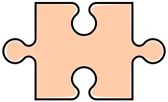
Entropy of quasiperiodic subshifts

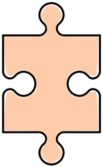
Guilhem Gamard

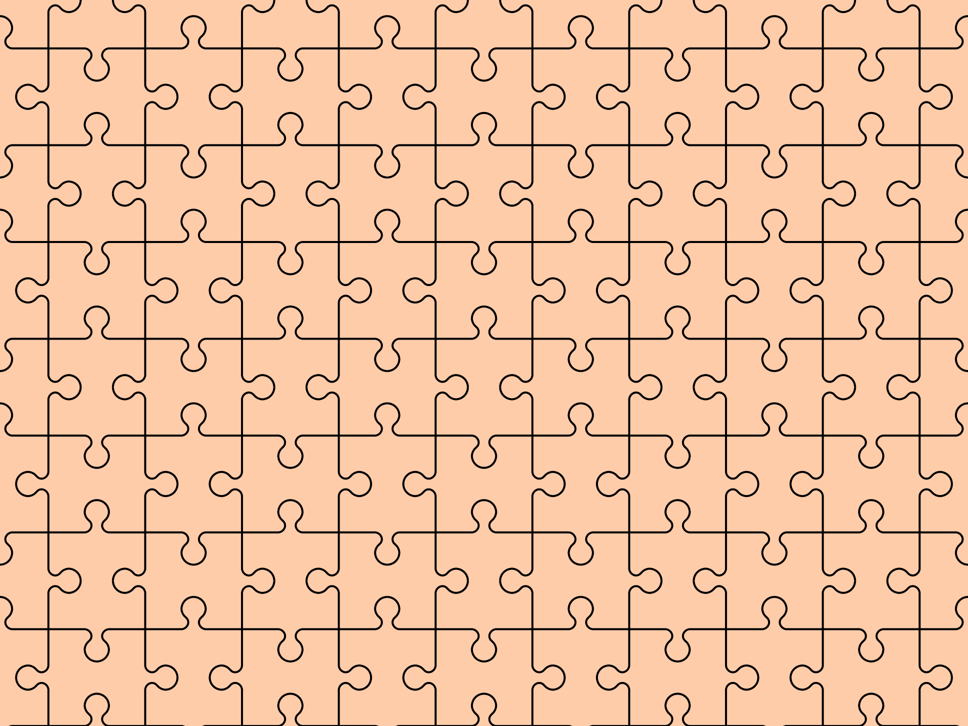
26th October 2018

Tilings and crystals

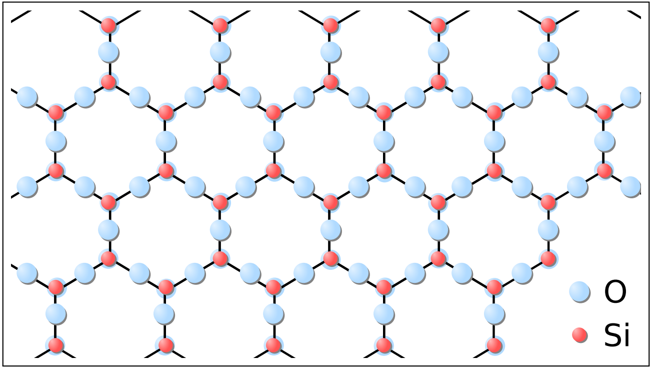
Tiles



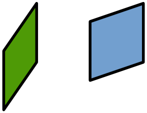


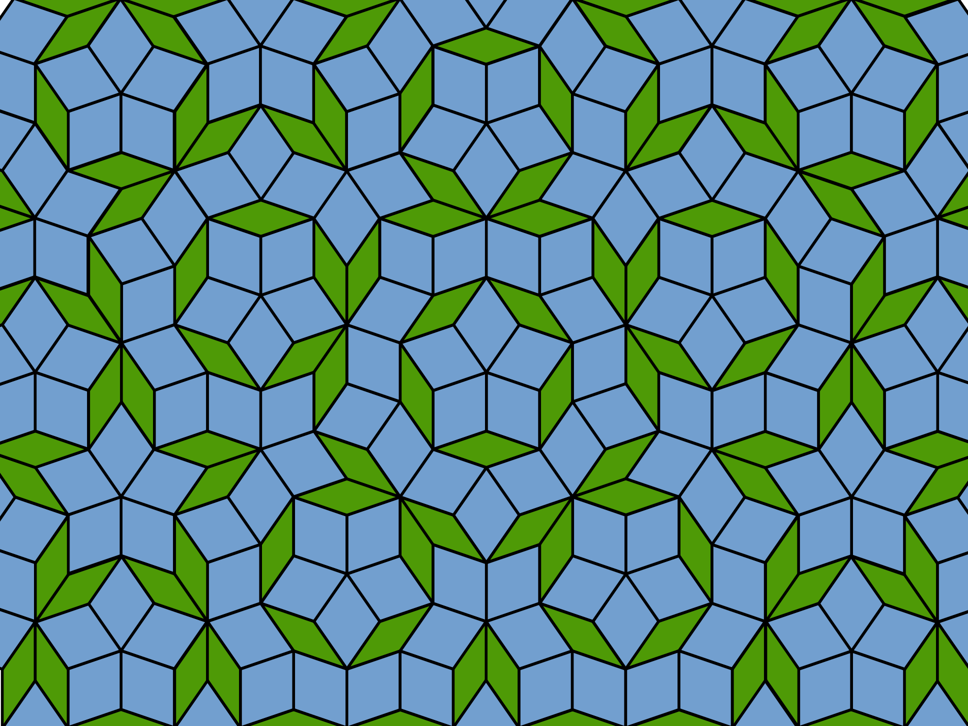


Crystals

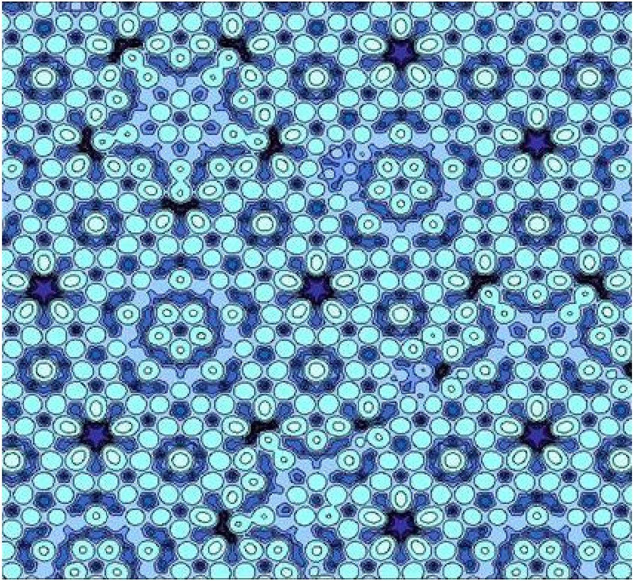


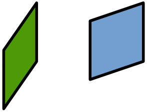
Penrose tiles





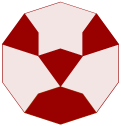
Quasicrystals

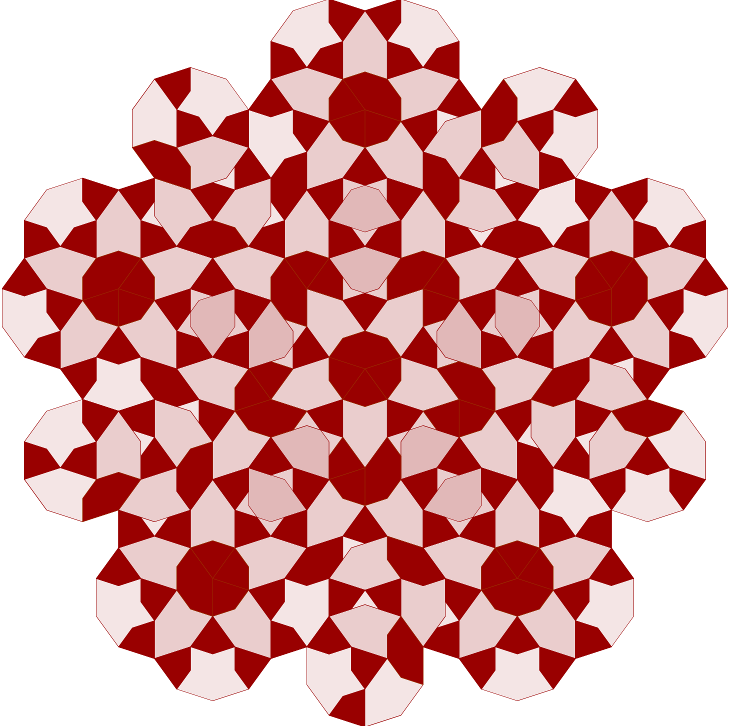




What's the physical meaning of Penrose tiles?

Gummelt's decagon





Gummelt $\simeq$ Penrose

Gummelt tiling rule

Tiles may overlap if decorations match; each tile must overlap.

Theorem (Gummelt, 1996)

Each Gummelt-tiling is isomorphic to a Penrose-tiling, and vice-versa.

Physical interpretation

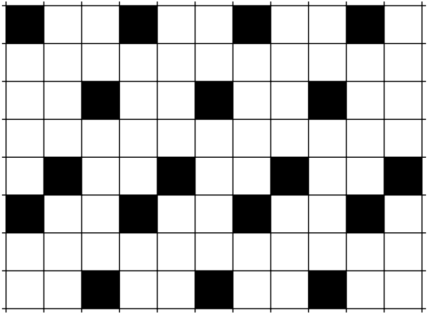
Gummelt decagon has locally minimal energy.

Back to subshifts

Let q denote a pattern.

Quasiperiodic

A configuration has *quasiperiod* q when it is covered with copies of q (possibly overlapping).



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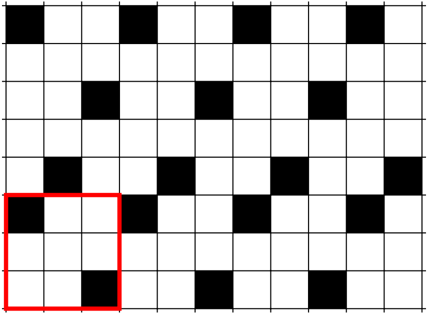

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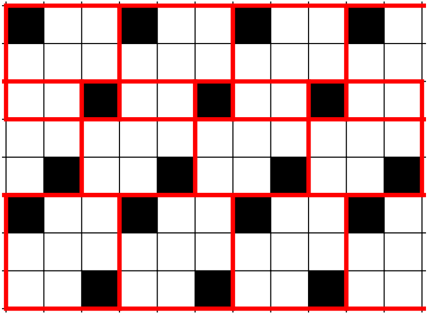


Lemma

The set of q-quasiperiodic configurations is an SFT, called X_q.

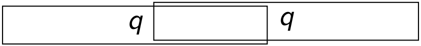
What is the entropy of X_q?

The case of $\mathbb{Z}$-subshifts

Let q denote a finite word.

Remark

Consider two overlapping copies of q; the overlap is a prefix and suffix of q.



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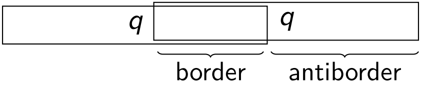

Border

A **border** is a proper suffix and prefix.
An **antiborder** is the right-complement of a border.

(Note: ε is a border and q is an antiborder.)

Let q, w denote finite words.

Theorem (Mouchard, 2000)

The word w has quasiperiod q iff $w = qu_0 \dots u_k$, where each u_i is an antiborder of q.

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Corollary

The biinfinite word $\mathbf{w}$ has quasiperiod q iff $\mathbf{w} = \dots u_{-2}\, u_{-1}\, u_0\, u_1\, u_2\, \dots$ where each u_i is an antiborder of q.

This decomposition is unique iff q is **not** quasiperiodic.

Fix q a quasiperiod. Let $\ell(n) = \#$ q-quasiperiodic words of length n.

Let $r_0, \ldots, r_{k-1}$ be the antiborders of q.

- If $n < |q|$, then $\ell(n) = 0$.
- If $n = |q|$, then $\ell(n) = 1$.
- If $n > |q|$, then $\ell(n) = \sum_{i=0}^{k-1} \ell(n - |r_i|)$.

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Let

$$P(x) = x^{|q|} - \sum_{i=0}^{k-1} x^{|q|-|r_i|}$$

and λ the largest real root of P.

Lemma

For large n: $\quad c_1\lambda^n \leq \ell(n) \leq c_2\lambda^n.$

Theorem (Polley, Staiger, 2010)

Let q denote a non-quasiperiodic word with antiborders $r_0, \ldots, r_{k-1}$, and

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If λ is the largest root of P, then $\boxed{Ent(X_q) = \log(\lambda)}$.

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X_q is finite iff q has no border besides ε.

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X_q is finite iff q has no border besides ε. Otherwise, $\mathrm{Ent}(X_q) > 0$.

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Remark

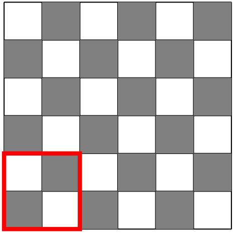
$\mathrm{Ent}(X_q)$ is maximal for $q = 010$.

The case of $\mathbb{Z}^2$-subshifts

We consider **rectangular** patterns only.

Root

The block r is a **root** of q iff $q = r^{m \times n}$ for $m, n \in \mathbb{N}$.



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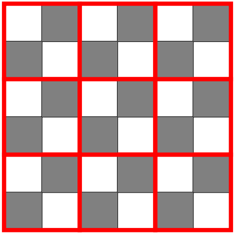

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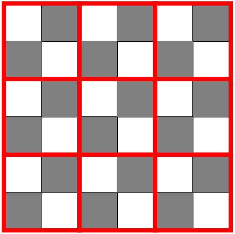

Primitive

The block q is **primitive** if it has no roots besides itself.

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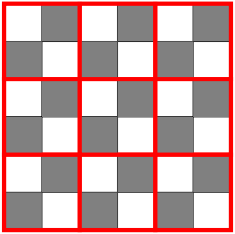

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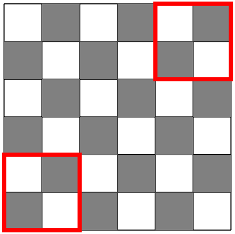
The block q is **primitive** if it has no roots besides itself.

Lemma (G, Richomme, 2015)

Each block has a unique primitive root.

Border

A **border** of q is a subblock that appears in two opposite corners.



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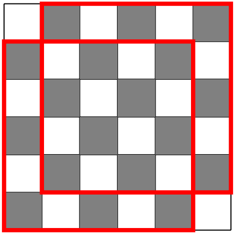

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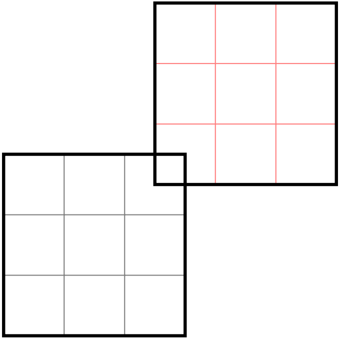

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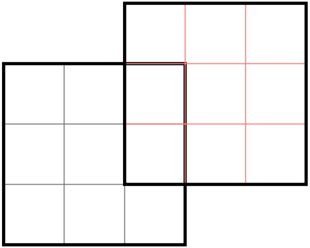

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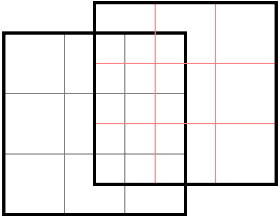

Let q denote a block and r its primitive root.

Theorem (G, Richomme, 2015)

The shift X_q is finite iff the primitive root of q has no border besides ε.

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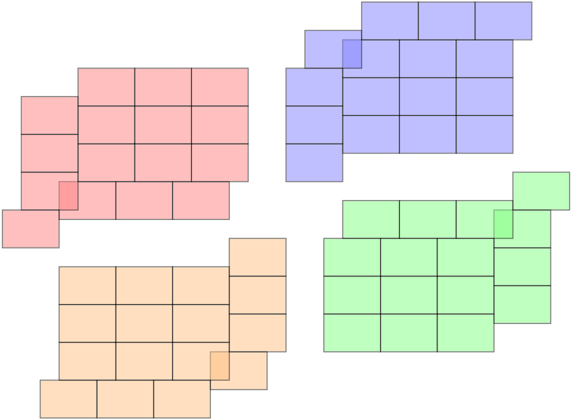
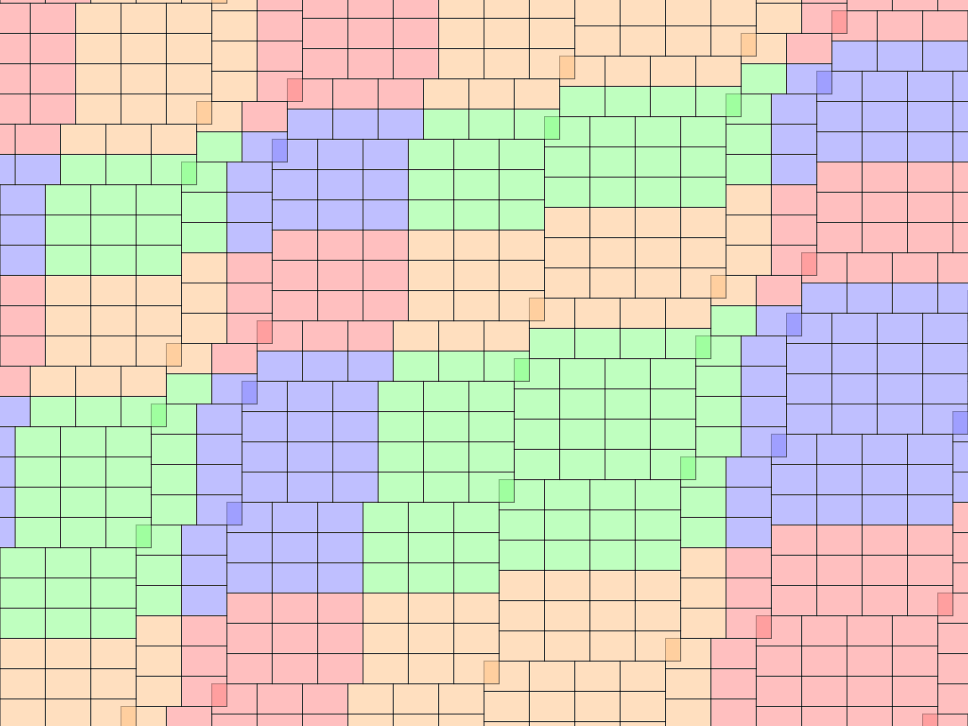
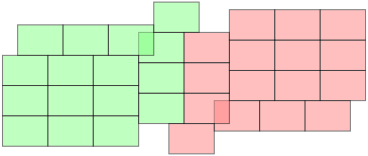




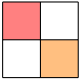
Tiles have local constraints:



Problem

Those constraints do not allow positive entropy!

We can prove that X_q is infinite, but not that $\mathrm{Ent}(X_q) > 0$.

Is it bad? Consider the block $q =$ .

Proposition

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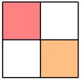
Is it bad? Consider the block $q =$ .

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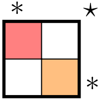
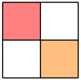


Is it bad? Consider the block $q =$ .

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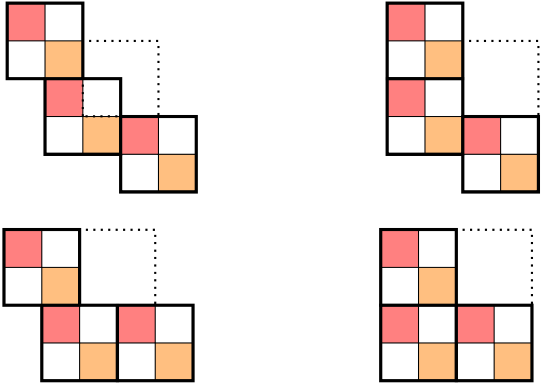
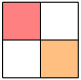


Is it bad? Consider the block $q =$ .

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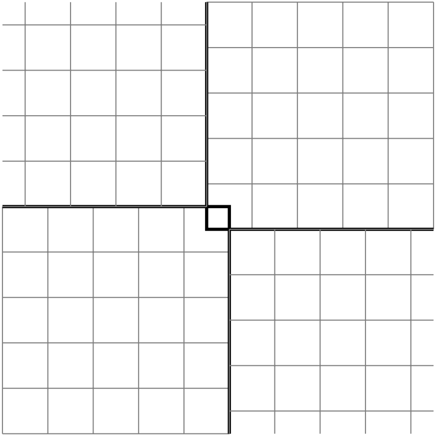
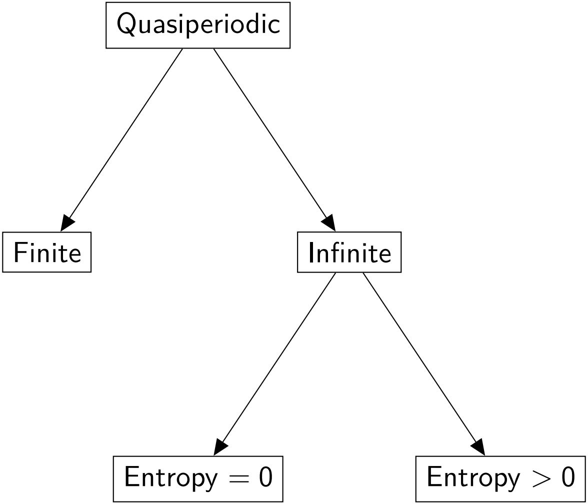
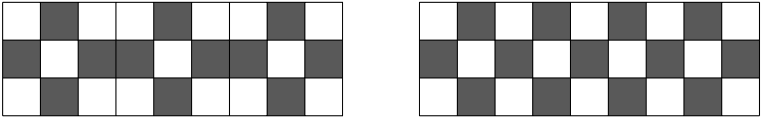


Mindmap so far



Interchangeable pairs

An **interchangeable pair** is a pair of q-quasiperiodic patterns, with the same shape, but different.



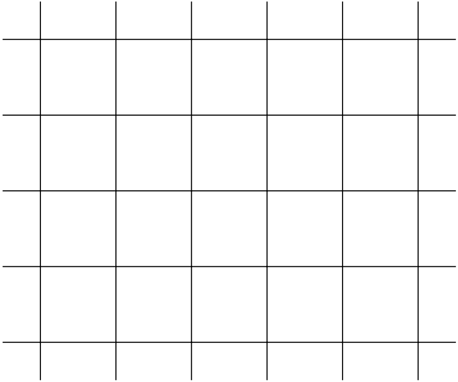
Valid

A pair is **valid** if it appears in a configuration of X_q.

Lemma

If there is an interchangeable pair whose size is a $k \times k$ square, then

$$Ent(X_q) \geq 1/k$$



Lemma

If there is no interchangeable pair for q, then $Ent(X_q) = 0$.

Idea: if there are no interchangeable pair, then the shape of a pattern defines the contents of the pattern.

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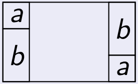
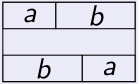
Theorem

We have $Ent(X_q) > 0$ if and only if there is a valid pair for q.

(Note: this is also true for Wang tiles.)

Proposition

If q is of the form 

or 

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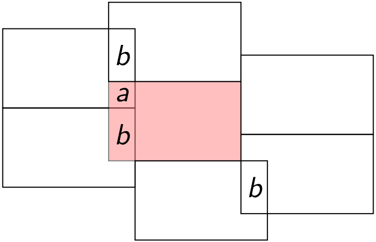

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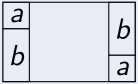
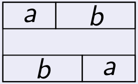
If q is of the form 

or 

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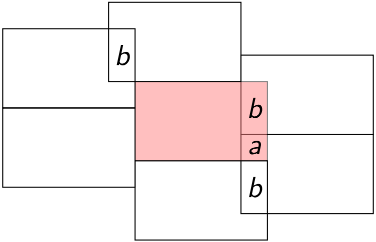

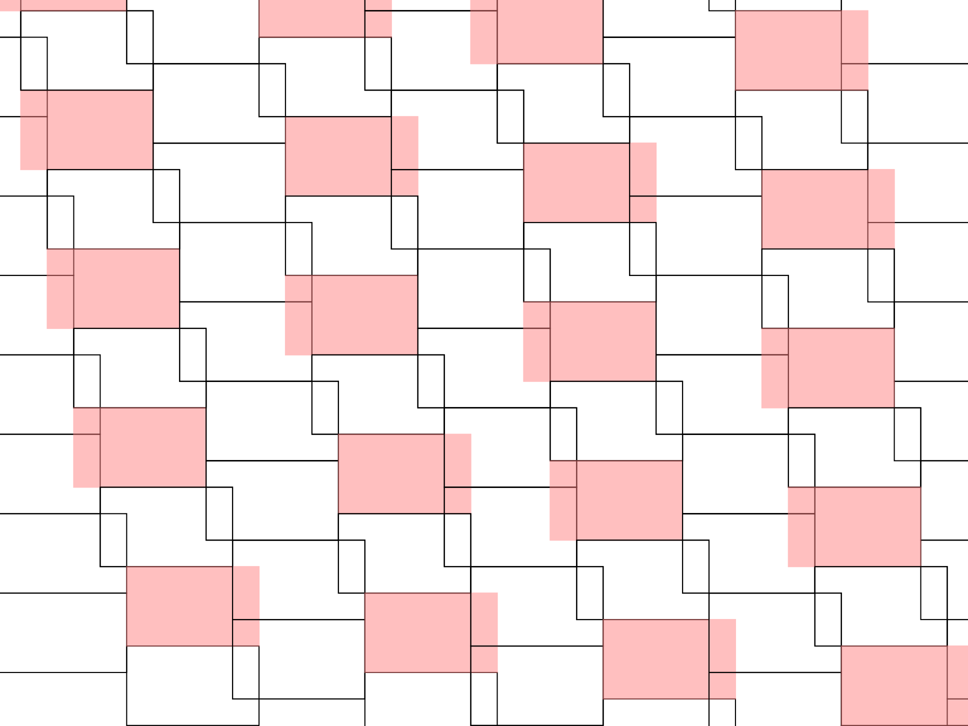

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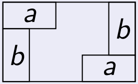
$\mathrm{Ent}(X_q) = 0$ unless q is of the form .

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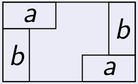
$\mathrm{Ent}(X_q) = 0$ unless q is of the form 

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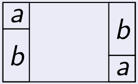
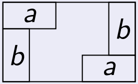
Special case

If all borders of q are in the same two corners, then $\mathrm{Ent}(X_q) = 0$.

Recap



$\Longrightarrow$ nonzero entropy

Not 

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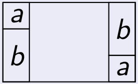
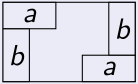


$\Longrightarrow$ nonzero entropy

Not 

$\Longrightarrow$ zero entropy

Rather frustrating. (Help appreciated.)

What's next?

What's next?

- Finish the characterization
- Exact values (or more precise bounds)
- Extend to other shapes
- Relax the definition (pre-quasiperiodic?)

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Thank you!

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