Research Project: frontiers of uncomputability of entropy for multidimensional subshifts of finite type

Silvère Gangloff

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Uncomputability of entropy (statement of the problem):

Multidimensional subshifts of finite type are sets of d-dimensional infinite words (elements of $\mathcal{A}^{\mathbb{Z}^d}$, where $d \geq 1$ and $\mathcal{A}$ is a finite set). These objects are studied not only as topological dynamical systems (in the field of symbolic dynamics), but also as statistical physics models (lattice models), or as computation models and related to decidability. An important problem related to subshifts of finite type is to compute (or understand) their entropy. This quantity is interesting from the three point of views, as a topological invariant (in the project of classification of dynamical systems), a thermodynamical quantity, and a quantity of information.

While any unidimensional subshift of finite type is described by a simple automaton, and thus entropy can be computed uniformly with an algorithm, this statement breaks down for multidimensional ones ($d \geq 2$). It was already known that there exist some multidimensional subshifts of finite type having uncomputable entropy [CHK92] (meaning that there is no algorithm which on input n outputs a rational number which is at distance less than 2^{-n} to the entropy), but a recent theorem, due to M. Hochman and T. Meyerovitch [HM10], states moreover that the possible values of the entropy of d-dimensional subshifts of finite type for any $d \geq 2$ are characterized by a computability condition: they are exactly the numbers that can be approximated from above by the outputs of an algorithm (without knowledge on the speed of convergence). The meaning of this result is that the study of these dynamical systems is intricated with computability theory.

On the other hand, the entropy of very particular simple bidimensional subshifts of finite type has been computed (exactly solvable models), such as the dimer model and the six vertex model. The exact value of the entropy of the hard cores (which consists in a subshift on $\{0, 1\}$ defined by forbidding two 1 to be neighbors) model is still not known, but is computable. This comes simply from its block gluing property [PS15] (this property states that two observable square patterns can be observed in the same element of the subshift in any relative position, provided that the distance is greater than a fixed one), but a more efficient algorithm was also proposed, based on percolation techniques [P12].

The main aim of this project is to understand the frontier between uncomputability and computability (in the exact and in the algorithmical sense) of the entropy in the class of bidimensional subshifts of finite type. Eventually, I would like to understand these systems in more dimensions, but this problem is very hard, and nothing is known.

Computational threshold phenomena:

There is no definite way to approach this frontier. A particular one is through a quantified version of the block gluing property, for which the minimal distance of gluing depends on the size of the square patterns. In this restricted context, approaching the frontier means characterizing a threshold for this minimal distance of gluing.

In a first long article [GS17a], I studied this directly for bidimensional subshifts of finite type, approaching the threshold from below, by generalising the computability of entropy to $o(\log(n))$-block gluing. Approaching from above, I proved that the possible values of the entropy do not change when restricting by linear block gluing. In particular, it answers the problem 9.1 of [HM10]. It is also interesting for other reasons, since it provides a combinatorial view on entropy transformation of subshifts of finite type through particular operators on them. Moreover, it provides solutions for embedding Turing computation (on which the result of M. Hochman and T. Meyerovitch relies) under strong dynamical restrictions, under which the construction of M. Hochman and T. Meyerovitch, like other ones of this type, breaks down on several points. For instance, I adapted these techniques in two other long articles in order to provide a characterization of the possible values of another topological invariant, the entropy dimension, under minimality restriction [GS17b], and adapt another classical result about simulation of effective systems by subshifts of finite type to this minimality restriction [GS18]. An important aspect for me of this approach is the complexification of the way Turing computations are embedded when getting near the frontier, that can be interpreted as information processing strategies under constraint. The constructed systems are thus analogically related to biological systems. I developed this analogy in my doctoral thesis.

The obstacle in fulfilling the gap from above for subshifts of finite type comes from the lack of tools for embedding Turing computations, since this relies on the classical structure of R. Robinson [Rob71], which prevents gluing at distance $o(n)$. From under, this comes from combinatorial reasons. In order to understand the frontier in this gap, I considered a more flexible class of subshifts: decidable ones. This means that there is an algorithm which given a pattern, decides if this pattern can be observed in a configuration of the system. For this class, I were able to characterize a threshold for the distance function of a quantified block gluing restriction under which the entropy becomes computable and above which the set of possible values of entropy is not affected. This threshold is formulated as a summability condition on the minimal distance function. The tools involved are combinatorial, and an important difficulty was to understand again how certain operators on subshifts change the entropy.

Robustness of computation methods:

Another strategy to approach the frontier is to consider classical exactly solvable lattice models in order to understand what are the properties of these models that make their entropy exactly computed (and in particular algorithmically computable) and extend the class of subshifts for which the entropy can be computed (in the exact and algorithmical senses), by relaxing these properties.

At first sigh, the most interesting of these model for me is the square ice (or six vertex model), since it is the conjonction of numerous properties that allow computation of the entropy through the so-called transfer matrix method. Moreover, some obstacles still prevent from a rigorous proof of the value of topological entropy. After overcoming these obstacles, my current project is to extend the computation to subshifts of finite type obtained to some particular subshifts of square ice.

References

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