Exact Inference in Bayesian Networks

Outline

I. Probabilistic query using a BN

II. Variable Elimination

III. Variable ordering and relevance

* Figures are either from the textbook site.

I. Probabilistic Query

X: query variable

$\boldsymbol{E} = \{E_1, \ldots. E_m\}$: evidence variables

$\boldsymbol{e} = \{e_1, \ldots. e_m\}$: an observed event

$\boldsymbol{Y} = \{Y_1, \ldots. Y_l\}$: hidden variables

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We will discuss exact algorithms for posterior probability computation.

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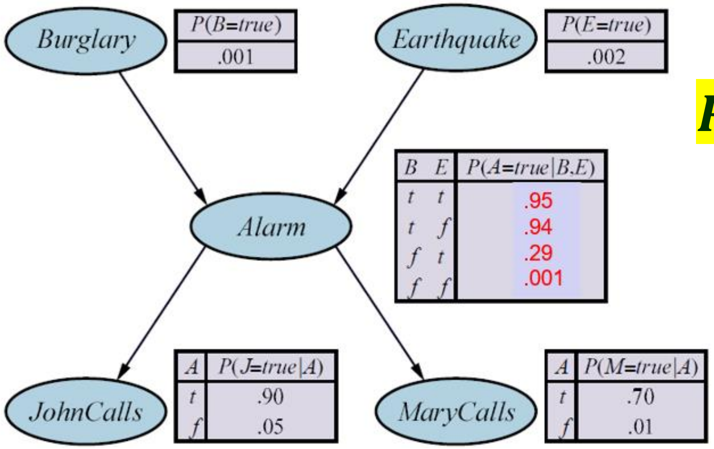
- Answer the query $\boldsymbol{P}(X \mid \boldsymbol{e})$ using a BN.

Burglary Example (revisited)



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Hidden variables: $Y = \{Earthquake, Alarm\}$

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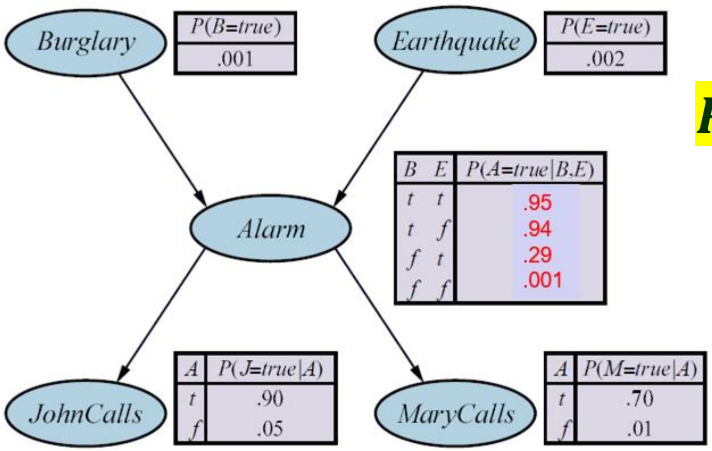

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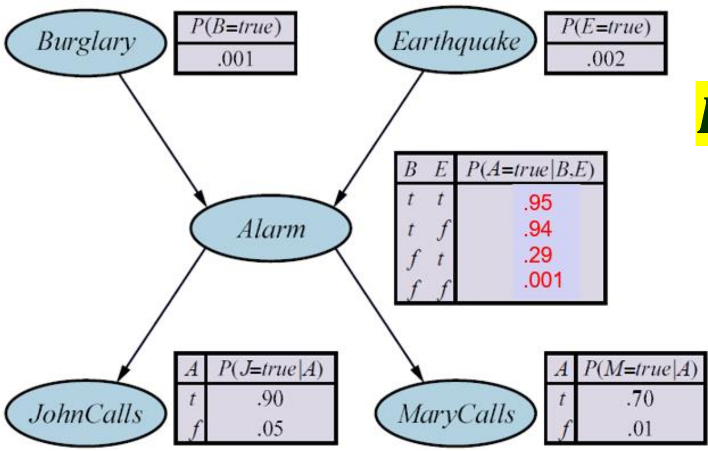

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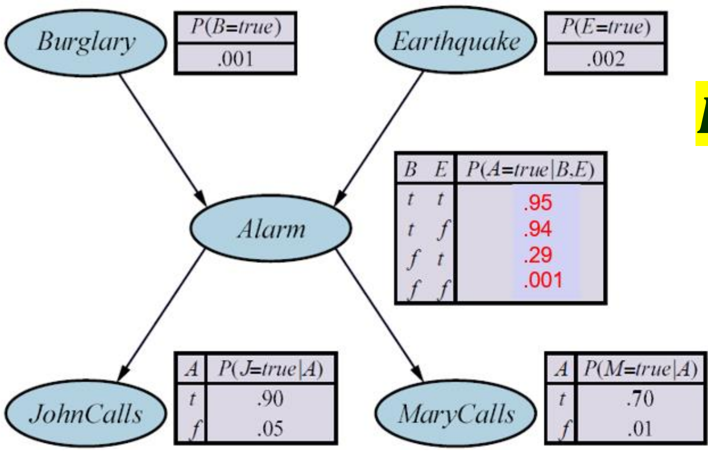

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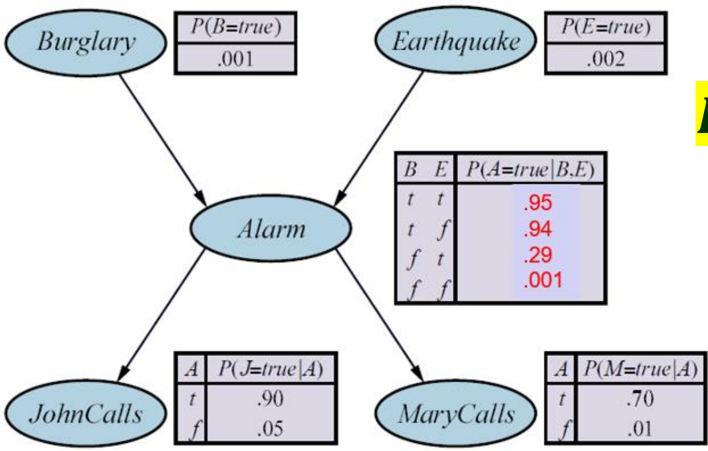

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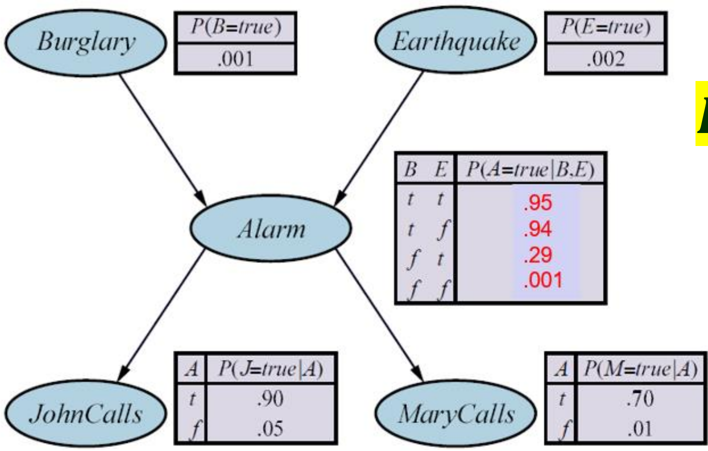

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In the general case with n variables, there are 2^n summands, each as a product requires $O(n)$ computation time.

Expression Tree

Take advantage of the nested structure to move summations inwards as far as possible.

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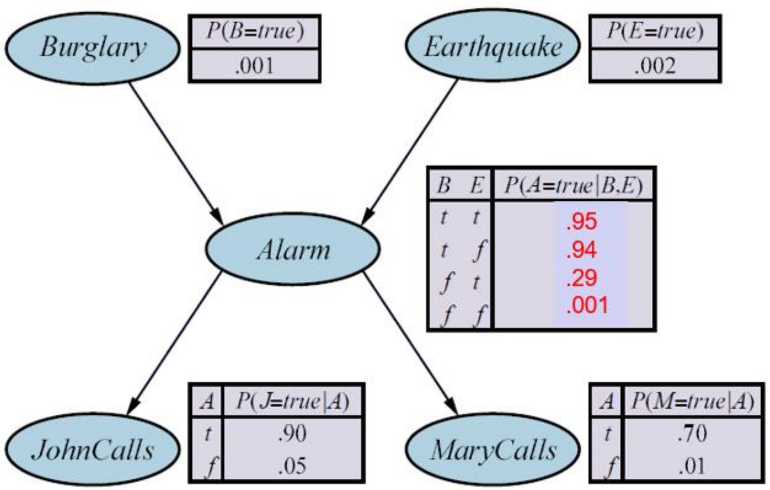

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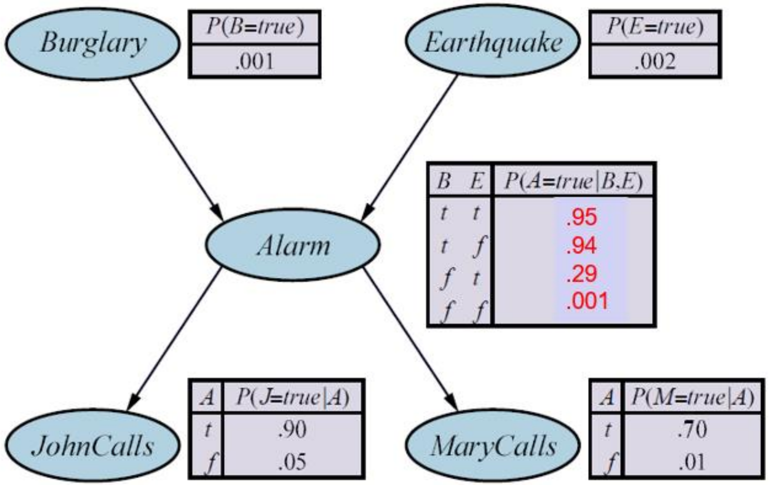

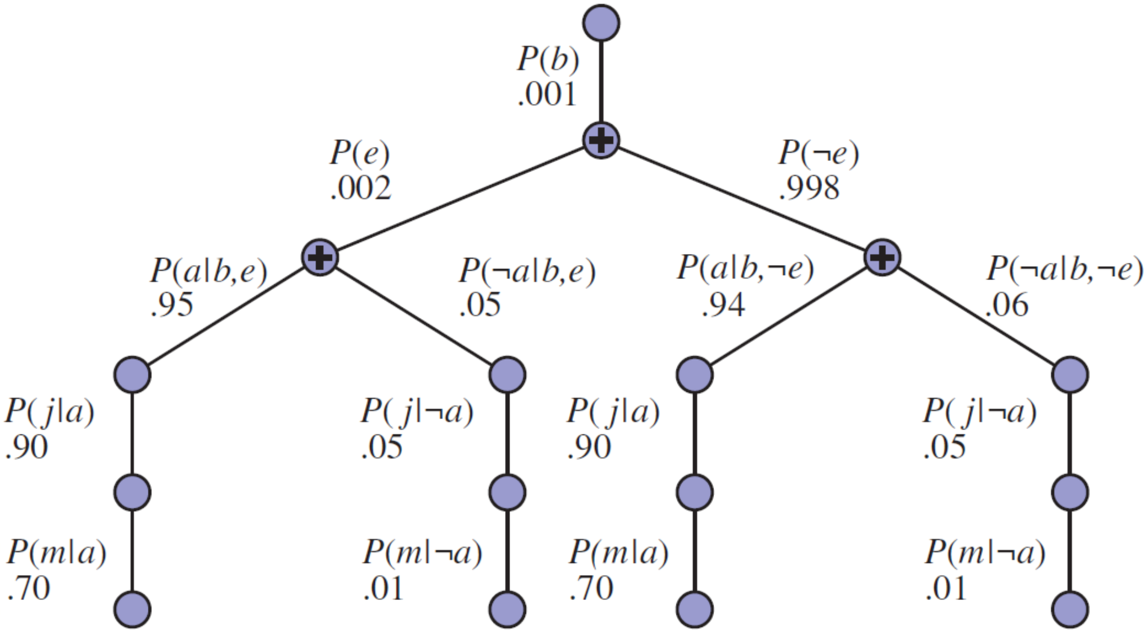

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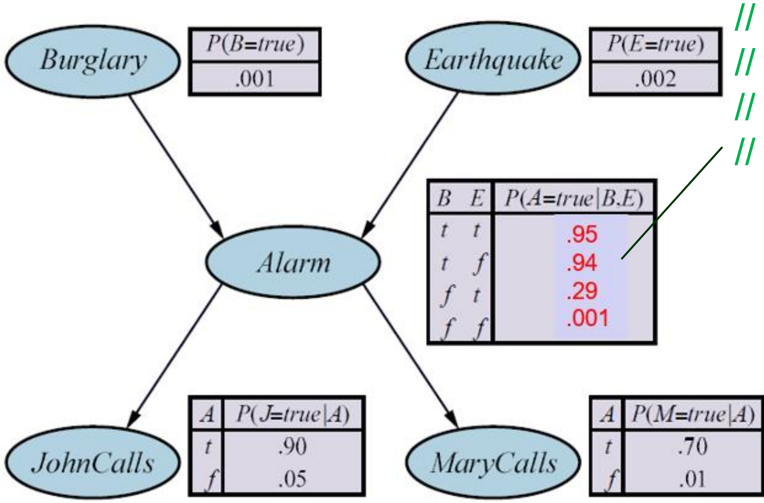

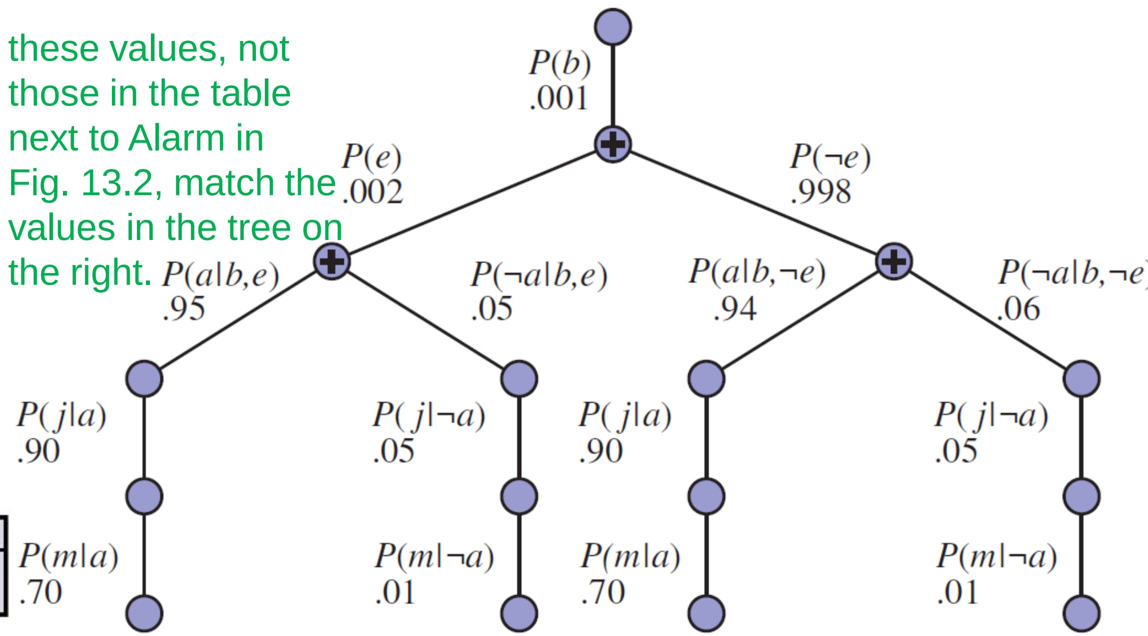

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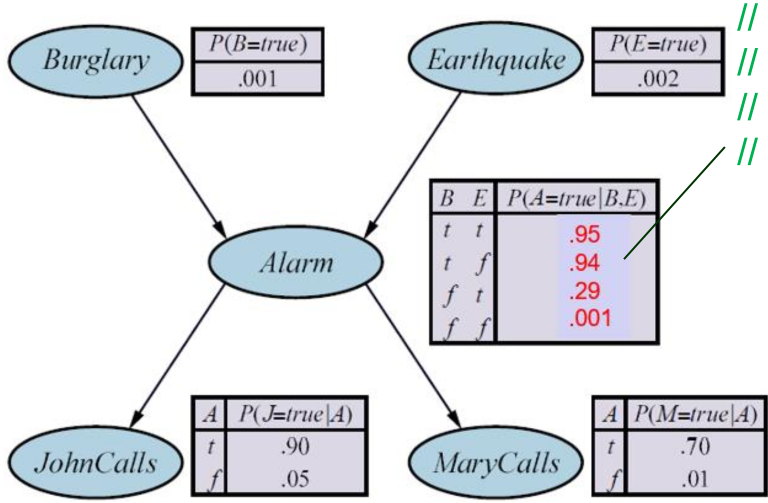

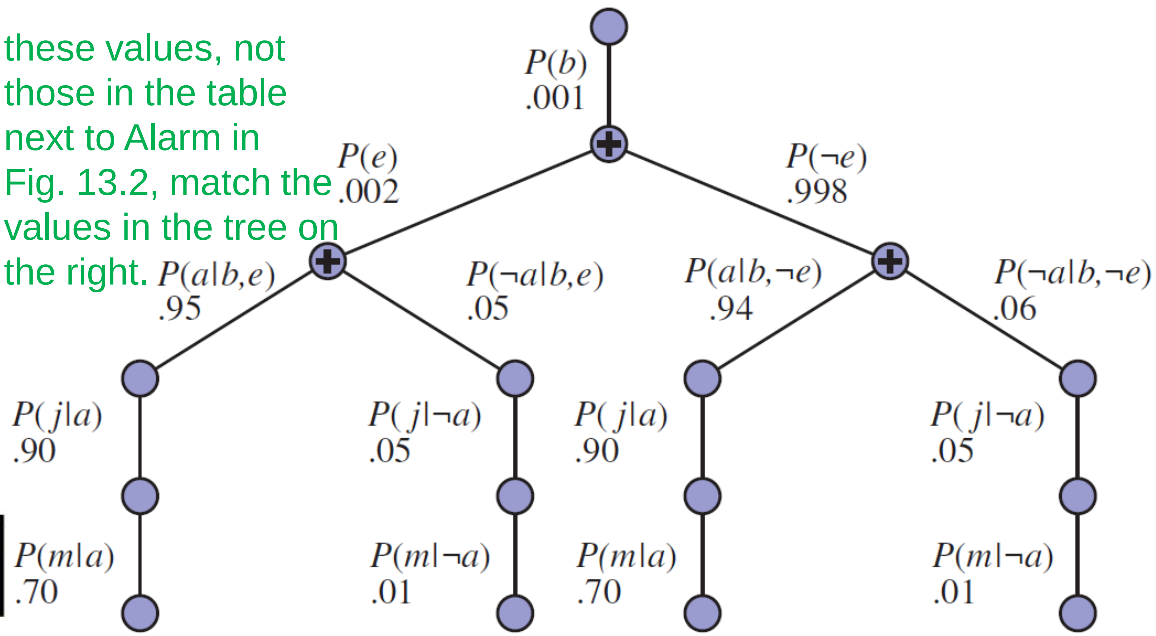

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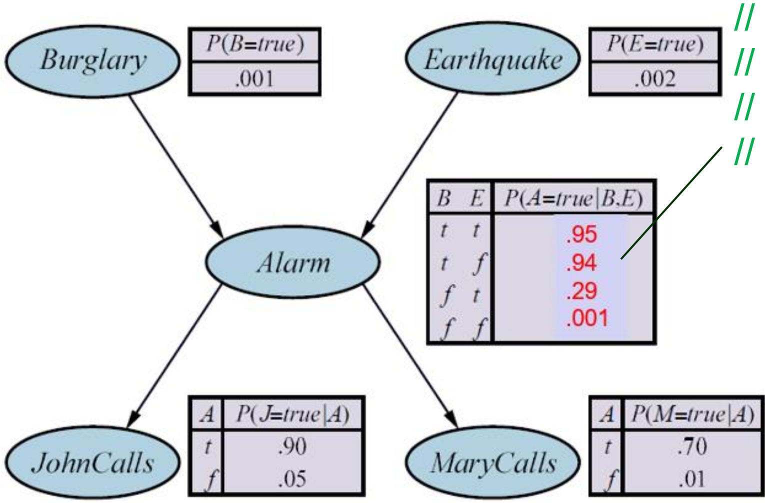

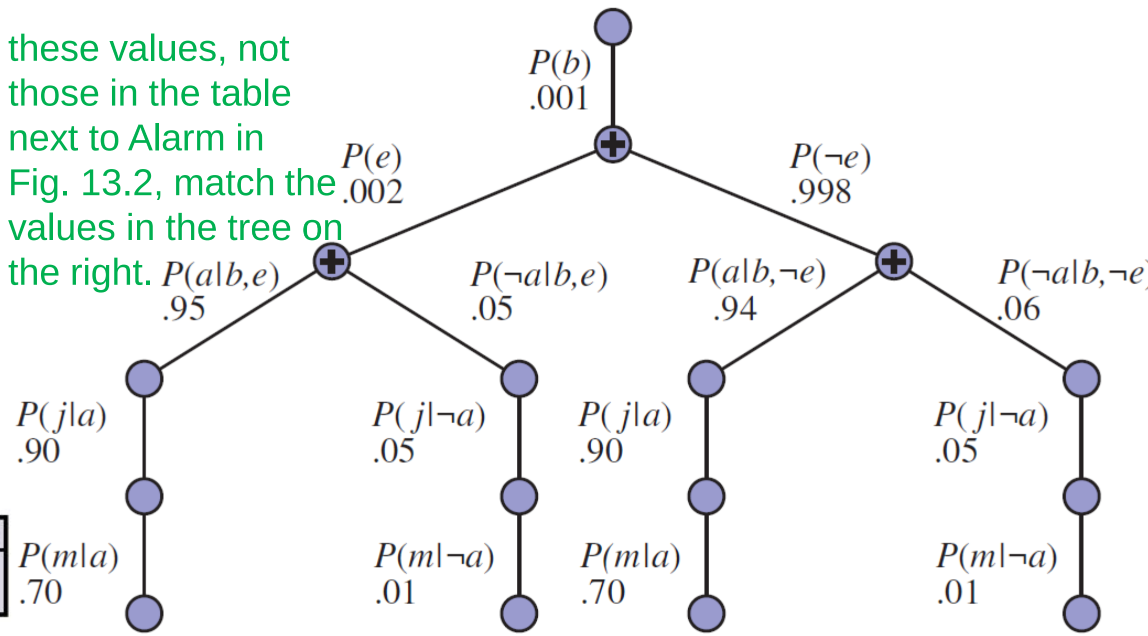

$$P(b \mid j, m) = \alpha \times 0.00059224$$

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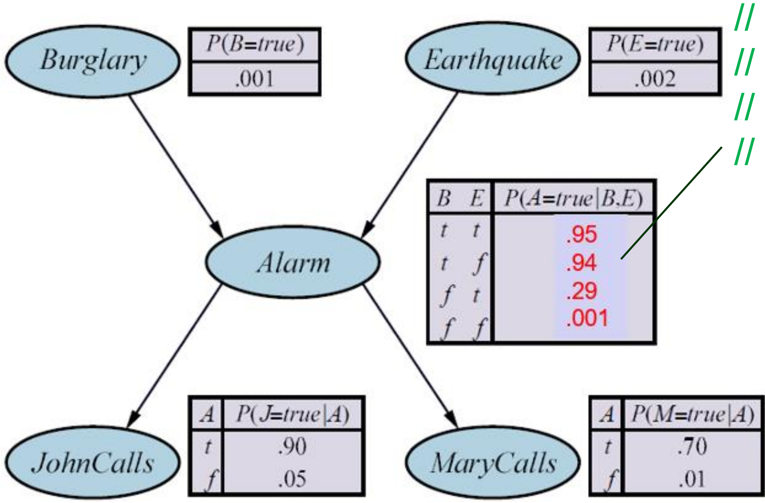

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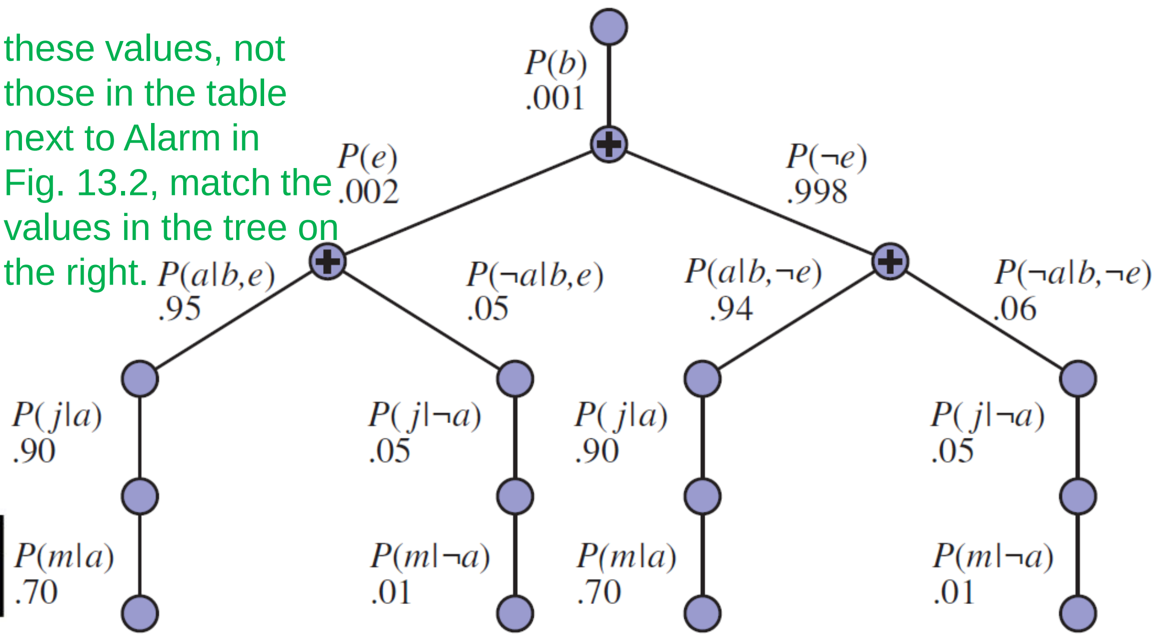

$$\left.\begin{aligned} P(b \mid j, m) &= \alpha \times 0.00059224 \\ P(\neg b \mid j, m) &= \alpha \times 0.0014919 \end{aligned}\right\} \Longrightarrow \mathbf{P}(B \mid j, m) = \alpha \langle 0.00059224, 0.0014919 \rangle \approx \langle 0.284, 0.716 \rangle$$

Enumeration Algorithm

function ENUMERATION-ASK(X, $\mathbf{e}$, bn) **returns** a distribution over X
  inputs: X, the query variable
    $\mathbf{e}$, observed values for variables $\mathbf{E}$
    bn, a Bayes net with variables $vars$

  $\mathbf{Q}(X) \leftarrow$ a distribution over X, initially empty
  for each value x_i of X **do**
    $\mathbf{Q}(x_i) \leftarrow$ ENUMERATE-ALL($vars$, $\mathbf{e}_{x_i}$)
      where $\mathbf{e}_{x_i}$ is $\mathbf{e}$ extended with $X = x_i$
  return NORMALIZE($\mathbf{Q}(X)$)

function ENUMERATE-ALL($vars$, $\mathbf{e}$) **returns** a real number
  if EMPTY?($vars$) **then return** 1.0
  $V \leftarrow$ FIRST($vars$)
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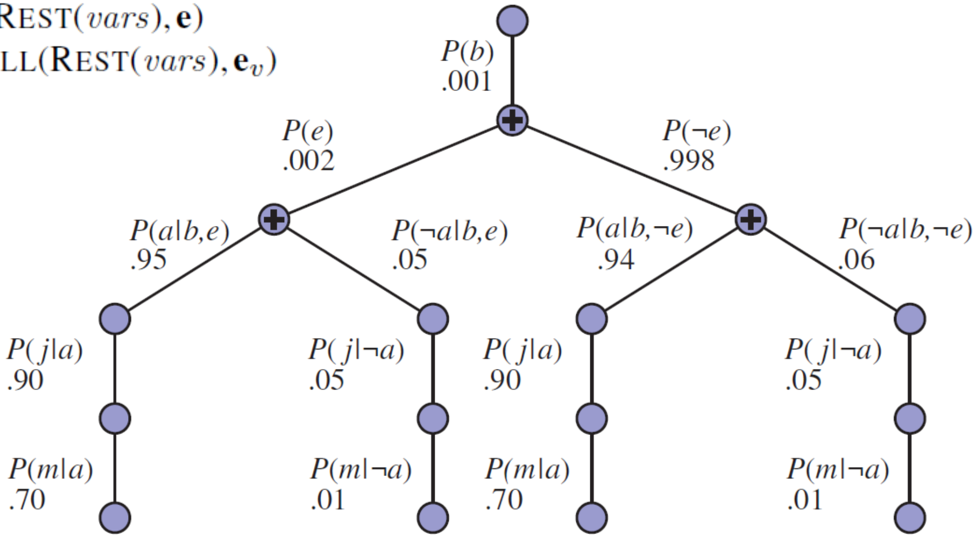

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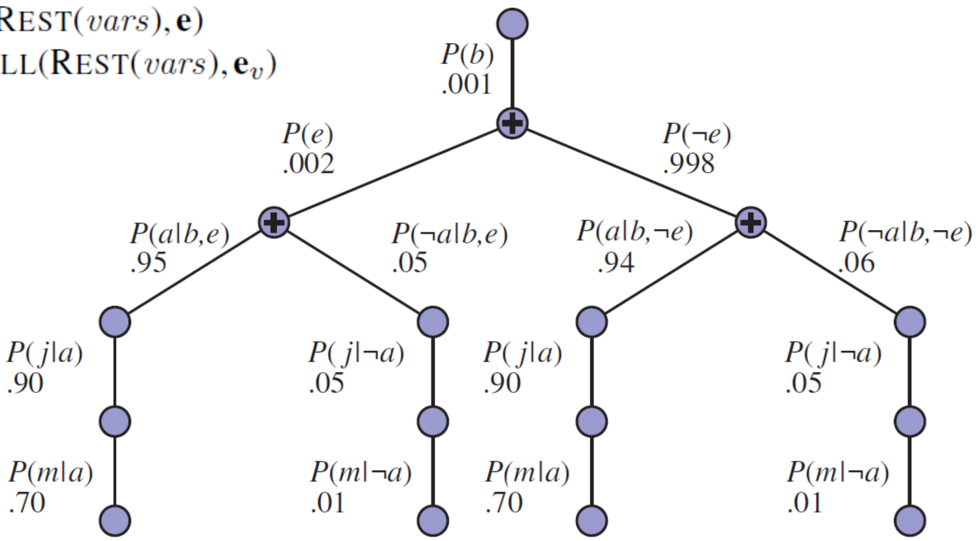

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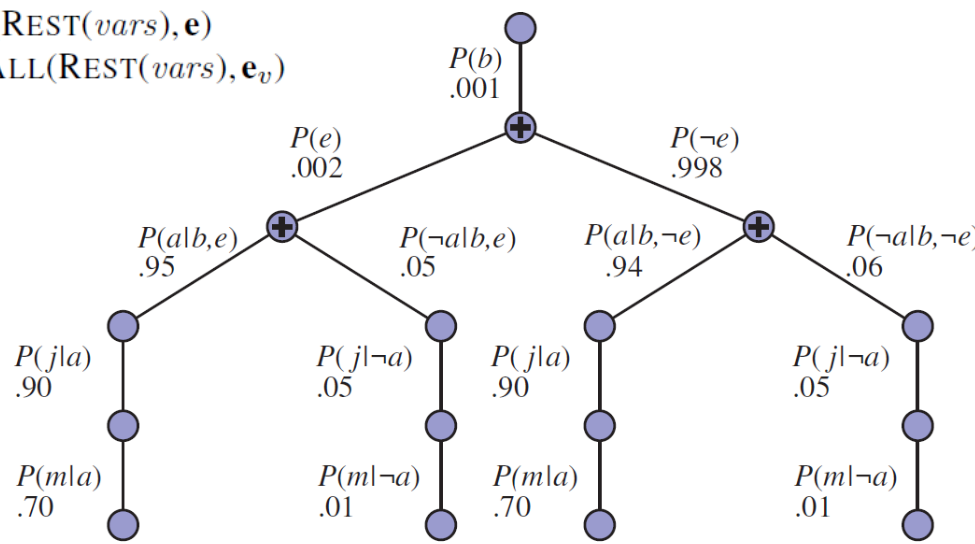
- Depth-first recursion
- $O(2^n)$ time for n variables



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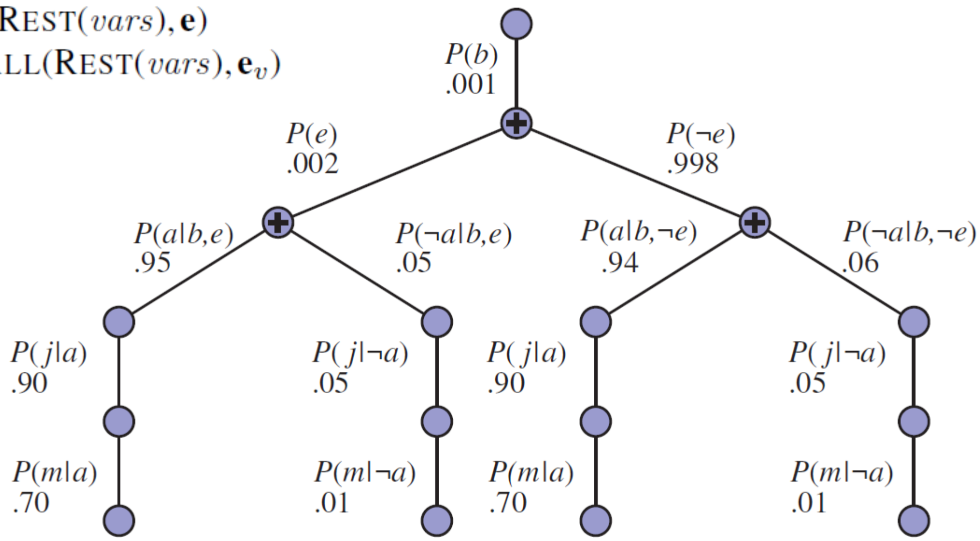

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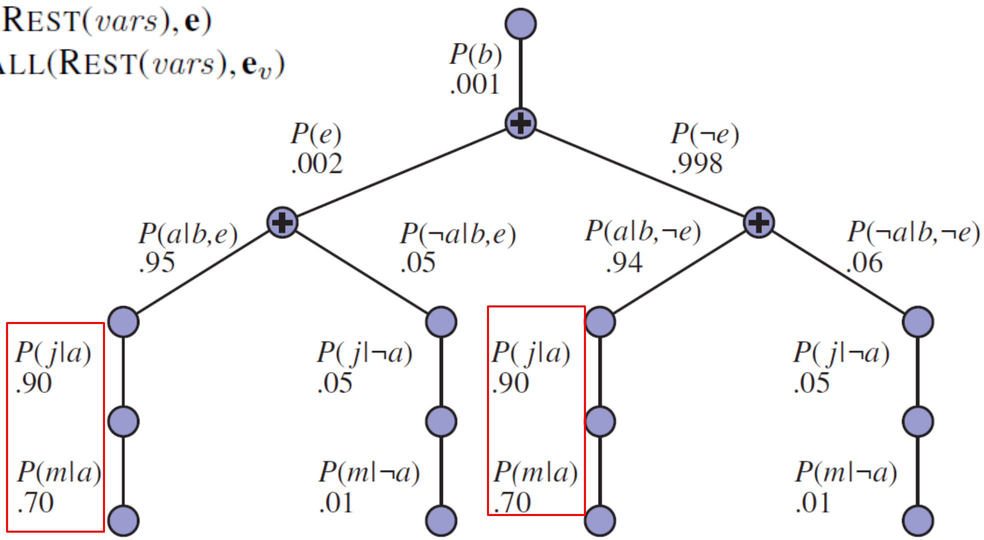

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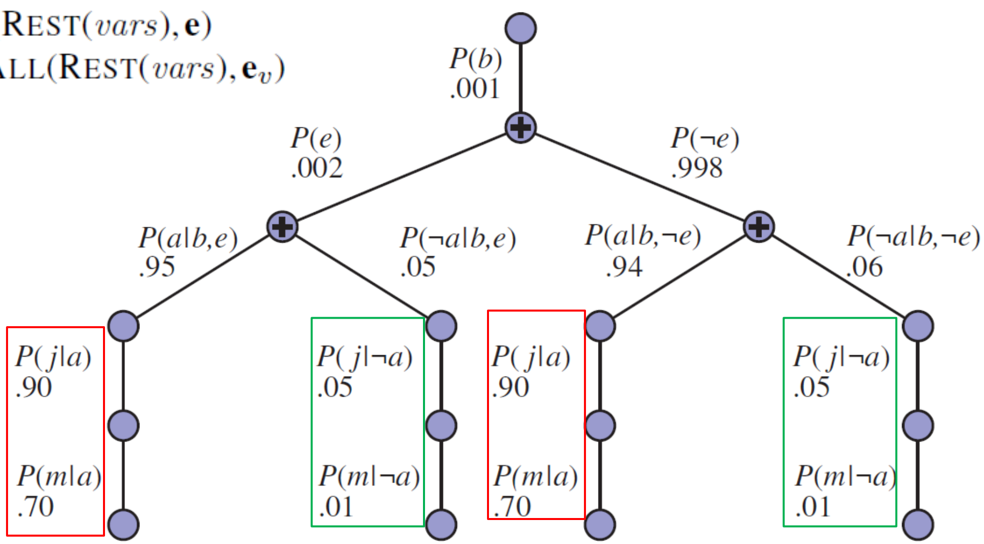

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factors:	$\boldsymbol{f}_1(B)$	$\boldsymbol{f}_2(E)$	$\boldsymbol{f}_3(A, B, E)$	$\boldsymbol{f}_4(A)$	$\boldsymbol{f}_5(A)$
dimensions:	2×1	2×1	$2 \times 2 \times 2$	2×1	2×1

Evaluation Example

$$\boldsymbol{P}(B \mid j, m) = \alpha \boldsymbol{f_1}(B) \times \sum_{E} \boldsymbol{f_2}(E) \times \sum_{A} \boldsymbol{f_3}(A, B, E) \times \boldsymbol{f_4}(A) \times \boldsymbol{f_5}(A)$$

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$$\boldsymbol{P}(B \mid j,m) = \alpha \boldsymbol{f}_{\mathbf{1}}(B) \times \sum_{E} \boldsymbol{f}_2(E) \times \boldsymbol{f}_6(B,E)$$

- Sum out E from the product of $\boldsymbol{f}_2$ and $\boldsymbol{f}_6$.

$$\boldsymbol{f}_7(B) = \sum_{E\in\{e,\neg e\}} \boldsymbol{f}_{\mathbf{2}}(E) \times \boldsymbol{f}_{\mathbf{6}}(B,E) = P(e) \times \underset{2\times 1}{\boldsymbol{f}_6(B,e)} + P(\neg e) \times \underset{2\times 1}{\boldsymbol{f}_6(B,\neg e)}$$

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- Finally, carry out the following pointwise product:

$$\boldsymbol{P}(B \mid j, m) = \alpha \boldsymbol{f_1}(B) \times \boldsymbol{f_7}(B)$$

Pointwise Product of Two Factors

$$\boldsymbol{f}(X_1, \dots, X_j, Y_1, \dots, Y_k) \times \boldsymbol{g}(Y_1, \dots, Y_k, Z_1, \dots, Z_l) = \boldsymbol{h}(X_1, \dots, X_j, Y_1, \dots, Y_k, Z_1, \dots, Z_l)$$

X	Y	$\mathbf{f}(X,Y)$	Y	Z	$\mathbf{g}(Y,Z)$	X	Y	Z	$\mathbf{h}(X,Y,Z)$
t	t	.3	t	t	.2	t	t	t	$.3 \times .2 = .06$
t	f	.7	t	f	.8	t	t	f	$.3 \times .8 = .24$
f	t	.9	f	t	.6	t	f	t	$.7 \times .6 = .42$
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  inputs: X, the query variable
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    bn, a Bayesian network with variables $vars$

$factors \leftarrow [\,]$
for each V **in** ORDER($vars$) **do**
  $factors \leftarrow$ [MAKE-FACTOR(V, **e**)] + $factors$
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return NORMALIZE(POINTWISE-PRODUCT($factors$))

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- Use a greedy heuristic: eliminate whichever variable minimizes the size of the next factor to be constructed.

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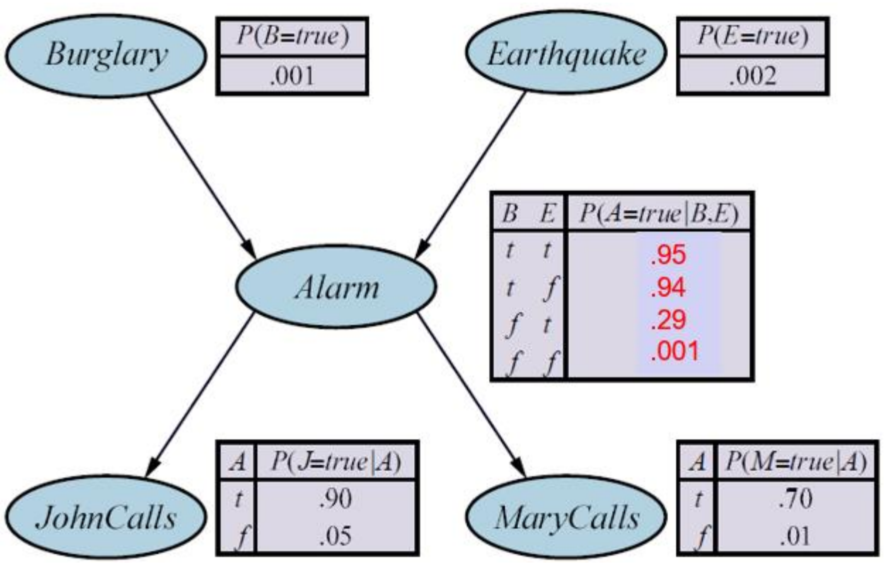

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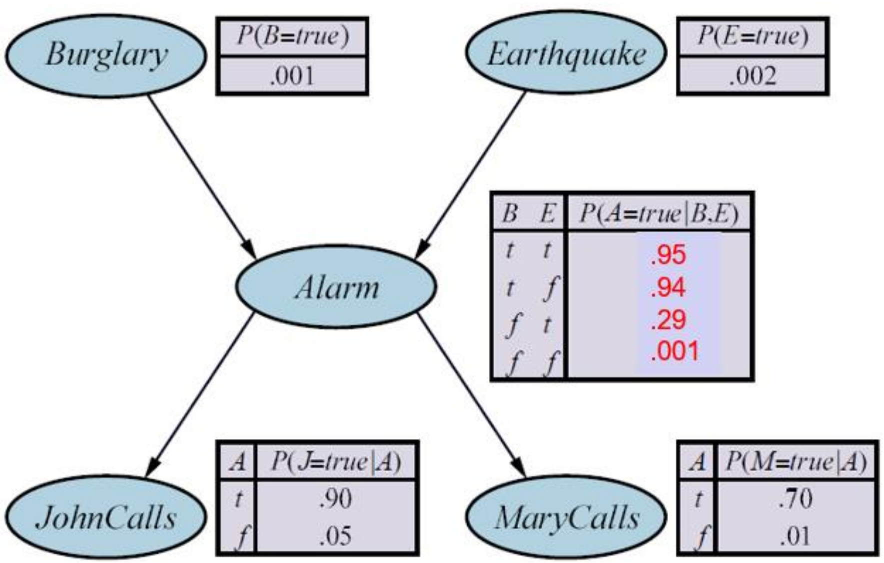

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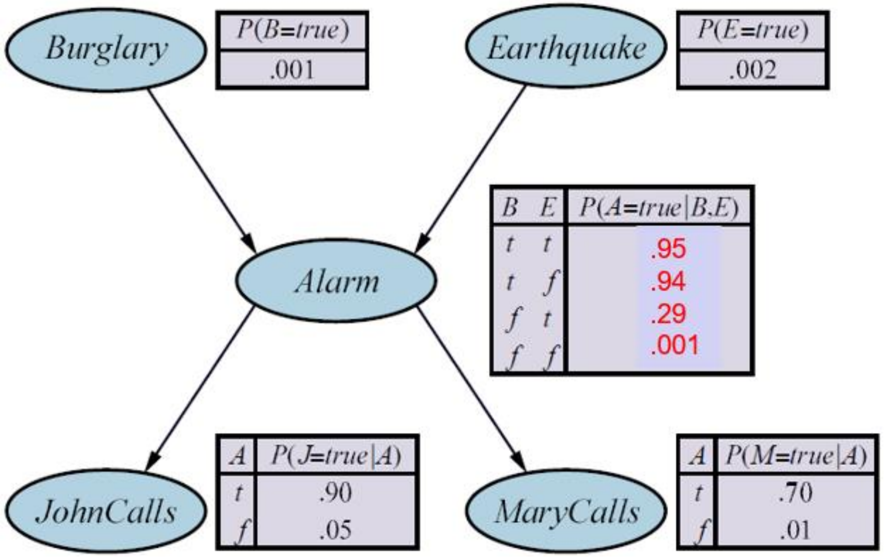

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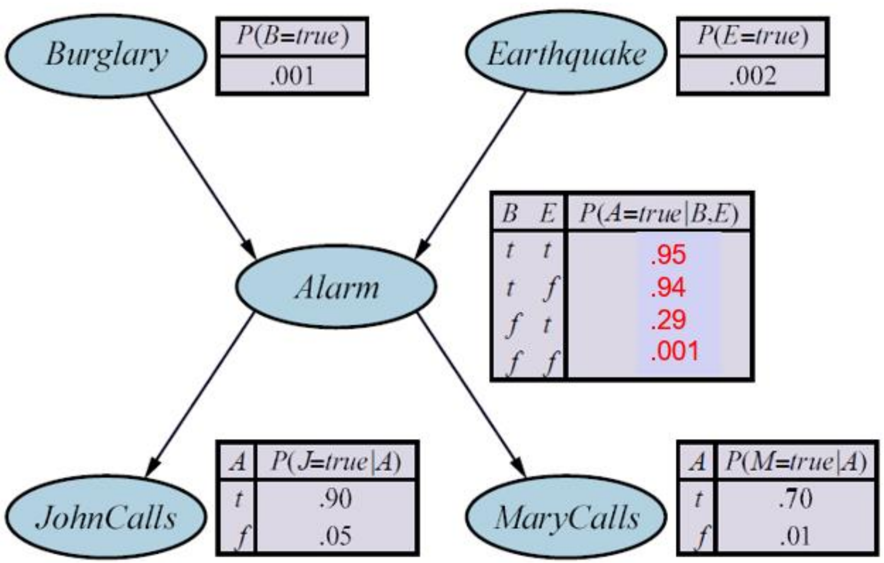

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$$\Downarrow$$

The variable M is irrelevant to the query.

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- ♦ Using *reverse topological order* for variables, exact inference with elimination can be 1,000 times faster than the enumeration algorithm.

- ♦ If we want to compute posterior probabilities for all the variables rather than answer individual queries, we can use clustering algorithms (i.e., join tree algorithms).