Alpha-Beta Pruning

Outline

I. Alpha-beta pruning

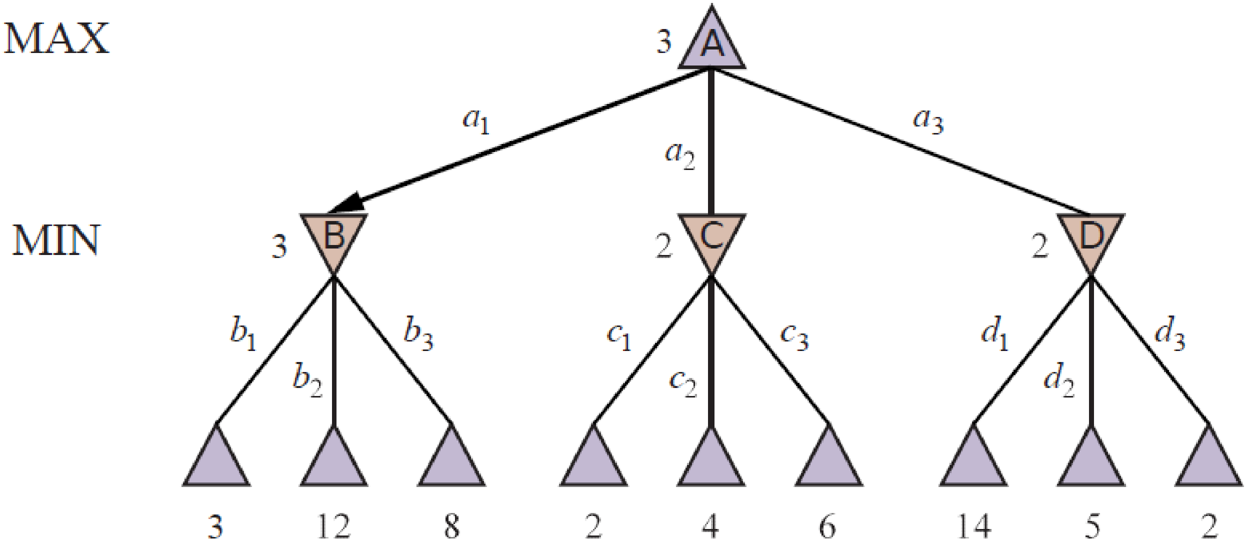
II. Heuristic alpha-beta tree approach

* Figures/images are from the textbook site (or by the instructor). Otherwise, the source is cited unless such citation would make little sense due to the triviality of generating the image.

I. Alpha-Beta Cutoff

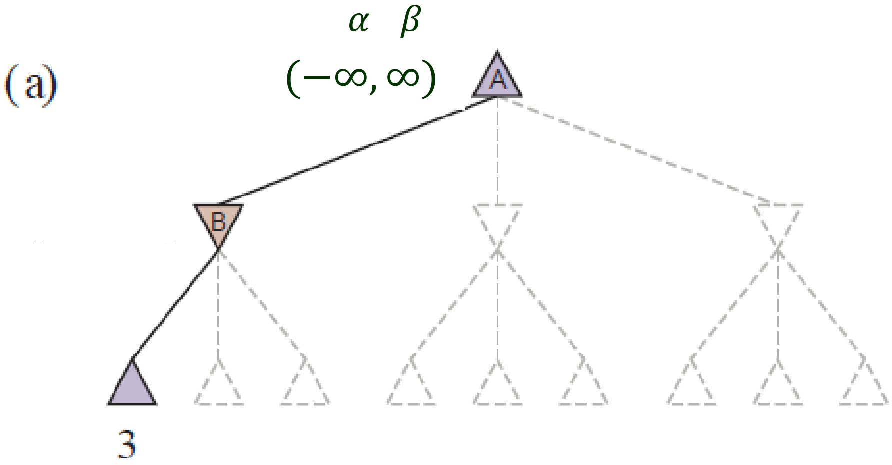
♣ #states is exponential in the depth of the game tree.

♦ But we can often compute the correct minimax decision by pruning large parts of the tree that do not affect the outcome.



Re-examining the Game Tree

Fig. 5.5 in the textbook *incorrectly executes* the algorithm in Fig. 5.7.



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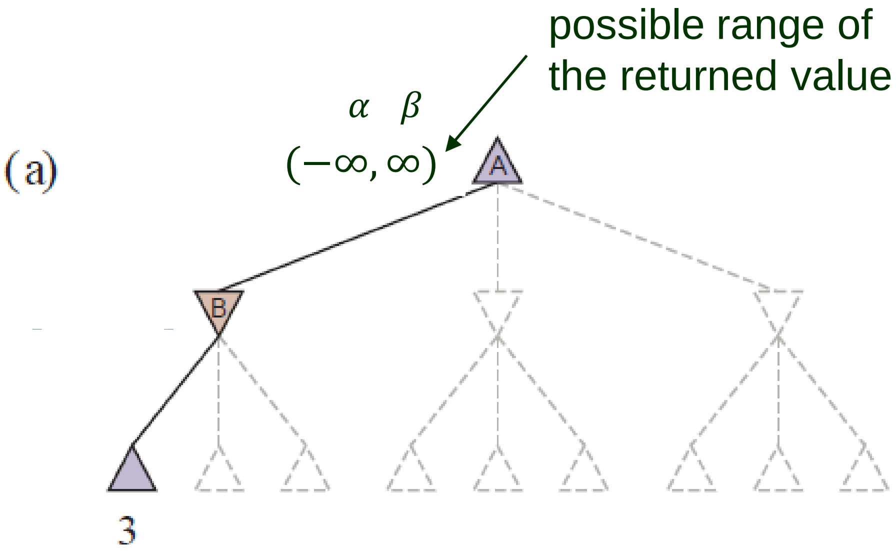

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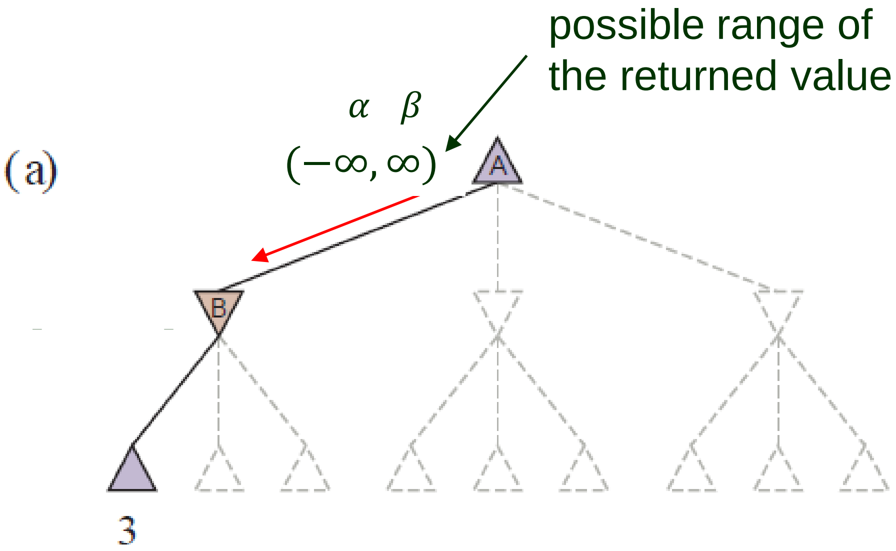

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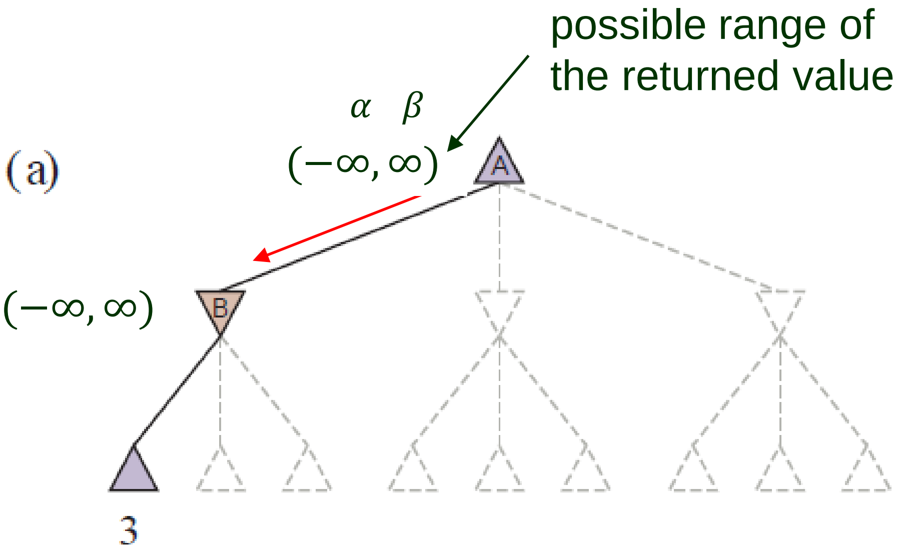

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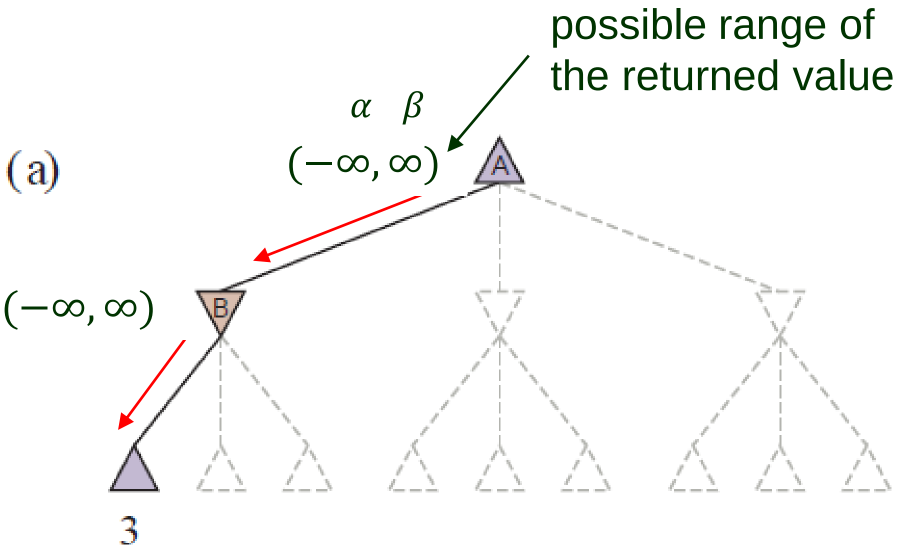

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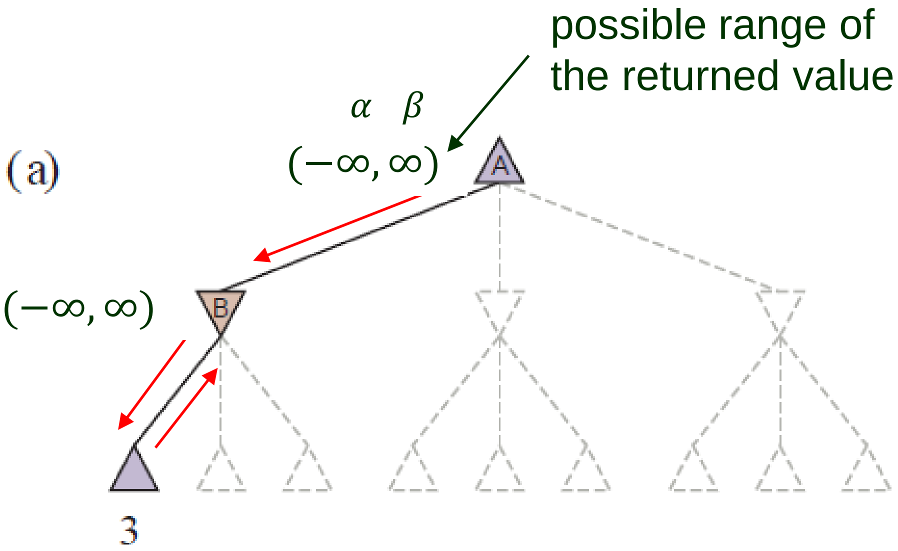

possible range of
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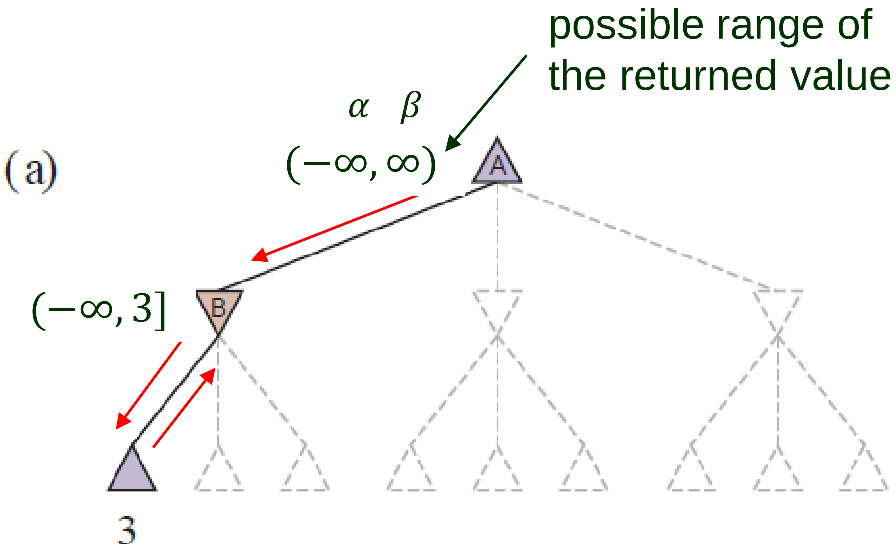

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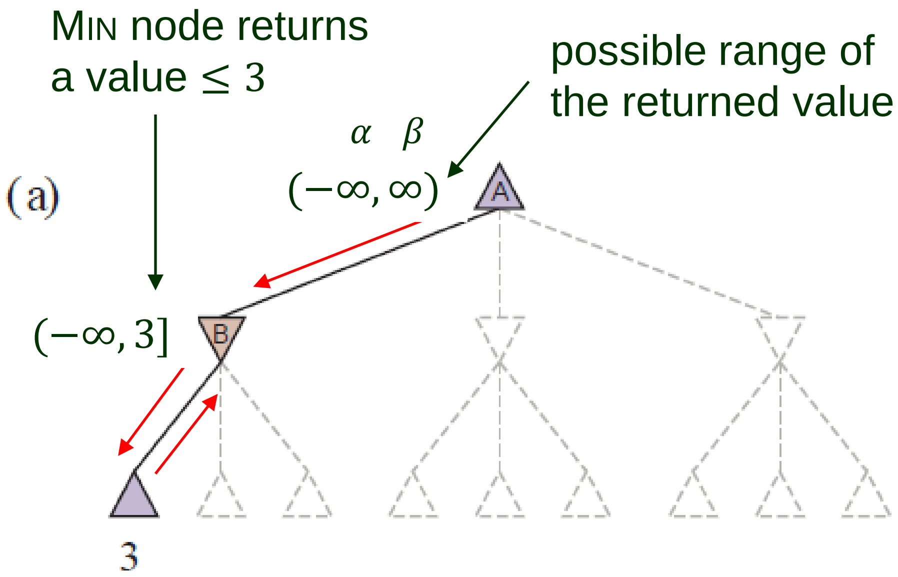

MIN node returns a value ≤ 3

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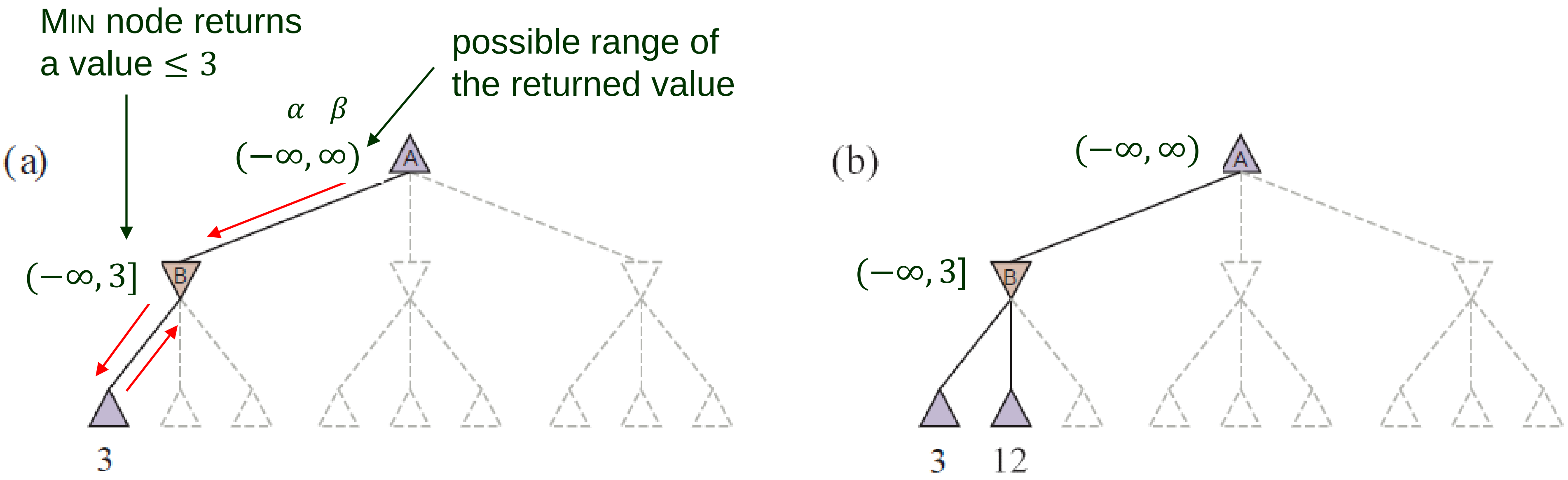

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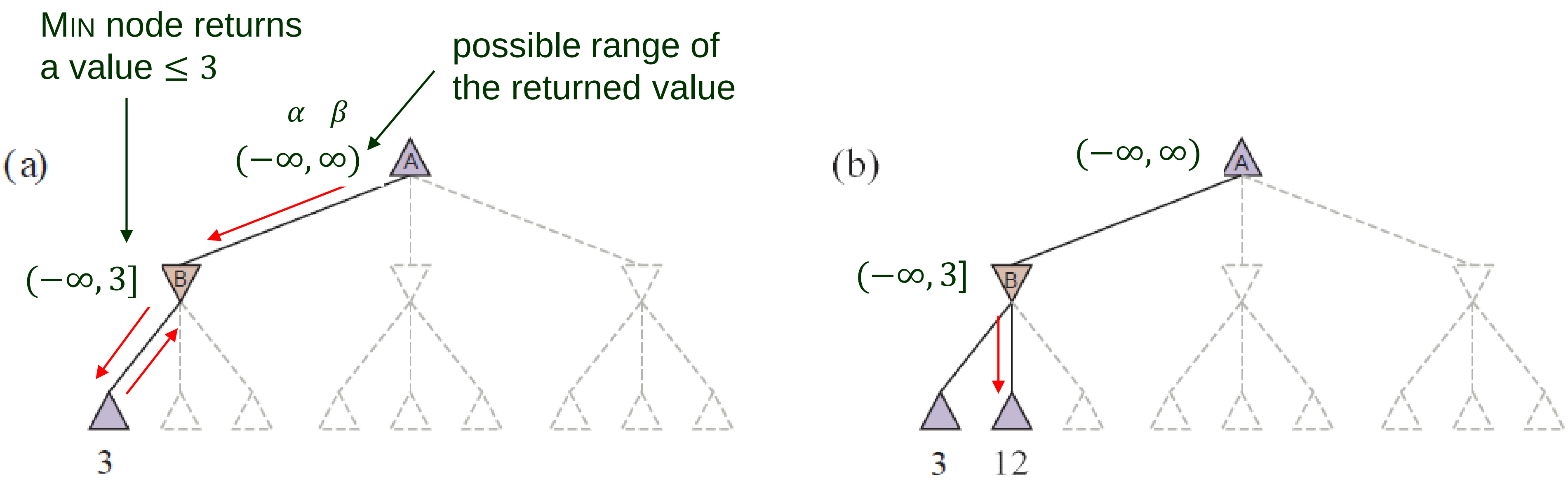

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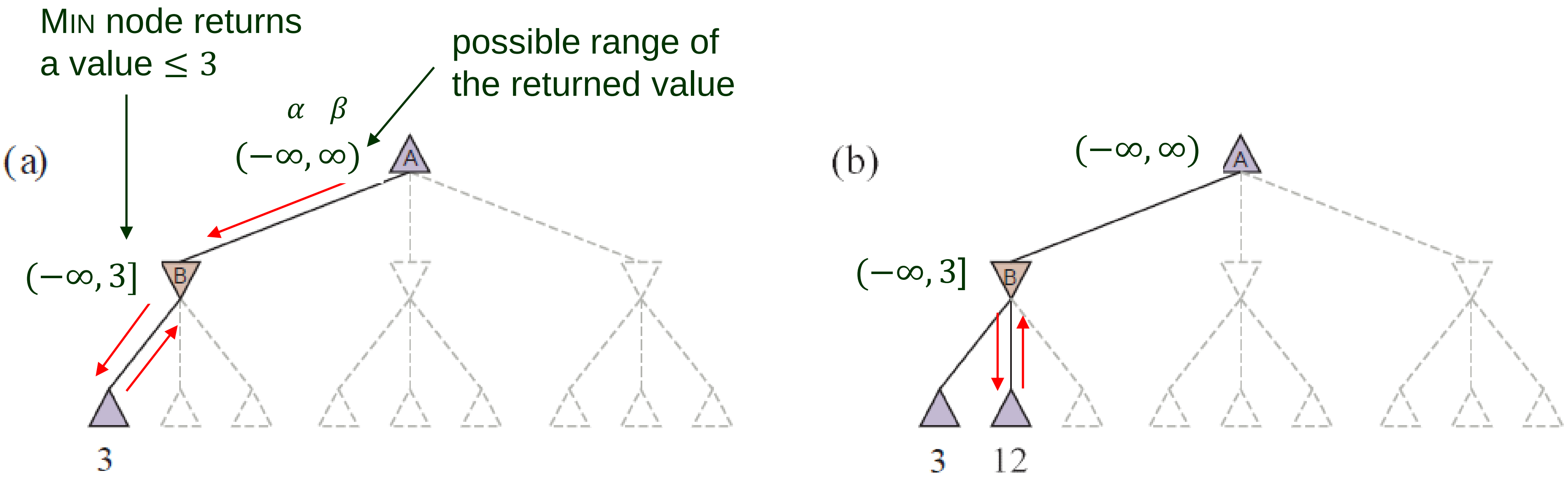

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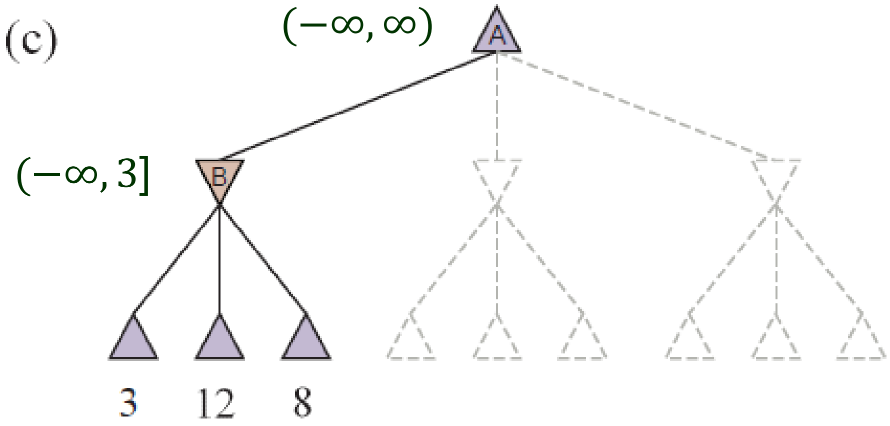

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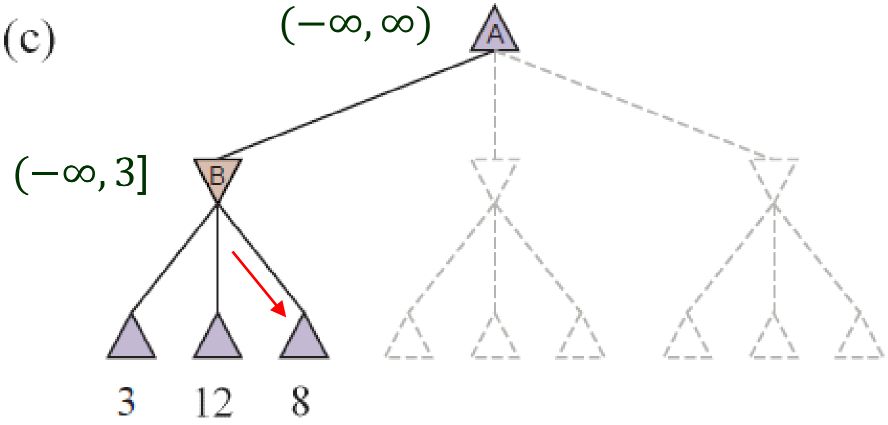

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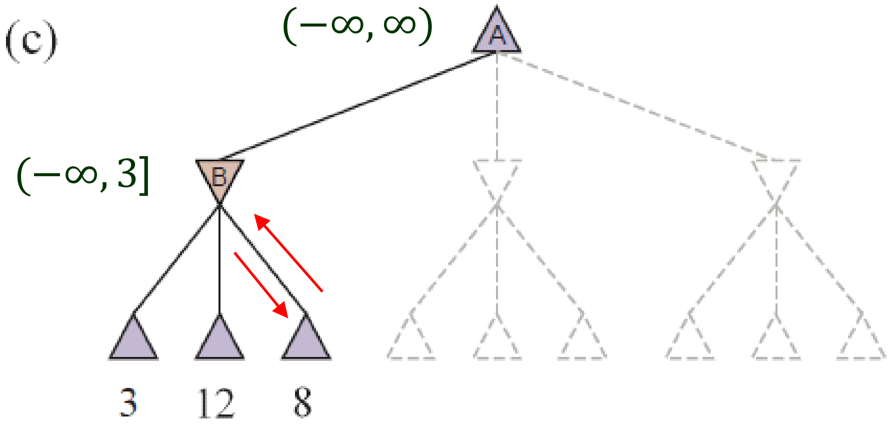

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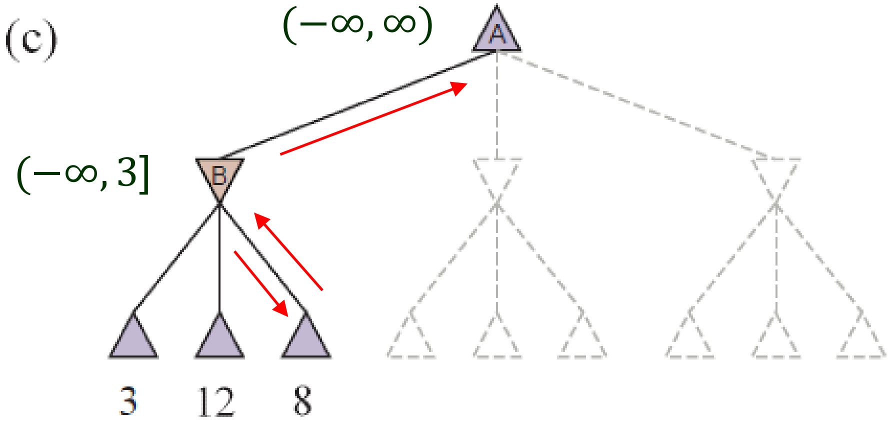

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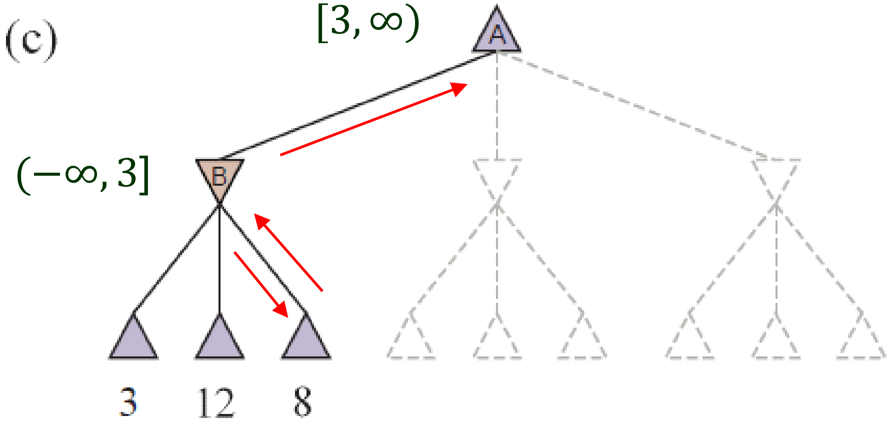

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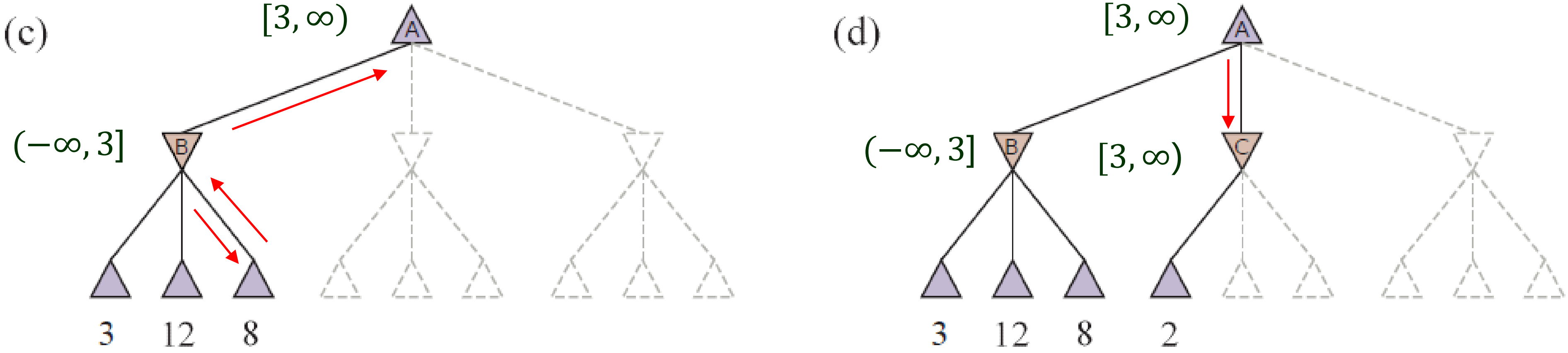

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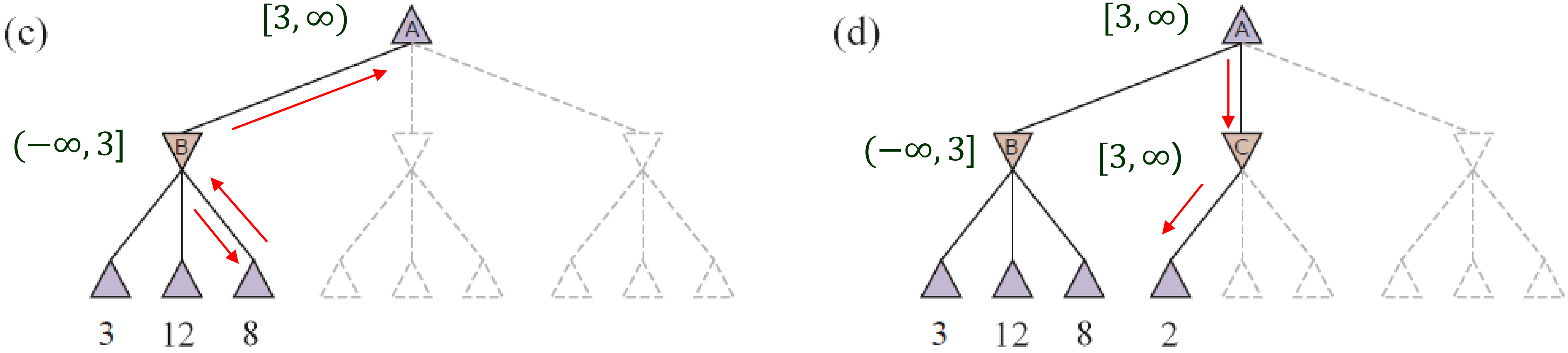

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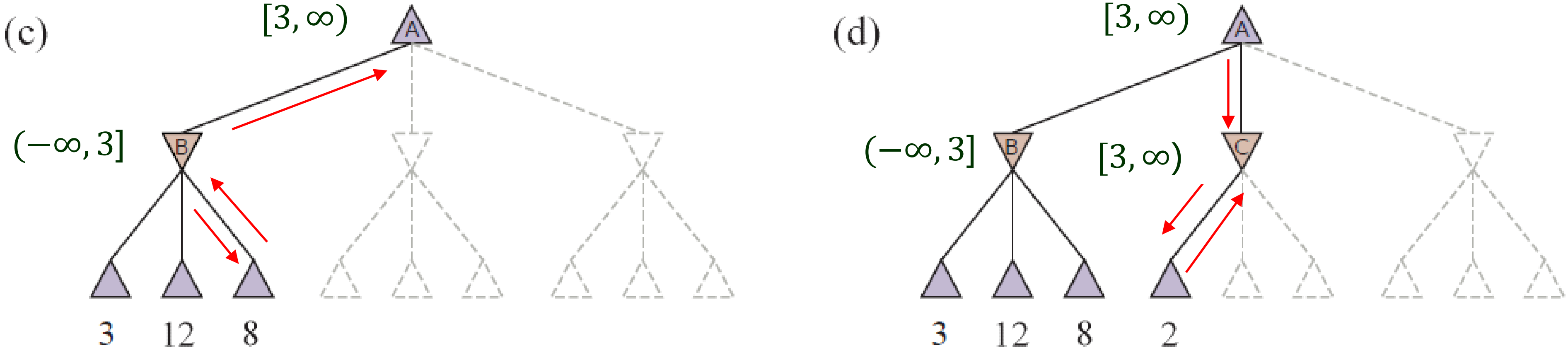

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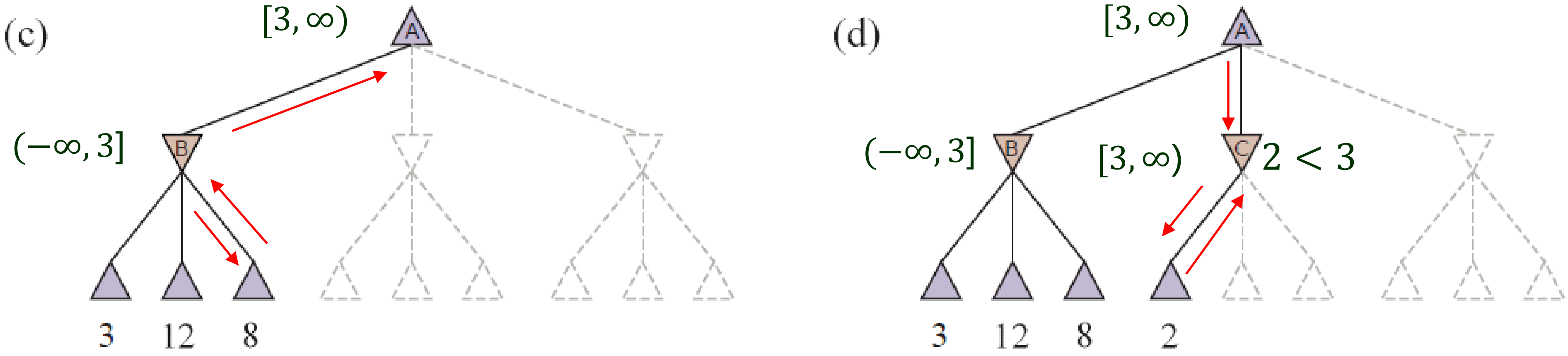

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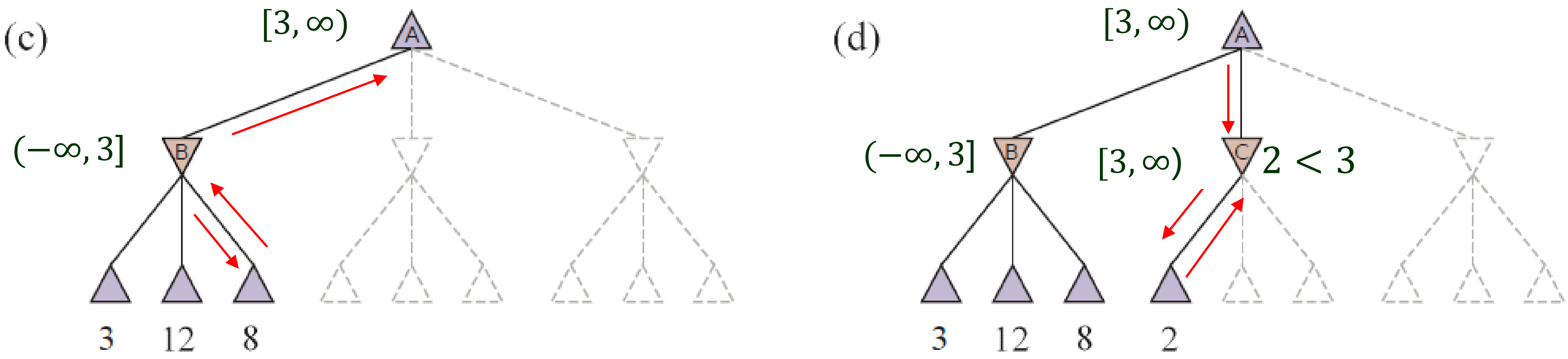

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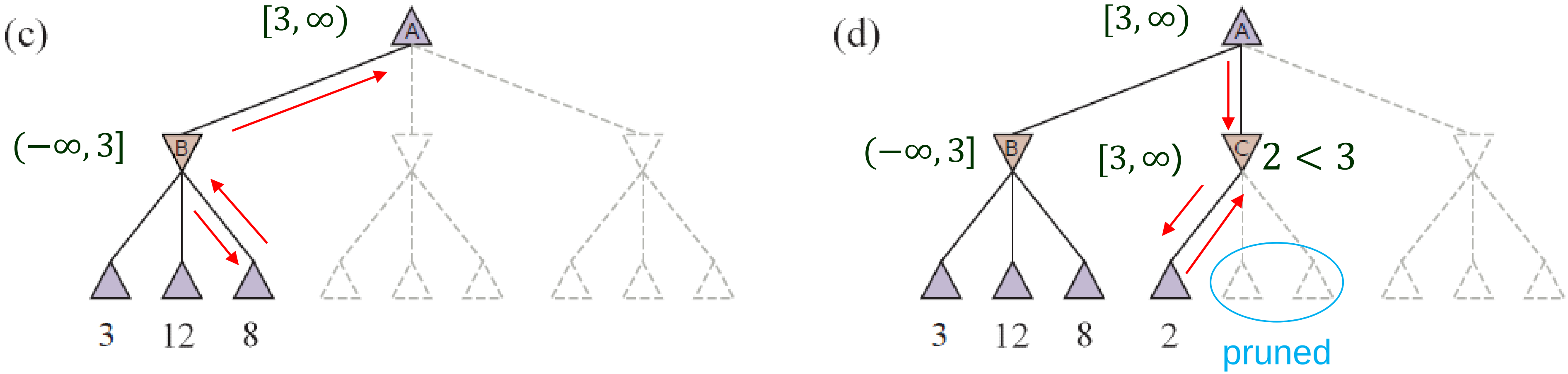

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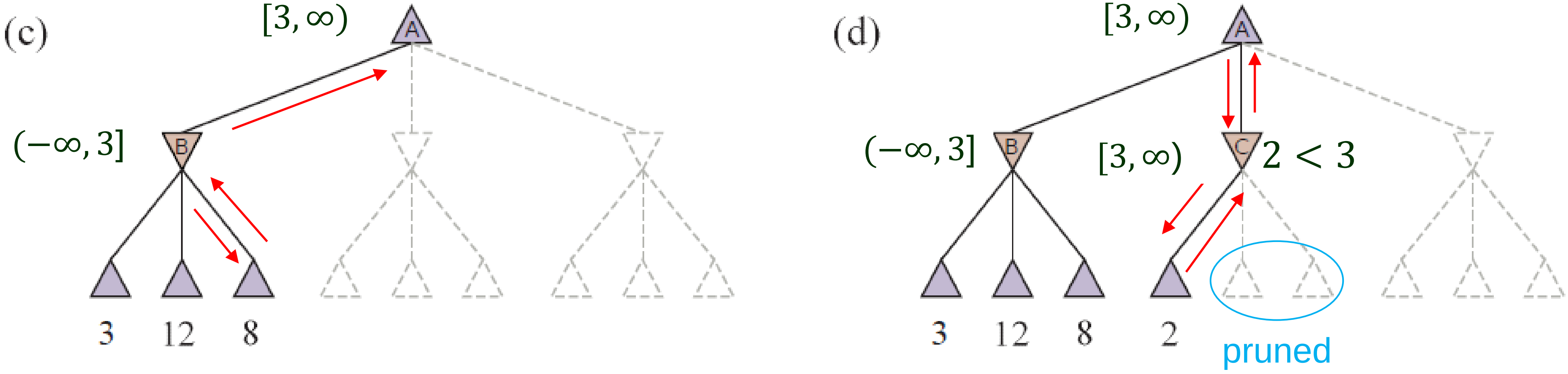

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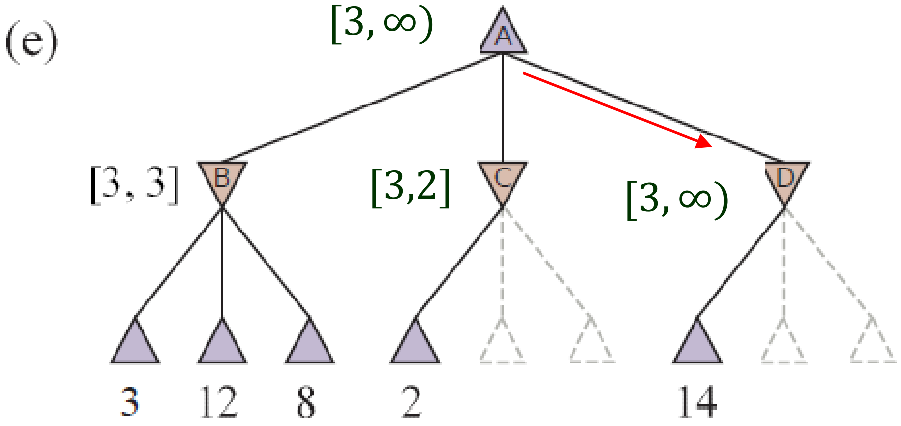

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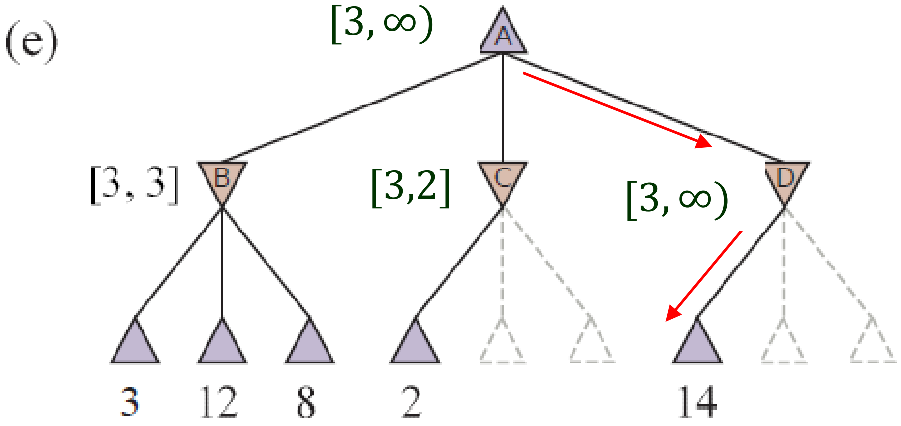

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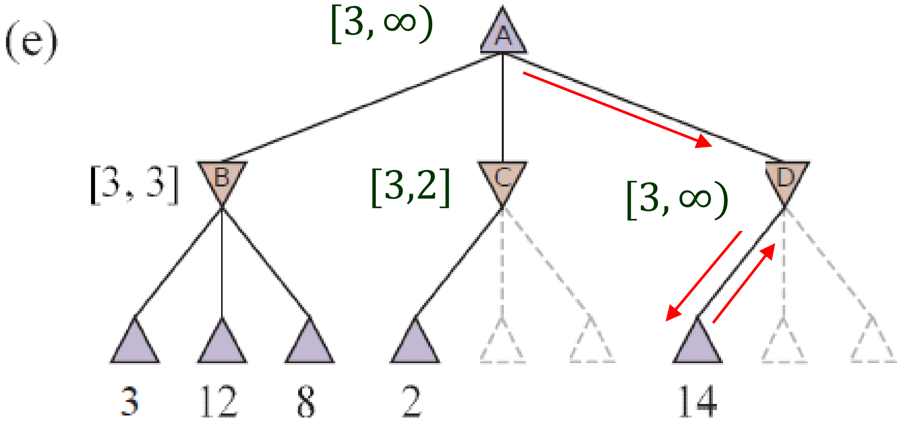

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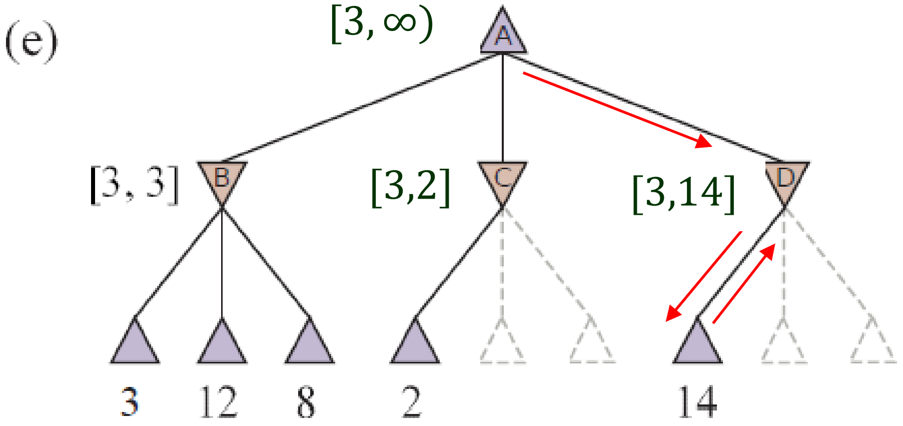

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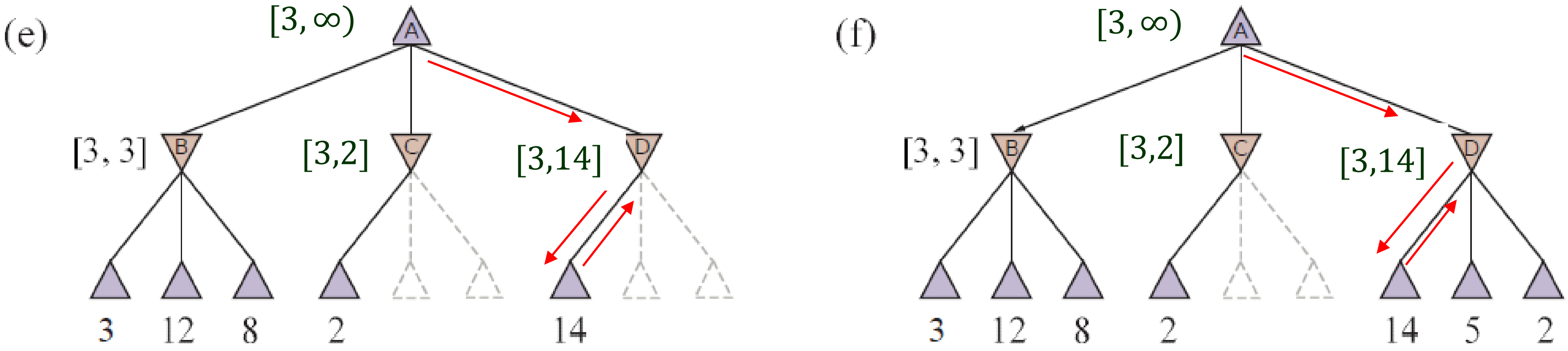

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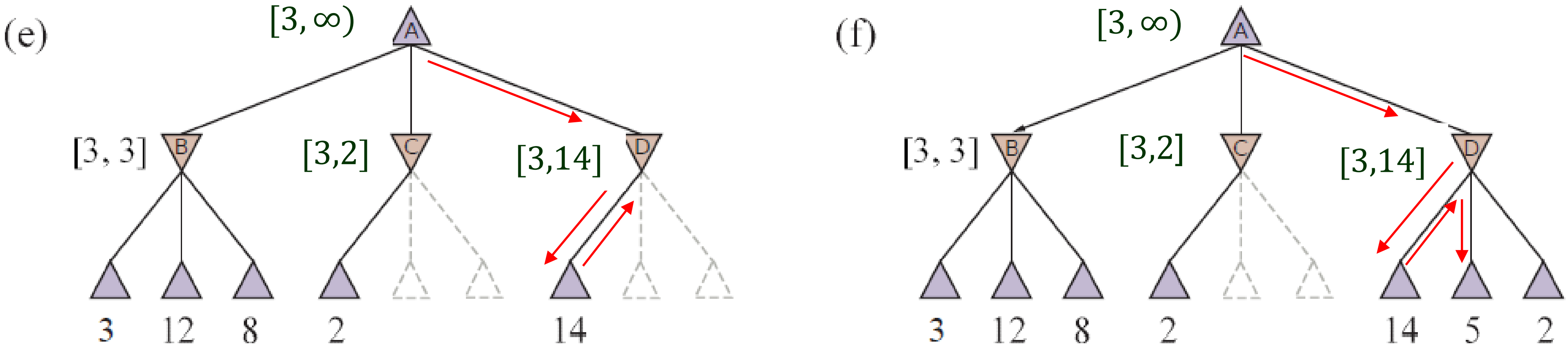

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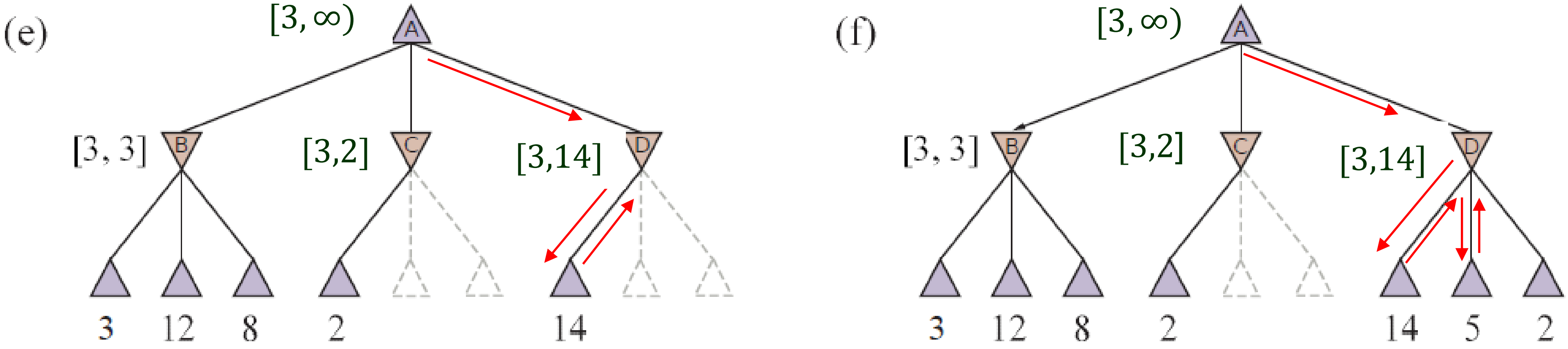

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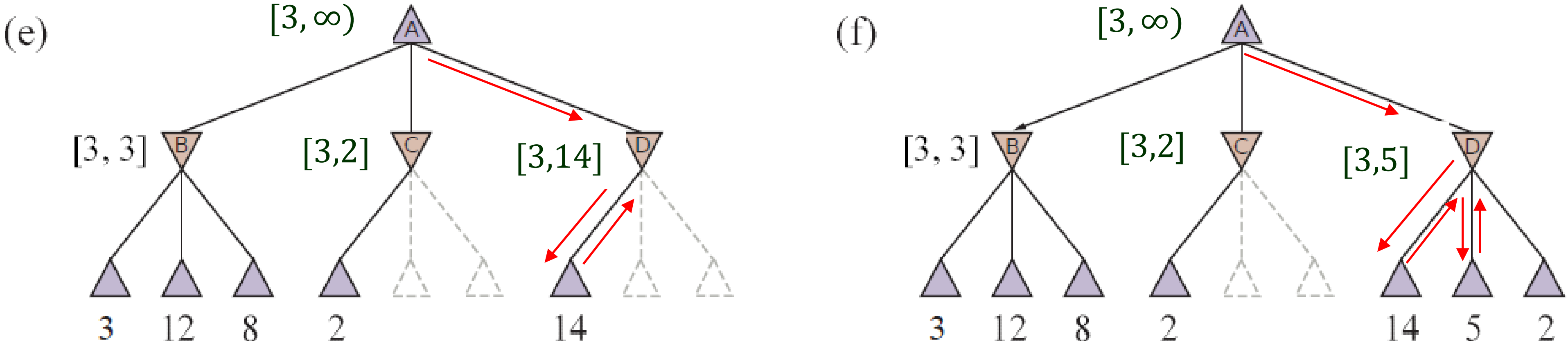

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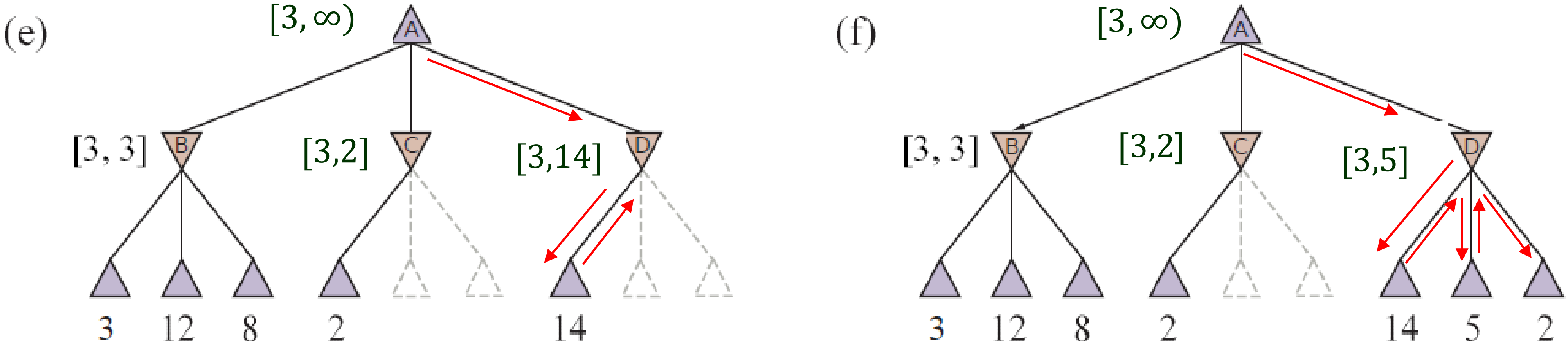

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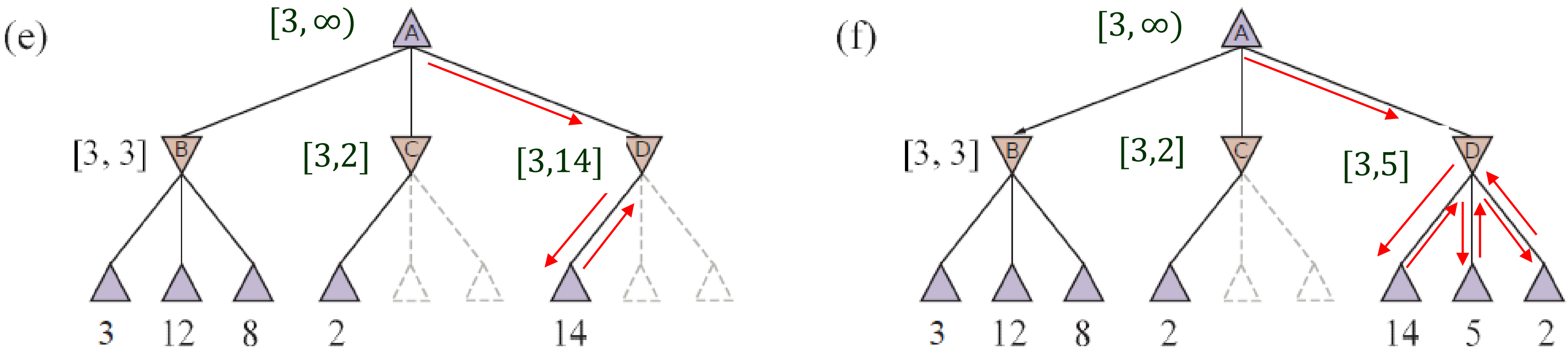

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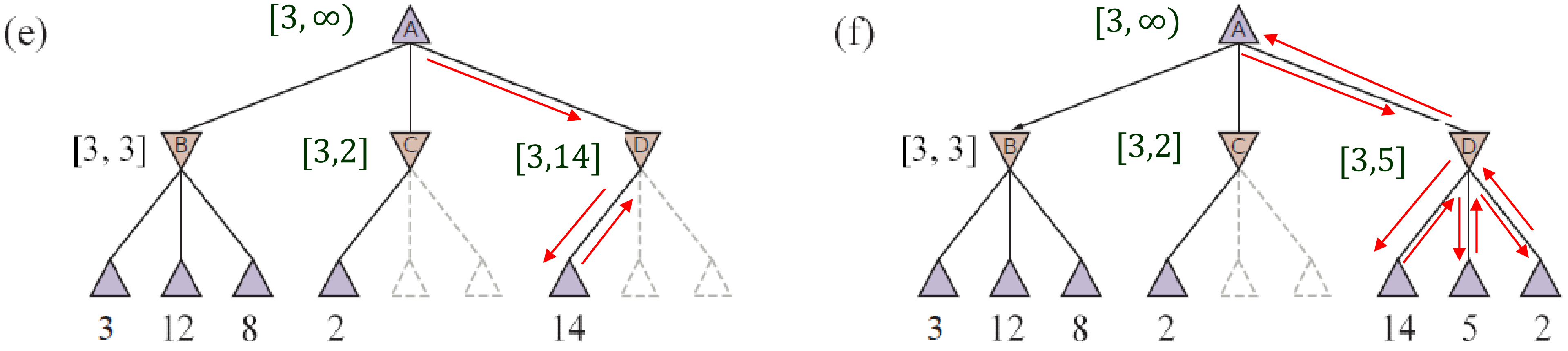

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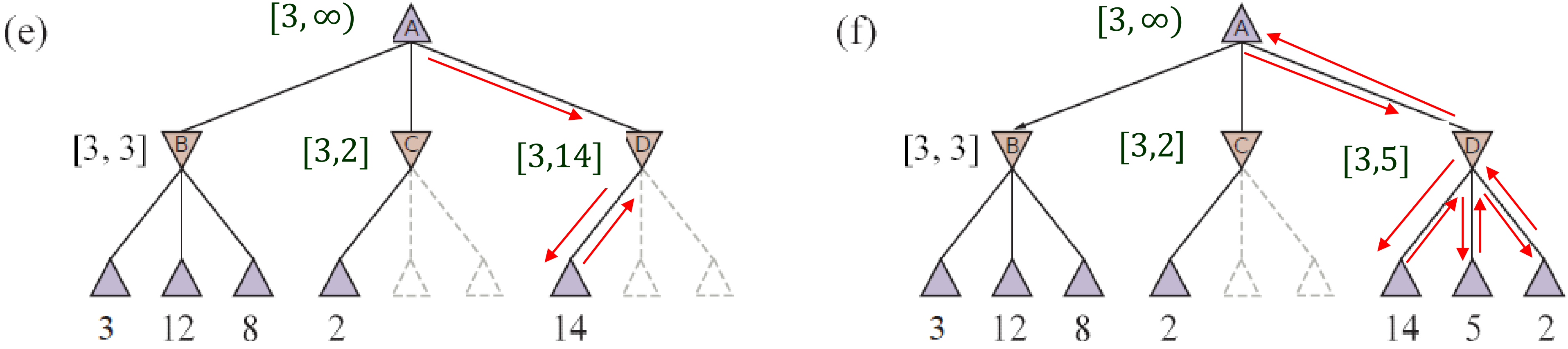
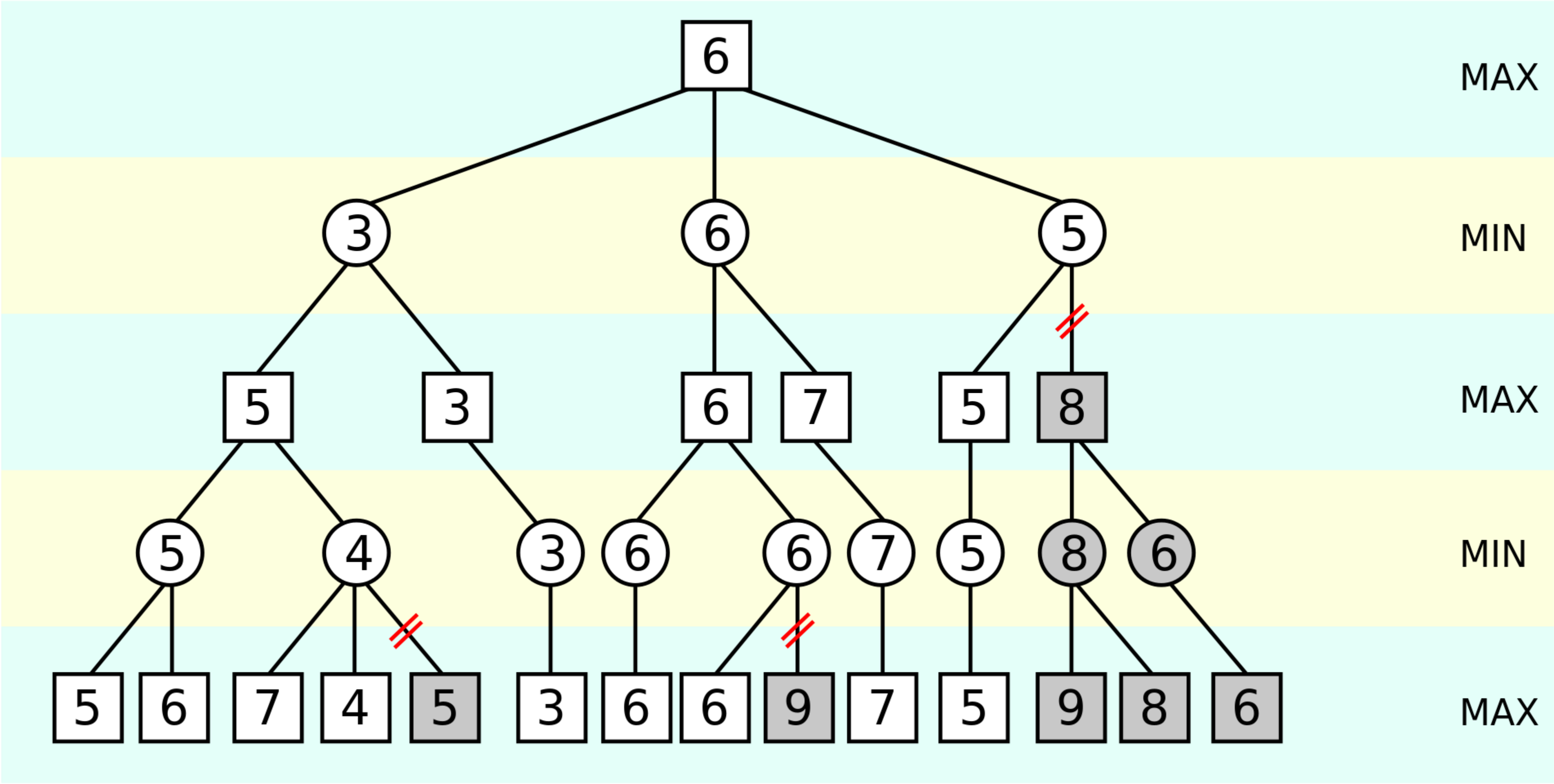


$$
\begin{aligned}
\text{MINIMAX}(\text{root}) &= \max(\min(3,12,8), \min(2,x,y), \min(14,5,2)) \\
&= \max(3, \min(2,x,y), 2) \\
&= \max(3, z, 2) \qquad \text{where } z = \min(2,x,y) \le 2 \\
&= 3.
\end{aligned}
$$

A Larger Example (Wikipedia)



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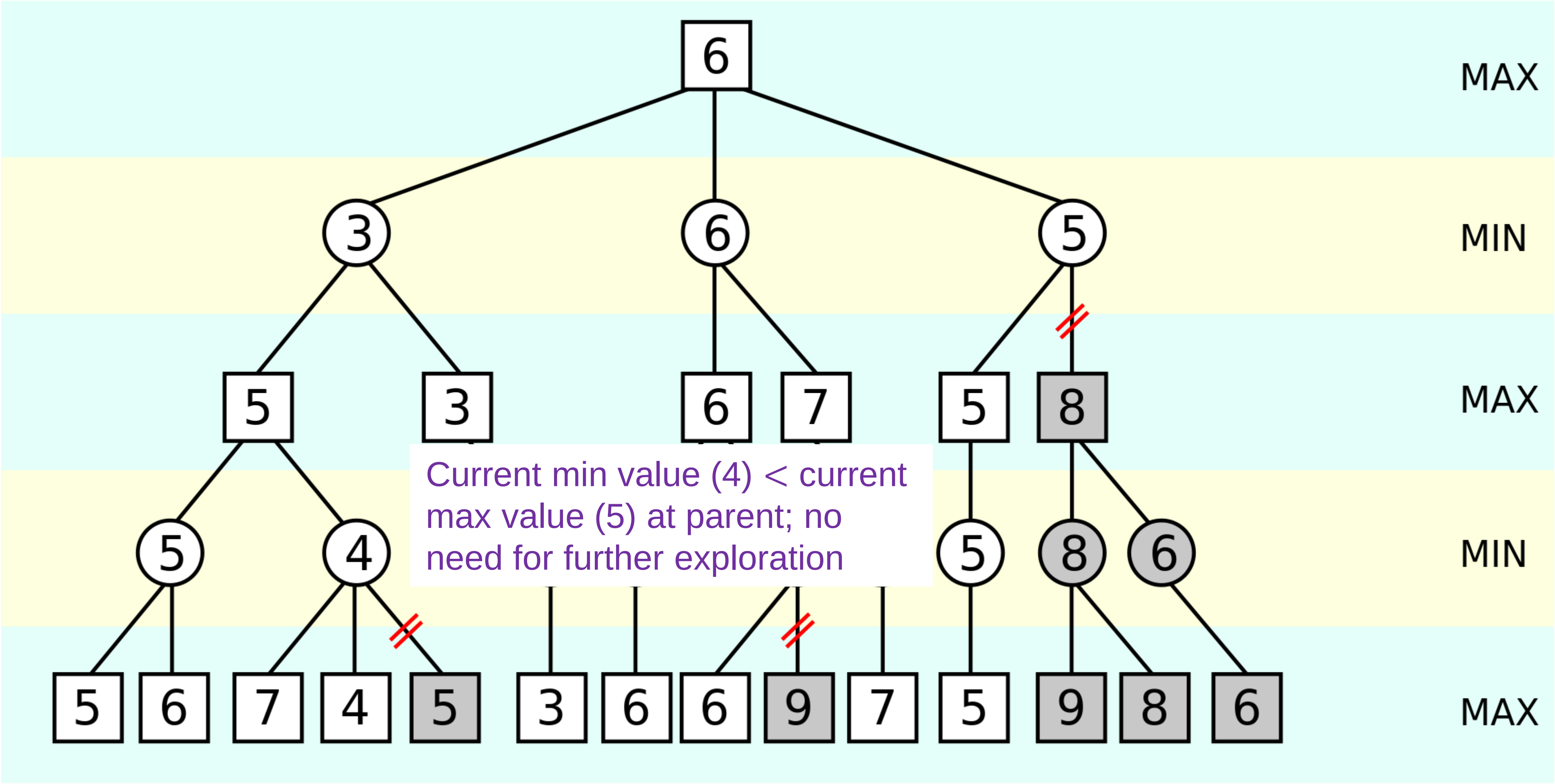

Current min value (4) < current max value (5) at parent; no need for further exploration

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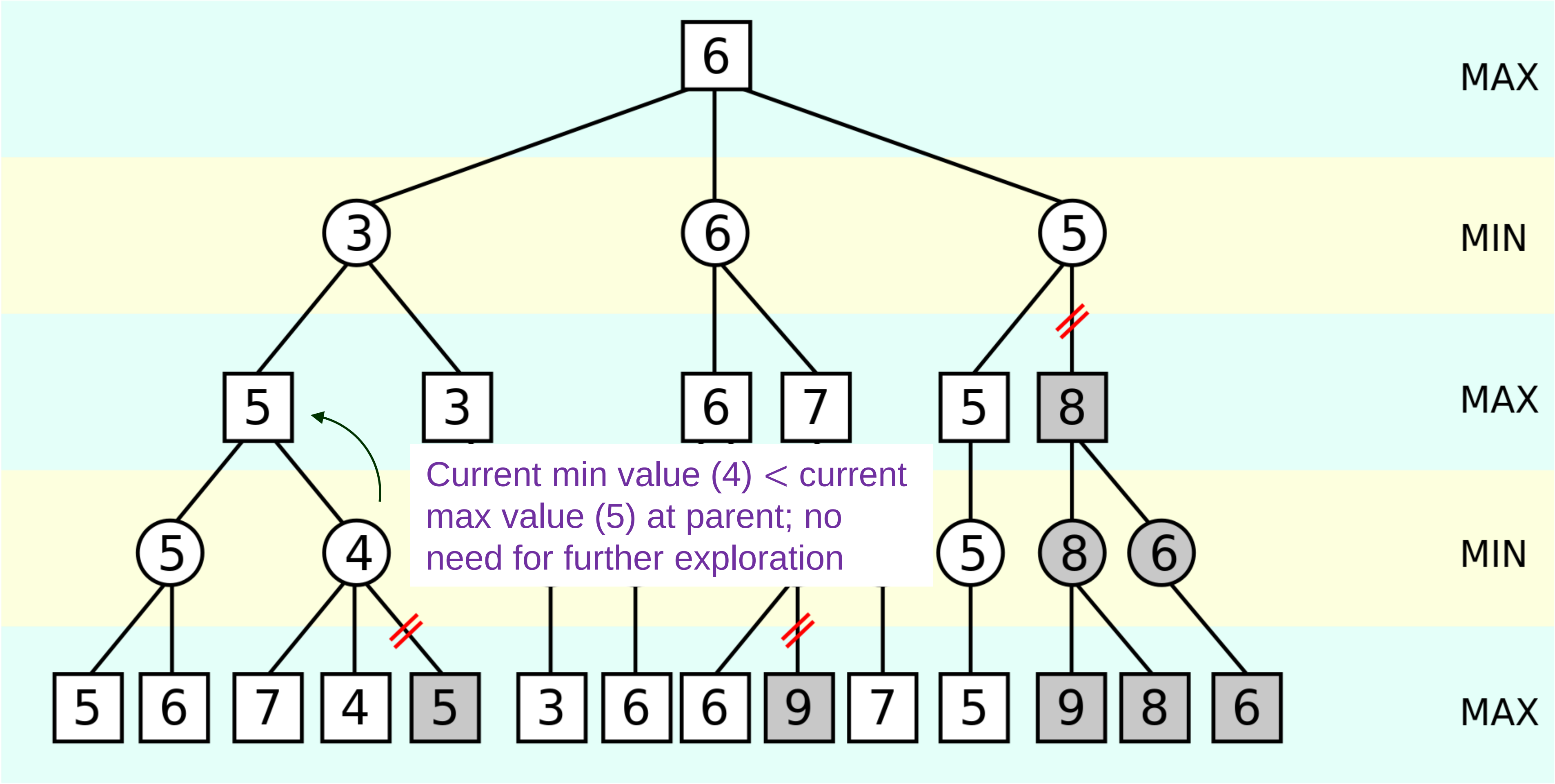

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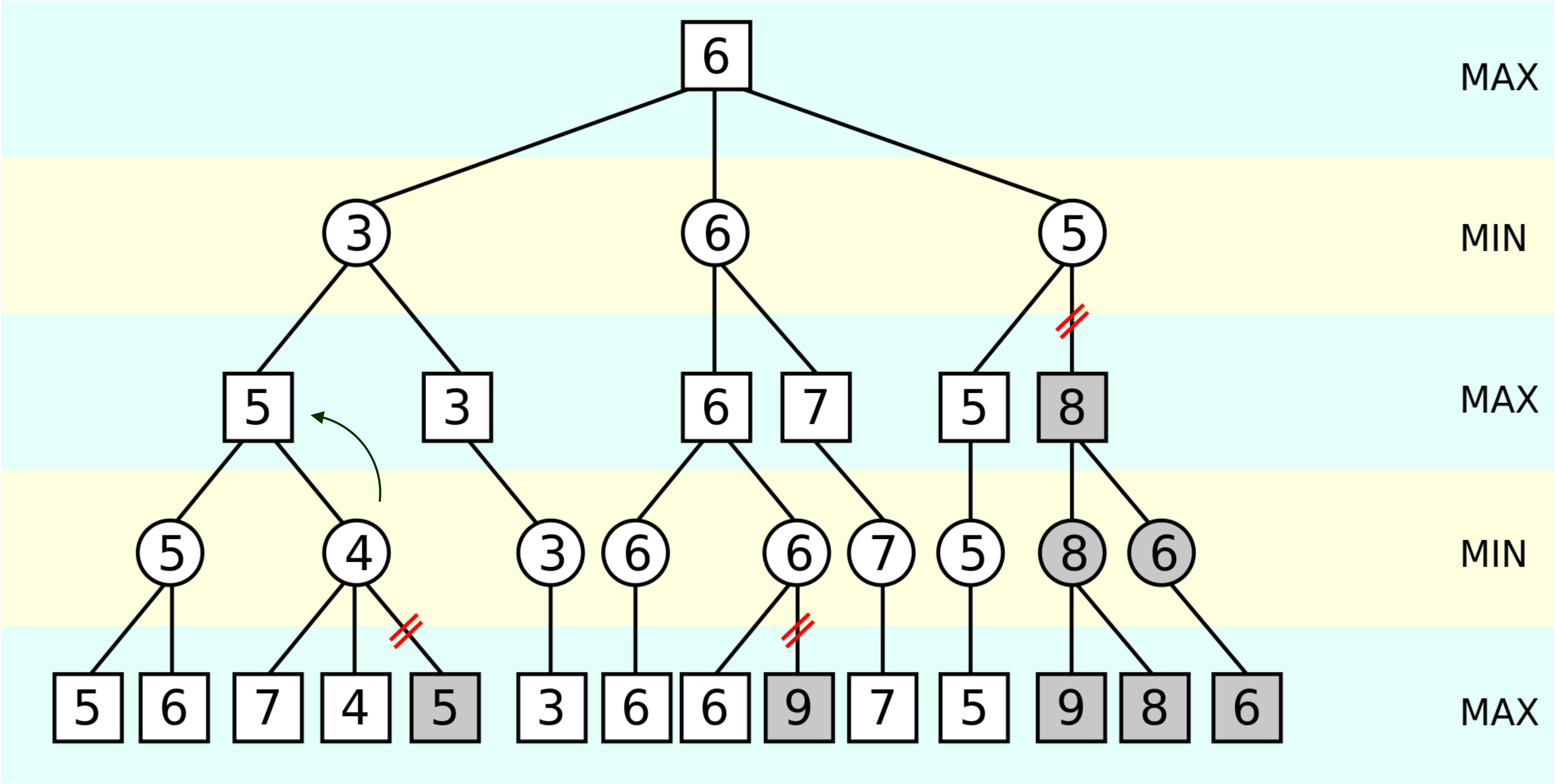

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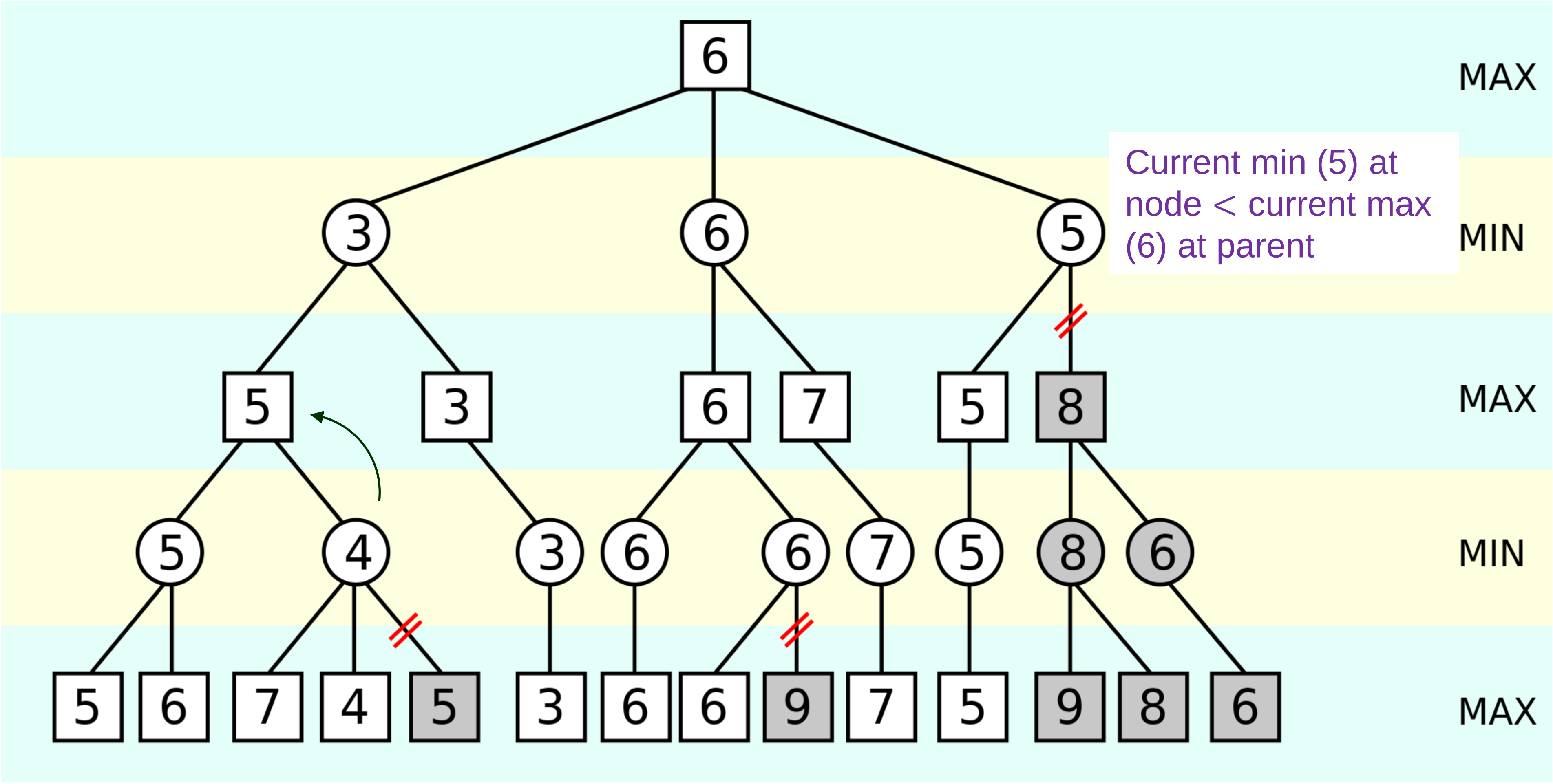

Current min (5) at node < current max (6) at parent

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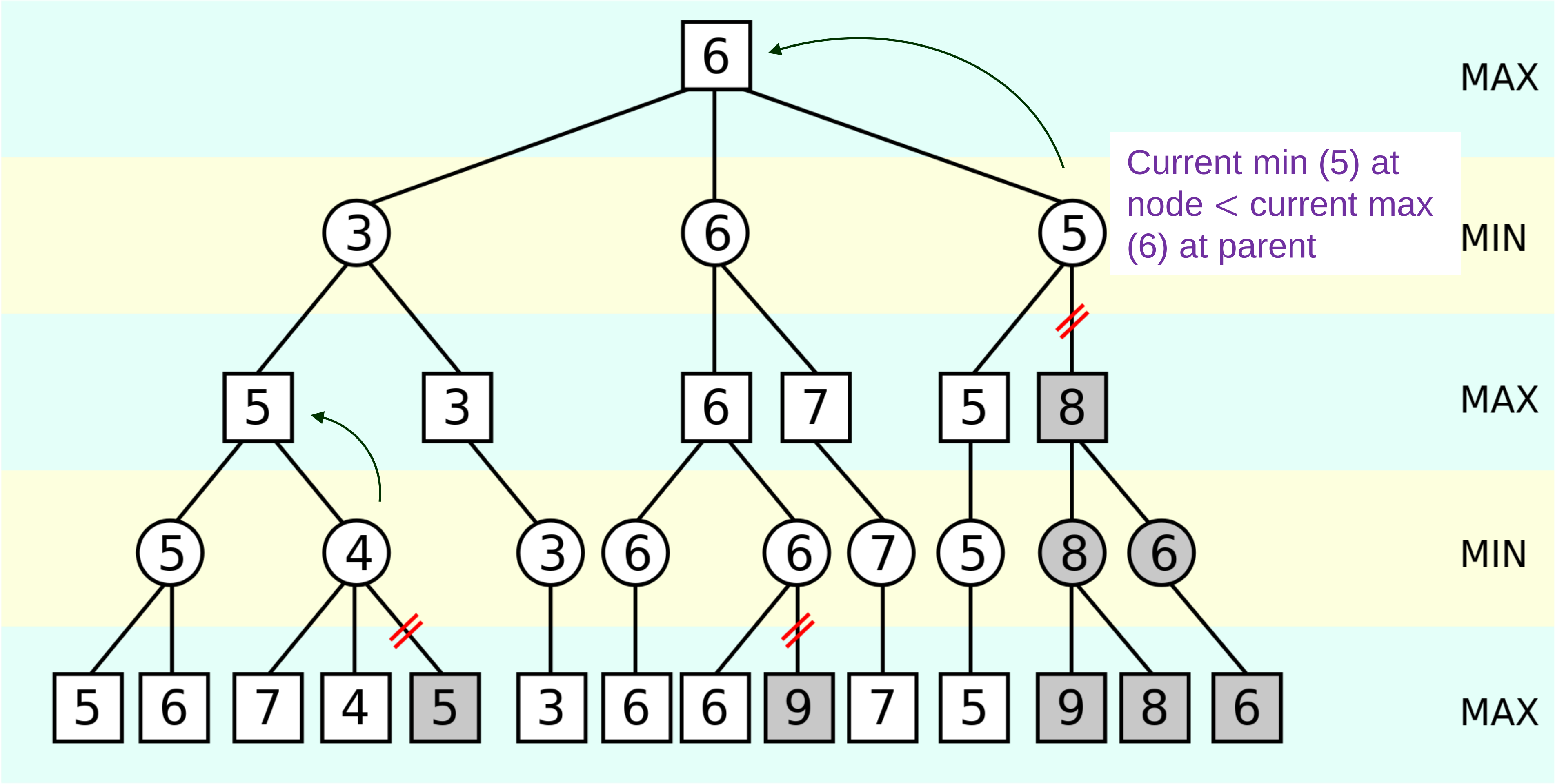

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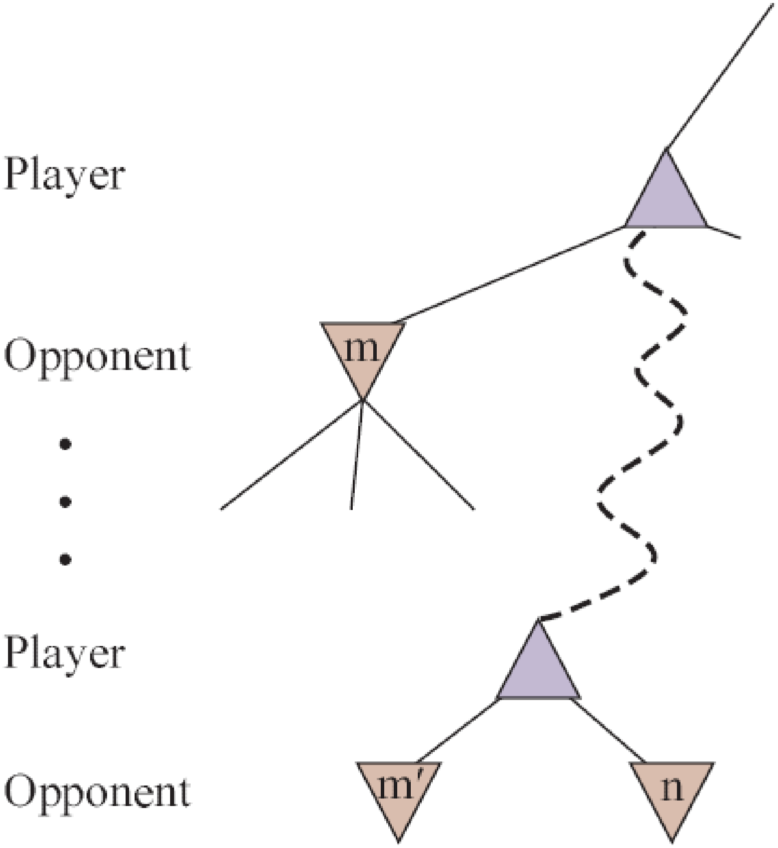
General Case



The player will not move to node n if it has a better choice

- either at the same level (e.g., m')
- or at any node (e.g., m) higher up in the tree.

Prune n once we have found enough about it to reach the above conclusion.

Alpha and Beta Values

Alpha-beta pruning gets its name from two extra parameters α, β

$\alpha =$ the highest-value (i.e., the best choice) so far along a path for MAX.

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- ♦ Update the values of α and β as the search goes along.
- ♦ Prune the remaining branches at a MIN node with current value $\leq \alpha$ or a MAX node with current value $\geq \beta$.

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Alpha-Beta Search Algorithm (3rd Edition of Textbook)

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function ALPHA-BETA-SEARCH(state) returns an action
    $v \leftarrow$ MAX-VALUE(state, $-\infty, +\infty$)
    return the action in ACTIONS(state) with value $v$


function MAX-VALUE (state,$\alpha, \beta$) returns a utility value
    if TERMINAL-TEST(state) then return UTILITY(state)
    $v \leftarrow -\infty$
    for each $a$ in ACTIONS(state) do
        $v \leftarrow$ MAX($v$, MIN-VALUE(RESULT($s, a$),$\alpha, \beta$))
        if $v \geq \beta$ then return $v$
        $\alpha \leftarrow$ MAX($\alpha, v$)
    return $v$


function MIN-VALUE (state,$\alpha, \beta$) returns a utility value
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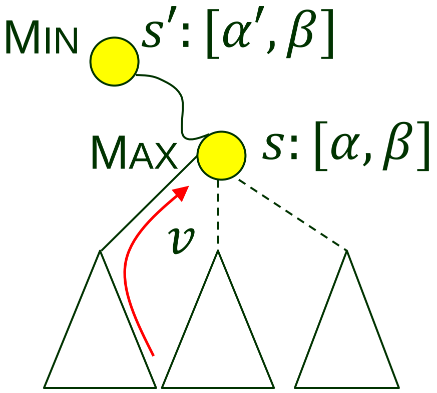

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        if $v \geq \beta$ then return $v$      // pruning
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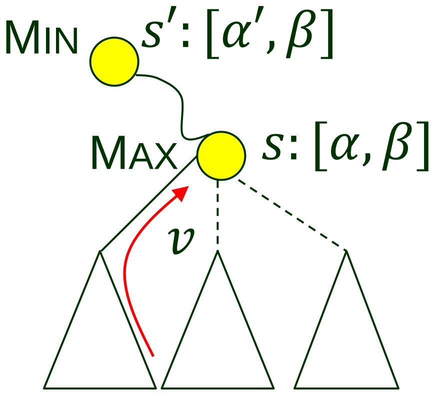

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    $v \leftarrow -\infty$
    for each $a$ in ACTIONS(state) do
        $v$ ←MAX($v$, MIN-VALUE(RESULT($s, a$),$\alpha, \beta$))
        if $v \geq \beta$ then return $v$    // pruning
        $\alpha$ ←MAX($\alpha, v$)  // new $\alpha$ to be passed on to the rest MIN-VALUE
    return $v$                      // calls within the for loop.

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        $\beta$ ←MIN($\beta, v$)
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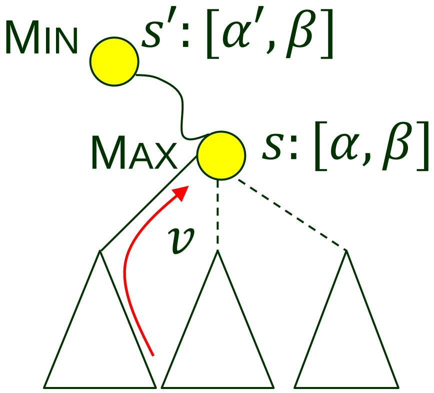

Alpha-Beta Search Algorithm (3rd Edition of Textbook)

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function ALPHA-BETA-SEARCH(state) returns an action
    $v$ ←MAX-VALUE(state, $-\infty, +\infty$)
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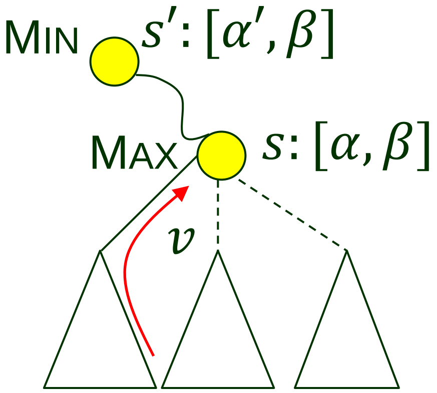

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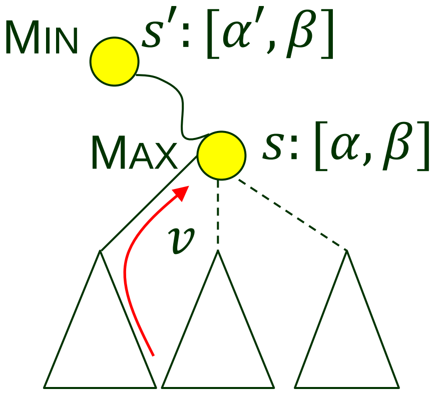
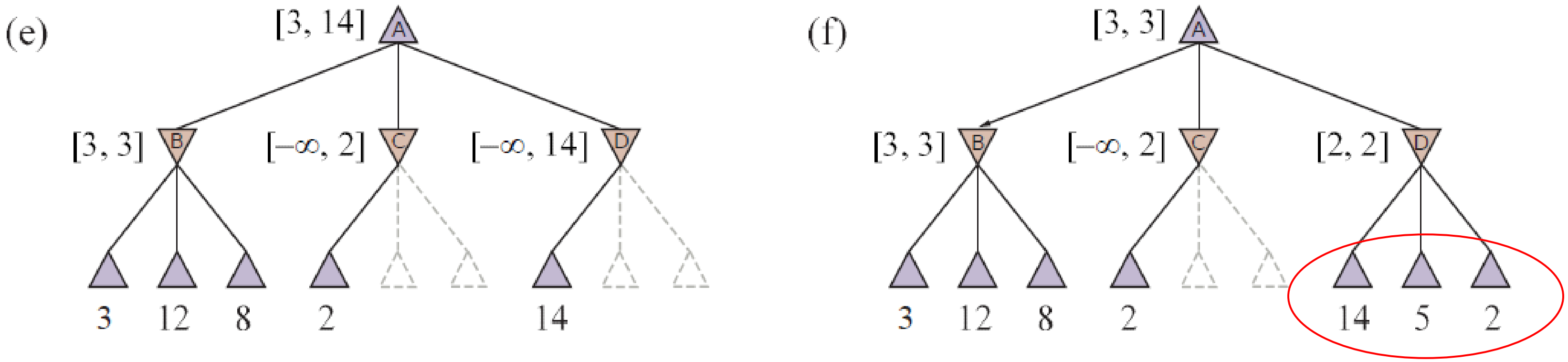


II. Move Ordering



Successors 14 and 5 would’ve been pruned had 2 been generated first.

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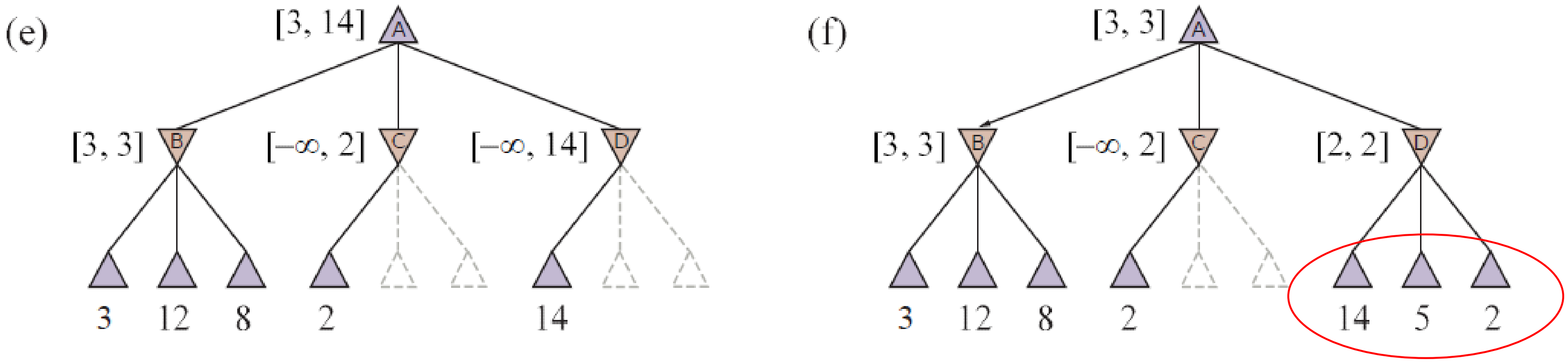

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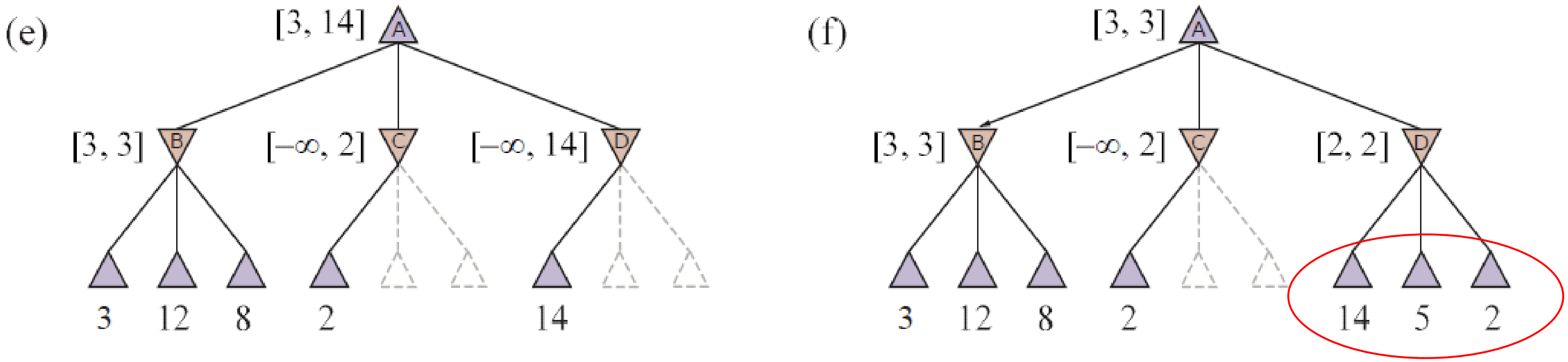

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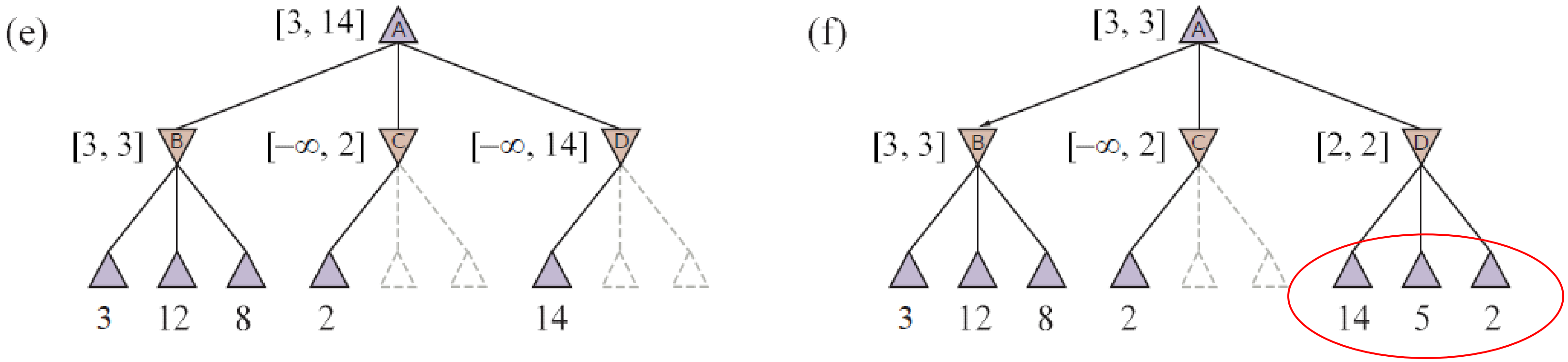

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- $O(b^{3m/4})$ nodes for random move ordering.

Heuristic Alpha-Beta Tree Search

- Cut off the search early by applying a heuristic evaluation function.
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Heuristic Alpha-Beta Tree Search

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$$\text{H-MINIMAX}(s, d) =$$

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Evaluation Functions

EVAL(s, p) returns an estimate of the expected utility s to player p.

- EVAL$(s, p) =$ UTILITY(s, p) if s is terminal;
- UTILITY$(loss, p) \leq$ EVAL$(s, p) \leq$ UTILITY(win, p) if s is nonterminal.

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Criteria:

- ♦ No excessive computation time.
- ♦ Strong correlation with actual chances of winning.

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e.g., two-pawns vs. one-pawn

1 ↤	wins	82%
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- Too many categories and too much dependence on experience.

Eval Function: Feature Combination

Compute separate numerical contributions from each feature and combine them.

$$\boxed{\textsc{Eval}(s) = \underset{\substack{\uparrow \\ \text{weight}}}{w_1} \underset{\substack{\uparrow \\ \text{e.g., \#pawns in chess}}}{f_1}(s) + w_2 f_2(s) + \cdots + w_n f_n(s)}$$

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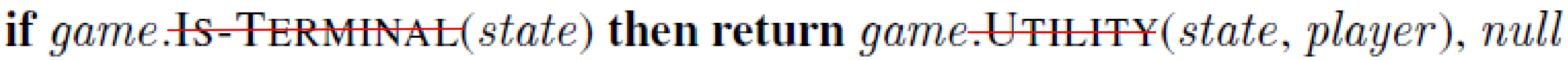
♠ Assumes independent feature contributions.

♦ Use a nonlinear feature combination.

e.g., two bishops might be worth more than twice the value of a single bishop.

Cutting off Search

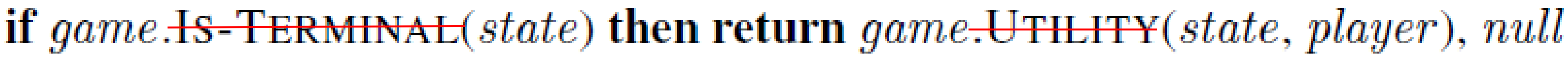


```
if $game$.~~IS-TERMINAL~~($state$) then return $game$.~~UTILITY~~($state$, $player$), $null$
```

IS-CUTOFF

EVAL

Cutting off Search



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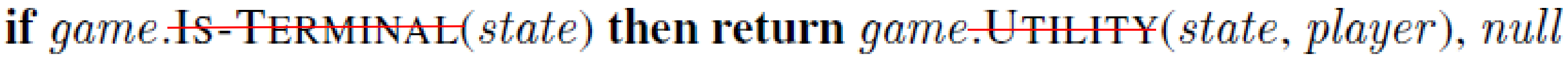
IS-CUTOFF EVAL

Some strategies:

- Set a fixed depth limit d to control the amount of search.

IS-CUTOFF returns true if depth $> \ d$.

Cutting off Search



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Is-Cutoff   Eval

Some strategies:

- Set a fixed depth limit d to control the amount of search.

  Is-Cutoff returns true if depth $> d$.

- Apply iterative deepening:

  When time runs out, returns the move selected by the deepest completed search.

Real-Time Decisions

♦ Minimax with alpha-beta pruning.

♦ Extensively tuned evaluation function.

♦ Pruning heuristics.

♦ A transposition table of repeated states and evaluations.

- To avoid re-searching the game tree below that state.

♦ A large database of optimal opening and endgame moves.

- Table lookup instead of search.
- Chess endgames with up to 7 pieces solved.

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