6.1 Power Series

We will use power series to solve differential equations by assuming that the solution function y is a power series (recall looks like "an infinite" polynomial), and proceed in a similar manner to the method of undetermined coefficients to find the coefficients of the sought series.

We will recall some definitions and facts about infinite series.

Definition

A power series centered at a is an infinite series of the form:

$$\sum_{n=0}^{\infty} C_n(x-a)^n$$

*We will mostly consider power series centered at $a = 0$.

Some Facts about Series:

- Recall $\sum_{n=0}^{\infty} C_n(x-a)^n = \lim_{N\to\infty} \sum_{n=0}^{N} C_n(x-a)^n$, and the series converges if the limit exists.
- The series might converge only for some values of x, or for x in some interval I, which is called interval of convergence.
- The radius of convergence R, is the radius of the interval of convergence, that is, the series converges for x such that $|x-a| < R$

  E.g. $R=0$, the series converges only for $x=a$

  $R=\infty$, the series converges for all real values of x.
- Absolute Convergence: A series $\sum_{n=0}^{\infty} C_n(x-a)^n$ converges absolutely if $\sum_{n=0}^{\infty} |C_n(x-a)^n|$ converges. Recall: Absolute Convergence => converges

- If $f(x) = \sum_{n=0}^{\infty} C_n(x-a)^n$, then $f(x)$ is a continuous differentiable function whenever x lies in the interval of convergence of the series and $f'(x)$ can be found term by term, that is,

$$f'(x) = \sum_{n=0}^{\infty} nC_n(x-a)^{n-1} = \sum_{n=1}^{\infty} n\,C_n(x-a)^{n-1}$$

- Ratio Test: (main tool to find R and the interval of convergence) With all $C_n \neq 0$, if the following limit exists:

$$\lim_{n\to\infty}\left|\frac{C_{n+1}(x-a)^{n+1}}{C_n(x-a)^n}\right| = |x-a|\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right| = L,$$

we have:
If $L<1$, the series converges absolutely,
if $L>1$, the series diverges,
if $L=1$, the test is inconclusive.
To find R, we assume $L<1$ and find for which values of x that holds.

Example. Find the radius of convergence of $\sum_{n=1}^{\infty}\frac{2^n}{n}(x-2)^n$. (& interval of conv.)

set $L<1$

$$|x-2|\lim_{n\to\infty}\frac{2^{n+1}}{n+1}\cdot\frac{n}{2^n} = |x-2|\,2\lim_{n\to\infty}\frac{n}{n+1} = 2|x-2| < 1$$

$$|x-2| < \underbrace{1/2 = R.}$$

$$-1/2 < x-2 < 1/2 \iff 3/2 < x < 5/2 \leftarrow \text{open int. of conv.}$$

Check Endpts

At $x = 3/2$ we have $\sum_{n=1}^{\infty}\frac{2^n}{n}(-1/2)^n = \sum_{n=1}^{\infty}(-1)^n\cdot\frac{1}{n}$ By A.S.T it converges

At $x = 5/2$ we have $\sum_{n=1}^{\infty}\frac{2^n}{n}(1/2)^n = \sum_{n=1}^{\infty}\frac{1}{n}$ Harmonic Series, diverges.

$\therefore$ Interval of Convergence $[3/2, 5/2)$

- Identity Property. If $\sum_{n=0}^{\infty} C_n(x-a)^n = 0$ for all x in some open interval then $C_n = 0$ for all n.

- Special Power Series Representations. Infinitely differentiable functions, such as, $e^x, \sin x, \cos x, \ln x$, etc., can be represented by

$$\text{Taylor series of } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

or

$$\text{Maclauring Series of } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

- Addition of Power Series (term by term) same

$\sum_{n=k}^{\infty} a_n x^n + \sum_{n=k}^{\infty} b_n x^n = \sum_{n=k}^{\infty}(a_n + b_n)x^n$

Same

Example. Write $S = \sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+1}$ as one power series.

Need
1. index starts at the same value k
2. Nee the powers of x to be in phase.

Rewrite $S = 2\cdot 1\, C_2 X^0 + \sum_{n=3}^{\infty} n(n-1)\, C_n X^{n-2} + \sum_{n=0}^{\infty} C_n X^{n+1}$

Re index, let $i = n-3 \Rightarrow n-2 = i+1 \quad n-1 = i+2$
$n = i+3 \quad$ when $n=3,\ i=0$

$S = 2C_2 + \sum_{i=0}^{\infty}(i+3)(i+2)C_{i+3} X^{i+1} + \sum_{n=0}^{\infty} C_n X^{n+1}$

$S = 2C_2 + \sum_{n=0}^{\infty}\left[(n+3)(n+2)\, C_{n+3} + C_n\right] X^{n+1}$ > same

OR $S = 2C_2 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)\, C_{n+2} + C_{n-1}\right] X^n$

Example. Find a power series solution $y = \sum_{n=0}^{\infty} C_n x^n$ of the differential equation: $y' + y = 0$

reindex

$$y' = \sum_{n=0}^{\infty} n C_n x^{n-1} = \sum_{n=1}^{\infty} n C_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) C_{n+1} x^n$$

$$+ \quad y = \sum_{n=0}^{\infty} C_n x^n$$

$$y' + y = \sum_{n=0}^{\infty} \left((n+1) C_{n+1} + C_n\right) x^n = 0$$

By Identity property: for all n we have: $(n+1) C_{n+1} + C_n = 0$, that is, $\boxed{C_{n+1} = -\frac{C_n}{n+1}}$ (we obtain a recurrence relation.)

$n=0 \quad C_1 = -\frac{C_0}{1} = -\frac{1}{1} C_0$

$n=1 \quad C_2 = -\frac{C_1}{2} = -\frac{1}{2}\left(-\frac{C_0}{1}\right) = \left(-\frac{1}{1}\right)\left(-\frac{1}{2}\right) C_0$

$n=2 \quad C_3 = -\frac{C_2}{3} = \left(-\frac{1}{1}\right)\left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right) C_0$

$n=3 \quad C_4 = -\frac{C_3}{4} = \left(-\frac{1}{1}\right)\left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{4}\right) C_0$

$\vdots$

Then $C_n = (-1)^n \frac{1}{n!} C_0$ and $y = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C_0 x^n$

initial condition

Note that $C_0 = y(0)$, and in this exaple we can recognize $\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$ so $y = C_0 e^{-x}$