On exact computations of topological entropy of multidimensional subshifts of finite type

Silvère Gangloff

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The aim of this short note is to provide a support for a work group around exactly solved lattice models in statistical mechanics and the computation of their entropy. We focus the approach of this research theme by symbolic dynamics community, and assume elementary knowledge of this field. The notion of lattice model is equivalent to the notion of subshift of finite type in symbolic dynamics, and this work group aims at providing tools to compute, where computing means computing with a formula, the entropy of subshifts of finite type without consideration on the physical meaning of these dynamical systems. We also expect a dynamical point of view to provide ideas in this theme.

In this text, we provide a raw overview of known techniques coming from statistical mechanics and some following mathematical developments [Section 1]. For more details on these techniques, we refer to the original articles and an elementary introduction on this matter in [Jac] and a review in [Bax82]. Some research directions are proposed in Section 2.

1 Subshifts of finite type in statistical mechanics

Statistical mechanics consist in the study of sets of colorings of (infinite) lattices with a finite set of symbols in a finite alphabet, ruled by a finite set of forbidden patterns. However, there is no notion of topological dynamical system in this setting, corresponding to the shift action in symbolic dynamics and its invariant closed sets.

However there is a notion of entropy, and other important tools, such as **partition functions** that are usually not involved in symbolic dynamics. The principle of a partition function is to consider a symbol in the alphabet as a physical entity (an atom for instance) in a particular state to which corresponds a quantity (depending on its energy, or a theoretical ersatz for energy) called **activity**. When the energy is zero, this number is often equal to 1. In statistical mechanics, a pattern is often called a **configuration**, and such a configuration is associated with the product of the activities of the entities composing it. The partition function of a finite subset of the lattice which to the set of energies of the physical entities associates the sum of the quantities associated to configurations over this subset. We introduce this tool more formally for the six vertex model [Section 1.1.1].

In particular, if the lattice is $\mathbb{Z}^2$, and the subset is $[\![1,n]\!]^2$, the value of the partition function when all the energies are zero is equal to $N_n(X)$, which denotes the number of globally admissible patterns of the model, denoted X, over $[\![1,n]\!]^2$. As a consequence, the complexity sequence and the entropy are registered into the partition functions.

The computation of the entropy in this setting often relies on the computation of particular partition functions. However, the reasons why these partition functions work in general seem not to be known, and their choice is not canonical. A natural question we would like to address here is the following:

Problem 1. *Is there a canonical way, at least for a sub-class of the general class of models, to associate a partition function that allows computations ?*

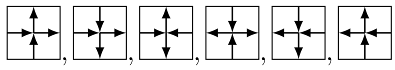
In Section 1.1, we extract some ideas underlying computation of the entropy for two models on the square lattice $\mathbb{Z}^2$: the six-vertex model and the dimer model. For completeness, we present the approach related to computability on the hard squares model) In Section 1.2, we do a similar treatment to known models on other (regular) lattices, on which more is known (the equivalent of the hard squares model on the triangular lattice is solved while the hard squares is not). We include self-avoiding walks on the honeycomb lattice, which do not form a subshift of finite type, since the techniques used for this model are similar.

1.1 On the square lattice

1.1.1 The six-vertex model and the Bethe ansatz

The six vertex model comes from the study of possible arrangements of atoms in (three-dimensional) ice, and was proposed by Pauling [Pau35]; the entropy of a two-dimensional equivalent model was computed later by Lieb [Lie67], motivated by the confirmation of the model. However, the value of the entropy is still not rigorously proved.

1.1.1.1 Description The model is on symbols



with rule that the direction of the arrows have to match between two adjacent positions.

One can see that any locally admissible square pattern can be extended into a greater locally admissible square pattern and thus is globally admissible. See an example of such pattern on Figure 1.

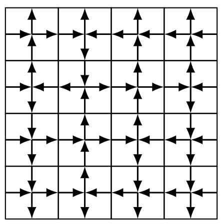


Figure 1: An example of locally and thus globally admissible pattern of the six vertex model.

1.1.1.2 Computing the entropy: from one dimensional subshifts to bidimensional ones The entropy of one-dimensional subshift can be computed through a matrix representation: the entropy is the Perron-Frobenius eigenvalue of this matrix. In order to compute the entropy of a subshift X on $\mathbb{Z}^2$, it is rather natural to rely on the sequence $(h_n)_n$, where for all n the number $h_n(X)$ is the entropy of the subshift

$$X_n = \{((x_{k,l})_{-n \leq k \leq n})_l : x \in X\}.$$

It is known that the topological entropy of X is equal to

$$h(X) = \lim_n \frac{h_n(X)}{n},$$

which means that the Cesaro mean of the sequence $(h_{n+1} - h_n)_n$ converges towards $h(X)$ (however, this sequence does not necessarily converge [Pie08]).

As a consequence, we reduce the computation of the entropy to the asymptotics of the number $N_{n,m}(X)$ of patterns over the rectangle $[\![1,n]\!] \times [\![1,m]\!]$ when m tends to $+\infty$.

1.1.1.3 Counting the rectangle patterns with transfer matrix In order to count these patterns, one needs only information on possible vertically adjacent length n segments of symbols (since the subshift is nearest-neighbor and every locally admissible pattern is globally admissible). This information is captured by the **transfer matrix** T_n acting on the vectorial space generated whose canonical basis is formed by the set of the length n words on the alphabet of the subshift. The entry of the matrix on (ω, ω') is 1 if the word ω' can be put over the word ω and 0 else. The number $N_{n,m}(X)$ is thus equal to

$$N_{n,m}(X) = \mathrm{Tr}(T_n^m).$$

As a consequence,

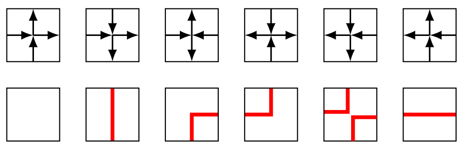
$$h_n(X) = \lim_m \frac{\log_2(N_{n,m}(X))}{m} = \log_2(\lambda_n),$$

where λ_n is the Perron-Frobenius eigenvalue of T_n.

In order to compute λ, one can replace the vectorial space by the one generated by the words $\omega_1 ... \omega_n$ such that $\omega_n \omega_1$ is not a forbidden pattern. This was assumed without verification by Lieb, and verified in a subsequent mathematical treatment of the six vertex model [DCGH$^+$b].

Remark 1. *It is possible that this method, consisting in taking into account a direction after another in the entropy computation, is adapted to the six vertex model for the reason that its configurations admit a natural "orientation", that one can see in the discrete curves representation: this is the direction of shifts of this curves.*

1.1.1.4 Representation by curves This model is also often represented with discrete curves, representing possible particle trajectories. This representation consists in changing the symbols in the following way:



The pattern on Figure 1 can be represented as on Figure 2.

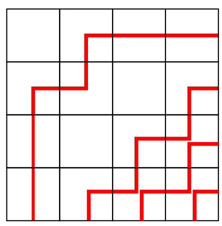
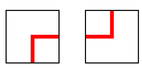


Figure 2: Representation of pattern on Figure 1.

1.1.1.5 Choice of partition function Constructing a partition function often corresponds to the association of a coefficient to each symbol. For instance, the **isotropic** six vertex model associates a coefficient $c > 0$ to the symbols



and 1 to the others.

The coefficient attached to a square pattern is c^k, where k is the number of symbols having coefficient c. Thus, the nth partition function of the model is

$$P_n : c \mapsto \sum_k N_k^{(k)} c^k,$$

where $N_k^{(k)}$ is the number of globally admissible n-square patterns containing exactly k symbols with coefficient c.

The interest of these functions regarding the entropy is that

$$h = \lim_n \frac{\log_2(P_n(1))}{n}.$$

Moreover, one can use analysis to relate the asymptotic behaviour (when simple) of these functions to the values $P_n(1)$, through a rescaling and extension by continuity to the value $+\infty$ of this function.

Remark 2 (Addition of random bits)**.** *Attributing these coefficients correspond in symbolic dynamics to the addition of random bits when the curves shift towards the right, except that these bits can be fractioned. When the coefficient c tends to infinity, the square patterns covered by curves constantly shifting, as on Figure 3, are the most numerous ones and counting them is sufficient to compute the entropy.*

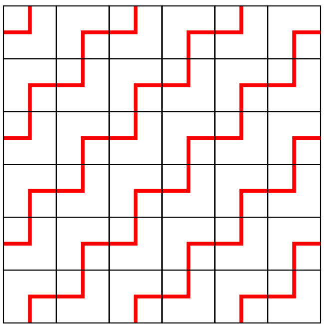


Figure 3: Example of ground state for the isotropic six-vertex model.

*This technique is very similar to the one used in [GS17] to compute the entropy of distortion subshifts, except that in this context the combinatorial simplicity is obtained by focusing on straight curves adding random bits in $\{0,1\}$. The configurations with maximal contribution to the partition function are often called **ground states** in statistical mechanics.*

Remark 3 (On the choice of the coefficients)**.** *When this method is used, the coefficients are often chosen to be coherent with symmetries of the symbols. However, there is no mathematical reason to impose such symmetries in order to compute the entropy.*

On the other hand, one can recode the subshift with higher block presentation. Thus one can consider that these blocks are attached with a coefficient equal to the coefficient of the symbol on which they are centered. However, there are partition functions for this recoded subshift that can not be obtained this way. As a consequence, there is a possibility that an adequate recoding simplifies this method.

1.1.1.6 The ansatz With this addition of random bits, to each value of $c > 0$ corresponds a Perron-Frobenius eigenvalue $\lambda_n(c)$ of the rescaled transfer matrix.

We can restrict this search on each of the stable subspaces defined by the quantity of curves crossing a row, which is conserved through the transfer operation.

Assuming an intuitive form of the eigenvector (the Bethe ansatz) and writing the eigenvalue equation, we obtain a value and a vector. We however don't know if this value is the Perron-Frobenius eigenvalue and if the vector is non zero. Moreover, the existence relies on the existence of a system of equations on a finite sequence of integers $\mathbf{k}$.

1.1.1.7 Mathematical treatment All these properties, and the fact that the Perron-Frobenius eigenvalue is obtained in the subspace defined by a quantity of $n/2$ curves [assuming n is even] is possible asymptotically in n and c, where all the equations are simplified, and the sequence $\mathbf{k}$ is regularly displayed in its range, the equations ruling it being reduced with an integral convolution equation on its limit density. This equation can be solved with Fourier analysis.

By analytic extension, it is possible to deduce the value of

$$\lim_{n} \frac{\log_2(P_n(c))}{n}$$

when $c > 2$, but at this point the asymptotic display of $\mathbf{k}$ according to a density breaks down.

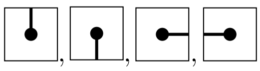
For further details on this mathematical account on the Bethe ansatz, we refer to [DCGH$^+$a]. A subsequent mathematical treatment of the six-vertex model can be found in [DCGH$^+$b]. This article proves a formula for the partition function in a range which excludes the entropy parameter.

Remark 4. *Besides the model presented in this section, the entropy has been computed or conjectured for other ones, such as the eight vertex model [Bax82], whose formulation consist of a slight modification of the six vertex model, adding possible assembling of arrows in the set of symbols. Moreover, in [Bax82] is proposed another solution for the six-vertex model.*

1.1.2 The dimer model and determinant method

In this section, we present the dimer model and the method used for the computation of its entropy.

1.1.2.1 Description This model has the following symbols:



The rules impose that through adjacent positions a segment is continued. See Figure 4 for a globally admissible square pattern of this model.

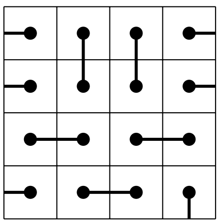


Figure 4: An example of globally admissible square pattern of the dimer model.

1.1.2.2 Equality of the partition function to a determinant modulo an orientation matrix Here the partition function is defined in a slightly different way, since edges are attributed with a coefficient $\epsilon \cdot x$ or $\epsilon \cdot z$, where $\epsilon \in \{-1, 1\}$. The coefficients ϵ follow the rule that each square has an odd number of -1.

Remark 5. *The conditions on the coefficients ϵ seem to be equivalent to the six-vertex rules.*

The second factor of the coefficient is x when the edge is vertical, and z when the edge is horizontal.

Problem 2. *Is there a way to formulate a method which covers the six-vertex model and the dimer model ?*

Another formulation could be: is it possible to solve the six vertex model with the method used for the dimer model and conversely? Moreover, is it possible to relate the two methods? Is it possible to formulate the two models in a similar way? Note for instance that E.H. Lieb proposed a solution of the dimer model based on the transfer matrix method [**?**] (Part X.2).

One could for instance make a systematic comparison of reformulations of the two models.

Suggestion 1. *One can recover the construction of partition function by attributing coefficients to the vertices by changing the lattice $\mathbb{Z}^2$ to the one whose vertices are the edges of $\mathbb{Z}^2$ and whose edges are the vertices of $\mathbb{Z}^2$. On this lattice, the dimer model can be reformulated as a hard core model: the symbols are $0, 1$ and the rules are that two 1 can not appear on neighbor vertices and exactly one of the vertices connected by an edge has symbol 1. As a consequence, this model admits a formulation similar to the ice formulation of the six-vertex model.*

Suggestion 2. *More generally, a lot of works related different various models in order to analyse one from knowledge of the other (for instance [DCGH$^+$b] relates the six vertex model to percolation problems). There exist simple tools in symbolic dynamics allowing to relate one model to another (isomorphism, factor, projective subaction, etc.): can we exhibit relations between these models in a more formal way using these tools?*

The associated partition function on the size n square is the product of the coefficients. This is equal to the determinant of the Kasteylen matrix, acting on the vectorial space generated by the vertices in the square. The coefficient (u, v) of this matrix is equal to the coefficient of the edge that links u and v when there is one. The equality with the determinant of this matrix comes from the fact that the contribution of a permutation in this determinant is non zero if and only if this permutation corresponds to a dimer configuration. Moreover, this contribution is the coefficient of this configuration.

Simple computation give the eigenvalues of this matrix, and the asymptotics of the partition functions involves elementary analysis.

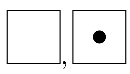
Remark 6. *The specificity of the dimer model on the modified lattice, relative to the hard core model on the same lattice, is that a condition of compactness is added. It seems that the hard core rules and the compactness rule together are a simple way to ensure a density of the 1 symbols in all the configurations.*

More recently, complexified versions of this model are considered, diluting the dimers (thus suppressing the compactness condition). This corresponds to the monomer-dimer model. For a review on this problem, one can see the introduction of [DPE]. An exact formula has been proved for the partitiion functions by A. Giuliani, I. Jauslin and E.H. Lieb [**?**].

1.1.3 Hard squares

In this section we present a last model on the square lattice. The particularity of this model is that the value of its entropy is still not even conjectured.

1.1.3.1 Description This model is on symbols



with rule that two non-blank symbols can not be adjacent. See Figure 5 for an example of globally admissible square pattern of this model.

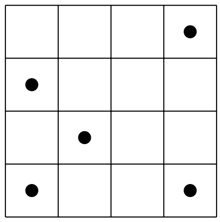
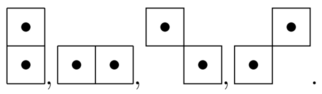


Figure 5: Example of globally admissible for the hard squares model.

Remark 7. *As for the six vertex model, there are variations on the formulation of the hard squares model. For instance, one could replace the forbidden patterns by*



One can also change the underlying lattice to the triangle lattice: this is the hard hexagons model, presented in Section 1.2.1.

1.1.3.2 Computability of the entropy Although we can not compute exactly the entropy of this model, it is known to be computable in an algorithmic sense. This means that there exists an algorithm which on input n outputs a rational number whose distance to the entropy is smaller than 2^{-n}.

This is a consequence of a theorem of R. Pavlov [Pav12], who proved that the entropy of this subshift is computable with tolerance ϵ in time which is polynomial in $1/\epsilon$. This derives also from the fact that this model is block gluing.

Suggestion 3. *Is it possible to change the model into a discrete curves model without entropy change ? This could further provide an example of constraint on curves to impose in order to explore the class of simple subshifts of finite type.*

1.2 On the other lattices

Problem 3. *Can we extract the exact properties of the honeycomb and triangular lattices which make the models on these lattices solvable, while the corresponding ones on the square lattice are not ?*

1.2.1 Hard hexagons

The hard hexagons is an equivalent of the hard squares on the triangular lattice. With a distortion of the lattice, one can see it as the subshift on lattice $\mathbb{Z}^2$ with additional edges (the ones which link the north west and south east corners of elementary squares of $\mathbb{Z}^2$) and on symbols $0, 1$, forbidding neighbors to have both 1 symbol.

Suggestion 4. *Is it possible to apply the same method in order to compute the entropy of similar models when the additional edges have longer range ? This would provide approximation from above of the entropy of hard squares.*

Question 1. *Is there a relation between the technique of additional edges and additional random bits ? How the two techniques can be combined ?*

1.2.2 Self avoiding walks

1.2.2.1 Description Self-avoiding walks on lattices were considered initially by the chemist P. Flory [Flo53] as a model for polymer chains. A walk on a lattice is a finite sequence of vertices such that two consecutive vertices are neighbors in the lattice ; it is self-avoiding when all these vertices are different. Denoting $c_n(\Lambda)$ the number of length n self avoiding walks on a lattice Λ, this sequence is submultiplicative and one can define an equivalent of the entropy called **connective constant**. The value $\sqrt{2+\sqrt{2}}$ was conjectured by B. Nienhuis [B.84] for the honeycomb lattice, and this was proved later by H. Duminil-Copin and S. Smirnov [HS12].

Remark 8. *Although this does not define a subshift of finite type, we present this model for the methods with which the value of the connective constant was proved. On the other hand, it is possible to define a bidimensional SFT whose configurations form covering self-avoiding walks of* $\mathbb{Z}^2$*. Its entropy is also not known.*

1.2.2.2 Partition function that relies total rotation angle This proof still relies on the definition of a partition function. To each symbol of a walk is attributed the coefficient x, and a length n walk is attributed with the coefficient x^n. This time the partition function is unique (this is not a family indexed by an integer n) and equal to

$$x \mapsto Z(x) = \sum_n c_n \cdot x^n,$$

where c_n denotes the number of length n self avoiding walks for the honeycomb lattice.

Suggestion 5. *We observe that partition functions can be defined for different objects (self-avoiding walks and square patterns) but similar and for similar purposes. However, a general definition of a partition function, that the difference in the use of these tools in the two contexts justifies, seems lacking.*

The connectivity constant is the radius of convergence of this series.

The proof of the value of the connective constant relies also on an auxiliary partition function such that each element of a walk is attributed with the coefficient $x \cdot e^{-i5\theta/8}$, where θ is the rotation angle of the walk at this point. The coefficient attributed to a finite walk γ is thus $x^{l(\gamma)} \cdot e^{-i5W(\gamma)/8}$, where $W(\gamma)$ is the total rotation angle of the walk and $l(\gamma)$ its length. The auxiliary partition function is, given a finite domain Ω of $\mathbb{Z}^2$,

$$H(z,x) = \sum_{\gamma \subset \Omega : \mapsto z} e^{-i5W_\gamma/8} \cdot x^{l(\gamma)}.$$

The constant 5/8 is chosen such that when $x = \sqrt{2+\sqrt{2}}$ there is a relation between the values of this function on neighbors points.

Suggestion 6. *This is one of the points where the proof breaks down on the square lattice. One could repair this defining coefficients taking into account the behavior of the walk in a larger range around the points, taking into account at each point, not only the rotation angle and the length addition but the evolution of more complex information transmitted along the walk.*

This auxiliary function allows the proof of a relation between the contribution to the partition function due to walks going in the top, bottom, left or right part of a strip domain. These relations, together with an evaluation of the evolution of these terms when the strip grows, allows the proof of the divergence of the series for $x = \sqrt{2+\sqrt{2}}$.

The convergence for x under this threshold relies on the possibility to decompose of any self-avoiding walk into maximal bridges, where a bridge is a walk which is included in a strip, start from the left side and end at the right side. It relies also on the convergence of the series for bridges, which comes from the relation proved earlier on the contributions to the partition function and the fact that the contribution of the right side is increasing in x.

Remark 9. *In the perspective of an adaptation of the proof to the square lattice, the notion of bridge seems to demand an adaptation, as well as this decomposition. Although, the possibility of this adaptation seems reasonable.*

Suggestion 7. *Self-avoiding walks considered by H. Duminil-Copin and S. Smirnov are walks from one half-edge to another. This does not change the asymptotic behavior of c_n but is useful for the method used. In other words, the walks are considered on the lattice whose edges are the vertices of the honeycomb lattice $\mathbb{H}$ and whose vertices are the edges of this lattice. This minor reformulation is similar to the one of the dimer model: is there some common reason for these changes?*

Suggestion 8. *In the same spirit as distortion configurations [Section 2.3] constructed with fixed length rigid segments, one could consider self-avoiding walks composed with fixed length rigid segments.*

1.2.2.3 Recent works on self-avoiding walks The most recent works on self-avoiding walks explore two directions.

The first one is concerned with extending the problem to groups and infinite graphs with a particular property: virtually indicable [possessing height function] groups [GZ] [here the walks are also weighted], graphs whose automorphism group has a transitive nonunimodular subgroup [T.] for instance. The main theorems relate the connective constant relative to bridges self-avoiding walks and the constant relative to general ones.

The second one focuses on constraints applied on self-avoiding walks: self-attracting ones [AT], weighting adjacent parallel edges, and prudent self-avoiding walks [M.10], extending the avoidance neighborhood around an already visited vertex.

2 Symbolic dynamics

In this section we present some research directions related the computation of the entropy of multidimensional subshifts of finite type, besides a systematic search of simple subshifts of finite type the computation of their entropy (in which one can focus for instance on particular subshifts exhibiting symetrical rules [CM18]).

In Section 2.1, Section 2.3 and Section 2.4, we expose some examples of subshifts coming from symbolic dynamics whose entropy is unknown. We expose in Section 2.2 a question related to the (algorithmic) computability of subshifts of finite type under constraints.

2.1 Aperiodic subshifts

Aperiodic subshifts of finite type appeared together with undecidability of the emptiness of plane tilings due to R. Berger. His construction, involving the implementation of Turing machines in a hierarchical structure. It was later simplified by R. Robinson, reducing the number of symbols involved (the interest of this reduction lying in the manipulation of the subshift for other constructions). Further reduction were done by J. Kari and K. Culik. Their construction consists in a coding of horizontal infinite sequences of symbols whose frequency are constrained and transformed in the vertical direction by multiplicating alternatively by a factor 2 or 3, using a Mealy machine (whose reading head moves in only one direction). The aperiodicity is ensured by the aperiodicity of this frequency, since none of the fractions $2^n/3^p$ can be equal to one. Let us note that the minimal number of symbols was reached recently by E. Jeandel and M. Rao, also using Mealy machines.

In constrast with hierarchical subshifts of finite type, and unexpectedly, the subshift exhibited by J. Kari and K. Culik has positive entropy [DGG14], using the notion of exchangeable pairs, whose appearance with positive frequency in at least a configuration ensure the positivity of the entropy. This opens the question:

Question 2. *What is the value of entropy for the subshift of J. Kari and K. Culik ?*

More generally, it would be interesting to understand the entropy of subshifts of finite type generated by the action of a Mealy machine on biinfinite words. There is a hope to adapt techniques developed for the six vertex model, since the subshift of J. Kari and K. Culik is ruled by the deterministic evolution of a simple global quantity.

2.2 Dynamical constraints and computability

It is well known that entropy of bidimensional subshifts of finite type is not computable in general [HM10], and that particular dynamical constraints (for instance block gluing, which consists in mixing with constant rate, verified for instance by the hard squares model), the entropy is a computable number (in an algorithmic sense).

Recently, S. Gangloff and M. Sablik [GS17] proposed a quantified version of this property with a rate function and proved that even under the constraint of block gluing with linear rate, the entropy is still uncomputable in general. The aim of this work was to characterize a threshold on the rate function separating a computability regime and an uncomputability one. This characterization seems highly difficult since we lack constructive methods in the sub-linear regime: the known construction use hierarchical structures which impose at least linearity.

It is noteworthy that the models for which a value is conjectured or computed (six vertex model, dimer model) are linearly block gluing, but not block gluing.

Question 3. *Is there another property of these models which added to linear block gluing ensures the computability of topological entropy?*

Suggestion 9. *One can for instance see that for these models, some pairs of square patterns can be glued with any intermediate minimal distance. It is possible that a constraint on the 'density function' of this minimal distance ensures the computability. One could also make an ordering according to computation difficulty in order to understand the properties adding difficulty in order to approach the computability threshold 'from below'. On can also consider the hard core model on any infinite graph and the contraints on the graph ensuring the computability of this model. This suggestion is related to the question of the rigidity induced by border conditions. For instance, for very rigid conditions, one is able to compute the growth rate (see on the conjecture on alternating sign matrices).*

For the dimer model, the minimal distance is k if and only if the maximal number of partial dimers on the sides is k. We have thus to estimate the numbers of patterns with this restriction.

2.3 Distortion subshifts

The proof of the characterization of the entropies of linear block gluing bidimensional SFT in [GS17] involves a family of operators allowing the linear block gluing property out of an other one which is simpler to obtain.

These operators consist in distorting the lines and and columns of the lattice $\mathbb{Z}^2$ through a subshift defining this distortion. The idea is to adapt the representation of the six-vertex model with discrete curves by imposing that two consecutive curves can not be spaced with a large gap. Moreover, these curves are made with rigid segments having constant length, which are superimposed with a word in $\{0^k 1^{r-k} : k \leq r\}$. To each length r corresponds a subshift denoted Δ_r. This formulation of the distortion operators allow them to transform the entropy adding a constant computable term, which is the entropy of Δ_r. It is non trivial to prove that this entropy is equal to $\frac{\log_2(1+r)}{r}$.

Suggestion 10. *One could modify the formulation of the subshifts Δ_r by adding constraints on the curves and attempt to compute entropy and extend the famility of subshifts of finite type whose entropy can be exactly computed.*

2.4 Density subshifts

In another work [GH17], a threshold was characterized for a quantified version of the irreducibility constraint one subshifts whose language is decidable. The proof of this theorem involves one-dimensional bounded density subshifts on alphabet $\{0,1\}$, previously defined by B. Stanley, whose rules consist in forbidding the number of 1 symbols in a length n sub-word to be greater than p_n. By controling the sequence $(p_n)_n$, one can have control over entropy.

Suggestion 11. *When the sequence (p_n) is periodic, the corresponding bounded density subshift is of finite type. Extending the definition to the bidimensional setting, this principle provides another family of natural examples.*

2.5 Symbolic dynamics on groups

Recently, developments have been made on symbolic dynamics on more general groups than $\mathbb{Z}^d$, $d \geq 1$. Examples include surfaces groups, Baumslag-Solitar group, free groups. We know for instance how to define the entropy on general infinite finitely generated groups [Bowen, then Seward]. It is rather easy to compute the entropy of the equivalent of the hard core model on free groups. This is an extreme counterpart to the additional edges technique, with difference here that the edges are suppressed. On the other hand, the extreme case when the graph is complete is also simple, since the entropy is zero. [Another way: reproduce the proof of Baxter on more various graphs] Another approach to compute the entropy of this subshift on $\mathbb{Z}^2$ would be to compute it for equivalent models on intermediate groups. The notion of intermediate group should rely on the 'quantity' of loops, where the rules of the hard core model make the combinatorics difficult.

Note: the entropy according to Bowen relies on pseudo-orbits. How is related this notion to additional random bits in symbolic dynamics ? Maybe cohomology would be a way to measure intermediacy of groups.

References

[AT] Hammond A. and Helmut T. Self-attracting self-avoiding walks. Prepublication.

[B.84] Nienhuis B. Critical behavior of two-dimensional spin models and charge asymmetry in the coulomb gas. *Journal of statistical physics*, 34:731–761, 1984.

[Bax82] R.J. Baxters. *Exactly solved models in statistical mechanics.* Dover Publications, 1982.

[CM18] N. Chandgotia and B. Marcus. Mixing properties for hom-shifts and the distance between walks on associated graphs. *Pacific journal of mathematics*, 294(1):41–69, 2018.

[DCGH+a] H. Duminil-Copin, M. Gagnebin, M. Harel, I. Manolescu, and V. Tassion. The bethe ansatz for the six-vertex and xxz models: an exposition. Prebublication.

[DCGH+b] H. Duminil-Copin, M. Gagnebin, M. Harel, I. Manolescu, and V. Tassion. Discontinuity of phase transition for planar random-cluster and potts models with q>4. Prepublication.

[DGG14] B. Durand, G. Gamard, and A. Grandjean. Aperiodic tilings and entropy. In *Developments in Language Theory*, pages 166–177, 2014.

[DPE] Alberici D., Contucci P., and Mingione E. Mean-field monomer-dimer models. a review. Prepublication.

[Flo53] P. Flory. *Principles of Polymer Chemistry.* Cornell University Press, 1953.

[GH17] S. Gangloff and B. Hellouin. Computability of the entropy of mixing subshifts. Dynamical systems, 2017.

[GS17] S. Gangloff and M. Sablik. Block gluing intensity of bidimensional sft: computability of the entropy and periodic points. Dynamical systems, 2017.

[GZ] Grimmett G.R. and Li Z. Weighted self-avoiding walks. Prepublication.

[HM10] M. Hochman and T. Meyerovitch. A characterization of the entropies of multidimensional shifts of finite type. *Annals of Mathematics*, 171:2011–2038, 2010.

[HS12] Duminil-Copin H. and Smirnov S. The connective constant of the honneycomb lattice equals $\sqrt{2+\sqrt{2}}$. *Annals of mathematics*, 175:1653–1665, 2012.

[Jac] J. L. Jacobsen. Lecture notes on exactly solved models.

[Lie67] E.H. Lieb. Residual entropy of square ice. *Physical Review*, 162(1):461–471, 1967.

[M.10] Bousquet-M. M. Families of prudent self-avoiding walks. *Journal of combinatorial theory*, 117:313–344, 2010.

[Pau35] L. Pauling. The structure and entropy of ice and of other crystals with some randomness of atomic arrangement. *Journal of the American Chemical Society*, 57:2680–2684, 1935.

[Pav12] R. Pavlov. Approximating the hard square entropy constant with probabilistic methods. *Annals of Probability*, 40:2362–2399, 2012.

[Pie08] L.A. Pierce. *Computing entropy for Z2-actions.* PhD thesis, Oregon state university, 2008.

[T.] Hutchcroft T. Self-avoiding walk on nonunimodular transitive graphs. Prepublication.