Sample Midterm Exam 1 Solution

- Total duration: 60 minutes.
- Please **write your name, netid, and recitation section** on the top of this page.
- You **can** use two pages as cheat sheets.
- You **cannot** consult your notes, textbook, your neighbor, or Google.
- Maximum points: 50 (any score above 50 will be rounded to 50).

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1. **(6 points)** Consider the propositional variables:

   q: You can ride the rollercoaster.

   r: You are under 4 feet tall.

   s: You are older than 16.

Express the following natural language statements in terms of these variables and appropriate logical connectives.

(a) You are under 4 feet tall and you are not older than 16.

$r \wedge \neg s$

(b) If you are under 4 feet tall, then you cannot ride the rollercoaster.

$r \implies \neg q$

(c) You can ride a rollercoaster only if you are over 4 ft tall or you are older than 16.

$q \implies (\neg r \vee s)$

2. **(4 points)** For each of the below propositions:

   (i) $p : \forall x, \exists y, 2x - y = 0$

   (ii) $q : \forall x, \exists y, 2y - x = 0$

indicate (no explanations necessary) which propositions are true when:

(a) the domain of discourse is the nonnegative integers.

Answer (i) True (ii) False

(b) the domain of discourse is the real numbers.

Answer (i) True (ii) True

3. **(10 points)** This is a 3-part question.

(a) (4 points) Express each of the three sentences below using propositional variables: “If the shoe does not fit, then he is acquitted. He isn’t acquitted. Therefore, the shoe fits.”

(b) (1 point) Identify the rule of inference implicitly used in this argument.

(c) (5 points) Prove that this particular rule of inference is *valid* in general using a truth table.

Solution

(a) p: The shoe does not fit. q: He is acquitted.

$$\begin{array}{r} p \implies q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

(b) Modus Tollens

(c)

p	q	$p \implies q$	$\neg q$	$\neg p$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Looking at the fourth row, it shows that when $p \implies q$ and $\neg q$ are both true, $\neg p$ is also true. Therefore Modus Tollens is valid.

4. **(10 points)** Prove that $\sqrt{\frac{1}{5}}$ is irrational.

Points will be given for (a) clearly stating your proof technique in the beginning, (b) listing assumptions, if any, and (c) finishing your proof with a concluding statement. You are free to use any familiar facts that you know about integer arithmetic.

Solution

Suppose, to the contrary, that $\sqrt{\frac{1}{5}}$ is not irrational; hence, it must be rational.

By the definition of a rational number, then $\sqrt{\frac{1}{5}}$ can be written as $\frac{a}{b}$ for $a, b \in Z$ and $b \neq 0$. Let $\frac{a}{b}$ be in lowest terms, which means that a and b have no common factors.

Thus, $\sqrt{\frac{1}{5}} = \frac{a}{b}$, so we square both sides to get that $\frac{1}{5} = \frac{a^2}{b^2}$.

Then, we cross multiply to obtain that $b^2 = 5a^2$. This means that b^2 is a multiple of 5. By Euclid's lemma, we can then conclude that b is also a multiple of 5 also since 5 is a prime number.

Since b is a multiple of 5, we can express b as $5k$ for some $k \in Z$. If we substitute this in the equation above, we get that $(5k)^2 = 5a^2$.

Then, $25k^2 = 5a^2$, so $5k^2 = a^2$. However, by the same logic as above, this implies that a is a multiple of 5.

This is a contradiction because a and b share the common factor 5.

Therefore, our initial assumption was incorrect; in fact, $\sqrt{\frac{1}{5}}$ is irrational. **QED.**

Note: Another way to solve this question would be to prove this in two steps: first prove that $\sqrt{5}$ is irrational and then prove that the reciprocal of an irrational number is also irrational. However, if you tried to do it this way, you must **prove** that $\sqrt{5}$ is irrational, which many people failed to do. The point of this question was to **prove** that it is irrational, so you must write some **relevant** proof for this problem.

5. **(10 points)** The following table lists driving distances between some California cities (in miles):

- Barstow to Fresno: 245 miles
- Eureka to Fresno: 450 miles
- Barstow to LA: 115 miles
- Eureka to LA: 645 miles
- Fresno to LA: 220 miles
- San Diego to Barstow: 175 miles
- San Diego to LA: 125 miles

Draw the above information in the form of a graph. Clearly label the nodes, define what you mean by an edge, and draw all relevant edges. Assign "weights" to the edges according to the driving distance information provided.

Using the graph, figure out a round trip (no proof needed) starting from San Diego and visiting all other cities using as few miles as possible.

solution: A node represents a city. An edge represents the path between two different cities with a weight of actual distance in miles.

The shortest round trip of visiting every cities starting from San Diego: "San Diego - LA - Eureka - Fresno - Barstow - San Diego". (Reverse path also works.)

Many different looking drawings, one potential one is shown below:

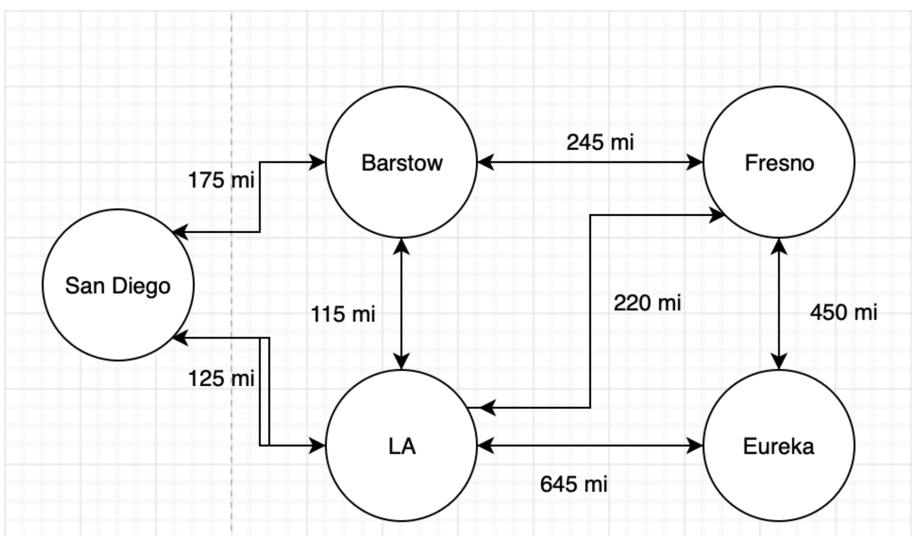

Figure 1: cities.png

6. **(5 points)** Which of the following is true and which is false? No explanations or proofs necessary.
   (a) $3 \in \{1, 2, 3\}$. **True**
   (b) $\{2\} \in \{1, 2\}$. **False**
   (c) $1 \in \{1\}$. **True**
   (d) $\{1\} \subseteq \{1, 2\}$. **True**
   (e) $\{1\} \subseteq \{1, \{2\}\}$. **True**
   (f) $\{1\} \subseteq \{1\}$. **True**
   (g) $\mathbb{Z}^+ \subseteq \mathbb{Q}$. **True**
   (h) $\mathbb{Q} \subseteq \mathbb{Z}$. **False**

(i) $\emptyset \subset \mathbb{R}$ **True**

(j) $0 \in \mathbb{Z}^+$ **False**

7. **(10 points)** Let a, b, c be three arbitrary positive real numbers. Define their *arithmetic mean* as $\frac{a+b+c}{3}$ and their *geometric mean* as $\sqrt[3]{abc}$. Prove that if the arithmetic mean is different from the geometric mean, then the three numbers cannot all be equal to each other.

Solution We shall prove this by contradiction. Let's say that if the arithmetic mean and geometric mean are differnet the three numbers are equal to each other.

If $a = b = c$ we can write the arithemetic mean as:

$\frac{a+b+c}{3} = \frac{3a}{3} = a$

The geometric mean will be: $\sqrt{3}abc = \sqrt{3}a^3 = a$ So the arithemetic mean is equal to the geometric mean. This is a contradiction. Therefore if the means are different, the three numbers cannot be equal.