7.4 Continued (and 7.5)

Example. Evaluate $\mathscr{L}^{-1}\left\{\dfrac{s}{(s^2+9)^2}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\underbrace{\dfrac{1\cdot 3}{s^2+9}}_{F(s)}\ \underbrace{\dfrac{s}{s^2+9}}_{G(s)}\right\}$

$= \frac{1}{3}\sin 3t * \cos 3t$

Then we would compute the convolution, however in this case it is easier to see that:

$\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+9)^2}\right\} = \frac{1}{6}\mathcal{L}^{-1}\left\{\dfrac{6s}{(s^2+9)^2}\right\} = \frac{1}{6}t\sin 3t$ (using $\mathcal{L}\{tf\} = -\frac{d}{ds}F(s)$)

Example. Evaluate $F(s) = \mathscr{L}\left\{\int_0^t \tau^2\cos(t-\tau)d\tau\right\} = \mathcal{L}\{t^2 * \cos t\}$

$= \mathcal{L}\{t^2\}\mathcal{L}\{\cos t\} = \frac{2}{s^3}\cdot\frac{s}{s^2+1}$

Example. Evaluate $\mathscr{L}\{f * 1\} = \mathcal{L}\{f\}\mathcal{L}\{1\} = \dfrac{F(s)}{s}$

$\parallel$

$$\boxed{\mathcal{L}\left\{\int_0^t f(\tau)\cdot d\tau\right\} = \frac{F(s)}{s}} \bigstar$$

Example. Evaluate $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{\frac{1}{s^2+1}}{s}\right\}$

i.e. $F(s) = \dfrac{1}{s^2+1} \Rightarrow f(t) = \sin t$

(By $\bigstar$) then $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s^2+1)}\right\} = \int_0^t \sin\tau\, d\tau = -\cos\tau\Big|_0^t = -\cos t + 1$

Example. Evaluate $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2(s^2+1)}\right\} = \mathscr{L}^{-1}\left\{\dfrac{\frac{1}{s(s^2+1)}}{s}\right\}$

$$= \int_0^t -\cos\tau + 1 \, d\tau = -\sin\tau + \tau \Big|_0^t = -\sin t + t$$

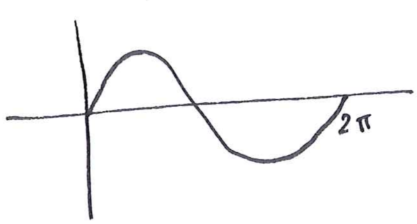
(and so on...)

Definition

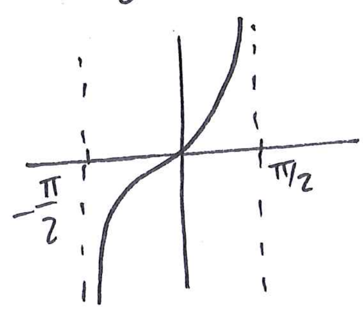
A function is periodic, if $f(t+T) = f(t)$ for all t in the domain of f. If T is the smallest value for which the equality holds, then T is the period of f.

Examples of Periodic Functions

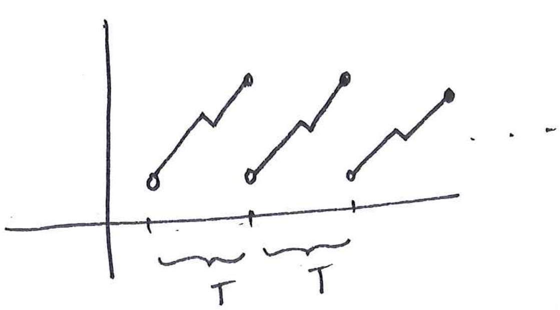
sine and cosine $(T = 2\pi)$

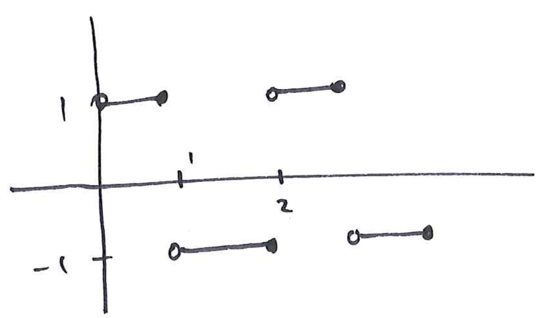


tangent $(T = \pi)$



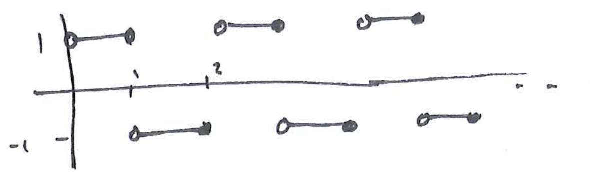
other periodic functions.





Consider the following periodic function $f(t)$:

$$f(t) = \begin{cases} 1, & 0 < t \le 1 \\ -1, & 1 < t \le 2 \end{cases}, \quad T = 2$$

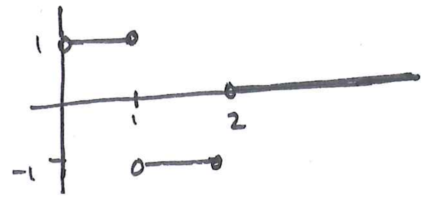


And define $f_T(t)$ (one period of f)

$$f_T(t) = \begin{cases} 1, & 0 < t \le 1 \\ -1, & 1 < t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

$\parallel$

f_2



Then we can see that $\mathscr{L}\{f_T\} = \int_0^\infty e^{-st} f_T \, dt = \int_0^T e^{-st} f \, dt =: F_T$

Theorem ($\mathscr{L}$-Transform of a Periodic Function)

If $f(t)$ is piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period T, then

$$\mathscr{L}\{f\} = \frac{F_T(s)}{1 - e^{-sT}} = \frac{\mathscr{L}\{f_T\}}{1 - e^{-sT}} = \frac{1}{1 - e^{-st}} \int_0^T e^{-st} f \, dt$$

Example. From our the previous example we can write $f_T = 1 - 2u(t-1) + u(t-2)$

$$(T=2) \quad \mathscr{L}\{f\} = \frac{\mathscr{L}\{f_T\}}{1 - e^{-2s}} = \frac{1}{1 - e^{-2s}} \mathscr{L}\{1 - 2u(t-1) + u(t-2)\}$$

$$= \frac{1}{1 - e^{-2s}} \left(\frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} \right).$$

Recall

$$\boxed{\mathscr{L}\{u(t-a)\} = \frac{e^{-sa}}{s}}$$

← By 2nd translation theorem

Example. (LR-Series Example) The differential equation for the current $i(t)$ in a single loop LR-Series circuit is:

$$L\frac{di}{dt}+Ri=E(t),\quad \text{where: } i(0)=0,\quad \text{and } E(t)=\begin{cases}1, & 0\le t<2\\ 0, & 1\le t<2\end{cases},\quad T=2$$

$E_T = 1-u(t-1)$; let $\mathcal{L}\{i\}=I$

$E_T \Rightarrow$

(Apply $\mathcal{L}$) $L\mathcal{L}\{\frac{di}{dt}\}+R\mathcal{L}\{i\}=\mathcal{L}\{E\}$

$$L(sI-\cancel{i(0)}^{\,0})+RI=\frac{1}{1-e^{-2s}}\mathcal{L}\{\overbrace{1-u(t-1)}^{E_T}\}$$

$$I(Ls+R)=\frac{1}{1-e^{-2s}}\left(\frac{1}{s}-\frac{e^{-s}}{s}\right)=\frac{1}{\cancel{(1-e^{-s})}(1+e^{-s})}\left(\frac{\cancel{1-e^{-s}}}{s}\right)$$

$$I=\frac{1}{1+e^{-s}}\,\frac{1}{L(s+R/L)s}=\frac{1}{R(1+e^{-s})}\left(\frac{1}{s}-\frac{1}{s+R/L}\right)$$

(↑ split into partial fractions & simplify)

Recall $\frac{1}{1+x}=\sum_{k=0}^{\infty}(-x)^k$, so $\frac{1}{1+e^{-s}}=\sum_{k=0}^{\infty}(-e^{-s})^k=1-e^{-s}+e^{-2s}-e^{-3s}\cdots$

$$I=\frac{1}{R}\left(\frac{1}{s}-\frac{1}{s+R/L}\right)\left(1-e^{-s}+e^{-2s}-e^{-3s}+\cdots\right)$$

Recall $1=u(t-0)$

$$\mathcal{L}^{-1}\left\{\frac{1}{R}\left(\frac{1}{s}-\frac{1}{s+R/L}\right)\right\}=\frac{1}{R}\left(1-e^{-R/L\,t}\right)(1)$$

By 2nd translation theorem.

$$\mathcal{L}^{-1}\left\{\frac{1}{R}\left(\frac{1}{s}-\frac{1}{s+R/L}\right)(-e^{-s})\right\}=-\frac{1}{R}\left(1-e^{-R/L(t-1)}\right)u(t-1)$$

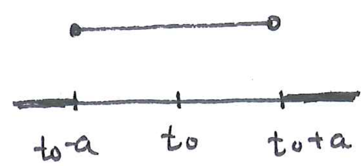
$$\mathcal{L}^{-1}\left\{\frac{1}{R}\left(\frac{1}{s}-\frac{1}{s+R/L}\right)(e^{-2s})\right\}=\frac{1}{R}\left(1-e^{-R/L(t-2)}\right)u(t-2)\quad \ldots\text{etc.}$$

$$\Rightarrow \mathcal{L}^{-1}\{I\}=i(t)=\sum_{k=0}^{\infty}(-1)^k\left(1-e^{-R/L(t-k)}\right)u(t-k)$$

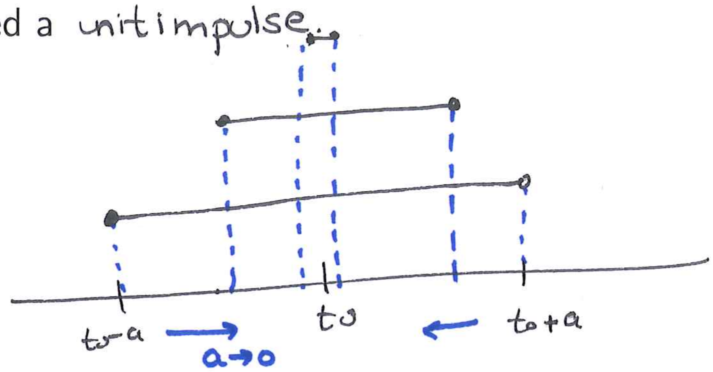
7.5 The Dirac Delta Function

Let $a > 0$, $t_0 > 0$ and consider the function

$$\delta_a(t-t_0) = \begin{cases} 0, & 0 \le t < t_0 - a \\ \dfrac{1}{2a}, & t_0 - a \le t < t_0 + a \\ 0, & t \ge t_0 + a \end{cases}$$



$\delta_a(t-t_0)$ is called a unit impulse.



Note that for all a: $\displaystyle\int_{t_0-a}^{t_0+a} \delta_a(t) = 1$ (area under the curve always $2a \cdot \frac{1}{2a} = 1$)

Definition

The Dirac delta function is defined as $\delta(t-t_0) = \lim_{a\to 0} \delta_a(t-t_0)$ and it is characterized by the following two properties:

(i) $\delta(t-t_0) = \begin{cases} 0, & t \neq t_0 \\ \infty, & t = t_0 \end{cases}$

(ii) $\displaystyle\int_0^\infty f(t)\,\delta(t-t_0)\,dt = f(t_0)$

Laplace Transform of the Dirac Delta Function

$$\mathscr{L}\{\delta(t-t_0)\} = e^{-s\,t_0}$$

In particular:

$$\mathscr{L}\{\delta(t)\} = 1$$