Stat 330 (Spring 2020): Homework 5 Due: Feb 26, 2020

Show all of your work, and *please* staple your assignment if you use more than one sheet. Write your name, the course number and the section on every sheet. Problems marked with * will be graded and one additional randomly chosen problem will be graded.

1. For each of the following three scenarios, do the following. 1.) Define a random variable for what the question is referring to. 2.) Write down the distribution of the random variable using the $X \sim$ notation, giving the parameter value(s). 3.) Write down a probability statement in terms of the random variable, $\mathbb{P}(X \leq \cdots)$ etc, for what the question of interest is. *You do not have to find the probability*

   (a) In a town, there is a stop sign that 85% of cars make a complete stop at. A police officer starts watching cars go through the stop sign. What is the probability less than ten cars go by before the officer finds one that *doesn't* make a complete stop at the stop sign?

   Answer:
   1.) Let $X =$ the number of cars that go by until one doesn't completely stop
   2.) $X \sim \text{geo}(.15)$
   3.) We want $\mathbb{P}(X < 10)$

   (b) A student takes a 20 question multiple choice exam where each question has four possible answers. The student will randomly guess on every question. The student needs to get at least a 60% on the exam (12 questions correct) in order to pass the class. What is the probability the student gets at least 12 questions correct?

   Answer:
   1.) Let $X =$ the number of questions the student guesses correctly out of 20
   2.) $X \sim \text{Bin}(20, .25)$
   3.) We want $\mathbb{P}(X \geq 12)$

   (c) At a certain Bank, customers arrive at an average rate of 16 per hour. What is the probability that exactly 14 customers arrive between 1:00pm and 2:00pm on a random day?

   Answer:
   1.) Let $X =$ the number of arriving customers in that 1 hour period
   2.) $X \sim \text{Pois}(16)$
   3.) We want $\mathbb{P}(X = 14)$

2. * In the board game *Monopoly*, a player in jail gets out by rolling "doubles" on their turn, i.e., by rolling the same number on the two dice thrown. Since the only outcome that allows the player to move his piece is doubles, the player will be less interested in the sum of the two dice and more interested in whether or not the he or she rolled doubles.

   Let X be the outcome of a single roll of the two dice, with "success" considered to be rolling doubles and "failure" rolling anything else. Then $X \sim \text{Bernoulli}(p)$.

   (a) What is p?

   (b) What is $\mathbb{E}(X)$ and $Var(X)$?

   Now, suppose we roll two dice 5 times. Define Y to be the number of times doubles are rolled in the 5 trials.

   (c) What is the distribution of Y? Give its name and parameter values.

   (d) What is $\mathbb{E}(Y)$?

   (e) Find $\mathbb{P}(Y = 3)$

   Now let's look at the number of turns that are needed until doubles are rolled (and a player "gets out of jail"). Let Z be a random variable representing the number of rolls of the dice until doubles comes up. Then $Z \sim \text{Geometric}(p)$.

   (f) Using your answer to part (2a) for p, what is the expected number of turns a player will need to get out of jail?

(g) What is the probability that a player will need four or more rolls to get out of jail?

Answer:

(a) Using the probability under equally likely outcomes, there are 6 doubles outcomes in 36 possibilities, so $p = \frac{1}{6}$.

(b) The expected value is $p = \frac{1}{6}$ and variance is $p(1-p) = \frac{5}{36}$

(c) $Y \sim \text{Bin}(n, p)$, where $n = 5$ and $p = \frac{1}{6}$

(d) $\mathbb{E}(Y) = np = \frac{5}{6}$

(e) $\mathbb{P}(Y = 3) = \binom{5}{3}(\frac{1}{6})^3(\frac{5}{6})^2 = .032$

(f) $\mathbb{E}(Z) = \frac{1}{p} = \frac{1}{1/6} = 6$

(g) $\mathbb{P}(Z \geq 4) = 1 - \mathbb{P}(Z \leq 3) = 1 - (1 - (\frac{5}{6})^3) = .579$ (Using the CDF of a geometric random variable)

3. Before a computer is assembled, its motherboard goes through a special inspection. Assume only 85% of motherboards pass this inspection.

(a) What is the probability that at least 13 of the next 15 motherboards pass inspection?

(b) On the average, how many motherboards should be inspected until a motherboard that passes inspection is found?

Answer:

(a) Let X be the number of motherboards that pass the inspection. It is the number of successes in 15 Bernoulli trials, thus it has Binomial distribution with n = 15 and p = 0.85.

$$\begin{aligned}
\mathbb{P}(X \geq 13) &= \mathbb{P}(X = 13) + \mathbb{P}(X = 14) + \mathbb{P}(X = 15) \\
&= \binom{15}{13}.85^{13}(.15)^2 + \binom{15}{14}.85^{14}(.15)^1 + \binom{15}{15}.85^{15}(.15)^0 \\
&= 0.6042
\end{aligned}$$

(b) Let Y be the number of motherboards that should be inspected until a motherboard that passes inspection is found. It is the number of trials needed to see the first success, thus it has Geometric distribution with p = 0.85.
Thus, we want $\mathbb{E}(Y) = \frac{1}{p} = \frac{1}{.85} \approx 1.18$

4. * Suppose that jobs are sent to a printer at an average rate of 10 per hour.

(a) Let X = the number of jobs sent in an hour. What is the distribution of X? Give the name and parameter values.

(b) What is the probability that exactly 8 jobs are sent to the printer in an hour?

(c) Let X = the number of jobs sent in a 12 min period. What is the distribution of X? Give the name and parameter values.

(d) What is the probability that at least 3 jobs will be sent to the printer in a 12 min period?

(e) How many jobs do you expect to be sent to the printer in a 12 min period?

Answer:

(a) $X \sim \text{pois}(10)$

(b) $\mathbb{P}(X = 8) = \frac{e^{-10}10^8}{8!} = .113$

(c) 12 mins is 1/5 of an hour, so if we get 10 per hour on average, in 12 mins we should expect $10/5 = 2$ jobs. Thus $X \sim \text{pois}(2)$

(d) $\mathbb{P}(X \geq 3) = 1 - \mathbb{P}(X \leq 2) = 1 - .677 = .323$ (Using the poisson cdf table with $\lambda = 2$ and x = 2)

(e) $\mathbb{E}(X) = \lambda = 2$ (Where $X \sim \text{pois}(2)$).

5. The number of goals scored in a game by a soccer team has a Poisson distribution, averaging 1.1 goals per game.

   (a) What is the probability of the team scoring more than 3 goals combined in the next five games?

   (b) During the next five games, what is the probability of having less than 2 games (out of 5) with exactly 0 goals in each. (Hint: Let Y = number of games (out of 5) with exactly 0 goals. Then Y follows a binomial distribution with $n = 5$. You need to find the appropriate value for p.)

   Answer:

   (a) Let X be the number of goals in five games. X follows a Poisson distribution with $\lambda = (1.10)(5) = 5.5$.
   The required probability is $\mathbb{P}(X > 3) = 1 - \mathbb{P}(X \leq 3) = 1 - .202 = .798$ (Use poisson table with $\lambda = 5.5$ and $x = 3$)

   (b) Let Y be the number of games (out of 5) with exactly 0 goals. Assume the event that there is exactly 0 goals in one game is independent of that in the other games. In addition, assume that the probability of exactly 0 goals is the same for all games. The probability of exactly 0 goals in a game is found using the poison distribution.
   This is $e^{-1.1} = .3329$. Then Y follows a Binomial distribution with $n = 5$ and $p = 0.3329$. Thus the required probability is

$$\begin{aligned}\mathbb{P}(Y < 2) &= \mathbb{P}(Y = 0) + \mathbb{P}(Y = 1) \\ &= \binom{5}{0} 0.3329^0 (1 - 0.3329)^{5-0} + \binom{12}{5} 0.3329^1 (1 - 0.3329)^{4-1} \\ &= 0.4618\end{aligned}$$

6. An insurance company divides its customers into 2 groups. Twenty percent are in the high-risk group, and eighty percent are in the low-risk group. The high-risk customers make an average of 1 accident per year while the low-risk customers make an average of 0.1 accidents per year. Eric had no accidents last year. What is the probability that he is a high-risk driver?

   Answer:

   3.29 Denote the events: $H = \{\text{high risk}\}$, $L = \{\text{low risk}\}$, $N = \{\text{no accidents}\}$. The number of accidents is the number of "rare events", discrete, ranging from 0 to infinity, thus it has a Poisson distribution. We have:

$$\boldsymbol{P}\{H\} = 0.2,\ \boldsymbol{P}\{L\} = 0.8,\ \boldsymbol{P}\{N|H\} = 0.368,\ \boldsymbol{P}\{N|L\} = 0.905$$

   (from Table A3, with $\lambda = 1$ and $\lambda = 0.1$).

   By the Bayes' Rule,

$$\begin{aligned}\boldsymbol{P}\{H|N\} &= \frac{\boldsymbol{P}\{N|H\}\,\boldsymbol{P}\{H\}}{\boldsymbol{P}\{N|H\}\,\boldsymbol{P}\{H\} + \boldsymbol{P}\{N|L\}\,\boldsymbol{P}\{L\}} \\ &= \frac{(0.368)(0.2)}{(0.368)(0.2) + (0.905)(0.8)} = \boxed{0.0923}\end{aligned}$$