Solutions. Final Exam 2015

1. $y' = (y^2+1)\cos x \qquad y(\pi/2) = 1$

$$\int \frac{1}{y^2+1}\,dy = \int \cos x\,dx$$

$$\arctan y = \sin x + C \Rightarrow y = \tan(\sin x + C)$$

plug in initial condition:

$$\arctan(1) = \sin \pi/2 + C \Rightarrow C = \frac{\pi}{4} - 1$$

$$\therefore \underline{y = \tan(\sin x + \pi/4 - 1)}$$

2. $y' + \dfrac{2y}{x} = \dfrac{\sin 2x}{x}$ (Linear)

$P = \dfrac{2}{x} \qquad \mu = e^{\int 2/x\,dx} = e^{2\ln|x|} = e^{\ln x^2} = x^2$

Equation becomes $(x^2 \cdot y)' = x \sin 2x$

$$\Rightarrow x^2 y = \int x \sin 2x\,dx = -\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x + C$$

$$\therefore \underline{y = -\frac{1}{2x}\cos 2x + \frac{1}{4x^2}\sin 2x + \frac{C}{x^2}}$$

3. $P_0 = 281.4 \qquad P_{10} = 308.7 \quad P_{16}$?

$$P(t) = P_0 e^{kt}$$

$P(0) = 281.4$

$P(10) = 281.4e^{10k} = 308.7 \Rightarrow e^{10k} = 308.7/281.4$

$$\Rightarrow 10k = \ln\left(\frac{308.7}{281.4}\right) \Rightarrow k = \frac{1}{10}\ln\left(\frac{308.7}{281.4}\right)$$

Ans

$P(16) = 281.4e^{16k} = 281.4e^{0.148148} \simeq$ 326 mill.

4. $y' = \frac{y^2+xy}{x^2} = \left(\frac{y}{x}\right)^2 + \frac{y}{x}$ (subst. homogeneous)

Let $u = y/x$ or $y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$

New Equ: $\cancel{u} + x\frac{du}{dx} = u^2 + \cancel{u} \Rightarrow \frac{du}{u^2} = \frac{1}{x}dx$

$-u^{-1} = \ln|x| + c \Rightarrow -\frac{x}{y} = \ln|x| + c$

$\Rightarrow y = -\frac{x}{\ln|x|+c}$

See 5 at the end →

5. $y'' + 2y' + 6y = 4x + e^{-2x}$

Aux Equ: $m^2 + 2m + 6 = 0 \iff m^2 + 2m + 1 = -5$

$(m+1)^2 = -5 \Rightarrow m+1 = \pm\sqrt{5}\,i \quad m_{1,2} = -1 \pm \sqrt{5}\,i$

$\alpha = -1 \quad \beta = \sqrt{5}$

$y_c = c_1 e^{-x}\cos(\sqrt{5}x) + c_2 e^{-x}\sin(\sqrt{5}x)$

$y_{p_1} = Ax + B \quad y_{p_1}' = A \quad y_{p_1}'' = 0$ plug in:

$0 + 2A + 6Ax + 6B = 4x \Rightarrow 6A = 4 \quad 2A + 6B = 0$

$A = 2/3 \quad B = -\frac{A}{3} = -\frac{2}{9} \quad \therefore y_{p_1} = \frac{2}{3}x - \frac{2}{9}$

$y_{p_2} = Ae^{-2x} \quad y_{p_2}' = -2Ae^{-2x} \quad y_{p_2}'' = 4Ae^{-2x}$

$\cancel{4Ae^{-2x}} - \cancel{4Ae^{-2x}} + 6Ae^{-2x} = e^{-2x} \Rightarrow 6A = 1 \Rightarrow A = 1/6$

$y_{p_2} = \frac{1}{6}e^{-2x} \Rightarrow y_p = y_{p_1} + y_{p_2} = \frac{2}{3}x - \frac{2}{9} + \frac{1}{6}e^{-2x}$

Sol $y = y_c + y_p = c_1 e^{-x}\cos(\sqrt{5}x) + c_2 e^{-x}\sin(\sqrt{5}x) + \frac{2}{3}x - \frac{2}{9} + \frac{1}{6}e^{-2x}$

7. $A = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$ $\vec{X}_0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $\vec{f}(t) = \begin{bmatrix} e^t \\ 1 \end{bmatrix}$

a) Char Eqn: $|A - \lambda I| = (-4-\lambda)(5-\lambda) + 18 = 0$

$\lambda^2 - \lambda - 2 = 0 \Rightarrow (\lambda+1)(\lambda-2) = 0 \Rightarrow$ Eigenvalues: $\lambda_1 = -1, \lambda_2 = 2$

find Eigenvectors:

For $\lambda_1 = -1$ Solve $(A - \lambda_1 I)\vec{k}_1 = \begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $3k_1 = 6k_2$, $k_1 = 2k_2$

let $\vec{K}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\therefore \vec{X}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}$

For $\lambda_2 = 2$ Solve

$(A - \lambda_2 I)\vec{k}_2 = \begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ $-6k_1 + 6k_2 = 0$, $k_1 = k_2$

let $\vec{k}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\therefore \vec{X}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$

General Sol $\vec{X} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$

b) Solve IVP $\vec{x}(0) = \begin{pmatrix} 2c_1 + c_2 \\ c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Rightarrow$

$2c_1 + c_2 = 0$
$-(c_1 + c_2 = 2)$
$c_1 = -2$

$c_2 = -2c_1 \Rightarrow c_2 = -2 \cdot -2 = 4$

$\therefore \vec{x} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} e^{-t} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} e^{2t}$

c) Find gral Sol. $\vec{x}' = A\vec{x} + f(t)$.
Using variation of par we have:

Find Matrix: $\Phi = \begin{pmatrix} 2e^{-t} & e^{2t} \\ e^{-t} & e^{2t} \end{pmatrix}$ $\det \Phi = (2-1)e^{t} = e^{t}$

$\Phi^{-1} = e^{-t}\begin{pmatrix} e^{2t} & -e^{2t} \\ -e^{-t} & 2e^{-t} \end{pmatrix} = \begin{pmatrix} e^{t} & -e^{t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}$

$$\vec{x}_p = \Phi\int \Phi^{-1}\vec{f}\,dt = \Phi\int \begin{pmatrix} e^{t} & -e^{t} \\ -e^{-2t} & 2e^{-2t}\end{pmatrix}\begin{pmatrix} e^{t} \\ 1\end{pmatrix} dt$$

$$= \Phi\int\begin{pmatrix} e^{2t}-e^{t} \\ -e^{-t}+2e^{-2t}\end{pmatrix}dt = \Phi\begin{pmatrix}\frac{1}{2} e^{2t}-e^{t} \\ e^{-t}-e^{-2t}\end{pmatrix}$$

$$=\begin{pmatrix}2e^{-t} & e^{2t}\\ e^{-t} & e^{2t}\end{pmatrix}\begin{pmatrix}\frac{1}{2} e^{2t}-e^{t}\\ e^{-t}-e^{-2t}\end{pmatrix} = \begin{pmatrix} e^{t}-2+e^{t}-1 \\ \frac{1}{2} e^{t} - 1 + e^{t} - 1\end{pmatrix} = \begin{pmatrix}2e^{t}-3\\ \frac{3}{2} e^{t}-2\end{pmatrix}$$

Gral Sol: $$\vec{X} = \begin{pmatrix}2e^{-t} & e^{2t}\\ e^{-t} & e^{2t}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} + \begin{pmatrix}2e^{t}-3\\ \frac{3}{2}e^{t}-2\end{pmatrix}$$

8. Find the general sol of $\vec{x}\,' = \begin{pmatrix}1&0&0\\2&3&-5\\6&2&-3\end{pmatrix}\vec{X}$

Char Equ. $|A-\lambda I| = \begin{vmatrix}1-\lambda&0&0\\2&3-\lambda&-5\\6&2&-3-\lambda\end{vmatrix} = 0$

$$\Rightarrow (1-\lambda)\left[(3-\lambda)(-3-\lambda)+10\right]+0+0 = 0$$
$$(1-\lambda)\left(-(9-\lambda^2)+10\right) = (1-\lambda)(\lambda^2+1) = 0$$

$\lambda = 1 \quad \lambda = \pm i$

For $\lambda = 1$

find $\vec{K}_1$ $\quad (A-\lambda_1 I)\vec{K}_1 = \begin{pmatrix}0&0&0\\2&2&-5\\6&2&-4\end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ $\quad 2k_1+2k_2-5k_3=0$, $\quad 6k_1+2k_2-4k_3=0$

Eliminating k_1 we get $4k_2 - 11k_3 = 0 \Rightarrow 4k_2 = 11k_3$

$\therefore \vec{K}_1 = \begin{pmatrix}-1\\11\\4\end{pmatrix}$ $\quad$ from eqn 1: $k_1 = \frac{5}{2}k_3 - k_2 = 10 - 11 = -1$

Find Complex eigenvector (use $\lambda_2 = i$)

$$\begin{pmatrix}1-i&0&0\\2&3-i&-5\\6&2&-3-i\end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} > \begin{array}{l}(1-i)k_1 = 0 \Rightarrow k_1 = 0\\ (3-i)k_2 - 5k_3 = 0 \Rightarrow k_2 = 5 \quad k_3 = 3-i\end{array}$$

$$\Rightarrow \vec{K} = \begin{pmatrix}0\\5\\3-i\end{pmatrix} = \underbrace{\begin{pmatrix}0\\5\\3\end{pmatrix}}_{\vec{a}} + i\underbrace{\begin{pmatrix}0\\0\\-1\end{pmatrix}}_{\vec{b}}$$

Note in our lecture notes I used B_1 instead of a and B_2 instead of b….

Sol:
$$\vec{X} = C_1\begin{pmatrix}-1\\11\\4\end{pmatrix}e^{t} + C_2\left(\vec{a}\cos t - \vec{b}\sin t\right) + C_3\left(\vec{a}\sin t + \vec{b}\cos t\right)$$

$$\vec{X} = C_1\begin{pmatrix}-1\\11\\4\end{pmatrix}e^{t} + C_2\begin{pmatrix}0\\5\cos t\\3\cos t + \sin t\end{pmatrix} + C_3\begin{pmatrix}0\\5\sin t\\3\sin t - \cos t\end{pmatrix}$$

9. Use Laplace transform to solve

$$y'' + y' - 2y = 3e^{t} \quad y(0) = 1 \; ; \; y'(0) = 0$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 3\mathcal{L}\{e^{t}\}$$

$$s^2 Y - s - 0 + sY - 1 - 2Y = 3\,\frac{1}{s-1}$$

$$Y(s^2+s-2) = \frac{3}{s-1} + s + 1$$

$$s^2+s-2 = (s+2)(s-1)$$

$$Y = \frac{3}{(s-1)^2(s+2)} + \frac{(s+1).(s-1)}{(s-1)(s+2)\,(s-1)}$$

$$Y = \frac{3+s^2-1}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$$

$$\mathcal{L}\{y\} = Y = \frac{1/3}{s-1} + \frac{1}{(s-1)^2} + \frac{2/3}{s+2} \Rightarrow y = \mathcal{L}^{-1}(Y)$$

$$y = \frac{1}{3}e^{t} + te^{t} + \frac{2}{3}e^{-2t}$$

10. $y'' - (x+1)y = 0$, $y(0) = 1$, $y'(0) = 3$

$\Rightarrow \quad C_0 = 1 \quad C_1 = 3$

assume $y = \sum_{n=0}^{\infty} C_n x^n$

$$y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n$$

$$xy = \sum_{n=0}^{\infty} C_n x^{n+1} = \sum_{n=1}^{\infty} C_{n-1} x^n$$

$$y'' - (x+1)y = 0$$

$$2C_2 + \sum_{n=1}^{\infty} (n+2)(n+1) C_{n+2} x^n - \sum_{n=1}^{\infty} C_{n-1} x^n - C_0 - \sum_{n=1}^{\infty} C_n x^n = 0$$

$$2C_2 - C_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)C_{n+2} - C_{n-1} - C_n\right] x^n = 0$$

$$\Rightarrow 2C_2 - C_0 = 0 \quad \Rightarrow C_2 = \frac{C_0}{2} = 1/2$$

Recurrence Relation is: $C_{n+2} = \dfrac{C_n + C_{n-1}}{(n+2)(n+1)} \quad (n \geq 1)$

$$n=1 \Rightarrow C_3 = \frac{C_1 + C_0}{3 \cdot 2} = \frac{1+3}{6} = 4/6 = 2/3$$

$$n=2 \Rightarrow C_4 = \frac{C_2 + C_1}{4 \cdot 3} = \frac{1/2 + 3}{12} = \frac{7}{24}$$

$$\therefore \quad y = \underbrace{1 + 3x + \frac{1}{2}x^2 + \frac{2}{3}x^3}_{\text{only these are asked...}} + \frac{7}{24}x^4$$

5. Solve IVP: $y''' - 2y'' - 24y' = 0$; $y(0)=1, y'(0)=2, y''(0)=0$

aux Eqn: $m^3 - 2m^2 - 24m = 0$

$m(m^2 - 2m - 24) = 0$

$m(m-6)(m+4) = 0 \Rightarrow m = 0, 6, -4$

l.i sols: $y_1 = e^{0x}$ $\quad y_2 = e^{6x}$ $\quad y_3 = e^{-4x}$

General Sol: $y = c_1 + c_2 e^{6x} + c_3 e^{-4x}$

$y' = 6c_2 e^{6x} - 4c_3 e^{-4x}$

$y'' = 36c_2 e^{6x} + 16c_3 e^{-4x}$

$\Rightarrow y(0) = c_1 + c_2 + c_3 = 1$

$y'(0) = 6c_2 - 4c_3 = 2$

$y''(0) = 36c_2 + 16c_3 = 0$

$-(36c_2 - 24c_3 = 12)$

$36c_2 + 16c_3 = 0$

$0 \quad 40c_3 = -12 \Rightarrow c_3 = -12/40$

$c_3 = -\frac{3}{10}$ $\qquad c_2 = \frac{-16c_3}{36} = -\frac{16}{36}\left(-\frac{3}{10}\right) = \frac{.8}{12.5} = \frac{8}{60}$

and $c_1 = 1 - c_2 - c_3 = 1 - \frac{8}{60} + \frac{3}{10} = \frac{60 - 8 + 18}{60} = \frac{70}{60} = \frac{7}{6}$

$\therefore y = \frac{7}{6} + \frac{8}{60} e^{6x} - \frac{3}{10} e^{-4x}$