define du, ∂ and $\overline{\partial}$ as continuous linear operators from $\mathscr{D}^1(D)$ to $\mathbb{C}$ by

$$\langle du, \varphi\rangle := -\int u d\varphi, \quad \langle \partial u, \varphi\rangle := -\int u\partial\varphi, \quad \text{and} \quad \langle \overline{\partial} u, \varphi\rangle := -\int u\overline{\partial}\varphi$$

for φ in $\mathscr{D}^1(D)$.

Exercise 2.1.3.19. *Show that ∂u vanishes on $\mathscr{D}^{1,0}(D)$ and $\overline{\partial} u$ vanishes on $\mathscr{D}^{0,1}(D)$. In term of currents, we say that ∂u is a current of bidegree $(1,0)$ and $\overline{\partial} u$ is a current of bidegree $(0,1)$.*

Denote by $W^1(D)$ the space of functions u in $L^2_{loc}(D)$ such that du is equal to a 1-form with coefficients in $L^2_{loc}(D)$. This space was introduced by Sobolev. Of course, these functions are defined almost everywhere and we identify two functions if they are equal almost everywhere. We will use latter the Sobolev space $W^1(\mathbb{P}^1)$ of functions u in $L^2(\mathbb{P}^1)$ such that du is in $L^2(\mathbb{P}^1)$.

Exercise 2.1.3.20. *Let u be a function in $W^1(\mathbb{P}^1)$. Show that $i\partial u \wedge \overline{\partial} u$ is a positive measure.*

We will not give the proof of the following inequality.

Theorem 2.1.3.21 (Poincaré-Sobolev)**.** *There is a constant $c > 0$ such that if u is a function in $W^1(\mathbb{P}^1)$ then*

$$\|u\|_{L^2} \leq c\Big(\Big|\int u\omega\Big| + \|i\partial u \wedge \overline{\partial} u\|\Big).$$

Define for u in $W^1(\mathbb{P}^1)$

$$\|u\|_{W^1} := \Big|\int u\omega\Big| + \|i\partial u \wedge \overline{\partial} u\|.$$

Exercise 2.1.3.22. *Show that $\|\cdot\|_{W^1}$ is a norm of $W^1(\mathbb{P}^1)$.*

We will not use the topology induced by this norm but the following weak topology.

Definition 2.1.3.23. Let u_n and u be functions in $W^1(\mathbb{P}^1)$. We say that $u_n \to u$ in $W^1(\mathbb{P}^1)$ if $u_n \to u$ in L^1 and $\|u_n\|_{W^1}$ is uniformly bounded.

2.2 Equilibrium measure

2.2.1 Constructions of the equilibrium measure

In this section we will give several methods to construct the canonical totally invariant measure associated to a rational fraction. We first for simplicity consider the case where f is a polynomial of degree $d \geq 2$. Define

$$G_n(z) := d^{-n} \log^+ |f^n(z)|.$$

It is clear that the functions G_n are continuous subharmonic, see Exercise 2.1.1.11.

Theorem 2.2.1.1. *The sequence (G_n) is almost decreasing and converges uniformly to a continuous subharmonic function G. More precisely, there is a sequence of real numbers (c_n) decreasing to 0 such that $(G_n + c_n)$ decreases to G. Moreover, $G(z) - \log|z|$ is bounded near infinity.*

Proof. Observe that there is a constant $c > 0$ such that

$$|d^{-1} \log^+ |f(z)| - \log^+ |z|| \leq c.$$

We deduce from this inequality that

$$|G_{n+1}(z) - G_n(z)| \leq cd^{-n}.$$

So, (G_n) converges uniformly to a continuous function G. Since G_n are subharmonic, G is also subharmonic. We also have that $G - G_0$ is bounded. It follows that $G - \log|z|$ is bounded nears infinity.

Define $c_n := c(d^{-n} + d^{-n-1} + \cdots)$. It is clear that (c_n) decreases to 0. The latter inequality implies

$$G_{n+1} + c_{n+1} \leq G_n + c_n.$$

Hence, $(G_n + c_n)$ is decreasing. □

Exercise 2.2.1.2. *Show that we can choose a constant c depending continuously on the coefficients of f. Let $\widetilde{G}(z, a_2, \ldots, a_d)$ be the Green function of $f(z) = z^d + a_2 z^{d-2} + \cdots + a_d$. Show that $\widetilde{G}$ is continuous.*

Exercise 2.2.1.3. *Show that $G = 0$ on the filled Julia set. Show that*

$$d^{-1} G \circ f = G.$$

Exercise 2.2.1.4. *Let u be a subharmonic on $\mathbb{C}$ such that $u(z) - \log^+|z|$ is bounded. Show that $d^{-n}(u \circ f^n)$ converge pointwise to G. Hint: consider $d^{-n}(u \circ f^n) - G_n$.*

Define

$$\mu := dd^c G \qquad \text{on} \qquad \mathbb{C}.$$

Theorem 2.2.1.5. *μ is a totally invariant probability measure and* $\operatorname{supp}(\mu) = J$.

Proof. Since $G = 0$ on the filled Julia set $\mathbb{C} \setminus \Omega_\infty$, we have $\mu = 0$ on the interior of $\mathbb{C} \setminus \Omega_\infty$, i.e. $\mathbb{C} \setminus \overline{\Omega}_\infty$. Fix a constant $R > 0$ large enough. We have for $|z| > R$ and for some constant $c > 0$

$$|f(z)| \geq c|z|^d > |z| > R.$$

So, f sends $\{|z| > R\}$ into itself. It follows that $|f^n(z)| > 1$ for $|z| > R$ and $n \geq 0$. Therefore, $G_n = d^{-n} \log |f^n(z)|$ is harmonic on $\{|z| > R\}$. We deduce that G is harmonic on $\{|z| > R\}$. Exercise 2.1.2.7 implies that μ is a probability measure. The invariance of G implies that μ is totally invariant.

We also have $G = d^{-n} G \circ f^n$. Hence G is harmonic on $f^{-n}\{|z| > R\}$. We deduce that G is harmonic on the basin Ω_∞ of ∞ since this basin is the union of $f^{-n}\{|z| > R\}$. It follows that $\mu = 0$ on Ω_∞ and then $\operatorname{supp}(\mu) \subset J$. On the other hand, $G_0(z) - \log^+ |z|$ is bounded and (G_n) converges uniformly to G. It follows that $G(z) - \log |z|$ is also bounded. Hence, G is positive near ∞. The invariance of G implies that $G > 0$ on Ω_∞. The maximum principle implies that $-G$ is not harmonic on any open set which intersects $J = b\Omega_\infty$. Hence, $\operatorname{supp}(\mu) \supset J$. This completes the proof. □

Definition 2.2.1.6. We call G *the Green function* and μ *the equilibrium measure* of f.

Now, we extend this construction of the equilibrium measure to the case where f is a rational fraction of degree $d \geq 2$. We will use quasi-potentials. Recall that ω is a probability measure. The measure $d^{-n}(f^n)^*(\omega)$ is also a probability measure since $(f^n)_* 1 = d^n$.

Theorem 2.2.1.7. *The sequence $\big(d^{-n}(f^n)^*(\omega)\big)$ converges to a totally invariant measure μ. Moreover, we have* $\operatorname{supp}(\mu) = J$.

Proof. Let g_0 be a quasi-potential of $d^{-1} f^*(\omega)$, i.e. a qsh function such that

$$d^{-1} f^*(\omega) = \omega + dd^c g_0.$$

Subtracting from g_0 a constant allows us to assume that g_0 is negative. Since ω and $f^*(\omega)$ are smooth, g_0 is also smooth, see Exercise 2.1.2.11. Pulling-back the previous identity gives

$$\begin{aligned} d^{-2}(f^2)^*(\omega) &= d^{-1} f^*(\omega) + dd^c(d^{-1} g_0 \circ f) \\ &= \omega + dd^c(g_0 + d^{-1} g_0 \circ f). \end{aligned}$$

We iterate this procedure and obtain

$$d^{-n}(f^n)^*(\omega) = \omega + dd^c(g_0 + d^{-1}g_0 \circ f + \cdots + d^{-n+1}g \circ f^{n-1}) = \omega + dd^c g_n$$

where

$$g_n := g_0 + d^{-1}g_0 \circ f + \cdots + d^{-n+1}g \circ f^{n-1}.$$

Since g_0 is negative, (g_n) is decreasing. Moreover, we have

$$|g_{n+1} - g_n| = d^{-n}|g_0 \circ f^n| \leq d^{-n}\|g_0\|_\infty.$$

So, (g_n) converges uniformly to a continuous function g. Since $dd^c g_n \geq -\omega$ we also have $dd^c g \geq -\omega$. So, g is qsh and $d^{-n}(f^n)^*(\omega)$ converge to the probability measure

$$\mu := \omega + dd^c g.$$

Write

$$d^{-n-1}(f^{n+1})^*(\omega) = d^{-1}f^*\big(d^{-n}(f^n)^*(\omega)\big).$$

Letting $n \to \infty$, we obtain

$$\mu = d^{-1}f^*(\mu).$$

So, μ is totally invariant.

It remains to prove that $\mathrm{supp}(\mu) = J$. On the Fatou set, since (f^n) is normal, the smooth forms $(f^n)^*(\omega)$ are locally bounded uniformly on n. It follows that $d^{-n}(f^n)^*(\omega)$ converge to 0 on F. This implies $\mathrm{supp}(\mu) \subset J$. Since μ is totally invariant, $\mathrm{supp}(\mu)$ is also totally invariant. Exercise 1.2.2.7 implies that $\mathrm{supp}(\mu) = J$. □

Definition 2.2.1.8. We call μ *the equilibrium measure* of f.

The previous theorem implies that the quasi-potentials of g are continuous. Exercise 2.1.2.3 implies the following result.

Corollary 2.2.1.9. *The Julia set is perfect.*

We now give another method to construct the equilibrium measure. Define *the Perron-Frobenius operator* Λ on functions φ by

$$\Lambda\varphi(z) := d^{-1}f_*(\varphi)(z) := d^{-1}\sum_{w\in f^{-1}(z)} \varphi(w)$$

where the points in $f^{-1}(z)$ are counted with multiplicities.

Proposition 2.2.1.10. *Λ is a bounded positive operator from* $\mathrm{DSH}(\mathbb{P}^1)$ *to itself and it is continuous with respect to the weak topology on* $\mathrm{DSH}(\mathbb{P}^1)$.

Proof. The positivity of the operator is clear. So, $|\Lambda\varphi| \leq \Lambda|\varphi|$. Since $f^*(\omega)$ is smooth, there is a constant $c > 0$ such that $f^*(\omega) \leq c\omega$. Hence

$$\int |\Lambda\varphi|\omega \leq \int \Lambda|\varphi|\omega = d^{-1}\int |\varphi| f^*(\omega) \leq c\int |\varphi|\omega.$$

It follows that

$$\|\Lambda\varphi\|_{L^1} \leq c\|\varphi\|_{L^1}.$$

On the other hand, write

$$dd^c\varphi = \nu^+ - \nu^-$$

where $\nu^\pm$ are positive measures.

Exercise 2.2.1.11. *Show that*

$$dd^c\Lambda\varphi = d^{-1}\Big(f_*(\nu^+) - f_*(\nu^-)\Big).$$

Suppose for the moment that $\Lambda\varphi$ is dsh. The above estimates imply

$$\begin{aligned} \|\Lambda\varphi\|_{\mathrm{DSH}} &\leq |\Lambda\varphi|_{L^1} + \min d^{-1}\|f_*(\nu^\pm)\| \\ &= |\Lambda\varphi|_{L^1} + \min d^{-1}\|\nu^\pm\| \\ &\leq \mathrm{const}\|\varphi\|_{\mathrm{DSH}}. \end{aligned}$$

Hence Λ is bounded.

If (φ_n) converges to φ in $\mathrm{DSH}(\mathbb{P}^1)$, it is clear that $\|\Lambda\varphi_n\|_{\mathrm{DSH}}$ is bounded uniformly on n. We also obtain as above that

$$\|\Lambda\varphi_n - \Lambda\varphi\|_{L^1} \leq c\|\varphi_n - \varphi\|_{L^1}.$$

So, $\Lambda\varphi_n$ converge to $\Lambda\varphi$ in $\mathrm{DSH}(\mathbb{P}^1)$. This is the continuity of Λ.

It remains to prove that $\Lambda\varphi$ is dsh. For this purpose, we can assume that φ is qsh with $dd^c\varphi \geq -\omega$ since it is a combination of such functions. In this case, $\Lambda\varphi$ is strongly u.s.c. Exercise 2.1.3.7 implies the result. □

Theorem 2.2.1.12. *Let φ be a dsh function on $\mathbb{P}^1$. Then there is a constant c_φ depending linearly and continuously on φ such that*

$$\|\Lambda^n\varphi - c_\varphi\|_{\mathrm{DSH}} \leq Ad^{-n}\|\varphi\|_{\mathrm{DSH}}$$

where $A > 0$ is a constant independent of φ.

Proof. Define

$$c_0 := \int \varphi\omega \quad \text{and} \quad \varphi_0 := \varphi - c_0$$

and by induction

$$c_{n+1} := \int (\Lambda\varphi_n)\omega \quad \text{and} \quad \varphi_{n+1} := \Lambda\varphi_n - c_{n+1}.$$

The functions φ_n are in $\mathrm{DSH}_0(\mathbb{P}^1)$ and by Exercise 2.1.1.28

$$dd^c\varphi_n = d^{-n}(f^n)_*(\nu^+) - d^{-n}(f^n)_*(\nu^-).$$

Since $\int \varphi_n\omega = 0$,

$$\|\varphi_n\|_{\mathrm{DSH}} \leq \min_{\nu^\pm} d^{-n}\|(f^n)_*(\nu^\pm)\| = \min_{\nu^\pm} d^{-n}\|\nu^\pm\| = d^{-n}\|\varphi\|_{\mathrm{DSH}}.$$

The operator Λ is bounded and continuous on $\mathrm{DSH}(\mathbb{P}^1)$, c_n depends linearly and continuously on φ. Moreover, there is a constant $A' > 0$ independent of φ such that

$$|c_n| = \left|\int(\Lambda\varphi_{n-1})\omega\right| \leq \|\Lambda\varphi_{n-1}\|_{L^1} \leq A'\|\varphi_{n-1}\|_{\mathrm{DSH}} = A'd^{-n}\|\varphi\|_{\mathrm{DSH}}.$$

It follows that $\sum_{n\geq 0} c_n$ converges to a constant c_φ which depends linearly and continuously on φ. Since Λ preserves constant functions, we have

$$\begin{aligned}
\Lambda^n\varphi &= \Lambda^n(c_0+\varphi_0) = c_0 + \Lambda^n\varphi_0 \\
&= c_0 + \Lambda^{n-1}(\Lambda\varphi_0) = c_0 + \Lambda^{n-1}(c_1+\varphi_1) \\
&= c_0 + c_1 + \Lambda^{n-1}\varphi_1.
\end{aligned}$$

By induction, we obtain

$$\Lambda^n\varphi = c_0 + \cdots + c_n + \varphi_n.$$

Hence

$$\|\Lambda^n\varphi - c_\varphi\|_{\mathrm{DSH}} \leq \|\varphi_n\|_{\mathrm{DSH}} + \sum_{k\geq n+1}|c_k| \leq Ad^{-n}\|\varphi\|_{\mathrm{DSH}}$$

for a constant $A > 0$ independent of φ. □

Corollary 2.2.1.13. *Let ν be a PB probability measure on $\mathbb{P}^1$. Then $d^{-n}(f^n)^*(\nu)$ converge to a PC measure μ which is totally invariant, independent of ν, and satisfies $\langle\mu,\varphi\rangle = c_\varphi$ for every dsh function φ. The convergence is uniform on bounded sets in* $\mathrm{DSH}(\mathbb{P}^1)$.

Proof. Let φ be a dsh function. We have

$$\langle d^{-n}(f^n)^*(\nu),\varphi\rangle = \langle\nu,\Lambda^n\varphi\rangle = c_\varphi + \langle\nu,\Lambda^n\varphi - c_\varphi\rangle.$$

Since ν is PB, there is constant $c > 0$ depending on ν and φ such that if A is the constant introduced in Theorem 2.2.1.12 then

$$|\langle\nu,\Lambda^n\varphi - c_\varphi\rangle| \leq c\|\Lambda^n\varphi - c_\varphi\|_{\mathrm{DSH}} \leq cAd^{-n}\|\varphi\|_{\mathrm{DSH}}.$$

We deduce that $\langle d^{-n}(f^n)^*(\nu),\varphi\rangle$ converge to c_φ. In particular, for φ smooth, this convergence implies that $d^{-n}(f^n)^*(\nu)$ converge to a probability measure μ such that $\langle\mu,\varphi\rangle = c_\varphi$. This measure can be extended to a linear continuous form on $\mathrm{DSH}(\mathbb{P}^1)$ by

$$\langle\mu,\varphi\rangle := c_\varphi \quad \text{for } \varphi \in \mathrm{DSH}(\mathbb{P}^1).$$

Exercise 2.1.3.16 implies that μ is PC. In order to prove the total invariance of μ, it is enough to observe that

$$d^{-n-1}(f^{n+1})^*(\nu) = d^{-1}f^*\big(d^{-n}(f^n)^*(\nu)\big)$$

and let $n \to \infty$. $\square$

Remark 2.2.1.14. For $\nu = \omega$, we obtain $\mu = \lim d^{-n}(f^n)^*(\omega)$. So, μ is the equilibrium measure of f. The convergence in Corollary 2.2.1.13 is also uniform on bounded sets of PB measures.

Exercise 2.2.1.15. *Show that J is locally non-polar and that μ gives no mass to polar sets. Hint: use the property that μ is PB.*

We can construct the measure μ using the Sobolev space instead of $\mathrm{DSH}(\mathbb{P}^1)$. We follow the same lines.

Exercise 2.2.1.16 (Cauchy-Schwarz's inequality)**.** *Let φ be an L^2 function on $\mathbb{P}^1$. Show that $\Lambda\varphi$ is in L^2 and there is a constant $c > 0$ independent of φ such that*

$$\|\Lambda\varphi\|_{L^2} \leq c\|\varphi\|_{L^2}.$$

Exercise 2.2.1.17. *Let D be the unit disc in $\mathbb{C}$. Show that there are smooth functions χ_n such that $0 \leq \chi_n \leq 1$, $\chi_n(z) = 1$ for $|z| > 1/n$, $\chi_n(z) = 0$ near 0 and $\|d\chi_n\|_{L^2}$ bounded. Let u be an L^2 function on D. Assume that, on $D \setminus \{0\}$, du is an L^2 form. Show that du is an L^2 form on D. Hint: write $\partial u = hdz$; show that $\langle\partial u, \chi_n\Phi\rangle$ converge to $\langle\partial u, \Phi\rangle$ and also to $\langle hdz, \Phi\rangle$ for $\Phi \in \mathscr{D}^{0,1}(D)$.*

Exercise 2.2.1.18. *If φ is a function in $W^1(\mathbb{P}^1)$, show that $\Lambda\varphi$ is also in $W^1(\mathbb{P}^1)$ and $\|d(\Lambda\varphi)\|_{L^2} \leq c\|d\varphi\|_{L^2}$ where $c > 0$ is a constant independent of φ. Hint: show that*

$$i\partial(\Lambda\varphi) \wedge \overline{\partial}(\Lambda\varphi) \leq \Lambda(i\partial\varphi \wedge \overline{\partial}\varphi)$$

outside the set of critical values of f; then use Exercise 2.2.1.17.

Exercise 2.2.1.19. *Show that for $\varphi \in W^1(\mathbb{P}^1)$ there is a constant c_φ depending linearly and continuously on φ such that*

$$\|\Lambda^n\varphi - c_\varphi\|_{W^1} \leq Ad^{-n/2}\|\varphi\|_{W^1}$$

where $A > 0$ is a constant independent of φ. Deduce that $d^{-n}(f^n)^*(\omega)$ converge to a totally invariant probability measure μ.

2.2.2 Equidistribution of preimages

In this section, we prove that the preimages of a non-exceptional point are equidistributed with respect to the equilibrium measure. More precisely, we have the following result.

Theorem 2.2.2.1 (Brolin, Lyubich, Freire-Lopès-Mañé)**.** *Let a be a non-exceptional point in $\mathbb{P}^1$. Then $d^{-n}(f^n)^*(\delta_a)$ converge to μ.*

We will present two different proofs of this theorem. The first one uses some geometrical arguments due to Lyubich and the second one is based on potential theory, see [64]. We first prove a theorem of Sodin following the approach in [35].

Theorem 2.2.2.2 (Sodin)**.** *Let f_n be rational fractions of degree d_n. Assume $\sum_{n\geq 0} 1/d_n$ is finite. Then there is a polar set $\mathcal{E}^*$ such that if a and b are in $\mathbb{P}^1 \setminus \mathcal{E}^*$ then*

$$d_n^{-1}(f_n)^*(\delta_a) - d_n^{-1}(f_n)^*(\delta_b) \to 0.$$

Exercise 2.2.2.3. *Let u_n be functions in $\mathscr{P}_2(\mathbb{P}^1)$ and c_n be positive numbers such that $\sum_{n\geq 0} c_n < +\infty$. Show that the functions $c_0u_0 + \cdots + c_nu_n$ are qsh and almostly decrease to a qsh function.*

Exercise 2.2.2.4. *Let u_n be dsh functions such that $\sum \|u_n\|_{\mathrm{DSH}} < +\infty$. Show that $\sum u_n$ converges pointwise outside a polar set to a dsh function. Hint: use Exercises 2.1.3.8 and 2.2.2.3.*

Proof. Define $\Lambda_n := d_n^{-1}(f_n)_*$. Let φ be a smooth function. Define

$$\varphi_n := \Lambda_n\varphi, \quad c_n := \int \varphi_n\omega \quad \text{and} \quad \psi_n := \varphi_n - c_n.$$

We have

$$\begin{aligned} \langle d_n^{-1}(f_n)^*(\delta_a) - d_n^{-1}(f_n)^*(\delta_b), \varphi\rangle &= \langle \delta_a - \delta_b, \Lambda_n\varphi\rangle = \langle \delta_a - \delta_b, \psi_n + c_n\rangle \\ &= \langle \delta_a - \delta_b, \psi_n\rangle = \psi_n(a) - \psi_n(b). \end{aligned}$$

As in the proof of Theorem 2.2.1.12, we obtain that

$$\|\psi_n\|_{\mathrm{DSH}} \leq d_n^{-1}\|\varphi\|_{\mathrm{DSH}}.$$

Exercise 2.2.2.4 implies that $\sum \psi_n$ converges outside a polar set $\mathcal{E}'$ to a dsh function ψ. We can take $\mathcal{E}'$ containing the set where ψ is infinite. So, for a and b outside $\mathcal{E}'$ we have $\psi_n(a) \to 0$ and $\psi_n(b) \to 0$; hence

$$\langle d_n^{-1}(f_n)^*(\delta_a) - d_n^{-1}(f_n)^*(\delta_b), \varphi\rangle \to 0.$$

Consider a dense sequence $(\varphi^{(k)})$ in the space of smooth functions. Let $\mathcal{E}'_k$ denote the polar set obtained as above for $\varphi^{(k)}$ instead of φ. Define $\mathcal{E}^* := \cup_{k\geq 0}\mathcal{E}'_k$. If a and b are out of $\mathcal{E}^*$, we have

$$\langle d_n^{-1}(f_n)^*(\delta_a) - d_n^{-1}(f_n)^*(\delta_b), \varphi^{(k)}\rangle \to 0$$

for every k. The density of $(\varphi^{(k)})$ implies that

$$\langle d_n^{-1}(f_n)^*(\delta_a) - d_n^{-1}(f_n)^*(\delta_b), \varphi\rangle \to 0$$

for every φ smooth. Hence

$$d_n^{-1}(f_n)^*(\delta_a) - d_n^{-1}(f_n)^*(\delta_b) \to 0.$$

This completes the proof. $\square$

Exercise 2.2.2.5. *Let f_n be rational fractions of degree d_n. Assume $d_n/\log n \to \infty$. Show that there is a Borel set $\mathcal{E}^*$ of Lebesgue measure 0 such that if a and b are in $\mathbb{P}^1 \setminus \mathcal{E}^*$ then*

$$d_n^{-1}(f_n)^*(\delta_a) - d_n^{-1}(f_n)^*(\delta_b) \to 0.$$

Hint: use Exercise 2.1.2.22.

Exercise 2.2.2.6. *Show that there is a polar set $\mathcal{E}^*$ such that $d^{-n}(f^n)^*(\delta_a) \to \mu$ for every $a \in \mathbb{P}^1 \setminus \mathcal{E}^*$.*

We continue the proof of Theorem 2.2.2.1. We call *holomorphic disc* any simply connected domain in $\mathbb{P}^1$. If D is a domain in $\mathbb{P}^1$ we call *an inverse branch of order n* of D a biholomorphic map $g : D \to D'$ such that $f^n \circ g = \mathrm{id}$ where D' is a domain in $\mathbb{P}^1$. The diameter of D' is called the *size of the inverse branch.*

Exercise 2.2.2.7. *Let $g_1 : D \to D_1$ and $g_2 : D \to D_2$ be inverse branches of order n of f. Show either $g_1 = g_2$ or $D_1 \cap D_2 = \varnothing$.*

Let PC_n denote the set of critical values of f^n. They satisfy the relation

$$\mathrm{PC}_n = \mathrm{PC}_{n-1} \cup f^n(\mathrm{C})$$

where C is the critical set of f.

Proposition 2.2.2.8. *If D is a holomorphic disc such that $D \cap \mathrm{PC}_{k+1} = \varnothing$, then D admits at least $d^n(1 - 4d^{-k})$ inverse branches for every $n \geq 0$.*

Proof. Let δ_n denote the number of inverse branches of order n of D. For $n \leq k+1$, since $D \cap \mathrm{PC}_n = \varnothing$, D admits the maximal number of inverse branches, i.e. $\delta_n = d^n$.

Consider $n \geq k+1$. Let $g_i : D \to D_i$, $i = 1, \ldots, \delta_n$, denote the inverse branches of order n of D. Since $\#\mathrm{PC}_1 < 2d$, Exercise 2.2.2.7 implies that there are at least $\delta_n - 2d$ inverse branches $g_i : D \to D_i$ such that $D_i \cap \mathrm{PC}_1 = \varnothing$. For such an i, D_i admits the maximal number d of inverse branches of order 1. It follows that D admits at least $(\delta_n - 2d)d$ inverse branches of order $n+1$, that is, $\delta_{n+1} \geq (\delta_n - 2d)d$. Hence

$$d^{-n-1}\delta_{n+1} \geq d^{-n}\delta_n - 2d^{1-n}.$$

These inequalities imply by induction that

$$d^{-n}\delta_n \geq 1 - 2d^{-k} - 2d^{-k-1} - \cdots - 2d^{-n+2} \geq (1 - 4d^{-k}).$$

This completes the proof. $\square$

Proposition 2.2.2.9. *Let D be as above and $\Omega \Subset D$ be an open set. Let $\epsilon > 0$ be a positive constant. Then Ω admits at least $d^n(1 - 5d^{-k})$ inverse branches of size $\leq d^{-n(1-\epsilon)/2}$ for n large enough.*

Proof. Exercise 2.2.2.7 and Proposition 2.2.2.8 imply, for n large enough, that D admits at least $d^n(1-5d^{-k})$ inverse branches $g_i : D \to D_i$ such that $\text{area}(D_i) \leq d^{-n(1-\epsilon/2)}$. Observe that $g_i : \Omega \to g_i(\Omega)$ is an inverse branch of Ω. Exercise 2.2.2.10 below implies that $\text{diam} g_i(\Omega) \leq c d^{-n(1-\epsilon/2)/2}$ where $c > 0$ is a constant. If n is large enough, we have $cd^{-n(1-\epsilon/2)/2} \leq d^{-n(1-\epsilon)/2}$. The proposition follows. $\square$

Exercise 2.2.2.10. *Let D be an open set in $\mathbb{P}^1$ and K be a compact subset of D Show that there is a constant $c > 0$ such that* $\text{diam}(h(K)) \leq c[\text{area}(h(D))]^{1/2}$ *for every injective holomorphic map $h : D \to \mathbb{P}^1$. Hint: if not there are h_n such that* $\text{diam}(h_n(K)) \geq n[\text{area}(h_n(D))]^{1/2}$*; show, using Montel's theorem, that the only possible limits of (h_n) are constant; then use Cauchy's formula.*

For $a \in \mathbb{P}^1$ define

$$m(a) := \sup_{\nu} \|\nu - \mu\|$$

where ν is a limit value of $d^{-n}(f^n)^*(\delta_a)$. We have $0 \leq m(a) \leq 2$. Also $m(a) = 0$ if and only if $d^{-n}(f^n)^*(\delta_a) \to \mu$. Exercise 2.2.2.6 shows that $m(a) = 0$ outside a polar set and we want to prove that $m(a) = 0$ outside the exceptional set $\mathcal{E}$.

Proposition 2.2.2.11. *If $a \notin \text{PC}_{k+1}$, then $m(a) \leq 10d^{-k}$. In particular, $m(a) = 0$ if $a \notin \cup_{k\geq 0}\text{PC}_k$.*

Proof. Let D be a holomorphic disc containing a such that $D \cap \text{PC}_{k+1} = \varnothing$. Let $\Omega \Subset D$ be an open set containing a. Choose a point $b \in \Omega$ such that $m(b) = 0$. We have $d^{-n}(f^n)^*(\delta_b) \to \mu$. Proposition 2.2.2.9 implies that Ω admits at least $d^n(1 - 5d^{-k})$ inverse branches of order n and of small size (their diameters are $\leq d^{-n(1-\epsilon)/2}$). For such a branch $g_i : \Omega \to \Omega_i$, observe that Ω contains exactly one point of $f^{-n}(a)$ and one of $f^{-n}(b)$ which are close since the size of Ω_i is small. We divide $d^{-n}(f^n)^*(\delta_a)$ and $d^{-n}(f^n)^*(\delta_b)$ into two parts : the first one is supported in the union of Ω_i and the second one is supported in the complement

$$d^{-n}(f^n)^*(\delta_a) = \nu_{a,n} + \nu'_{a,n} \quad \text{and} \quad d^{-n}(f^n)^*(\delta_b) = \nu_{b,n} + \nu'_{b,n}.$$

Since the number of the branches g_i is at least equal to $d^n(1 - 5d^{-k})$ we have $\|\nu_{a,n}\| = \|\nu_{b,n}\| \geq 1 - 5d^{-k}$. Since the size of Ω_i tends to 0 when $n \to \infty$, we have $\nu_{a,n} - \nu_{b,n} \to 0$. Therefore, any limit value of $d^{-n}(f^n)^*(\delta_a) - d^{-n}(f^n)^*(\delta_b)$ has mass $\leq 10d^{-k}$. This and the fact that $d^{-n}(f^n)^*(\delta_b) \to \mu$ imply that $m(a) \leq 10d^{-k}$. $\square$

End of the proof of Theorem 2.2.2.1. Since each PC_n is finite, there is $a \notin \mathcal{E}$ such that

$$m := m(a) = \max_{z \in \mathbb{P}^1 \setminus \mathcal{E}} m(z).$$

We have

$$d^{-n}(f^n)^*(\delta_a) = d^{-1} \sum_{b \in f^{-1}(a)} d^{1-n}(f^{n-1})^*(\delta_b),$$

hence

$$m := m(a) \leq d^{-1} \sum_{b \in f^{-1}(a)} m(b).$$

It follows from the maximality of $m(a)$ that $m(b) = m$ for every $b \in f^{-1}(a)$. By induction, we have $m(b) = m$ for $b \in f^{-n}(a)$ for every n. We have seen that the closure of $\cup_{n \geq 0} f^{-n}(a)$ contains J. So, $\cup_{n \geq 0} f^{-n}(a)$ is infinite. For every k, it contains a point outside PC_{k+1}. Proposition 2.2.2.11 implies that $m \leq 10 d^{-k}$ for every k. It follows that $m = 0$ and this completes the proof. $\square$

Exercise 2.2.2.12. *Let ν be a probability measure on $\mathbb{P}^1$. Show that $d^{-n}(f^n)^*(\nu) \to \mu$ if and only if $\nu(\mathcal{E}) = 0$.*

We now give another proof using potential theory.

Exercise 2.2.2.13. *Show that for every $a \notin \mathcal{E}$ the multiplicity of f^3 at a is $\leq d^3 - 1$.*

Exercise 2.2.2.14 (Lojasiewicz's inequality)**.** *Show that there is a constant $c > 0$ such that*

$$\mathrm{dist}_{\mathbb{P}^1}(f(a), b) \geq c\big[\mathrm{dist}_{\mathbb{P}^1}(a, f^{-1}(b))\big]^d$$

for $a, b \in \mathbb{P}^1$. Deduce that if D is a disc of radius r then $f^n(D)$ contains a disc of radius $(Ar)^{d^n}$ where $A > 0$ is a constant independent of D, r and n. Hint: write the factors of $f(z) - b$ and estimate the value at a.

Exercise 2.2.2.15. *Let K be a compact subset of $\mathbb{P}^1 \setminus \mathcal{E}$. Show that there is a constant $c > 0$ such that*

$$\mathrm{dist}_{\mathbb{P}^1}(f^3(a), b) \geq c\big[\mathrm{dist}_{\mathbb{P}^1}(a, f^{-3}(b))\big]^{d^3-1}$$

for $a \in K$ and $b \in \mathbb{P}^1$. Assume that K is the complement of the basin of $\mathcal{E}$. Let D be a disc of radius r with center in K. Deduce that $f^n(D)$ contains a disc of radius $(Ar)^{(d^3-1)^{n/3-1}}$ where $A > 0$ is a constant independent of D, r and n. Hint: there is a point in $f^{-3}(b)$ which is far from a.

Definition 2.2.2.16. Let ν be a probability measure on $\mathbb{P}^1$. We call *μ-potential* of ν the strongly u.s.c. function $u : \mathbb{P}^1 \to \mathbb{R} \cup \{-\infty\}$ such that

$$dd^c u = \nu - \mu \quad \text{and} \quad \int u \, d\mu = 0.$$

Exercise 2.2.2.17. *Show that ν admits a unique μ-potential u. Hint: use the fact that μ has continuous quasi-potentials. Show that u is dsh and u is bounded from above by a constant independent of ν.*

Exercise 2.2.2.18. *Extend the compactness theorem, Hartogs' lemma and Exercise 2.1.2.22 to μ-potentials.*

Exercise 2.2.2.19. *Let u be the μ-potential of ν. Show that $d^{-1}u \circ f$ is the μ-potential of $d^{-1}f^*(\nu)$. Show that $\limsup d^{-n}u \circ f^n \leq 0$. Show that any limit value u' of $d^{-n}u \circ f^n$ is a μ-potential. Hint: use the fact that μ is PC. Deduce that $u' = 0$ on J. Hint: use the upper semi-continuity of u'.*

Exercise 2.2.2.20. *Let v be the μ-potential of δ_a. Show that $v : \mathbb{P}^1 \to \mathbb{R} \cup \{-\infty\}$ is a continuous map, $v^{-1}(-\infty) = a$ and $v(z) - \log \mathrm{dist}_{\mathbb{P}^1}(z, a)$ is bounded.*

Let ν be the μ-potential of δ_a where a is a point in $\mathbb{P}^1 \setminus \mathcal{E}$. We want to prove that $d^{-n}v \circ f^n$ converge to 0 in L^1. This implies that

$$d^{-n}(f^n)^*(\delta_a) = \mu + dd^c(d^{-n}v \circ f^n) \to \mu.$$

Assume there is a subsequence $(d^{-n_i}v \circ f^{n_i})$ converging in L^1 to a μ-potential $v' \neq 0$. Exercise 2.2.2.19 implies that $v' \leq 0$.

Exercise 2.2.2.21. *Show that $d^{-n}v \circ f^n$ converge to 0 pointwise in the basin of $\mathcal{E}$. In particular, $v' = 0$ there.*

Let K denote the complement of the basin of $\mathcal{E}$.

Exercise 2.2.2.22. *Show that there is a constant $\alpha > 0$ and a disc D in K such that*

$$d^{-n_i}v \circ f^{n_i} < -\alpha$$

for n_i large enough. Hint: use Hartogs' lemma for μ-potentials.

End of the second proof of Theorem 2.2.2.1. We deduce that $v < -\alpha d^{n_i}$ on $f^{n_i}(D)$ which contains a disc of radius $(Ar)^{(d^3-1)^{n_i/3-1}}$, see Exercise 2.2.2.15. It follows that

$$\begin{aligned} \int e^{-v}\omega &\geq e^{\alpha d^{n_i}}\left[(Ar)^{(d^3-1)^{n_i/3-1}}\right] \\ &= \exp\left[\alpha d^{n_i} + \log(Ar)(d^3-1)^{n_i/3-1}\right]. \end{aligned}$$

The last term tend to $+\infty$ when $n_i \to \infty$. This contradicts Exercises 2.1.2.22 and 2.2.2.18. The proof is now complete. □

2.2.3 Equidistribution of periodic points

In this section, we show that the repelling periodic points are equidistributed with respect to the equilibrium measure.

Theorem 2.2.3.1 (Fatou-Brolin-Lyubich). *Let P_n denote the set of repelling periodic points of period n. Then*

$$\mu_n := d^{-n} \sum_{w \in P_n} \delta_w \to \mu.$$

Proof. First, observe that $\# P_n \leq d^n + 1$. Hence, if ν is a limit of (μ_n) then $\|\nu\| \leq 1$. Let U be a connected open set such that $\mu(U) > 0$. It is enough to show that $\liminf \mu_n(U) \geq \mu(U)$.

Fix a constant $\epsilon > 0$. We only have to show that $\mu_n(U) \geq \mu(U) - 3\epsilon$ for n large enough. Fix $k > 0$ such that $5d^{-k} < \epsilon$. Since μ has no mass on PC_{k+1}, we can choose simply connected open sets U_i such that $U_3 \Subset U_2 \Subset U_1 \subset U \setminus \mathrm{PC}_{k+1}$ and $\mu(U_3) \geq \mu(U) - \epsilon$. Proposition 2.2.2.9 implies that U_2 admits at least $d^n(1-\epsilon)$ inverse branches $g_i : U_2 \to D_i$ of order n with size $\leq d^{-n/2}$. Observe that if n_i is large enough we have $\operatorname{diam}(D_i) < \operatorname{dist}(U_3, bU_2)$ which implies that $D_i \subset U_2$ if $D_i \cap U_3 \neq \varnothing$. In this case, Exercise 1.1.1.15 implies that g_i admits a unique attracting fixed point in D_i which is repelling for $g_i^{-1} = f^n$. Denote by $\mathscr{B}_n$ the family of g_i satisfying the previous property. We have constructed $\# \mathscr{B}_n$ repelling periodic points of period n in U_2. By Exercise 2.2.2.7, they are distinct. It remains to show that $\# \mathscr{B}_n \geq d^n(\mu(U) - 3\epsilon)$ for n large enough.

Fix a point $a \in U_3 \setminus \mathcal{E}$. Since $\nu_{a,n} := d^{-n}(f^n)^*(\delta_a) \to \mu$, we have

$$\nu_{a,n}(U_3) \geq \mu(U_3) - \epsilon \geq \mu(U) - 2\epsilon$$

for n large enough. So, the number of points in $f^{-n}(a) \setminus U_3$ is bounded by $d^n(1 - \mu(U) + 2\epsilon)$. This and the fact that each branch in $\mathscr{B}_n$ contains a point of $f^{-n}(a)$ imply that

$$\# \mathscr{B}_n \geq d^n(1-\epsilon) - d^n(1 - \mu(U) + 2\epsilon) = d^n(\mu(U) - 3\epsilon).$$

This completes the proof. □

2.2.4 Mixing and rate of mixing

The following result says that μ is exponentially mixing.

Theorem 2.2.4.1. *The equilibrium measure is mixing. Moreover, for $\varphi \in L^\infty(\mu)$ and $\psi \in \mathrm{DSH}(\mathbb{P}^1)$, we have*

$$\Big| \langle \mu, (\varphi \circ f^n)\psi \rangle - \langle \mu, \varphi \rangle \langle \mu, \psi \rangle \Big| \leq c \|\varphi\|_\infty \|\psi\|_{\mathrm{DSH}} d^{-n}$$

where $c > 0$ is a constant independent of φ and ψ.

Proof. Recall that Theorem 2.2.1.12 implies that $\|\Lambda^n\psi - c_\psi\| \lesssim \|\psi\|_{\rm DSH} d^{-n}$ where $c_\psi := \langle\mu, \psi\rangle$. Since μ is totally invariant, we have

$$\begin{aligned} &\Big|\langle\mu, (\varphi\circ f^n)\psi\rangle - \langle\mu,\varphi\rangle\langle\mu,\psi\rangle\Big| = \Big|\langle d^{-n}(f^n)^*(\varphi\mu), \psi\rangle - c_\psi\langle\mu,\varphi\rangle\Big| \\ = \;& \Big|\langle\varphi\mu, \Lambda^n\psi\rangle - c_\psi\langle\mu,\varphi\rangle\Big| = \Big|\langle\mu, \varphi(\Lambda^n\psi)\rangle - c_\psi\langle\mu,\varphi\rangle\Big| \\ = \;& \Big|\langle\mu, \varphi(\Lambda^n\psi - c_\psi)\rangle\Big| \le \|\varphi\|_\infty\langle\mu, |\Lambda^n\psi - c_\psi|\rangle \\ \lesssim \;& \|\varphi\|_\infty\|\Lambda^n\psi - c_\psi\|_{\rm DSH} \lesssim \|\varphi\|_\infty\|\psi\|_{\rm DSH}d^{-n}. \end{aligned}$$

Note that the above bound of $\langle\mu, |\Lambda^n\psi - c_\psi|\rangle$ is obtained using that μ is PC. This is the desired inequality. □

Exercise 2.2.4.2. *Show that $\Lambda : L^2(\mu) \to L^2(\mu)$ is well-defined and of norm 1. Show that for $\psi \in L^2(\mu)$ we have $\|\Lambda^n\psi - c_\psi\|_{L^2(\mu)} \to 0$. Prove that μ is K-mixing, i.e. for every $\psi \in L^2(\mu)$*

$$\sup_{\|\varphi\|_{L^2(\mu)}\le 1} \Big|\langle\mu, (\varphi\circ f^n)\psi\rangle - \langle\mu,\varphi\rangle\langle\mu,\psi\rangle\Big| \to 0.$$

Hint: apply Schwarz's inequality to $\langle\mu, \varphi(\Lambda^n\psi - c_\psi)\rangle$.

For $\alpha \ge 0$ denote by $[\alpha]$ the integer part of α. If D is a domain in $\mathbb{R}^2$ let $\mathscr{C}^\alpha(D)$ denote the space of functions which admit partial derivatives of order $\le [\alpha]$ and these derivatives are Hölder continuous of order $\alpha - [\alpha]$. If $0 \le \alpha \le 1$, define

$$\|\psi\|_{\mathscr{C}^\alpha} := \sup_{a,b\in D} \frac{|\psi(b) - \psi(b)|}{|a-b|^\alpha}$$

and for $\alpha \ge 1$

$$\|\psi\|_{\mathscr{C}^\alpha} := \|\psi\|_{\mathscr{C}^{[\alpha]}} + \sup_{m+n=[\alpha]} \Big\|\frac{\partial^{m+n}\psi}{\partial x^m\partial y^n}\Big\|_{\mathscr{C}^{\alpha-[\alpha]}}.$$

Cover $\mathbb{P}^1$ by a finite number of open sets D_i and fix local real coordinates on a neighbourhood of each $\overline{D}_i$. A function belongs to the space $\mathscr{C}^\alpha(\mathbb{P}^1)$ if its restriction to each D_i belongs to $\mathscr{C}^\alpha(D_i)$. Define the $\mathscr{C}^\alpha$-norms on $\mathbb{P}^1$ as the sum of the $\mathscr{C}^\alpha$-norms on each D_i using the given local coordinates. The space $\mathscr{C}^\alpha(\mathbb{P}^1)$ does not depend on the choice of D_i and the local coordinates. The norm depends on the choice of D_i and of the coordinates but when one uses another covering and other local coordinates one obtains an equivalent norm.

Exercise 2.2.4.3. *There is a constant $c > 0$ such that $\|\cdot\|_{\rm DSH} \le c\|\cdot\|_{\mathscr{C}^2}$ and $\|\cdot\|_{W^1} \le c\|\cdot\|_{\mathscr{C}^1}$.*

Corollary 2.2.4.4. *For $\varphi \in L^\infty(\mu)$ and $\psi \in \mathscr{C}^\alpha$, $0 \le \alpha \le 2$, we have*

$$\Big|\langle\mu, (\varphi\circ f^n)\psi\rangle - \langle\mu,\varphi\rangle\langle\mu,\psi\rangle\Big| \le c_\alpha\|\varphi\|_\infty\|\psi\|_{\mathscr{C}^\alpha}d^{-n\alpha/2}$$

where $c_\alpha > 0$ is a constant independent of φ and ψ.

Proof. For $\alpha = 0$, it is clear that

$$\Big|\langle\mu, (\varphi\circ f^n)\psi\rangle - \langle\mu,\varphi\rangle\langle\mu,\psi\rangle\Big| \leq 2\|\varphi\|_\infty\|\psi\|_{\mathscr{C}^0} = 2\|\varphi\|_\infty\|\psi\|_\infty.$$

For $\alpha = 2$, Theorem 2.2.4.1 and Exercise 2.2.4.3 imply that

$$\Big|\langle\mu, (\varphi\circ f^n)\psi\rangle - \langle\mu,\varphi\rangle\langle\mu,\psi\rangle\Big| \leq c_2\|\varphi\|_\infty\|\psi\|_{\mathscr{C}^2}d^{-n}$$

for some constant $c_2 > 0$. We obtain the general case using interpolation theory, see Theorem 2.2.4.5 below. □

The following result holds for general Banach spaces but we only give the statement in our context.

Theorem 2.2.4.5 (Interpolation)**.** *Let X be a real compact manifold (in particular for $X = \mathbb{P}^1$). Let $L : \mathscr{C}^0(X) \to \mathbb{C}$ be a linear continuous form and $k \geq 0$ be a real number. Assume that there are positive constants A_0 and A_k such that*

$$|L(\psi)| \leq A_0\|\psi\|_\infty \quad \textit{for } \psi \in \mathscr{C}^0(X)$$

and

$$|L(\psi)| \leq A_k\|\psi\|_{\mathscr{C}^k} \quad \textit{for } \psi \in \mathscr{C}^k(X).$$

Then for $0 \leq \alpha \leq k$ there is a constant $c_\alpha > 0$ independent of L, A_0 and A_k such that

$$|L(\psi)| \leq c_\alpha A_0^{(k-\alpha)/k}A_k^{\alpha/k}\|\psi\|_{\mathscr{C}^\alpha} \quad \textit{for } \psi \in \mathscr{C}^\alpha(X).$$

Sketch of the proof. Using a convolution we obtain a function ψ' with some estimates on $\|\psi - \psi'\|_\infty$ and on $\|\psi'\|_{\mathscr{C}^k}$ in term of $\|\psi\|_{\mathscr{C}^\alpha}$. Then

$$|L(\psi)| \leq |L(\psi - \psi')| + |L(\psi')| \leq A_0\|\psi - \psi'\|_\infty + A_2\|\psi'\|_{\mathscr{C}^k}.$$

We obtain a bound using the estimates on $\|\psi - \psi'\|_\infty$ and $\|\psi'\|_{\mathscr{C}^k}$. For the sharp estimate one have to use an infinite number of convolutions. □

2.2.5 Lyapounov exponent and entropy

The following result shows that the Julia set is repelling in some sense.

Theorem 2.2.5.1 (Lyubich)**.** *The Lyapounov exponent of μ is larger or equal to $\frac{1}{2}\log d$.*

Proof. We have to show that

$$\int \log|f'|_{\mathbb{P}^1}d\mu \geq \frac{1}{2}\log d.$$

It is clear that for $a \in P_n$ we have $|(f^n)'(a)|_{\mathbb{P}^1} \geq 1$. We have seen in Theorem 2.2.3.1 that $\mu_n \to \mu$.

Exercise 2.2.5.2. *For the $d^n(1-\epsilon)$ points a in P_n that we constructed with inverse branches, show that $|(f^n)'(a)|_{\mathbb{P}^1} \geq cd^{n(1-\epsilon)/2}$, $c > 0$.*

Exercise 2.2.5.3. *Let $p_0, \ldots, p_{n-1}$ be a periodic cycle of period n. Show that*

$$\prod_{i=0}^{n-1} |f'(p_i)|_{\mathbb{P}^1} = |(f^n)'(p_0)|_{\mathbb{P}^1} = \cdots = |(f^n)'(p_{n-1})|_{\mathbb{P}^1}.$$

We deduce from previous exercises that

$$\int \log |f'|_{\mathbb{P}^1} d\mu_n \geq \frac{(1-\epsilon)^2}{2} \log d + \frac{\log c}{n}.$$

Since $\log |f'|$ is u.s.c. we have

$$\int \log |f'|_{\mathbb{P}^1} d\mu \geq \limsup_{n\to\infty} \int \log |f'|_{\mathbb{P}^1} d\mu_n.$$

This implies the desired inequality. □

Theorem 2.2.5.4 (Parry-Lyubich)**.** *μ is the unique invariant measure of maximal entropy $\log d$.*

Proof. We will not give the proof of the unicity. Recall that μ has no mass on finite set, in particular, on the critical values of f. If we connect the critical values with appropriate paths, we can choose a simply connected domain D which does not contain critical values of f such that $\mu(D) = 1$. This domain admits d inverse branches $g_i : D \to D_i$ where the open sets D_i are disjoint. Define $E := \mathbb{P}^1 \setminus D$ and $E' := \cup_{n,m\in\mathbb{N}} f^{-m}(f^n(E))$. Consider the partition $\xi := \{A_i\}$ with $A_i := D_i \setminus E'$.

Exercise 2.2.5.5. *Show that E' is totally invariant and $\mu(E') = 0$. Show that ξ_n^f contains exactly d^n elements and they have the same μ-measure $1/d^n$. Deduce that $h_f(\mu) \geq h_f(\mu, \xi) = \log d$.*

So, μ has maximal entropy $\log d$. □

Notes for Chapter 2. The books by Hörmander [56, 57] provide a good source for the classical theory on subharmonic functions. The normed space of DSH-functions was systematically used by the authors in order to study equidistribution problems in several variables [35], see also [29]. The idea to use subharmonic and quasi-subharmonic functions as test funtions was introduced in [29, 35]. One important point is that subharmonic and dsh functions are invariant under f_* and have good compactness properties. We have given here the one variable version.

Sodin's theorem is due to Sodin [69], see [63, 62] for the case of $\mathbb{P}^k$ and [35] for the case of any Kähler manifolds. The proof given here follows the authors proof for arbitrary Kähler manifolds in [35]. For Theorem 2.2.2.1, Brolin considered only the polynomial case [13]. Lyubich, Freire-Lopès-Mañé, proved independently the case of rational maps [60, 50]. The proof given here is the one variable version of results in [29]. For the second proof, the use of Lojasiewicz type inequality for maps in $\mathbb{P}^2$, is due to Fornæss and the second author [48]. The proof follows closely the approach of the authors of equidistribution towards Green currents for holomorphic maps in $\mathbb{P}^k$ [39]. The first result on the rate of mixing for μ was obtained by Fornæss and the second author for holomorphic maps in $\mathbb{P}^k$ [47]. The method used here follows [29, 35] which give better estimates. The use of periodic points to prove Lyubich's estimate on Lyapounov exponent gives a simple proof.

Chapter 3

Dynamics in higher dimension

3.1 Pluripotential theory

3.1.1 Differential forms and currents

Let U be an open set in $\mathbb{R}^k$. A *differential p-form* on U or a *form of degree p on U*, is a smooth map assigning to each point $a \in U$ an alternating p-form on $\mathbb{R}^k$. So, a differential 1-form can be written as

$$\phi = \sum_{i=1}^{k} \phi_i dx_i$$

where ϕ_i are functions and dx_i are the linear forms $(x_1, \ldots, x_k) \mapsto x_i$ defined on $\mathbb{R}^k$. A differential p-form is expressed as

$$\phi = \sum_{I} \phi_I dx_I$$

where $I = (i_1, \ldots, i_p)$, $1 \leq i_1 < \cdots < i_p \leq k$, $dx_I := dx_{i_1} \wedge \ldots \wedge dx_{i_p}$ and ϕ_I are functions. In practice, the functions ϕ_i and ϕ_I are bounded, continuous, in L^p or $\mathscr{C}^k$. We don't recall here the basic operations on differential forms: wedge product, Poincaré's exterior derivative d, pull-back, integration.

A current of degree p on U can be seen as a differential form with distributions as coefficients. It is then useful to extend the operations of differental calculus to such objects. One of the advantages is to have in the same class, subvarieties and differentials forms. More formally, we are going to introduce currents as the dual of smooth test forms, just like distributions are introduced as the dual space of smooth test functions.

Let $\mathscr{D}^p(U)$ denote the space of smooth forms of degree p with compact support in U. A sequence (φ_j) converges to 0 in $\mathscr{D}^p(U)$ if the supports of the φ_j lie in a fixed compact set $K \subset U$ and the coefficients of φ_j and their derivatives converge to 0 uniformly.

Exercise 3.1.1.1. *Let $\mathscr{D}^p_K(U)$ be the space of forms in $\mathscr{D}^p(U)$ with support in the compact set K. Define on $\mathscr{D}^p_K(U)$ the family of norms*

$$P_{K,l}(\phi) := \sup_{x\in K} \sup_{\substack{|\alpha|\leq l \\ |I|\leq p}} |D^\alpha \phi_I(x)|.$$

Show that $\mathscr{D}^p_K(U)$ is a Fréchet space. Show that a sequence is convergent in $\mathscr{D}^p(U)$ if and only if it belongs to some $\mathscr{D}^p_K(U)$ and converges there.

Definition 3.1.1.2. Let $\mathscr{D}'_{k-p}(U)$ (or $\mathscr{D}'^p(U)$) be the space of continuous linear forms on $\mathscr{D}^{k-p}(U)$. We will say that an element $S \in \mathscr{D}'_{k-p}(U)$ is a *current of degree p and of dimension $k-p$*. It acts on forms of degree $k-p$.

Exercise 3.1.1.3. *Let S be a current of degree p on U. Show that S can be identified with a differential form $\sum S_I dx_I$ of degree p with distribution coefficients. Hint: if I and J are disjoint and $I \cup J = \{1,\dots,k\}$ define*

$$\langle S_I, \psi\rangle := (-1)^{\sigma(I,J)} \langle S, \psi dx_J\rangle$$

where $\sigma(I,J)$ denotes the signature of the permutation $\{I,J\}$ of $\{1,\dots,k\}$.

Exercise 3.1.1.4. *Show that any differential form of degree p with distribution coefficients defines a current of degree p on U. If $Y \subset U$ is a variety of dimension $k-p$ with locally finite $(k-p)$-dimensional volume, show that*

$$\langle [Y], \phi\rangle := \int_Y \phi$$

defines a current of degree p on U. Write the expression of this current in coordinates when $Y = \{x_1 = \cdots = x_p = 0\}$.

Exercise 3.1.1.5.

1. *If S is a current of degree p, and ψ is a smooth form of degree q, show that*

$$\langle S \wedge \psi, \phi\rangle := \langle S, \psi \wedge \phi\rangle, \qquad \phi \in \mathscr{D}^{k-p-q}(U)$$

defines a current of degree $p+q$.

2. *If S is a current of degree p, define dS by setting*

$$\langle dS, \phi\rangle := (-1)^{p+1} \langle S, d\phi\rangle, \qquad \phi \in \mathscr{D}^{k-p-1}(U).$$

Show that dS is a current of degree $p+1$ and that the above definition extends the definition of the operator d on smooth forms.

3. *Suppose S is the current of integration on a smooth oriented submanifold N with boundary bN of dimension $k-p$. Give the relation between dS and the current of integration $[bN]$ on the boundary bN of N.*

Definition 3.1.1.6. A current S is *closed* if $dS = 0$.

Definition 3.1.1.7. Let (S_j) be a sequence in $\mathscr{D}'^p(U)$. We say that $S_j \to S$ if

$$\langle S_j, \phi\rangle \to \langle S, \phi\rangle$$

for every $\phi \in \mathscr{D}'^p(U)$.

Exercise 3.1.1.8. *Show that if $S_j \to S$ then $S_j \wedge \psi \to S \wedge \psi$ and $dS_j \to dS$ for every smooth form ψ.*

Exercise 3.1.1.9. *Show that there is a largest open subset V of U where S vanishes, i.e. $\langle S, \phi\rangle = 0$ for every $\phi \in \mathscr{D}'^p(V)$.*

Definition 3.1.1.10. We call *support* of S the closed subset $\mathrm{supp}(S) := U \setminus V$ of U.

Definition 3.1.1.11. A current S in $\mathscr{D}'^p(U)$ is *of order* $\leq l$ if for every compact subset K of U, when $\mathrm{supp}(\phi_j) \subset K$ and $\phi_j \to 0$ in $\mathscr{C}^l$, we have $\langle S, \phi_j\rangle \to 0$. The *order* of S is the smallest l satisfying this property.

Exercise 3.1.1.12. *Show that a current is of order* 0 *if and only if it has measure coefficients. Prove that a current with compact support is of finite order. Give exemples of currents with infinite order.*

Exercise 3.1.1.13. *Show that any current is the limit of currents with compact support and any current is limit of smooth forms. Hint: use a convolution. Suppose S closed, is S a limit of closed smooth forms?*

Some of the basic operations on smooth forms extend "by continuity" to currents. Define the direct image of a current. Let $f : U \to V$ be a smooth proper map from $U \subset \mathbb{R}^k$ to $V \subset \mathbb{R}^{k'}$. Recall that f is *proper* if the inverse image of every compact subset of V is compact in U. Let $S \in \mathscr{D}'_{k-p}(U)$, define $f_*(S)$ as an element in $\mathscr{D}'_{k-p}(V)$ by the following relation

$$\langle f_*(S), \phi\rangle = \langle S, f^*(\phi)\rangle.$$

Exercise 3.1.1.14. *Show that $f_*(S)$ is a well-defined current and that the following properties hold*

1. $\mathrm{supp}(f_*(S)) \subset f(\mathrm{supp}(S))$. *Is it always true that* $\mathrm{supp}(f_*(S)) = f(\mathrm{supp}(S))$*?*
2. *If ψ is a smooth test form then*

$$f_*(S \wedge f^*(\psi)) = f_*(S) \wedge \psi.$$

3. $d(f_*(S)) = f_*(dS)$.
4. *f_* is continuous, that is, if $S_j \to S$ then $f_*(S_j) \to f_*(S)$.*

5. *Show that in order to define* $f_*(S)$ *it is enough that* f *restricted to the support of* S *is proper.*

6. *Show that if* $g : V \to W \subset \mathbb{R}^{k''}$ *is a smooth map and if* $g \circ f$ *is proper on* $\mathrm{supp}(S)$ *then* $g_*(f_*(S)) = (g \circ f)_*(S)$.

Now, define the pull-back of a current by a submersion $f : U \to V$.

Exercise 3.1.1.15. *Let* π_i *be the projections from* $U_1 \times U_2$ *on* U_i *where* U_i *are open sets in* $\mathbb{R}^{k_i}$. *Let* ϕ *be a smooth form of degree* $k_1 + k_2 - p$ *with compact support on* $U_1 \times U_2$ *with* $p \leq k_1$. *Show that* $(\pi_1)_*(\phi)$ *is a smooth form of degree* $k_1 - p$ *with compact support on* U_1. *If* ϕ *is a form of degree* $k-p$ *on* U *with* $p \leq k'$, *show that* $f_*(\phi)$ *is a smooth form of degree* $k' - p$ *on* V. *Moreover,* $\phi \to f_*(\phi)$ *is continuous with respect to the topology on smooth forms. Hint: use a partition of unity in appropriate charts.*

Let T be a current of degree p on V, define the current $f^*(T)$ on U by

$$\langle f^*(T), \phi\rangle = \langle T, f_*(\phi)\rangle.$$

The previous exercise shows that the definition makes sense.

Exercise 3.1.1.16. *Show the following properties*

1. $\deg f^*(T) = \deg T$.

2. *If* ψ *is smooth,* $f^*(T \wedge \psi) = f^*(T) \wedge f^*(\psi)$.

3. $d(f^*(T)) = f^*(dT)$.

4. $\mathrm{supp}(f^*(T)) = f^{-1}(\mathrm{supp}(T))$.

5. f^* *is continuous, that is, if* $T_j \to T$ *then* $f^*(T_j) \to f^*(T)$.

3.1.2 Holomorphic maps and analytic sets

Definition 3.1.2.1. Let U be an open set in $\mathbb{C}^k$. A function $f : U \to \mathbb{C}$ is holomorphic if it is of class $\mathscr{C}^1$ and satisfies the Cauchy-Riemann equations $\partial f/\partial \overline{z}_j = 0$ for $j = 1, \dots, k$. Recall that

$$\frac{\partial f}{\partial \overline{z}_j} := \frac{1}{2}\Big(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\Big), \qquad \text{if } z_j = x_j + iy_j.$$

Exercise 3.1.2.2. *Consider the closed polydisc*

$$\overline{D}(a, r) := \overline{D}(a_1, r_1) \times \cdots \times \overline{D}(a_k, r_k)$$

where $D(a_i, r_i)$ denotes the disc of center a_i and of radius r_i in $\mathbb{C}$. Using Cauchy's formula in one variable show that if f is holomorphic in a neighbourhood of $\overline{D}(a, r)$ then

$$f(w) = \frac{1}{(2\pi i)^k} \int_{bD(a_1,r_1)} \cdots \int_{bD(a_k,r_k)} \frac{f(z_1, \ldots, z_k)}{(z_1 - w_1) \ldots (z_k - w_k)} dz_1 \ldots dz_k$$

for $w \in D(a, r)$. Deduce that f is smooth and we have in $D(a, r)$

$$f(w) = \sum_{\alpha \in \mathbb{N}^k} a_\alpha (w - a)^\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_k)$, $(w - a)^\alpha := (w_1 - a_1)^{\alpha_1} \ldots (w_k - a_k)^{\alpha_k}$ and

$$a_\alpha := \frac{1}{(2\pi i)^k} \int_{bD(a_1,r_1)} \cdots \int_{bD(a_k,r_k)} \frac{f(z_1, \ldots, z_k)}{(z_1 - a_1)^{\alpha_1+1} \ldots (z_k - a_k)^{\alpha_k+1}} dz_1 \ldots dz_k.$$

Let $\mathscr{H}(U)$ denote the space of holomorphic functions in U. Many properties of holomorphic functions in one variable extend to several variables, without difficulty. For example, the uniqueness of analytic continuation and the maximum principle. A function f which is a uniform limit on compact sets of holomorphic functions is also holomorphic. On $\mathscr{H}(U)$ we can define a family of semi-norms. For $K \Subset U$, define

$$p_K(f) := \sup_K |f(z)|.$$

Then $\mathscr{H}(U)$ is a Fréchet space. It also satisfies a useful compactness property: *if (f_j) is uniformly bounded on compact sets, then it admits convergent subsequences.*

However, in one variable the zero sets of a non-constant holomorphic function is discrete. This is not the case in higher dimension.

Exercise 3.1.2.3. *Consider the complex hyperplane defined as the zero set $\{z_1 = 0\}$ of the holomorphic function z_1. Show that the complement of this hyperplane is connected.*

The implicit function theorem is valid for holomorphic maps. Let $F = (f_1, \ldots, f_k)$ be a map from U to $\mathbb{C}^k$. We say that F is *holomorphic* if the components of F are holomorphic functions.

Theorem 3.1.2.4. *If $df_1 \wedge \ldots \wedge df_k \neq 0$ at a then there is a polydisc $D(a, r)$ on which F is invertible. That is, there is a bijective holomorphic mapping G of a neighbourhood of $F(a)$ onto $D(a, r)$ such that $G \circ F = \mathrm{id}$ on $D(a, r)$.*

Recall here that

$$df_1 \wedge \ldots \wedge df_k = \det \left(\frac{\partial f_j}{\partial z_l} \right)_{1 \leq j, l \leq k} dz_1 \wedge \ldots \wedge dz_k.$$

So, the hypothesis in the theorem is that the complex Jacobian of F does not vanish at a.

Exercise 3.1.2.5. *Let $F = (f_1, \dots, f_m) : U \to \mathbb{C}^m$ be a holomorhic map with $m \le k$. Assume $df_1 \wedge \ldots \wedge df_m \neq 0$ at a. Show $F^{-1}(0)$ is a complex submanifold in a neighbourhood of a. More precisely, there is a polydisc $D(a,r)$ such that $F^{-1}(0) \cap D(a,r)$ is a graph of a holomorphic map over a polydisc $D(a_{i_1}, r_1) \times \cdots \times D(a_{k-m}, r_{k-m})$.*

If we consider the common zero set of a family of holomorphic functions without any condition on the rank, we obtain the notion of analytic sets.

Definition 3.1.2.6. Let U be an open set in $\mathbb{C}^k$. A subset A of U is *analytic* if, for every $a \in U$, there exists a neighbourhood V of a and finitely many holomorphic functions $f_1, \dots, f_m$ on V such that

$$A \cap V = \{f_1 = \cdots = f_m = 0\}.$$

Exercise 3.1.2.7. *Show that A is closed. Assume that U is connected. If $A \neq U$ then $U \setminus A$ is a dense connected open subset of U. Hint: show that every point a admits a neighbourhood B such that $B \setminus A$ is connected; for this purpose, consider the restriction of f_j to a complex line.*

We are going to describe locally the zero set of a holomorphic function g. We can assume that $g(0) = 0$ and $g(0, \dots, 0, z_k)$ is not identically zero. Assume also that g is defined in a neighbourhood of $\overline{D}(0,r) = \overline{D}(0,r') \times \overline{D}(0,r_k)$ where $\overline{D}(0,r') := \overline{D}(0,r_1) \times \cdots \times \overline{D}(0,r_{k-1})$. Reducing r_k, we can assume that $g(0, \dots, 0, z_k)$ does not vanish on $\{|z_k| = r_k\}$.

Exercise 3.1.2.8.

1. *Show that reducing r' allows us to assume that $g(z', z_k)$ does not vanish on $\{|z_1| \le r_1, \dots, |z_{k-1}| \le r_{k-1}, |z_k| = r_k\}$.*

2. *Prove that*

$$\sigma_j(z') := \frac{1}{2\pi i} \int_{bD(0,z_k)} \frac{1}{g(z', z_k)} \frac{\partial g(z', z_k)}{\partial z_k} z_k^j dz_k$$

are holomorphic on $D(0, r')$.

3. *Show that σ_0 is a constant function which is equal to the number of zeros in $\{|z_k| < r_k\}$ of the function $z_k \mapsto g(z', z_k)$ counted with multiplicities. Hint: use the residue theorem in one variable.*

4. *What is the meaning of $\sigma_l(z')$ in terms of the zeros of the above function?*

5. *Let $w_l(z')$ denote the zeros of the function $z_k \mapsto g(z', z_k)$ and define*

$$W(z', z_k) := \prod_{l=1}^{s} (z_k - w_l(z')), \qquad \text{where } s := \sigma_0.$$

Prove that W is a holomorphic function and g/W is a holomorphic function without zero on $D(0,r)$.

We have proved that near 0 the set $(g = 0)$ is given as the zero set of the Weierstrass polynomial

$$W(z) := z_k^s + \sum_{j=0}^{s-1} a_j(z')z_k^j$$

where a_j are holomorphic functions near 0 vanishing at 0. If we denote by $R(z')$ the resultant of W and $\partial W/\partial z_k$ then if $R(a') \neq 0$ we have $\partial W/\partial z_k(a', \cdot) \neq 0$. Hence the analytic set $(g = 0)$ is *smooth* at all points in $\{g = 0, z' = a'\}$, i.e. $(g = 0)$ is a manifold near these points. In this case, we observe that if π is the projection of $(g = 0)$ on $D(0, r')$ and if S is the zero set of R then π defines a covering over $D(0, r') \setminus S$. In order to describe the analytic set over S, we need to introduce the notion of germ of analytic set and the notion of irreducibility.

Definition 3.1.2.9. Let $a \in U$ and V_1, V_2 be two neighbourhoods of a. Suppose that A_1, A_2 are two analytic subsets of V_1, V_2 respectively. We say that *they have the same germ at a* if they are equal in a neighbourhood of a. We denote by A_a the germ of an analytic set at a. We say that *A_a is irreducible* if when $A_a = A_{1,a} \cup A_{2,a}$ with $A_{i,a}$ analytic germs, then $A_a = A_{1,a}$ or $A_a = A_{2,a}$.

Using the properties of the ring of germs of holomorphic functions, one proves

Proposition 3.1.2.10. *Let A_a be a germ of analytic set at a. Then there is a unique finite decomposition of A_a into irreducible analytic germs $A_{j,a}$:*

$$A_a = \cup_{j=1}^m A_{j,a} \quad \textit{with} \quad A_{j,a} \not\subset \cup_{i \neq j} A_{i,a}.$$

Definition 3.1.2.11. The germs $A_{j,a}$ are called *the irreducible components* of A_a.

Theorem 3.1.2.12 (structure of analytic sets)**.** *Let A be an analytic set in U with irreducible germ at 0. There is a system of coordinates $(z_1, \dots, z_k)$ in $\mathbb{C}^k$, and integers a, p and polydiscs D_m, D_{k-m} in $\mathbb{C}^m$ and $\mathbb{C}^{k-m}$ respectively, and an analytic subset S in D_m with the following properties. Let $\pi : D = D_m \times D_{k-m} \to D_m$ denote the canonical projection $(z_1, \dots, z_k) \mapsto (z_1, \dots, z_m)$ on D_m. Then*

1. *$A \setminus \pi^{-1}(S)$ is a complex submanifold of dimension m in $D \setminus \pi^{-1}(S)$.*
2. *$\pi : A \cap D \setminus \pi^{-1}(S) \to D_m \setminus S$ is a holomorphic covering of degree p over $D_m \setminus S$.*
3. *$\pi : A \cap D \to D_m$ is proper and surjective.*
4. *$A \cap D \setminus \pi^{-1}(S)$ is connected and dense in $A \cap D$.*
5. *If f is holomorphic in a neighbourhood of 0, vanishes on A_0 and depends only on $z_1, \dots, z_m$, then $f = 0$.*

The system of coordinates with above properties can be chosen arbitrary closed to any given system of holomorphic coordinates. The number m is called *the dimension of A at* 0.

3.1.3 Positive forms and positive currents

If we identify $\mathbb{C}^k$ with $\mathbb{R}^{2k}$ we can write $z_n := x_n + ix_{n+k}$ with x_n real. Hence linear forms dx_n, dx_{n+k} can be written in a unique way as linear combinations with complex coefficients of $dz_n := dx_n + idx_{n+k}$ and of $d\overline{z}_n := dx_n - idx_{n+k}$. If $I = (i_1, \ldots, i_r) \subset \{1, \ldots, k\}$ is a multi-index define $dz_I := dz_{i_1} \wedge \ldots \wedge dz_{i_r}$ and $d\overline{z}_I := d\overline{z}_{i_1} \wedge \ldots \wedge d\overline{z}_{i_r}$. Then any r-form α on $U \subset \mathbb{C}^k$ can be written in a unique way as

$$\alpha = \sum_{|I|+|J|=r} \alpha_{IJ} dz_I \wedge d\overline{z}_J$$

where α_{IJ} are functions with complex values. We say that α is *a form of bidegree* (p, q) if $\alpha_{IJ} = 0$ when either $|I| \neq p$ or $|J| \neq q$. We have the following decomposition of spaces of r-forms as direct sums of spaces of (p, q)-forms

$$\mathscr{D}^r(X) = \bigoplus_{p+q=r} \mathscr{D}^{p,q}(X)$$

and by duality we get

$$\mathscr{D}'^r(X) = \bigoplus_{p+q=r} \mathscr{D}'^{p,q}(X).$$

Currents in $\mathscr{D}'^{p,q}(X)$ are *of bidegree* (p, q) *and of bidimension* $(k-p, k-q)$. They act trivially on forms of bidegree $(k - p', k - q')$ when $(p', q') \neq (p, q)$.

Exercise 3.1.3.1. *Let $f : U \to U'$ be a holomorphic map. If α is a (p, q)-form on U' show that $f^*(\alpha)$ is a (p, q)-form on U. If T is a current of bidimension (p, q) on U and if f is proper on* $\mathrm{supp}(T)$ *show that $f_*(T)$ is a current of bidimension (p, q). If T is a current of bidegree (p, q) on U' and if f is a submersion show that $f^*(T)$ is a current of bidegree (p, q) on U.*

If T is a (p, q)-current on U then we can decompose dT as the sum of a $(p + 1, q)$-current ∂T and of a $(p, q + 1)$-current $\overline{\partial} T$, that is, $d = \partial + \overline{\partial}$.

Exercise 3.1.3.2. *Show that $\partial(\partial T) = 0$, $\overline{\partial}(\overline{\partial} T) = 0$ and $\partial\overline{\partial} T = -\overline{\partial}\partial T$. Prove that if α is a test form we have*

$$\langle \partial T, \alpha \rangle = (-1)^{p+q+1} \langle T, \partial\alpha \rangle \quad \text{and} \quad \langle \overline{\partial} T, \alpha \rangle = (-1)^{p+q+1} \langle T, \overline{\partial}\alpha \rangle.$$

Exercise 3.1.3.3. *Check that f^*, f_* commute with ∂ and $\overline{\partial}$.*

Exercise 3.1.3.4. *Let $\tau(z) := idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_k \wedge d\overline{z}_k$. Show that*

$$\tau(z) = 2^k dx_1 \wedge dy_1 \wedge \ldots \wedge dx_k \wedge dy_k = i^{k^2} dz_1 \wedge \ldots \wedge dz_k \wedge d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_k.$$

Suppose $(w_1, \ldots, w_k)$ *is another system of coordinates in* U *which is holomorphic, i.e.* w_l *are holomorphic functions on* z_j*. Check that*

$$dw_1 \wedge \ldots \wedge dw_k = \det\Big(\frac{\partial w_l}{\partial z_j}\Big) dz_1 \wedge \ldots \wedge dz_k$$

and

$$\tau(w) = \Big|\det\Big(\frac{\partial w_l}{\partial z_j}\Big)\Big|^2 \tau(z).$$

Deduce that any complex manifold has a canonical orientation.

We define also *the conjugate* of a form or a current by the formulas

$$\overline{\alpha} := \sum \overline{\alpha}_{IJ} d\overline{z}_I \wedge dz_J \quad \text{and} \quad \langle \overline{T}, \alpha\rangle := \overline{\langle T, \overline{\alpha}\rangle}$$

where $\alpha = \sum \alpha_{IJ} dz_I \wedge d\overline{z}_J$.

Exercise 3.1.3.5. *Show that* $\overline{T}$ *is a* (q,p)*-current.*

We say that T is *real* if $\overline{T} = T$. In this case, we have $p = q$. Define $d^c := \frac{1}{2i\pi}(\partial - \overline{\partial})$, then $dd^c = \frac{i}{\pi}\partial\overline{\partial}$ is a real operator.

Exercise 3.1.3.6. *Show that* d^c *is a real operator in the sense that* $d^c u = \overline{d^c \overline{u}}$*. Show that* $dd^c = \frac{i}{\pi}\partial\overline{\partial}$*.*

We now introduce a notion of positivity on the exterior algebra $\Lambda\mathbb{C}^k$. A form σ of maximal bidegree is *positive* if $\sigma = \lambda\tau$ with $\lambda \geq 0$. Recall that $\tau = idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_k \wedge d\overline{z}_k$. It is a consequence of Exercise 3.1.3.4 that the notion is independent of the basis. So, it makes sense in any complex vector space of dimension k. Let E be a complex vector space. A (q,q)-form in $\Lambda^{q,q}E$ is *positive* if it belongs to the convex cone generated by the forms $iu_1 \wedge \overline{u}_1 \wedge \ldots \wedge iu_q \wedge \overline{u}_q$ with $u_j \in \Lambda^{1,0}E$. A (q,q)-form ϕ is *weakly positive* if $\phi \wedge iu_1 \wedge \overline{u}_1 \wedge \ldots \wedge iu_{k-q} \wedge \overline{u}_{k-q}$ is positive for all $(1,0)$-forms $u_1, \ldots, u_{k-q}$. A smooth (q,q)-form ϕ is *positive* (resp. *weakly positive*) if $\phi(z)$ is positive (resp. weakly positive) for every $z \in U$.

Exercise 3.1.3.7. *Let* $\Lambda^{p,p}_{\mathbb{R}}E$ *denote the real part of* $\Lambda^{p,p}E$*. Show that the cone* $W^{p,p}$ *of weakly positive forms in* $\Lambda^{p,p}_{\mathbb{R}}E$ *is the dual of the cone* $S^{k-p,k-p}$ *of positive forms in* $\Lambda^{k-p,k-p}_{\mathbb{R}}E$*. Show that* $S^{1,1} = W^{1,1}$ *and* $S^{k-1,k-1} = W^{k-1,k-1}$*.*

Exercise 3.1.3.8. *Check that* $dz_j \wedge d\overline{z}_l$ *is in the span of* $W^{1,1}$*. Hint: let* $\alpha = dz_j + idz_l$ *and* $\beta = dz_j + dz_l$*. Compute* $i\alpha\wedge\overline{\alpha}$*,* $i\beta\wedge\overline{\beta}$*; show that* $\mathrm{Re}(dz_j \wedge d\overline{z}_l)$ *and* $\mathrm{Im}(dz_j \wedge d\overline{z}_l)$ *are in the span of* $W^{1,1}$*. Show by induction that* $S^{p,p}$ *spans* $\Lambda^{p,p}_{\mathbb{R}}\mathbb{C}^k$*.*

Exercise 3.1.3.9. *Prove that weakly positive forms are real. Hint: if* u *is weakly positive then for* $v \in S^{k-p,k-p}$*, show that* $u \wedge v = \overline{u} \wedge \overline{v}$*; use the previous exercise.*

Proposition 3.1.3.10. *A form* $u \in \Lambda^{p,p}\mathbb{C}^k$ *is weakly positive if the restriction of* u *to any complex subspace of dimension* p*, is positive. In particular,* $u = i\sum u_{jl} dz_j \wedge d\overline{z}_l$ *is positive if and only if the Hermitian form* $\sum u_{jl}\xi_j\overline{\xi}_l$ *is positive.*

Proof. Suppose $F = \{z_{p+1} = \cdots = z_k = 0\}$ is a subspace of dimension p. Then $u_{|F} = \lambda_F i dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge i dz_p \wedge d\bar{z}_p$ where λ_F is a constant. On the other hand

$$\lambda_F \tau(z) = u \wedge i dz_{p+1} \wedge d\bar{z}_{p+1} \wedge \ldots \wedge i dz_k \wedge d\bar{z}_k.$$

So $\lambda_F \geq 0$. The converse is clear because forms $i dz_{p+1} \wedge d\bar{z}_{p+1} \wedge \ldots \wedge i dz_k \wedge d\bar{z}_k$ generate strongly positive forms when we allow complex linear changes of coordinates.

When u is of bidegree $(1,1)$ the restriction to the complex lines parametrized by $t \mapsto t\xi$ with $t \in \mathbb{C}$, $\xi \in \mathbb{C}^k$ is

$$u_{|F} = \Big(\sum u_{jl}\xi_j\bar{\xi}_l\Big) i dt \wedge d\bar{t}.$$

So, the coefficient λ_F is precisely the value of the Hermitian form at ξ. □

Exercise 3.1.3.11. *Let $f : E_1 \to E_2$ be a complex linear map between complex vector spaces. If α is a weakly positive form of bidegree (p,p), then $f^*(\alpha)$ is weakly positive. Hint: consider first the case where f is an isomorphism. In the general case, decompose f.*

Definition 3.1.3.12. A (p,p)-current T on U is *weakly positive* if

$$\langle T, iu_1 \wedge \overline{u}_1 \wedge \ldots \wedge iu_{k-p} \wedge \overline{u}_{k-p}\rangle \geq 0$$

for all smooth test $(1,0)$-forms $u_1, \ldots, u_{k-p}$. The current T is *(strongly) positive* if

$$\langle T, \phi\rangle \geq 0$$

for all weakly positive forms $\phi \in \mathscr{D}^{k-p,k-p}(U)$.

Exercise 3.1.3.13. *Let U_1, U_2 be two open sets in $\mathbb{C}^k$ and $\mathbb{C}^{k'}$ respectively. Let $f : U_1 \to U_2$ be a holomorphic map. Let $T \in \mathscr{D}'_{k-p,k-p}(U_1)$ such that the restriction of f to* supp(T) *is proper. If T is weakly positive (resp. positive) show that $f_*(T)$ is weakly positive (resp. positive).*

Exercise 3.1.3.14. *Let $Z = \{z \in U,\ z_{p+1} = \cdots = z_k = 0\}$. Show that the current of integration on Z, denoted by $[Z]$, is positive. Hint: let $\phi \in \mathscr{D}^{p,p}(U)$ and $i : (z_1, \ldots, z_p) \mapsto (z_1, \ldots, z_p, 0, \ldots, 0)$, then*

$$\langle [Z], \phi\rangle = \int_{\mathbb{C}^p} i^*(\phi).$$

Show that $d[Z] = 0$. Prove the same result for a complex submanifold of dimension p. Hint: use a partition of unity and holomorphic changes of coordinates.

Exercise 3.1.3.15. *Assume the current T is defined by a (p,p)-form ψ. Show that T is positive (resp. weakly positive) if and only if ψ is positive (resp. weakly positive).*

Exercise 3.1.3.16. *Let $f : U_1 \to U_2$ be a holomorphic submersion. Show that the operator $f^* : \mathscr{D}'^{p,p}(U_2) \to \mathscr{D}'^{p,p}(U_1)$ is continuous and maps positive (resp. weakly positive) currents into positive (resp. weakly positive) currents.*

The following theorem is useful in dynamics, the proof is much more tricky.

Theorem 3.1.3.17 ([37])**.** *Let U_1, U_2 be open sets in $\mathbb{C}^k$ and $\mathbb{C}^{k'}$ respectively. Let $f : U_1 \to U_2$ be a holomorphic map. Assume each fiber of f is either empty or an analytic set of dimension $k - k'$. Then the pull-back operator can be extended to positive closed currents of bidegree (p,p) on U_2. If T is such a current, then $f^*(T)$ is positive closed and depends continuously on T for the weak topology of currents. Moreover, if T has no mass on a Borel set $A \subset U_2$ then $f^*(T)$ has no mass on $f^{-1}(A)$.*

Exercise 3.1.3.18. *Let $f(z_1, z_2) = (z_1^2 + c, z_2^2 + c)$. Show that f is not a submersion, but f satisfies the assumption of the above theorem.*

Exercise 3.1.3.19. *Let $T = \sum i^{p^2} T_{IJ} dz_I \wedge d\overline{z}_J$ be a positive current. Show that T is of order zero. In particular, T_{IJ} are (can be identified to) distributions of order zero, i.e. complex measures.*

Define for T positive

$$\beta := \frac{i}{\pi}\partial\overline{\partial}\|z\|^2 \quad \text{and} \quad \sigma_T := \frac{1}{2^{k-p}(k-p)!} T \wedge \beta^{k-p}.$$

Then σ_T is a positive measure that we call *the trace measure* of T. The mass of σ_T on a Borel set B is called *mass of T on B* and is also denoted by $\|T\|_B$. Recall that we are in $\mathbb{C}^k$.

Exercise 3.1.3.20. *Let $T = \sum i^{p^2} T_{IJ} dz_I \wedge d\overline{z}_J$ be a positive current and let σ_T be the trace measure of T. Show that the total variations $|T_{IJ}|$ of T_{IJ} verify $|T_{IJ}| \leq 2^k \sigma_T$.*

The following important theorem is due to Wirtinger.

Theorem 3.1.3.21. *Let $V \subset \mathbb{C}^k$ be a manifold of pure dimension p. Then the volume form on V associated to the standard metric in $\mathbb{C}^k$ is equal to $\frac{1}{2^p p!}\beta^p_{|V}$. In particular, we have*

$$\mathrm{volume}(V) = \frac{1}{2^p p!}\int_V \beta^p.$$

One says that an analytic subset V of U has *pure dimension p* if the germ V_a of V at a is of dimension p for every $a \in V$. The following Lelong's theorem gives an important class of positive closed currents.

Theorem 3.1.3.22. *Let V be an analytic subset of pure dimension p of U and $\mathrm{reg}(V)$ be the regular part of V. Define the current $[V]$ of bidimension (p,p) by*

$$\langle [V], \alpha\rangle := \int_{\mathrm{reg}(V)} \alpha, \qquad \alpha \in \mathscr{D}^{p,p}(X).$$

Then the current $[V]$ is well defined and is positive and closed.

This theorem says in particular that the volume of $\mathrm{reg}(V)$ near the singular points is locally bounded.

Exercise 3.1.3.23. *Let α be a positive (k,k)-form on U. Show that α can be written as $\alpha = \varphi(idz_1 \wedge d\overline{z}_1) \wedge \ldots \wedge (idz_k \wedge d\overline{z}_k)$ where φ is a positive function.*

Exercise 3.1.3.24. *Let T be a weakly positive (p,p)-current and let α be weakly positive (q,q)-form on U. Show that T is positive if $p = 0, 1, k-1$ or k. Find an example of a weakly positive form α which is not positive. Show that $T \wedge \alpha$ is weakly positive if either T or α is positive, and $T \wedge \alpha$ is positive if both T and α are positive. Find an example so that $T \wedge \alpha$ is not weakly positive.*

Exercise 3.1.3.25. *Let T be a (weakly) positive (p,p)-current on an open set U of $\mathbb{C}^k$. Let ρ be a smooth positive function with support in $\mathbb{C}^k$ depending only on $\|z\|$ such that $\int \rho d\mathrm{Leb} = 1$. Define $\rho_\epsilon(z) := \epsilon^{-2k}\rho(\epsilon^{-1}z)$ an approximation of the identity in $\mathbb{C}^k$. Show that $T * \rho_\epsilon$ is a (weakly) positive (p,p)-form on the open set where it is defined. If T is closed show that $T * \rho_\epsilon$ is closed. Prove that $T * \rho_\epsilon \to T$ in the sense of currents as $\epsilon \to 0$.*

Exercise 3.1.3.26. *Let T and T' be two positive closed (p,p)-currents on an open subset U of $\mathbb{C}^k$ with $p \leq k-1$. Let K be a compact subset of U. Assume that $T = T'$ outside K. Show that $\|T\|_K = \|T'\|_K$.*

Exercise 3.1.3.27. *Let $T = i^{p^2} \sum T_{IJ} dz_I \wedge d\overline{z}_J$ be a positive closed (p,p)-current on an open subset U of $\mathbb{C}^k$. Show that*

$$\sigma_T = \frac{1}{2^{k-p}} \Big(\sum_{|I|=p} T_{II} \Big) idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_k \wedge d\overline{z}_k.$$

3.1.4 Plurisubharmonic functions

Plurisubharmonic functions were introduced by Lelong and Oka.

Definition 3.1.4.1. Let $u : U \to \mathbb{R} \cup \{-\infty\}$ be an u.s.c. function on an open set $U \subset \mathbb{C}^k$ which is not identically $-\infty$ on any component of U. We say that u is *plurisubharmonic* (psh for short) if its restriction to any complex line L is either subharmonic or identically $-\infty$ on each compenent of $U \cap L$. We say that u is *pluriharmonic* if u and $-u$ are psh.

Theorem 3.1.4.2. *A function $u : U \to \mathbb{R} \cup \{-\infty\}$ in $L^1_{loc}(X)$ is psh if and only if it is strongly u.s.c. and $dd^c u$ is a positive $(1,1)$-current.*

Let ρ be a smooth positive function with compact support in the unit ball of $\mathbb{C}^k$ depending only on $\|z\|$ such that $\int \rho d\mathrm{Leb} = 1$. Define $\rho_\epsilon(z) := \epsilon^{-2k}\rho(\epsilon^{-1}z)$.

Theorem 3.1.4.3. *If u is a psh function on an open set $U \subset \mathbb{C}^N$ then $u * \rho_\epsilon$ is smooth psh on $U_\epsilon := \{z \in U,\ \mathrm{dist}\,(z, \partial U) > \epsilon\}$ and decrease to u when ϵ decreases to* 0.

Example 3.1.4.4. If a is a point in the unit ball B of $\mathbb{C}^k$ then $\log \|z - a\|$ is psh on $\mathbb{C}^k$. If (λ_n) is a sequence of positive numbers and $(a_n) \subset$ B such that $\sum \lambda_n \log \|a_n\| > -\infty$, then $\sum \lambda_n \log \|z - a_n\|$ defines a psh function on B. The sequence (a_n) can be dense in B.

Proposition 3.1.4.5. *The set* $\mathrm{PSH}(U)$ *of psh functions on a domain U is a convex cone with the following properties.*

1. *If a function $\chi : (\mathbb{R} \cup \{-\infty\})^p \to \mathbb{R} \cup \{-\infty\}$ is convex and increasing in each variable and if $u_1, \ldots, u_p$ are psh functions on U then $\chi(u_1, \ldots, u_p)$ is psh or is identically $-\infty$ on U. In particular,* $\max(u_1, \ldots, u_p)$ *is psh.*

2. *If u_n are psh functions decreasing to u then either $u = -\infty$ or u or is psh.*

3. *If (u_n) is a sequence of psh functions locally bounded from above then $(\sup_n u_n)^*$ is a psh function and $(\sup_n u_n)^* = \sup_n u_n$ almost everywhere.*

4. *If $f : U' \to U$ is a holomorphic map and if u is psh on U then on each component of U', $u \circ f$ is either psh or identically $-\infty$.*

Here, v^* denote the upper-regularization of v, that is, $v^*(a) := \limsup_{z \to a} v(a)$. The compactness theorem 2.1.1.29 extends to psh functions.

Theorem 3.1.4.6 (compactness)**.** *Let (u_n) be a sequence of psh functions on a domain U in $\mathbb{C}^k$. Assume (u_n) is locally bounded from above. Then either (u_n) converges locally uniformly to $-\infty$ or there is a subsequence (u_{n_i}) converging to a psh function u in all $L^p_{loc}(U)$, $1 \le p < \infty$.*

Exercise 3.1.4.7. *Let u be a psh function on U and K be a compact subset of U. Show that there is $\alpha > 0$ such that $e^{-\alpha u}$ is integrable on K. Hint: use Fubini's theorem and Exercise 2.1.2.8.*

The following result is called the *Lelong-Poincaré formula*.

Theorem 3.1.4.8. *Let f be a holomorphic function on a domain U which is not identically zero. Then* $\log|f|$ *is a psh function which is pluriharmonic outside the zero set $f^{-1}(0)$ of f. Moreover, we have*

$$dd^c \log|f| = \sum m_i [V_i]$$

where V_i are irreducible components of $f^{-1}(0)$ and m_i are their multiplicities.

Definition 3.1.4.9. A subset E of U is *locally pluripolar* if for every point $a \in U$ there is a neighbourhood W of a and a psh function u on W such that $E \cap W \subset \{u = -\infty\}$. The set E is said to be *locally complete pluripolar* if u can be chosen so that $E \cap U = \{u = -\infty\}$. A subset E is *pluripolar* (resp. *complete pluripolar*) if there is a psh function u on U such that $E \subset \{u = -\infty\}$ (resp. $E = \{u = -\infty\}$)

Proposition 3.1.4.10. *Any proper analytic subset of a domain U of $\mathbb{C}^k$ is complete pluripolar. A countable union of complete pluripolar (resp. pluripolar) sets in U is complete pluripolar (resp. pluripolar). The Hausdorff dimension of a pluripolar set is smaller or equal to $2k-2$.*

Exercise 3.1.4.11. *Let E be a closed locally pluripolar subset of U and let u be a psh function on $U \setminus E$. Assume that u is locally bounded from above near E. Then there is a psh function $\widetilde{u}$ on U such that $\widetilde{u} = u$ outside E.*

Exercise 3.1.4.12. *Prove the Hartogs' lemma for psh functions.*

Exercise 3.1.4.13. *Construct a psh function on $\mathbb{C}^k$ which is nowhere continuous.*

Exercise 3.1.4.14. *Construct two psh functions u, v on $\mathbb{C}^k$ which are equal outside a ball* B *but are not equal. Show that there are two positive closed $(1,1)$-currents T, S which are equal outside* B *but are not equal. It is possible to choose* $\mathrm{supp}(T)$ *and* $\mathrm{supp}(S)$ *connected.*

Exercise 3.1.4.15. *Let $z_n = x_n + ix_{n+k}$ be the coordinates of $\mathbb{C}^k$ and let $\pi(z) := (x_1, \ldots, x_k)$ denote the projection on $\mathbb{R}^k$. Let v be a real-valued function on an open set V of $\mathbb{R}^k$ such that $u := v \circ \pi$ is psh on $\pi^{-1}(V)$. Show that v is convex in V. Hint: assume first that v is smooth.*

Exercise 3.1.4.16. *Let u be a strictly negative psh function on U. Show that $v := -\log(-u)$ is psh. Prove that ∇v is in L^2_{loc}.*

Exercise 3.1.4.17. *Let u be a psh function on U. Show that ∇u is in $L^{2-\epsilon}_{loc}$ for $0 < \epsilon \leq 1$. Show that ∇u is in L^2_{loc} if u is locally bounded.*

Exercise 3.1.4.18. *Let u be a pluriharmonic function on a ball* B *in $\mathbb{C}^k$. Show that there is a holomorphic function f on* B *such that $u = \mathrm{Re}(f)$. Deduce that u is real analytic.*

Exercise 3.1.4.19. *Let u_n be a sequence of pluriharmonic functions which are locally uniformly bounded on U. Show that there is a subsequence converging locally uniformly to a pluriharmonic function.*

Exercise 3.1.4.20. *In $\mathbb{C}^2$ describe geometrically the currents*

$$dd^c \log^+ |z_1| \quad \textit{and} \quad dd^c \log \max(|z_1|, |z_2|).$$

Compute the mass of $dd^c \log^+ \|z\|$ on the unit sphere of $\mathbb{C}^2$.

Exercise 3.1.4.21. *Let E be a complete pluripolar set in $\mathbb{C}^k$ and let V be a connected complex submanifold of $\mathbb{C}^k$. Show that either $E \cap V$ is complete pluripolar in V or $V \subset E$.*

Exercise 3.1.4.22. *Construct a pluripolar subset of $\mathbb{C}^k$ which is not complete pluripolar.*

3.1.5 Intersection of currents

Let T be a positive (p,p)-current on U with $p \leq k-1$. Consider a $\mathscr{C}^2$ psh function u on U. Then $dd^c u$ is a positive continuous form and it makes sense to consider the $(p+1,p+1)$-current $dd^c u \wedge T$ which is positive. We want to extend this to psh functions which are locally σ_T-integrable, in particular, to u continuous or locally bounded.

Assume that T is **positive** and **closed**. Define

$$dd^c u \wedge T := dd^c(uT).$$

Theorem 3.1.5.1. *Let T and u be as above. Then*

1. *The current $dd^c u \wedge T$ is well defined and is positive closed. If (u_n) is a sequence of psh functions decreasing to u then $dd^c u_n \wedge T \to dd^c u \wedge T$ in the sense of currents.*
2. *If u_n are psh continuous functions converging locally uniformly towards u and if T_n are positive closed (p,p)-currents converging to T then $dd^c u_n \wedge T_n \to dd^c u \wedge T$.*

We have the following Chern-Levine-Nirenberg inequalities.

Theorem 3.1.5.2. *Let $L \Subset K$ be two compact subsets of U. Let T, u be as above and let $u_1, \ldots, u_m$, v be psh functions with $m+p \leq k$. Assume that u_n are locally bounded functions and v is locally σ_T-integrable. Then there is a constant c_{LK} independent of T, u, v and u_n such that*

1. $\|dd^c v \wedge T\|_L \leq c_{LK} \|T\|_K \|v_{|K}\|_{L^1(\sigma_T)}$.
2. $\|dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge T\|_L \leq c_{LK} \|T\|_K \|u_1\|_{L^\infty(K)} \ldots \|u_m\|_{L^\infty(K)}$.
3. $\|v dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge T\|_L \leq c_{LK} \|vT\|_K \|u_1\|_{L^\infty(K)} \ldots \|u_m\|_{L^\infty(K)}$.

We have seen that the assumption that u is σ_T-integrable is verified when u is continuous or locally bounded. The following result gives another situation where this assumption is easy to check.

Theorem 3.1.5.3. *Let U be an open set in $\mathbb{C}^k$ and let K be a compact subset of U. Let T be a positive closed (p,p)-current on U with $p \leq k-1$. If a psh function u on U is locally bounded on $U \setminus K$ then it is locally σ_T-integrable. If $u_1, \ldots, u_m$ are psh on U and locally bounded on $U \setminus K$ with $m+p \leq k$ then*

1. *$dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge T$ is a positive closed $(m+p, m+p)$-current on U.*

2. *If u_j^n are psh on X, $u_j^n \geq u_j$ and $u_j^n \to u_j$ in L^1_{loc}, then*

$$dd^c u_1^n \wedge \ldots \wedge dd^c u_m^n \wedge T \to dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge T.$$

In particular $dd^c u_1 \wedge \ldots \wedge dd^c u_m \wedge T$ is symmetric with respect to $u_1, \ldots, u_m$.

Exercise 3.1.5.4. *Show that the current $dd^c \log|z_1| \wedge dd^c \log|z_2|$ is well defined and compute it.*

Exercise 3.1.5.5. *Show that the measure $(dd^c \log\|z\|)^k$ is well defined and compute it.*

Exercise 3.1.5.6. *Let $L := \{(z_1, z_2) \in \mathbb{C}^2,\ |z_1| = |z_2|\}$. Define for $n \geq 1$*

$$v_n(z) := |z_1^{2n} + z_2^{2n}|^{1/2n} \quad \text{and} \quad v(z) := \max(|z_1|, |z_2|).$$

1. *Show that $0 \leq v_n(z) \leq 2^{1/n} v(z)$ and that $v_n \to v$ locally uniformly on $\mathbb{C}^2 \backslash L$.*

2. *Prove that $(dd^c v_n)^2 = 0$ and $(dd^c v)^2 \neq 0$. Hint: prove it first out of the origin, and show that $(dd^c v_n)^2$ has no mass at 0.*

3. *Show that $v_n \to v$ pointwise except on a set of Hausdorff dimension 2. Hint: write $z_2 = z_1 e^{i\theta}$ for $z \in L$. Show that the following set has Hausdorff dimension zero*
$$E := \Big\{\theta,\ \liminf_{n\to\infty} \big(\cos(n\theta)\big)^{1/n} < 1\Big\}.$$

Exercise 3.1.5.7. *Let u_1 and u_2 be two psh functions in $\mathbb{C}^2$. Define $T_i := dd^c u_i$. Show that u_1 is locally σ_{T_2}-integrable if and only if u_2 is locally σ_{T_1}-integrable.*

Exercise 3.1.5.8. *Let U, L, K and T be as in Theorem 3.1.5.2. Let u be a smooth psh function on U. Show that there is a constant c_{LK} independent of u and T such that*
$$\|i\partial u \wedge \overline{\partial} u \wedge T\|_L \leq c_{LK} \|u\|^2_{L^\infty(K)} \|T\|_K.$$

If u is a continuous psh function then the currents $\partial u \wedge T$ and $\overline{\partial} u \wedge T$ are well-defined. If u_n are continuous psh functions converging locally uniformly to u show that $\partial u_n \wedge T \to \partial u \wedge T$ and $\overline{\partial} u_n \wedge T \to \overline{\partial} u \wedge T$.

3.1.6 Skoda's extension theorem

Let E be a closed subset of U. Let T be a current on $U \setminus E$. Assume that the mass of T is locally bounded near E. More precisely, we assume that at every point $z \in E$ there is a neighbourhood U of z such that

$$\|T\|_{U \setminus E} < \infty.$$

Then one can consider the trivial extension $\widetilde{T}$ of T in U. Basically since T has measure coefficients we extend the measure by putting the mass zero on E. One can also define $\widetilde{T}$ as follows. Let $0 \leq \chi_n \leq 1$ be a sequence of smooth functions vanishing near E and such that χ_n increase to $\mathbf{1}_{U \setminus E}$ locally uniformly on $U \setminus E$. Then if α is a test form with compact support in U we define

$$\langle \widetilde{T}, \alpha \rangle := \lim_{n \to \infty} \langle T, \chi_n \alpha \rangle.$$

The following theorem was proved by Skoda [67].

Theorem 3.1.6.1. *Let E be an analytic subset of U. Let T be a positive closed (p,p)-current on $U \setminus E$. Assume that the mass of T is locally bounded near E. Then the trivial extension $\widetilde{T}$ of T is a* **positive closed** *(p,p)-current on U.*

This implies the following result due to Bishop.

Theorem 3.1.6.2. *Let E and U be as above. Let V be an analytic subset of pure dimension p of $U \setminus E$. Assume that V has locally finite volume near E. Then $\overline{V}$ is an analytic subset of U.*

Remark 3.1.6.3. The previous results still hold when E is a locally complete pluripolar set of U and when T is a positive current such that $dd^c T \leq S$ on $X \setminus E$ where S is a current of order zero on U, see [1, 65, 18, 37].

When E is an analytic subset of small dimension the assumption on the mass of T is not necessary. The following theorem is due to Harvey-Polking.

Theorem 3.1.6.4. *Let E be a closed subset of U such that $\mathscr{H}^{2k-2p-1}(E) = 0$ where $\mathscr{H}^{\alpha}$ denote the Hausdorff measure of dimension α. Let T be a positive closed (p,p)-current on $U \setminus E$. Then T has finite mass near E and the trivial extension $\widetilde{T}$ of T is positive closed on U.*

This gives in particular the Remmert-Stein theorem. Let A be an analytic subset of U of dimension less or equal to $k-p-1$. If V is an analytic subset of pure dimension $k-p$ of $U \setminus A$ then $\overline{V}$ is an analytic subset of U.

Exercise 3.1.6.5. *Find a (non-complete) pluripolar set E on U and a positive closed current T such that T has finite mass near E but $\widetilde{T}$ is not closed.*

Exercise 3.1.6.6. *Let T be a positive closed $(1,1)$-current in $\mathbb{C}^2 \setminus \{z_1 = 0\}$. Assume that T has finite mass near $\{z_1 = 0,\ |z_2| > 1\}$. Show that T has locally finite mass near $\{z_1 = 0\}$. Deduce that the trivial extension of T is positive closed on $\mathbb{C}^2$.*

Exercise 3.1.6.7. *Find a positive dd^c-closed $(1,1)$-current on $\mathbb{C}^2 \setminus \{z_1 = 0\}$ with mass locally bounded near $\{z_1 = 0\}$ so that its trivial extension is not dd^c-closed.*

3.1.7 Lelong number and Siu's theorem

Let σ be a positive measure on U. It is possible to define the *upper* and *lower* m*-densities* of σ at $a \in U$. More precisely

$$\Theta^*(m,a) := \limsup_{r\to 0} \frac{\sigma(\mathrm{B}(a,r))}{c_m r^m} \quad \text{and} \quad \Theta_*(m,a) := \liminf_{r\to 0} \frac{\sigma(\mathrm{B}(a,r))}{c_m r^m}$$

where c_m is the volume of the unit ball in $\mathbb{R}^m$. We have $c_{2p} = \pi^p/p!$. In general the limit does not exist. When it exists we call it simply m*-density* of σ at a. It turns out that the trace measure σ_T of a positive closed (p,p)-current T has this remarkable property.

Definition 3.1.7.1. The $(2k-2p)$-density of σ_T at a is called *Lelong number* of T at a and is denoted by $\nu(T,a)$.

The following results were proved by Siu.

Theorem 3.1.7.2 (Siu)**.** *Let U and T be as above. Then the Lelong number $\nu(T,a)$ is a non-negative finite number which does not depend on the local coordinates on U.*

Theorem 3.1.7.3 (Siu)**.** *Let U and T be as above. Then the level set $\{\nu(T,a) \geq c\}$ is an analytic subset of dimension $\leq k-p$ of U for every $c > 0$.*

Corollary 3.1.7.4. *Let U and T be as above. Then there is a finite or countable family of irreducible analytic sets V_i of dimension $k-p$ of U and positive constants λ_i such that $T' := T - \sum \lambda_i [V_i]$ is a positive closed current such that $\{\nu(T',a) > 0\}$ is a finite or countable union of analytic subsets of dimension $< k-p$.*

Exercise 3.1.7.5. *Let T be a positive closed (p,p)-current on U. Assume that $T = dd^c V$ where V is a locally bounded $(p-1,p-1)$-form. Show that the Lelong number of T is zero at every point of U.*

Exercise 3.1.7.6. *Let V be an analytic subset of pure dimension $k-p$ of U. Compute the Lelong number of $[V]$ at a point a of U.*

Exercise 3.1.7.7. *Compute the Lelong number of $(dd^c \log \|z\|)^p$ at all points of $\mathbb{C}^k$.*

3.1.8 Projective spaces

We first give the general definition of a complex manifold.

Definition 3.1.8.1. Let M be a topological space which is a countable union of compact sets. M is a *complex manifold* of dimension k if there is a covering of M by open sets $(U_i)_{i\in I}$ with homeomorphisms $\phi_i : U_i \to V_i \subset \mathbb{C}^k$, on open sets V_i in $\mathbb{C}^k$ such that the maps

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

are holomorphic for all $i, j \in I$.

We can identify U_i with V_i via ϕ_i and consider the coordinates in $V_i \subset \mathbb{C}^k$ as local holomorphic coordinates on U_i. It follows that the coordinates changes $\phi_j \circ \phi_i^{-1}$ are biholomorphic. A map $f : M \to \mathbb{C}^n$ is *holomorphic* if for all i, $f \circ \phi_i^{-1}$ is holomorphic on V_i. A map $g : M' \to M$ between complex manifolds is *holomorphic* if $g \circ \phi_i$ is holomorphic for every i.

An important example of complex manifold is the projective space $\mathbb{P}^k$. Consider on $\mathbb{C}^{k+1} \setminus \{0\}$ the following equivalence relation

$$z = (z_0, \ldots, z_k) \sim z' = (z'_0, \ldots, z'_k)$$

if there is $\lambda \in \mathbb{C}^*$ such that $\lambda z = z'$. The quotient space is denoted by $\mathbb{P}^k$ and is called *projective space of dimension k*. Let $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ denote the canonical map. If $z = (z_0, \ldots, z_k)$, write $\pi(z) = [z] = [z_0 : \cdots : z_k]$ which we call *homogeneous coordinates* of $\pi(z)$. The space $\mathbb{P}^k$ has the quotient topology, i.e. U is open in $\mathbb{P}^k$ if and only if $\pi^{-1}(U)$ is open in $\mathbb{C}^{k+1} \setminus \{0\}$.

We consider on $\mathbb{P}^k$ the open sets $U_i := \{[z],\ z_i \neq 0\}$ and the map $\phi_i : U_i \to \mathbb{C}^k$ with

$$\phi_i([z]) := (z_0/z_i, \ldots, z_{i-1}/z_i, z_{i+1}/z_i, \ldots, z_k/z_i).$$

Exercise 3.1.8.2. *Show that ϕ_i is a homeomorphism and that the charts (U_i, ϕ_i), $0 \leq i \leq k$, define a structure of a complex manifold on $\mathbb{P}^k$. Show that $\mathbb{P}^k$ is compact and connected. Prove that holomorphic functions on $\mathbb{P}^k$ are constant.*

The notion of analytic set can be extended to any complex manifold M. A subset A of M is analytic if and only if for every $p \in M$ there is a neighbourhood U_p of p such that $A \cap U_p = \{f_1 = \cdots = f_l = 0\}$ where the functions f_j are holomorphic on U_p.

Let $P_1, \ldots, P_l$ be holomorphic homogeneous polynomials in $\mathbb{C}^{k+1}$ of degree $d_1, \ldots, d_l$. This means that for $\lambda \in \mathbb{C}$

$$P_j(\lambda z) = \lambda^{d_j} P_j(z).$$

Define

$$X := \{[z] \in \mathbb{P}^k,\ P_1(z) = \cdots = P_l(z) = 0\}.$$

It is clear that A is an analytic set. Since it is defined by the vanishing of polynomials, one call such sets *algebraic*. The following is a deep theorem due to Chow.

Theorem 3.1.8.3 (Chow). *Every analytic subset in $\mathbb{P}^k$ is algebraic.*

Exercise 3.1.8.4. *Let P be a homogeneous polynomial of degree d. Suppose $\partial P/\partial z_j$, $0 \leq j \leq k$, have no common zero in $\mathbb{C}^{k+1} \setminus \{0\}$. Show that $X := \{[z],\ P(z) = 0\}$ is a submanifold in $\mathbb{P}^k$, so it has the structure of a complex manifold. We say that X is a hypersurface in $\mathbb{P}^k$. When $d = 1$, X is a projective hyperplane in $\mathbb{P}^k$.*

Exercise 3.1.8.5. *Let $A \in \mathrm{GL}(k+1, \mathbb{C})$ be a linear invertible self-map on $\mathbb{C}^{k+1}$. Show that A induces a biholomorphic self-map $\overline{A}$ on $\mathbb{P}^k$ such that $\pi \circ A = \overline{A} \circ \pi$.*

Exercise 3.1.8.6. *Points $[p_i]$ are linearly independent if the corresponding points p_i are linearly independent in $\mathbb{C}^{k+1}$. Show that such a family contains at most $k+1$ points. We say that $k+2$ points $[p_i]$ are in general position if any subfamily of $k+1$ points are linearly independent. Show that $[1 : 0 : \cdots : 0]$, $[0 : 1 : \cdots : 0]$, $\ldots$, $[0 : \cdots : 0 : 1]$ and $[1 : 1 : \cdots : 1]$ are in general position. If $[p_i]$ and $[q_i]$ are two families of $k+2$ points in general position. Show that there is a map A such that $\overline{A}([p_i]) = [q_i]$. Hint: treat first the case of dimension 1 and use Exercise 3.1.8.5.*

Consider now holomorphic and meromorphic self-maps on $\mathbb{P}^k$. Let $F = (F_0, \ldots, F_k) : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ be a polynomial map where all the F_j are homogeneous polynomials of the same degree $d \geq 1$ and without common factors. Since for $\lambda \in \mathbb{C}^*$ we have $F(\lambda z) = \lambda^d F(z)$, the map F induces a holomorphic map $f : \mathbb{P}^k \to \mathbb{P}^k$ as soon as $F(z) = 0$ is equivalent to $z = 0$. In this case, the map f satisfies $\pi \circ F = f \circ \pi$. We say that such a map is *holomorphic of (algebraic) degree d*. We denote the space of such maps by $\mathscr{H}_d(\mathbb{P}^k)$.

In general, let $I := \{[z],\ F(z) = 0\}$ then F induces a holomorphic map $f : \mathbb{P}^k \setminus I \to \mathbb{P}^k$ and we still have $\pi \circ F = f \circ \pi$ on $\mathbb{C}^{k+1} \setminus \{\pi^{-1}(I) \cup 0\}$. We call such map *rational* and I is the *indeterminacy set of f*. Since $F_0, \ldots, F_k$ have no common factors, the set I is of codimension ≥ 2. A *meromorphic map from $\mathbb{P}^k$ to $\mathbb{P}^k$* is a holomorphic map defined on $\mathbb{P}^k \setminus I$ where I is an analytic set with codimension ≥ 2. One can show that the graph

$$\Gamma_f := \{([z], f[z]),\ z \in \mathbb{P}^k \setminus I\}$$

has analytic closure in $\mathbb{P}^k \times \mathbb{P}^k$. It is a consequence of Chow's theorem that any meromorphic self-map on $\mathbb{P}^k$ is rational and also any holomorphic self-map of $\mathbb{P}^k$ is obtained as described above.

Exercise 3.1.8.7. *Let p be a one variable polynomial in $\mathbb{C}$ of degree $d \geq 2$. Show that $f(z, w) = (p(z) + aw, z)$, $a \in \mathbb{C}^*$, is a polynomial automorphism of $\mathbb{C}^2$. Write the rational extension to $\mathbb{P}^2$ with homogeneous coordinates $[z : w : t]$ for f and*

f^{-1}. Describe the action of f and f^{-1} on the hyperplane at infinity $L_\infty := \{[z : w : t],\ t = 0\}$. Hint: the only indeterminacy point of f is $I_+ := [0 : 1 : 0]$, and for f^{-1}, it is $I_- := [1 : 0 : 0]$.

Exercise 3.1.8.8. *Let $f[z : w : t] = [z^d : w^d : t^d]$, $d \geq 2$. Show that f is holomorphic. Find the attractive fixed points of f and describe their bassins of attraction. Find the periodic points of f.*

We have the following Bézout theorem.

Theorem 3.1.8.9 (Bézout)**.** *Let $P_1, \ldots, P_k$ be homogeneous polynomials in $\mathbb{C}^{k+1}$ of degrees $d_1, \ldots, d_k$ respectively. Let Z denote the set of common zeros of P_j in $\mathbb{P}^k$, i.e. the set of $[z]$ such that $P_j(z) = 0$ for every j. If Z is discrete then the number of points in Z counted with multiplicities is equal to $d_1 \ldots d_k$.*

Proposition 3.1.8.10. *Let $f \in \mathscr{H}_d(\mathbb{P}^k)$. Then for every $a \in \mathbb{P}^k$, $\# f^{-1}(a) = d^k$ counting with multiplicities.*

Proof. Let $f = [F_0 : \cdots : F_k]$ be the expression of f in the homogeneous coordinates. The polynomials F_j are homogeneous with the same degree d and have no common zero in $\mathbb{P}^k$. Consider a point $a = [a_0 : \cdots : a_k] \in \mathbb{P}^k$. Without loss of generality, we can assume $a_0 = 1$, hence $a = [1 : a_1 : \cdots : a_k]$. The points in $f^{-1}(0)$ are the common zeros of polynomials $F_j - a_j F_0$, $j = 1, \ldots, k$ in $\mathbb{P}^k$. In order to apply Bézout theorem, we only have to show that this set is discrete. If not, the set of common zeros of $F_j - a_j F_0$ is of dimension ≥ 2 in $\mathbb{C}^{k+1}$. This implies that the common zeros of F_j is of dimension ≥ 1 which is impossible since f is holomorphic. Applying Bézout theorem implies the result. □

Exercise 3.1.8.11. *If $f \in \mathscr{H}_d(\mathbb{P}^k)$ and $g \in \mathscr{H}_{d'}(\mathbb{P}^k)$, what is the algebraic degree of $f \circ g$?*

Exercise 3.1.8.12. *Let $f \in \mathscr{H}_d(\mathbb{P}^k)$ show that the number of fixed points counted with multiplicities is $(d^{k+1} - 1)/(d - 1)$. What is the number of periodic points of order n? Hint: apply Bézout theorem to $F_j(z) - t^{d-1} z_j = 0$ in $\mathbb{P}^{k+1}$ with homogeneous coordinates $[z : t]$ and observe that $[0 : \cdots : 0 : 1]$ is a solution.*

Exercise 3.1.8.13.

1. *Show that the space $\mathscr{R}_d(\mathbb{P}^k)$ of rational self-maps $f = [F_0 : \cdots : F_k]$ of degree d in $\mathbb{P}^k$ where F_j are homogeneous polynomials of degree d without common divisors, can be identified to a Zariski open set (the complement of an analytic set) in $\mathbb{P}^N$ with $N := \frac{(k+1)!(d+k)!}{d!k!} - 1$. Hint: count the dimension of the vector space of homogeneous polynomials of degree d.*

2. *Show that the set*

$$\Sigma := \{(f, [z]) \in \mathscr{R}_d(\mathbb{P}^k) \times \mathbb{P}^k,\ F(z) = 0\}$$

is analytic in $\mathscr{R}_d(\mathbb{P}^k) \times \mathbb{P}^k$.

3. *Using that the image under a proper holomorphic analytic map of an analytic set is analytic, show that the complement of $\mathscr{H}_d(\mathbb{P}^k)$ in $\mathbb{P}^N$ is analytic.*

4. *Deduce that $\mathscr{H}_d(\mathbb{P}^k)$ is connected.*

5. *For $f \in \mathscr{H}_d(\mathbb{P}^k)$, let $J(F(z)) := \det(\partial F_j/\partial z_i)_{0\leq i,j\leq k}$. Show that $J(F(z))$ is a homogeneous polynomial. What is its degree when $F = [z_0^d : \cdots : z_k^d]$? Using the connectedness of $\mathscr{H}_d(\mathbb{P}^k)$ prove that $\deg J(F) = (k+1)(d-1)$ for every $f \in \mathscr{H}_d(\mathbb{P}^k)$. The zero set $C := \{[z],\ J(F(z)) = 0\}$ is called the critical set of f. What is the critical set of f^n?*

Exercise 3.1.8.14. *Give examples of rational maps f, g on $\mathbb{P}^k$ such that $\deg(f \circ g) \leq (\deg f)(\deg g)$.*

Exercise 3.1.8.15. *Let $f[z : w : t] = [w^d + \lambda z t^{d-1} : z^d : t^d + P(z, w)]$ where P is a homogeneous polynomial of degree d.*

1. *Show that f is holomorphic.*

2. *Let $p = [0 : 0 : 1]$, then $f^{-1}(p) = p$. Show that the eigenvalues of $(Df)(p)$ are 0 and λ, hence p is not necessarily attractive.*

Exercise 3.1.8.16. *Let $f \in \mathscr{H}_d(\mathbb{P}^k)$. For φ a continuous function on $\mathbb{P}^k$, define*

$$\Lambda\varphi(a) := d^{-k} f_*\varphi(a) := d^{-k} \sum_{b\in f^{-1}(a)} \varphi(b).$$

Show that $\Lambda\varphi$ is continuous.

Exercise 3.1.8.17. *Let $f_\alpha[z : w : t] := [z^d : w^d : t^d + \alpha z^{d-l} w^l]$ with $\alpha \in \mathbb{C}$. Describe the parameters α such that f_α is postcritically finite, i.e. the orbit of the critical set is an algebraic set.*

Exercise 3.1.8.18. *Describe the orbit of the critical set of the map $f[z : w : t] := [z^2 : w^2 - 2zt : t^2]$.*

Exercise 3.1.8.19. *Consider $f : \mathbb{P}^2 \to \mathbb{P}^2$ defined by*

$$f([z : w : t]) = [(z - 2w)^2 : z^2 : t^2].$$

Compute the critical set C of f and its orbit $\cup_{n\geq 0} f^n(C)$. Is this orbit algebraic?

3.1.9 Quasi-psh functions and positive currents on $\mathbb{P}^k$

The notion of (p, q)-forms and currents extends without any difficulty on complex manifolds. We can also consider the operators ∂, $\overline{\partial}$, d and d^c using local coordinates. The notion of positivity of a current being also invariant by change of holomorphic coordinates makes sense in an arbitrary complex manifold.

A complex manifold M is Kähler if there is a $(1,1)$-form ω which is strictly positive and closed. In local coordinates

$$\omega = \frac{i}{2}\sum h_{lj}(z)dz_l \wedge d\overline{z}_j.$$

This means that ω acts on tangent vectors ξ, η as

$$\omega(\xi,\eta) = \frac{i}{2}\sum h_{lj}(\xi_l\overline{\eta}_j - \eta_l\overline{\xi}_j).$$

Moreover the form ω is closed, i.e. $d\omega = 0$.

An important example is $\mathbb{C}^k$ with the form $\beta := idd^c\|z\|^2$. Clearly it is strictly positive and closed. Another important example is $\mathbb{P}^k$ with the Fubini-Study form ω. Indeed, the form $dd^c \log\|z\|$ on $\mathbb{C}^{k+1}\setminus\{0\}$ is the pull-back under π of a Kähler form ω in $\mathbb{P}^k$. In the coordinate chart $\{z_i \neq 0\}$, consider the section $s_j([z]) := (z_0/z_i, \ldots, z_k/z_i)$ of π, i.e. $\pi \circ s_i = \mathrm{id}$ on U_i. Define on U_i

$$\omega_i := dd^c \log\|s_i\|.$$

On U_j we have $\omega_j = dd^c\log\|s_j\|$. But $s_j = s_i z_i/z_j$ on $U_i \cap U_j$ and $dd^c \log|z_i/z_j| = 0$. Hence, $\omega_i = \omega_j$ on $U_i \cap U_j$. Therefore, the form ω is well defined on $\mathbb{P}^k$ by setting $\omega := \omega_i$ on U_i. Clearly, it is positive and closed. On the chart $\{z_0 \neq 0\}$, we can assume $z_0 = 1$. We have

$$\omega = dd^c\log(1+|z_1|^2+\cdots+|z_k|^2)^{1/2} = \frac{i}{2\pi}\Big(\frac{\partial\overline{\partial}|z|^2}{1+|z|^2} - \frac{\partial|z|^2\wedge\overline{\partial}|z|^2}{(1+|z|^2)^2}\Big).$$

Exercise 3.1.9.1. *1. Show that ω is strictly positive and ω^k is a volume form.*

2. Show that in the above coordinate chart

$$\omega^k = \Big(\frac{i}{2\pi}\Big)^k \frac{k!}{(1+\|z\|^2)^{k+1}}\prod_{j=1}^k dz_j\wedge d\overline{z}_j.$$

3. Prove by induction on k that $\int_{\mathbb{P}^k}\omega^k = 1$.

Exercise 3.1.9.2. *Let u be a real-valued function on $\mathbb{C}^k$. We say that u is log-homogeneous if*

$$u(\lambda z) = \log|\lambda| + u(z) \quad \textit{for } \lambda \in \mathbb{C}^* \textit{ and } z \in \mathbb{C}^k.$$

1. *Prove that if u is such a function then there is a constant $c > 0$ such that $u(z) \leq \log|z| + c$. If P is a homogeneous polynomial of degree $d \geq 1$ find a positive constant α such that $\alpha\log|P|$ is log-homogeneous.*

2. *Let u be a psh function on $\mathbb{C}^k$ such that $u(z) \leq \log|z|+c$ with $c > 0$. Define*

$$v(z_0, z_1, \dots, z_k) := u\Big(\frac{z_1}{z_0}, \dots, \frac{z_k}{z_0}\Big) + \log|z_0| \quad \textit{if } z_0 \neq 0.$$

Show that v can be extended to a psh log-homogenenous function on $\mathbb{C}^{k+1}$.

Exercise 3.1.9.3. *Let $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ be the canonical projection. A holomorphic section of π over an open set $U \subset \mathbb{P}^k$ is a holomorphic map $s : U \to \mathbb{C}^{k+1} \setminus \{0\}$ such that $\pi \circ s = \mathrm{id}$.*

1. *If s and s' are two holomorphic sections of π over U, show that there is a non-vanishing holomorphic function h on U such that $s' = hs$.*

2. *If u is a psh log-homogeneous function on $\mathbb{C}^{k+1}$, define $T_U := dd^c(u \circ s)$. Show that T_U is a positive closed $(1,1)$-current independent of s. Show that there is a unique positive closed $(1,1)$-current T on $\mathbb{P}^k$ such that $\pi^*(T) = dd^c u$.*

Definition 3.1.9.4. A function $u : \mathbb{P}^k \to \mathbb{R} \cup \{-\infty\}$ is *quasi-psh* if for every $a \in \mathbb{P}^k$ there is a neighbourhood U_a and a $\mathscr{C}^2$ function v_a in U_a such that $u_a + v_a$ is psh on U_a.

Exercise 3.1.9.5. *Show that if u is quasi-psh in $\mathbb{P}^k$, then u is strongly u.s.c. and there exists a constant $c > 0$ such that $dd^c u \geq -c\omega$. Prove the converse.*

Exercise 3.1.9.6. *Define quasi-psh functions for an arbitrary complex manifold. Extend the previous exercise to compact Kähler manifolds.*

Let

$$\mathscr{P}_1 := \{u \text{ quasi-psh on } \mathbb{P}^k,\ dd^c u \geq -\omega, \max_{\mathbb{P}^k} u = 0\}$$

and

$$\mathscr{P}_2 := \{u \text{ quasi-psh on } \mathbb{P}^k,\ dd^c u \geq -\omega, \int_{\mathbb{P}^k} u\omega^k = 0\}.$$

Exercise 3.1.9.7. *Extend Hartogs' lemma for quasi-psh functions. Prove that $\mathscr{P}_1$ and $\mathscr{P}_2$ are compact in L^p for $1 \leq p < \infty$.*

Exercise 3.1.9.8.

1. *Let u be a quasi-psh function on $\mathbb{P}^k$ such that $dd^c u \geq -c\omega$. Show that the function $v(z) := u \circ \pi + c \log\|z\|$ and $v(0) := -\infty$ is psh in $\mathbb{C}^{k+1}$ and that v is log-homogeneous, i.e.*

$$v(\lambda z) = v(z) + c \log|\lambda| \quad \textit{for } \lambda \in \mathbb{C}^*.$$

Conversely starting from a psh log-homogeneous function v satisfying the above equation construct a quasi-psh function u in $\mathbb{P}^k$ such that $dd^c u \geq -c\omega$.

2. *Show that if u is quasi-psh with $dd^c u \geq -c\omega$, there is a decreasing sequence (u_ϵ) of smooth function with $dd^c u_\epsilon \geq -c\omega$, converging to u in L^p. Hint: consider $v'(z) := v(1, z)$ in $\mathbb{C}^k$, regularize it and construct psh log-homogeneous functions v_ϵ on $\mathbb{C}^{k+1}$.*

Definition 3.1.9.9. A subset E in $\mathbb{P}^k$ is *pluripolar* if E is contained in $\{u = -\infty\}$ where u is a quasi-psh function. It is *complete pluripolar* if there is a quasi-psh function u such that $E = \{u = -\infty\}$.

Exercise 3.1.9.10. *Show that analytic sets are complete pluripolar and countable unions of pluripolar sets are pluripolar. Give an example of a closed set E which is pluripolar but is not complete pluripolar.*

Exercise 3.1.9.11 (Capacity)**.** *Define the capacity of a Borel set E in $\mathbb{P}^k$ by*

$$\operatorname{cap}(E) := \inf_{\varphi \in \mathscr{P}_1} \exp\Big(\sup_E \varphi\Big).$$

Show that E is pluripolar if and only if $\operatorname{cap}(E) = 0$.

Exercise 3.1.9.12. *Show that there is a constant $c > 0$ such that* $\operatorname{cap}(E) \geq \exp(-c/\operatorname{volume}(E))$. *Hint: use the compactness of $\mathscr{P}_1$ in L^1.*

Exercise 3.1.9.13. *Let B denote the unit ball in $\mathbb{C}^{k+1}$ and B_r the ball of center 0 and of radius r. If E is a Borel set in $\mathbb{P}^k$ define $\widehat{E} := B \cap \pi^{-1}(E)$. If P is a holomogeneous polynomial in $\mathbb{C}^{k+1}$ show that*

$$\sup_{B_r} |P| \leq \sup_{\widehat{E}} |P|$$

where $r := \operatorname{cap}(E)$. *Hint: we can assume* $\sup_B |P| = 1$*; show that* $d^{-1}\log|P| - \log(1 + \|z\|^2)^{1/2} + (\log 2)/2$ *induces a function in $\mathscr{P}_1$.*

Exercise 3.1.9.14. *Assume that E is non-pluripolar. Show that any holomorphic function g in a neighbourhood of $\widehat{E} \cup \{0\}$ can be extended to a function $\widetilde{g}$ which is holomorphic in B_r. Moreover, we have $\|\widetilde{g}\|_{\infty, B_r} \leq \|g\|_{\infty, \widehat{E}}$. Hint: write g as a serie $\sum P_n$ of homogeneous polynomials P_n of degree n; using the Cauchy formula in one variable, find a bound of $|P_n|$ on the discs $B \cap \pi^{-1}(a)$ with $a \in E$ in term of $|g|$; then use the previous exercise.*

3.1.10 Green quasi-potentials of positive closed currents

We have given in Exercise 3.1.9.8 examples of positive closed $(1,1)$-currents of the form $dd^c u + c\omega$. It is clear that when $T_1, \ldots, T_p$ are smooth positive closed $(1,1)$-currents then $T = T_1 \wedge \ldots \wedge T_p$ is a positive closed current of bidegree (p, p). We denote by $\omega^p := \omega \wedge \ldots \wedge \omega$ with p factors.

For positive (p, p)-currents, we have introduced the notion of mass of a current. It is easy to show that the mass is comparable to

$$\|T\| := \langle T, \omega^{k-p} \rangle.$$

We often identify these two quantities for positive currents. Let $\mathscr{C}_p$ denote the convex set of positive closed currents of bidegree (p, p) of mass 1 on $\mathbb{P}^k$.

Exercise 3.1.10.1. *Show that $\mathscr{C}_p$ is compact for the weak topology on currents. Suppose $T = \omega^p + dd^c U$ where U is a smooth $(p-1, p-1)$-form. Compute $\|T\|$. Compute $\|T\|$ if we only assume that U has coefficients in L^1. Hint: use a partition of unity in order to regularize of U.*

Exercise 3.1.10.2.

1. *Let $f \in \mathscr{H}_d(\mathbb{P}^k)$. Show that there is u quasi-psh such that $d^{-1} f^*(\omega) = \omega + dd^c u$. Compute the mass of $f^*(\omega^p)$ and $f_*(\omega^p)$. Hint: the function $\log(|F(z)|/\|z\|^d)$ is defined on $\mathbb{P}^k$.*

2. *Show that $d^{-p} f^*$ and $d^{-(k-p)} f_*$ act continuously on $\mathscr{C}_p$. Hint: use Theorem 3.1.3.17 and the description of the fibers of f.*

3. *For every $1 \leq p \leq k$, find a current T in $\mathscr{C}_p$ such that $f^*(T) = d^p T$. Hint: use Cesàro mean. Show that $f_*(T) = d^{k-p}(T)$.*

Exercise 3.1.10.3. *Let S be a current in $\mathscr{C}_1$. Show that $S = \omega + dd^c u$ where u is a quasi-psh function. Hint: use Exercises 3.1.9.2 and 3.1.9.3.*

Let R be a current in $\mathscr{C}_p$ with $p \geq 1$. If U is a $(p-1, p-1)$-current such that $dd^c U = R - \omega^p$, we say that U is a *quasi-potential* of R. The integral $\langle \omega^{k-p+1}, U \rangle$ is *the mean* of U. Such currents U exist but they are not unique. When $p = 1$ the quasi-potentials of R differ by constants, when $p > 1$ they differ by dd^c-closed currents which can be singular. Moreover, for $p > 1$, U is not always defined at every point of $\mathbb{P}^k$. This is one of the difficulties in the study of positive closed currents of higher bidegree. We have the following result.

Proposition 3.1.10.4. *Let R be a current in $\mathscr{C}_p$. Then, there is a negative quasi-potential U of R depending linearly on R such that for every r, s with $1 \leq r < k/(k-1)$ and $1 \leq s < 2k/(2k-1)$ we have*

$$\|U\|_{L^r} \leq c_r \quad \textit{and} \quad \|dU\|_{L^s} \leq c_s$$

for some constants $c_r, c_s > 0$ independent of R. Moreover, U depends continuously on R with respect to the L^r topology on U and the weak topology on R

Definition 3.1.10.5. We call U *the Green quasi-potential* of R.

3.1.11 Space of dsh functions

We introduce in this section two functional spaces useful in complex dynamics.

Definition 3.1.11.1. A function u defined on $\mathbb{P}^k$ minus a pluripolar set E with values in $\mathbb{R} \cup \{\pm\infty\}$ is *dsh* if it is equal in $\mathbb{P}^k \setminus E$ to a difference of two quasi-psh functions. We identify dsh functions which are equal out of a pluripolar set.

Let $\mathrm{DSH}(\mathbb{P}^k)$ denote the space of dsh functions on $\mathbb{P}^k$.

Proposition 3.1.11.2. *Let u be a dsh function. Then $dd^c u = T^+ - T^-$ where $T^\pm$ are positive closed $(1,1)$-currents of the same mass. Conversely given $T^\pm$ positive closed $(1,1)$-currents of the same mass, there is a dsh function u such that $dd^c u = T^+ - T^-$.*

Proof. Assume $u = u^+ - u^-$ with $u^\pm$ quasi-psh with $dd^c u^\pm \geq -c\omega$. Observe that u is in L^p. Define $T^\pm := dd^c u^\pm + c\omega$. We have $dd^c u = T^+ - T^-$. Exercise 3.1.10.1 implies that $T^\pm$ have the same mass.

Conversely, assume for simplicity that $T^\pm$ are in $\mathscr{C}_1$. Choose $u^\pm$ such that $dd^c u^\pm = T^\pm - \omega$. The function $u^\pm$ are constructed using Proposition 3.1.10.4. We can assume that $u^\pm$ are quasi-psh. Define $u := u^+ - u^-$. Then $dd^c u = T^+ - T^-$. $\square$

For u dsh, define

$$\|u\|_{\mathrm{DSH}} := \left| \int u\omega^k \right| + \min \|T^\pm\|$$

where the minimum is taken on all $T^\pm$ positive closed such that $dd^c u = T^+ - T^-$.

Exercise 3.1.11.3. *Show that $\|\cdot\|_{\mathrm{DSH}}$ is a norm on $\mathrm{DSH}(\mathbb{P}^k)$.*

We will mostly use the following *weak topology*.

Definition 3.1.11.4. Let (u_n) be a sequence of dsh functions. We say that $u_n \to u$ in $\mathrm{DSH}(\mathbb{P}^k)$ if $u_n \to u$ in L^1 and if $\|u_n\|_{\mathrm{DSH}}$ is a bounded sequence.

Exercise 3.1.11.5. *Let $\mathrm{DSH}_0(\mathbb{P}^k)$ denote the subspace in $\mathrm{DSH}(\mathbb{P}^k)$ of functions satisfying $\int u\omega^k = 0$. Show that any bounded set in $\mathrm{DSH}_0(\mathbb{P}^k)$ is relatively compact for the weak topology.*

Exercise 3.1.11.6. *Show that smooth functions are dense in $\mathrm{DSH}(\mathbb{P}^k)$. Hint: use the regularization of quasi-psh functions.*

The notion of PB and PC measures on $\mathbb{P}^1$ and theirs properties extend without any difficulty to the case of $\mathbb{P}^k$.

Exercise 3.1.11.7.
1. *Let ν be a probalility measure on $\mathbb{P}^k$. Suppose $\nu - \omega^k = dd^c U$ with U a continuous $(k-1,k-1)$-form. Show that ν is PC. Hint: write $\langle \nu, \varphi \rangle - \langle \omega^k, \varphi \rangle = \langle U, dd^c\varphi \rangle$.*

2. *Suppose $\nu - \omega^k = dd^c U$ with $U \geq 0$ (resp. a bounded form U). Show that $\langle \nu, \varphi\rangle \geq \int \varphi\omega^k - c\int U\wedge\omega$ if φ is smooth such that $dd^c\varphi \geq -c\omega$. Prove that ν is PB.*

Exercise 3.1.11.8. *Show that the Lebesgue measure on the torus $\{|z_1| = \cdots = |z_k| = 1\}$ in $\mathbb{C}^k \subset \mathbb{P}^k$ is a PC measure. Hint: show that this measure is equal to $dd^c \log^+ |z_1| \wedge \ldots \wedge dd^c \log^+ |z_k|$.*

3.1.12 Complex Sobolev space

We introduce now the complex Sobolev space which is a subspace of the standard Sobolev space. It has good invariance properties with respect to meromorphic maps. The classical Sobolev space $W := W^{1,1}(\mathbb{P}^k)$ is the space of real-valued L^2 functions such that $d\varphi$ has coefficients in L^2 or equivalently

$$\int i\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{k-1} < \infty.$$

This space is endowed with the norm

$$\|\varphi\|_W := \Big[\int \varphi^2\omega^k + \int i\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{k-1}\Big]^{1/2}.$$

It has the structure of a Hilbert space and it can be defined as the completion of smooth functions with respect to the previous norm. For φ in L^1, define

$$m(\varphi) := \int \varphi\omega^k.$$

Theorem 3.1.12.1 (Poincaré-Sobolev). *There is a constant $c > 0$ such that for φ in W we have*

$$\|\varphi\|_{L^2}^2 \leq \|\varphi\|_W^2 \leq c\Big[\Big|\int \varphi\omega^k\Big|^2 + \int i\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{k-1}\Big].$$

Define the subspace W^* of W as follows. The function φ is in W^* if there is a positive closed $(1,1)$-current T such that

$$i\partial\varphi\wedge\overline{\partial}\varphi \leq T.$$

Let

$$\|\varphi\|_{W^*} := |m(\varphi)| + \inf\{\|T\|^{1/2},\ T \text{ as above}\}.$$

Exercise 3.1.12.2. *Show that $\|\cdot\|_{W^*}$ is a norm on W^*.*

We introduce a *weak topology* on W^*.

Definition 3.1.12.3. A sequence (u_n) converges to u in W^* if $u_n \to u$ in L^1 and $\|u_n\|_{W^*}$ is bounded.

Exercise 3.1.12.4. *1. Let χ be a Lipschitz function on $\mathbb{R}$. Show that for $\varphi \in W^*$, we have $\chi(\varphi) \in W^*$ and $\|\chi(\varphi)\|_{W^*} \leq c\|\varphi\|_{W^*}$ with $c > 0$ independent of φ. Hint: $|\chi(x)| \leq a|x| + b$.*

2. Show that W^* is stable under the operation max, min *and that $\varphi \mapsto \max(\varphi, 0)$ is bounded and continuous for the weak topology.*

3. Show that bounded functions are dense in W^*. Hint: use $\max(\varphi, -n)$ and $\min(\varphi, n)$.

Exercise 3.1.12.5. *Show that the closed unit ball in W^* is compact for the weak topology. Hint: the injection of W in L^2 is compact by Rellich theorem.*

Exercise 3.1.12.6. *Let u be a strictly negative quasi-psh function on $\mathbb{P}^k$. Show that $\log(-u)$ is in W^*. Show that $\mathrm{DSH}(\mathbb{P}^k) \cap L^\infty \subset W^*$ and the inclusion is continuous. Hint: for u in $\mathrm{DSH}(\mathbb{P}^k) \cap L^\infty$ write $2i\partial u \wedge \overline{\partial} u = -\partial\overline{\partial} u^2 - 2ui\partial\overline{\partial} u$.*

Proposition 3.1.12.7. *Let $f \in \mathscr{H}_d(\mathbb{P}^k)$. Then $\Lambda := d^{-k} f_*$ defines a linear bounded continuous operator from W^* to itself.*

Proof. We have $f^*(\omega^k) \lesssim \omega^k$. Hence

$$\int |f_*\varphi| \wedge \omega^k \leq \int f_*|\varphi| \wedge \omega^k = \int |\varphi| f^*(\omega^k) \lesssim \int |\varphi|\omega^k.$$

This implies that if $\varphi_n \to \varphi$ in W^* then $\Lambda\varphi_n \to \Lambda\varphi$ in L^1. By Exercise 3.1.12.4, we can assume $\varphi \geq 0$. We obtain from the above estimate that $m(\Lambda\varphi) \lesssim m(\varphi)$. We also have outside the critical values of f

$$i\partial f_*\varphi \wedge \overline{\partial} f_*\varphi \leq d^{2k} f_*(i\partial\varphi \wedge \overline{\partial}\varphi) \leq d^{2k} f_*(T).$$

So, the restriction of $df_*\varphi$ out of the critical values is in L^2. One checks easily that this restriction is equal to $d\Lambda\varphi$ in the sense of currents. It follows that $f_*\varphi$ is in W^* since $f_*(T)$ is positive closed. Since f_* is continuous on positive closed currents, we deduce that f_* is bounded and continuous on W^*. □

3.2 Equilibrium measure and Green currents

3.2.1 Construction of the equilibrium measure

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism (self-map) of algebraic degree $d \geq 2$. Recall there is a lift $F : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ with $F = (f_0, \dots, f_k)$ where the f_j's are homogeneous polynomials of degree d such that $F^{-1}(0) = \{0\}$. Moreover, the *topological degree* d_t is equal to d^k, i.e. the number of points in $\#f^{-1}(a)$ is equal to d^k counting with multiplicities.

As in one variable, we introduce the Perron-Frobenius operator

$$\Lambda\varphi(z) := d_t^{-1} f_*\varphi(z) := d_t^{-1} \sum_{w\in f^{-1}(z)} \varphi(w).$$

It is a positive continuous operator on the space $\mathscr{C}(\mathbb{P}^k)$ of continuous function on $\mathbb{P}^k$. However, since f is proper, it is easy to check that this operator admits an extension to L^1 functions.

Proposition 3.2.1.1. $\Lambda : \mathrm{DSH}(\mathbb{P}^k) \to \mathrm{DSH}(\mathbb{P}^k)$ *is a bounded positive operator which is continuous with respect to the weak topology on* $\mathrm{DSH}(\mathbb{P}^k)$*. Moreover, if* φ *is dsh and* $m := \int \Lambda\varphi\omega^k$ *then* $|m| \leq c\|\varphi\|_{\mathrm{DSH}}$ *and* $\|\Lambda\varphi - m\|_{\mathrm{DSH}} \leq d^{-1}\|\varphi\|_{\mathrm{DSH}}$ *where* $c > 0$ *is a constant independent of* φ*.*

Proof. When φ is positive, it is clear that $\Lambda\varphi$ is positive. It follows that $|\Lambda\varphi| \leq \Lambda|\varphi|$. We also have since $f^*(\omega^k)$ is smooth that

$$\int |\Lambda\varphi|\omega^k \leq \int \Lambda|\varphi|\omega^k = d_t^{-1}\int |\varphi| f^*(\omega^k) \lesssim \int |\varphi|\omega^k.$$

So, $\|\Lambda\varphi\|_{L^1} \lesssim \|\varphi\|_{L^1} \lesssim \|\varphi\|_{\mathrm{DSH}}$.

Suppose $dd^c\varphi = T^+ - T^-$ with $T^\pm$ positive closed $(1,1)$-currents. Then using that f_* preserves positive closed currents gives

$$dd^c\Lambda\varphi = d_t^{-1}[f_*(T^+) - f_*(T_-)].$$

We compute the mass of $f_*(T^\pm)$:

$$\int f_*(T^\pm)\wedge\omega^{k-1} = \int T^+\wedge f^*(\omega^{k-1}) = d^{k-1}\int T^+\wedge\omega^{k-1} = d^{k-1}\|T^\pm\|.$$

Hence, $d_t^{-1}\|f_*(T^\pm)\| = d^{-1}\|T^\pm\|$. It follows that $\|\Lambda\varphi\|_{\mathrm{DSH}} \lesssim \|\varphi\|_{\mathrm{DSH}}$. Hence, Λ is bounded on $\mathrm{DSH}(\mathbb{P}^k)$. We also get

$$\|\Lambda\varphi - m\| \leq \|d_t^{-1} f_*(T^\pm)\| \leq d^{-1}\|T^\pm\|$$

which implies the estimate in the proposition.

Suppose $\varphi_n \to 0$ in $\mathrm{DSH}(\mathbb{P}^k)$ then $\|\Lambda\varphi_n\|$ is bounded and since $\varphi_n \to 0$ in L^1, we obtain that $\Lambda\varphi_n \to 0$ in L^1. So, Λ is continuous with respect to the weak topology on $\mathrm{DSH}(\mathbb{P}^k)$. □

As in Theorem 2.2.1.12, we prove that

Theorem 3.2.1.2. *Let* φ *be a dsh function. Then there is a constant* c_φ *such that*

$$\|\Lambda^n\varphi - c_\varphi\| \leq Ad^{-n}\|\varphi\|_{\mathrm{DSH}}$$

where $A > 0$ *is independent of* φ*. Moreover,* c_φ *depends continuously on* φ *with respect to the weak topology on* $\mathrm{DSH}(\mathbb{P}^k)$*.*

As in Corollary 2.2.1.13, we obtain

Corollary 3.2.1.3. *Let ν be a PB probability measure on $\mathbb{P}^k$. Then $d^{-kn}(f^n)^*(\nu)$ converge to a PC measure μ which is totally invariant, independent of ν, and satisfies $\langle \mu, \varphi\rangle = c_\varphi$ for every dsh function φ. The convergence is uniform on bounded sets in* $\mathrm{DSH}(\mathbb{P}^k)$.

Definition 3.2.1.4. The measure μ is called *the equilibrium or the Green measure* of f.

Exercise 3.2.1.5. *Define on* $\mathrm{DSH}(\mathbb{P}^k)$

$$\|\varphi\|_\mu := \Big|\int \varphi d\mu\Big| + \min \|T^\pm\|, \quad \textit{with } T^\pm \textit{ positive closed and } dd^c\varphi = T^+ - T^-.$$

Show that $\|\cdot\|_\mu$ is a norm which is equivalent to the usual norm on $\mathrm{DSH}(\mathbb{P}^k)$. Hint: write $\varphi = m + \varphi^+ - \varphi^-$ with $\varphi^\pm$ quasi-psh, $\int \varphi^\pm\omega^k = 0$ and $dd^c\varphi^\pm \geq -\|\varphi\|_{\mathrm{DSH}}$; use the compactness of quasi-psh functions.

Exercise 3.2.1.6. *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of algebraic degree $d \geq 2$. Show that there is a pluripolar set $\mathcal{E}^*$ such that for a in $\mathbb{P}^k \setminus \mathcal{E}^*$*

$$d^{-kn} f^*(\delta_a) - \mu \to 0.$$

Hint: repeat the proof of Sodin's theorem and use that μ has no mass to pluripolar sets.

Exercise 3.2.1.7. *Show that if ν is a probability measure with no mass on pluripolar sets then $d^{-kn}(f^n)^*(\nu) \to \mu$. Hint: use the above result.*

3.2.2 Mixing and rate of mixing

Let φ and ψ be test functions. Define

$$I_n(\varphi, \psi) := \langle \mu, (\varphi \circ f^n)\psi\rangle - \langle \mu, \varphi\rangle\langle \mu, \psi\rangle.$$

As in Theorem 2.2.4.1 and Corollary 2.2.4.4, we prove the following result.

Theorem 3.2.2.1. *The equilibrium measure of f is exponentially mixing. More precisely, we have for φ bounded and ψ dsh*

$$|I_n(\varphi, \psi)| \leq A d^{-n} \|\varphi\|_\infty \|\psi\|_{\mathrm{DSH}}$$

where $A > 0$ is a constant independent of φ and ψ. We also have for φ bounded and ψ in $\mathscr{C}^\alpha$ with $0 \leq \alpha \leq 2$ that

$$|I_n(\varphi, \psi)| \leq A d^{-\alpha n/2} \|\varphi\|_\infty \|\psi\|_{\mathscr{C}^\alpha}$$

where $c_\alpha > 0$ is a constant independent of φ and ψ.

Recall that μ is mixing means that $I_n(\varphi,\psi) \to 0$ for φ and ψ smooth. We also have the following result.

Theorem 3.2.2.2. *The equilibrium measure of f is K-mixing. More precisely, we have for every ψ in $L^2(\mu)$*

$$\sup_{\|\varphi\|_{L^2(\mu)}\leq 1} |I_n(\varphi,\psi)| \to 0.$$

Proof. As in the case of one variable, if $c_\psi := \langle \mu, \psi\rangle$, we have

$$I_n(\varphi,\psi) = \langle \mu, \varphi(\Lambda^n\psi - c_\psi)\rangle.$$

Hence,

$$\sup_{\|\varphi\|_{L^2(\mu)}\leq 1} |I_n(\varphi,\psi)| \leq \|\Lambda^n\psi - c_\psi\|_{L^2(\mu)}.$$

So, it is sufficient to show that $\|\Lambda^n\psi - c_\psi\|_{L^2(\mu)} \to 0$. By Exercise 3.2.2.3 below, it is enough to prove this convergence for a dense family of ψ in $L^2(\mu)$.

Exercise 3.2.2.3. *Show that Λ defines an operator from $L^2(\mu)$ to itself and that its norm is equal to* 1.

So, we can assume ψ continuous. Subtracting from ψ a constant allows us to assume that $c_\psi = 0$. Observe that $(\Lambda^n\psi)$ are uniformly bounded by $\|\psi\|_\infty$. Moreover, Exercise 3.2.1.6 implies that they converge outside a pluripolar set $\mathcal{E}^*$ to $\langle\mu,\psi\rangle = c_\psi = 0$. Since μ has no mass on pluripolar sets, $\Lambda^n\psi$ converge to 0 μ-almost everywhere. The Lebesgue's convergence theorem implies that $\Lambda^n\psi$ converge to 0 in $L^p(\mu)$ for $1 \leq p < \infty$. This implies the result. □

Exercise 3.2.2.4. *Let A be a Borel set such that $f^{-n}(f^n(A)) = A$ for every $n \geq 0$. Show that $\mu(A) = 0$ or 1. Hint: consider the characteristic function of $f^n(A)$. If B is a Borel set such that $\mu(B) > 0$ show that* $\lim \mu(f^n(B)) = 1$. *Hint: show that $\mu(B_n) = \mu(f^n(B))$ and consider the union A of $B_n := f^{-n}(f^n(B))$.*

Exercise 3.2.2.5. *Let $L^2_0(\mu)$ be the space of functions $\varphi \in L^2(\mu)$ such that $\langle\mu,\varphi\rangle = 0$ and define $V_n := \{\varphi \in L^2(\mu),\ \Lambda^n\varphi = 0\}$. Show that $H_0 := \cup_{n\geq 0} V_n$ is dense in $L^2_0(\mu)$. Hint: $H_0^\perp = \mathbb{R}$ since it is generated by the characteristic functions of Borel sets A such that $f^{-n}(f^n(A)) = A$.*

3.2.3 Equidistribution of preimages and exceptional set

We define the *exceptional set* associated to f as

$$\mathcal{E} := \{a,\ d^{-kn}(f^n)^*\delta_a \not\to \mu\}.$$

We have seen that $\mathcal{E}$ is pluripolar.

Theorem 3.2.3.1. *The exceptional set $\mathcal{E}$ is the maximal proper analytic subset of $\mathbb{P}^k$ which is totally invariant.*

We study first the exceptional set associated to an arbitrary analytic set X in $\mathbb{P}^k$. We define some functions associated to X. For $z \in X$ let

$$\mathscr{F}^n_X(z) := \{w \in f^{-n}(z) : f^i(w) \in X \text{ for } 0 \leq i \leq n\}$$

and

$$\mathscr{N}^n_X(z) := \#\mathscr{F}^n(z) \quad \text{and} \quad \tau^n_X(z) := d^{-kn}\mathscr{N}^n_X(z).$$

Note that the points in $\mathscr{F}^n_X(z)$ are counted with multiplicities and $\mathscr{N}^n_X(z)$ is the number of n-histories of z which are completely contained in X. Since $f(\mathscr{F}^{n+1}_X(z)) \subset \mathscr{F}^n_X(z)$ we have $\mathscr{N}^{n+1}_X(z) \leq d^k\mathscr{N}^n_X(z)$. It follows that $\tau^{n+1}_X(z) \leq \tau^n_X(z)$. So, we can define

$$\tau_X(z) := \lim_{n\to\infty} \tau^n_X(z).$$

Observe that the set

$$\mathcal{E}_X := \{z \in X,\ \tau_X(z) = 1\}$$

is the set of points z such that $f^{-n}(z) \subset X$ for every n.

Exercise 3.2.3.2. *Show that $\mathcal{E}_X$ is an analytic set. Show that $\{\mathscr{N}^n_X(z) \geq c\}$ is analytic and hence $\{\tau_X(z) \geq \theta\}$ is analytic for $\theta > 0$.*

Theorem 3.2.3.3. *The exceptional set $\mathcal{E}_X$ is totally invariant and is equal to $\{z,\ \tau_X(z) > 0\}$.*

Proof. It is clear that $f^{-1}(\mathcal{E}_X) \subset \mathcal{E}_X$, so the sequence of analytic sets $f^{-n}(\mathcal{E}_X)$ is decreasing and hence stationary. Therefore, there is n such that $f^{-n-1}(\mathcal{E}_X) = f^{-n}(\mathcal{E}_X)$. Since f is surjective, we get $\mathcal{E}_X = f^{-1}(\mathcal{E}_X)$. Hence, $\mathcal{E}_X$ is totally invariant.

Define $\mathcal{E}_X(\theta) := \{z,\ \tau_X(z) > \theta\}$ for $\theta > 0$. We want to show that $\mathcal{E}'_X := \cup_{\theta>0}\mathcal{E}_X(\theta)$ is equal to $\mathcal{E}_X$. Assume that $\mathcal{E}'_X \neq \mathcal{E}_X$. Observe that $\mathscr{E}_X(\theta)$ is an increasing family of analytic sets as θ decreases to 0. Hence, there is a $\theta_0 < 1$ maximal such that $\mathcal{E}_X(\theta_0) \neq \mathcal{E}_X$. Let $\mathcal{E}^*_X(\theta_0)$ be the union of components of $\mathcal{E}_X(\theta_0)$ which are not contained in $\mathcal{E}_X$. Let z be a point in $\mathcal{E}^*_X(\theta_0) \setminus \mathcal{E}_X$.

Exercise 3.2.3.4. *Show that $f^{-1}(z) \cap \mathcal{E}_X = \varnothing$ and that*

$$\tau_X(z) \leq d^{-k} \sum_{w\in f^{-1}(z)} \tau_X(w).$$

Deduce that $f^{-1}(z) \subset \mathcal{E}^*_X(\theta_0)$.

We obtain from the previous exercise that $f^{-1}(\mathcal{E}^*_X(\theta_0)) \subset \mathcal{E}^*_X(\theta_0)$. This implies that $f^{-n}(\mathcal{E}^*_X(\theta_0)) \subset X$ for every $n \geq 0$. Hence, $\tau_X = 1$ on $\mathcal{E}^*_X(\theta_0)$. This is a contradiction. □

Exercise 3.2.3.5. *Extend Brody's lemma to holomorphic maps with images in* $\mathbb{P}^k$.

Exercise 3.2.3.6. *Let* $g : D^* \to \mathbb{P}^k$ *be a holomorphic map where* D^* *is the pointed unit disc in* $\mathbb{C}$. *Assume that the area of* $g(D^*)$ *is finite where the points of* $g(D^*)$ *are counted with multiplicities. Show that* g *can be extended to a holomorphic map from* D *to* $\mathbb{P}^k$. *Hint: use the Bishop's theorem for the graph of* g *and that the graph is irreducible. Deduce that if* $g : \mathbb{C} \to \mathbb{P}^k$ *is a non-constant holomorphic map then the area of* $g(\mathbb{C})$ *is at least equal to* 1. *Hint: if not consider the extension of* g *to* $\mathbb{P}^1$.

Exercise 3.2.3.7. *Let* $0 < r < 1$ *be a real number and* D_r *denote the disc of center* 0 *and of radius* r *in* $\mathbb{C}$. *Show that there is a constant* $c_r > 0$ *such that for every holomorphic injective map* $g : D \to \mathbb{P}^k$ *we have*

$$\operatorname{diam}(g(D_r)) \leq c_r \operatorname{area}(g(D))^{1/2}.$$

Hint: if $g_n : D \to \mathbb{P}^k$ *are holomorphic maps such that* $\operatorname{area}(g_n(D)) \to 0$, *using the previous exercises, prove that there is a subsequence* (g_{n_i}) *which converges locally uniformly to a constant map; use local chart of* $\mathbb{P}^k$ *and Cauchy's formula.*

Proposition 3.2.3.8. *Let* l *be an integer large enough. Let* B *be a ball of center* a *and of radius* r *such that* $B \cap \mathrm{PC}_{l+1} = \varnothing$. *Then for every* $\epsilon > 0$, *the ball* B' *of center* a *and of radius* $r/2$ *admits at least* $d^{kn}(1 - 10d^{-l})$ *inverse branches of order* n *and of size* $\leq d^{-(1-\epsilon)n/2}$ *if* n *is large enough.*

Exercise 3.2.3.9. *Let* L *be a projective line which is not contained in* PC_∞. *Show that* $f^{-n}(L)$ *is of degree* $d^{(k-1)n}$. *Deduce that* $f^{-n}(L)$ *contains at most* $(k+1)(d-1)d^{(k-1)n}$ *points in* PC_1. *Hint: use Bézout theorem.*

Exercise 3.2.3.10. *Let* Δ *denote the disc* $D \cap B$. *Show that* Δ *admits at least* $d^{kn}(1 - 5d^{-l})$ *inverse branches of size* $\leq d^{-(1-\epsilon)n/2}$ *if* n *is large enough. Hint: follow the proof of Proposition 2.2.2.9.*

Exercise 3.2.3.11. *Prove Proposition 3.2.3.8. Hint: use Exercises 3.2.3.10 and 3.1.9.14.*

End of the proof of Theorem 3.2.3.1. For $a \in \mathbb{P}^k$, define

$$m(a) := \sup_{\nu} \|\nu - \mu\|$$

where ν is a limit value of $d^{-kn}(f^n)^*(\delta_a)$. As in Section 2.2.2, we prove that $m(a) \leq 20d^{-l}$ if $a \notin \mathrm{PC}_{l+1}$. Moreover, this function satisfies $m \leq d^{-k} f_*(m)$.

Exercise 3.2.3.12. *Let* X_n *denote the exceptional set associated to* PC_n. *Show that the sequence* (X_n) *is increasing and that* $m = 2$ *on* X_n. *Hint:* μ *has no mass on analytic sets. Prove that* $m(a) \leq 20d^{-l}$ *for* $a \notin X_{l+1}$.

Fix l_0 large enough such that $20d^{-l_0} < 2$. We deduce from the previous exercise that X_n is contained in X_{l_0} for every n. Hence, if $a \notin X_{l_0}$, $m(a) \leq 20d^{-l}$ for every l. This implies that $\mathcal{E} = X_{l_0}$. In particular, $\mathcal{E}$ is analytic and totally invariant. If $\mathcal{E}'$ is a proper analytic subset of $\mathbb{P}^k$ which is totally invariant, it is clear that $m = 2$ on $\mathcal{E}'$. Hence $\mathcal{E}' \subset \mathcal{E}$. This implies the maximality of $\mathcal{E}$ and completes the proof.

□

Theorem 3.2.3.13. *The convex of totally invariant probability measures of f is of finite dimension.*

Sketch of the proof. By Theorem 3.2.3.1, this convex is generated by μ and totally invariant measures with support in $\mathcal{E}_1 := \mathcal{E}$ which is of dimension $\leq k-1$. We can prove an analogous result for the restriction of f to $\mathcal{E}_1$. This map admits an equilibrium measure μ_1 and an exceptional set $\mathcal{E}_2$ of dimension $\leq k-2$. The considered convex is generated by μ, μ_1 and the totally invariant measures with support in $\mathcal{E}_2$. We obtain the result by induction on the dimension.

Observe here that the $\mathcal{E}_i$ may be singular. Therefore, we should extend the notion of quasi-psh and dsh functions to this case. The construction of inverse branches of a ball has also some technical difficulties. Indeed, using some properties of positive closed currents, we can work with local coordinates and local holomorphic discs instead of the global projective lines. □

Exercise 3.2.3.14. *Show that the following properties are equivalent:*

1. *$a \in \mathrm{supp}(\mu)$.*

2. *For every neighbourhood B of a, we have $\cup_{n\geq 0} f^n(B) \supset \mathbb{P}^k \setminus \mathcal{E}$.*

3. *For every neighbourhood B of a, $\cup_{n\geq 0} f^n(B)$ is pluripolar.*

Hint: μ has no mass to pluripolar sets.

Exercise 3.2.3.15. *Consider $f[z : w : t] := [z^d : w^d : t^d]$, $d \geq 2$. Determine the measure μ, its support and the exceptional set $\mathcal{E}$.*

Exercise 3.2.3.16. *Let $g : B \to B$ be a holomorphic map where B is a ball in $\mathbb{C}^k$. If $g(B) \Subset B$, show that g admits a unique fixed point p in B. Show that all the eigenvalues of Dg at p have module strictly smaller than* 1.

Exercise 3.2.3.17 (Ueda's example)**.** *Let $\Phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ be the holomorphic map defined by*

$$\phi([z_0 : z_1], [w_0 : w_1]) := [z_0w_0 : z_1w_1 : z_0w_1 + z_1w_0].$$

Show that ϕ is a ramified covering of degree 2 *and that the locus of ramification is the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$. If $h : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree d, show that there is a holomorphic map $\widehat{h} : \mathbb{P}^2 \to \mathbb{P}^2$ of algebraic degree d such that*

$\phi(h(z), h(w)) = \widehat{h} \circ \phi$. What is the relation between the equilibrium measures of h and $\widehat{h}$? Same question for the critical sets and the exceptional sets? Extend the construction to $\mathbb{P}^k$. Give an example of a map $\widehat{h}$ such that the support of the equilibrium measure is $\mathbb{P}^k$.

Exercise 3.2.3.18. *Fix an integer N large enough. Show that there is a rational fraction h of degree $d \geq 2$ such that $h^{-N}(a)$ contains at least $d^N/2$ distinct points for every $a \in \mathbb{P}^1$. Show that $\widehat{h}^{-N}(b)$ admits at least $d^{2N}/8$ points for every $b \in \mathbb{P}^2$. Show that the set $\mathscr{H}_d^*(\mathbb{P}^2)$ of the maps f in $\mathscr{H}_d(\mathbb{P}^2)$ satisfying the previous property is a non-empty Zariski open set in $\mathscr{H}_d(\mathbb{P}^2)$, that is, $\mathscr{H}_d^*(\mathbb{P}^2)$ is the complement of a proper analytic subset of $\mathscr{H}_d(\mathbb{P}^2)$.*

Exercise 3.2.3.19. *Let f be a map in $\mathscr{H}_d(\mathbb{P}^2)$ and V be an invariant curve. Show that the topological degree of the restriction of f to V is equal to d. Deduce that if f is in $\mathscr{H}_d^*(\mathbb{P}^2)$, the exceptional set of f is empty. Extend the result to $\mathbb{P}^k$.*

3.2.4 Equidistribution of periodic points

In this section we give the proof of the following result.

Theorem 3.2.4.1 (Briend-Duval). *Let P_n denote the set of repelling periodic points of period n of f. Then*

$$d^{-kn} \sum_{w \in P_n} \delta_w \to \mu.$$

Using Proposition 3.2.3.8, we can follow the proof of Theorem 2.2.2.1. Nevertheless, Riemann theorem is not valid in higher dimension: a domain in $\mathbb{C}^k$ homeomorphic to a ball is not always biholomorphic to a ball, see Exercise 3.1.9.14. We need the following Proposition where the proof uses a trick due to Buff. The rest of the proof of Theorem 3.2.4.1 follows Theorem 2.2.2.1.

Proposition 3.2.4.2. *Let l, B, B', ϵ be as in Proposition 3.2.3.8. Let U be an open set in $\mathbb{P}^k$. Then for n large enough, B' admits at least $d^{kn}(1 - 20d^{-l})\mu(U)$ inverse branches $g_i : B' \to B'_i$ of order n and of size $\leq d^{-(1-2\epsilon)n/2}$ with $B'_i \subset U$.*

Fix $\epsilon' > 0$ small enough and $l' > l$ large enough. Since μ has no mass on $\mathrm{PC}_{l'}$, we can choose neighbourhoods V_1 and V_2 of $\mathrm{PC}_{l'}$ such that $\mu(V_1) < \epsilon'$ and $V_2 \Subset V_1$.

Exercise 3.2.4.3. *Show that for m large enough, B' admits at least $d^{km}(1 - 10d^{-l} - 2\epsilon')$ inverse branches $g_i : B' \to B'_i$ of order n and of size $\leq d^{-(1-\epsilon)m/2}$ such that $B'_i \subset \mathbb{P}^k \setminus V_2$. Hint: if B'_i meet $\mathbb{P}^k \setminus V_1$, it is contained in $\mathbb{P}^k \setminus V_2$.*

Fix an open set $U' \Subset U$ such that $\mu(U') \geq \mu(U) - \epsilon'$.

Exercise 3.2.4.4. *Show that for n large enough, each B_i admits at least $(1 - 10d^{-l'})d^{k(n-m)}$ inverse branches $g_{i,j} : B'_i \to B'_{i,j}$ of order $n - m$ and of size $\leq d^{-(1-2\epsilon)n/2}$. Hint: apply Proposition 3.2.3.8 to l' instead of l. Deduce that there are at least $(\mu(U) - 10d^{-l'} - \epsilon')d^{k(n-m)}$ inverse branches with $B'_{i,j} \subset U$. Hint: if $B'_{i,j}$ meets U', it is contained in U.*

Exercise 3.2.4.5. *Prove Proposition 3.2.4.2.*

3.2.5 Lyapounov exponents and entropy

We have seen various results showing that f is mostly expanding with respect to the measure μ. In particular, if A is a Borel set with positive μ-measure then $\lim \mu(f^n(A)) = 1$.

There is an important result due to Oseledec stating, in a quite general context, the existence of Lyapounov exponents for a map acting on a measured space. We give here a very special case.

Let $g : M \to M$ be a $\mathscr{C}^1$ map on a compact real manifold M. Let ν be a probability measure on M ergodic invariant under g_*. Assume also that

$$\int \log^+ \|Dg(x)\| d\nu(x) < \infty \quad \text{and} \quad \int \log^+ \|Dg^{-1}(x)\| d\nu(x) < \infty.$$

Then there is $X \subset M$ with $\nu(X) = 1$ such that for each $x \in X$ there is a measurable decomposition of the tangent space

$$T_x M = \oplus_{i=1}^m E_i(x)$$

invariant under Dg. There are $\lambda_1 > \cdots > \lambda_m$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|Dg^n(x)v\| = \lambda_j \quad \text{for } v \in E_i, \ \nu\text{-almost everywhere.}$$

When the λ_i are non-zero, the measure ν is said to be *hyperbolic*. In the general real case, it is quite difficult to decide when a measure is hyperbolic. The exponent λ_m measures the minimal rate of expansion.

Exercise 3.2.5.1. *Show that*

$$\lambda_m \geq \liminf_{n\to\infty} \int -\frac{1}{n} \log \|(Dg^n)^{-1}(x)\| d\nu(x).$$

Observe that the hypothesis in Oseledec's theorem are satisfied for f and μ. Indeed, μ is a PC measure and hence quasi-psh functions are μ-integrable and we can use the following exercise.

Exercise 3.2.5.2. *Show that there is a quasi-psh function φ such that* $\log \|Df\|$ *and* $\log \|(Df)^{-1}\|$ *are bounded by $-\varphi$. Hint: use the homogeneous coordinates.*

Theorem 3.2.5.3 (Briend-Duval). *The Lyapounov exponents of μ are larger than or equal to $\frac{1}{2}\log d$.*

Proof. By Exercise 3.2.5.1, we have to estimate

$$\int \frac{1}{n}\log \|(Df^n)^{-1}(x)\|^{-1}d\mu(x).$$

The basic idea is to construct enough inverse branches with estimated size. Fix $\delta > 0$ and $\epsilon > 0$ small. According to Proposition 3.2.3.8, we can choose a family of disjoint balls $B_1, \ldots, B_m$ such that

1. $\mu(B_1 \cup \ldots \cup B_m) > 1 - \delta$.
2. $\mu(bB_i) = 0$.
3. For n large enough, each B_i admits $(1-\delta)d^{kn}$ inverse branches $g_{i,j} : B_i \to B_{i,j}$ of order n and of size $\leq d^{-(1-\epsilon)n/2}$.

Reducing the size of B_i allows us to assume that $\|Dg_{i,j}\| \leq d^{-(1-2\epsilon)n/2}$ on B_i. It follows that $\|(Df^n)^{-1}\| \leq d^{-(1-\epsilon)n/2}$ on the union A_n of $B_{i,j}$.

Exercise 3.2.5.4. *Let $E_n := \mathbb{P}^k \setminus A_n$. Show that $\mu(f^{-s}(E_n)) \leq \delta$ for every $s \geq 0$.*

Exercise 3.2.5.5. *Let $M \geq 1$ be a constant such that $\|Df\| \leq M$ on $\mathbb{P}^k$. Let J_{f^n} denote the real Jacobian of f^n. Show that*

$$\|(Df^n)^{-1}\|^{-1} \geq J_{f^n}M^{-2kn}.$$

Exercise 3.2.5.6. *Let J denote the real Jacobian of f. Show that there is a constant $c_\delta > 0$ such that $c_\delta \to 0$ when $\delta \to 0$ and*

$$\left|\int_E \log J d\mu\right| \leq c_\delta$$

for every Borel set E such that $\mu(E) \leq \delta$.

Exercise 3.2.5.7. *Using the previous exercises, show that*

$$\limsup \int_{E_n} \frac{1}{n}\log \|(Df^n)^{-1}(x)\|^{-1}d\mu(x) \leq c_\delta.$$

Hint: $J_{f^n} = J(J\circ f)\ldots(J\circ f^{n-1})$.

We deduce from the previous exercises that

$$\begin{aligned}
&\liminf \int \frac{1}{n}\log \|(Df^n)^{-1}(x)\|^{-1}d\mu(x)\\
&\geq \liminf \int_{A_n} \frac{1}{n}\log \|(Df^n)^{-1}(x)\|^{-1}d\mu(x) - c_\delta\\
&\geq \liminf \int_{A_n} \frac{(1-2\epsilon)\log d}{2}d\mu(x) - c_\delta\\
&= \frac{(1-\delta)(1-2\epsilon)\log d}{2} - c_\delta.
\end{aligned}$$

Letting $\delta \to 0$ and $\epsilon \to 0$ gives the result. $\square$

Theorem 3.2.5.8. *The topological entropy of f is equal to $k \log d$. The measure μ is the unique invariant measure of maximal entropy.*

The inequality $h_t(f) \leq k \log d$ is due to Gromov, the converse is due to Misiurewicz-Przytycki. In fact, their theorem is more general.

Theorem 3.2.5.9 (Misiurewicz, Przytycki)**.** *Let M be a compact orientable real manifold. Let $g : M \to M$ be a $\mathscr{C}^1$ map preserving the orientation. Assume that generic fibers of g contain at least δ points. Then $h_t(g) \geq \log \delta$.*

The uniqueness of the measure of maximal entropy is due to Briend-Duval. The proof of Theorem 2.2.5.4 can be easily extended to the case of higher dimension and give the inequality $h_f(\mu) \geq k \log d$. The variational principle implies that $h_f(\mu) \leq h_t(f)$. We present here the proof of Gromov's inequality $h_t(f) \leq k \log d$.

Proof of Gromov's inequality. We cover $\mathbb{P}^k$ by a finite family of charts: let U_i be open sets of $\mathbb{P}^k$ that we identify to open sets in $\mathbb{C}^k$. We can use on U_i the canonical distance $\mathrm{dist}_{\mathbb{P}^k}$ on $\mathbb{P}^k$ or the Euclidian distance dist_i induced by the asociated local coordinates. Reducing U_i allows us to assume that the Euclidian distance is defined in a neighbourhood of $\overline{U}_i$.

Choose compact sets K_i and L_i in U_i such that $\mathbb{P}^k = \cup K_i$ and $K_i \Subset L_i$. Fix a constant $\delta > 0$ small enough such that

$$\delta \mathrm{dist}_{\mathbb{P}^k} \leq \mathrm{dist}_i \leq \delta^{-1} \mathrm{dist}_{\mathbb{P}^k}, \quad \mathrm{dist}_i(K_i, bU_i) > \delta \text{ and } \mathrm{dist}_i(L_i, bK_i) > \delta.$$

Let $\beta := dd^c \|z\|^2$ denote the canonical Kähler form on $\mathbb{C}^k$. We recall an inequality of Lelong (see the definition of Lelong number). If V is an analytic subset of pure dimension k of a ball B of radius r in $\mathbb{C}^N$ containing the center a of B, then the volume of V is at least equal to $r^{2k}/k!$. The last quantity is independent of N.

Consider now the graph of $(f, \ldots, f^{n-1})$, i.e. the set $\Gamma_n \subset (\mathbb{P}^k)^n$ of points $(z, f(z), \ldots, f^{n-1}(z))$. Consider an (n, ϵ)-separated family $\{z_1, \ldots, z_m\}$ in $\mathbb{P}^k$ and the points $a_j := (z_j, f(z_j), \ldots, f^{n-1}(z_j))$ in Γ_n.

Exercise 3.2.5.10. *Show that for each j there are compacts $K_{i_0}, \ldots, K_{i_{n-1}}$ such that a_j belongs to $K_{i_0} \times \cdots \times K_{i_{n-1}}$. Consider the balls B_j of center a_j and of radius $\delta\epsilon/2$ with respect to the Euclidian distance on $K_{i_0} \times \cdots \times K_{i_{n-1}}$. Show that the B_j are disjoint. Hint: use the fact that z_j are separated.*

Consider now the metric on $(\mathbb{P}^k)^n$ induced by the product of the canonical metrics on the factors $\mathbb{P}^k$. That is, we use the Kähler form $\omega_n := \sum \pi_i^*(\omega)$ where π_i is the projections of $(\mathbb{P}^k)^n$ on the factors with $0 \leq i \leq n-1$.

Exercise 3.2.5.11. *Show that there is a constant $c > 0$ depending only on δ and k such that* $\mathrm{volume}(B_j) \geq c\epsilon^{2k}$. *Hint: use Lelong's inequality. Deduce that m is at most equal to $c^{-1}\epsilon^{-2k}\mathrm{volume}(\Gamma_n)$.*

By definition of topological entropy, it is enough to show that $m \lesssim n^k d^{kn}$. We only have to check that volume$(\Gamma_n) \lesssim n^k d^{kn}$. According to Wirtinger's theorem, we have

$$\begin{aligned} k!\mathrm{volume}(\Gamma_n) &= \int_{\Gamma_n} \omega_n^k = \sum_{0\leq i_1,\ldots,i_k\leq n-1} \int_{\Gamma_n} \pi_{i_1}^*(\omega)\wedge\ldots\wedge\pi_k^*(\omega) \\ &= \sum_{0\leq i_1,\ldots,i_k\leq n-1} \int_{\mathbb{P}^k} (f^{i_1})^*(\omega)\wedge\ldots\wedge(f^{i_k})^*(\omega). \end{aligned}$$

Exercise 3.2.5.12. *Show that there is a smooth $(k-1,k-1)$-form U such that*

$$(f^{i_1})^*(\omega)\wedge\ldots\wedge(f^{i_k})^*(\omega) = d^{i_1+\cdots+i_k}\omega^k + dd^cU.$$

Deduce that

$$\int_{\mathbb{P}^k} (f^{i_1})^*(\omega)\wedge\ldots\wedge(f^{i_k})^*(\omega) = d^{i_1+\cdots+i_k}.$$

Hint: write $(f^i)^*(\omega) = d^i\omega + dd^c u_i$ with u_i smooth.

It follows that volume$(\Gamma_n) \leq n^k d^{kn}$. This completes the proof. □

Note that the inequality $h_f(\mu) \geq k\log d$ holds in a more general context. Let $g : M \to M$ be a $\mathscr{C}^1$ map on a real oriented compact manifold. Let Ω be a volume form on M. Recall that the Jacobian of g is the function J_g defined by

$$g^*(\Omega) = J_g\Omega.$$

It measures the distortion of g with respect to Ω. The previous relation is equivalent to

$$\int_{g^{-1}(B)} J_g\Omega = \int_B \Omega$$

for every Borel set B.

If g is a ramified covering and ν is a probability measure on M we can define the measure $g^*(\nu)$ by

$$\langle g^*(\nu),\varphi)\rangle := \langle \nu, g_*(\varphi)\rangle$$

for φ continuous function. When $g^*(\nu) \ll \nu$, then there is a function $J_{g,\nu}$ such that

$$g^*(\nu) = J_{g,\nu}\nu.$$

This is *the Jacobian of g with respect to ν*. If g is injective on a Borel set B then $g_*(1_B) = 1_{g(B)}$ and hence

$$\nu(g(B)) = \int_B J_{g,\nu}d\nu.$$

Theorem 3.2.5.13 (Parry)**.** *Let ν be a measure as above. Assume it is invariant (under g_*). Then*

$$h_g(\nu) \geq \int \log J_{g,\nu}d\nu.$$

For the measure μ associated to f, we have $f^*(\mu) = d^k\mu$. Since μ has no mass to the critical set of f, the Jacobian of μ in the above sense is equal to d^k. The Parry's theorem implies $h_f(\mu) \geq k\log d$. Hence, μ is of maximal entropy.

3.2.6 Green currents and equidistribution problems

Green currents are fundamental objects associated to a holomorphic maps. They are totally invariant. We have the following result, see e.g [64].

Theorem 3.2.6.1. *The sequence $d^{-n}(f^n)^*(\omega)$ converges to a current T of mass 1 such that $d^{-1}f^*(T) = T$. The support of T is equal to the Julia set of f.*

Definition 3.2.6.2. The wedge-product T^p is called *the Green current of order p of f*. This is a positive closed (p,p)-current of mass 1 satisfying $f^*(T^p) = d^pT^p$.

We sketch here a construction of T^p using the dd^c-method as in the case of dimension 1. Let $\mathrm{DSH}^{k-p,k-p}(\mathbb{P}^k)$ denote the space of real currents Φ of bidegree $(k-p,k-p)$ of ordre 0 on $\mathbb{P}^k$ such that

$$dd^c\Phi = \Omega^+ - \Omega^-$$

where $\Omega^\pm$ are positive closed of bidegree $(k-p+1, k-p+1)$. This space is endowed with the following norm

$$\|\Phi\|_{\mathrm{DSH}} := \|\Phi\| + \min \|\Omega^\pm\|$$

with $\Omega^\pm$ as above. The weak topology on this space is defined exactly as in the case of functions. We say that a positive closed (p,p)-current S is *PB* if there is a positive constant $c > 0$ such that $|\langle S, \Phi\rangle| \leq c\|\Phi\|_{\mathrm{DSH}}$ for Φ smooth. We say that S is *PC* if it can be extended to a linear form on $\mathrm{DSH}^{k-p,k-p}(\mathbb{P}^k)$ which is continuous with respect to the weak topology.

Theorem 3.2.6.3 ([40])**.** *It S is a PB (p,p)-current then $d^{-pn}(f^n)^*(S)$ converge to a PC current which does not depend on S.*

Of course, the limit is equal to T^p. The proof of this result follows the case of the equilibrium measure μ. There are some extra difficulties. The following conjecture is still open.

Conjecture 3.2.6.4. *There are proper analytic subsets $\mathcal{E}_1, \ldots, \mathcal{E}_m$ of $\mathbb{P}^k$ totally invariant by f such that if V is an analytic subset of codimension p which intersects $\mathcal{E}_i$ properly, then $d^{-pn}(f^n)^*[V]$ converge to* $\deg(V)T^p$.

We say that V intersects $\mathcal{E}_i$ properly if for any component E of $\mathcal{E}_i$ we have $\dim(V \cap E) = \dim E - p$ if $\dim E \geq p$ and $V \cap E = \varnothing$ if $\dim E < p$.

We have seen that the conjecture holds for the case of measures, i.e. $p = k$. It holds also for the case of bidegree $(1,1)$.

Theorem 3.2.6.5 ([39])**.** *There is a proper analytic subset $\mathcal{E}'$ of $\mathbb{P}^k$ totally invariant by f such that if V is a hypersurface that contains no components of $\mathcal{E}'$, then $d^{-n}(f^n)^*[V]$ converge to* $\deg(V)T$.

The result is due to Favre-Jonsson in the case $k = 2$ [45]. The following result was proved by Fornaess and the second author in the case $p = 1$ [48].

Theorem 3.2.6.6 ([40]). *There is a dense Zariski open set $\mathscr{H}_d^*(\mathbb{P}^k)$ in $\mathscr{H}_d(\mathbb{P}^k)$ such that if f is in $\mathscr{H}_d^*(\mathbb{P}^k)$ we have $d^{-pn}(f^n)^*(S) \to T^p$ uniformly on positive closed (p,p)-currents S of mass 1.*

We can also hope that the convergence in the Conjecture 3.2.6.4 is exponentially fast. This is true for the case of measures [41]. Recently, Taflin proved in [71] that when $p = 1$ and $1/d < \lambda < 1$, there are invariant proper analytic subsets $E_1, \ldots, E_m$ such that if V is a hypersurface which does not contain E_i, then $d^{-n}(f^n)^*[V]$ converge to $\deg(V)T$ with rate λ^n with respect to $\mathscr{C}^2$ test forms.

3.2.7 Some other open problems

Let f be a holomorphic endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$. When $k = 1$, Sullivan's theorem gives the non-existence of wandering Fatou component. It is unknown if a similar result holds in $\mathbb{P}^k$ for $k \geq 2$.

If U is an invariant Fatou component. Assume that $f^n(z) \to bU$ for $z \in U$. Is there a fixed point on bU? For $k = 1$, this is the case.

The Ueda's construction gives examples where $\mathrm{supp}(\mu) = \mathbb{P}^k$. Construct different classes of maps f such that $\mathrm{supp}(\mu) = \mathbb{P}^k$. In the parameter space $\mathscr{H}_d(\mathbb{P}^k)$ how big is such set? Does it have positive Lebesgue measure?

What is the Hausdorff dimension of μ? Recall that this dimension is defined by

$$\dim_H(\mu) := \inf\{\text{Hausdorff dimension of Borel sets } B \text{ with } \mu(B) = 1\}.$$

Notes for Chapter 3. The standard references for currents are Federer's and de Rham books [46, 22]. The book by Demailly [21] covers complex analytic geometry. It contains in particular the basic facts on (p,p)-currents, analytic sets and holomorphic functions in several variables. The developments in dynamics require the extension of basic operators: pull-back of currents by meromorphic maps, wedge-product of currents, convergence theorems. Theorem 3.1.3.17 gives the simplest case; for other results see [37, 40].

The spaces DSH were introduced in [29, 32, 35] with the duality method and the use of good solutions to the dd^c-equation. The construction of the equilibrium measure given here follows [29, 35]. The exponential decay of correlations was proved by Fornæss and the second author in [47] for holomorphic maps. The authors obtained sharp results in this case and also for meromorphic maps in [29, 35]. The theory of Sobolev space W^* was developed by the authors in [34], see also Vigny [74].

For a holomorphic map on $\mathbb{P}^k$, Fornæss and the second author showed that the exceptional set $\mathcal{E}$ is pluripolar [47]. Briend-Duval showed that $\mathcal{E}$ is contained in the orbit of the critical set [12]. A proof that $\mathcal{E}$ is algebraic is given in [29], see also [39]. This is the approach we follow.

Equidistribution of repelling periodic points in the general case is due to Briend-Duval [11]. We have used a simplification due to Buff. The estimate of Lyapounov exponents is due to Briend-Duval [11], we have followed the proof given in [29]. The estimate on entropy by the growth of the volume is due to Gromov [52]. It can be extended to arbitrary meromorphic maps or correspondences [30, 31, 38]. Briend-Duval have showed that μ is the unique invariant measure of maximal entropy [12].

The reader will find in the bibliography a list of references on dynamics in higher dimension.

Exam 2007

A. The aim of this part is to prove results by Brolin and others, using the pluripotential theory.

Theorem 1. *Let f be a polynomial of degree $d \geq 2$ and let Q_n denote the set of periodic points of period n of f. The points in Q_n are repeated according to their multiplicities. Then the measures*

$$\mu_n := d^{-n} \sum_{a \in Q_n} \delta_a$$

converge weakly to the equilibrium measure μ of f.

1. Define $u_n := d^{-n} \log |f^n(z) - z|$. Show that u_n is subharmonic and $dd^c u_n = \mu_n$.

2. Let K be the filled Julia set of f. Show that u_n converge pointwise on $\mathbb{C} \setminus K$ to the Green function G of f.

3. Deduce that (u_n) is relatively compact in $L^p_{loc}(\mathbb{C})$ for $1 \leq p < \infty$.

4. Let (n_i) be an increasing sequence of integers such that u_{n_i} converge to a subharmonic function v in $L^1_{loc}(\mathbb{C})$. Show that $v = G$ on $\mathbb{C} \setminus K$ and $v \leq 0$ on K.

5. Show that $v = 0$ on the boundary of K.

6. In the questions 6, 7, 8, assume that $v \neq G$ on a bounded Fatou component U of f. Show that for every open set $U' \Subset U$, there is $\alpha > 0$ such that $v_{n_i} < -\alpha$ on U' for i large enough.

7. Deduce that U is a periodic Siegel disc.

8. Deduce that U contains a unique periodic point and then v is harmonic on U.

9. Deduce that $v = G$ and conclude that $\mu_n \to \mu$. For this question, we use the following classical result: *if a potential u of a positive measure ν, restricted to* supp(ν)*, is continuous, then u is continuous.*

10. Using the same method, prove the following result.

Theorem 2. *Let f be a polynomial of degree $d \geq 2$ and g be a non-constant polynomial. Let R_n denote the set of solutions of the equation $f^n = g$. The points in R_n are repeated according to their multiplicities. Then the measures*

$$\nu_n := d^{-n} \sum_{a \in R_n} \delta_a$$

converge weakly to the equilibrium measure μ of f.

B. Polynomial-like maps: some properties. In this part, we consider some properties of the following family of maps.

Definition. Let U, V be open sets in $\mathbb{C}$ such that V is connected and $U \Subset V$. Let $f : U \to V$ be a proper holomorphic map. Then $f : U \to V$ is a (possibly ramified) covering. Let d be the degree of this covering and assume that $d \geq 2$. We say that f is a *polynomial-like map* of *(topological) degree d*.

1. Let g be a polynomial of degree $d \geq 2$. Let D denote a disc of center 0 and of radius R with R large enough. Define $W := g^{-1}(D)$. Show that $g : W \to D$ is a polynomial-like map.

2. Let h be a bounded holomorphic function defined in a neighbourhood of $\overline{W}$. Define $g_\epsilon := g + \epsilon h$ and $W_\epsilon := g_\epsilon^{-1}(D)$. Show that if ϵ is small enough, $g_\epsilon : W_\epsilon \to D$ is a polynomial-like map of degree d.

3. From now on, $f : U \to V$ is a polynomial-like map of degree $d \geq 2$. Define $U_{-n} := f^{-n}(V)$. Show that (U_{-n}) is a decreasing sequence of open sets and $K := \cap_{n \geq 0} U_{-n}$ is a compact set.

4. Show that K is the set of points $x \in U$ such that $x, f(x), f^2(x), \ldots$ are defined and belong to U.

5. Show that K and bK are totally invariant by f and there is a totally invariant probability measure with support in bK. We call K *the filled Julia set* and $J := bK$ *the Julia set.*

6. Assume that V is simply connected. Prove that f admits exactly d fixed points counted with multiplicity and all these points are in K. Hint: consider first the case where V is the unit disc and use Rouché's theorem.

C. Polynomial-like maps: equilibrium measure. We will construct a canonical measure μ associated to f.

1. From now on, v denote a subharmonic function on V. Define $\Lambda := d^{-1}f_*$. If v is constant, then $\Lambda(v) = v$. If $v \leq 0$, show that $\Lambda(v) \leq 0$.
2. Show that $\Lambda(v)$ is subharmonic on V. Hint: consider first the case where v is continuous.
3. Show that
$$\sup_V \Lambda(v) \leq \sup_U v.$$
Deduce that if $\Lambda(v) = v$ then v is constant.
4. Define
$$c_v := \lim_{n\to\infty} \sup_V \Lambda^n(v).$$
Show that the limit exists and $c_v \neq +\infty$. We want to prove that $\Lambda^n(v) \to c_v$.
5. If $c_v = -\infty$, it is clear that $\Lambda^n(v) \to -\infty$ uniformly on V. From now on, assume that c_v is finite. Show that $(\Lambda^n(v))$ is relatively compact in $L^p_{loc}(V)$ for $1 \leq p < \infty$.
6. Let (n_i) be an increasing sequence of integers such that $\Lambda^{n_i}(v)$ converge in $L^1_{loc}(V)$ to a subharmonic function u. Show that $u \leq c_v$.
7. Assume $u \neq c_v$. Let $U' \Subset V$ be a neighbourhood of $\overline{U}$. Show that there is $\epsilon > 0$ such that $u < c_v - 2\epsilon$ on U'.
8. Show that there is an i such that $\Lambda^{n_i}(v) < c_v - \epsilon$ on U.
9. Show that $\Lambda^n(v) < c_v - \epsilon$ on V for every $n > n_i$.
10. Find a contradiction. Conclude that $\Lambda^n(v) \to c_v$ in $L^p_{loc}(V)$ for $1 \leq p < \infty$.
11. Let Ω be a smooth positive $(1,1)$-form on U which defines a probability measure, that is, $\int_U \Omega = 1$. Show that
$$\langle d^{-n}(f^n)^*(\Omega), v\rangle = \langle \Omega, \Lambda^n(v)\rangle.$$
12. Deduce that $d^{-n}(f^n)^*(\Omega)$ converge to a probability measure μ which does not depend on Ω.

D. Polynomial-like maps: some properties of the equilibrium measure. We show that μ has no mass on countable sets and we give a lower bound for the entropy of μ.

1. Show that the previous convergence holds when Ω is a form on V.

2. Show that μ has support in the Julia set and is totally invariant.

3. Show that $\langle \mu, v\rangle = c_v$ for every v subharmonic continuous or not (the value $-\infty$ is allowed).

4. For the questions 4, 5, 6, assume that v is harmonic. Show that $\Lambda^n(v)$ converge to c_v uniformly on compact sets.

5. Let a be a totally invariant point. Show that $c_v = v(a) = (\Lambda^n v)(a)$.

6. If a and b are two different points in $\mathbb{C}$, construct a harmonic function v such that $v(a) \neq v(b)$.

7. Deduce that there is a maximal set $\mathcal{E}$ which is finite and totally invariant, and that $\mathcal{E}$ is empty or contains only 1 point.

8. If $\mathcal{E}$ contains 1 point, show that this point is critical and does not belong to the Julia set. Deduce that $\mu(\mathcal{E}) = 0$.

9. For every point a in $V\backslash\mathcal{E}$, show that the negative orbit $\mathrm{O}^-(a) := \cup_{n\geq 0} f^{-n}(a)$ of a is infinite and contains a non-periodic point.

10. Deduce that there is a point $a' \in \mathrm{O}^-(a)$ such that $f^{-n}(a')$ contains exactly d^n points and that $f^{-n}(a') \cap f^{-m}(a') = \varnothing$ if $n \neq m$.

11. Show that $\mu(f^{-n}(a')) = \mu\{a'\}$. Deduce that $\mu\{a'\} = 0$ and $\mu\{a\} = 0$. Deduce that μ has no mass on countable sets.

12. Show that $h_f(\mu) \geq \log d$.

E. Polynomial-like maps: mixing. We will show that μ is mixing. More precisely, if φ and ψ are smooth functions then

$$\lim_{n\to\infty} \langle \mu, (\varphi \circ f^n)\psi\rangle = \langle \mu, \varphi\rangle\langle \mu, \psi\rangle.$$

1. Prove the convergence when φ or ψ is a constant function.

2. Deduce that it is enough to consider φ positive and ψ such that $c_\psi = \langle \mu, \psi\rangle = 0$.

3. Show that there are two smooth subharmonic functions ψ_1 and ψ_2 on $\mathbb{C}$ such that $\psi = \psi_1 - \psi_2$ on U.

4. Deduce that we can assume that ψ is subharmonic. From now on, φ is smooth positive, ψ is smooth subharmonic and $c_\psi = \langle \mu, \psi\rangle = 0$.

5. Show that
$$\langle \mu, (\varphi \circ f^n)\psi \rangle = \langle \mu, \varphi(\Lambda^n \psi) \rangle.$$

6. Show that $\limsup \Lambda^n \psi \leq 0$. Deduce that
$$\limsup_{n\to\infty} \langle \mu, \varphi(\Lambda^n \psi) \rangle \leq 0.$$

7. Let $A > 0$ be a constant such that $\varphi < A$ on U. Show that
$$\limsup_{n\to\infty} \langle \mu, (A - \varphi)(\Lambda^n \psi) \rangle \leq 0.$$

8. Prove that
$$\lim_{n\to\infty} \langle \mu, \varphi(\Lambda^n \psi) \rangle = 0$$
and conclude.

F. Polynomial-like maps: topological entropy. We will show the Gromov's type inequality $h_t(f) \leq \log d$. The variational principle will imply that
$$h_f(\mu) = h_t(f) = \log d.$$
Note that the topological entropy of f is defined as in the case of polynomials but we only consider the orbits in the filled Julia set K. Let $\Gamma_{[n]}$ denote the set of points $(x, f(x), \ldots, f^{n-1}(x))$ in $V^n \subset \mathbb{C}^n$ with $x \in U_{-n+1}$. Let $z = (z_1, \ldots, z_n)$ denote the canonical coordinates in $\mathbb{C}^n$ and $\pi_1, \ldots, \pi_n$ denote the projections on the factors, i.e. $\pi_i(z) := z_i$. Define
$$\mathrm{lov}(f) := \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{volume}(\Gamma_{[n]} \cap \pi_n^{-1}(U)).$$
Let $W \Subset V$ be a neighbourhood of $\overline{U}$. Fix a constant $\delta > 0$ small enough.

1. Show that $\pi_1 : \Gamma_{[n]} \to U_{-n+1}$ is bijective.

2. Show that $\pi_n : \Gamma_{[n]} \to V$ defines a (possibly ramified) covering of degree d^{n-1}.

3. Define $\Pi_0 := \pi_n$ and $\Pi_i(z) := \Pi_0(z) + \delta z_i$. Show that $\Gamma_n \cap \Pi_i^{-1}(W)$ contains $\Gamma_{[n]} \cap \pi_n^{-1}(U)$ for $0 \leq i \leq n-1$.

4. Show that $\Pi_i : \Gamma_{[n]} \cap \Pi_i^{-1}(W) \to W$ defines a (possibly ramified) covering of degree d^i. Hint: use in particular Rouché's theorem.

5. Show that there is a constant $c > 0$ independent of n such that
$$\int_{\Gamma_{[n]} \cap \pi_n^{-1}(U)} \Pi_i^*(dd^c |z_n|^2) \leq c d^n.$$

6. Prove that there is a contant $A > 0$ independent of n such that

$$An \sum_{i=0}^{n-1} \Pi_i^*(dd^c|z_n|^2) \geq dd^c\|z\|^2.$$

7. Deduce that $\mathrm{volume}(\Gamma_{[n]} \cap \pi_n^{-1}(U)) \lesssim nd^n$ and that $\mathrm{lov}(f) \leq \log d$.

8. Fix $\epsilon > 0$ a small constant. If $\mathscr{F} \subset K$ is an (n, ϵ)-separated family, show that $\#\mathscr{F} \lesssim \mathrm{volume}(\Gamma_{[n]} \cap \pi_n^{-1}(U))$.

9. Deduce that $h_t(f) \leq \log d$.

G. Polynomial-like maps: case of higher dimension. Study the case of higher dimension.

H. Entropy outside the support of Green currents. Let f be a holomorphic endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. Let T^p denote the Green current of bidegree (p, p) of f. Prove the following result due to de Thélin in the case of dimension $k = 2$.

Theorem. *If K is a compact set in $\mathbb{P}^k \setminus \mathrm{supp}(T^p)$, show that the topological entropy of f on K satisfies $h_t(f, K) \leq (p-1)\log d$ for $p \geq 1$.*

We can first consider the case $p = 1$ and $k = 1$ or 2. Hint: we can write

$$d^{-n}(f^n)^*(\omega) = T + dd^c u_n$$

with u_n continuous and $\|u_n\|_\infty = O(d^{-n})$.

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