Homework: Lambda Calculus

Learning Objectives:

1. Understand evaluation order

2. Understand church encoding

3. Learn to perform β-reduction

Instructions:

- Total points: 49 pt

- Early deadline: Oct 30 (Wed) 2019 at 11:59 PM; Regular deadline: Nov 1 (Fri) 2019 at 11:59 PM (you can continue working on the homework till TA starts to grade the homework).

- Submit one pdf file on Canvas under Assignments, Homework 6 submission. You are encouraged to use latex. But we will accept a scanned copy as well.

Questions:

1. (9 pt) Perform β-reduction for the following λ expressions.

    (a) (3 pt) $(((\lambda(x)(x\ x))(\lambda(y)(y\ x)))\ z)$
    (b) (3 pt) $(((\lambda(a)(\lambda(b)(a\ b)))((\lambda(c)\ c)\ x))\ y)$
    (c) (3 pt) $(((\lambda(x)(x\ x))(\lambda(y)\ y))(\lambda(y)\ y))$

    Sol.

    (a) (3pt)

$$
\begin{align}
& (((\lambda(x)(x\ x))(\lambda(y)(y\ x)))\ z) \tag{1}\\
=\ & (((\lambda(y)(y\ x))(\lambda(y)(y\ x)))\ z) \tag{2}\\
=\ & (((\lambda(y)(y\ x))\ x)\ z) \tag{3}\\
=\ & ((x\ x)\ z) \tag{4}
\end{align}
$$

    (b) (3 pt)

$$
\begin{align}
& (((\lambda(a)(\lambda(b)(a\ b)))((\lambda(c)\ c)\ x))\ y) \tag{1}\\
=\ & (((\lambda(a)(\lambda(b)(a\ b)))\ x)\ y) \tag{2}\\
=\ & ((\lambda(b)(x\ b))\ y) \tag{3}\\
=\ & (x\ y) \tag{4}
\end{align}
$$

(c) (3pt)

$$(((\lambda(x)(x\ x))(\lambda(y)\ y))(\lambda(y)\ y)) \tag{1}$$
$$= (((\lambda(y)y)(\lambda(y)\ y))(\lambda(y)\ y)) \tag{2}$$
$$= ((\lambda(y)\ y)(\lambda(y)\ y)) \tag{3}$$
$$= (\lambda(y)\ y) \tag{4}$$

2. (6 pt) The goal of this problem is to help you understand the evaluation order of lambda calculus. In the following, show the steps of β-reduction for the lambda expression using two types of evaluation orders

$$((\lambda(x)\ p)((\lambda(y)(y\ y))(\lambda(z)(z\ z))))$$

Sol.

(a) (3pt)

$$((\lambda(x)p)((\lambda(y)(y\ y))(\lambda(z)(z\ z)))) \tag{1}$$
$$= p \tag{2}$$

(b) (3pt)

$$((\lambda(x)p)((\lambda(y)(y\ y))(\lambda(z)(z\ z)))) \tag{1}$$
$$= ((\lambda(x)p)((\lambda(z)(z\ z))\ (\lambda(z)(z\ z)))) \tag{2}$$
$$= ((\lambda(x)p)((\lambda(z)(z\ z))\ (\lambda(z)(z\ z)))) \tag{3}$$
$$= ((\lambda(x)p)((\lambda(z)(z\ z))\ (\lambda(z)(z\ z)))) \tag{4}$$
$$= ... \tag{5}$$

3. (4 pt) Encode the logic Boolean operations of *and a b* using *true*, *false* and *ite* given in the lecture.

Sol.
AND:

```
(ite  a
      (ite  b  true  false)
      false
)
```

4. (18 pt) Using *zero*, *one* and *two* as well as *succ*, *true* and *false* provided in the lecture, answer the following two questions:

   (a) (4 pt) What is the result of $((\lambda(z)((two\ f)\ z))\ (succ\ zero))$?
   (b) Suppose we define *unknown*: $(\lambda(x)(\lambda(y)(\lambda(z)\ y)))$ and *g*: $(\lambda(n)((n\ unknown)\ false))$, what is the result of:

    i. (4 pt) $(g\ zero)$

   ii. (3 pt) $(g\ one)$

  iii. (3 pt) $(g\ two)$

   iv. (4 pt) What mathematical/logical operation is computed by g?

Sol.

(a) (4 pt)

$$((\lambda(z)((two\ f)\ z))\ (succ\ zero)) =$$
$$((\lambda(z)(((\lambda(f)(\lambda(x)(f\ (f\ x))))\ f)\ z))\ (succ\ zero)) =$$
$$((\lambda(z)((\lambda(x)(f\ (f\ x)))\ z))\ (succ\ zero)) =$$
$$((\lambda(z)(f\ (f\ z)))\ (succ\ zero)) =$$
$$((\lambda(z)(f\ (f\ z)))\ ((\lambda(n)(\lambda(f)(\lambda(x)(f((n\ f)\ x)))))\ zero)) =$$
$$((\lambda(z)(f\ (f\ z)))\ (\lambda(f)(\lambda(x)(f((zero\ f)\ x))))) =$$
$$((\lambda(z)(f\ (f\ z)))\ (\lambda(f)(\lambda(x)(f(((\lambda(f)(\lambda(x)x))\ f)\ x))))) =$$
$$((\lambda(z)(f\ (f\ z)))\ (\lambda(f)(\lambda(x)(f((\lambda(x)x)\ x))))) =$$
$$((\lambda(z)(f\ (f\ z)))\ (\lambda(f)(\lambda(x)(f\ x)))) =$$
$$((\lambda(z)(f\ (f\ z)))\ one)) =$$
$$(f\ (f\ one))$$

(b)   i. (4 pt)

$$(g\ zero) =$$
$$((\lambda(n)((n\ unknown)\ false))zero) =$$
$$((zero\ unknown)\ false) =$$
$$(((\lambda(f)(\lambda(x)x))\ unknown)\ false) =$$
$$((\lambda(x)x)false) =$$
$$false$$

  ii. (3 pt)

$$(g\ one) =$$
$$((\lambda(n)((n\ unknown)\ false))\ one) =$$
$$((one\ unknown)\ false) =$$
$$(((\lambda(f)(\lambda(x)(f\ x)))\ unknown)\ false) =$$
$$((\lambda(x)(unknown\ x))\ false) =$$
$$((\lambda(x)((\lambda(x)(\lambda(y)(\lambda(z)\ y)))\ x))\ false) =$$
$$((\lambda(x)(\lambda(y)(\lambda(z)\ y)))\ false) =$$
$$((\lambda(x)true)\ false) =$$
$$true$$

  iii. (3 pt)

$$(g\ two) =$$
$$((\lambda(n)((n\ unknown)\ false))\ two) =$$
$$((two\ unknown)\ false) =$$
$$(((\lambda(f)(\lambda(x)(f\ (f\ x))))\ unknown)\ false) =$$
$$((\lambda(x)(unknown\ (unknown\ x)))\ false) =$$
$$(unknown\ (unknown\ false)) =$$
$$(unknown\ ((\lambda(x)(\lambda(y)(\lambda(z)\ y)))\ false)) =$$
$$(unknown\ (\lambda(y)(\lambda(z)\ y))) =$$

$$(unknown\ true) =$$
$$((\lambda(x)(\lambda(y)(\lambda(z)\ y)))\ true) =$$
$$(\lambda(y)(\lambda(z)\ y)) =$$
$$true$$

iv. (4 pt) g performs the computation g != 0.

5. (12 pt) Given:
   g: $(\lambda(a)(\lambda(b)(\lambda(c)((a\ b)\ ((a\ b)\ c)))))$
   $zero$: $(\lambda(f)(\lambda(x)x))$
   one: $(\lambda(f)(\lambda(x)(f\ x)))$.
   two: $(\lambda(f)(\lambda(x)(f\ (f\ x))))$.
   $three$: $(\lambda(f)(\lambda(x)(f\ (f\ (f\ x)))))$.
   $four$: $(\lambda(f)(\lambda(x)(f\ (f\ (f\ (f\ x))))))$.

   (a) (4 pt) What is the result of (g one)?
   (b) (4 pt) What is the result of (g two)?
   (c) (4 pt) What computation does g performs?

   Sol.

   (a) (4 pt)
       $(g\ one) =$
       $((\lambda(a)(\lambda(b)(\lambda(c)((a\ b)\ ((a\ b)\ c)))))\ one) =$
       $(\lambda(b)(\lambda(c)((one\ b)\ ((one\ b)\ c)))) =$
       $(\lambda(b)(\lambda(c)((one\ b)\ (((\lambda(f)(\lambda(x)(f\ x)))\ b)\ c)))) =$
       $(\lambda(b)(\lambda(c)((one\ b)\ (b\ c)))) =$
       $(\lambda(b)(\lambda(c)(((\lambda(f)(\lambda(x)(f\ x)))\ b)\ (b\ c)))) =$
       $(\lambda(b)(\lambda(c)(((\lambda(x)(b\ x)))\ (b\ c)))) =$
       $(\lambda(b)(\lambda(c)(b\ (b\ c)))) =$
       two

   (b) (4 pt)
       $(g\ two) =$
       $((\lambda(a)(\lambda(b)(\lambda(c)((a\ b)\ ((a\ b)\ c)))))\ two) =$
       $(\lambda(b)(\lambda(c)((two\ b)\ ((two\ b)\ c)))) =$
       $(\lambda(b)(\lambda(c)((two\ b)\ (((\lambda(f)(\lambda(x)(f\ (f\ x))))\ b)\ c)))) =$
       $(\lambda(b)(\lambda(c)((two\ b)\ (b\ (b\ c))))) =$
       $(\lambda(b)(\lambda(c)(((\lambda(f)(\lambda(x)(f\ (f\ x))))\ b)\ (b\ (b\ c))))) =$
       $(\lambda(b)(\lambda(c)(((\lambda(x)(b\ (b\ x))))\ (b\ (b\ c))))) =$
       $(\lambda(b)(\lambda(c)(b\ (b\ (b\ (b\ c)))))) =$
       four

   (c) (4 pt) g performs a multiplication of a given number by 2. e.g. (g three) = 2 * 3 = 6