Research Project: exploring computational threshold phenomena

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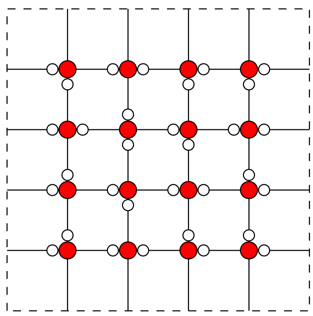
Background material

Subshifts are sets of colorings of the infinite grid $\mathbb{Z}^k$ with colors in a finite set $\mathcal{A}$. These objects appeared in dynamical systems theory as symbolic codings of dynamical systems, according to a topological partition for instance, but also in statistical mechanics where they are called **lattice models**, and as computation models in information theories.

A **pattern** p on shape $\mathbb{U} \subset \mathbb{Z}^k$ is an element of $\mathcal{A}^{\mathbb{U}}$ and it appears in an element (called configuration) of a subshift X when the restriction of this configuration to some $\mathbf{v} + \mathbb{U}$ is p. A subshift X is **decidable** when there is an algorithm that decides if a pattern on alphabet $\mathcal{A}$ appears in a configuration of X or not. It is said to be **of finite type** when it is defined by forbidding a finite set of patterns on shape $\mathbb{U}$ finite.

A widely studied **example** of subshift of finite type is the **square ice**. An example of representation of a configuration in this subshift is as follows:



The elements of the alphabet corresponds to local configurations of molecules, and forbidden patterns to interactions between them.

For a subshift X, we denote $N_n(X)$ the number of patterns on $[\![1, n]\!]^k$ that appear in a configuration of X. The **entropy of** X is

$$h(X) = \lim_n \frac{\log_2(N_n(X))}{n^k}.$$

This invariant appeared in dynamical systems to be a tool in the project of classification of systems. This quantity corresponds to the thermodynamics entropy, and to a quantity of information in information theories.

We say that a real number x is **computable** when there is an algorithm that on input n outputs a rational number q_n such that $|x - q_n| \leq 2^{-n}$. It is said to be **right recursively computable** when there is an algorithm which on input n outputs q_n such that $\inf_n q_n = x$.

State of the art

1. **Intertwinement of dynamics and computability**

   The intertwinement between dynamical systems theory and computability theory, and more widely computation theories, has been developped during the last decades. A particularly expressive result in this direction was the characterization of entropies of multidimensional subshifts of finite type, by M. Hochman and T. Meyerovitch [HM10], with a simple recursion-theoretic criterion: they are exactly the non-negative right recursively computable real numbers. In particular, entropy is algorithmically uncomputable on this class of systems, meaning that there is no algorithm that allows to approximate this number up to arbitrary precision.

2. **Towards an understanding between the computable and uncomputable worlds**

   More generally, results of this field typically state the uncomputability of a dynamical quantity over a large enough class of systems, or the computability on more restricted classes, often defined by dynamical restrictions. However, a gap always remains between computable and uncomputable regimes. The general **long-term aim of this project** is to explore various aspects of the frontier between computable and uncomputable worlds in dynamics.

I. An exact computational threshold phenomenon

There is no definite way to approach the frontier. A particular one is through a quantified version of mixing property, that we introduced: a subshift X is **f-mixing** when for any couple of patterns on $[\![1,n]\!]^k$, whenever put on the grid with relative distance $\geq f(n)$, they can be completed in a configuration of X. This is a generalisation of a notion introduced in [PS15].

1. **Saillant result [GH18] : Exhibition of a sharp computational threshold phenomenon**

   When the gap function f verifies

   $$\sum_n \frac{f(2^n)}{2^n} < +\infty,$$

   the entropy of a f-mixing decidable subshift is computable. When this sum is infinite, the possible entropies are the non-negative right recursively computable numbers.

2. **New perspectives**

   To our knowledge, this is the first time that one is able to characterise a threshold (no gap) between a computability regime and a (general) uncomputability one. We hope that this type of result could be found for other types of dynamical systems.

3. **Research direction: what happens on the frontier?**

   (a) **Potential benefits of interactions** with other theories about thresholds phenomena. For instance: percolation theory.

   (b) **Example of exploratory question**

   Can we make more precise the statement about computability of entropy when one get closer and closer to the frontier? [Example: computability speed].

   (c) **A more precise question**

   Characterisation of the entropies of f-mixing subshifts with $\sum f(2^n)/2^n < +\infty$?

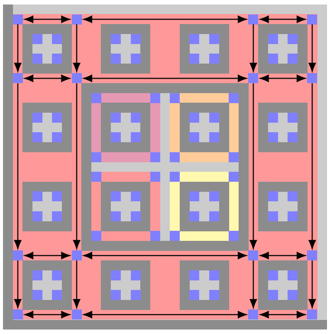
4. **Research direction: other types of dynamical systems?**

(a) **Formalisation work done**
Development of computability notions for topological dynamical systems. General computability properties of entropy, and some computability restriction under dynamical constraints.

(b) **Specific systems: on the interval** A characterization of entropies with computability criterion, and characterization for systems under constraints. Similar quantified constraints as f-mixing ?

(c) **Potential interactions with complex dynamics:** example of work done in this direction: computability and uncomputability of Julia sets.

II. Extension of the *uncomputable* world in statistical physics:



We got interested with M. Sablik in what happens of subshifts of finite type, which are more pertinent for physicists. Amongst the results that we obtained is the following:

1. **Saillant result [GS17a]**

   The characterization of M. Hochman and T. Meyerovitch is still true under $O(n)$-mixing for multidimensional SFT (lattice models in statistical physics).

   (a) **Interpretation**
   This condition is verified by all known models in statistical physics. This work is interpreted as a **localisation of computable statistical physics world** and an extension of the knowledge on its uncomputable part.

   (b) **Acquired skills**
   Ability to **manage highly complex constructions**, constructive methods of dynamical systems involving Turing machines. These constructions are complex in the sense that any modification of local arguments involve complete change of the whole construction. **Formalisation of complex material.**

   (c) **Developement of new tools**
   Multiple refinements of embedding Turing computations constructions that led to new tools for uncomputability results under dynamical constraints.

   (d) **Application and refinements of these tools**
   We complexified further the methods to prove a characterization of entropy dimensions and dynamical systems than can be simulated by minimal multidimensional SFT [GS17b],[GS18].

2. **Research directions**

(a) **Minimal multidimensional SFT**

These systems are not very well understood, and our work is the first to provide construction tools to explore this class of systems. We expect more constructions to come, related to other aspects of multidimensional dynamics.

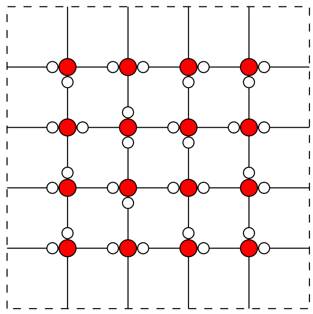
(b) **Short-term research direction**

Comparison with other uncomputability results obtained independantly for the spectral gap [CPGW], which was proven interesting for physicists [K17]. Possible simplification using our tools?

(c) **Long-term direction**

Uncomputability of other physically pertinent quantities and influence of physically pertinent constraints on their computability?

III. Extension of the *computable* world in statistical physics



After my doctoral thesis, N. Aubrun and M. Rao pointed out that some computations methods were developped by physicists to compute exactly entropy for some lattice models. I got naturally interested in them and obtained the following:

1. **Saillant result [G19]**

   Proof of the exact value of square ice entropy.

2. **Acquired skills**

   Tradional **tools of statistical physics** (partition functions, transfer matrices, etc). **Dealing with non-rigorous arguments**, and strong **bibliography abilities** (form of the literature in physics,etc).

3. **Research directions**

   (a) **Purely algebraic proof**

   The proof presented has, besides algebraic arguments, a heavy analytical part, that we would like to avoid. We expect that some generalisation of R. Baxter method of commuting matrices would lead to a simplification of this part.

   (b) **More proofs of physicists computations**

   Still a lot of work to be done in this direction, on eight-vertex model and hard hexagons. R. Baxter himself admits the lack of rigour of these developments. Highly probable profit of understanding of computation methods for square ice.

   (c) **Extension to models outside the scope of physicists**

   Examples of such models involve Kari-Culik (known to have positive entropy) or Jeandel-Rao tilings, that appeared in interactions between mathematics and computer science, subshifts of square ice, that appeared in our work on uncomputability results, and many other multidimensional SFT.

IV. Approaching the frontier: a specific notion of complexity

This axis is still in developement, but I have a good knowledge of bibliography, and some ideas of developments, in which I am highly interested.

1. **Context: a notion of complexity for organised systems**

   G. Tononi et al. published in 1994 [TSE] a seminal paper that defines a quantity called neural complexity, based on information theory, that aims at measuring the balance in a system between independance of its parts and their integration into a whole that have *organised sytems* such as the brain. This quantity has then started to be studied by mathematicians, who generalized it into the notion of intricacies, defined for general dynamical systems (see [BZ12] and [PW18]).

2. **Research directions**

   (a) **Understanding existing quantites:** Concepts far from understood by the community, still a lot of work. Examples of tasks:

   - computing the neural complexity or intricacies for simple systems (would benefit skills acquired in part III).
   - and understanding their maxima and minima.

   (b) **A formalisation work to be done**

   This question is interesting for me since the notion of *organisation* of dynamical systems matches intuitively with the constructions of multidimensional SFT under constraints (thus approaching the frontier) that I have done with M. Sablik. In particular, they exhibit complex information processing phenomena, with natural analogies with biological systems: functional segregation, multiple exchanges of informations between functional parts, inhibition of information transport, etc. An apparently possible work would be to provide formal definitions for these phenomena.

   (c) **Other possible definitions**

   We expect that, based on a better understanding of these phenomena in our constructions and their formal counterparts, we would be able to develop a quantity that charges this type of systems, intuitively analogical to biological ones, and would bring new ideas to this research area.

References

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