# Sample Midterm Exam 2 Solution

- There are 6 questions in this exam, totaling 50 points.
- Total duration: 60 minutes.
- Please **write your name and netid** on the top of this page.
- You **can** use two pages as cheat sheets.
- You **cannot** consult your notes, textbook, your neighbor, or Google.

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1. **(10 points)** Consider the following relations defined on $\mathbb{Z}^+$:

   $R_1 = \{(x, y) \mid x + y > 10\}$

   $R_2 = \{(x, y) \mid y \text{ divides } x\}$

   $R_3 = \{(x, y) \mid x \text{ and } y \text{ have no common divisors}\}$

Indicate which of these relations is/are:

**Solution**

(a) reflexive.

$R_2$ only. Consider $x = 4$ as a counterexample for $R_1$, and consider any $z \in Z$ as a counterexample for $R_3$.

(b) symmetric.

$R_1$ and $R_3$. Consider any $x \neq y \in Z$ as a counterexample for $R_2$.

(c) antisymmetric.

$R_2$ only. Consider $x = 1$ and $y = 10$ as a counterexample for $R_1$, and consider any $x \neq y \in Z$ as a counterexample for $R_3$. Note that antisymmetry is defined only on non-equal pairs, so the reflexive property of $R_2$ does not affect its antisymmetry.

(d) transitive.

$R_2$ only. Consider $a = 1$, $b = 10$, and $c = 1$ as a counterexample for $R_1$, and consider $x = 2$, $y = 3$, and $z = 4$ as a counterexample for $R_3$.

(e) irreflexive.

$R_3$ only. Consider $x = 6$ as a counterexample for $R_1$, and consider any $z \in Z$ as a counterexample for $R_2$.

*Note that it was not necessary to disprove the incorrect answers on the tests, but I have provided some counterexamples here for your convenience.

2. **(5 points)** A sequence $a_0, a_1, a_2, ...$ is defined by letting $a_0 = 3$ and $a_k = a_{k-1}^2$ for all integers $k \geq 1$.

(a) Evaluate the first four elements ($a_0$ through $a_3$) of this sequence.

(b) Write down a closed form expression for $a_n$. (No proof necessary.)

**Solution**

(a) First, we have $a_0 = 3$ by definition.

Then $a_1 = 3^2 = 9$, and

$a_2 = 9^2 = 81$.

Lastly, $a_3 = 6561$.

(b) If we look closely at the previous exercise, we can find the following pattern: $a_0 = 3 = 3^1 = 3^{2^0}$,

$a_1 = 9 = 3^2 = 3^{2^1}$,

$a_2 = 81 = 9^2 = 3^4 = 3^{2^2}$, and

$a_3 = 81^2 = 9^4 = 3^8 = 3^{2^3}$.

From this pattern, we find that for any $n \in N$, $a_n = 3^{2^n}$.

3. **(10 points)** The set $A = \{2, 4, 5, 10, 12, 20, 25, 30\}$ is partially ordered with respect to the "divides" relation.

(a) Draw the Hasse diagram representation of the above relation.

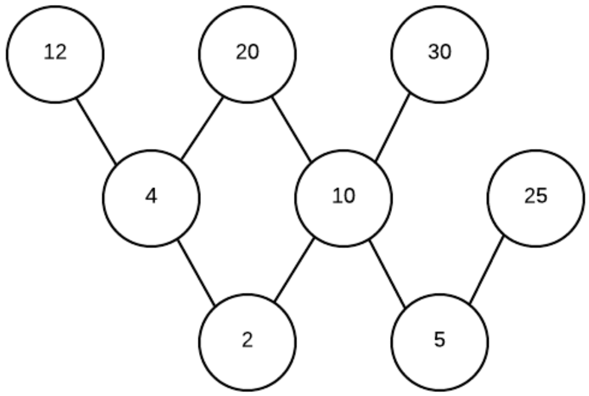



Figure 1: Hasse

(b) List all minimal and maximal elements.

Minimal: 2, 5

Maximal: 12, 20, 25, 30

(c) Run topological sort on the Hasse diagram to obtain a compatible total ordering of the elements.

Many topological sort exists and this is one of them.

2, 5, 4, 10, 12, 20, 25, 30

4. **(10 points)** Let $S = \{1, 2, 3, 4, 5\}$ and let $A = S \times S$ (i.e., $A$ consists of all pairs of elements from $S$). Define the following relation $R$ on $A$ as follows: $(a, b)R(a', b')$ if and only if $ab' = a'b$.

Prove that $R$ is an equivalence relation.

**Solution**

The relation $(a, b)R(c, d)$ is also satisfied if $a/b = c/d$ from a simple algebra.

- R is reflexive because $(a, b)R(a, b)$ since $ab = ab$.
- R is symmetric because if $a/b = c/d$ then $c/d = a/b$:

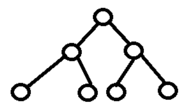
$(a, b)R(c, d) \to (c, d)R(a, b)$

- R is transitive because if $a/b = c/d$ and $c/d = e/f$ then $a/b = e/f$:

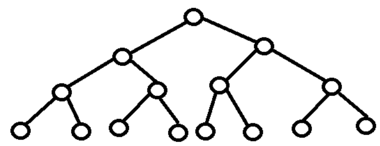
Hence, $(a, b)R(c, d) \wedge (c, d)R(e, f) \to (a, b)R(e, f)$

5. **(5 points)** Recall that a (complete) binary tree is a graph that is constructed by starting from the *root* connected to a pair of children nodes (called leaves), and recursively adding a pair of children nodes to each leaf node. The number of layers in a complete binary tree (excluding the root) is called the *depth* of the tree.

(a) **(1 point)** Draw the complete binary trees of depth $n$ for $n = 2$ and $n = 3$, and count the **total** number of nodes in each of these trees.



N=2; Total 7 nodes.



N=3; Total 15 nodes.

(b) **(4 points)** Via mathematical induction, provide a (very simple) proof of the fact that a tree of depth $n$ contains $2^n$ **leaf** nodes.

Base case: Tree of depth 1 has two leaf nodes = 2^1. Theorem holds.

Induction Hypothesis: Let us assume this is true for a tree with depth k. That is, a tree with depth k has 2^k nodes.

Inductive step: We shall prove that this is true for a tree with depth k+1.

Every node at depth k would have two children. As a result, the total childern of the nodes of depth k is 2(2^k) = 2^(k+1). Therefore, every tree with depth k+1 also has 2^(k+1) leaf nodes.

6. **(10 points)** A *complete bipartite* graph is an undirected graph with $m + n$ nodes, where each of the first $m$ nodes are connected with each of the last $n$ nodes. Assume for the sake of this problem that $m$ and $n$ are both greater than 2.

(a) Use the First Degree theorem to count the number of edges in this graph.

In order to calculate the total number of edges, we need to calculate the total number of degree.

Using the first degree theorem, $\sum_{v \in V} deg(v) = 2 \cdot |E|$,

$$
\begin{aligned}
mn + nm &= 2 \cdot |E| \\
|E| &= mn
\end{aligned}
$$

(b) What is the minimum number of colors needed to color the vertices of such a graph so that no adjacent vertices have the same color?

2 colors are the minimum number to satisfy the condition.

(c) Under what conditions on $m$ and $n$ does this graph admit an Euler path?

Since $m$ and $n$ are both greater than 2, both $m$ and $n$ has to be even number to satisfy an Euler path. (Additional explanation: m=3, n=2 also satisfies the Euler path)

## SCRATCH