Recitation 5 Solutions

- Feel free to solve this collaboratively during recitation, and ask (and answer) questions about this problem set on Piazza.
- This is an **optional** problem set; do not turn this in for grading.
- While you don't have to turn this in, be warned that this material **can** appear in a quiz or exam.

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1. Indicate which of the following relationships are true and which are false, together with a brief explanation in words why you think that is the case:

(a) $Z^+ \subseteq Q$.
(b) $Q \subseteq Z$.

(c) $Q \cap R = Q$.
(d) $Z^+ \cap R = Z^+$.
(e) $\emptyset \subset \mathbb{N}$.

Solution

(a) True

(b) False. $1/2 \in Q$ but $1/2 \notin Z$

(c) True

(d) True

(e) True

2. Prove each of the following for all sets A, B, C in a universal set $\mathbb{U}$:

   (a) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$.
   (b) $A \oplus \emptyset = A$.
   (c) $A \oplus A = \emptyset$
   (d) If $A \oplus C = B \oplus C$, then $A = B$.

   Solution

   (a) Observe that A $\oplus$ B = $\{x|(x \in A \land x \notin B) \lor (x \notin A \land x \in B)\}$ = $\{x|x \in A \oplus x \in B\}$

   Therefore:

   A $\oplus$ (B $\oplus$ C) = $\{x|x \in A \oplus (x \in B \oplus x \in C)\}$

   = $\{x|(x \in A \oplus x \in B) \oplus x \in C\}$ (Because XOR is associative)

   = (A $\oplus$ B) $\oplus$ C

   (b) A $\oplus$ $\emptyset$ = (A - $\emptyset$)$\cup$($\emptyset$ - A) = A $\cup$ $\emptyset$ = A.

   (c) A $\oplus$ A = (A - A) $\cup$ (A - A) = $\emptyset \cup \emptyset = \emptyset$.

(d) Proof by contradiction:

Assume that A $\neq$ B, then there exists an element x such that x $\in$ A and x $\notin$ B.

- Case 1: x $\in$ C:

Because x $\in$ C and x $\in$ A, x $\notin$ (A $\oplus$ C)

Because x $\notin$ (A $\oplus$ C) and A $\oplus$ C = B $\oplus$ C, x $\notin$ (B $\oplus$ C).

Because x $\notin$ (B $\oplus$C) and x $\in$ C, x $\in$ B which contradicts to the assumption that x $\notin$ B.

- Case 2: x $\notin$ C:

Because x $\notin$ C and x $\in$ A, x $\in$ (A $\oplus$ C)

Because x $\in$ (A $\oplus$ C) and A $\oplus$ C = B $\oplus$ C, x $\in$ (B $\oplus$ C).

Because x $\in$ (B $\oplus$C) and x $\notin$ C, x $\in$ B which contradicts to the assumption that x $\notin$ B.

3. Either prove the following statement, or disprove it using a counter-example.

   For all sets A, B, C, if $B \bigcap C \subseteq A$, then $(A - B) \bigcap (A - C) = \emptyset$.

Solution

Counter-example: $A = \{1, 2, 3, 4\}$ and $B = C = \emptyset$. Then $(A - B) \bigcap (A - C) = \{1, 2, 3, 4\}$

4. A pair of sets are called *disjoint* if their intersection is the empty set. Prove that two finite sets A and B are disjoint if and only if $|A| + |B| = |A \cup B|$.

   Solution

   A and B are disjoint $\implies |A| + |B| = |A \cup B|$.

   If A and B are disjoint, $|A \cap B| = 0$.

   Using the principle of inclusion - exclusion, $|A| + |B| = |A \cup B| + |A \cap B|$.

   Substituting $|A \cap B| = 0$ to the above, we get $|A| + |B| = |A \cup B|$.

   On the other hand, we also need to prove following:

   $|A| + |B| = |A \cup B| \implies A$ and B are disjoint.

   If $|A| + |B| = |A \cup B|$ is satisfied, we can substitute this to the principle of inclusion - exclusion and get following:

   $|A \cap B| = 0$. Since the cardinality of $|A \cap B| = 0$, A and B are disjoint.