Central Limit Theorem Example

A study involving stress is done on a college campus among the students. The stress scores follow a uniform distribution with the lowest stress score equal to 1 and the highest equal to 5.

1. What is the probability that a student has a stress score of more than 3.5?

   Answer:

$$\text{Let } X = \text{stress score} \quad X \sim Unif(a = 1, b = 5)$$
$$\text{Recall that the CDF of X is } F_X(t) = \frac{t-1}{5-1} = \frac{t-1}{4}$$
$$\mathbb{P}(X > 3.5) = 1 - F_X(3.5) = 1 - \frac{3.5-1}{4} = 1 - 0.625 = 0.375$$

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2. Consider taking a sample of $n = 40$ of the college students, and calculating the sample mean stress score $\bar{X}$. Give the distribution of $\bar{X}$

   Answer:
   Since X follows a uniform distribution, it's mean $\mu = E(X) = \frac{a+b}{2} = \frac{1+5}{2} = 3$, and variance $\sigma^2 = Var(X) = \frac{(b-a)^2}{12} = \frac{(5-1)^2}{12} = \frac{4}{3}$

   Then, by central limit theorem, $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \equiv N\left(3, \frac{4/3}{40}\right) \equiv N\left(3, 0.0333\right)$

3. What is the probability that the sample mean stress score is greater than 3.5?

   Answer:

$$\begin{aligned}
\mathbb{P}(\bar{X} > 3.5) &= \mathbb{P}\left(\frac{\bar{X}-3}{\sqrt{0.0333}} > \frac{3.5-3}{\sqrt{0.0333}}\right) \\
&= \mathbb{P}(Z > 2.74) \\
&= 1 - P(Z < 2.74) \\
&= 1 - 0.9969 = 0.0031
\end{aligned}$$

4. Consider taking a sample of $n = 40$ of the college students, and calculating the sample sum stress score S_n. Give the distribution of S_n.

   Answer:
   By the central limit theorem, $S_n \sim N\left(n \cdot \mu, n \cdot \sigma^2\right) \equiv N\left(40 \cdot 3, 40 \cdot \frac{4}{3}\right) \equiv N\left(120, 53.3333\right)$

5. What is the probability that the sample sum stress score is less than 115?

   Answer:

$$\begin{aligned}
\mathbb{P}(S_n < 115) &= \mathbb{P}\left(\frac{S_n-120}{\sqrt{53.3333}} < \frac{115-120}{\sqrt{53.3333}}\right) \\
&= \mathbb{P}\left(Z < -0.68\right) \\
&= 0.2483
\end{aligned}$$