# Stat 330 (Spring 2020): Homework 2 Due: January 29, 2020

Show all of your work, and *please* staple your assignment if you use more than one sheet. Write your name, the course number and the section on every sheet. Problems marked with * will be graded and one additional randomly chosen problem will be graded.

1. Suppose we have two events, $A$ and $B$, defined on some sample space. Let $|A|$ denote the number of outcomes in event $A$ etc. The classical definition of probability says that if all outcome are equally likely, then for any event $A$, $\mathbb{P}(A) = \frac{|A|}{|\Omega|}$. For a finite sample space $\Omega$, show that the classical definition satisfies the three Axioms of Probability.

   **Answer:**
   Axiom 1) First, $|A| \geq 0$ for any event $A$ and $|\Omega| > 0$. Second, since $A \subset \Omega$, $|A| \leq |\Omega|$ and thus for any event $A$, $0 \leq \frac{|A|}{|\Omega|} \leq 1$.

   Axiom 2) $\mathbb{P}(\Omega) = \frac{|\Omega|}{|\Omega|} = 1$

   Axiom 3) Let $A$ and $B$ be disjoint events. Thus $|A \cup B| = |A| + |B|$.
   So, $\mathbb{P}(A \cup B) = \frac{|A \cup B|}{|\Omega|} = \frac{|A|+|B|}{|\Omega|} = \frac{|A|}{|\Omega|} + \frac{|B|}{|\Omega|} = \mathbb{P}(A) + \mathbb{P}(B)$

2. Twelve athletes compete in an archery event at the Olympics.

   (a) How many ways are there to award the Gold, Silver, and Bronze medals to these athletes?

   (b) How many ways are there to award 3 medals if we do not care about the color of the medal?

   (c) If we know the three individuals who got a medal, how many ways are there to distribute the Gold, Silver, and Bronze to these three individuals?

   **Answer:**

   (a) This is an ordered sample without replacement. There are
   $$P(12,3) = \frac{12!}{(12-3)!} = 12 \times 11 \times 10 = 1320$$
   ways.

   (b) This is an unordered sample without replacement. There are
   $$\binom{12}{3} = \frac{12!}{3!(12-3)!} = 220$$
   ways.

   (c) This is an ordered sample without replacement. There are
   $$P(3,3) = \frac{3!}{(3-3)!} = 3! = 6$$
   ways.

3. The AccessPlus system at ISU has the following policy for creating a password:

   - Passwords must be exactly 8 characters in length.
   - Passwords must include at least one letter (a-z, A-Z) or supported special character (@, #, $ only). All letters are case-sensitive.
   - Passwords must include at least one number (0-9).
   - Passwords cannot contain spaces or unsupported special characters.

According to this policy, how many possible AccessPlus passwords are available? Round to the nearest trillion. (Hint: Count up the number of 8 character passwords that could be made, and then subtract off the number that don't meet the requirement above)

**Answer:** A way to do this is to count all the passwords we could make with all the useable characters and then subtract off the number of invalid passwords. There are 52 letters (26 + 26), 3 supported characters, and 10 numbers. So we could make $65^8$ passwords using those characters. But some of those include passwords like 12345678 which is invalid because it doesn't contain at least on letter or supported special character. Also abcdef$ is invalid because it doesn't contain at least one number. We have to subtract off the all letter/special character passwords and the all number passwords.

There are $55^8$ passwords that don't contain at least one number and $10^8$ passwords that don't include at least one letter/special character. So, the number of valid passwords is $65^8 - 55^8 - 10^8 \approx 235$ trillion passwords.

4. * Consider rearranging the letters in the word "COMPUTER"

   (a) Find the number of 8 letter "words" that can be formed by considering all possible permutations of the letters in the word "COMPUTER"

   (b) How many of these words begin with "C" and end with "R"?

   (c) What is the probability of forming a eight letter word that begins with "C" and ends with "R" by randomly rearranging the letters in "COMPUTER"?

   **Answer:**

   (a) This is a permutation without replacement, so the answer is $P(8,8) = \frac{8!}{(8-8)!} = \frac{8!}{0!} = 8! = 40320$.
   $\underline{8} \cdot \underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 40320$

   (b) We start with C and end with R. We then have to permute the 6 remaining letters. $1 \cdot P(6,6) \cdot 1 = 1 \cdot \frac{6!}{(6-6)!} \cdot 1 = \frac{6!}{0!} = 6! = 720$.
   $\underline{1} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} \cdot \underline{1} = 720$

   (c) $P(\text{word begins with C and ends with R}) = \frac{|A|}{|\Omega|} = \frac{720}{40320} = 0.018$

5. A famous problem in probability is the Birthday Problem. The problem is, *How many people do you need in a room so that the probability that at least two people share the same birthday is at least 0.50?* Assuming 365 days a year, no twins in the room, and each day is equally likely, we can answer the problem as follows:

   First, it is easier to work with the compliment. We will find the probability that in a room full of $n$ people, *none* share a birthday. Number the days 1 - 365 (Jan 1st = 1, …, Dec 31st = 365). For each person in the room, they could have one of the 365 birthdays. The sample size is then the sequence of birthdays for $n$ people. $|\Omega| = 365^n$. For no one to share a birthday, the first person could be born on any of the 365 days. The next person has to be born on one of the 364 remaining days, and the $n$th person born on one of the remaining $365-(n-1)$ days. The total number of outcomes in the event "No one shares a birthday" is $P(365, n)$. Thus $\mathbb{P}(\text{at least two people share}) = 1 - \mathbb{P}(\text{Nobody shares}) = 1 - \frac{P(365,n)}{365^n}$. We could then plug in numbers for $n$ to find the answer to the original problem.

   Go to a computer on campus with excel. Make two columns. In the first column make a list from 1 - 40 representing rooms with $n = 1, \ldots, 40$ people. In the second column, use the formula above to find the probability at least two people in each sized room share a birthday. (Hint: **permut** is the excel function for the permutation number) What is the minimum value of $n$ so that the probability is at least 0.50? Take a screen shot of your excel table and print and turn in with the homework.

   **Answer:** The minimum value for $n$ is 23. This yields a probability of 0.507.

6. *Harry Potter's closet contains 12 brooms. 7 brooms are *Comet 260*s, 4 brooms are *Nimbus 2000*s, and 1 broom is a *Firebolt*. Harry, Ron, George and Fred want to sneak out in the middle of the night for a

game of Quidditch. They are afraid to turn on the light in case they get caught. Harry reaches into the closet and randomly pulls 4 brooms out at once without looking.

(a) What is the probability that all 4 chosen brooms are *Comet 260*s?

(b) What is the probability that Harry pulls out 1 *Comet 260*, 2 *Nimbus 2000*s, and 1 *Firebolt* broom?

(c) What is the probability that at least 1 of the 4 chosen brooms is a *Comet 260*?

**Answer:** First, this is sampling without replacement and order does not matter. $|\Omega|$ = number of ways to select 4 objects from 12 which is $\binom{12}{4} = 495$.

(a) We need to choose 4 of the 7 *Comet 260*s and 0 from the others. The number of ways to do this is $\binom{7}{4} \cdot \binom{4}{0} \cdot \binom{1}{0} = 35$. So the probability is $\frac{35}{495} = 0.073$

(b) This is done in $\binom{7}{1} \cdot \binom{4}{2} \cdot \binom{1}{1} = 42$ ways. So the probability is $\frac{42}{495} = 0.085$.

(c) You could do $\mathbb{P}(1\textit{Comet 260}) + ... + \mathbb{P}(4\textit{Comet 260s})$ and proceed as above. But, it is quicker to find the compliment and subtract from 1. To get no *Comet 260*s we need to choose 0 from the 7 *Comet 260*s and then 4 brooms from the 5 non *Comet 260*s.

$$\begin{aligned}
\mathbb{P}(\text{at least one C}) &= 1 - \mathbb{P}(\text{no } C) \\
&= 1 - \left[\frac{\binom{7}{0} \cdot \binom{5}{4}}{\binom{12}{4}}\right] \\
&= 1 - \frac{5}{495} \\
&= 0.9899
\end{aligned}$$