Stat 330 (Spring 2020): Homework 4

Due: February 19, 2020

Show all of your work, and *please* staple your assignment if you use more than one sheet. Write your name, the course number and the section on every sheet. Problems marked with * will be graded and one additional randomly chosen problem will be graded.

1. A box contains seven marbles. Four of them are red and three of them are green. You reach in and choose three at random without replacement. Define a random variable X as: $X =$ the number of red marbles selected.

   (a) What are the possible values X can take on? (i.e. give $Im(X)$)
   Answer: $Im(X) = \{0, 1, 2, 3\}$

   (b) Find $\mathbb{P}(X = x)$ for all x in $Im(X)$.
   Answer:
   $\mathbb{P}(X = 0) = \frac{\binom{4}{0}\cdot\binom{3}{3}}{\binom{7}{3}} = \frac{1}{35}$

   $\mathbb{P}(X = 1) = \frac{\binom{4}{1}\cdot\binom{3}{2}}{\binom{7}{3}} = \frac{12}{35}$

   $\mathbb{P}(X = 2) = \frac{\binom{4}{2}\cdot\binom{3}{1}}{\binom{7}{3}} = \frac{18}{35}$

   $\mathbb{P}(X = 3) = \frac{\binom{4}{3}\cdot\binom{3}{0}}{\binom{7}{3}} = \frac{4}{35}$

   (c) Make a table for the probability distribution of X as shown in lecture. (Leave probabilities as fractions)
   Answer:

x	0	1	2	3
$p_X(x)$	$\frac{1}{35}$	$\frac{12}{35}$	$\frac{18}{35}$	$\frac{4}{35}$

2. * Let X be a random variable with image $Im(X) = \{-2, -1, 0, 1, 2\}$.

   (a) Fill in the blank in the table below to make it a valid probability mass function:

x	-2	-1	0	1	2
$p_X(x)$	0.1	0.3	0.3	0.1	

   (b) Add the cumulative distribution function, $F_X(x)$ to the table.

   (c) Using $p_X(x)$, determine the probabilities that...
      i. X is at least 1.
      ii. X is greater than -1 and at most 1
      iii. X is a negative value

   (d) Using $F_X(x)$, find...
      i. $F_X(1)$
      ii. $F_X(.5)$
      iii. $\mathbb{P}(X \geq 0)$ (*rewrite this first in terms of* $F_X(x)$)

   (e) Find the expected value and variance of X.

   Answer:

   (a) Since the sum of the probabilities has to be 1 for a probability mass function, $p_X(2) = 1 - 0.1 - 0.3 - 0.3 - 0.1 = 0.2$.

   (b)

x	-2	-1	0	1	2
$p_X(x)$	0.1	0.3	0.3	0.1	0.2
$F_X(x)$	0.1	0.4	0.7	0.8	1

(c) i. $\mathbb{P}(X \geq 1) = \mathbb{P}(X = 1) + \mathbb{P}(X = 2) = 0.1 + 0.2 = 0.3$

ii. $\mathbb{P}(-1 < X \leq 1) = \mathbb{P}(X = 0) + \mathbb{P}(X = 1) = 0.3 + 0.1 = 0.4$

iii. $\mathbb{P}(X \leq -1) = \mathbb{P}(X = -1) + \mathbb{P}(X = -2) = 0.3 + 0.1 = 0.4$

(d) i. $F_X(1) = 0.8$

ii. $F_X(.5) = \mathbb{P}(X \leq .5) = \mathbb{P}(X \leq 0) = F_X(0) = .7$

iii. $\mathbb{P}(X \geq 0) = 1 - \mathbb{P}(X < 0) = 1 - \mathbb{P}(X \leq -1) = 1 - F_X(-1) = 1 - .4 = .6$

(e) $\mathbb{E}[X] = (-2)\mathbb{P}(X = -2) + (-1)\mathbb{P}(X = -1) + (0)\mathbb{P}(X = 0) + (1)\mathbb{P}(X = 1) + (2)\mathbb{P}(X = 2)$
$= (-2)(0.1) + (-1)(0.3) + (0)(0.3) + (1)(0.1) + (2)(0.2) = 0$

To find the variance, we will use the formula $Var[X] = E[X^2] - (E[X])^2$.

$\mathbb{E}[X^2] = (-2)^2\mathbb{P}(X = -2) + (-1)^2\mathbb{P}(X = -1) + (0)^2\mathbb{P}(X = 0) + (1)^2\mathbb{P}(X = 1) + (2)^2\mathbb{P}(X = 2)$
$= (-2)^2(0.1) + (-1)^2(0.3) + (0)^2(0.3) + (1)^2(0.1) + (2)^2(0.2) = 1.6$

$Var[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 1.6 - (0)^2 = 1.6.$

3. * Let Y be a random variable with $Y = 4 - 2X$ where X was defined in the previous problem.

(a) Determine the image of Y.

(b) Using the rules for computing expected values and variances of a linear function of a random variable, find the expected value and variance of Y, using the corresponding values of X.

Answer:

(a) Since X has image $Im(X) = \{-2, -1, 0, 1, 2\}$, the image of Y has to be $Im(Y) = \{8, 6, 4, 2, 0\}$ (the order of elements in a set does not matter).

(b) $\mathbb{E}[Y] = \mathbb{E}[4 - 2X] = 4 - 2\mathbb{E}[X] = 4 - (2)(0) = 4$
$Var[Y] = Var[4 - 2X] = Var[-2X] = (-2)^2 Var[X] = (4)(1.6) = 6.4$

4. Let X be a random variable and a be a constant. Using the "short-cut" definition of variance, prove that $Var(aX) = a^2 Var(X)$.

Answer:

$$
\begin{aligned}
Var(aX) &= \mathbb{E}((aX)^2) - (\mathbb{E}(aX))^2 \\
&= \mathbb{E}(a^2X^2) - (a\mathbb{E}(X))^2 \\
&= a^2\mathbb{E}(X^2) - a^2(\mathbb{E}(X))^2 \\
&= a^2[\mathbb{E}(X^2) - (\mathbb{E}(X))^2] \\
&= a^2 Var(X)
\end{aligned}
$$

5. A quality control engineer tests the quality of produced computers in a shipment of 6 computers. Suppose that 5% of computers have defects, and defects occur independently of each other.

(a) Find the probability of exactly 2 defective computers in the shipment.

(b) Find the probability of at most 2 defective computers in the shipment.

Answer:

$$X = \#\text{ of defective computers}$$
$$X \sim Bin(6, 0.05)$$

(a)

$$\mathbb{P}(X = 2) = p_X(2) = \binom{6}{2}(0.05)^2(0.95)^4 = 0.0305$$

(b)

$$\begin{aligned}
\mathbb{P}(X \leq 2) &= F_X(2) = 0.9978 \text{ (from Appendix A: Binomial Dist. Table)} \\
&\quad \textit{or} \\
\mathbb{P}(X \leq 2) &= \mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2) \\
&= \binom{6}{0}(0.05)^0(0.95)^6 + \binom{6}{1}(0.05)^1(0.95)^5 + \binom{6}{2}(0.05)^2(0.95)^4 \\
&= 0.9978
\end{aligned}$$

6. An internet search engine looks for a certain keyword in a sequence of independent web sites. It is believed that 20% of the sites contain this keyword.
   (a) Compute the probability that at least 5 of the first 10 sites contain the given keyword.
   (b) Compute the probability that the search engine had to visit at least 5 sites in order to find the first occurrence of a keyword.

Answer:

3.24 (a) Let X be the number of sites with a keyword, which is Binomial$(n = 10, p = 0.2)$. From Table A2,

$$\boldsymbol{P}\{X \geq 5\} = 1 - \boldsymbol{P}\{X \leq 4\} = 1 - 0.9672 = \boxed{0.0328}$$

(b) Let Y be the number of sites visited until a site with a keyword is found, which is Geometric$(p = 0.2)$.

$$\boldsymbol{P}\{Y \geq 5\} = (1 - p)^4 = \boxed{0.4096}$$