On exact computation of square ice entropy
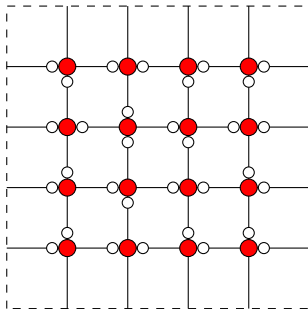
Silvère Gangloff

LIP, ENS Lyon
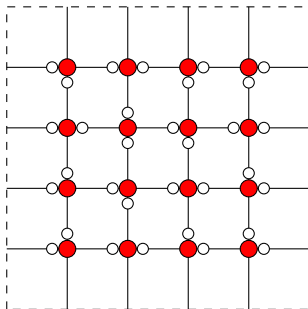
March 11, 2019

I. Representations of square ice
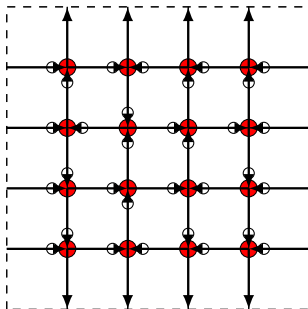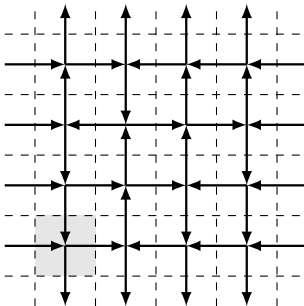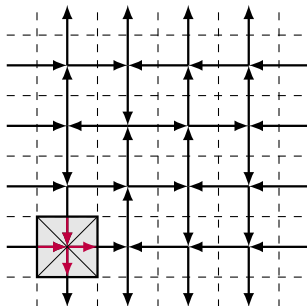
Square ice model [Pauling-Lieb]:

Wang tiles representation [Six-vertex model]

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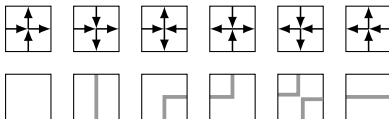**Wang tiles representation [Six-vertex model]**

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Discrete curves subshift $[X^s]$:

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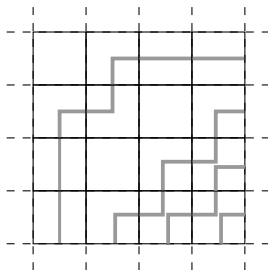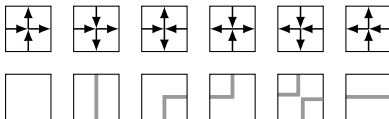
Example of apparition in symbolic dynamics:

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S. Gangloff, *A proof that square ice entropy is $\frac{3}{2}\log_2(4/3)$*, 2019.

II. Subshifts of finite type and entropy

SFT (subshift of finite type): subset of $\mathcal{A}^{\mathbb{Z}^2}$, defined by a finite set of **forbidden patterns**.

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Forbidden patterns $\begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline\end{array}$ et $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}$.

SFT (subshift of finite type): subset of $\mathcal{A}^{\mathbb{Z}^2}$, defined by a finite set of **forbidden patterns**.

Ex: **Hard square shift**, or hard core model.

Forbidden patterns $\begin{smallmatrix}1\\1\end{smallmatrix}$ et $\boxed{1\,|\,1}$.

0	0	0	0	0
0	0	0	1	0
0	0	0	0	1
0	0	0	1	0
1	0	0	0	0

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Ex: **Hard square shift**, or hard core model.

Forbidden patterns $\boxed{\begin{array}{c}1\\1\end{array}}$ et $\boxed{1\,|\,1}$.

0	0	0	0	0
0	0	0	1	0
0	0	0	0	1
1	0	0	1	0
0	0	0	0	0

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0	0	0	0	0
0	0	0	1	0
0	0	0	0	1
1	0	1	0	0
0	0	0	0	0

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Ex: **Hard square shift**, or hard core model.

Forbidden patterns $\boxed{\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}}$ et $\boxed{1 \mid 1}$.

0	0	0	0	0
0	0	0	0	0
0	0	0	1	1
1	0	1	0	0
0	0	0	0	0

oops

SFT (subshift of finite type): subset of $\mathcal{A}^{\mathbb{Z}^2}$, defined by a finite set of **forbidden patterns**.

Ex: **Hard square shift**, or hard core model.

Forbidden patterns $\boxed{\begin{matrix} 1 \\ 1 \end{matrix}}$ et $\boxed{1 \mid 1}$.

0	0	0	0	0
0	0	0	1	0
0	0	0	0	1
1	0	1	0	0
0	0	0	0	0

Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles

0	0
0	0

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$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles

0	0
1	0

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Free tiles

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0	1

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Free tiles

1	1
1	1

Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles

$$\begin{array}{|c|c|}
\hline
1 & 1 \\
\hline
1 & 1 \\
\hline
\end{array}$$

$$\mathcal{N}_2(X) = 2^{2^2}$$

Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles

$$\begin{array}{|c|c|}\hline 1 & 1 \\\hline 1 & 1 \\\hline\end{array}$$

$$\mathcal{N}_2(X) = 2^{2^2}$$
$$\mathcal{N}_N(X) = 2^{N^2}$$

Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array}$$

$$\mathcal{N}_2(X) = 2^{2^2}$$
$$\mathcal{N}_N(X) = 2^{N^2}$$

Hard core

$$\begin{array}{|c|c|} \hline 0 & 0 \\ \hline 0 & 0 \\ \hline \end{array}$$

Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array}$$

Hard core

$$\begin{array}{|c|c|} \hline 0 & 0 \\ \hline 1 & 0 \\ \hline \end{array}$$

$$\mathcal{N}_2(X) = 2^{2^2}$$
$$\mathcal{N}_N(X) = 2^{N^2}$$

Entropy: "quantity of possible states of the system".

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Free tiles	**Hard core**

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$$

$$\mathcal{N}_2(X) = 2^{2^2}$$
$$\mathcal{N}_N(X) = 2^{N^2}$$

Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles

1	1
1	1

$$\mathcal{N}_2(X) = 2^{2^2}$$
$$\mathcal{N}_N(X) = 2^{N^2}$$

Hard core

1	0
0	0

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Free tiles

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array}$$

Hard core

$$\begin{array}{|c|c|} \hline 0 & 1 \\ \hline 0 & 0 \\ \hline \end{array}$$

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1	1

Hard core

0	1
1	0

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Free tiles

1	1
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Hard core

1	0
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Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array}$$

$$\mathcal{N}_2(X) = 2^{2^2}$$
$$\mathcal{N}_N(X) = 2^{N^2}$$

Hard core

$$\begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$$

$$\mathcal{N}_2(X) = 7$$

Entropy: "quantity of possible states of the system".

$\mathcal{N}_N(X)$: number of size N square patterns observable in the system.

Free tiles	Hard core

$$\mathcal{N}_2(X) = 2^{2^2} \qquad \mathcal{N}_2(X) = 7$$
$$\mathcal{N}_N(X) = 2^{N^2} \qquad \mathcal{N}_N(X) = 2^{N^2(h(X)+o(1))}$$

Entropy of a SFT X:

$$h(X) = \inf_N \frac{\log_2(\mathcal{N}_N(X))}{N^2}$$

Free tiles

$h = 1$

Hard core

$h \geq 1/2$

Square ice [Lieb 67]

$h = \frac{3}{2} \log_2(4/3)$

Entropy of a SFT X:

$$h(X) = \inf_N \frac{\log_2(\mathcal{N}_N(X))}{N^2} = \inf_N \frac{\log_2(\mathcal{N}_N^{loc}(X))}{N^2}$$

Free tiles   **Hard core**   **Square ice [Lieb 67]**

$h = 1$ $h \geq 1/2$ $h = \frac{3}{2}\log_2(4/3)$

Entropy of a SFT X:

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Free tiles	**Hard core**	**Square ice [Lieb 67]**
$h = 1$	$h \geq 1/2$	$h = \frac{3}{2}\log_2(4/3)$

Computable from above:

Entropy of a SFT X:

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Free tiles   **Hard core**   **Square ice [Lieb 67]**
$h = 1$ $h \geq 1/2$ $h = \frac{3}{2}\log_2(4/3)$

Computable from above:

$$N = 4 \longrightarrow \boxed{\text{Algorithm}} \longrightarrow \begin{array}{c} h(X) \\ \overline{||} \\ r_4 \end{array}$$

II. Lieb transfer matrices approach

Entropy of square ice:

$$h(X^s) = \lim_{M,N} \frac{\log_2(\mathcal{N}_{M,N}(X^s))}{MN}.$$

Stripes subshifts:

$$h(X^s) = \lim_N \frac{h(X_N^s)}{N}$$

X^s

X_N^s
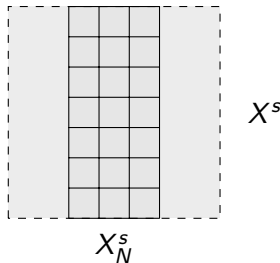
Cylindric stripes subshifts:
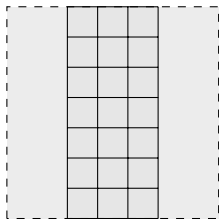
$$X_N^s \qquad \overline{X}_N^s$$

Cylindric stripes subshifts:

X_N^s
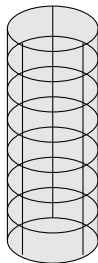
$\overline{X}_N^s$
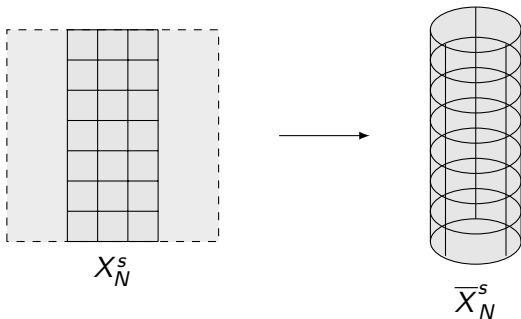
Symmetry properties of square ice imply:

$$h(X^s) = \lim_N \frac{h(\overline{X}_N^s)}{N}$$

Symmetry properties of square ice:
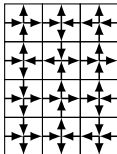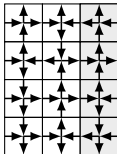
Symmetry properties of square ice:

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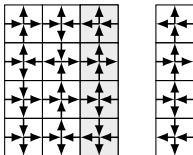
Symmetry properties of square ice:
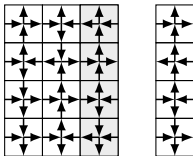
Symmetry properties of square ice:
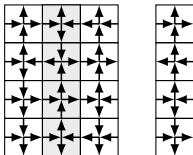
Symmetry properties of square ice:
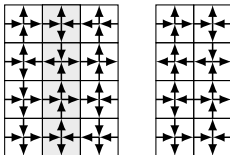
Symmetry properties of square ice:
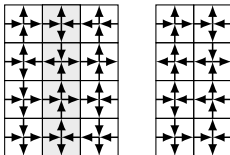
Symmetry properties of square ice:
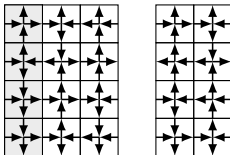
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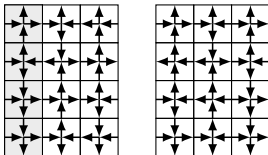
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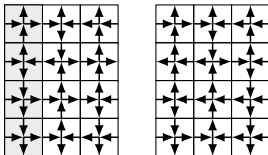
Symmetry properties of square ice:
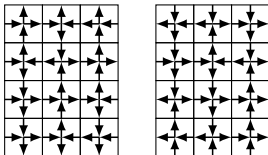
Symmetry properties of square ice:

Lieb transfer matrices:

$V_N(t)$

Lieb transfer matrices:

Lieb transfer matrices:

$V_N(t)$

$\equiv$

v
w
u

$\mathcal{R}[w]$
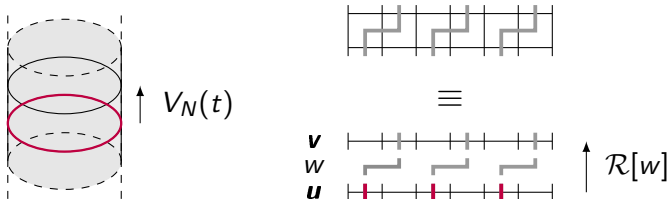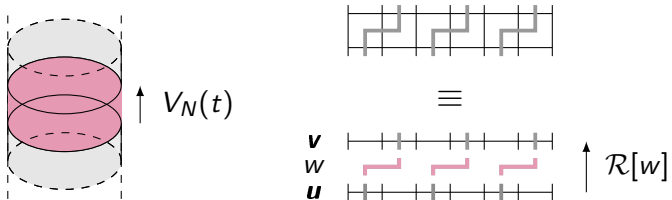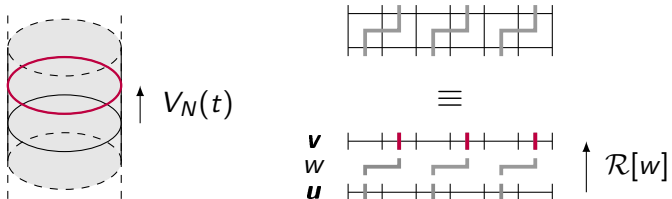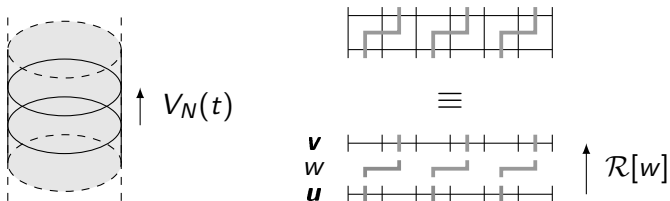
Lieb transfer matrices:

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Lieb transfer matrices:

Lieb transfer matrices:

$$V_N(t)[\boldsymbol{u}, \boldsymbol{v}] = \sum_{\boldsymbol{u}\mathcal{R}[w]\boldsymbol{v}} t^{|w|}.$$

where $|w| = \#$ of ⌐ and ⌐

Lieb transfer matrices:
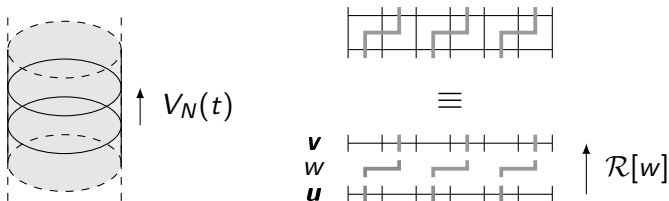
$$V_N(t)[\boldsymbol{u}, \boldsymbol{v}] = \sum_{\boldsymbol{u}\mathcal{R}[w]\boldsymbol{v}} t^{|w|}.$$

where $|w| = \#$ of ⬜ and ⬛

$$h(X^s) = \lim_N \frac{\log_2(\lambda_{\max}(V_N(1)))}{N}$$

Computing maximal eigenvalue of $V_N(1)$, strategy:

$V_N(1)$ •

Computing maximal eigenvalue of $V_N(1)$, strategy:
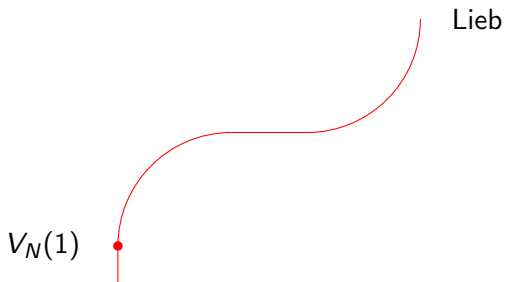
Lieb

$V_N(1)$

Computing maximal eigenvalue of $V_N(1)$, strategy:

Lieb

$V_N(t)$

$V_N(1)$
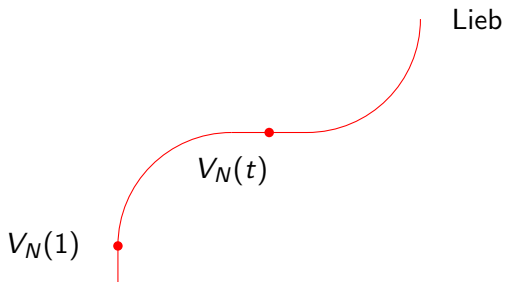
Computing maximal eigenvalue of $V_N(1)$, strategy:
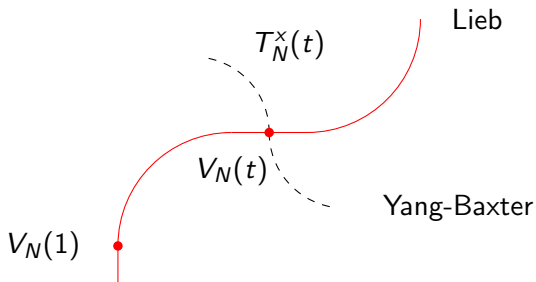
Computing maximal eigenvalue of $V_N(1)$, strategy:
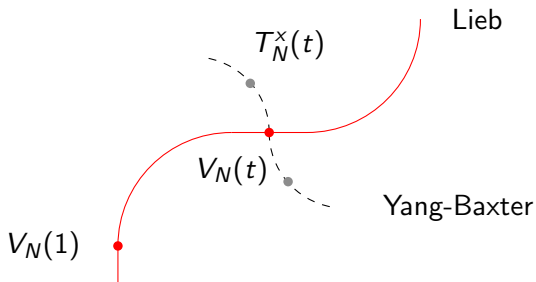
Computing maximal eigenvalue of $V_N(1)$, strategy:
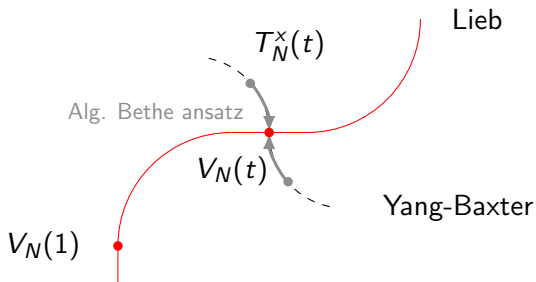
Computing maximal eigenvalue of $V_N(1)$, strategy:
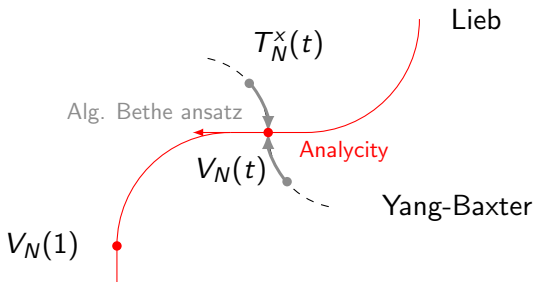
III. Yang-Baxter transfer matrices

R-matrices and monodromy matrices:

R-matrices and monodromy matrices: $\rightarrow$: input/output:

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R-matrices and monodromy matrices: $\rightarrow$: input/output:

$$R(0,1) = \begin{pmatrix} & \\ & \end{pmatrix}$$

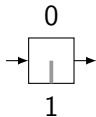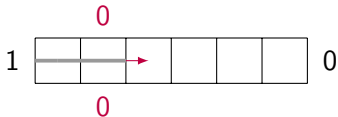**R-matrices and monodromy matrices:** $\rightarrow$: input/output:

$$0$$

$$1$$

$$R(0,1) = \begin{pmatrix} 0 & \\ & \end{pmatrix}$$

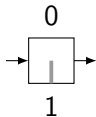**R-matrices and monodromy matrices:** $\to$: input/output:

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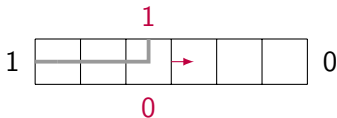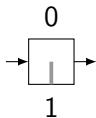**R-matrices and monodromy matrices:** $\rightarrow$: input/output:

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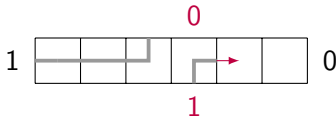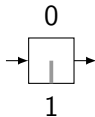**R-matrices and monodromy matrices:** $\rightarrow$: input/output:

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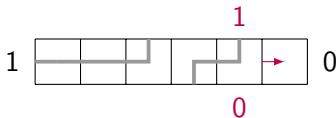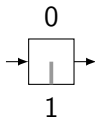**R-matrices and monodromy matrices:** $\to$: input/output:

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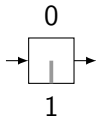**R-matrices and monodromy matrices:** $\rightarrow$: input/output:

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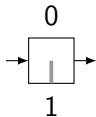**R-matrices and monodromy matrices:** $\rightarrow$: input/output:

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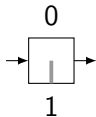**R-matrices and monodromy matrices:** $\rightarrow$: input/output:

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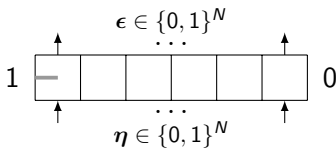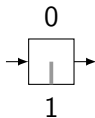**R-matrices and monodromy matrices:** $\rightarrow$: input/output:
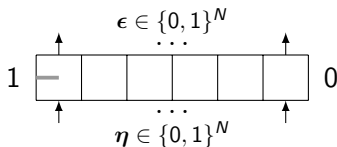

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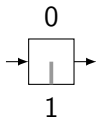**R-matrices and monodromy matrices:** $\rightarrow$: input/output:

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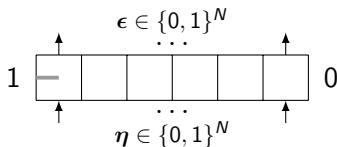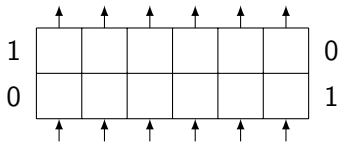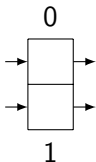$$M_N(1,0)[\epsilon, \eta]$$

Yang-Baxter transfer matrices:

$$T_N[\epsilon, \eta] = \sum_{u \in \{0,1\}} M_N(u, u)[\epsilon, \eta].$$

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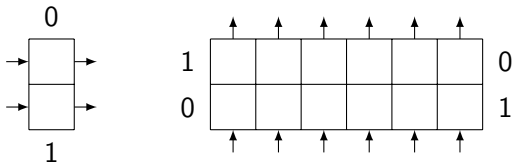
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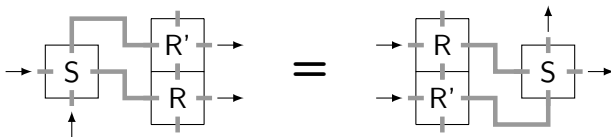
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IV. Asymptotics

Asymptotics of counting functions:

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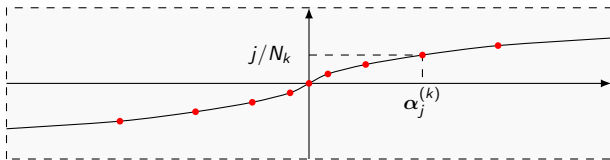
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$$\xi_t^{(k)} : \alpha \mapsto \frac{1}{2\pi}\kappa_t(\alpha) + \frac{n_k+1}{2N_k} + \frac{1}{2\pi N_k}\sum_{j=1}^{n_k}\theta_t(\alpha, \boldsymbol{\alpha}_j^{(k)})$$
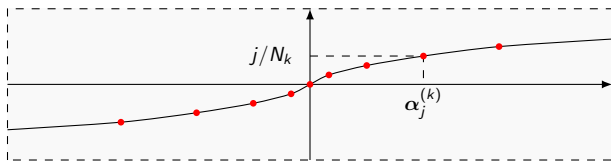
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$$\lim_N \frac{\log_2(\lambda_{\max}(V_N(1)))}{N} = \lim_k \frac{1}{N_k}\sum_{j=1}^{n_k} f(\boldsymbol{\alpha}_j^{(k)}) = \int_{\mathbb{R}} f(\alpha)\rho_t(\alpha)d\alpha.$$

Strategy:

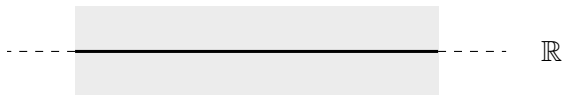1. Extend $\xi_t^{(k)}$ on a stripe including $\mathbb{R}$:

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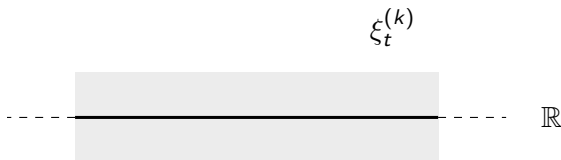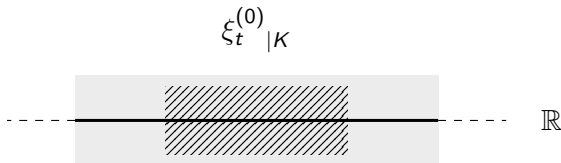
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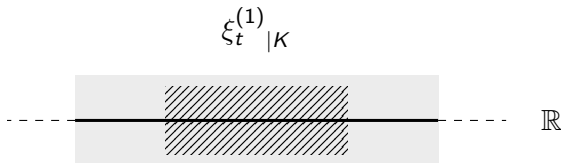① Extend $\xi_t^{(k)}$ on a stripe including $\mathbb{R}$:

Strategy:

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$$\xi_t^{(0)}{}_{|K}$$

$\mathbb{R}$

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$\mathbb{R}$

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$\mathbb{R}$

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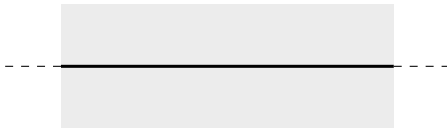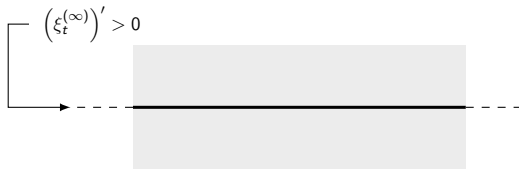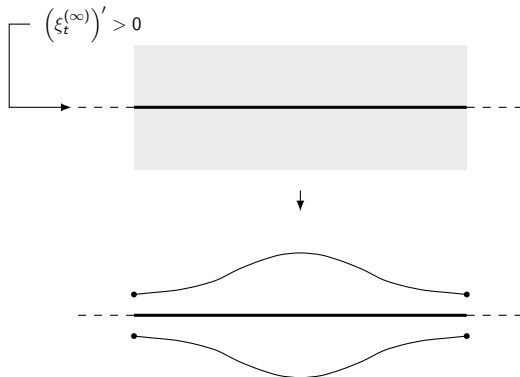4. Thus, $\xi_t^{(k)} \to \xi_t^{(\infty)}$.

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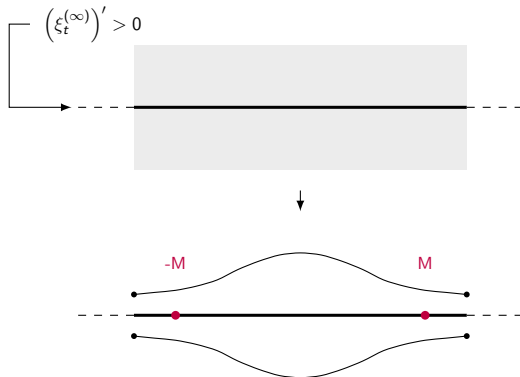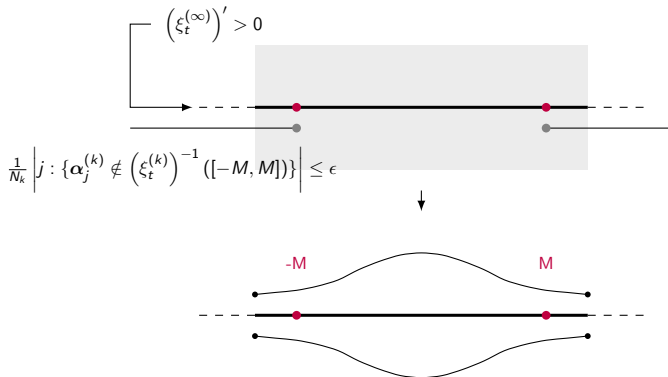

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$\frac{1}{N_k} \left| j : \{\boldsymbol{\alpha}_j^{(k)} \notin \left(\xi_t^{(k)}\right)^{-1} ([-M, M])\} \right| \leq \epsilon$
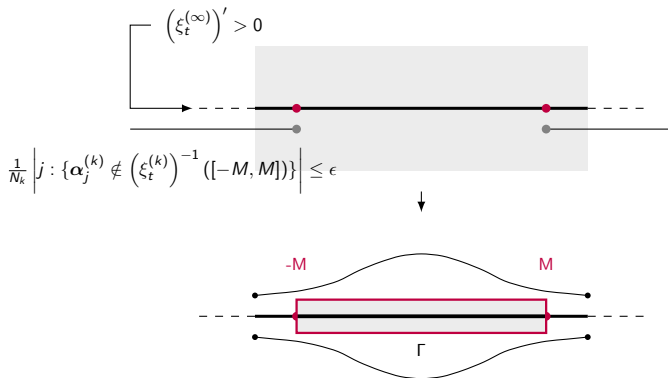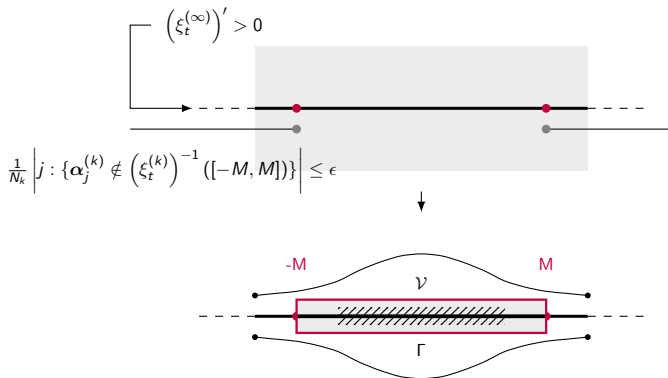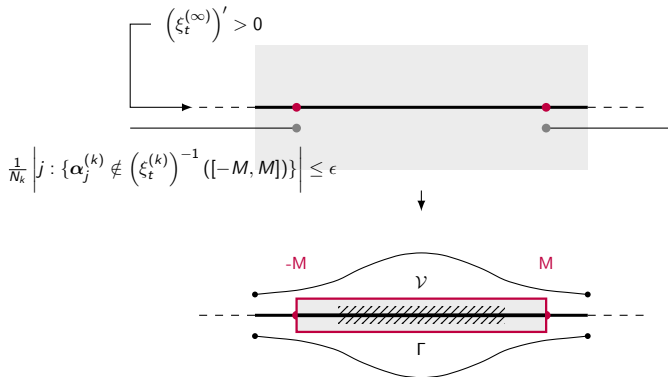
-M   M

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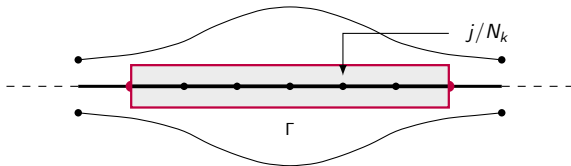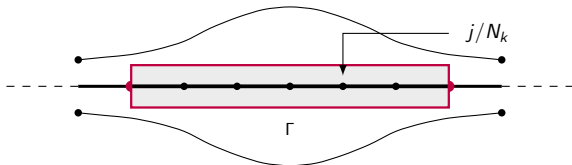**Rarefaction of the roots and $\xi_t^{(k)}$ biholomorphisms:** $\epsilon > 0$:

The functions have distinct values on $\mathcal{V}$ and Γ. Thus they are bihilomorphisms onto $\mathcal{V}$.

Lace integral expression of $\xi_t^{(k)}$:

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By residues theorem:

$$\xi_t^{(k)}(\alpha) = \frac{1}{2\pi}\kappa_t(\alpha) + \frac{n_k+1}{2N_k} + \oint_\Gamma \theta_t\left(\left(\xi_t^{(k)}\right)^{-1}(\alpha)\right)\frac{e^{2i\pi sN_k}}{e^{2i\pi sN_k}-1}ds + O(\epsilon).$$

Fredholm integral equation: Limit and change of variable:

$$\xi_t^{(\infty)}(\alpha) = \frac{1}{2\pi}\kappa_t(\alpha) + \frac{1}{4} + \int_0^{+\infty} \theta_t(\alpha) \left(\xi_t^{(\infty)}\right)'(\alpha)d\alpha.$$

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Solution by Fourier transforms.

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$$h(X^s) = \int_{\mathbb{R}} \log_2(2|\sin(\kappa_t(\alpha))/2|).\rho_t(\alpha)d\alpha.$$
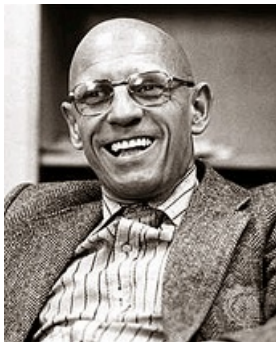
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Through an expression of $\rho_t = \left(\xi_t^{(\infty)}\right)'$ and lace integrals computations:

$$\boxed{h(X^s) = \frac{3}{2}\log_2(4/3)}.$$

V. Comments

Why mathematical physics are hard to read for mathematicians?

Archaeology of Knowledge,1969

Concept of *discursive formation*

Mathematics and mathematical physics are distinct discursive formations ; different conceptions of units of meaning, etc.

Further research:

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1 Extensions: eight-vertex model [Baxter], dimer model [Lieb]..

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3 Transformation of entropy by subshifts operators ?