

Markov Chain Example

1. An offspring of a spotted dog is spotted with probability 0.6 and solid color with probability 0.4. An offspring of a solid color dog is spotted with probability 0.2 and solid color with probability 0.8.

- (a) Write the transition probability matrix of this Markov chain.

Answer:

Define “1” = “spotted”; “2” = “solid”

X_t = color of dog

State space = $Im(X_t) = \{1 \ 2\}$

The transition probability matrix has elements p_{ij} where i = current state and j = future state.

- $p_{11} = 0.6 \leftarrow \mathbb{P}(\text{spotted puppy}|\text{spotted dog})$
- $p_{12} = 0.4 \leftarrow \mathbb{P}(\text{solid puppy}|\text{spotted dog})$
- $p_{21} = 0.2 \leftarrow \mathbb{P}(\text{spotted puppy}|\text{solid dog})$
- $p_{22} = 0.8 \leftarrow \mathbb{P}(\text{solid puppy}|\text{solid dog})$

The (1-step) transition probability matrix is

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$$

- (b) Rex is a solid dog. Compute the probability that his grandpuppy is spotted.

Answer:

State space: $\{1 \ 2\}$

Initial Distribution: $P_0 = [0 \ 1]$

2-step transition matrix: $P^{(2)} = P \cdot P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{bmatrix}$

Prediction: $P_2 = P_0 \cdot P^{(2)} = [0 \ 1] \begin{bmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{bmatrix} = [0.28 \ 0.72]$

The probability that Rex's grandpuppy is spotted is 0.28.

- (c) Suppose you're going to adopt a dog from the pound. There's a 60% probability that this dog is spotted, and 40% probability this dog is solid. What is the probability that this dog's grandpuppy will be solid color?

Answer:

Recall “1” = “spotted”; “2” = “solid”

State space: $\{1 \ 2\}$

Initial Distribution: $P_0 = [0.6 \ 0.4]$

Prediction: $P_2 = P_0 \cdot P^{(2)} = P_0 \cdot P \cdot P = [0.6 \ 0.4] \begin{bmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{bmatrix} = [0.376 \ 0.624]$

The probability that your adopted dog's grandpuppy is solid color is 0.624.

- (d) Find the steady state distribution π , and interpret its values.

Answer:

The steady state distribution $\pi = [\pi_1 \ \pi_2]$ is found by solving the system of equations given by

(i) $\pi P = \pi \rightarrow [\pi_1 \ \pi_2] \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} = [\pi_1 \ \pi_2]$

(ii) $\sum \pi_x = 1 \rightarrow \pi_1 + \pi_2 = 1$

(i) gives the relation $(0.6\pi_1 + 0.2\pi_2 = \pi_1)$ and $(0.4\pi_1 + 0.8\pi_2 = \pi_2) \rightarrow \pi_2 = 2\pi_1$

(ii) gives the relation $\pi_1 + \pi_2 = 1 \rightarrow \pi_2 = 1 - \pi_1$

Combining the above into a system of equations, we solve for π_1 and π_2 to obtain

$\pi_1 = 1/3$ is the long run probability that a dog is spotted

$\pi_2 = 2/3$ is the long run probability that a dog is solid color.

Alternatively, we can multiply $P \cdot P \cdot P \cdots P$ until convergence and arrive at the same answer.