Show all of your work, and *please* staple your assignment if you use more than one sheet. Write your name, student ID, the course number, and the section on every sheet. Problems marked with * will be graded and one additional randomly chosen problem will be graded.

1. * Consider the following joint distribution for the weather in two consecutive days. Let X and Y be the random variables for the weather in the first and the second days, with the weather coded as 0 for sunny, 1 for cloudy, and 2 for rainy.

X \ Y	0	1	2
0	0.3	0.1	0.1
1	0.2	0.1	0
2	0.1	0.1	0

(a) Find the marginal probability mass functions for X and Y.

(b) Calculate the expectation and variance for X and Y.

(c) Calculate the covariance and correlation between X and Y. Are they correlated?

(d) Are the weather in two consecutive days independent?

Answer:

(a) The marginal distributions for X and Y are

x	0	1	2
$p_X(x)$	0.5	0.3	0.2

y	0	1	2
$p_Y(y)$	0.6	0.3	0.1

(b) The expectation and variance are

$$\begin{aligned}
E(X) &= (0)(0.5) + (1)(0.3) + (2)(0.2) = 0.7 \\
E(X^2) &= (0)^2(0.5) + (1)^2(0.3) + (2)^2(0.2) = 1.1 \\
Var(X) &= E(X^2) - [E(X)^2] = 1.1 - 0.7^2 = 0.61 \\
E(Y) &= (0)(0.6) + (1)(0.3) + (2)(0.1) = 0.5 \\
E(Y^2) &= (0)^2(0.6) + (1)^2(0.3) + (2)^2(0.1) = 0.7 \\
Var(Y) &= E(Y^2) - [E(Y)]^2 = 0.7 - 0.5^2 = 0.45
\end{aligned}$$

(c) We have:

$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

$$\begin{aligned}
E(XY) &= (0)(0)(0.3) + (0)(1)(0.5) + (0)(2)(0.1) \\
&\quad + (1)(0)(0.2) + (1)(1)(0.1) + (1)(2)(0) \\
&\quad + (2)(0)(0.1) + (2)(1)(0.1) + (2)(2)(0) \\
&= 0.3 \\
Cov(X, Y) &= .3 - (.7)(.5) = -0.05 \\
\rho = Corr(X, Y) &= \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}} = \frac{-0.05}{\sqrt{(0.61)(0.45)}} = -0.095
\end{aligned}$$

(d) X and Y are not independent since $Cov(X, Y) = -0.05 \neq 0$,
or
X and Y are not independent since

$$0 = \mathbb{P}(X = 2, Y = 2) \neq \mathbb{P}(X = 2)\mathbb{P}(Y = 2) = 0.2 \times 0.1 = 0.02$$

2. * Using the joint distribution table given in problem 1, calculate the following probabilities:

   (a) $\mathbb{P}(X = Y)$
   (b) $\mathbb{P}(X < Y)$
   (c) $\mathbb{P}(X > Y)$
   (d) Probability that the weather is sunny on two consecutive days.
   (e) Probability that the weather is cloudy on the first day, and rainy on the second day.

   Answer:

   (a) $\mathbb{P}(X = Y) = 0.3 + 0.1 + 0 = 0.4$
   (b) $\mathbb{P}(X < Y) = 0.1 + 0.1 = 0.2$
   (c) $\mathbb{P}(X > Y) = 0.2 + 0.1 + 0.1 + 0 = 0.4$
   (d) $\mathbb{P}(X = 0, Y = 0) = 0.3$
   (e) $\mathbb{P}(X = 1, Y = 2) = 0$

3. Suppose a fair coin is tossed 3 times. Let X = the number of heads on the last toss, and let Y = the total number of heads in the 3 tosses.

   (a) Write down the joint PMF for X and Y in table form.
   (b) Give $p_X(x)$ and $p_Y(y)$ in table form.
   (c) Find $\mathbb{P}(Y = 1|X = 1)$.
   (d) Are X and Y independent? Explain your answer.

   Answer:

   (a) The joint PMF for X and Y is

X \ Y	0	1	2	3
0	1/8	2/8	1/8	0
1	0	1/8	2/8	1/8

   (b)

x	0	1
$p_X(x)$	1/2	1/2

y	0	1	2	3
$p_Y(y)$	1/8	3/8	3/8	1/8

   (c)

$$\mathbb{P}(Y = 1|X = 1) = \frac{\mathbb{P}(Y = 1, X = 1)}{\mathbb{P}(X = 1)} = \frac{1/8}{1/2} = \frac{1}{4}$$

   (d) No, X and Y are not independent since

$$p_{X,Y}(0, 3) = 0 \neq (1/2)(1/8) = p_X(0)p_Y(3)$$

4. Suppose X and Y are two random variables and their joint pmf is given by this table:

X \ Y	2	3	4
1	1/12	1/6	0
2	1/6	0	1/3
3	1/12	1/6	0

(a) Find the marginal probability mass functions for X and Y.

(b) Show that X and Y are dependent.

(c) Give a joint probability table (like we have above for X and Y) for random variables U and V that have the same marginal distributions as X and Y respectively *but are* independent.

Answer:

(a)

x	1	2	3
$p_X(x)$	1/4	1/2	1/4

y	2	3	4
$p_Y(y)$	1/3	1/3	1/3

(b) X and Y are dependent since

$$p_{X,Y}(2,3) = 0 \neq (1/2)(1/3) = p_X(2)p_Y(3)$$

(c) U and V have the same marginal distribution as X and Y. So, their marginal probabilities are:

u	1	2	3
$p_U(u)$	1/4	1/2	1/4

v	2	3	4
$p_V(v)$	1/3	1/3	1/3

In order for U and V to be independent, we need $p_{U,V}(u,v) = p_U(u)p_V(v)$ for <u>all</u> (u,v) pairs. So, we construct the joint probability table for U and V as

- $p_{U,V}(1,2) = p_U(1)p_V(2) = (1/4)(1/3) = 1/12$
- $p_{U,V}(1,3) = p_U(1)p_V(3) = (1/4)(1/3) = 1/12$
- $\dots$

U \ V	2	3	4
1	1/12	1/12	1/12
2	1/6	1/6	1/6
3	1/12	1/12	1/12

5. * Suppose a continuous random variable X has the following probability density function

$$f_X(x) = \begin{cases} cx & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of c that makes $f_X(x)$ a valid probability density function. (Recall a property that a PDF must have)

(b) Give the CDF, $F_X(x)$.

(c) Find $\mathbb{P}(0.5 \leq X \leq 1.5)$ using $f_X(x)$.

(d) Find $\mathbb{P}(1 \leq X \leq 2)$ using $F_X(x)$.

(e) Find the value of x such that the probability of being less than x is .75

(f) Find $\mathbb{E}(X)$.

(g) Find $Var(X)$.

Answer:

(a) For a PDF to be valid, $\int_{-\infty}^{\infty} f_X(x)dx = 1$. We have:

$$\int_0^2 cx = 1$$
$$\left.\frac{cx^2}{2}\right|_0^2 = 2c$$
$$\to 2c = 1 \to c = \frac{1}{2}$$

So the final valid PDF is

$$f_X(x) = \begin{cases} \frac{x}{2} & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(b) $X < 0 \to F_X(x) = 0, x > 2 \to F_X(x) = 1$. For $x \in [0, 2]$ we have:

$$\begin{aligned} F_X(t) &= P(X \le t) \\ &= \int_0^t \frac{x}{2} dx \\ &= \left.\frac{x^2}{4}\right|_0^t \\ &= \frac{t^2}{4} \end{aligned}$$

$$F_X(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{4} & 0 \le t \le 2 \\ 1 & t > 2 \end{cases}$$

(c) $\int_{.5}^{1.5} \frac{x}{2} dx = \left.\frac{x^2}{4}\right|_{.5}^{1.5} = \frac{1.5^2}{4} - \frac{.5^2}{4} = .5625 - .0625 = 0.50$

(d) $\mathbb{P}(1 \le X \le 2) = F_X(2) - F_X(1) = \frac{2^2}{4} - \frac{1^2}{4} = 1 - .25 = 0.75$

(e) We want x such that $\mathbb{P}(X \le x) = .75$. Set the CDF $= .75$ and solve for x.
$\frac{x^2}{4} = .75 \to x = \sqrt{4(.75)} = 1.73$

(f) $\mathbb{E}(X) = \int_0^2 x f_X(x)dx = \int_0^2 \frac{x^2}{2} dx = \left.\frac{x^3}{6}\right|_0^2 = \frac{8}{6}$

(g) We already have $\mathbb{E}(X)$, we just need $\mathbb{E}(X^2)$ and can use the short cut formula for variance.
$\mathbb{E}(X^2) = \int_0^2 x^2 f_X(x)dx = \int_0^2 \frac{x^3}{2} dx = \left.\frac{x^4}{8}\right|_0^2 = 2$
So, $Var(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = 2 - (\frac{8}{6})^2 = .222$

6. A supplier of kerosene has a 150-gallon tank that is filled at the beginning of each week. His weekly demand shows a relative frequency behavior that increases steadily up to 100 gallons and then levels off between 100 and 150 gallons. If X denotes weekly demand in hundreds of gallons, the relative frequency of demand can be modeled by

$$f_X(x) = \begin{cases} x & 0 \le x \le 1 \\ 1 & 1 < x \le 1.5 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $F_X(x)$. (Remember to cover all cases)

(b) Find $\mathbb{P}(0.5 \le X \le 1.2)$.

(c) Find $\mathbb{E}(X)$.

Answer:

(a) $X < 0 \to F_X(x) = 0, x > 1.5 \to F_X(x) = 1$.

For $x \in [0,1]$ we have $\int_0^t x dx = \frac{x^2}{2}\Big|_0^t = \frac{t^2}{2}$

For $x \in [1, 1.5]$ we have $\int_0^1 x dx + \int_1^t 1 dx = \frac{x^2}{2}\Big|_0^1 + x\Big|_1^t = t - \frac{1}{2}$

$$F_X(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t \le 1 \\ t - \frac{1}{2} & 1 < t \le 1.5 \\ 1 & t > 1.5 \end{cases}$$

(b) $\mathbb{P}(0.5 \le X \le 1.2) = F_X(1.2) - F_X(.5) = (1.2 - \frac{1}{2}) - (.5^2/2) = 0.575$

(c) We have to be careful because the PDF has two parts:

$$\begin{aligned} \mathbb{E}(X) &= \int_0^{1.5} x f_X(x) dx \\ &= \int_0^1 x^2 dx + \int_1^{1.5} x dx \\ &= \frac{x^3}{3}\Big|_0^1 + \frac{x^2}{2}\Big|_1^{1.5} \\ &= \frac{23}{24} = .9583 \end{aligned}$$