# Informed (Heuristic) Search

## Outline

I. Greedy best-first search

II. A* search

* Figures are from the textbook site (or drawn by the instructor) unless the source is specifically cited.

# Heuristic Function

Use of domain-specific hints about the location of a goal can find a solution more efficiently.

*Heuristic function*:

$h(n) =$ *estimated* cost of the cheapest path from the state at node $n$ to a goal state

e.g., straight-line distance $h_{SLD}$ on the map between two sites

| | | | |
|---|---|---|---|
| **Arad** | 366 | **Mehadia** | 241 |
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Straight-line distance to Bucharest

# Greedy Best-First Search

Evaluation function $f(n) = h(n)$

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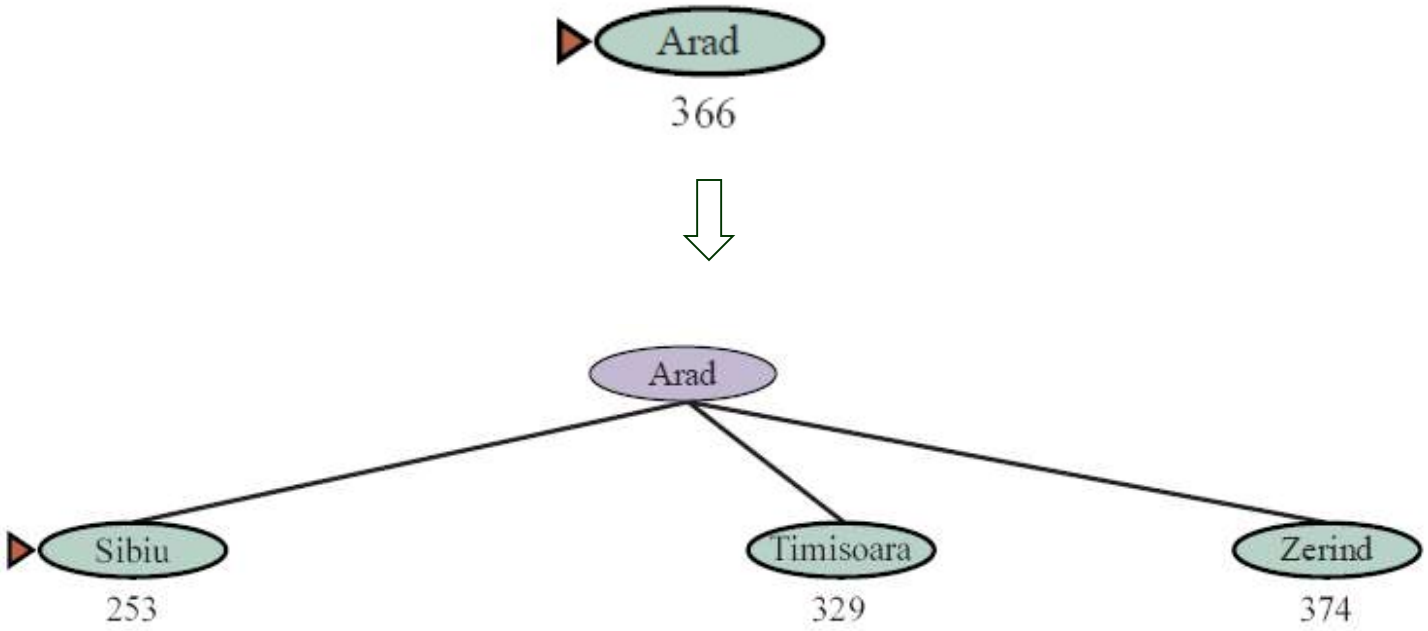
$$f(n) = h_{SLD}(n)$$

Good correlations with road distances

# Greedy Search for Bucharest




Closer to Bucharest
than is Zerind or Timisoara

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# Search for Bucharest (cont'd)

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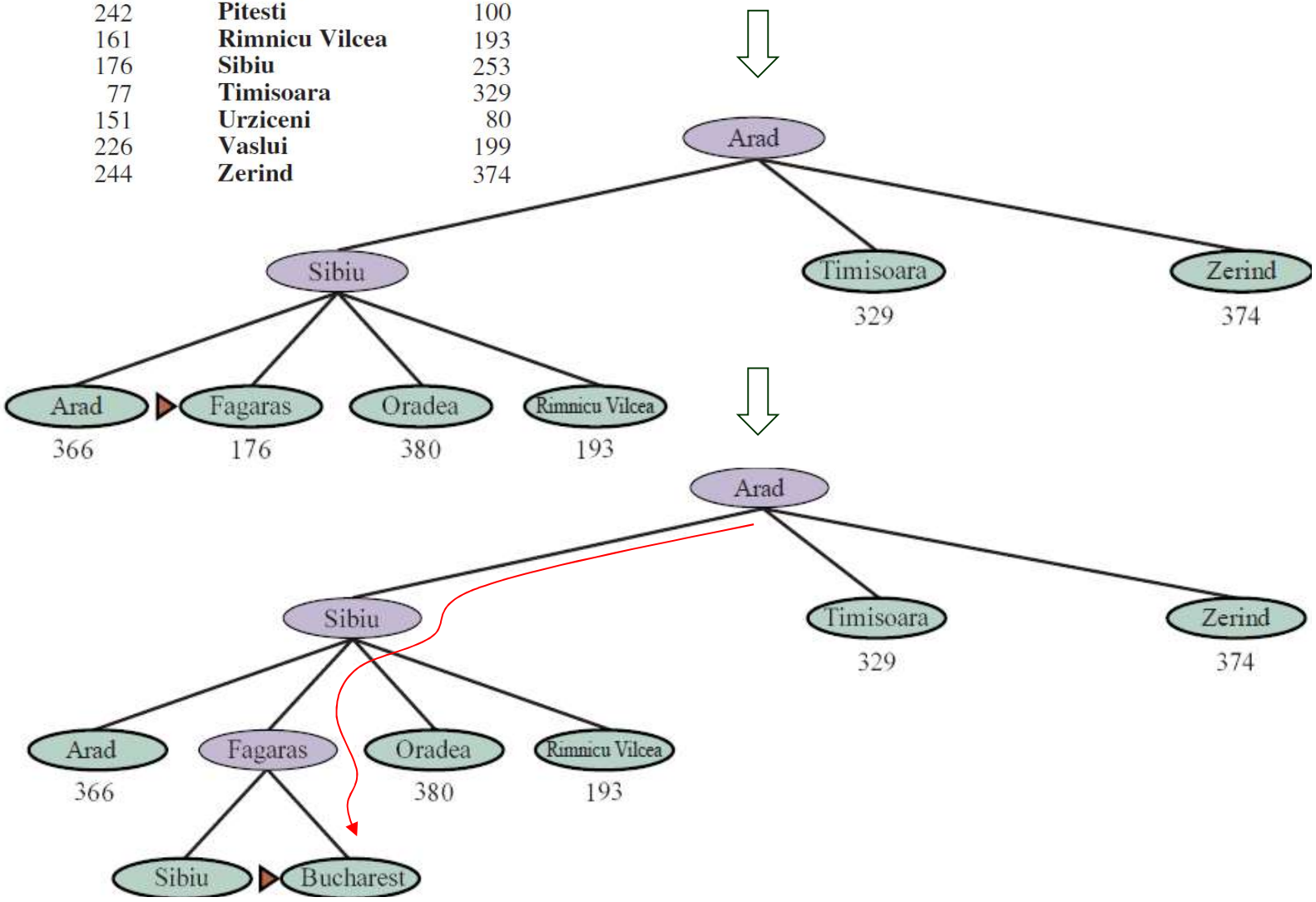
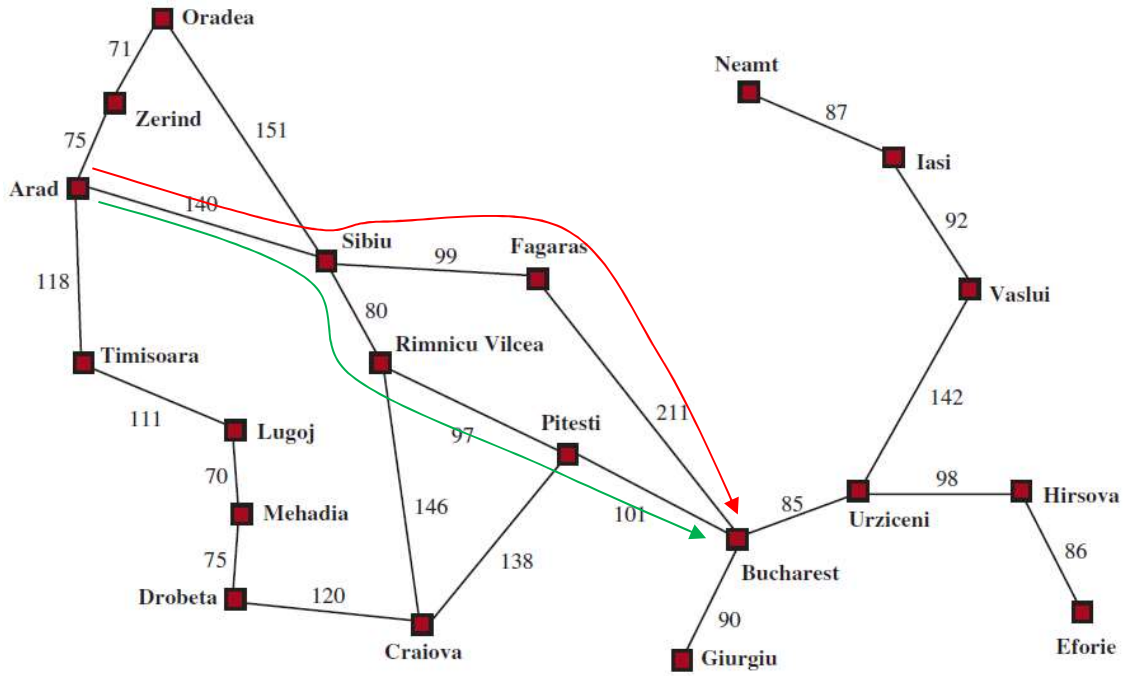


# Greedy ≠ Optimal




Greedy solution: Arad – Sibiu – Fagaras – Bucharest (450)

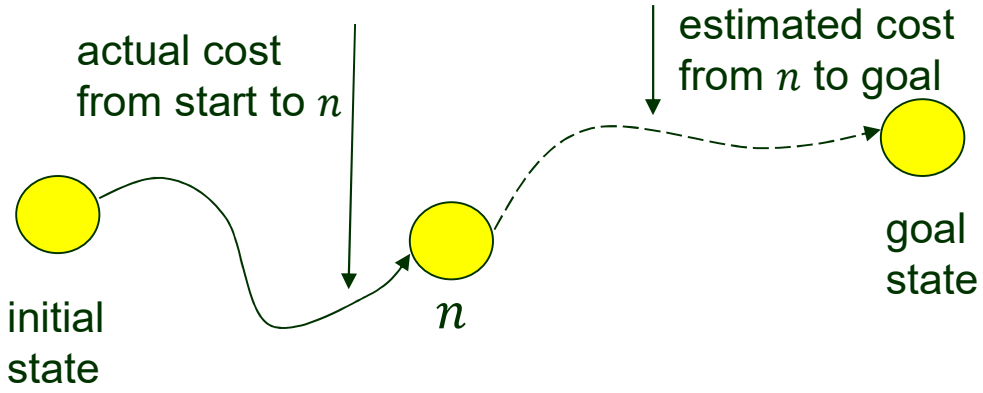
Optimal solution: Arad – Sibiu – Rimnicu Vilcea – Pitesti – Bucharest (418)

This greedy strategy does not account for the cost to get to the current state.

# A* Search

P. E. Hart, N. J. Nilsson, and B. Raphael (1968)

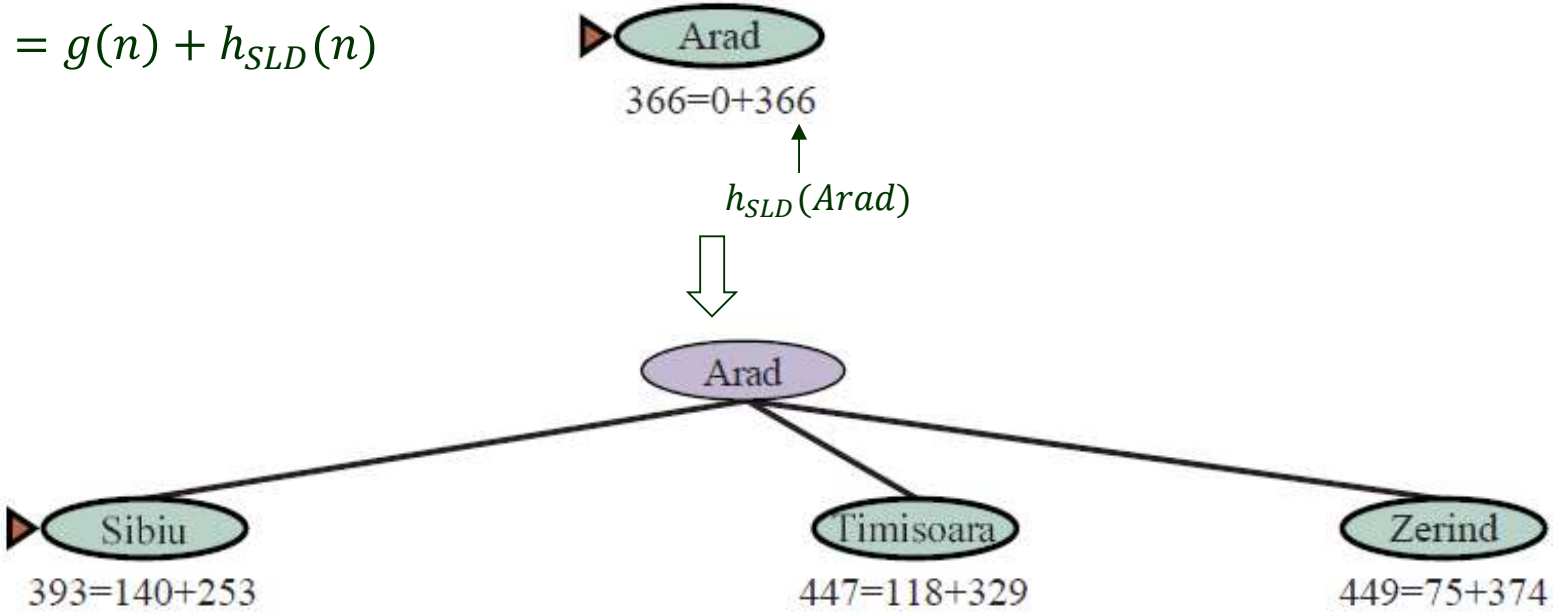
$$f(n) = g(n) + h(n)$$




$f(n)$: estimated cost of the best path that continues from $n$ to a goal.

# A* on Route Planning

$f(n) = g(n) + h_{SLD}(n)$




$h_{SLD}$: straight-line distance

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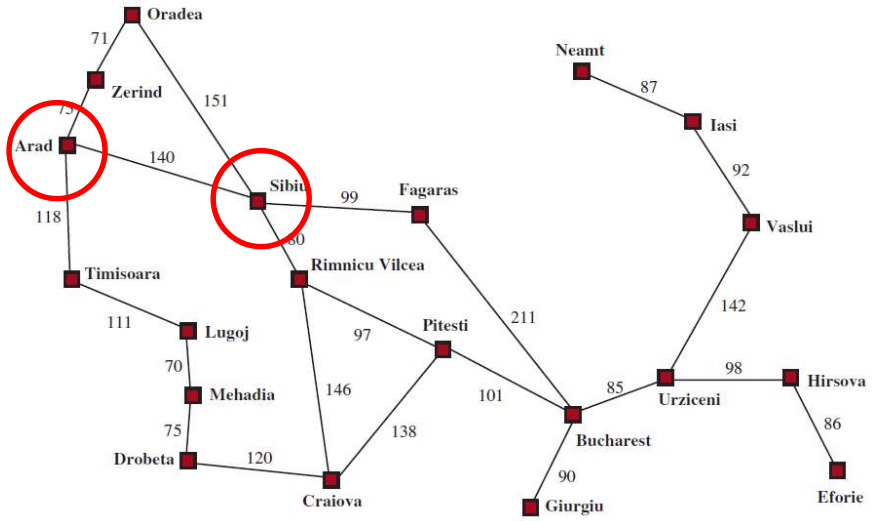
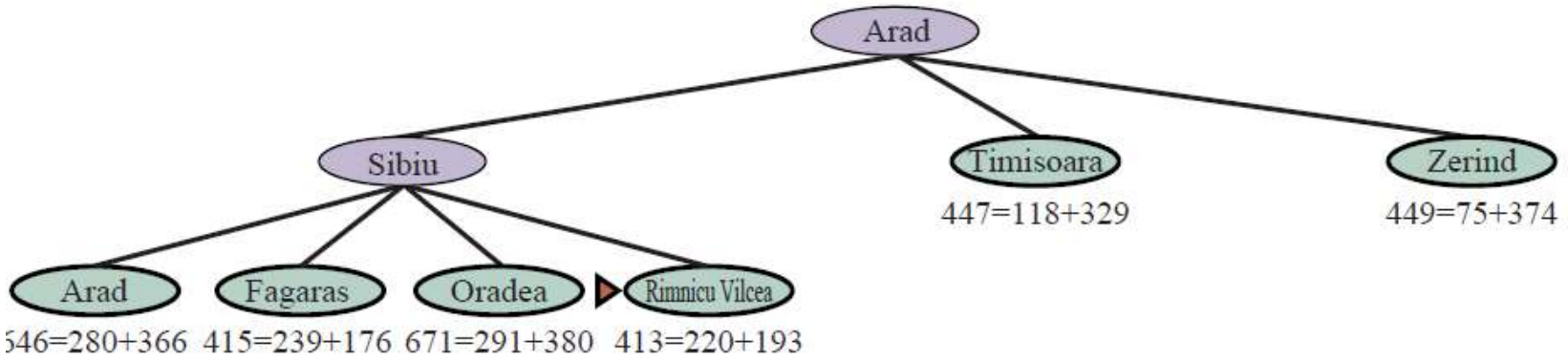


# Working of A* Search

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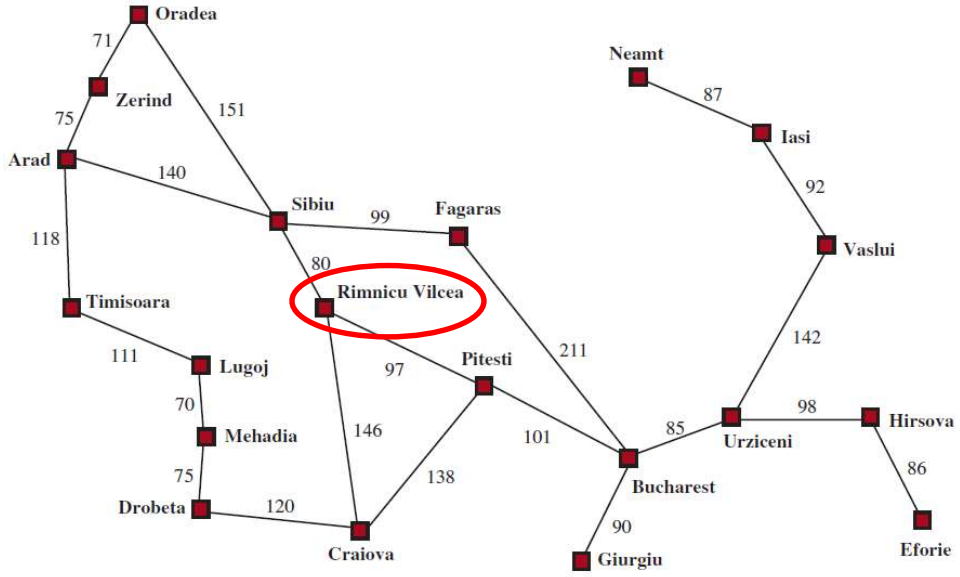
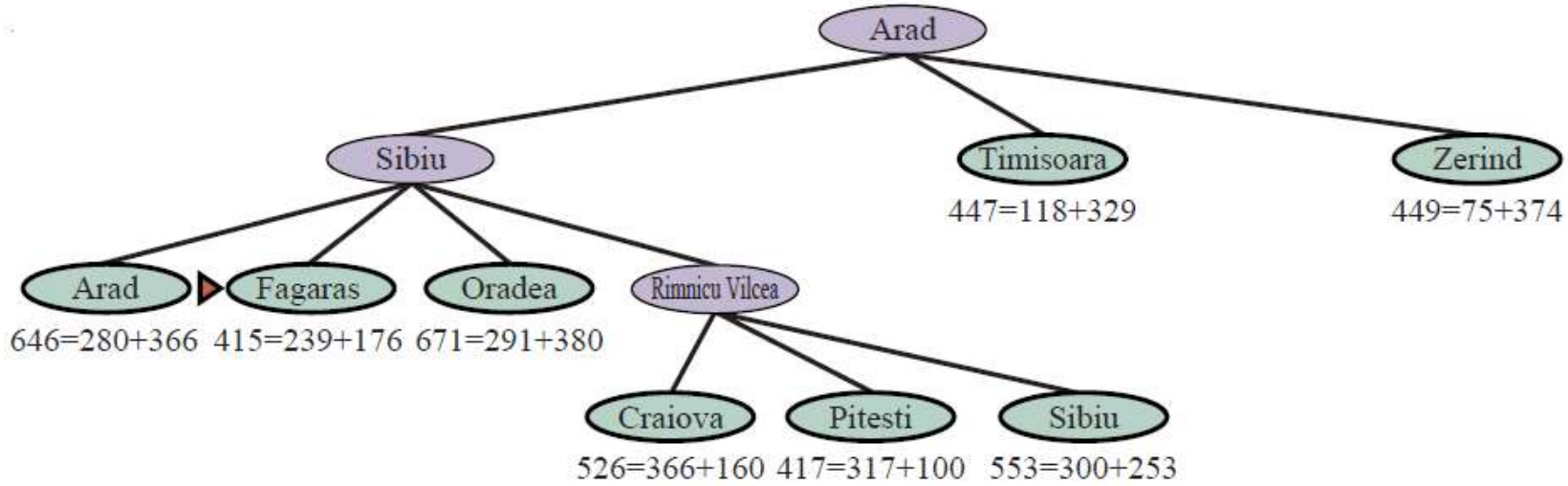


# Working of A*

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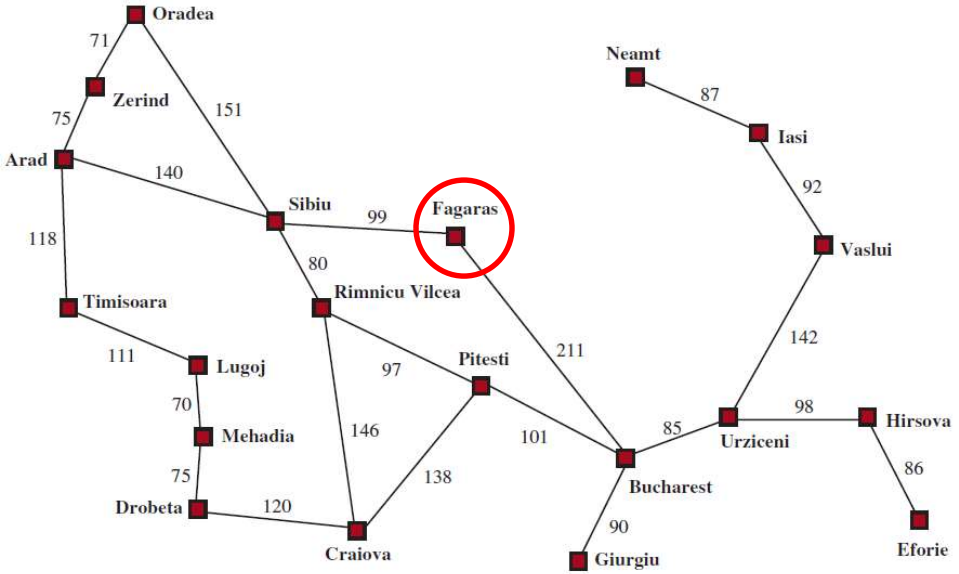
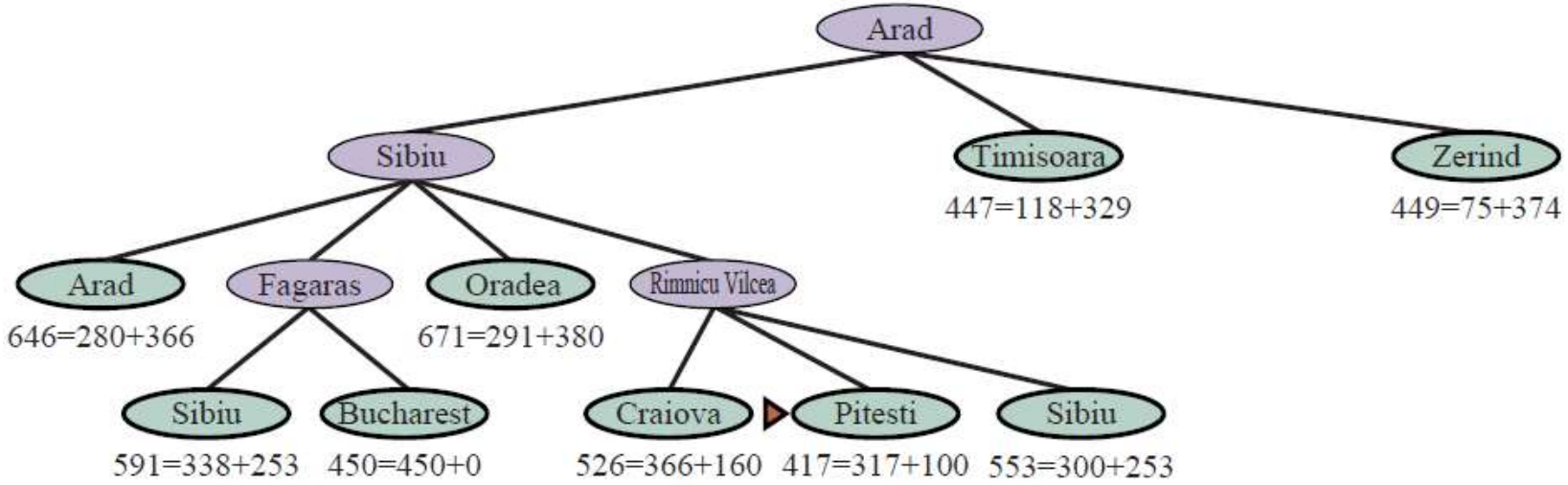


# Working of A*




$h_{SLD}$: straight-line distance

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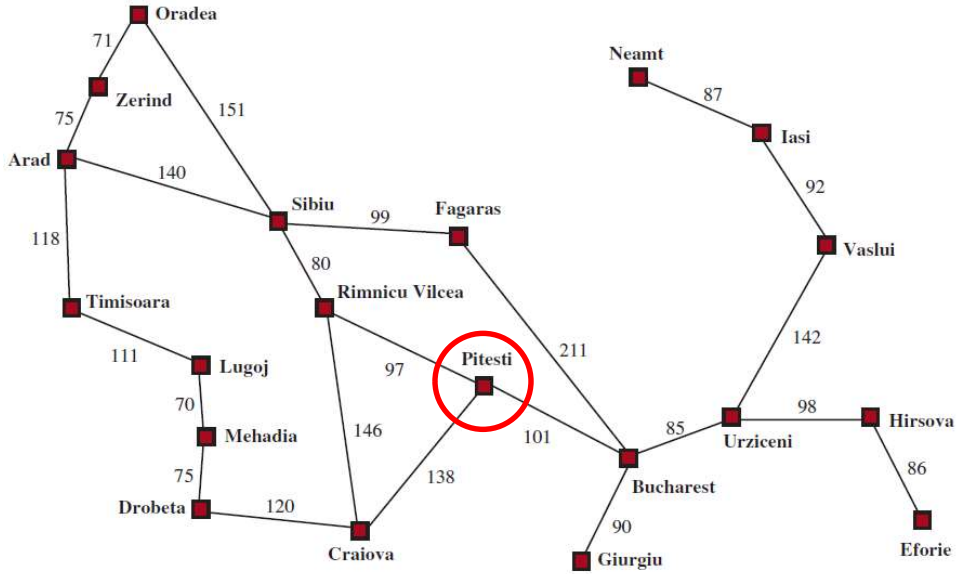
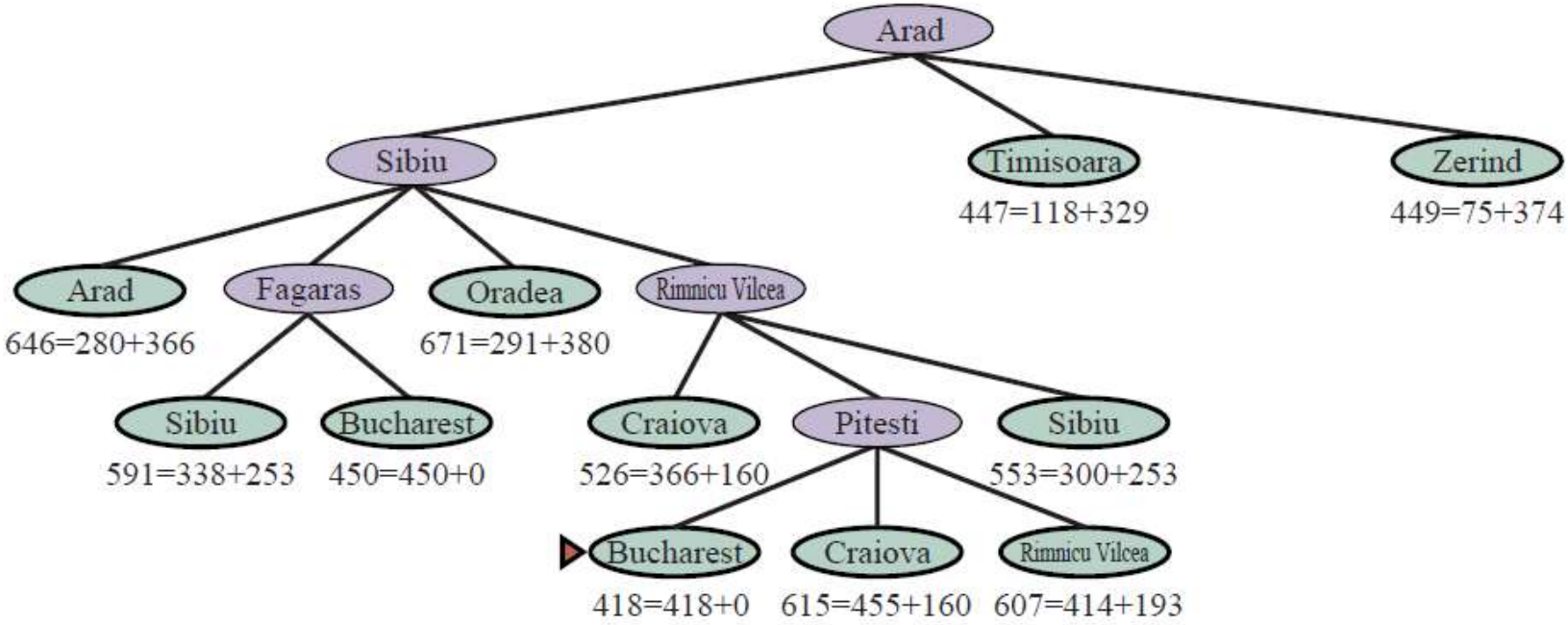


# Working of A*

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# Admissible Heuristic

- A* search is *complete* (when the state space either has a solution or is finite).
- Whether it is optimal depends on the heuristic.

A heuristic is *admissible* if it *never overestimates* the cost to reach a goal.

$h_{SLD}$: straight-line distance

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$h_{SLD}$ is admissible because the actual distance to Bucharest cannot be less than the straight-line distance.

# Optimality of A*

**Theorem** A* is cost-optimal with an admissible heuristic.

**Proof** By contradiction. Suppose the algorithm returns a path with cost $C$ greater than the optimal cost $C^*$.

⇩

Let $n$ be the first node on the optimal path that is *unexpanded*.

$g^*(n)$: optimal cost from start to $n$

$h^*(n)$: optimal cost from $n$ to a goal

⇩

$f(n) > C^*$ (otherwise $f(n) \leq C^* < C$ so $n$ would have been expanded)

But $f(n) = g(n) + h(n)$

$= g^*(n) + h(n)$ ($n$ on the optimal path)

$\leq g^*(n) + h^*(n)$ ($h(n) \leq h^*(n)$ due to admissibility)

$= C^*$

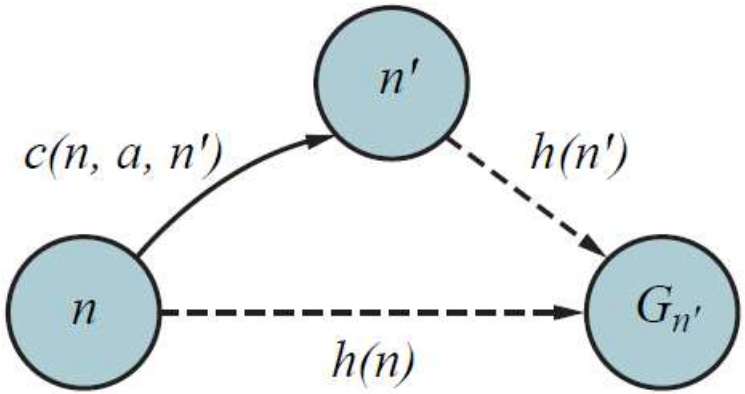
That is, $f(n) \leq C^*$, contradicting with $f(n) > C^*$.

# Consistent Heuristic

A heuristic $h$ is *consistent* if for every two nodes $n$ and $n'$ such that $n'$ is generated from $n$ by some action $a$, the following inequality holds:

$$h(n) \leq c(n, a, n') + h(n')$$

(*triangle inequality*)




- Every consistent heuristic is admissible.
- A* with a consistent heuristic is cost-optimal.

# What If the Heuristic Is Inadmissible?

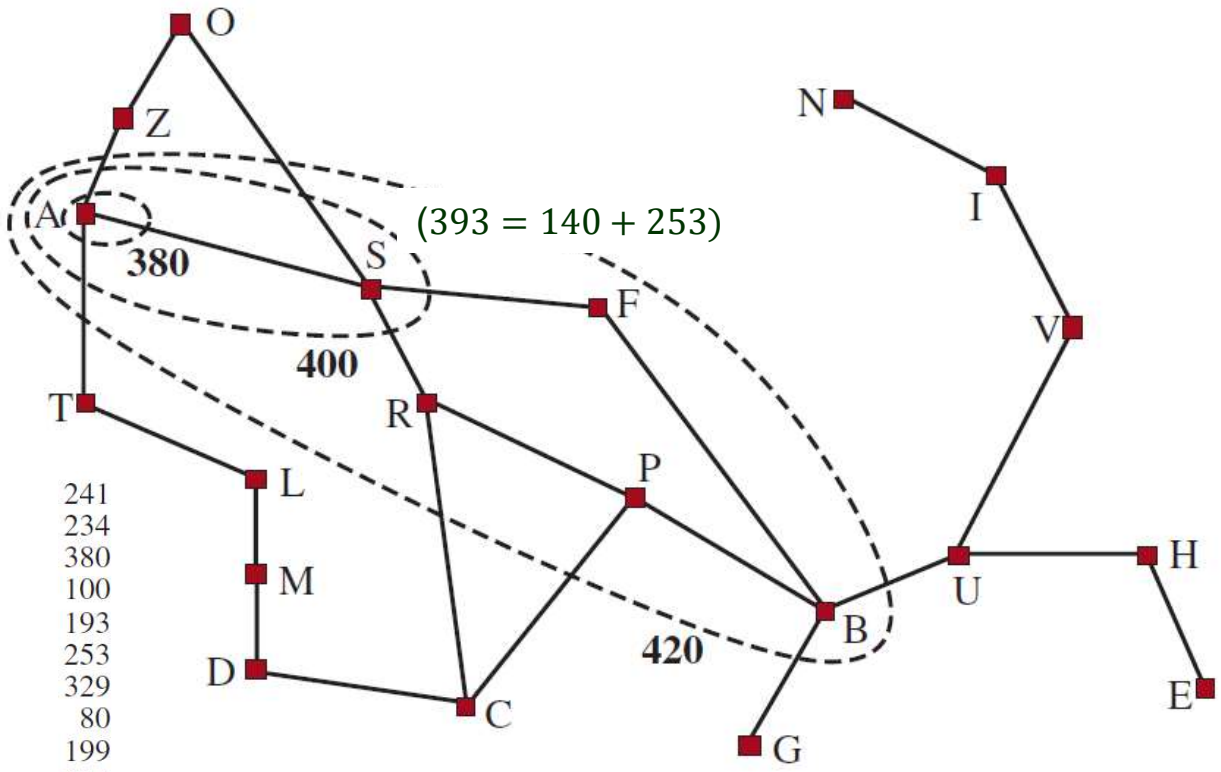
In such a situation, A* may or may not be cost-optimal.

Two of the cases where A* finds an optimal path:

- $h$ is admissible for all the nodes on one optimal path.
- $h(n)$ does not overestimate the cost on each node $n$ by more than the difference between the costs of the optimal and the second-best paths.

# Search Contours

A contour labeled by a cost $c$ encloses all the nodes $n$ with $f(n) = g(n) + h(n) \leq c$.

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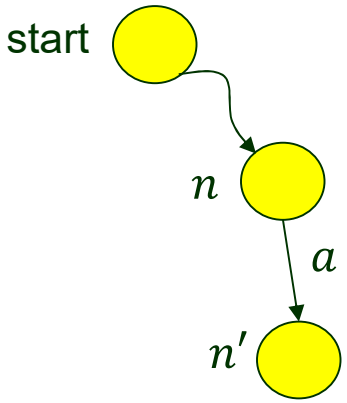
- Dijkstra's algorithm (i.e., uniform search) would have contours of $g$-cost to "circle" around the start state.
- With a good $h$, the $g + h$ bands will stretch toward a goal state.

# Monotonicity?

- The $g$ cost increases along a path because action costs are positive.




Cost at $n$: $g(n) + h(n)$

Cost at its successor $n'$: $\underbrace{g(n) + c(n, a, n')}_{= g(n')} + h(n')$

The path's cost increases monotonically iff

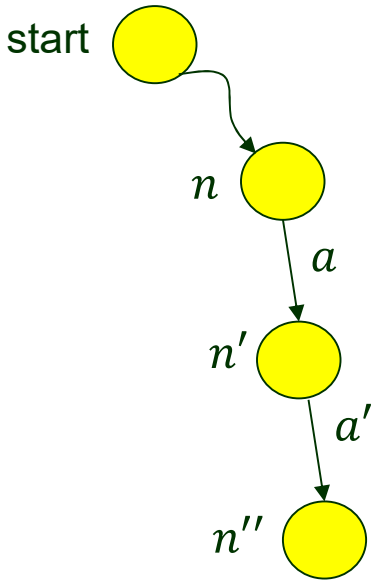
$$g(n) + h(n) \leq g(n) + c(n, a, n') + h(n')$$

$$\Updownarrow$$

$$h(n) \leq c(n, a, n') + h(n')$$

(consistent heuristic)

# Consecutive Nodes Scored the Same




$$g(n) + h(n) = \overbrace{g(n) + c(n, a, n')}^{= g(n')} + h(n')$$

$$= \underbrace{g(n) + c(n, a, n') + c(n', a, n'')}_{= g(n'')} + h(n'')$$

$$\Updownarrow$$

$$h(n) - h(n') = g(n') - g(n)$$
$$h(n') - h(n'') = g(n'') - g(n')$$

$h$ decreases as much as $g$ increases after an action.

# Efficiency of A*

$h$: admissible

$C^*$: cost of the optimal solution path

- A* will expand every node reachable via a sequence of nodes that have costs $< C^*$.
- A* will not expand any node $n$ with $f(n) > C^*$.
- A* might expand a node $n$ with cost $f(n) = C^*$ before selecting a goal node.

A* prunes away nodes unnecessary for finding an optimal solution.

- A* may take exponential time with a poor heuristic function.