Constraint Satisfaction Problems (CSPs)

Outline

I. Definition of a CSP

II. Map coloring

III. Job shop scheduling

IV. The diet problem and linear programming

* Figures/images are from the textbook site (or by the instructor). Otherwise, the source is cited unless such citation would make little sense due to the triviality of generating the image.

I. Definition of a CSP

- A set of variables $\mathcal{X} = \{X_1, \dots, X_n\}$.

- A set of domains $\mathcal{D} = \{D_1, \dots, D_n\}$.

  Domain $D_i = \{v_1, \dots, v_{k_i}\}$ is the set of allowable values for the variable X_i.

  e.g., $\{true, false\}$ for a Boolean variable.

- A set of constraints $\mathcal{C} = \{C_1, \dots, C_m\}$ that specifies allowable combination of values.

Relation

- C_j: $\langle (v_i, v_j)$, relation$\rangle$

  - a set of tuple of values for v_i and v_j

    If $D_1 = D_2 = \{1, 2, 3\}$, the relation "X_1 is greater than X_2":

$$\langle (X_1, X_2), \{(3,1), (3,2), (2,1)\} \rangle$$

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    $$\langle (X_1, X_2), \{(3,1), (3,2), (2,1)\} \rangle$$

  - a function that checks if a tuple satisfies the relation

    $$\langle (X_1, X_2), X_1 > X_2 \rangle$$

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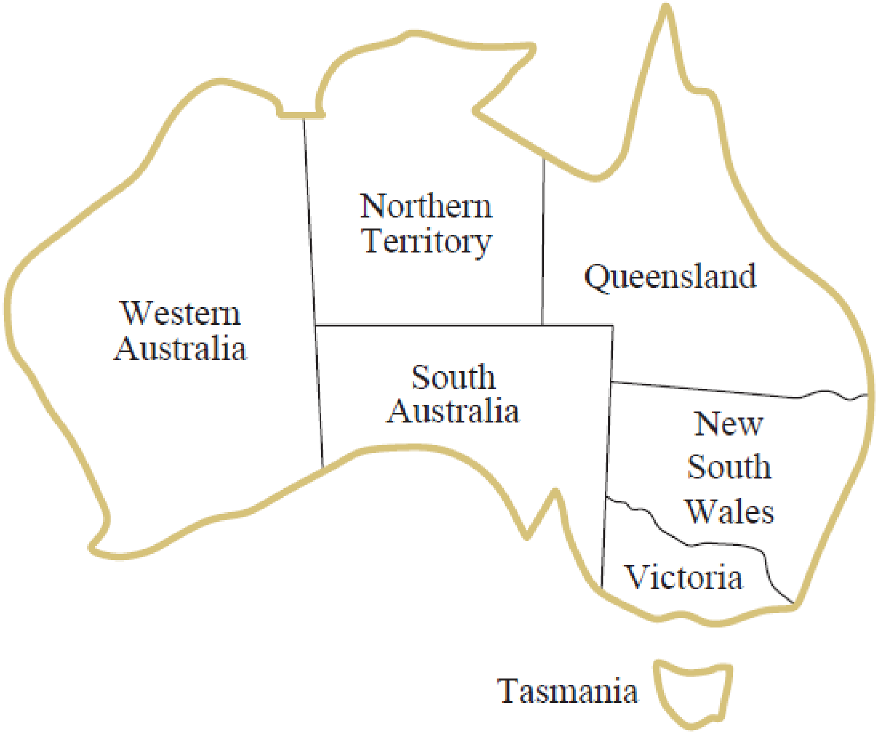
Solving a CSP is NP-complete in general!

II. Example 1: Map Coloring

Color the regions of Australia in red, green, or blue such that no two neighboring regions share the same color.



Variables: $\mathcal{X} = \{WA, NT, Q, NSW, V, SA, T\}$

Domains: $D_i = \{red, green, blue\}$

Map Coloring (cont'd)

Constraints: $\mathcal{C} = \{SA \neq WA,\ SA \neq NT,\ SA \neq Q,\ SA \neq NSW,\ SA \neq V$
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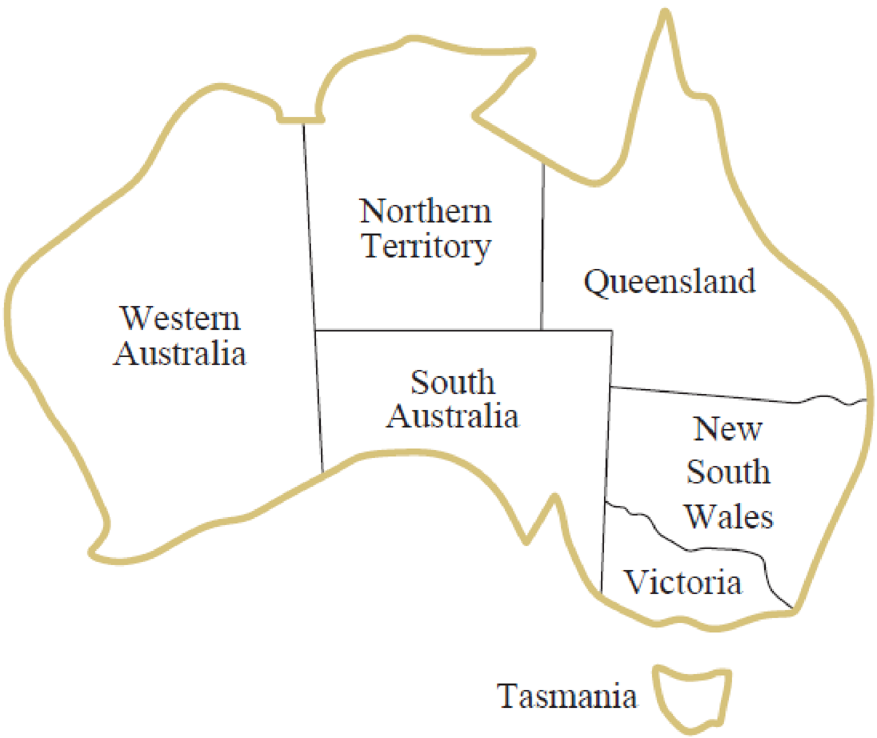

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$SA \neq WA$ by enumeration: { (*red*, *green*), (*red*, *blue*), (*green*, *red*),
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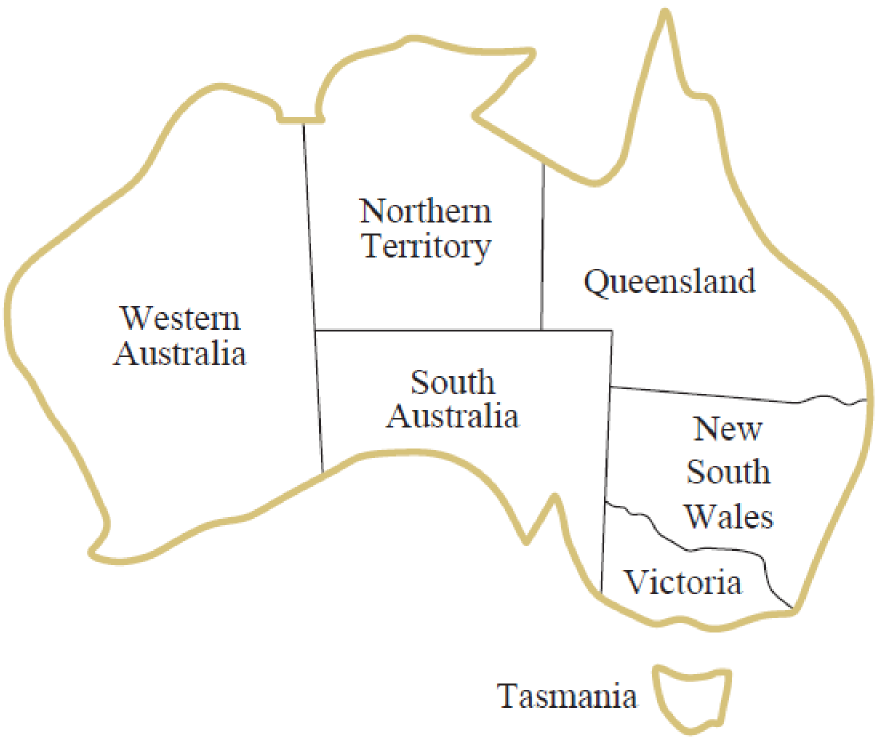

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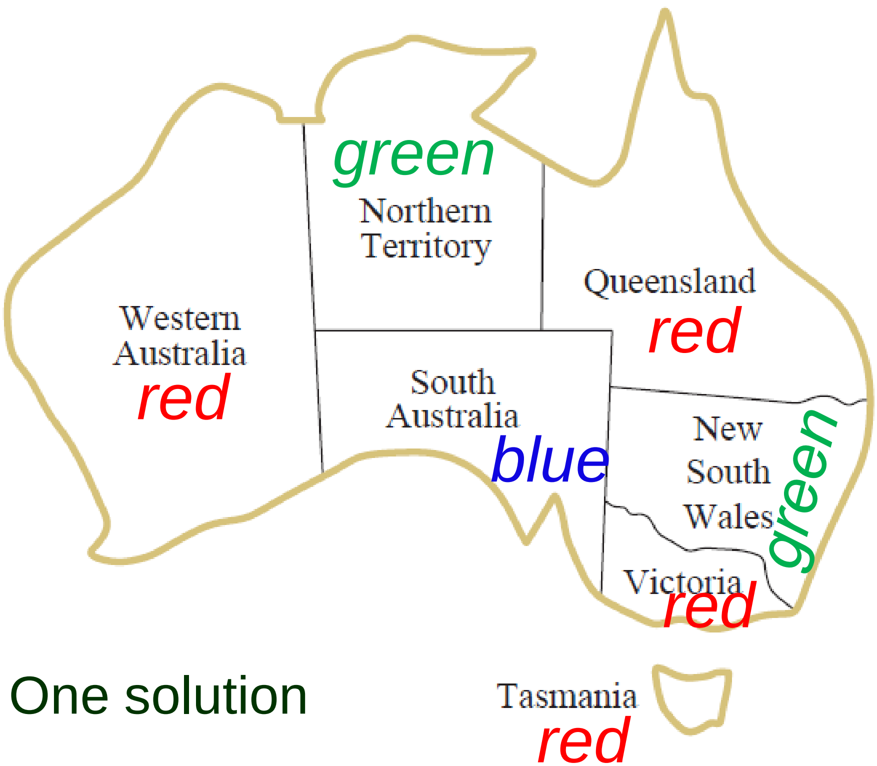
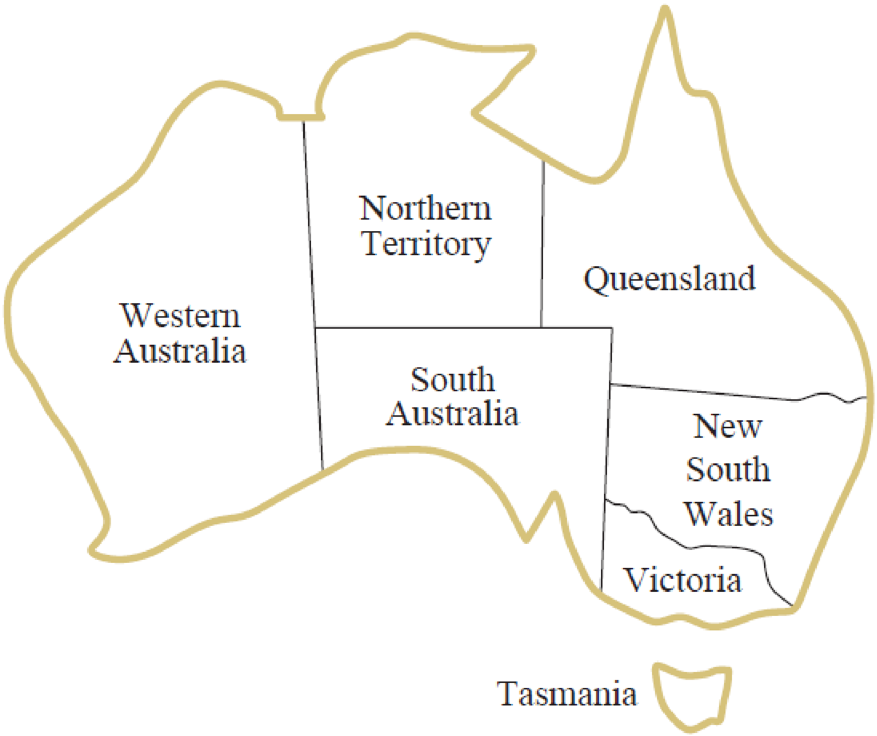


One solution

Constraint Graph

Binary constraints only.

variable ↔ vertex
constraint ↔ edge



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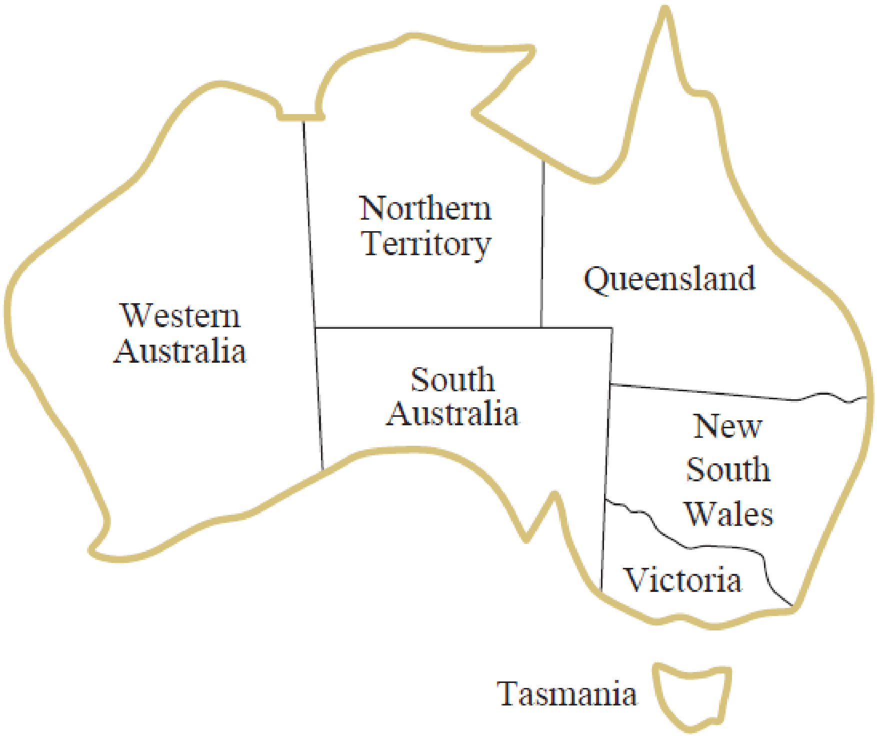

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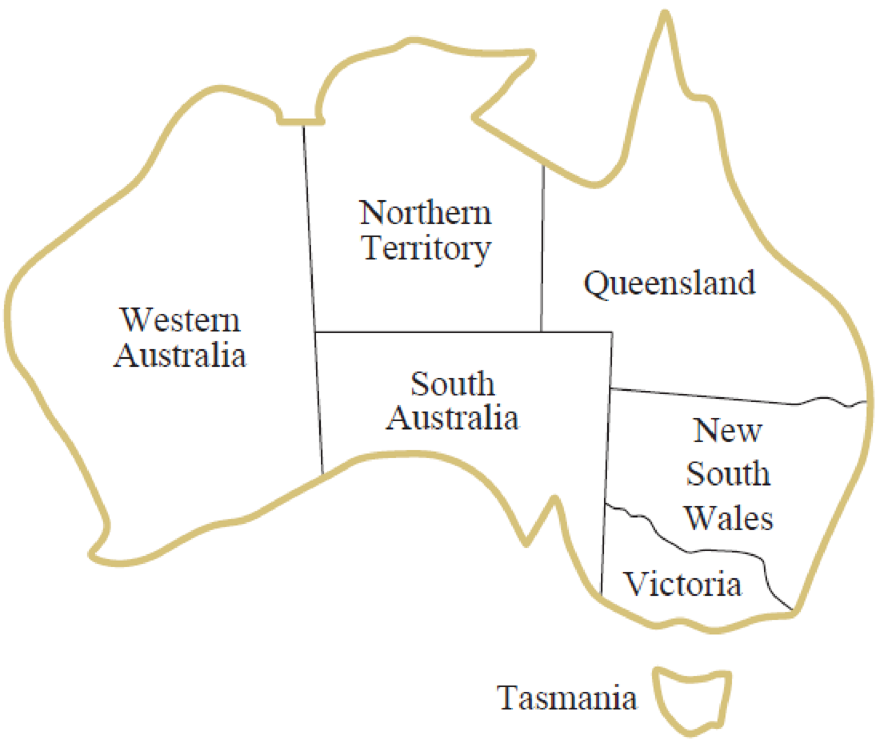

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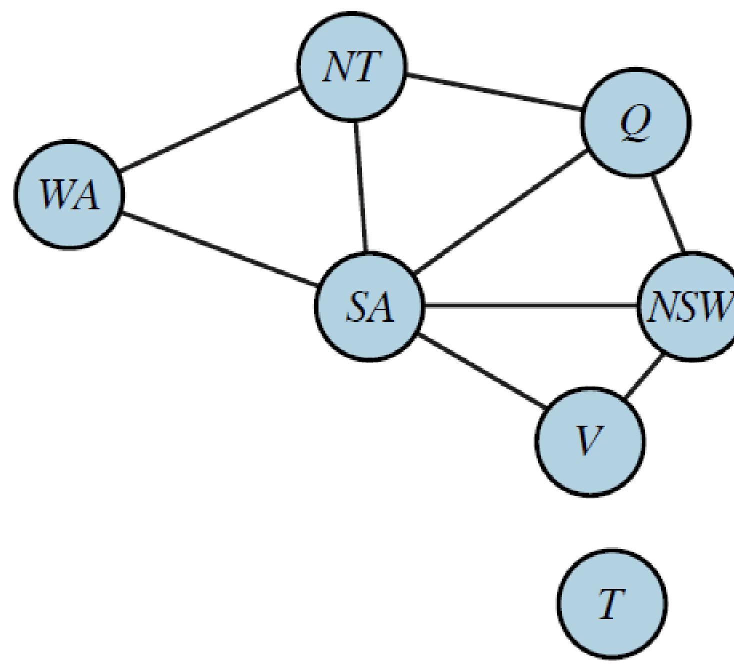
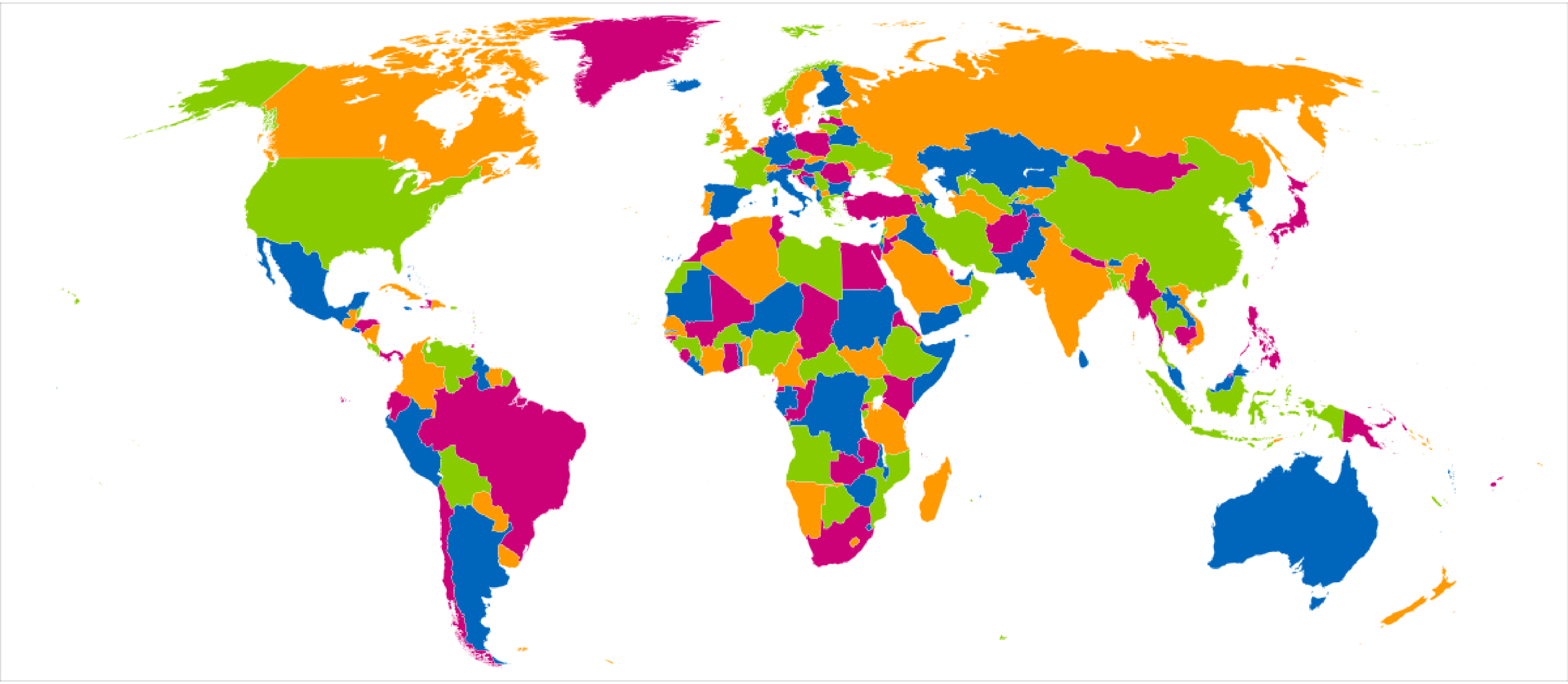


Four-Color Theorem

Theorem Any map in a plane can be colored using four colors in such a way that regions sharing a common boundary (other than a single point) do not share the same color.



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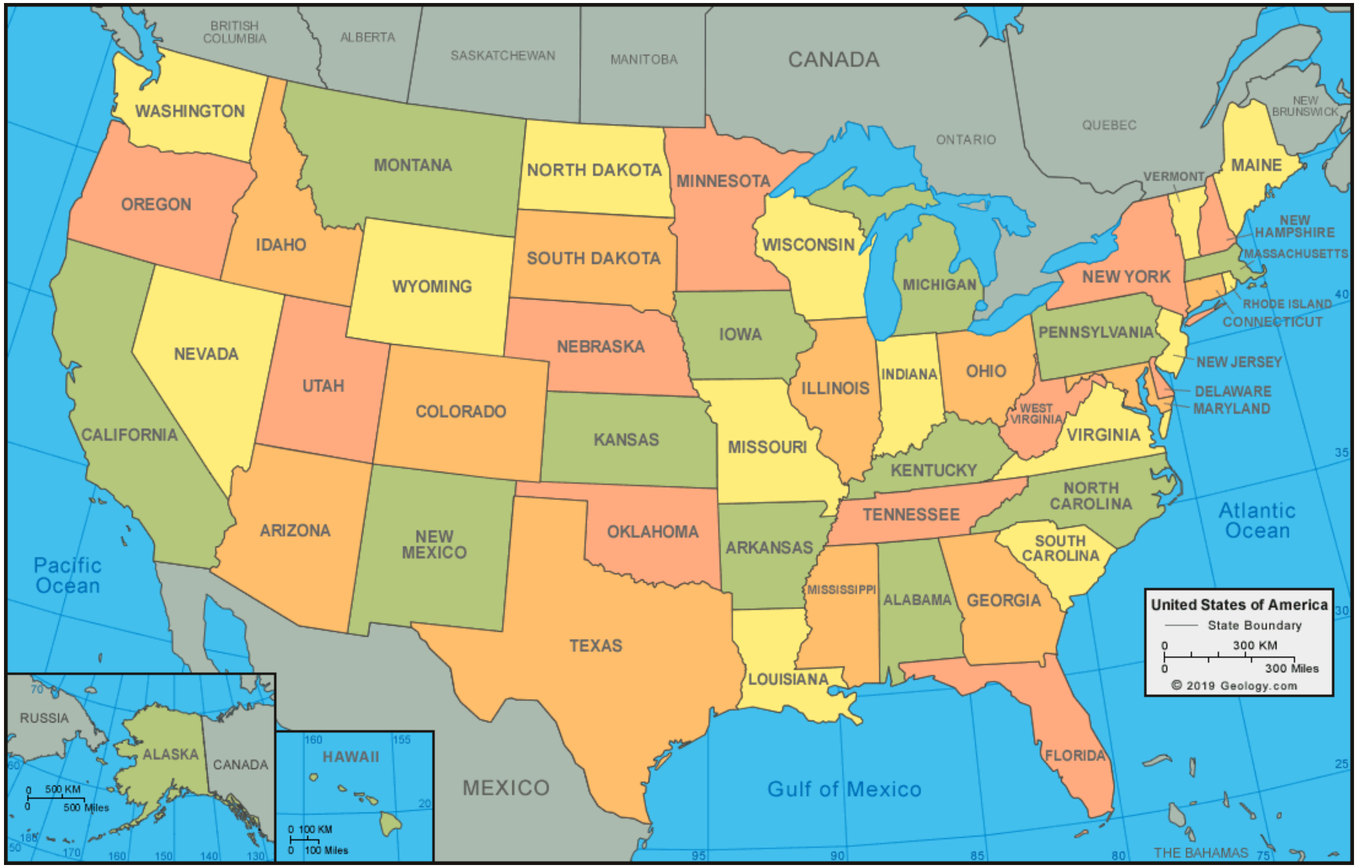

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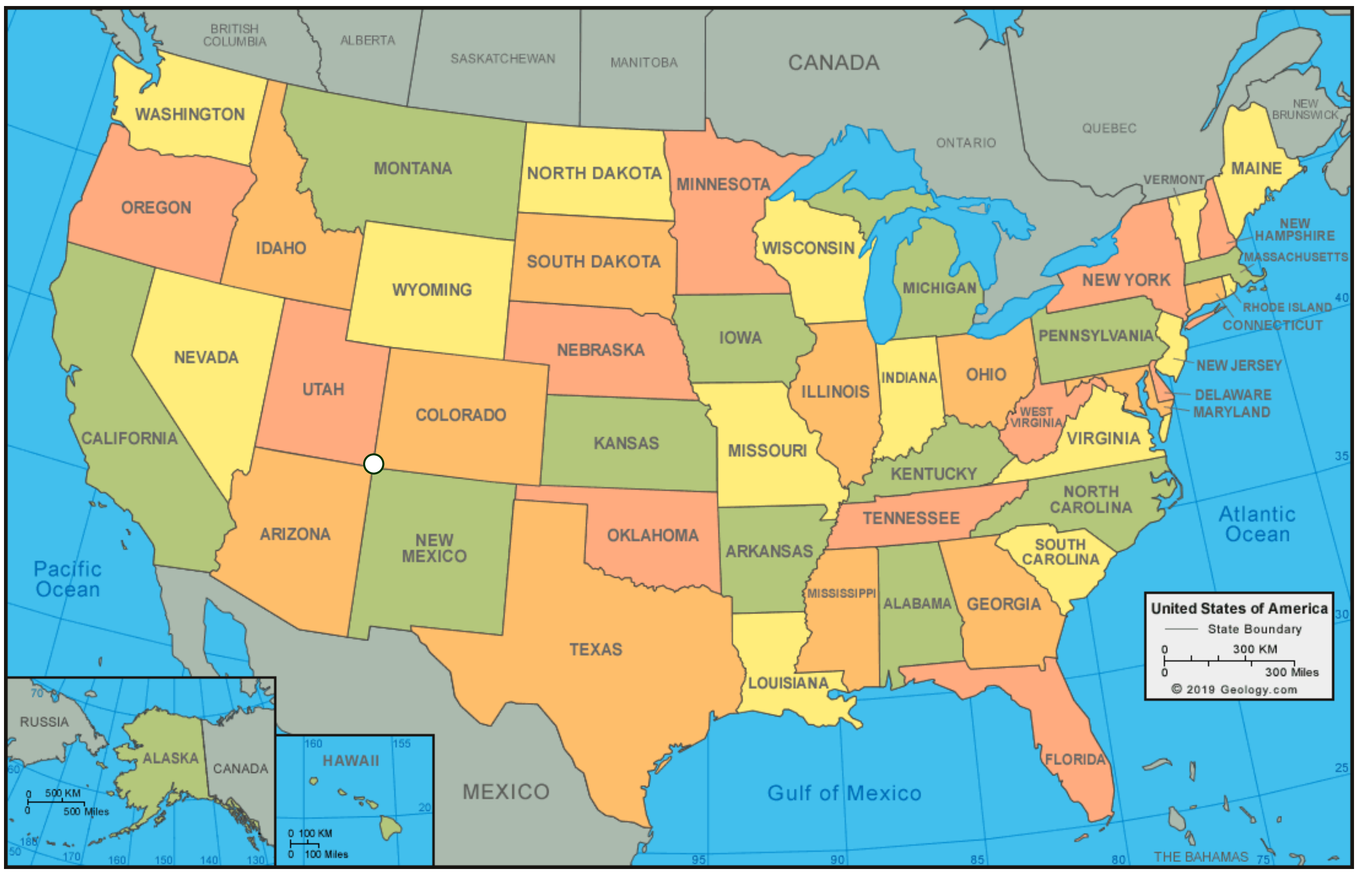

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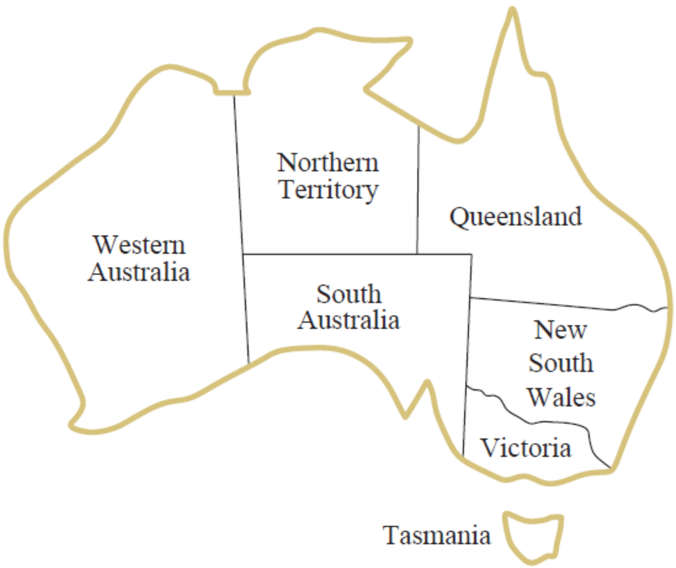

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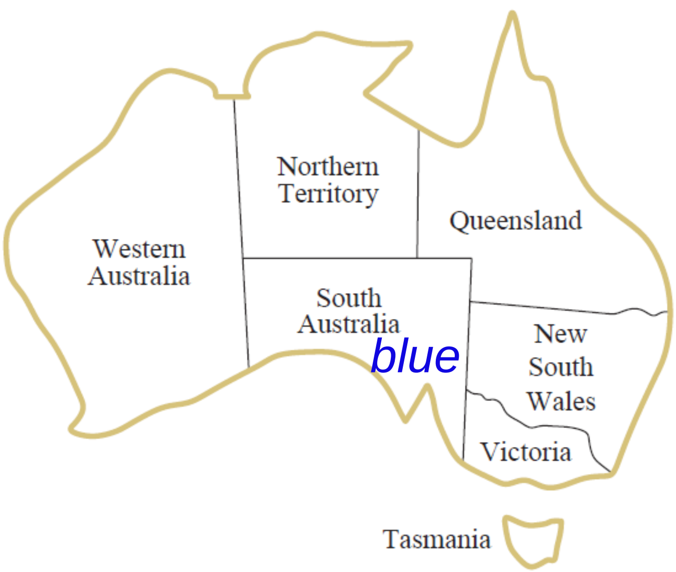

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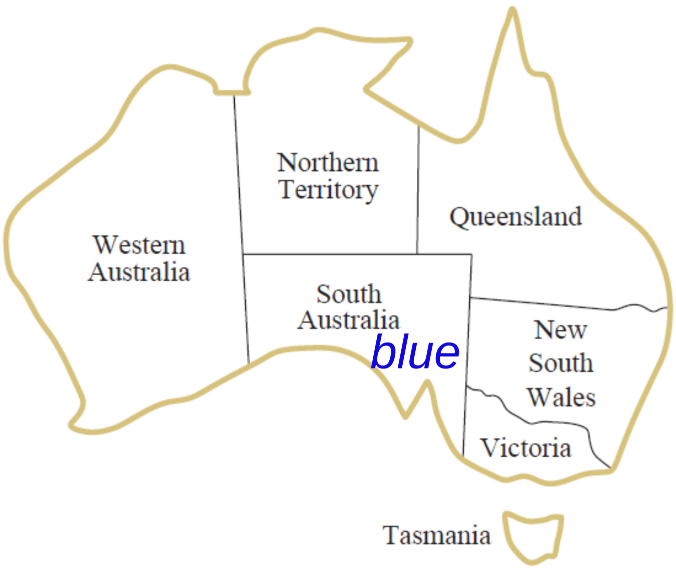

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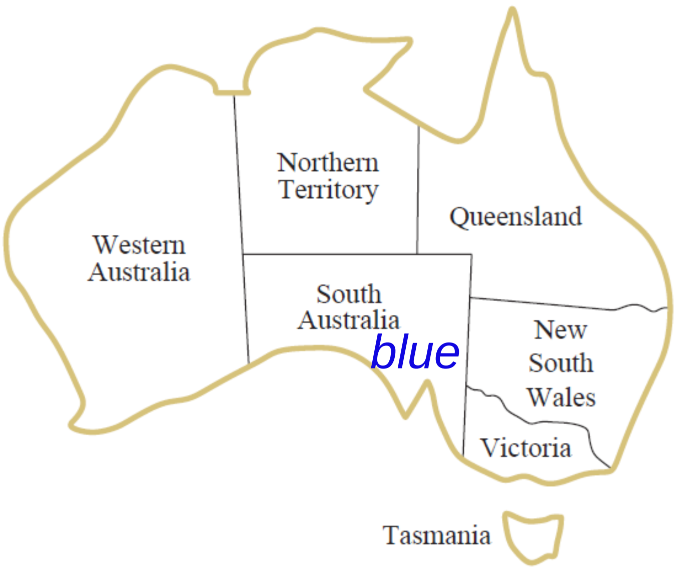

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A reduction from $3^5 = 243$ assignments by a search procedure not using the constraint.

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Is this specific state a goal?

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III. Example 2: Job-Shop Scheduling

Car assembly with 15 tasks:

- install axles (front and back): 2
- affix wheels (right and left, front and back): 4
- tighten nuts for each wheel: 4
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$$\mathcal{X} = \{Axle_F, Axle_B, Wheel_{RF}, Wheel_{LF}, Wheel_{RB}, Wheel_{LB}, Nuts_{RF}, Nuts_{LF}, Nuts_{RB}, Nuts_{LB}, Cap_{RF}, Cap_{LF}, Cap_{RB}, Cap_{LB}, Inspect\}.$$

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$Axle_F =$ starting time for installation of the front axle.

Precedence Constraints

$$T_1 + d_1 \leq T_2$$

starting time of task T_1 duration of task T_1

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- The axles have to be in place before the wheels are put on (axle installation takes 10 minutes).

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- Affix each wheel (1 minutes), then tighten the nuts (2 minutes), and finally attach the hubcap (1 minute, not represented)

$$Wheel_{RF} + 1 \leq Nuts_{RF} \qquad Nuts_{RF} + 2 \leq Cap_{RF}$$

$$Wheel_{LF} + 1 \leq Nuts_{LF} \qquad Nuts_{LF} + 2 \leq Cap_{LF}$$

$$Wheel_{RB} + 1 \leq Nuts_{RB} \qquad Nuts_{RB} + 2 \leq Cap_{RB}$$

$$Wheel_{LB} + 1 \leq Nuts_{LB} \qquad Nuts_{LB} + 2 \leq Cap_{LB}$$

More Constraints

- Inspection comes last and take 3 minutes

$$X + d_X \leq Inspect \quad \text{for every } X \in \mathcal{X}$$

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duration of task X

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Limit the domain of all variables (discretization)

$$\mathcal{D} = \{1, 2, \ldots, 27\}$$

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$(Axle_F + 10 \leq Axle_B)$ or $(Axle_B + 10 \leq Axle_F)$

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  Map coloring, 8-queens, scheduling (with time limits).

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IV. The Diet Problem*

How much money to spend in order to get what Polly needs every day?

- energy (2,000 kcal)
- protein (55 g)
- calcium (800 mg)

* V. Chvatal. *Linear Programming.* W. H. Freeman and Company, 1983.

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Food	Serving size	Energy (kcal)	Protein (g)	Calcium (mg)	Price per serving (cents)
Oat meal	28 g	110	4	2	3
Chicken	100 g	205	32	12	24
Eggs	2 large	160	13	54	13
Whole milk	237 cc	160	8	285	9
Cherry pie	170 g	420	4	22	20
Pork with beans	260 g	260	14	80	19

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Daily Serving Limits

	Servings at most per day
Oatmeal	4
Chicken	3
Eggs	2
Milk	8
Cherry pie	2
Pork with beans	2

Task: Design the *most economical* menu.

Formulating the Problem

x_1: servings of oatmeal
x_2: servings of chicken
x_3: servings of eggs
x_4: servings of whole milk
x_5: servings of cherry pie
x_6: servings of pork with beans

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$$\min \quad 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6$$

subject to

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\text{subject to} \quad & 0 \le x_1 \le 4 \\
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and

energy $\quad 110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 \ge 2000$

protein $\quad 4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 \ge 55$

calcium $\quad 2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 \ge 800$

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Objective function (linear)

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Linear Program!

$$\text{subject to} \quad \left.\begin{aligned} &0 \le x_1 \le 4 \\ &0 \le x_2 \le 3 \\ &0 \le x_3 \le 2 \\ &0 \le x_4 \le 8 \\ &0 \le x_5 \le 2 \\ &0 \le x_6 \le 2 \end{aligned}\right\} \text{Servings-per-day limits}$$

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energy $110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 \ge 2000$

protein $4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 \ge 55$

calcium $2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 \ge 800$

Constraints (linear)

Linear Programming (LP)

Max $c_1 x_1 + c_2 x_2 + \cdots + c_d\, x_d$

subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1d}x_d &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2d}x_d &\le b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nd}x_d &\le b_n \end{aligned}$$

Linear Programming (LP)

Max $c_1 x_1 + c_2 x_2 + \cdots + c_d\, x_d$

subject to
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1d}x_d \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2d}x_d \le b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nd}x_d \le b_n$$

$\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$

$\boldsymbol{c} = (c_1, c_2, \ldots, c_d)$

$\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$

Linear Programming (LP)

Max $c_1 x_1 + c_2 x_2 + \cdots + c_d\, x_d \qquad = \boldsymbol{c}\boldsymbol{x}^{\boldsymbol{T}}$

subject to

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1d}x_d &\le b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2d}x_d &\le b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nd}x_d &\le b_n
\end{aligned}
$$

$\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$

$\boldsymbol{c} = (c_1, c_2, \ldots, c_d)$

$\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$

Linear Programming (LP)

Max $c_1x_1 + c_2x_2 + \cdots + c_d\, x_d \qquad = \boldsymbol{cx}^{\boldsymbol{T}}$

subject to

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1d}x_d \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2d}x_d \le b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nd}x_d \le b_n \end{array}\right\} \quad A\boldsymbol{x}^T \le \boldsymbol{b}^T$$

$\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$

$\boldsymbol{c} = (c_1, c_2, \ldots, c_d)$

$\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$

Linear Programming (LP)

Max $c_1x_1 + c_2x_2 + \cdots + c_d\, x_d \qquad = \boldsymbol{cx^T}$

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$\boldsymbol{x} = (x_1, x_2, \dots, x_d)$

$\boldsymbol{c} = (c_1, c_2, \dots, c_d)$

$\boldsymbol{b} = (b_1, b_2, \dots, b_n)$

Solvable in time polynomial in d.

Linear Programming (LP)

Max $c_1x_1 + c_2x_2 + \cdots + c_d\,x_d$ $\qquad = \boldsymbol{c}\boldsymbol{x}^T$

subject to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1d}x_d \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2d}x_d \le b_2 \\ \quad\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nd}x_d \le b_n \end{array}\right\} A\boldsymbol{x}^T \le \boldsymbol{b}^T$$

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Max $c_1x_1 + c_2x_2 + \cdots + c_d\, x_d \qquad = \boldsymbol{c}\boldsymbol{x}^{\boldsymbol{T}}$

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$\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$

Solvable in time polynomial in d.

Simplex method $O(2^d)$
(best performance in practice)

Linear Programming (LP)

Max $c_1 x_1 + c_2 x_2 + \cdots + c_d\, x_d \qquad = \boldsymbol{c}\boldsymbol{x}^{\boldsymbol{T}}$

subject to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1d}x_d \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2d}x_d \le b_2 \\ \qquad\qquad\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nd}x_d \le b_n \end{array}\right\} \quad A\boldsymbol{x}^T \le \boldsymbol{b}^T$$

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