Minicourse on *information, complexity and organisation in multidimensional symbolic dynamics*

Exact computations of entropy for multidimensional SFT

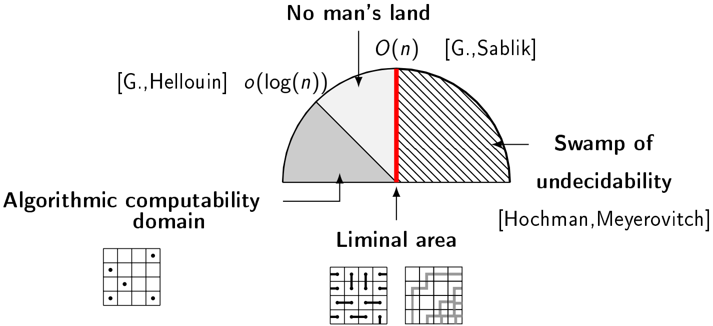
Silvere Gangloff

April 30, 2021

sgangloff@agh.edu.pl ; silvere.gangloff@gmx.com

Multidimensional SFT: a computational 'transition':

Reminder (third lecture):



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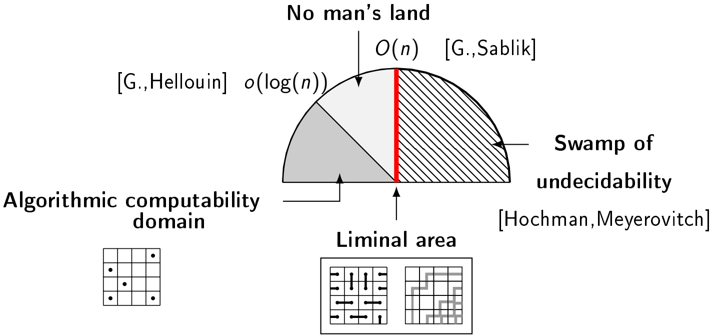

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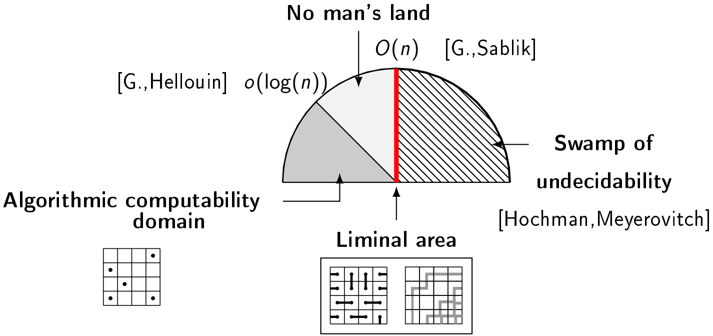

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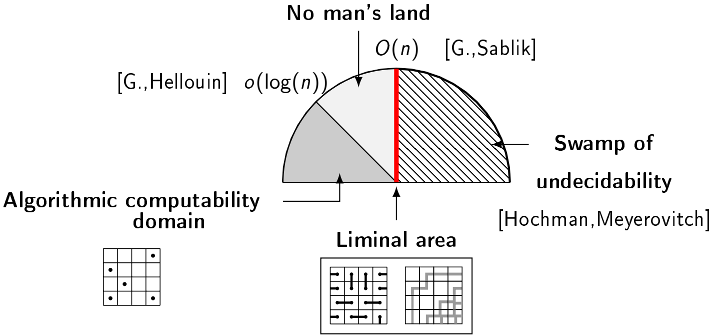

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Question: what makes the entropy of subshifts in the liminal area computable ?

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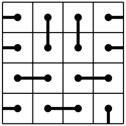

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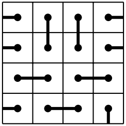

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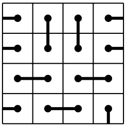

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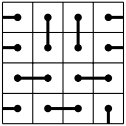

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(Called Catalan constant)

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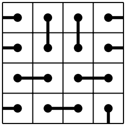

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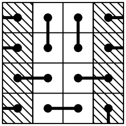

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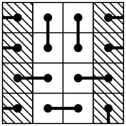

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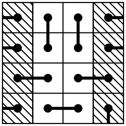

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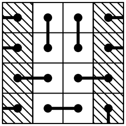

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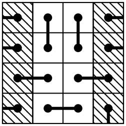

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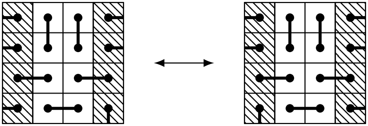

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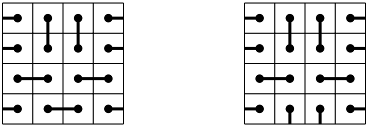

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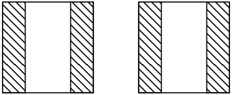

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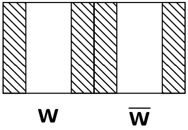

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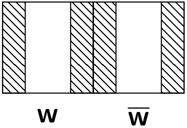

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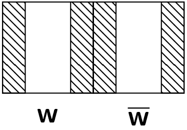

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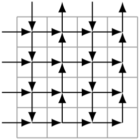

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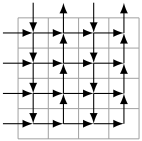

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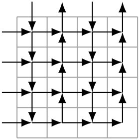

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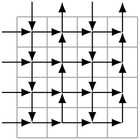

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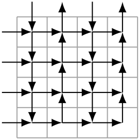

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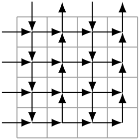

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Diagonalisation of $K^{(n)} \to$ formula for $N_n^t(X_0)$ as sum of trigonometric functions.

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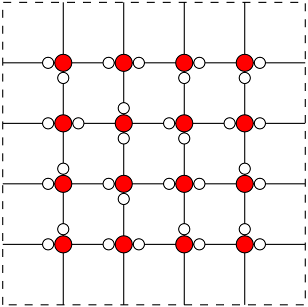

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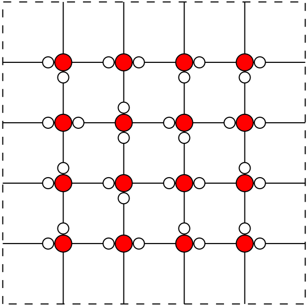

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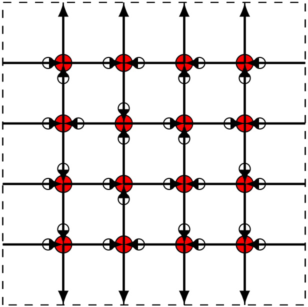

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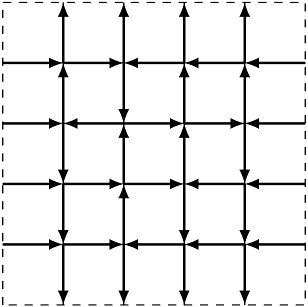

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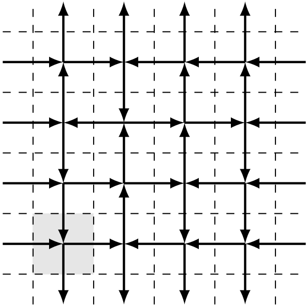

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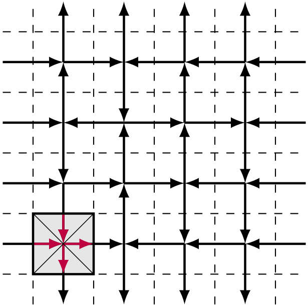

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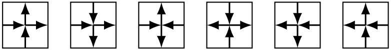

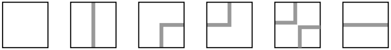

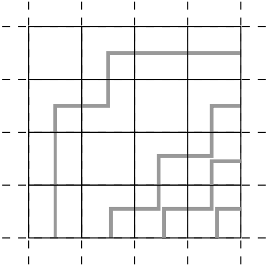

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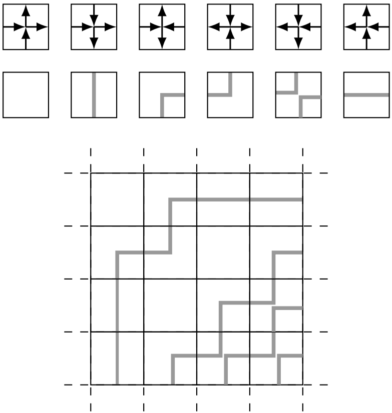

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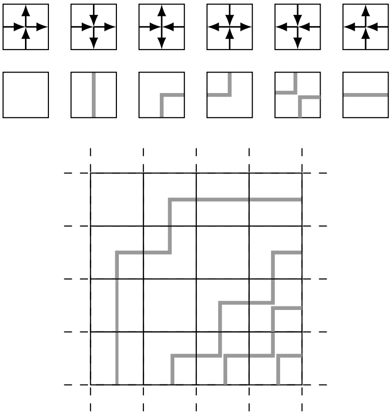

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S. Gangloff, *A proof that square ice entropy is* $\frac{3}{2}\log_2(4/3)$, 2019 (based on the work of R.Baxter, K.Kozlowski).

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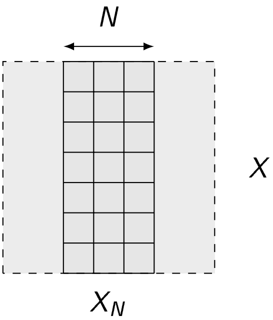

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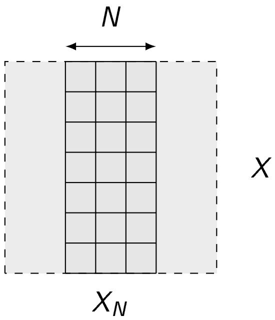

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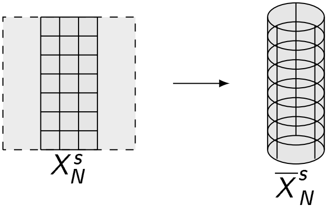

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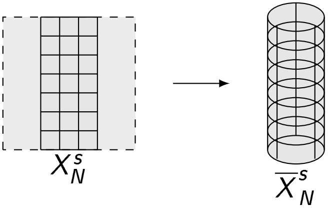

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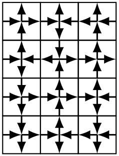

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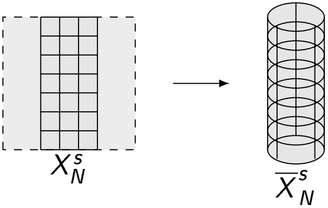

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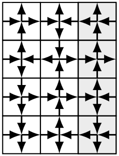

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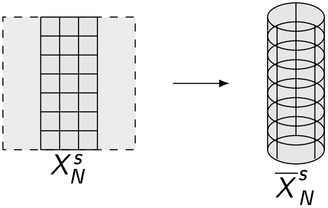

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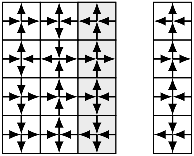

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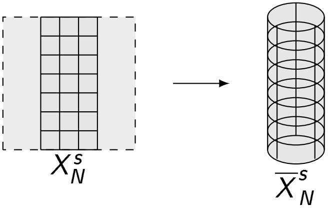
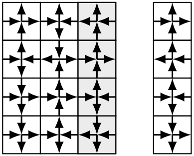


Symmetry properties:



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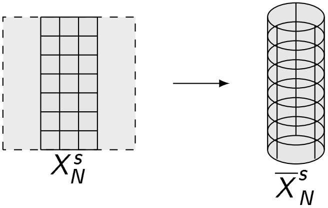

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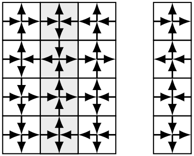

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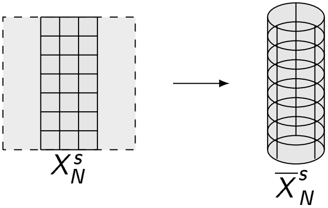

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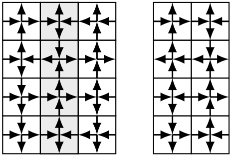

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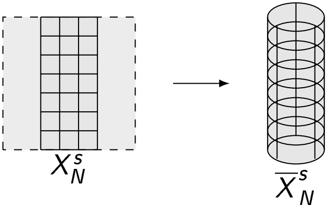

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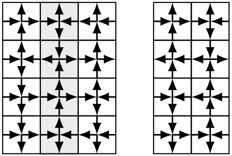

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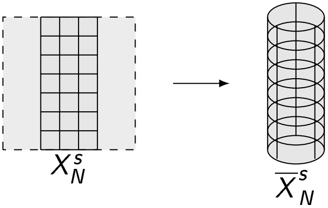

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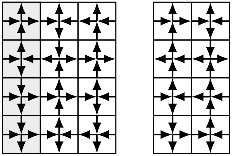

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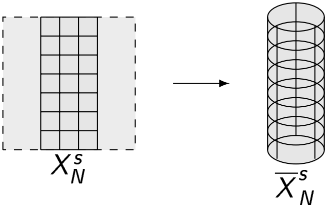

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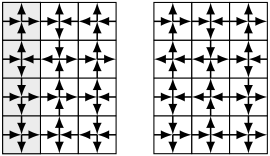

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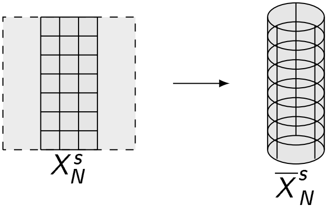

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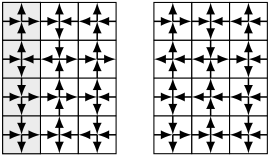

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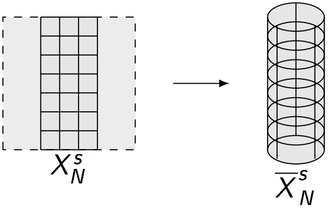

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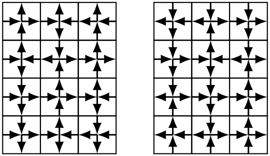

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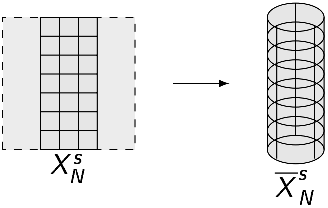

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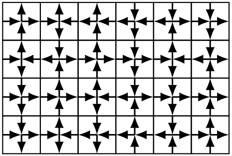

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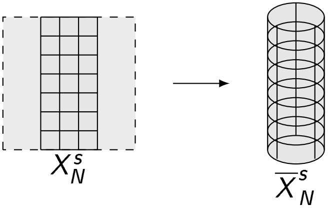

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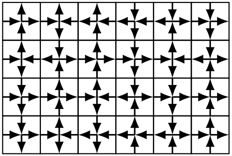
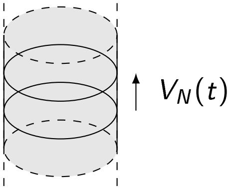


$$h(X^s) = \lim_N \frac{h(\overline{X}_N^s)}{N}$$

Square ice: Lieb transfer matrices:



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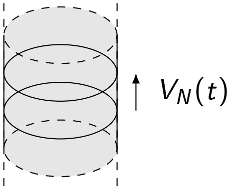

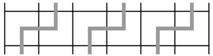

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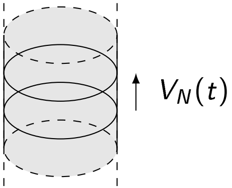

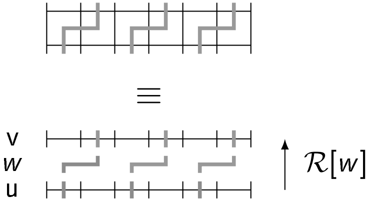

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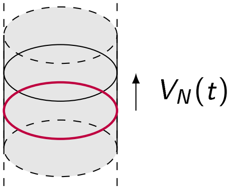

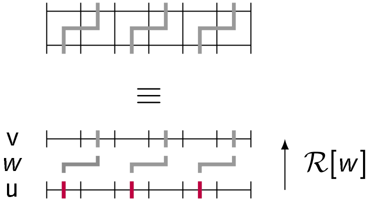

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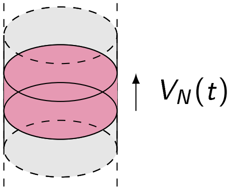

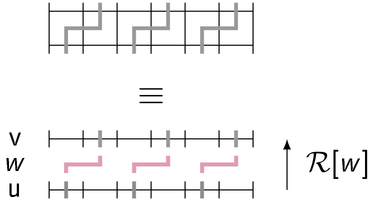

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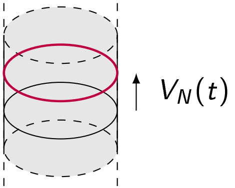

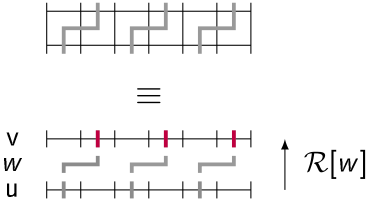

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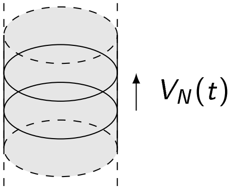

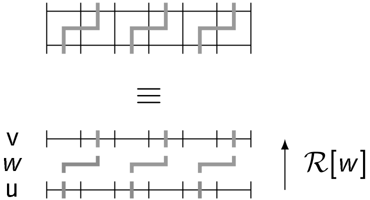

$$V_N(t)[\mathrm{u},\mathrm{v}] = \sum_{\mathrm{u}\mathcal{R}[w]\mathrm{v}} t^{|w|}.$$

where $|w| = \#$ of  and 

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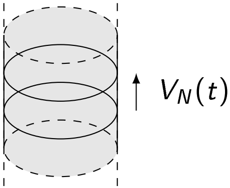

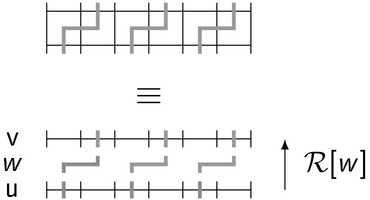

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Square ice: Computing maximal eigenvalue of $V_N(1)$, strategy:

$V_N(1)$ •

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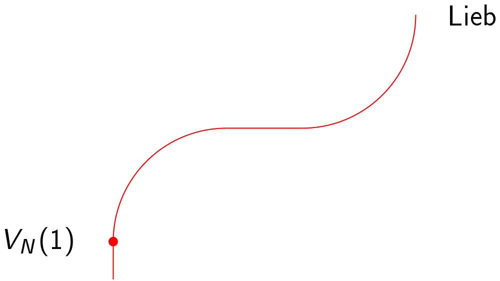

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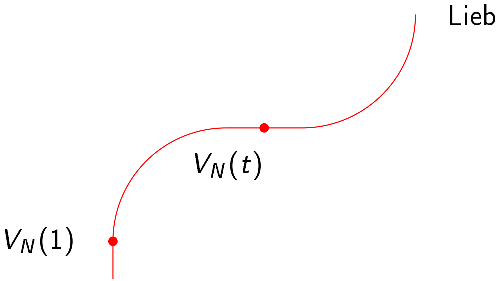

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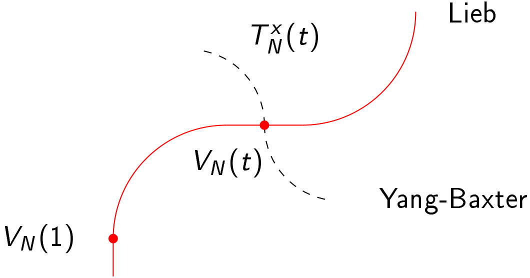

$$\forall x, y,\ T_N^y(t)\,T_N^x(t) = T_N^y(t)\,T_N^x(t).$$

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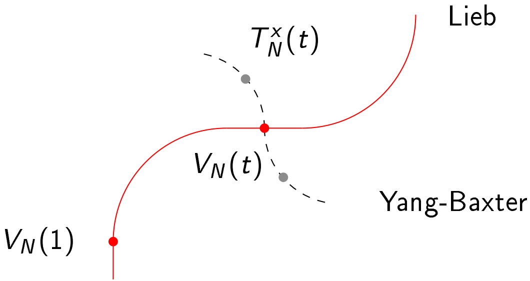

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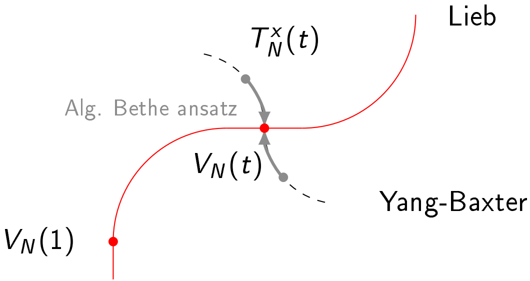

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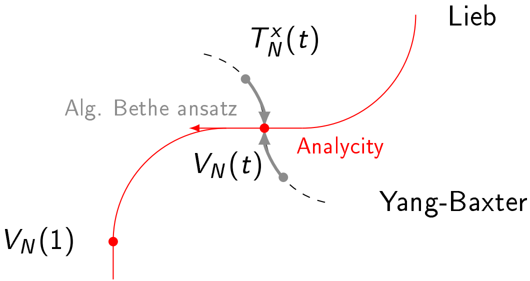

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Square ice: R-matrices and monodromy matrices:

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input/output:

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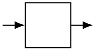

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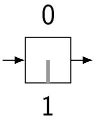

$$R(0,1) = \begin{pmatrix} & \\ & \end{pmatrix}$$

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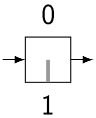

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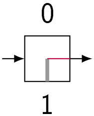

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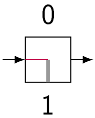

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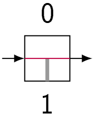

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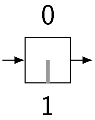

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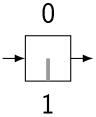

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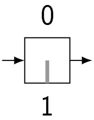

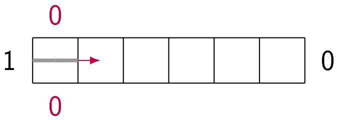

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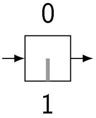

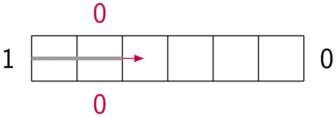

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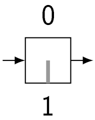

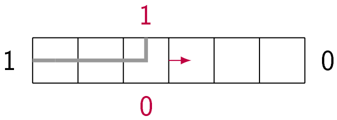

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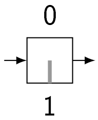

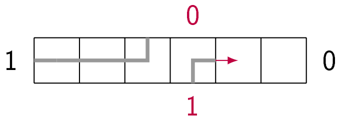

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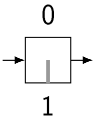

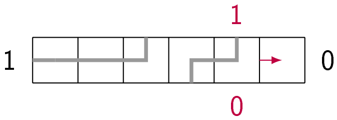

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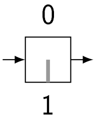

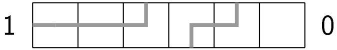

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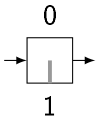

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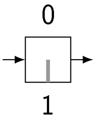

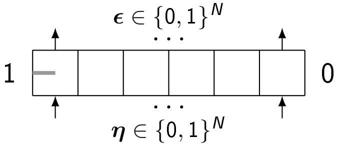

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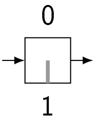

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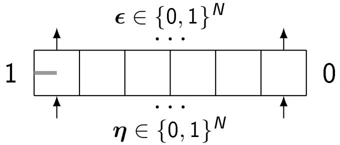

$$M_N(1,0)[\epsilon, \eta]$$

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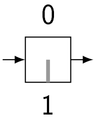

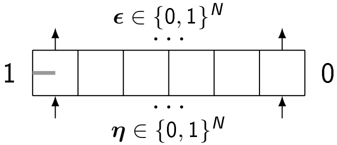

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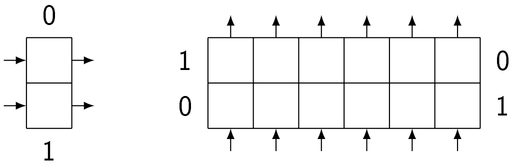
Yang-Baxter transfer matrices:

$$T_N[\epsilon, \eta] = \sum_{u \in \{0,1\}} M_N(u,u)[\epsilon, \eta].$$

Square ice: Commutation of Yang-Baxter matrices:

Composition of R-matrices and monodromy matrices:



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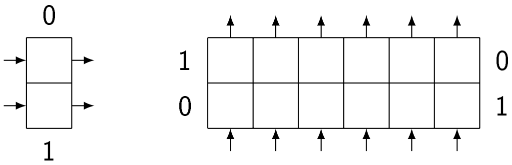

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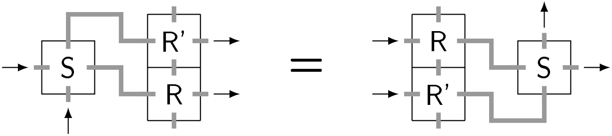

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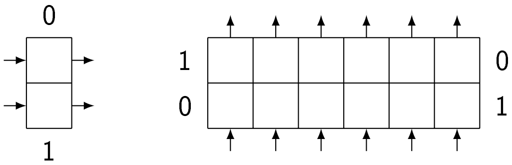

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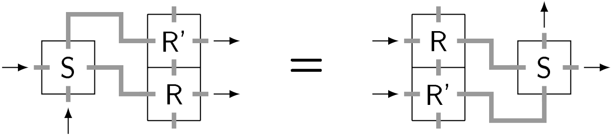

Yang-Baxter equation $\Rightarrow$ transfer matrices commute.

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where $R^x_{\mu_t}(0,0)$ is the up-left 2×2 part of this matrix, etc.

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If there exists $(p_j(t))_{j=1}^n$ solution of:

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construction of a candidate eigenvector for the transfer matrix for value:

$$\prod_{k=1}^{n} L_t(e^{ip_k}) + \prod_{k=1}^{n} M_t(e^{ip_k}).$$

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Lemma: for all t there exists $(p_j(t))_{j=1}^n$ solution of (E_t) and it is an analytic function in t.

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Theorem: there exists ρ_t s.t. for all $f \in L^1$:

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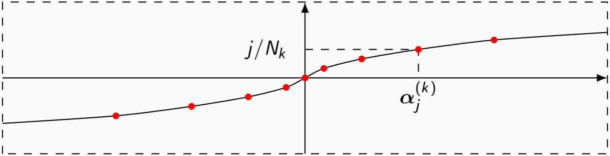
$$\xi_t^{(k)} : \alpha \mapsto \frac{1}{2\pi}\kappa_t(\alpha) + \frac{n_k + 1}{2N_k} + \frac{1}{2\pi N_k}\sum_{j=1}^{n_k} \theta_t(\alpha, \boldsymbol{\alpha}_j^{(k)})$$



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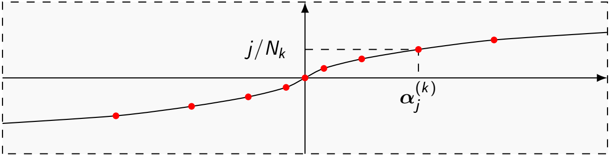

Equality with a Riemann sum:

$$\lim_k \frac{1}{N_k}\sum_{j=1}^{n_k} f(\boldsymbol{\alpha}_j^{(k)}) = \lim_k \frac{1}{N_k}\sum_{j=1}^{n_k} \left(\boldsymbol{\alpha}_{j+1}^{(k)} - \boldsymbol{\alpha}_j^{(k)}\right) \frac{\left(\xi_t^{(k)}(\boldsymbol{\alpha}_{j+1}^{(k)}) - \xi_t^{(k)}(\boldsymbol{\alpha}_j^{(k)})\right)}{\left(\boldsymbol{\alpha}_{j+1}^{(k)} - \boldsymbol{\alpha}_j^{(k)}\right)} f(\boldsymbol{\alpha}_j^{(k)})$$

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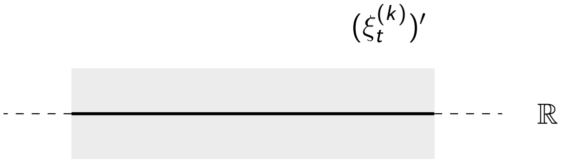

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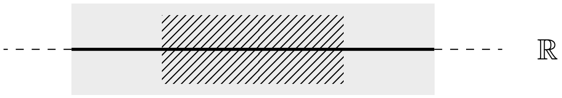

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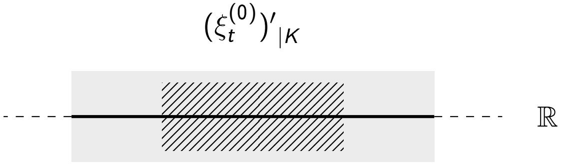

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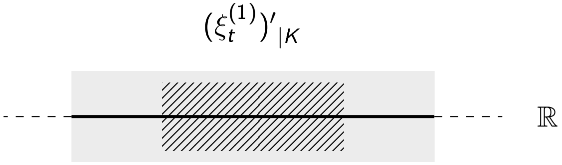

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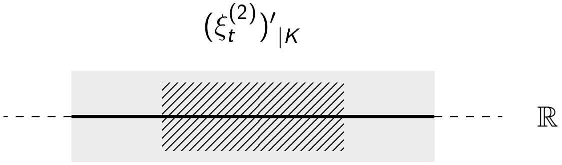

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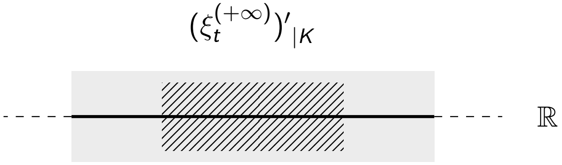

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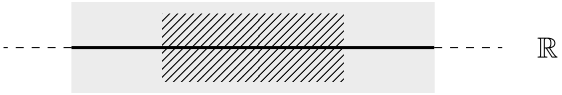

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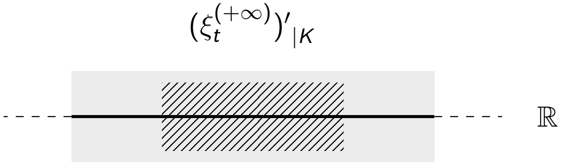

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4. Thus, $(\xi_t^{(k)})'$ converges to this solution.

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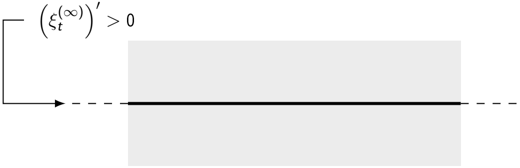

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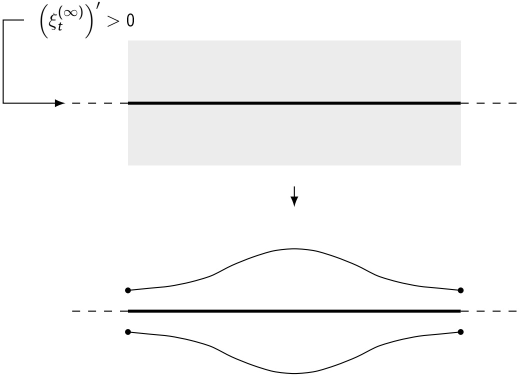

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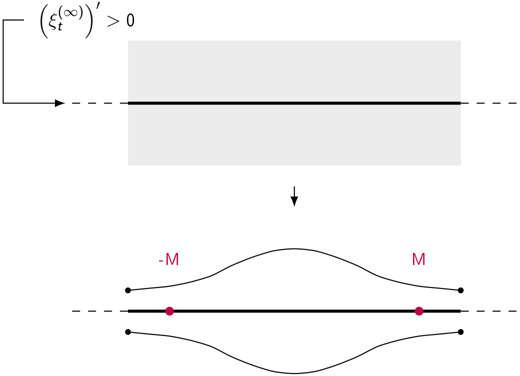

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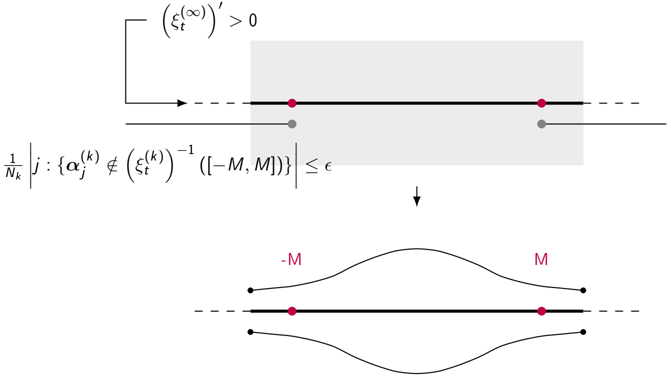

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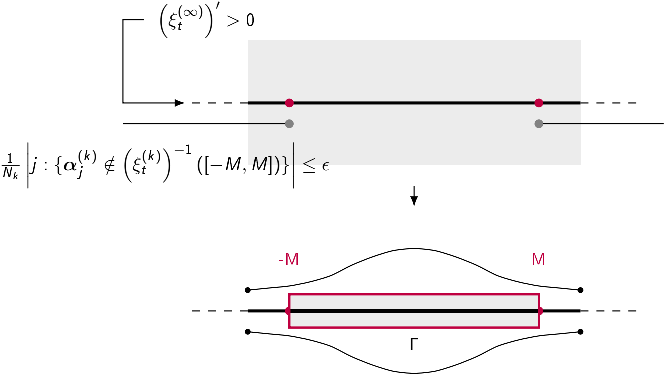

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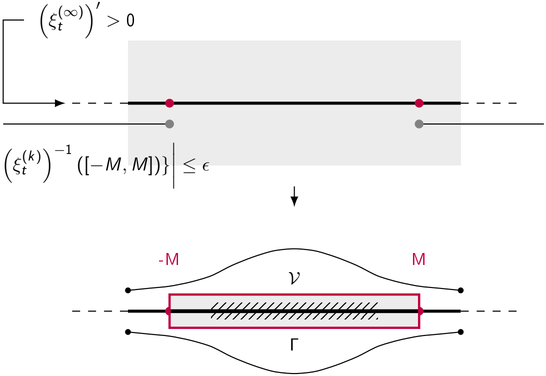

$$\frac{1}{N_k}\left|j : \{\boldsymbol{\alpha}_j^{(k)} \notin \left(\xi_t^{(k)}\right)^{-1}([-M, M])\}\right| \leq \epsilon$$

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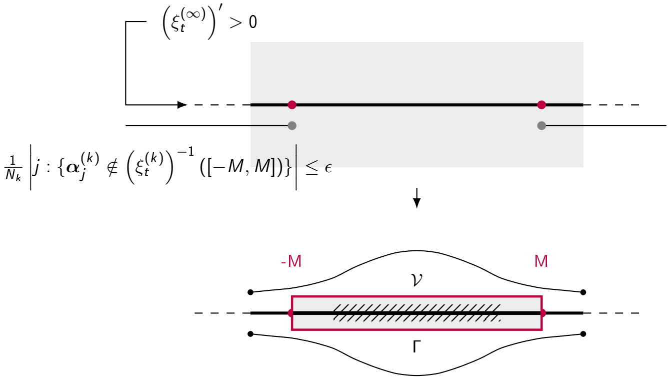
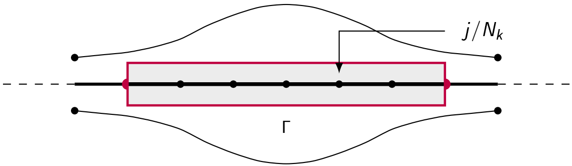


The functions $\xi_t^{(k)}$ have distinct values on $\mathcal{V}$ and Γ. Thus they are bihilomorphisms onto $\mathcal{V}$ (Cauchy formula).

Square ice: Lace integral expression of $\xi_t^{(k)}$:



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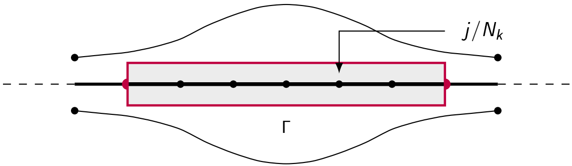

By residues theorem:

$$\xi_t^{(k)}(\alpha) = \frac{1}{2\pi}\kappa_t(\alpha) + \frac{n_k+1}{2N_k} + \oint_\Gamma \theta_t\left(\left(\xi_t^{(k)}\right)^{-1}(\alpha)\right)\frac{e^{2i\pi s N_k}}{e^{2i\pi s N_k}-1}ds + O(\epsilon).$$

Square ice: Fredholm integral equation: Limit and change of variable:

$$\xi_t^{(\infty)}(\alpha) = \frac{1}{2\pi}\kappa_t(\alpha) + \frac{1}{4} + \int_0^{+\infty} \theta_t(\alpha) \left(\xi_t^{(\infty)}\right)'(\alpha) d\alpha.$$

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Unique solution by Fourier transforms.

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Expression of $\rho_t = \left(\xi_t^{(\infty)}\right)'$ and lace integrals computations:

$$\boxed{h(X^s) = \frac{3}{2}\log_2(4/3)}.$$

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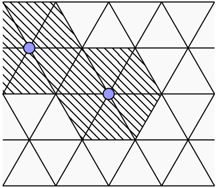
Question: what are the possible values of entropy for symmetric bidimensional subshifts of finite type ?

Baxter's hard hexagons:[Exactly solvable models in statistical physics]



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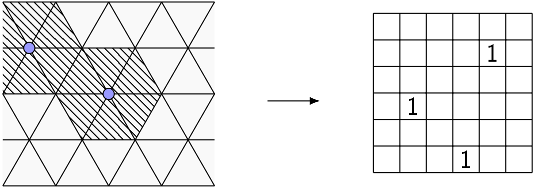

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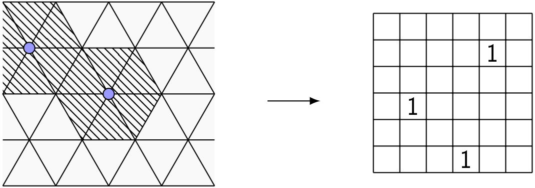

Formula for entropy as sum of a series:

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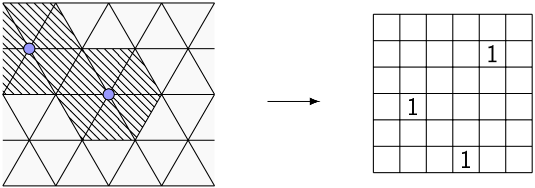

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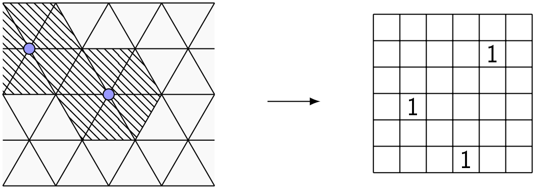

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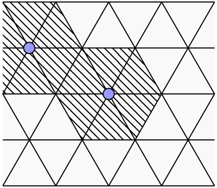

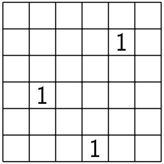

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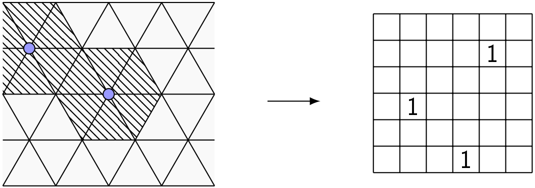

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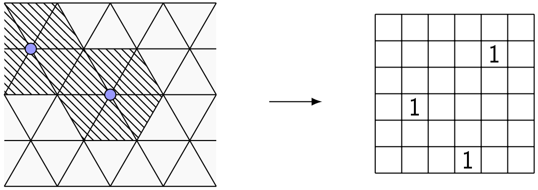

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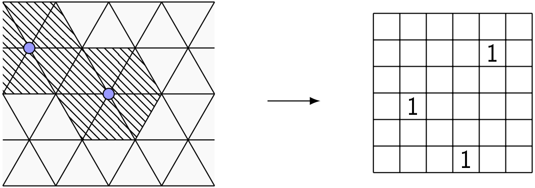

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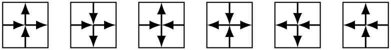
Main problems: points **3**, **4**.

Eight-vertex model:[Exactly solvable models in statistical physics]



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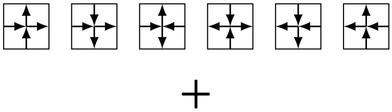

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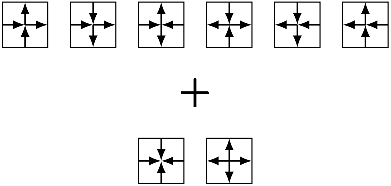

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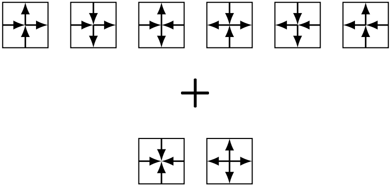

Entropy computation: similar to square ice; analytical part non verified.

Subsidiary questions:

Question: can we use similar methods to talk about invariant measures (for instance $\times 2$, $\times 3$ conjecture) ?

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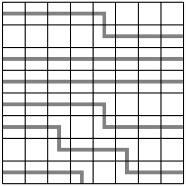
Question: can we find solutions of Yang-Baxter equations for other subshifts of finite type ? *Example*: Kari-Culik tilings (know: positive entropy [Durand,Gamard,Grandjean(2017)]).

Combinatorial methods

Definition subshift Δ_r; ex for $r = 3$:



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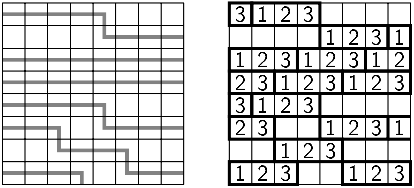

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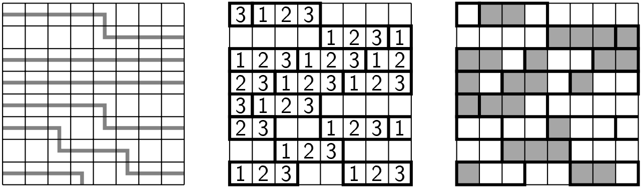

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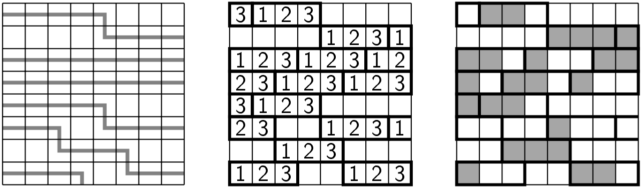

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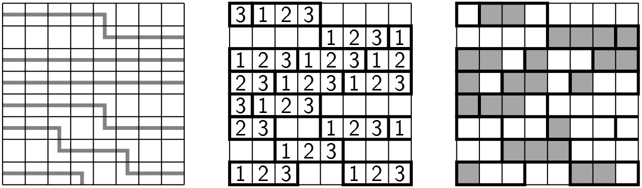

Theorem:[G.,Sablik] $h(\Delta_r) = \frac{\log_2(r+1)}{r}$.

Question: for what kind of subshifts can we compute entropy with similar methods ?

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Second layer is trivial: we consider only first and third.

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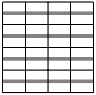

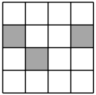

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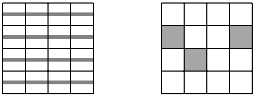

$$N_n(\Delta_1) \geq 2^{n^2}.$$

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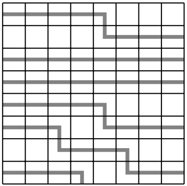

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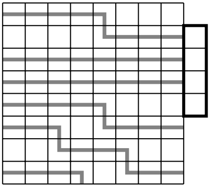

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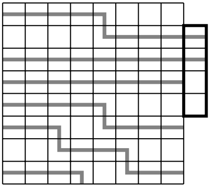

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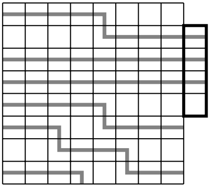

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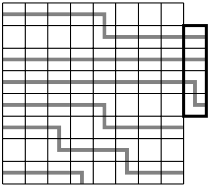

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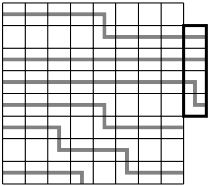

2^3 choices in 2nd layer

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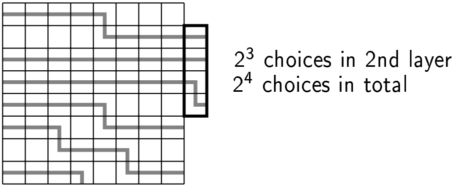

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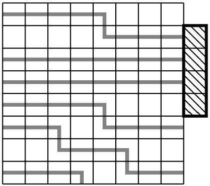

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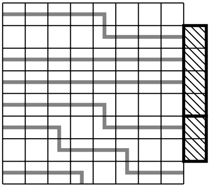

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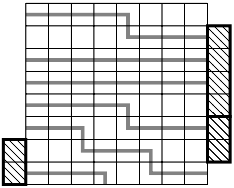

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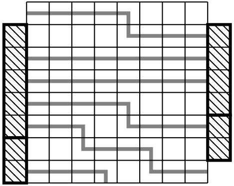

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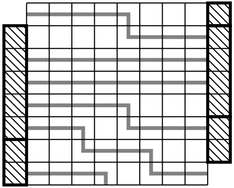

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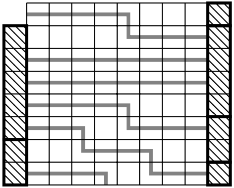

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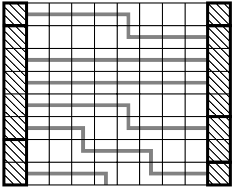

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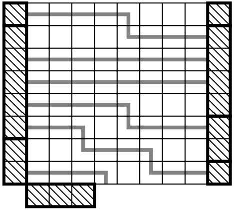

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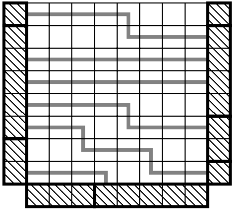

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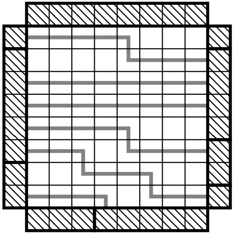

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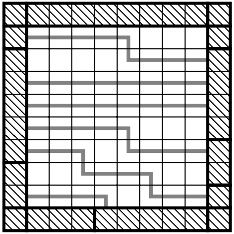

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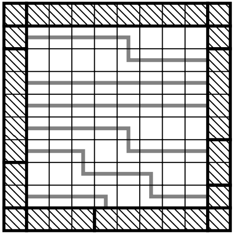

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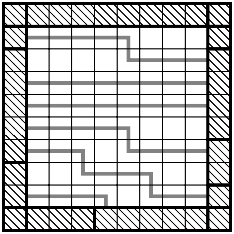

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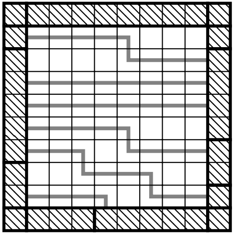

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3. Try Baxter's method for square ice; more precisely: do both transfer matrices of square ice have same maximal eigenvalues or not ?