Minicourse on *information, complexity and organisation in multidimensional symbolic dynamics*

On the limit between the computable and the uncomputable

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April 16, 2021

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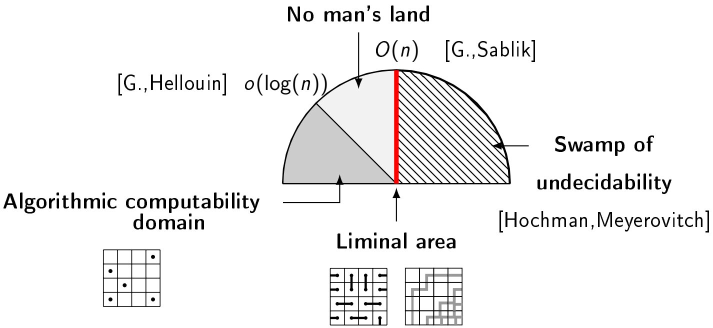
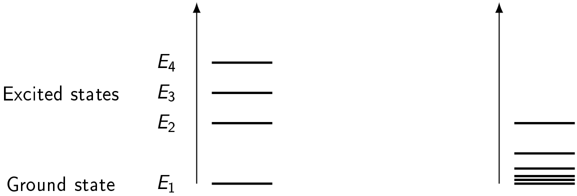
Multidimensional SFT: a computational ’transition’:

Reminder (third lecture):



Uncomputability in quantum physics:

Energy states:



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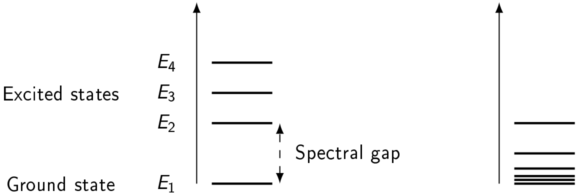

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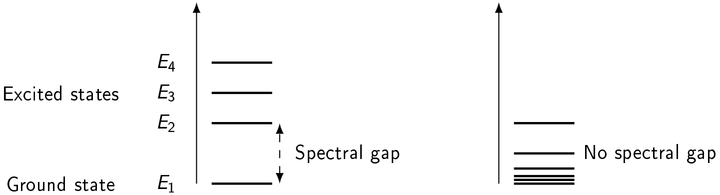

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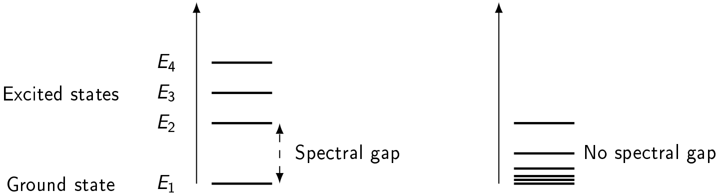

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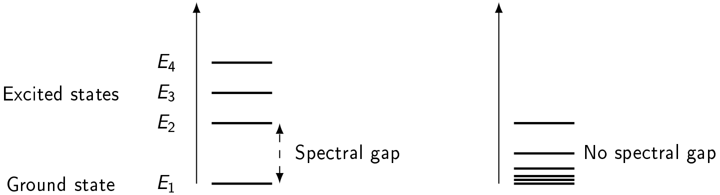
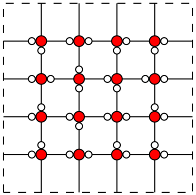


Cubitt,Perez-Garcia,Wolf(2015): The *spectral gap problem* is undecidable.

Kreinovich(2017): Why Some Physicists Are Excited About the Undecidability of the Spectral Gap Problem and Why Should We

Uncomputability in quantum physics: Exactly solvable models

Square ice model [Pauling(1935)]:



Uncomputability in quantum physics: **Exactly solvable models**

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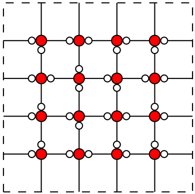

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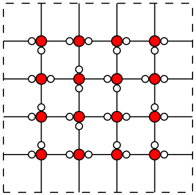

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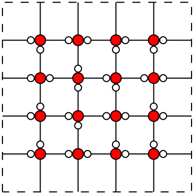

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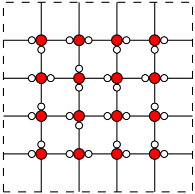

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Questions:

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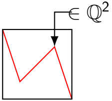
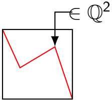
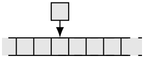
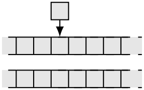
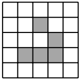
2. How does 'organisation' emerge from simple interactions between elements of matter ?
3. Are the models for which uncomputability occur physically significant ? Can we formulate a restriction which ensures computability ?

Computability of (topological) entropy:

Milnor (2002): is the *entropy* of a dynamical system effectively computable ?

Computable	Uncomputable
$[0,1]$ 	$[0,1]^4$  [Koiran 2001]
 [Jeandel 2014]	 [Delvenne Blondel 2004]
 Expansive [D'amico Manzini Margara 2003]	Non contraint [Hurd Kari Culik 1992]

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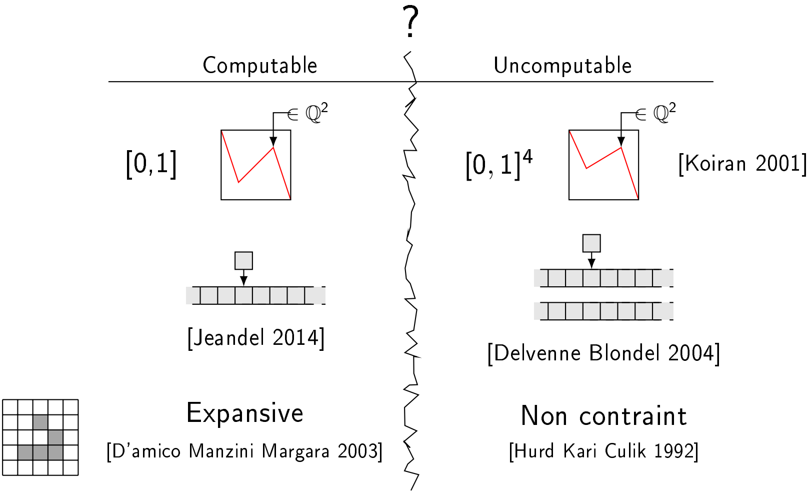


Reminders:

Alphabet $\mathcal{A}$ finite. **Patterns:**(d=1) elements of $\mathcal{A}^{\mathbb{U}}$, $\mathbb{U} \subset \mathbb{Z}$.

Subshifts(d=1): set of patterns $\mathcal{F}$.

$$X_{\mathcal{F}} = \{x \in \mathcal{A}^{\mathbb{Z}} \ : \ \forall \mathbb{U} \subset \mathbb{Z}, x_{|\mathbb{U}} \notin \mathcal{F}\}.$$

For every subshift X on alphabet $\mathcal{A}$ there exists $\mathcal{F}$ s.t. $X = X_{\mathcal{F}}$.

When $\mathcal{F}$ finite : of finite type; when $\mathcal{F}$ recursively enumerable (set of outputs of a computing machine): effective.

Reminders:

Language: $\mathcal{L}(X)$: set of patterns which appear in some $x \in X$.

Entropy(d=1): $N_n(X)$: number of words $w \in \mathcal{L}(X)$, $|w| = n$.

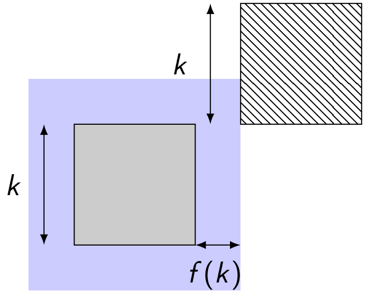
$$h(X) = \lim_{n \to +\infty} \frac{\log_2(N_n(X))}{n} \underset{\text{Th}}{=} \inf_{n \to +\infty} \frac{\log_2(N_n(X))}{n}$$

Π_1-computable: $x \in \mathbb{R}$: exists an algorithm $n \mapsto r_n$ with $r_n \downarrow x$.

Lemma: when X is effective, $h(X)$ is Π_1-computable.

Reminders:

f**-block gluing:**



When $d = 1$: square patterns $\rightarrow$ words.

Decidable subshifts: caracterization of a threshold

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$\boxed{\text{Algorithm}}$

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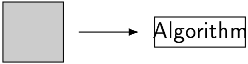

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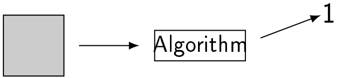

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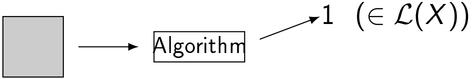

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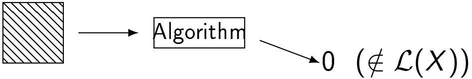

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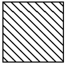 $\longrightarrow$ $\boxed{\text{Algorithm}}$ $\longrightarrow$ $0 \quad (\notin \mathcal{L}(X))$

Assume $\Sigma(f) \underset{Def}{=} \sum_{n} \frac{f(k)}{k^2}$ computable.

Decidable subshifts: caracterization of a threshold

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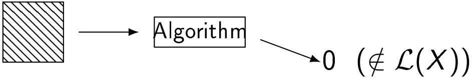

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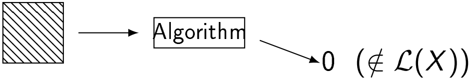

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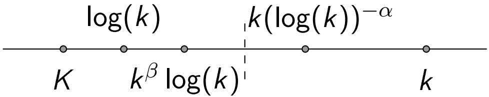

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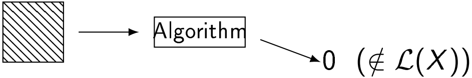

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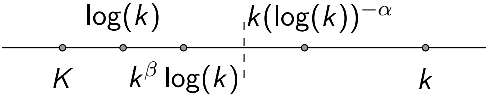

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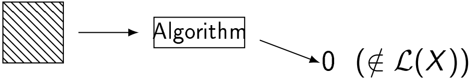

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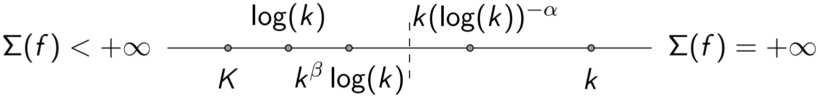

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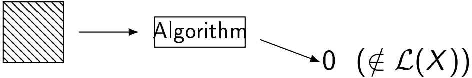

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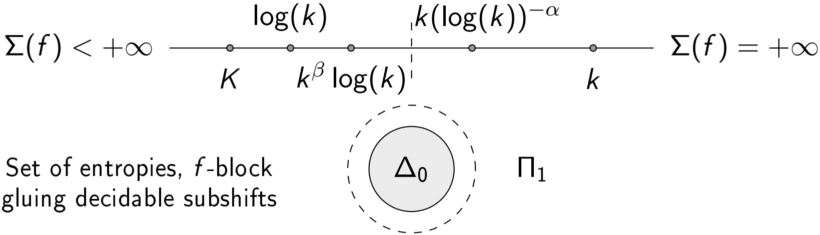

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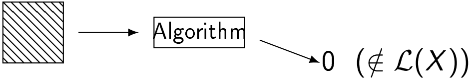

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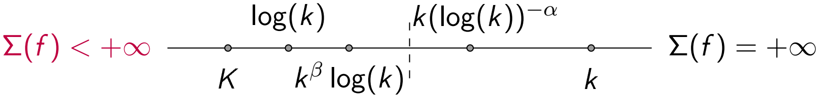

Set of entropies, f-block
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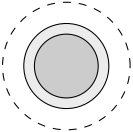

Π_1

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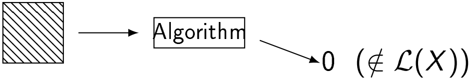

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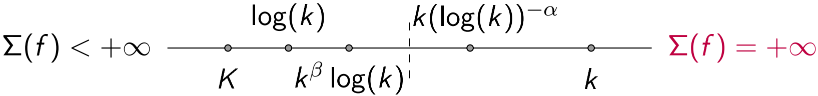

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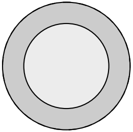

Π_1

Below the threshold : $\Sigma(f) < +\infty \Rightarrow h$ computable

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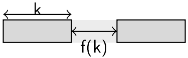

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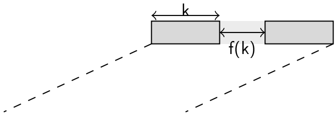

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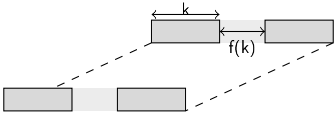

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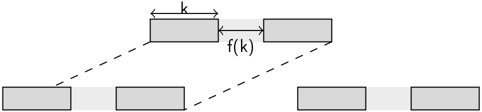

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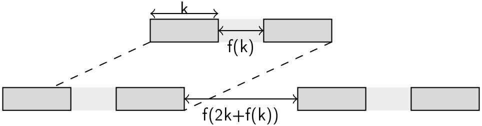

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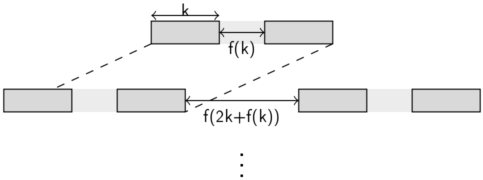

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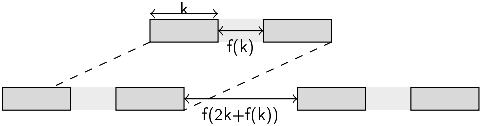

$$\vdots$$

$$\frac{\log(N_k(X))}{k} - |\mathcal{A}| \cdot \sum_{k}^{+\infty} \frac{f(2^l)}{2^l} \leq h \leq \frac{\log(N_k(X))}{k}$$

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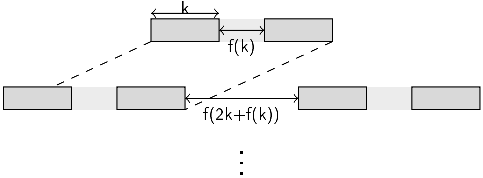

$$\frac{\log(N_k(X))}{k} - |\mathcal{A}| \cdot \sum_{k}^{+\infty} \frac{f(2^l)}{2^l} \leq h \leq \frac{\log(N_k(X))}{k}$$

Since X is decidable, $k \mapsto N_k(X)$ is computable, hence h is computable.

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Known: obstruction $\rightarrow$ let us prove realization

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Bounded density shifts.

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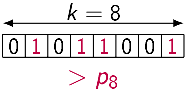

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$$\begin{array}{c} \overset{k=8}{\longleftrightarrow} \\ \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0&1&0&1&1&0&0&1\\\hline\end{array} \\ > p_8 = 3 \end{array}$$

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Bounded density shifts. Consider $(p_k)_k \in \mathbb{N}^{\mathbb{N}}$ non-decreasing and computable. Define $\mathcal{F}$ as set of:

$$\overset{k = 8}{\longleftrightarrow}$$

0	1	0	1	1	0	0	1

$$> p_8 = 3$$

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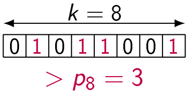

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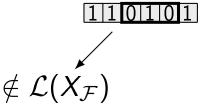

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$$\in \mathcal{L}(X_{\mathcal{F}})$$

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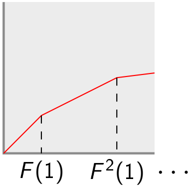

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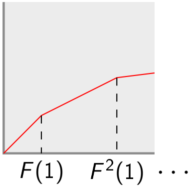

f-block gluing $\Leftrightarrow \forall n,\ p_{F(n)} \geq 2p_n + 4$

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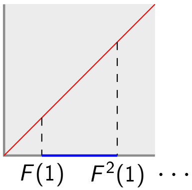

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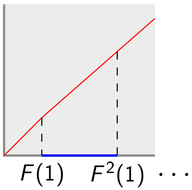

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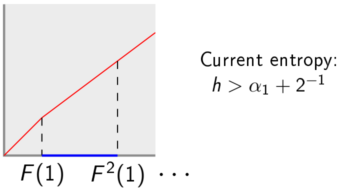

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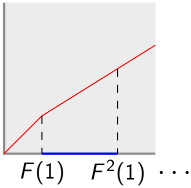

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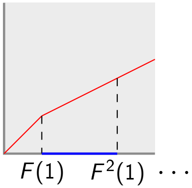

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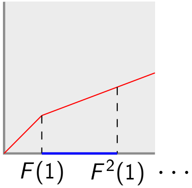

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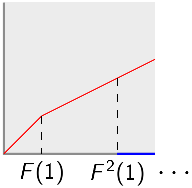

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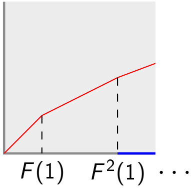

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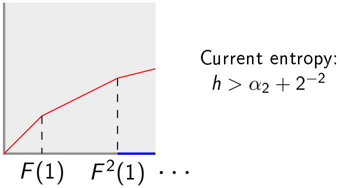

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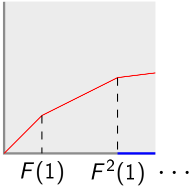

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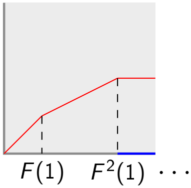

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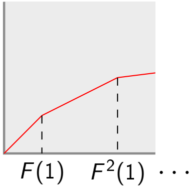

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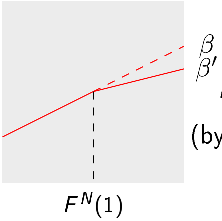
Above the threshold III. Tracking tool: $\Sigma(f) = +\infty$

Entropy change: $\beta = (\beta_1, \beta_2, ..)$ slopes:



$0 \geq \Delta h \geq -H(1/F^N(1))$

$H(\epsilon) = \epsilon \log(\epsilon) + (1-\epsilon)\log(1-\epsilon)$

(by bounding preimages of a transformation)

Above the threshold III. Tracking: $\Sigma(f) = +\infty$

Let us assume that $h_\infty < \alpha$.

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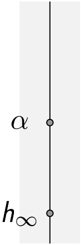

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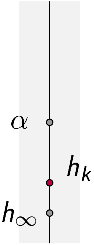

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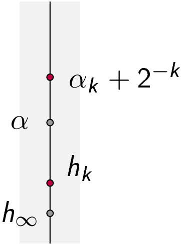

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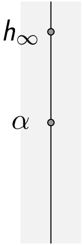

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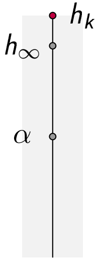

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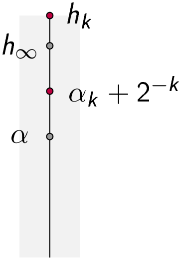

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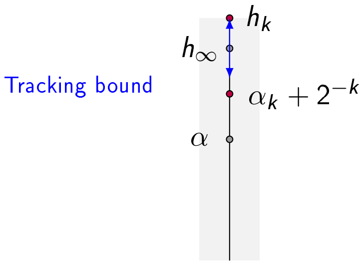

Block gluing condition is minimal:

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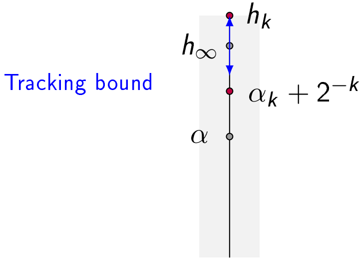

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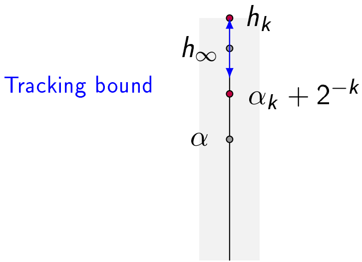

Block gluing condition is minimal: $\forall l \geq k$, $p_{F(l)} = 2p_l + 4$.

Repeat + limit:

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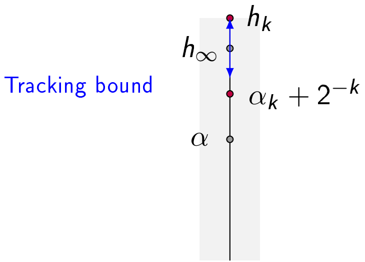

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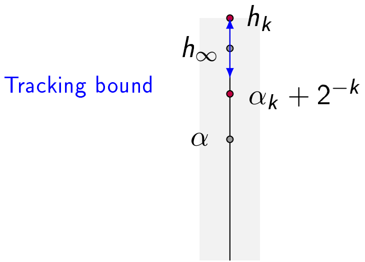

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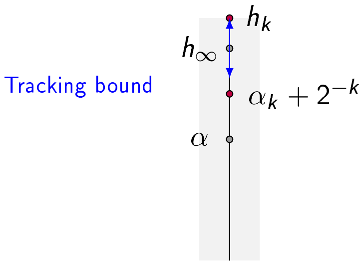

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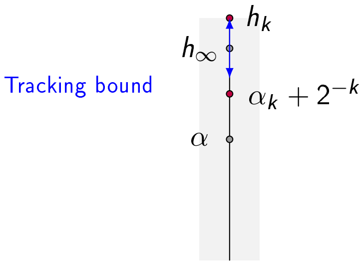
$\Sigma(f) = +\infty$: $\inf \frac{p_l}{l} = h_\infty = 0$

Above the threshold III. Tracking: $\Sigma(f) = +\infty$

Let us assume that $h_\infty > \alpha$.



Block gluing condition is minimal: $\forall l \geq k$, $p_{F(l)} = 2p_l + 4$.

Repeat + limit: $\inf \frac{p_l}{l} \cdot \Sigma(f) < +\infty$.

$\Sigma(f) = +\infty$: $\inf \frac{p_l}{l} = h_\infty = 0 \rightarrow$ contradiction.

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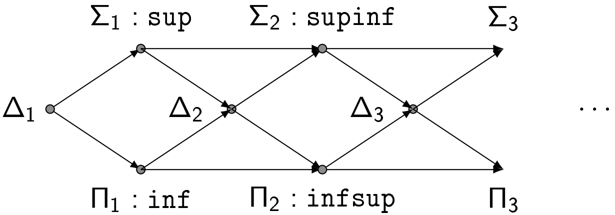
3. For other classes of dynamical systems ? [$\rightarrow$ better understanding of the threshold phenomenon]

Computability in general:

X.Zheng, K.Weihrauch: The arithmetical hierarchy of real numbers.



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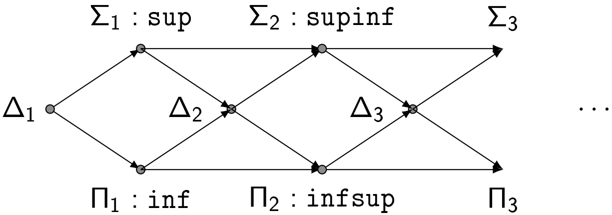

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Theorem: for all m, $\Sigma_m \subsetneq \Delta_{m+1}$, $\Pi_m \subsetneq \Delta_{m+1}$, $\Delta_m \subsetneq \Sigma_m$, $\Delta_m \subsetneq \Pi_m$.

General metric dynamics:

Question: Classification of classes of dynamical systems according to possible values of entropy ?

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Theorem[G.,Herrera,Rojas,Sablik(2019)]: the entropy of a topological **computable** dynamical system (X, f) is Σ_2-computable.

If in the arithmetical hierarchy, possible classes are: Δ_1, Σ_1, Π_1, Δ_2, Σ_2.

Definition of computable dynamical system

A topological dynamical system is some (X, f), where X is 'computable' and f is 'computable'.

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Computable metric space: $(X, d, \mathcal{S})$ with (X, d) metric space, $\mathcal{S} = \{s_i \ : \ i \geq 0\}$ a countable dense subset of X (**ideal points**), s.t. exists an algorithm which on input (i, j, n) outputs $r \in \mathbb{Q}$ s.t.:

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2. $\mathcal{A}^{\mathbb{N}}$, $\# \in \mathcal{A}$, $\mathcal{S} = \{w \cdot \#^{\infty} \ : |w| < +\infty\}$,

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Definition: A function $f : X \to X$ is **computable** when there exists an algorithm which on input m enumerates $I_m \subset \mathbb{N}$ such that

$$f^{-1}(B_m) = \bigcup_{n \in I_m} B_n,$$

Characterization of computable functions on Cantor sets

Lemma: a function $\mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ is computable when there exists a non-decreasing computable function $\varphi : \mathbb{N} \to \mathbb{N}$ and an algorithm which provided as input the $\varphi(n)$ first elements of some $x \in \mathcal{A}^{\mathbb{N}}$ outputs the n first elements of $f(x)$.

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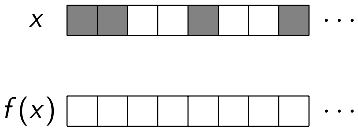

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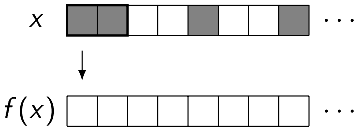

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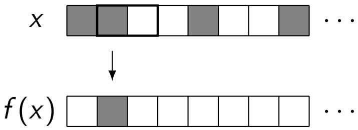

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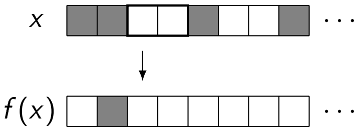

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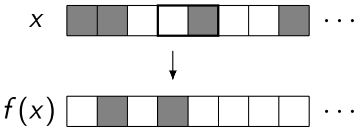

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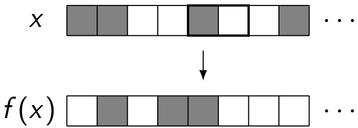

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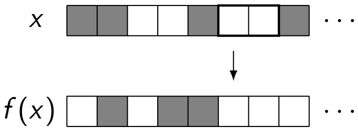

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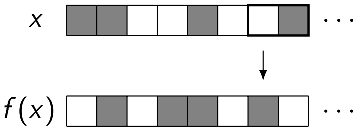

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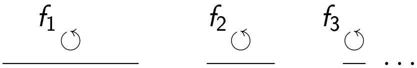

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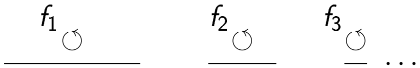

5. f surjective with entropy $\sup_n h_n$.

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Interval maps: realization

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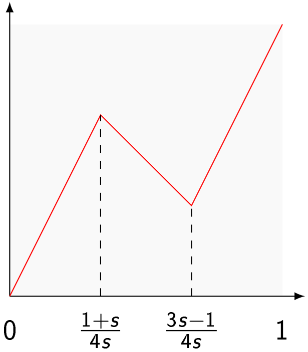

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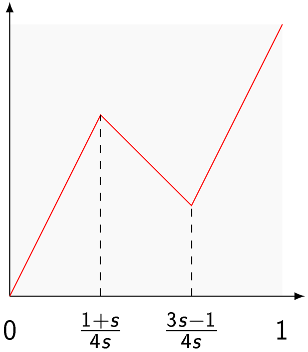

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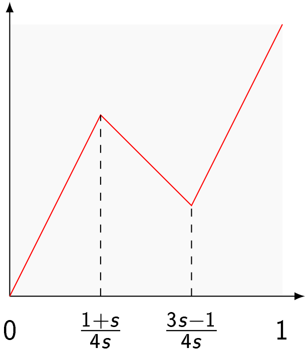

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Thus for all $s \in \mathbb{Q}$, the entropy of $([0,1], f_s)$ is s.

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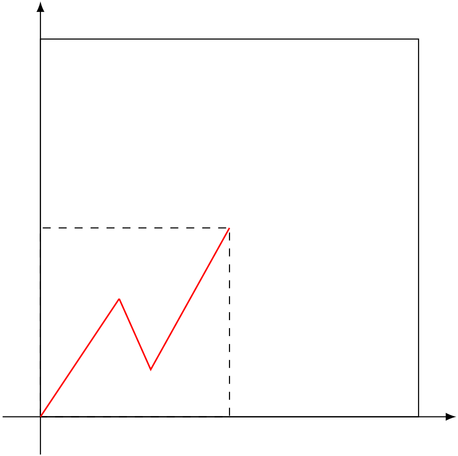

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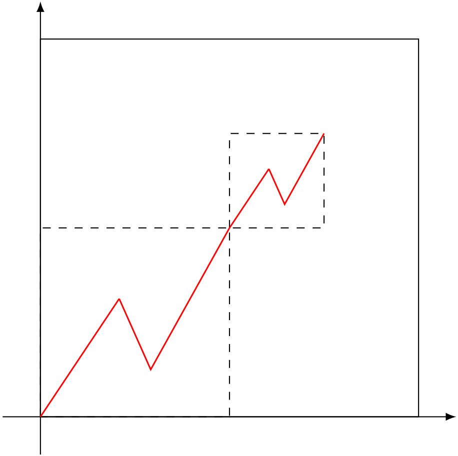

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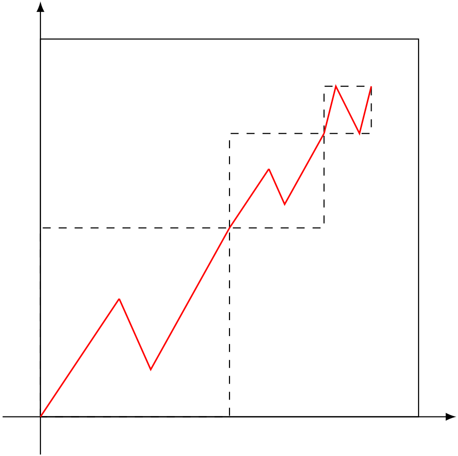

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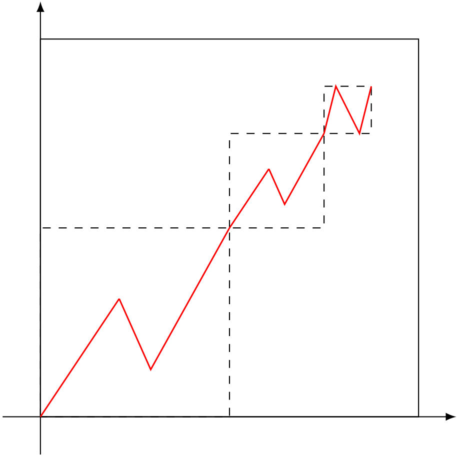

Computable map, entropy s.

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3. Do you have other ideas of classes of systems and dynamical constraints ?