Homework 6 Solutions

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1. **(10 points)** Prove, using mathematical induction, that for any $n \geq 1$:

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution

Let $P(n)$ be the proposition that $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \geq 1$.

Base Case:

$P(1)$ is true because:

$1^2 = \frac{1(1+1)(2+1)}{6} = 1$

Induction Hypothesis:

Assume that $P(k)$ is true:

$1^2 + 2^2 + 3^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$

Intductive Step:

Then $P(k+1)$ is true because:

$P(k+1) = 1^2 + 2^2 + 3^2 + ... + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$

$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$

$= \frac{(k+1)(2k^2+7k+6)}{6}$

$= \frac{(k+1)(k+2)(2k+3)}{6}$

$= \frac{(k+1)(k+2)[2(k+1)+1]}{6}$

taking $n = k+1$, the given expression is true.

2. **(10 points)** Prove the *Prime Factorization Theorem* (PFT) using strong induction. The statement of the PFT is

   Every positive integer n can be expressed as a product of 1 or more prime numbers.

   For example, $6 = 2 \times 3$, $7 = 7$, $8 = 2 \times 2 \times 2$, etc.

Solution

Predicate: $P(n)$: n can be factorized as a product of 1 or more prime numbers. $(n \geq 2)$

Base Case: $P(2) = 1 \times 2$. Therefore, it is true.

Strong Induction Hypothesis: Assume $P(m)$ is true for $\forall m \leq n$. (m is positive integer greater or equal to 2).

Induction Step:

Case 1: If $n+1$ is a prime number, it is trivial that $P(n+1)$ is true.

Case 2: If $n+1$ is not a prime number, then there exists a divisor x satisfying $1 \leq x \leq n+1$. Then, there also should exist an integer $1 \leq y \leq n+1$ such that $n+1 = x \cdot y$. Using the strong induction hypothesis, $n+1 = P(x) \cdot P(y)$. Since $P(x)$ and $P(y)$ are the prime factorization of x and y respectively, it implies that P(n+1) is true.

Therefore, every integer n can be expressed as a product of 1 or more prime numbers.

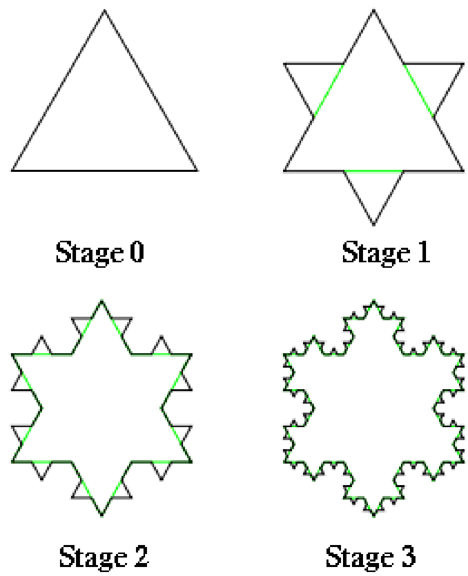

Figure 1: Koch snowflake

3. **(10 points)** A *Koch snowflake* is created by starting with an equilateral triangle with sides one unit in length. Then, on each side of the triangle, a new equilateral triangle is created on the middle third of that side. This process is repeated continuously, as shown in Figure 1.

   Prove that the number of sides (colored in black) for the n^{th} Koch snowflake is given by $3 \cdot 4^n$.

Solution

We will use induction to prove this theorem. Note that we consider the first Koch snowflake (the triangle) to be the 0^{th} snowflake.

Base case: For $n = 0$, $0^{th} numSides = 3 = 3(4^0)$.

Induction Hypothesis: For some $n \in \mathbb{N}$, $n^{th} numSides = 3(4^n)$.

Induction Step: Suppose the induction hypothesis is true. Then, for some $n \in \mathbb{N}$, we have the following:

$n^{th} numSides = 3(4^n)$.

The key here is to understand what happens to each side during this process. By adding an equilateral triangle along the middle third, we essentially create 4 new sides: the top third, the bottom third, and the two exposed sides of the added triangle. This occurs for each side, so the number of sides increases by a factor of 4 in this process.

By the above reasoning, we obtain the final result:

$(n+1)^{th} numSides = 4(n^{th} numSides) = 4(3(4^n)) = 3(4^{n+1})$.

As this equation simply replaces n by $n+1$ in the induction hypothesis, the proof is finished.

4. **(20 points)** Let us play the following single-player game.

   You begin with a single pile of n marbles. In the first move, you arbitrarily partition it into two piles (they could have unequal numbers of marbles – totally up to you).

   In the second move, you pick any one of the two piles and partition it. And so on: in the i^{th} move, you pick any pile and split into two piles as you wish.

   You are awarded **points** in each move as follows: if the pile you just split gives two new piles of sizes a and b, then you get ab points.

   The game continues until you cannot continue any more i.e., you have each of the marbles in its own separate file.

   Here is a trial of me playing the above game for $n = 5$.

- Initial pile size: 5, 0 points
- Move 1: Split the pile; pile sizes: $(3, 2)$; points tally: $3 \times 2 = 6$ points
- Move 2: Split second pile; pile sizes: $(3, 1, 1)$; points tally: $6 + (1 \times 1) = 7$ points
- Move 3: Split first pile; pile sizes: $(1, 2, 1, 1)$; points tally: $7 + (1 \times 2) = 9$ points
- Move 4: Split second pile; pile sizes: $(1, 1, 1, 1)$; points tally: $9 + (1 \times 1) = 10$ points

  So in this case, my final points tally is 10.

(a) Play the game out yourself two separate times. Start with $n = 7$ marbles. As I did above, clearly describe the sequence of moves that you carried out, and the points tally at the end of each move.

  If you did it correctly, you will find that *no matter what* your sequence of moves is, if you start with n marbles you will always end up with a score of $n(n-1)/2$.

  We prove this curious fact using **strong** induction.

(b) Clearly state the induction hypothesis in terms of a predicate $P(n)$.

(c) Prove the base cases $P(1)$ and $P(2)$.

(d) Assume the (strong) induction hypothesis that $P(a)$ is true for every $a \leq n$. Use this to prove that $P(n+1)$ is true.

Solution (a)

- (7); score = 0
- (4, 3); score = 0 + 4 × 3 = 12
- (2, 2, 3); score = 12 + 2 × 2 = 16
- (2, 2, 2, 1); score = 16 + 2 = 18

- (1, 1, 2, 2, 1); score = 18 + 1 = 19
- (1, 1, 1, 1, 2, 1); score = 19 + 1 = 20
- (1, 1, 1, 1, 1, 1, 1); score = 20 + 1 = 21 = $\frac{7\times 6}{2}$

(b) $P(k) = \frac{k(k-1)}{2}$, where P(k) is the score of a game with k marbles and $1 \le k \le n$.

(c) When playing with one marble, no piles can be split, so $P(1) = 0 = \frac{1\times 0}{2}$ When playing the game with two marbles, the pile of two can only be split once resulting in two piles of one, so $P(2) = 1 = \frac{2\times 1}{2}$.

(d) According to Strong Induction, assume $P(a)$ is true for all $a \le n$. Now, consider a game being played with $n+1$ marbles. On the first move, we split the marbles in two parts, a and b. Here, $a + b = n + 1$. Points scored with this move are $a \times b = ab$.

Now, as $a + b = n + 1$, both $a, b \le n$. So, according to our assumption, $P(a), P(b)$ is true. So, total points scored by further splitting the pile of a marbles till end would be $P(a)$. The same way, total points scored by further splitting the pile of b marbles till end would be $P(b)$. So, the final score of this game will then be $P(n+1) = ab + P(a) + P(b)$ where the first term is the score of the first move, and the following two terms are the scores that can be obtained from splitting the remaining piles.

So, $P(n+1) = ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} = \frac{2ab+a^2-a+b^2-b}{2} = \frac{(a^2+2ab+b^2)-(a+b)}{2} = \frac{(a+b)^2-(a+b)}{2}$

Now, substituting $a + b = n + 1$,

$= \frac{(n+1)^2-(n+1)}{2} = \frac{(n+1)(n+1-1)}{2} = \frac{(n+1)(n)}{2} = P(n+1)$

Hence Proved.