Entropy of shifts: a sharp computational threshold phenomenon.

Silvere Gangloff, Benjamin Hellouin

7 décembre 2018

Effect of quantified irreducibility on the computability of subshifts entropy, Gangloff, Hellouin, Discr. Cont. Dyn. Sys. (2018)

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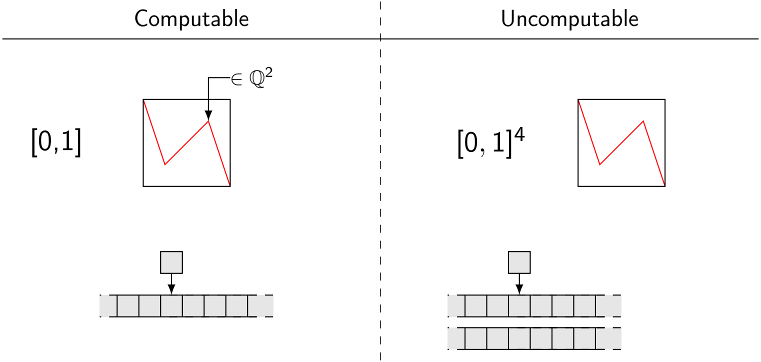


Computable

Uncomputable

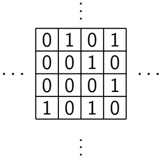
$\in \mathbb{Q}^2$

[0,1]

$[0,1]^4$

About multidimensional SFT :

Definition : set of



0	1	0	1
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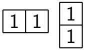
where are forbidden (**finite set**) :



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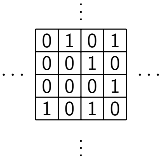

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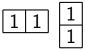

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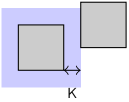

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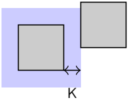
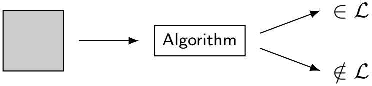


2017 : Gap function $K \rightarrow f(n)$ (G., Sablik) : f-block gluing

Decidable subshifts : a computational threshold for entropy

Decidable :



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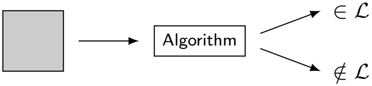

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Theorem[G.,Hellouin] : define $\Sigma(f) = \sum_n f(n)/n^2$, assumed computable.

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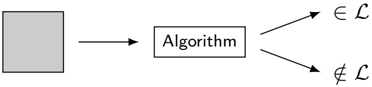

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Question : $\Sigma(f)$ not computable ?

Entropy :

Number of patterns : $\cdots \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&0&1&0&0&1&0&1\\\hline\end{array} \cdots$; forbidden : $\begin{array}{|c|c|}\hline 1&1\\\hline\end{array}$

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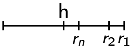
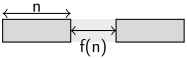


Computable $\equiv$ comp. speed

Under the threshold : $\Sigma(f) < +\infty$



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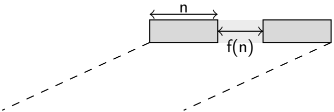

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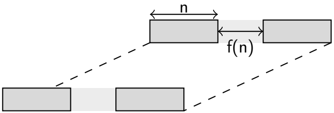

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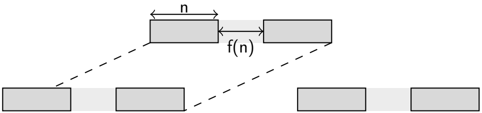

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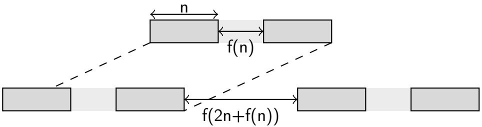

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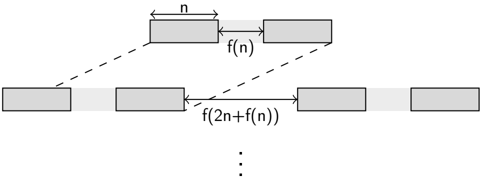

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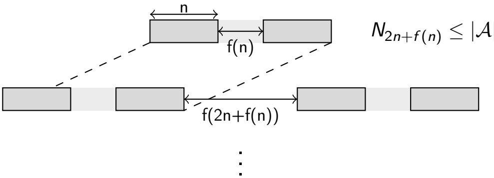

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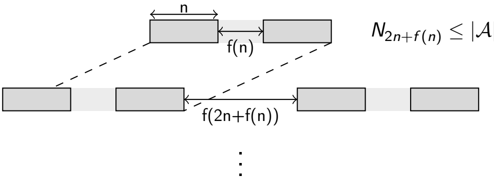

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Repetition+log :

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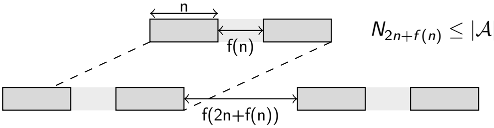

$$N_{2n+f(n)} \leq |\mathcal{A}|^{f(n)}.N_{2n}$$

$$\vdots$$

Repetition+log :

$$\frac{\log(N_n)}{n} - |\mathcal{A}|.\sum_{n}^{+\infty} \frac{f(2^k)}{2^k} \leq h \leq \frac{\log(N_n)}{n}$$

Above the threshold I. Objects : $\Sigma(f) = +\infty$

Bounded density shifts : $(p_n)_n \in \mathbb{N}^{\mathbb{N}}$ growing, forbidden :

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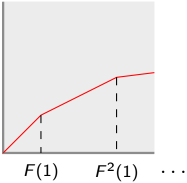

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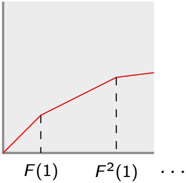

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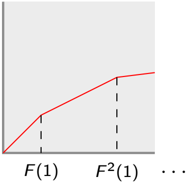

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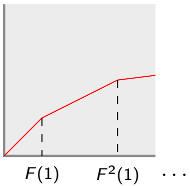

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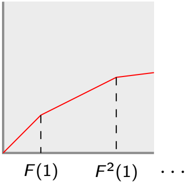

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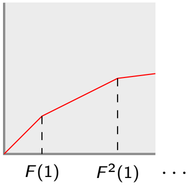

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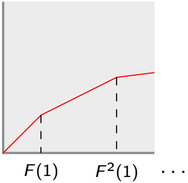

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2. f-gluing when $p_{F(n)} \geq 2p_n + 4$.

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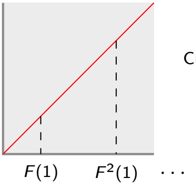

Current entropy :
$h > \alpha_1 + 2^{-1}$

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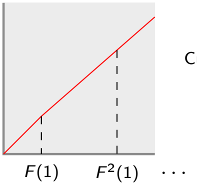

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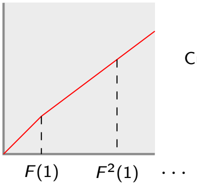

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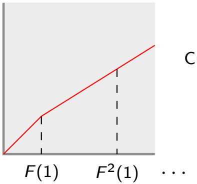

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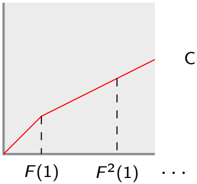

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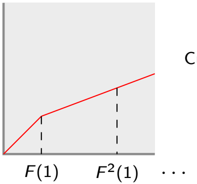

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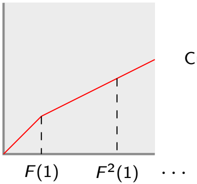

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$h > \alpha_2 + 2^{-2}$

Above the threshold II. Strategy : $\Sigma(f) = +\infty$

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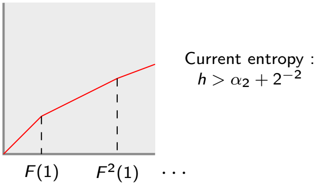

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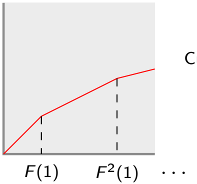

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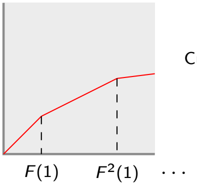

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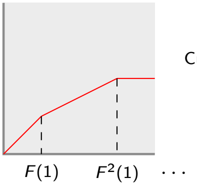

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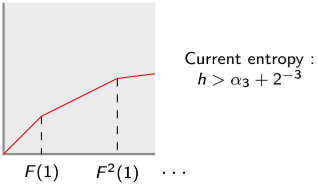

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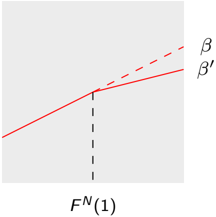

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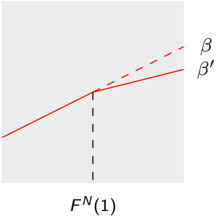

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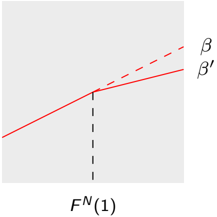

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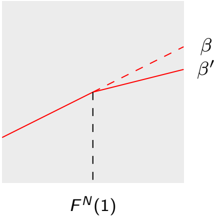

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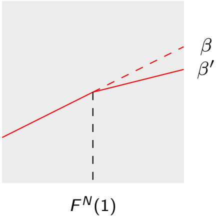

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