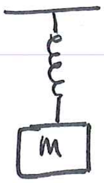
5.1 Linear Models (Spring-Mass Systems)

Spring-Mass System (Free undamped motion):



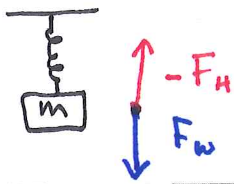
A spring-mass system consists of a mass m attached to one end of a spring that is fixed on the other end. If we displace m from its equilibrium position the system will oscillate. We will model this system with a DE, whose solution describes the motion (or position of at time t) of m.

Model's Ingredients

- Newton's Second Law: $F = m \cdot a = m \dfrac{d^2x}{dt^2}$
- Hooke's Law: The force needed to extend /compress a spring a distance s, is proportional to such distance

$$F = ks \qquad (k > 0)$$

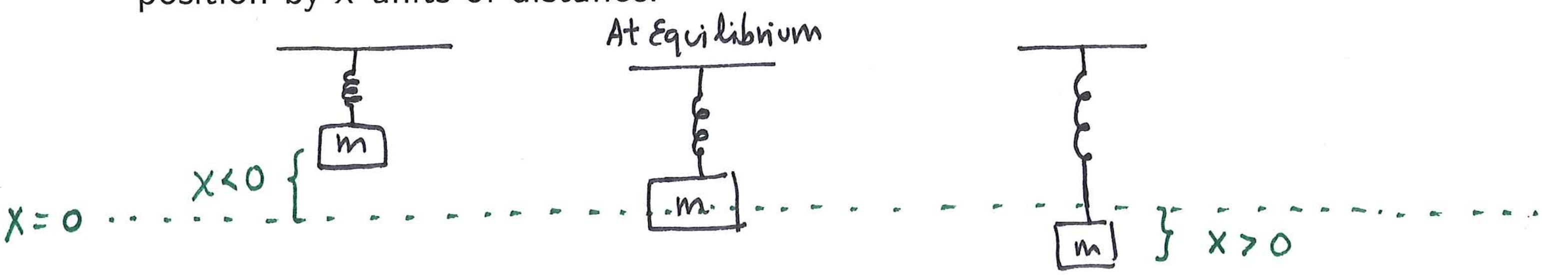
where k is called the spring constant. This is the restoring (reaction) force.



At equilibrium $F_w = F_H$

$$F_w - F_H = 0$$
$$mg - ks = 0$$

We want to describe the outcome when we shift m from its equilibrium position by x units of distance.



Restoring force is : $F = k(s+x) = ks + kx$

Weight is: $W = mg$

Resultant force is: $mg - ks - kx = -kx$

Newton's 2^{nd} Law $\Rightarrow$ $m \dfrac{d^2x}{dt^2} = -kx \iff m \dfrac{d^2x}{dt^2} + kx = 0$

$\dfrac{d^2x}{dt^2} + \dfrac{k}{m} x = 0$, we can call $\dfrac{k}{m} = \omega^2$ $\quad \underbrace{\dfrac{d^2x}{dt^2} + \omega^2 x = 0}$

DE that models simple harmonic motion (free undamped).

Solution: Auxiliary Equation: $r^2 + \omega^2 = 0 \Rightarrow r^2 = -\omega^2 \Rightarrow r = \pm \omega i$

$\Rightarrow$ General Sol: $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$.

Initial Conditions $x(0) = x_0$ (E.g. if >0, means below equilibrium)

$x'(0) = x_1$ (E.g. if <0, means upward velocity).

Definition

- The period of $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$ is $T = \frac{2\pi}{\omega}$

  (the time it takes to complete one cycle).

- The frequency of $x(t)$ is $f = \frac{1}{T} = \frac{\omega}{2\pi}$

  (Number of cycles per unit of time).

- The quantity $\omega = \sqrt{\frac{k}{m}}$ is called circular or angular frequency.

  (measured in radians).

Graph of $x(t)$

To sketch the graph of $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$ we will first rewrite $x(t)$ as

Recall:

$$A \sin(\omega t + \phi) = \underbrace{A \sin \phi}_{c_1} \cos \omega t + \underbrace{A \cos \phi}_{c_2} \sin \omega t$$

Note $c_1 = A \sin \phi$ and $c_2 = A \cos \phi$, so we can find A:

$c_1^2 + c_2^2 = A^2 \sin^2 \phi + A^2 \cos^2 \phi = A^2 \Rightarrow A = \sqrt{c_1^2 + c_2^2}$ and we could

find ϕ: $\frac{A \sin \phi}{A \cos \phi} = \tan \phi = \frac{c_1}{c_2} \Rightarrow \phi = \tan^{-1}\left(\frac{c_1}{c_2}\right)$ * just be

aware of the quadrant.

$\Rightarrow x(t) = A \sin(\omega(t + \phi/\omega))$.

Example

A force of 40 newtons stretches a spring 50 cm $= 1/2\,m$. A 20 kg mass is attached to the end of the spring and it is initially released from a point 1.5 meters above the equilibrium position with an upward velocity of 4 m/s. Determine the equation of motion for this mass (and sketch it).

Hooke's Law $\quad F = Ks \iff 40 = K\frac{1}{2} \implies K = 80$

Mass: $\quad m = 20\text{ kg} \implies$ DE is: $\dfrac{d^2x}{dt^2} + \dfrac{80}{20}x = 0$

Aux Eqn: $\quad r^2 + 4 = 0 \implies r = \pm 2i$

General Sol. $\quad x(t) = C_1\cos(2t) + C_2\sin(2t)$

<u>Initial Conditions:</u> $\quad x(0) = -1.5$ (initial position)

$x'(0) = -4$ (initial velocity).

Plug initial Conditions:

$x(0) = C_1 = -1.5$

$x'(0) = 2C_2 = -4 \implies C_2 = -2$

Sol. or Equation of motion: $x(t) = -1.5\cos(2t) - 2\sin(2t)$.

<u>Sketch</u> $\quad$ 1st Find $A = \sqrt{(-3/2)^2 + (-2)^2} = \sqrt{9/4 + 4} = \sqrt{\frac{25}{4}} = \frac{5}{2}$

2nd Find $\phi = \tan^{-1}\left(\frac{-3/2}{-2}\right) = \tan^{-1}(3/4) \simeq 0.6435$, but ϕ is in QIII

So $\quad \phi = \pi + 0.6435 \approx 3.78$

$\implies x(t) = \frac{5}{2}\sin(2t + 3.78) = \frac{5}{2}\sin(2(t + \underbrace{1.89}_{\text{shift}}))$