Recitation 11 Solution

- Here is a set of additional problems. They range from being very easy to very tough. The best way to learn the material in 310 is to solve problems on your own.
- Feel free to ask (and answer) questions about this problem set on Piazza.
- This is an **optional** problem set; do not turn this in for grading.
- While you don't have to turn this in, be warned that this material **can** appear in a quiz or exam.

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1. Prove by mathematical induction the following properties:

   a. The sum of the first n entries of the geometric progression $1, r, r^2, \ldots, r^{n-1}$ (for $r < 1$) is given by $\frac{1-r^n}{1-r}$. What is the answer if $r > 1$ What is the answer if $r = 1$?

   Solution

   We can express this as a sum to make this easier to write:

   Proposition: $\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$.

   Base case: For $n = 1$, $\sum_{i=0}^{n-1} r^i = r^0 = 1 = \frac{1-r^1}{1-r}$.

   Induction Hypothesis: For some $n \in \mathbb{N}$, $\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$.

   Induction Step: Suppose the induction hypothesis is true. Then, for some $n \in \mathbb{N}$, we have the following:

   $\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$.

   We are concerned with the case when the upper limit of the sum is n, so we should add the next term in the series to both sides. This term is r^n in this case. This gives us the following:

   $\sum_{i=0}^{n-1} r^i + r^n = \sum_{i=0}^{n} r^i = \frac{1-r^n}{1-r} + r^n$.

   Then the left side is finished, so we only need to work with the right side. Finding a common denominator on the right side gives us the following:

   $\sum_{i=0}^{n} r^i = \frac{1-r^n}{1-r} + \frac{r^n(1-r)}{1-r}$.

   Simplifying, we obtain the final result:

$\sum_{i=0}^{n} r^i = \frac{1-r^n+r^n-r^{n+1}}{1-r} = \frac{1-r^{n+1}}{1-r}$.

As this equation simply replaces n by $n+1$ in the induction hypothesis, the proof is finished.

Note that since this equation does not use the value of r, it is valid for any r for which we do not divide by 0. This means that it is valid for any $r \neq 1$, as $1 - r \neq 0$.

If $r = 1$, we have the the following:

$\sum_{i=0}^{n-1} 1^i = \sum_{i=0}^{n-1} 1$.

This sum is easier to evaluate. This is simply repeated addition of 1 n times, which evaluates to n.

b. The sum of the first n entries of the arithmetic progression $d, 2d, 3d, \ldots, nd$ (for $d > 0$) is given by $dn(n+1)/2$.

Solution

We can express this as a sum to make this easier to write:

Proposition: $\sum_{i=1}^{n} id = \frac{dn(n+1)}{2}$.

Base case: For $n = 1$, $\sum_{i=1}^{n} id = 1d = d = \frac{d(1)(1+1)}{2}$.

Induction Hypothesis: For some $n \in \mathbb{N}$, $\sum_{i=1}^{n} id = \frac{dn(n+1)}{2}$.

Induction Step: Suppose the induction hypothesis is true. Then, for some $n \in \mathbb{N}$, we have the following:

$\sum_{i=1}^{n} id = \frac{dn(n+1)}{2}$.

We are concerned with the case when the upper limit of the sum is $n+1$, so we should add the next term in the series to both sides. This term is $(n+1)d$ in this case. This gives us the following:

$\sum_{i=1}^{n} id + (n+1)d = \sum_{i=1}^{n+1} id = \frac{dn(n+1)}{2} + (n+1)d$.

Then the left side is finished, so we only need to work with the right side. Factoring the $d(n+1)$ term on the right side gives us the following:

$\sum_{i=1}^{n+1} id = d(n+1)(\frac{n}{2} + 1)$.

Simplifying, we obtain the final result:

$\sum_{i=1}^{n+1} id = d(n+1)(\frac{n+2}{2}) = \frac{d(n+1)((n+1)+1)}{2}$.

As this equation simply replaces n by $n+1$ in the induction hypothesis, the proof is finished.

2. Prove that every amount of postage that is at least 12c can be made from some combination of 4c and 5c stamps. (Hint: (i) strong induction. (ii) you need to check multiple base cases.)

   Solution

   Method 1: Using the ordinary induction.

   This question is equivalent to the proof such that $n = 4a + 5b$ where $n, a, b \in \mathbb{N}$ and $n \geq 12$.

   Base case: $n = 12$, $a = 3$, $b = 0$.

   Induction Hypothesis: For some k, there exists a and b such that $k = 4a + 5b$.

   Induction Step: Assuming the induction hypothesis is true, we need to think about two different cases: $a \geq 1$ and $a = 0$.

   If $a \geq 1$,

$$\begin{aligned} n + 1 &= 4a + 5b + 1 \\ &= 4a + 4 - 4 + 5b + 1 \\ &= 4a - 4 + 5b + 5 \\ &= 4(a-1) + 5(b+1) \end{aligned}$$

   Denoting $a^* = a - 1$ and $b^* = b + 1$, we get new a^* and b^* for the next sequence from previous a and b values from induction hypothesis.

   If $a = 0$,

$$\begin{aligned} n + 1 &= 5b + 1 \\ &= 5b + 1 + 16 - 16 \\ &= 16 + 5b - 15 \\ &= 4 \cdot 4 + 5(b-3) \end{aligned}$$

   Denoting $a^* = 4$ and $b^* = b - 3$, we can define new a^* and b^* with the same reasoning in the first case.

   Method 2: Using a strong induction,

   Define the predicate $P(n)$ that there exists non-negative integers a, b, and an integer n such that $4a + 5b = n$ where $n \geq 12$.

   Base case: Need to consider following 4 cases, and you can see that the pattern repeats every four steps by calculating further.

   a. $n = 12$, $a = 3$, $b = 0$

   b. $n = 13$, $a = 2$, $b = 1$

c. $n = 14$, $a = 1$, $b = 2$

d. $n = 15$, $a = 0$, $b = 3$

Strong Induction Hypothesis: $P(k)$ is true for all $k \in \{12, 13, \ldots, n\}$.

Induction Step: Assuming that the strong induction hypothesis is true, we now show that $P(n+1)$ is true. Since we know $P(n-3)$ is true, we have $n-3 = 4a+5b$ for some non-negative integers a and b. Therefore, $n+1 = 4(a+1)+5b$, which proves $P(n+1)$.

Therefore, by induction, $P(n)$ is true for all n.

3. The game of *Nim* is a two-player game involving a box of matchsticks. Two piles of n matchsticks each are placed on a table. Players take turns, and in each turn a player removes some (non-zero) number of matchsticks from one of the two piles. The player who removes the last matchstick wins.

   a. Find another student in your recitation class, and play the game using $n = 4$ and $n = 5$.

   b. The player who has the second move *always wins*. Figure out the winning strategy.

   c. Prove that the winning strategy always works using strong induction.

   Solution

   The winning strategy of *Nim* is that the second move player mimicks the opponent's move so that both piles have equal amount of matchsticks. For example, if the first player removed k matchsticks from the first pile, the second player needs to take k matchsticks from the other pile.

   Let $P(n)$ be "If both piles of matchsticks have n matchsticks and its the first player's turn, then the second player wins the game assuming that the player used the correct strategy."

   Base case: $n = 0$, then the second player wins the game.

   Induction Hypothesis: Let $P(i)$ is true such that $0 \leq i \leq n$ for some n.

   Induction Step: We need to show that P(n+1) holds.

   Let the first player took k matchsticks from one of the piles where $1 \leq k \leq n+1$ and the second player removed k matchsticks from the other piles. Then, both pile has $n+1-k$ matchsticks which $0 \leq n+1-k \leq n$.

   By the induction hypothesis, the second move player wins using the winning strategy.