Exploring Big Oh

1 Introduction

We denote the set of natural numbers by $\mathbb{N}$. The notation $n \in \mathbb{N}$ means that n is a natural number.

Definition 1. Let f and g be functions from $\mathbb{N}$ to $\mathbb{N}$, i.e., f and g maps natural numbers to natural numbers. We say that f *is big Oh of* g, written $f(n) \in O(g(n))$, if there is a constant real number $c > 0$ and a natural number N such that, for every $n > N$,

$$f(n) \leq cg(n).$$

Say you are given two functions, f and g, and are asked to prove that $f(n) \in O(g(n))$. How does someone solve this? Let's look at an analogy. You and a friend are talking, and you claim that there is a number greater than 12 which is divisible by 5. The friend is skeptical, how do you prove your claim? The most immediate thing that comes to mind is to show your friend that 15 (or 20, or 25, or...) is divisible by 5 and greater than 12.

The point here is that the big Oh problem is, at its core, the same problem as proving that there is a number greater than 12 which is divisible by 5. So, one way to prove that $f(n) \in O(g(n))$ is to simply

1. choose constants c and N, and
2. prove that $f(n) \leq cg(n)$ for every $n > N$.

Important: The second step, proving that the constants you choose in the first step work, is required!

Let's look at a big Oh example where we use this strategy.

Example 1. Prove that that $2n + 1 \in O(n)$.
Proof:

1. Let $c = 3$ and $N = 1$.
2. Then for every $n > 1$,

$$\begin{aligned} 2n + 1 &< 2n + n \text{ (because } n > 1) \\ &= 3n \text{ (grouping terms)} \\ &= cn \text{ (by definition of } c) \end{aligned}$$

The main difference between the divisible by 5 analogy and this example is the second step - proving that the choice is correct. In the divisible by 5 scenario, the second step is just showing that 15 is divisible by 5 and that it is greater than 12. In Example 1, it's a bit more complicated. Intuitively, this is because we have to show that the inequality

$$2n + 1 \leq 3n \tag{1}$$

is true **for every** n **greater than** 1. The way we proved this is to give a sequence of inequalities showing that inequality (1) is true *for arbitrary* n.

In this example, it was pretty easy to see what constants would work. But what if we are given a more complicated example for f and g? One option is to "guess and pray": guess values for the constants c and N, and hope that these work[1].

2 How to find good values?

The punchline of this section is to reduce the problem to something simpler. Here is another example.

Example 2. Prove that $3n^2 + 25n + 45 \in O(n^2)$.

It is a little less obvious what we should choose for c and N. However, regardless of what values we eventually settle on, we know that we need to show that

$$3n^2 + 25n + 45 \leq cn^2 \tag{2}$$

for all $n > N$. We can improve our situation by simplifying inequality (2). Notice that, as long as $n \geq 1$,

$$25n \leq 25n^2 \text{ and } 45 \leq 45n^2.$$

Therefore, for every $n \geq 1$,

$$\begin{aligned} 3n^2 + 25n + 45 &\leq 3n^2 + 25n^2 + 45n^2 \\ &= (3 + 25 + 45)\, n^2 \\ &= 73n^2. \end{aligned}$$

What we've done is to reduce finding c and N making inequality (2) true to finding c and N such that

$$73n^2 \leq cn^2 \tag{3}$$

for every $n \geq N$. This is a much simpler thing, and it becomes clear what we can choose for c and N, namely, $c = 73$ and $N = 1$.

We should stress that the above is just describing a strategy for finding good values for c and N. You are free to do this on a scratch piece of paper, and only turn in the following proof:

[1] It might seem like I am disparaging this approach, but I'm really not. As you get more experience, you will find that you actually are able to guess values that work, or are close enough that you are able to find values that do.

Proof of Example 2. Let $c = 73$ and $N = 1$. Then, for every $n > N = 1$,

$$\begin{aligned} 3n^2 + 25n + 45 &\leq 3n^2 + 25n^2 + 45n^2 \text{ (because } n > 1) \\ &= (3 + 25 + 45)\, n^2 \text{ (grouping terms)} \\ &= 73n^2 \\ &= cn^2 \text{ (by definition of } c). \end{aligned}$$

□

3 Tips and Tricks

When using the basic strategy of explicitly giving values of c and N to prove big Oh, there are a couple "short cuts" which make finding good values easier. We stress that these are non-rigorous[2], and are **not guaranteed to work**. In particular, you still have to perform step 2, proving that the values do work.

Trick 1. Try $N = 1$.

You might have already got a sense that, for typical problems, the trivial value of $N = 1$ works. There is a provable reason behind this, but we won't go into it here. There's also a more advanced version of this trick.

Trick 2. Try setting N to the first input n for which $g(n) \geq 1$.

This trick usually comes up when dealing with logarithms on the right hand side. When $n = 1$, then $\log n = 0$, and often there is no c such that $f(1) \leq cg(1)$.

Trick 3. Don't be afraid go big! What we mean is that you should always be on the look out for simple functions $h(n)$ such that $f(n) \leq h(n)$ for all $n > N$.

We use this trick several times in these notes. For example, consider $f(n) = 3n^2 + 25n + 45$ and $g(n) = n^2$. We noticed that, as long as $n \geq 1$, $f(n) \leq h(n) = 73n^2$. $h(n)$ is a much simpler function, and makes it easy to see how to find c and N.

Trick 4. Don't be afraid go small! What we mean is that if $g(n)$ is complicated, you should try to find a simple function $h(n)$ such that $h(n) \leq g(n)$ *for every* n. Then find c, N such that $f(n) \leq ch(n)$ for all $n > N$.

Here's a simple example of this trick in action.

Example 3. Prove $10n^2 \in O(n^2 + 35n + 5)$.

In this example, $n^2 + 35n + 5$ is making things too complicated. We can make our lives easier by finding a simple $h(n)$. Let's just remove the $35n + 5$ term, and look at the function $h(n) = n^2$. Our first step is to find c, N such that

$$10n^2 \leq n^2 \text{ for all } n > N.$$

We see that we can take $c = 10$ and $N = 1$. To complete the proof, we just notice that $h(n) \leq g(n)$ *for every* n. Therefore, we can take the same values for the constants to show $10n^2 \in O(n^2 + 35n + 5)$.

[2]Although it is possible to rigorously prove versions of them.

4 Using Calculus

The method of proving big Oh inclusions of the first two sections is not the only way. We can use the following lemma to bring techniques from calculus to bear on this problem.

Lemma 1. *Let f and g be functions from the natural numbers to the natural numbers. If*

$$\lim_{n\to\infty} \tfrac{f(n)}{g(n)} = 0,$$

then f is big Oh of g.

You won't be expected to prove this. This lemma will be helpful, especially in conjunction with L'Hopital's rule. In case you've forgotten, this is (a weaker version) of this rule.

Lemma 2 (L'Hopital's rule)**.** *Let f and g be differentiable functions from the real numbers to the real numbers. If*

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = 0,$$
$$\lim_{x\to\infty} |f'(x)| = \infty, \textit{ and}$$
$$\lim_{x\to\infty} |g'(x)| = \infty,$$

then

$$\lim_{x\to\infty} \tfrac{f(x)}{g(x)} = 0,$$

This turns out to give simple proofs of certain big Oh inclusions. We don't expect you to prove the facts you learned from calculus, you can simply state them. Here's an example of this in action.

Example 4. Prove that $\log n \in O(n^a)$, for every $0 < a$.

Proof. Let $a > 0$. First note that the real valued functions $f(x) = \log x$ and $g(x) = n^a$ are differentiable, and

$$\lim_{x\to\infty} |f'(x)| = \infty,$$
$$\lim_{x\to\infty} |g'(x)| = \infty.$$

We also know from calculus that

$$f'(x) = \frac{1}{\ln(2)x}$$
$$g'(x) = ax^{a-1}.$$

Therefore,

$$\begin{aligned}\frac{f'(x)}{g'(x)} &= \frac{\frac{1}{\ln(2)x}}{ax^{a-1}} \\ &= \frac{1}{\ln(2)x\, ax^{a-1}} \\ &= \frac{1}{a\ln(2)x^a}.\end{aligned}$$

As $a > 0$, it is easy to see that

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = 0.$$

We may therefore apply L'Hopital's rule, and conclude that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

Since this limit goes to 0 for every real number, the limit goes to 0 for every natural number and we see that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0,$$

Hence, by Lemma 1, $\log n \in O(n^a)$, for every $0 < a$. □

5 More Examples

Here's another example.

Example 5. For every $0 < a < b$, $n^a + 10 \in O(n^b)$.

Proof. Let $0 < a < b$. Let $c = 11$ and $N = 1$. Then for every $n > N = 1$,

$$\begin{aligned} n^a + 10 &\leq n^a + n^a 10 \\ &= n^a(1 + 10) \\ &= 11n^a \\ &\leq 11n^b \\ &= cn^b. \end{aligned}$$

□

Here's an example where we aren't given explicit functions. Instead we are trying to prove a general property of big Oh notation[3]

Example 6. Let f, g, h be functions from the naturals to the naturals. If $f(n) \in O(h(n))$ and $g(n) \in O(h(n))$, then $(f + g)(n) \in O(h(n))$ (where $(f + g)$ is the function which maps any n to $f(n) + g(n)$).

Proof. By our assumption there are constants c_f, N_f, c_g, N_g such that

$$\begin{aligned} f(n) &\leq c_f h(n) \text{ for every } n > N_f \\ g(n) &\leq c_g h(n) \text{ for every } n > N_g. \end{aligned}$$

Let $c = c_f + c_g$ and $N = \max\{N_f, N_g\}$. Then for every $n > N$,

$$\begin{aligned} f(n) &\leq c_f h(n) \text{ for every } n > N \\ g(n) &\leq c_g h(n) \text{ for every } n > N. \end{aligned}$$

Therefore,

$$\begin{aligned} (f + g)(n) &= f(n) + g(n) \\ &\leq c_f h(n) + c_g h(n) \\ &= (c_f + c_g)h(n) \\ &= ch(n). \end{aligned}$$

□

[3]In this example we're showing that big Oh is *closed under addition*.

Example 7. For every $1 < a$, $n \log n \in O(n^a)$.

Proof. We will use L'Hopital's rule and Lemma 1 to solve this example. Let $0 < a$. First note that the real valued functions $f(x) = n \log n$ and $g(x) = n^a$ are differentiable, and

$$\lim_{x\to\infty} |f'(x)| = \infty,$$
$$\lim_{x\to\infty} |g'(x)| = \infty.$$

We also know from calculus that

$$f'(x) = \log x + \frac{1}{\ln(2)}$$
$$g'(x) = ax^{a-1}.$$

Therefore,

$$\frac{f'(x)}{g'(x)} = \frac{\log x + \frac{1}{\ln(2)}}{ax^{a-1}}$$
$$= \frac{\log x}{ax^{a-1}} + \frac{\frac{1}{\ln(2)}}{ax^{a-1}}.$$

As $a > 1$, we know that $x^{a-1} > x$ for all $x \geq 1$. Therefore it is immediate that the term $\frac{\frac{1}{\ln(2)}}{ax^{a-1}}$ goes to 0 as x goes to infinity. We also know, from Example 4, that

$$\lim_{x\to\infty} \frac{\log x}{ax^{a-1}} = 0.$$

Combining these, we have

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = 0.$$

We may therefore apply L'Hopital's rule, and conclude that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

Since this limit goes to 0 for every real number, the limit goes to 0 for every natural number and we see that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0,$$

Hence, by Lemma 1, $n \log n \in O(n^a)$, for every $0 < 1$. □

6 Proving not big Oh

We are also interested in proving that, for some functions f, g, $f(n) \notin O(g(n))$. If you unpack the definition and negate it, this means that we need to show that, for all constants c and N, there is a natural number $n > N$ such that $f(n) > cg(n)$.

How would we go about proving this? In this class we'll use the technique of proof by contradiction.

Example 8. Prove that $2^n \notin O(n^4)$.

Proof. Assume that $2^n \in O(n^4)$. Then, by definition, there are constants c and N such that $2^n \leq cn^4$ for all $n > N$. By taking logarithms of both sides, this is implies that $n \leq \log c + 4 \log n$ for all $n > N$. Rearranging, this implies that $n - 4 \log n \leq \log c$ for all $n > N$. Note that $\log c$ is a constant. However, the left side, $n - 4 \log n$ is an increasing function (goes to ∞ as n increases), a contradiction. Therefore our assumption ($2^n \in O(n^4)$) is false. So $2^n \notin O(n^4)$. $\square$

Recall the general form of a proof by contradiction:

1. Start by assuming the negation of what you want to prove.
2. Show that this implies something.
3. Which implies something else.
4. and so on.
5. Eventually imply something false.
6. But this means that the assumption must be false!

In this example, we showed that our assumption, $n \leq \log c + 4 \log n$ for all $n > N$, implies via a chain of implications, that $n - 4 \log n \leq \log c$ for all $n > N$. To prove that this is false, we applied to the fact that $n - 4 \log n$ is an increasing function.