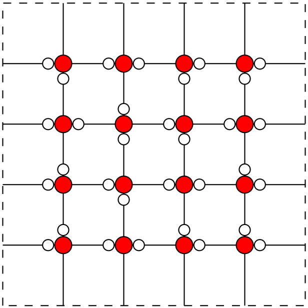
Calcul de l'entropie résiduelle de la glace carrée

Silvère Gangloff

LIP, ENS Lyon

October 4, 2018

États stables de la glace carrée [Pauling-Lieb]:



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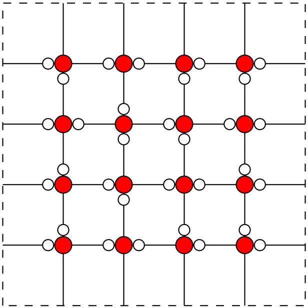

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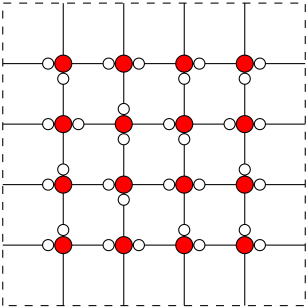

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Valeur de l'entropie?

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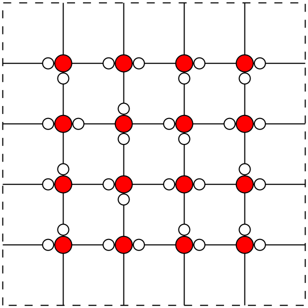
But de l'exposé: 'calcul' de la l'entropie de la glace.

Représentation par tuiles [Folklore]:



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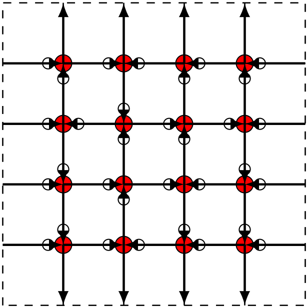

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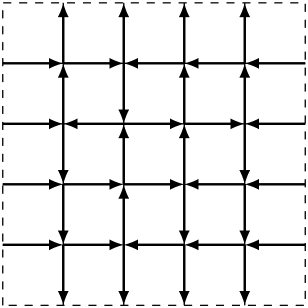

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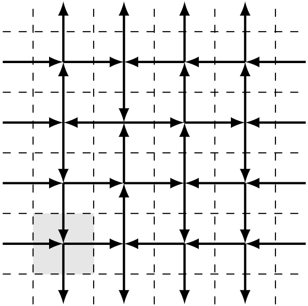

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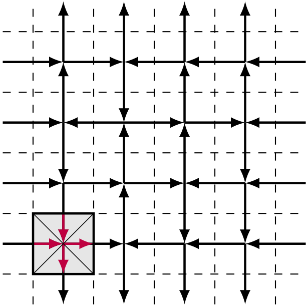

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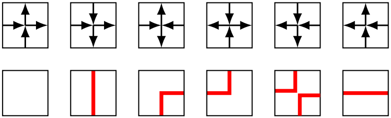

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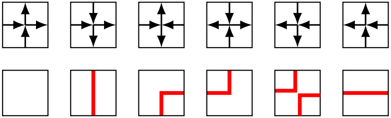
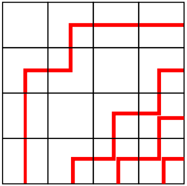
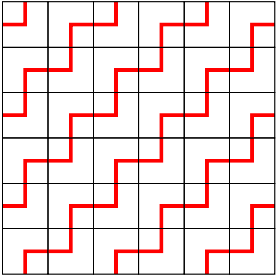




Condition toroïdale [Lieb, Preuve Duminil-Copin et al.]:



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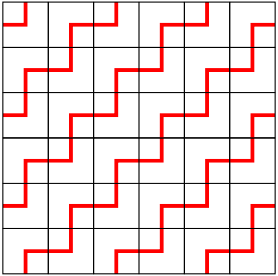

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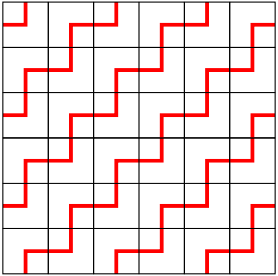

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Suffisant: compter les motifs valides sur un tore

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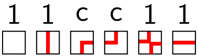

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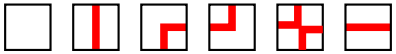

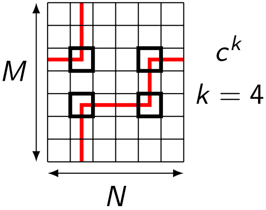

$$P_{N,M}(c) = \sum_{\text{motifs}} c^{k(\text{motif})}$$

$$P_{N,M}(c) = Tr(V_N(c)^M)$$

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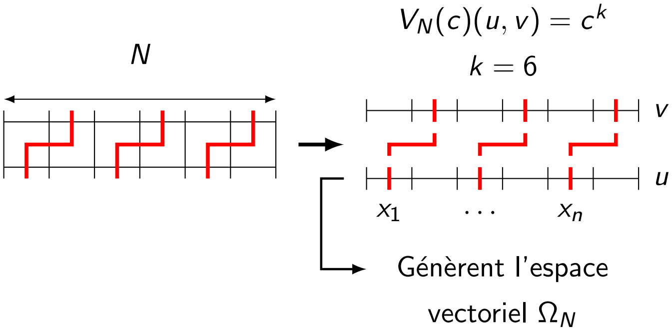

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$$V_N(c)(u,v) = c^k$$

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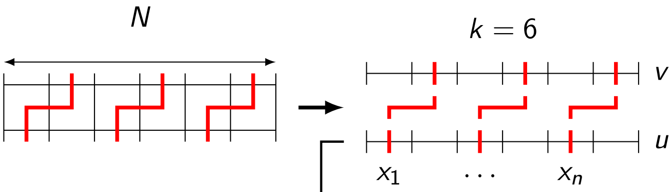

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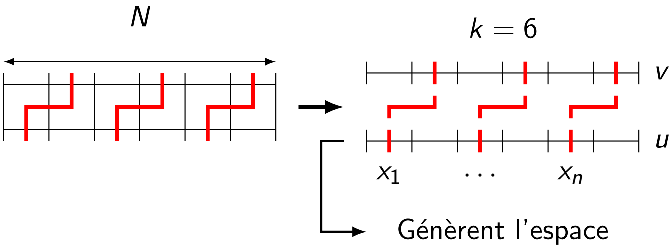

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$$h_c = \lim_N \lim_M \frac{\log(Tr(V_N(c)^M))}{M} \quad \text{et} \quad h_1 = h_{top}$$

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T.J. Baxter, *Exactly solved models in statistical mechanics*, 1982.

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Donc $\exists P$ inversible t.q. $\forall c > 0$,

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Perron-Frobenius [matrices symétriques irréductibles]: o.p.s. que $\lambda_1(c) = \lambda_{max}(V_N(c))$ pour tout $c > 0$.

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où $\Theta(0,0)$,

$$e^{-i\Theta(x,y)} = e^{i(x-y)}\frac{e^{ix} + e^{-iy} - 2\Delta}{e^{-ix} + e^{-iy} - 2\Delta},$$

et $\Delta = (2 - c^2)/2$,

$$\varphi_n(x_1, \dots, x_n) = \sum_{\sigma \in \Sigma_n} A_\sigma \prod_{k=1}^{n} \exp(i p_{\sigma(k)} x_k)$$

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Remarque: $\lambda_n \neq 0$?, $\varphi_n \neq 0$?

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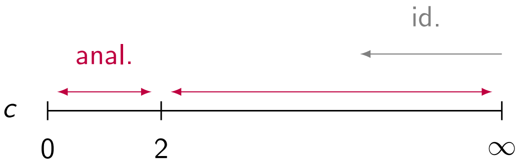
Problème: identification $c > 2$, on veut $c = 1$.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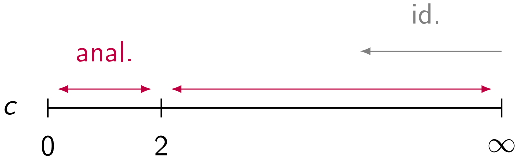

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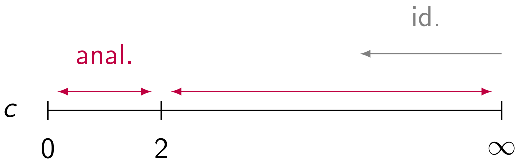

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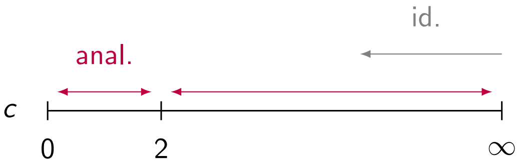

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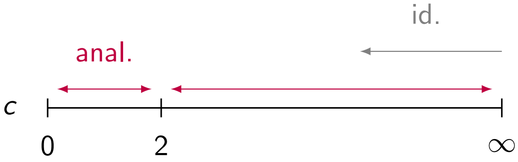

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$$\boxed{h_1 = (4/3)^{3/2}}$$

(contours dans $\mathbb{C}$)

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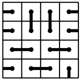

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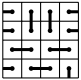

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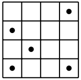

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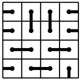

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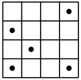

3. Perron-Frobenius: simplification de l'ansatz ?

"If all eigenvectors are given by the Bethe ansatz and span the 2^N *dimensional vectorial space (which is the case) [...]"*, Baxter.

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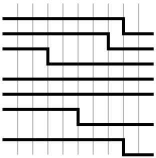

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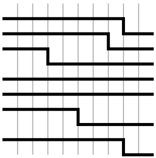

5. Relation de l'ansatz avec les Parafermonic observables [Duminil-Copin,Smirnov] ?