Homework 7

Please scan and upload your assignments to BBLearn on or before April 24, 2018.

- You must do your work independently and on your own. That means no collaborations!
- However, you *can* ask questions about the homework on Piazza. You can also answer others' questions. It is possible that your question is already answered there, so check Piazza regularly.
- Scores on late submissions will be penalized by 50% for every day submitted late. Be on time!

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1. **(10 points)** A computer programming team has 15 members.
   (a) How many ways can a group of seven be chosen to work on a project?
   (b) Suppose nine team members are SE students and six are CPRE students.
      i. How many groups of seven can be chosen that contain four SE and three CPRE students?
      ii. How many groups of seven can be chosen that contain at least one SE student?
      iii. How many groups of seven can be chosen that contain at most four CPRE students?

2. **(10 points)** If there are 4 colors of jellybeans and you are trying to fill up a jar that holds 100 beans, how many different color combinations exist (assuming no restrictions on the distributions of the colors)?

3. **(10 points)** In how many ways can you place 2 identical rooks on an 8×8 chessboard such that they will not be able to capture each other (i.e., they do not share the same row or column).

4. **(20 points)** Here, we prove a deep result in number theory known as *Fermat's Little Theorem*. However, our proof will require very little knowledge of number theory! Instead, we construct a proof that purely uses combinatorics.
   (a) Suppose there are beads available in a different colors for some integer $a > 1$, and let p be a prime number. How many different length p sequences of beads can be strung together?
   (b) How many of them contain beads of at least two different colors? (Hint: Calculate how many beads contain exactly 1 color, and subtract from the first answer.)
   (c) Each string of p beads with at least two colors can be made into a bracelet by winding it around a circle in a clockwise manner and tying the two ends of the string together. Two bracelets are the same if one can be rotated to form the other. "Flipping" bracelets or reflecting them is not allowed. Argue that for every bracelet, there are exactly p **distinct** strings of beads that yield it. (Here, you have to use the fact that p is a prime number.)
   (d) Use the above result, combined with the Division Rule, to argue Fermat's Little Theorem: > for any integer $a > 1$ and any prime number p, $a^p - a$ is a multiple of p.

   I recommend actually trying this out with (say) $p = 5$ and $a = 2$. Enumerate all the possible sequences and bracelets, and conclude that the logic in the above steps makes sense.