Lecture 7

Geometric and Poisson Distributions

STAT 330 - Iowa State University

Geometric Distribution

Geometric Distribution

Set up:

- Experiment where each trial is Bernoulli (only 2 outcomes) with P(success) $= p$
- Repeat the trials until you obtain the *first success*.

  F F F F $\cdots$ F S

$$X = \text{“ \# of trials until first success”}$$

This random variable X follows a *Geometric Distribution*

is distributed

$$X \sim Geo(p)$$

parameter $p = P(\text{success})$ for each trial

where p is the probability of success for each trial.

Geometric Random Variable

F F F F . . . . F S

$x-1$ failure 1 success

- Probability Mass Function (pmf)
  1. $Im(X) = \{1, 2, 3, 4, \ldots, \} = \mathbb{N}$
  2. $P(X = x) = \underbrace{(1-p)^{x-1}}_{\text{failures}}\underbrace{p}_{\text{success}}$

$$p_X(x) = (1-p)^{x-1}p \quad \text{for } x = 1, 2, 3, \ldots$$

- Cumulative distribution function (cdf)

$$\left[\; F_X(t) = P(X \le t) = 1 - (1-p)^{\lfloor t \rfloor} \;\right]$$

"floor" of t

$\lfloor t \rfloor =$

$\lfloor 5.999 \rfloor = 5$

$\lfloor 5.1 \rfloor = 5$

Why?

- $P(X > t) = (1-p)^{\lfloor t \rfloor}$ because this is the probability that the first $\lfloor t \rfloor$ trials are failures
- $P(X \le t) = 1 - P(X > t) = 1 - (1-p)^{\lfloor t \rfloor}$

Geometric Random Variable Cont.

- Expected Value

$$E(X) = \sum_{x=1}^{\infty} x(1-p)^x p = \cdots = \frac{p}{(1-[1-p])^2} = \frac{1}{p}$$

- Variance

$$Var(X) = \sum_{i=1}^{\infty} \left(i - \frac{1}{p}\right)^2 (1-p)^i p = \cdots = \frac{1-p}{p^2}$$

- "Memoryless" Property of Geometric Dist.

$$P(X \geq i + j | X \geq i) = P(X \geq j) \text{ for } i, j = 0, 1, 2, \ldots$$

[geometric distribution is the <u>only</u> discrete distribution that has this property]

Geometric Distribution Examples

Geometric Distribution Examples

Example 1: Suppose we have an unfair 2−sided coin with $P(Head) = 0.3$. We flip a coin until we get our first head, and stop flipping once we obtain the head.

What is the probability that ...

1. the first head occurs on the third flip?
2. we get the first head before the third flip?
3. we have to flip the coin at least 3 times, but at most 7 times to get the first head?
4. What is the expected number of flips until we obtain the first head?
5. What is the variance?

Geometric Distribution Examples

Start by defining the R.V and stating it's distribution.

Y = # of flips until 1^{st} head (success)

$Y \sim Geo(p) \equiv Geo(0.3)$

1. What is the probability that the first head occurs on the third flip?

$\underline{F}\ \underline{F}\ \underline{S}$

$$P(Y=3) = (1-0.3)^{3-1}(0.3) = 0.7^2 \cdot 0.3 = 0.147$$

2. What is the probability that we get the first head before the third flip?

(using PMF) $P(Y<3) = P(Y \leq 2) = P_Y(1) + P_Y(2)$
$= P(Y=1) + P(Y=2)$
$= 0.3 + (0.7)(0.3) = 0.51$

(using CDF) $P(Y<3) = P(Y \leq 2) = F_X(2) = 1-(1-0.3)^2$
$= 1 - 0.7^2$
$= 0.51$

Geometric Distribution Examples

3. What is the probability that we have to flip the coin at least 3 times, but at most 7 times?

$$P(3 \le Y \le 7) = P_Y(3) + P_Y(4) + P_Y(5) + P_Y(6) + P_Y(7)$$
$$= 0.7^2 0.3 + 0.7^3 0.3 + 0.7^4 0.3 + \dots + 0.7^6 0.3$$
$$= 0.4076$$

$$P(3 \le Y \le 7) = P(Y \le 7) - P(Y < 3)$$
$$= P(Y \le 7) - P(Y \le 2)$$
$$= F_Y(7) - F_Y(2)$$
$$= [1 - (1-0.3)^7] - [1 - (1-0.3)^2] = 0.9176 - 0.51$$
$$= 0.4076$$

4. What is the expected value?

$$EY = \frac{1}{P} = \frac{1}{0.3} = 3.33$$

5. What is the variance?

$$Var\, Y = \frac{1-P}{P^2} = \frac{1-0.3}{0.3^2} = 7.78$$

Poisson Distribution

Poisson Distribution

Set up: The Poisson distribution is used to model the number of ("rare") events occurring in a fixed interval of time.

Examples of Poisson R.Vs

- $Y = \#$ of meteorites that strike Earth in a year
- $Z = \#$ of patients arriving to emergency room from 10-11 pm

Define the random variable

$$X = \text{"\# of events occurring during an interval"}$$

This random variable X follows a *Poisson Distribution*

is distributed

$$X \sim Pois(\lambda)$$

parameter "Lambda"

where $\lambda > 0$ is called the rate parameter

Poisson R.V. Summary

- Probability Mass Function (pmf)

$$p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{for } x = 0, 1, 2, 3, \ldots$$

where $\lambda > 0$ is the rate parameter.

- Cumulative Distribution Function (cdf)

$$F_X(t) = P(X \leq t) = \sum_{x=0}^{\lfloor t \rfloor} p_X(x)$$

or use Poisson Table

- Expected Value: $E(X) = \lambda$

- Variance: $Var(X) = \lambda$

Poisson Distribution Examples

Poisson Distribution Examples

Example 2: Suppose the number of customers entering West Street Deli can be modeled using a Poisson distribution. Customers enter the deli at an average rate of 10 customers every 15 minutes during the lunch rush.

$\lambda = 10$ interval

Between 12pm and 12:15pm today, what is the probability that

1. exactly 3 customers enter? $P(X = 3) =$
2. at most 3 customers enter? $P(X \leq 3) =$
3. at least 4 customers enter? $P(X \geq 4)$
4. between 8 and 10 customers enter? (inclusive) $P(8 \leq X \leq 10)$
5. What is the expected value of the random variable? $EX = ?$
6. What is the variance of the random variable? $Var(X) = ?$

Poisson Distribution Examples

Start by defining the R.V and stating it's distribution.

X = # of customers entering W.S deli b/w 12-12:15pm

$X \sim Pois(\lambda = 10)$

1. What is the probability that exactly 3 customers enter?

$$P(X=3) = \frac{e^{-10}(10)^3}{3!} = 0.00757$$

2. What is the probability that at most 3 customers enter

(using PMF) $P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$$= p_X(0) + p_X(1) + p_X(2) + p_X(3)$$

$$= \frac{e^{-10}10^0}{0!} + \frac{e^{-10}10^1}{1!} + \frac{e^{-10}10^2}{2!} + \frac{e^{-10}10^3}{3!}$$

$$= 0.0103$$

$P(X \leq 3) = F_X(3) = 0.0103$ (Appendix A Poisson Table)

How to Use Poisson CDF Table (Appendix A)

Suppose we have random variable $X \sim Pois(\lambda = 10)$.

$P(X \leq 3) = ?$

- $P(X \leq 3)$ is found inside the table corresponding to $\lambda = 10$ (column) and $x = 3$ (row).

$\boxed{P(X \leq 3) = 0.01033605}$

x	λ=9	10	11	12	13
0	0.00012341	0.00004540	0.00001670	0.00000614	0.00000226
1	0.00123410	0.00049940	0.00020042	0.00007987	0.00003164
2	0.00623220	0.00276940	0.00121087	0.00052226	0.00022264
3	0.02122649	0.01033605	0.00491587	0.00229179	0.00105030
4	0.05496364	0.02925269	0.01510460	0.00760039	0.00374019

Poisson Distribution Examples

3. What is the probability that at least 4 customers enter?

$$P(X \geq 4) = 1 - P(X < 4)$$
$$= 1 - P(X \leq 3)$$
$$= 1 - F_X(3)$$
$$= 1 - 0.0103 \quad \text{(Appendix A Poisson Table)}$$
$$= 0.9897$$

Poisson Distribution Examples

4. What is the probability that between 8 and 10 customers enter (inclusive)

$$P(8 \leq X \leq 10) = P(X=8) + P(X=9) + P(X=10)$$
$$= \frac{e^{-10} 10^8}{8!} + \frac{e^{-10} 10^9}{9!} + \frac{e^{-10} 10^{10}}{10!} = 0.3628$$

$$P(8 \leq X \leq 10) = P(X \leq 10) - P(X < 8)$$
$$= P(X \leq 10) - P(X \leq 7)$$
$$= F_X(10) - F_X(7)$$
$$= 0.5830 - 0.2202 = 0.3628$$

5. What is the expected value of the random variable?

$$EX = \lambda = 10$$

6. What is the variance of the random variable?

$$\text{Var}\, X = \lambda = 10$$