Sample Midterm Exam 2

- There are 6 questions in this exam, totaling 50 points.
- Total duration: 60 minutes.
- Please **write your name and netid** on the top of this page.
- You **can** use two pages as cheat sheets.
- You **cannot** consult your notes, textbook, your neighbor, or Google.

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1. **(10 points)** Consider the following relations defined on $\mathbb{Z}^+$:

   $R_1 = \{(x, y) \mid x + y > 10\}$

   $R_2 = \{(x, y) \mid y \text{ divides } x\}$

   $R_3 = \{(x, y) \mid x \text{ and } y \text{ have no common divisors}\}$

Indicate which of these relations is/are:

(a) reflexive.

(b) symmetric.

(c) antisymmetric.

(d) transitive.

(e) irreflexive.

2. **(5 points)** A sequence $a_0, a_1, a_2, ...$ is defined by letting $a_0 = 3$ and $a_k = a_{k-1}^2$ for all integers $k \geq 1$.

(a) Evaluate the first four elements (a_0 through a_3) of this sequence.

(b) Write down a closed form expression for a_n. (No proof necessary.)

3. **(10 points)** The set $A = \{2, 4, 5, 10, 12, 20, 25, 30\}$ is partially ordered with respect to the "divides" relation.

(a) Draw the Hasse diagram representation of the above relation.

(b) List all minimal and maximal elements.

(c) Run topological sort on the Hasse diagram to obtain a compatible total ordering of the elements.

4. **(10 points)** Let $S = \{1, 2, 3, 4, 5\}$ and let $A = S \times S$ (i.e., A consists of all pairs of elements from S). Define the following relation R on A as follows: $(a, b)R(a', b')$ if and only if $ab' = a'b$.

Prove that R is an equivalence relation.

5. **(5 points)** Recall that a (complete) binary tree is a graph that is constructed by starting from the *root* connected to a pair of children nodes (called leaves), and recursively adding a pair of children nodes to each leaf node. The number of layers in a complete binary tree (excluding the root) is called the *depth* of the tree.

(a) **(1 point)** Draw the complete binary trees of depth n for $n = 2$ and $n = 3$, and count the **total** number of nodes in each of these trees.

(b) **(4 points)** Via mathematical induction, provide a (very simple) proof of the fact that a tree of depth n contains 2^n **leaf** nodes.

6. **(10 points)** A *complete bipartite* graph is an undirected graph with $m+n$ nodes, where each of the first m nodes are connected with each of the last n nodes. Assume for the sake of this problem that m and n are both greater than 2.

(a) Use the First Degree theorem to count the number of edges in this graph.

(b) What is the minimum number of colors needed to color the vertices of such a graph so that no adjacent vertices have the same color?

(c) Under what conditions on m and n does this graph admit an Euler path?

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