

# Lecture 2

## Combinatorics

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STAT 330 - Iowa State University

## Equally likely outcomes

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# Equally Likely Outcomes

Example 1: There are 4 chips in a box; 1 chip is defective. Randomly draw a chip from the box. What is the probability of selecting the defective chip?

- Common sense:  $P(\text{draw defective chip}) = \frac{1}{4}$  or 25%
- Using probability theory...

*Sample space:*

$$\Omega = \{g_1, g_2, g_3, d\}$$

$$|\Omega| = 4$$

*Event:*

$$A = \text{“draw defective chip”} = \{d\}$$

$$|A| = 1$$

*Probability of event:*  $P(A) = \frac{|A|}{|\Omega|} = \frac{1}{4}$

### Theorem

If events in sample space are equally likely (i.e.  $P(\{\omega\})$  is same for all  $\omega \in \Omega$ ), then the probability of an event  $A$  is given by:

$$P(A) = \frac{|A|}{|\Omega|},$$

where  $|A|$  is the number of elements in set  $A$  (cardinality of  $A$ ).

## Equally Likely Outcomes Cont.

Example 2: There are 4 chips in a box; 1 chip is defective. Randomly draw 2 chips from the box. What is the probability that defective chip is among the 2 chosen?

*Sample space:* (All possibilities for drawing 2 chips)

$$\Omega = \{(g_1, g_2), (g_1, g_3), (g_1, d), (g_2, g_3), (g_2, d), (g_3, d)\}$$

$$|\Omega| = 6$$

*Event:*

$$A = \text{"defective chip is among the 2 chips drawn"}$$

$$=$$

$$|A| =$$

*Probability of event:*  $P(A) = \frac{|A|}{|\Omega|} =$

# Multiplication Principle

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# Multiplication Principle

## Multiplication Principle

If a complex action can be broken down into a series of  $k$  component actions, performed one after the other, where ...

- first action can be performed in  $n_1$  ways
- second action can be performed in  $n_2$  ways
- $\vdots$
- last action can be performed in  $n_k$  ways

Then, the complex action can be performed in  $n_1 n_2 \cdots n_k$  ways.

## Multiplication Principle Cont.

Example 3: Your friend owns 4 shirts (red, blue, green, white), and 2 pants (blue, black). What are all the ways he can create an outfit by choosing a shirt and pants to wear?

Example 4: Suppose licence plates are created as a sequence of 3 letters followed by 3 numbers. What is  $|\Omega|$ ? (ie. how many license plates are in the sample space?)



# Sample selection

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# Sample Selection

Imagine picking  $k$  objects from a box containing  $n$  objects.

## Definitions

*with replacement*: After each selection, the object is put back in the box. It is possible to select the same object multiple times in the  $k$  selections.

*without replacement*: After each selection, the object is removed from the box. Cannot select the same object again.

*ordered sample*: Order of selected objects matters.

**Example**: Passwords  $\dots abc1 \neq c1ba$

*unordered sample*: Order of selected objects doesn't matter.

**Example**: Selecting people for a study.

$(\text{Mary, John, Susan}) = (\text{John, Mary, Susan})$

### 3 Main Scenarios

There are *3 main scenarios* we will deal with ...

Consider selecting 2 letters from a box containing “a”, “b”, “c”.

1. Ordered with replacement

- “with replacement” means repeat letters are allowed.

$$\Omega = \{(a, a), (b, b), (c, c), (a, b), (b, a), \\ (a, c), (c, a), (b, c), (c, b)\}$$

2. Ordered without replacement

- “without replacement” means repeat letters **not** allowed.

$$\Omega = \{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b)\}$$

3. Unordered without replacement

- “unordered” means  $(a, b)$  same as  $(b, a)$  - only written once.
- “without replacement” means repeat letters **not** allowed.

$$\Omega = \{(a, b), (a, c), (b, c)\}$$

Ultimately, we want to count up  $|\Omega|$  for these scenarios.

## Ordered With Replacement

A box has  $n$  items numbered  $1, \dots, n$ . Draw  $k$  items with replacement. (A number can be drawn twice).

*Sample Space:*  $\Omega = \{(x_1, \dots, x_k) : x_i \in \{1, \dots, n\}\}$

What is  $|\Omega|$ ?

Break complex action into a series of  $k$  single draws.

1.  $n$  possibilities for  $x_1$
2.  $n$  possibilities for  $x_2$
- $\vdots$
- $k$ .  $n$  possibilities for  $x_k$

Multiplication principle:  $|\Omega| = n \cdot n \cdot n \cdots n = n^k$

# Permutation

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## Ordered Without Replacement

A box has  $n$  items numbered  $1, \dots, n$ . Select  $k$  items **without** replacement. This means once a number is chosen, it can't be selected again.

*Sample Space:*  $\Omega = \{(x_1, \dots, x_k) : x_i \in \{1, \dots, n\}, x_i \neq x_j\}$

What is  $|\Omega|$ ?

Break complex action into a series of  $k$  single draws.

1.  $n$  possibilities for  $x_1$
2.  $n - 1$  possibilities for  $x_2$
3.  $n - 2$  possibilities for  $x_2$
- $\vdots$
- k.  $n - (k - 1)$  possibilities for  $x_k$

Multiplication principle:  $|\Omega| = n \cdot (n - 1) \cdot (n - 2) \cdots (n - (k - 1))$

This is equivalent to  $\frac{n!}{(n-k)!}$

# Permutation

## Definition

A *permutation* is an ordering of  $k$  distinct objects chosen from  $n$  objects. This is another name for the *ordered without replacement* scenario.

## Theorem

$P(n, k)$ , called the *permutation number*, is the number of permutations of  $k$  distinct objects out of  $n$  objects.

$$P(n, k) = \frac{n!}{(n - k)!}$$

**Note (factorials):**  $n! = n \cdot (n - 1) \cdot (n - 2) \cdots 3 \cdot 2 \cdot 1$

$$0! = 1$$

$$\text{Ex. } 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

## Permutation Example

### Example 5:

Out of a group of 10 students, I choose 3 distinct students to give prizes to. How many ways can I select 3 students?

$$n = 10 \quad k = 3$$

$$\begin{aligned} P(n, k) &= \frac{n!}{(n - k)!} \\ P(10, 3) &= \frac{10!}{(10 - 3)!} \\ &= \frac{10!}{7!} \\ &= \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} \\ &= 10 \cdot 9 \cdot 8 = 720 \end{aligned}$$



## Permutation Example

### Example 6:

University phone exchange starts with 641 — — — —

What is the probability that a randomly selected phone number contains 7 distinct digits?

*Sample space:* (All possibilities for 4 chosen numbers)

$$|\Omega| =$$

*Event:* (4 chosen numbers are distinct - no repeats!)

$$|A| =$$

$$P(A) = \frac{|A|}{|\Omega|} =$$

# Combination

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# Unordered Without Replacement

Select  $k$  objects out of  $n$  objects with *no replacement* where *order does not matter*.

$$\Omega = \{(x_1, \dots, x_k) : x_i \in \{1, \dots, n\}, x_i \neq x_j\}$$

To derive  $|\Omega|$  for this scenario, we can go back to how it was derived for permutations (where order mattered).

- **Step 1:** Select  $k$  objects from  $n$  (order doesn't matter)
- **Step 2:** Order the objects (there is  $k!$  ways to order objects)

$$P(n, k) = (\text{number of ways to select } k \text{ objects unordered}) \cdot k!$$

$$\text{Number of ways to select } k \text{ objects unordered} = \frac{P(n, k)}{k!} = \frac{n!}{(n-k)!k!}$$

## Definition

A *combination* is a subset of  $k$  objects from  $n$  objects. This is another name for *unordered without replacement* scenario.

## Theorem

$C(n, k)$ , called the *combination number*, is the number of combinations of  $k$  objects chosen from  $n$ .

$$C(n, k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

- $C(n, k)$  or  $\binom{n}{k}$  is read “ $n$  choose  $k$ ”

# Combination Example

## Example 7: Lottery (pick-five)

The lottery picks 5 numbers from  $\{1, \dots, 49\}$  without replacement as the “winning numbers”. You choose 5 numbers and win if you pick at least 3 of the winning numbers.

1. What is the probability you match all 5 winning numbers?
2. What is the probability you win?

Easiest way to do combination problems is to draw a picture of the problem by visualizing a box of items you are selecting from. Break the box into sections according to the problems.

Here, we break the box into “winning” and “non-winning” numbers.

## Combination Example

1. What is the probability you match all 5 winning numbers?

*Event:* To match all 5 winning numbers – we need to choose 5 numbers from “winning” and group, and 0 numbers from the “non-winning” group. This is done in ...

$$|A| = \binom{5}{5} \cdot \binom{44}{0} = \frac{5!}{(5-5)!5!} \frac{44!}{(44-0)!0!} = \frac{5!}{0!5!} \frac{44!}{44!0!} = \frac{5!}{1 \cdot 5!} \frac{44!}{44! \cdot 1} = 1$$

*Sample Space:* How many total ways are there to choose 5 numbers from 49 numbers (all possibilities). This is done in ...

$$|\Omega| = \binom{49}{5} = \frac{49!}{(49-5)!5!} = \frac{49!}{44!5!} = 1,906,884$$

$$P(\text{match all}) = \frac{\binom{5}{5} \cdot \binom{44}{0}}{\binom{49}{5}} = \frac{1}{1,906,884} = 0.000005$$

## Combination Example

2. What is the probability you win? (Recall that you win if you match at least 3 “winning” numbers.)

$$\begin{aligned} P(\text{win}) &= P(\text{match at least 3}) = \\ &P(\text{match 3}) + P(\text{match 4}) + P(\text{match 5}) \end{aligned}$$

# Combination Example



# Counting Summary

<u>Method</u>	<u># of Possible Outcomes</u>
<i>Ordered with replacement</i>	$n^k$
<i>Ordered without replacement</i>	$P(n, k) = \frac{n!}{(n-k)!}$
<i>Unordered without replacement</i>	$C(n, k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$