Homework 6

Please scan and upload your assignments to BBLearn on or before April 10, 2019.

- You must do your work independently and on your own. That means no collaborations!
- However, you *can* ask questions about the homework on Piazza. You can also answer others' questions. It is possible that your question is already answered there, so check Piazza regularly.
- Scores on late submissions will be penalized by 50% for every day submitted late. Be on time!

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1. **(10 points)** Prove, using mathematical induction, that for any $n \geq 1$:

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

2. **(10 points)** Prove the *Prime Factorization Theorem* (PFT) using strong induction. The statement of the PFT is

   Every positive integer $n \geq 2$ can be expressed as a product of 1 or more prime numbers.

   For example, $6 = 2 \times 3$, $7 = 7$, $8 = 2 \times 2 \times 2$, etc.

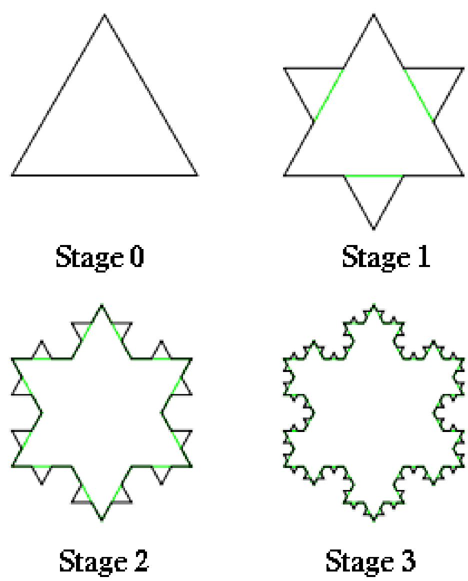

Figure 1: Koch snowflake

3. **(10 points)** A *Koch snowflake* is created by starting with an equilateral triangle with sides one unit in length. Then, on each side of the triangle, a new equilateral triangle is created on the middle third of that side. This process is repeated continuously, as shown in Figure 1.

Prove that the number of sides (colored in black) for the n^{th} Koch snowflake is given by $3 \cdot 4^n$.

4. **(20 points)** Let us play the following single-player game.

   You begin with a single pile of n marbles. In the first move, you arbitrarily partition it into two piles (they could have unequal numbers of marbles – totally up to you).

   In the second move, you pick any one of the two piles and partition it. And so on: in the i^{th} move, you pick any pile and split into two piles as you wish.

   You are awarded **points** in each move as follows: if the pile you just split gives two new piles of sizes a and b, then you get ab points.

   The game continues until you cannot continue any more i.e., you have each of the marbles in its own separate file.

   Here is a trial of me playing the above game for $n = 5$.

- Initial pile size: 5, 0 points
- Move 1: Split the pile; pile sizes: $(3, 2)$; points tally: $3 \times 2 = 6$ points
- Move 2: Split second pile; pile sizes: $(3, 1, 1)$; points tally: $6 + (1 \times 1) = 7$ points
- Move 3: Split first pile; pile sizes: $(1, 2, 1, 1)$; points tally: $7 + (1 \times 2) = 9$ points
- Move 4: Split second pile; pile sizes: $(1, 1, 1, 1)$; points tally: $9 + (1 \times 1) = 10$ points

   So in this case, my final points tally is 10.

(a) Play the game out yourself two separate times. Start with $n = 7$ marbles. As I did above, clearly describe the sequence of moves that you carried out, and the points tally at the end of each move.

   If you did it correctly, you will find that *no matter what* your sequence of moves is, if you start with n marbles you will always end up with a score of $n(n-1)/2$.

   We prove this curious fact using **strong** induction.

(b) Clearly state the induction hypothesis in terms of a predicate $P(n)$.

(c) Prove the base cases $P(1)$ and $P(2)$.

(d) Assume the (strong) induction hypothesis that $P(a)$ is true for every $a \leq n$. Use this to prove that $P(n+1)$ is true.