Lecture 9: Set theory

Set theory basics

Sets are simple to understand, but hard to define precisely.

Informally, a set is a bunch of objects; these objects are called the *elements*, or *members* of the set.

The definition of "object" is deliberately general, and could be just about anything: numbers, letters, points on a plane, strings, names of CPRE students, football teams, and so on. In fact, the elements of a set can even be *other sets*. Our convention is that whenever we list the elements of a set, we do so within curly braces $\{\}$.

Some examples of sets:

- The set of characters in the English alphabet: $A = \{a, b, c, \ldots, z\}$
- The set of even integers strictly between 5 and 10: $E = \{6, 8\}$
- The set of living ex-Presidents: $P = \{$Carter, Bush41, Clinton, Bush43, Obama$\}$
- A set of sets: $\{\{a, b\}, \{b, c\}, \{a, c\}\}$

We use the symbol "$\in$" to denote set membership; i.e., if x is a member of some set S, then we write it concisely as:

$$x \in S$$

Similarly, to say that x is not a member of some set S, we write it as:

$$x \notin S$$

Sets are typically described by identifying some specific property that is satisfied by all its elements. Such sets can be denoted using *set builder notation* by stating the property as a *predicate*. For instance, instead of listing out all the alphabets in the first example, we can alternatively specify A as:

$$A = \{x | x \text{ is a letter in the English alphabet}\}$$

Properties of sets

There are two main properties of sets to keep in mind:

1. The *order* of specifying the elements in a set is not important. For example, the set $\{6, 8\}$ is identical to the set $\{8, 6\}$, just written down in a different way. (Later, we will talk about *sequences*, where the order or elements will be vitally important.)

2. *Repetition* of elements is not important: there is no notion of an element appearing more than once in a set. For example, the set $\{6, 8\}$ is the same as the set $\{6, 6, 8, 6, 8\}$. By convention, we will remove all duplicates while enumerating the contents of a set.

Some important sets

In mathematical proofs, the following sets often come up over and over again:

- The set of integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- The set of nonnegative integers: $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$
- The set of positive integers: $\mathbb{Z}^+ = \{1, 2, \ldots\}$

Using predicates, we can also write $\mathbb{Z}^+$ as $\mathbb{Z}^+ = \{x | (x \in Z) \wedge (x > 0)\}$

- The set of real numbers: $\mathbb{R} = \{x | x \text{ is real}\}$
- The set of positive real numbers: $\mathbb{R}^+ = \{x | x \text{ is real} \wedge x > 0\}$
- The set of rational numbers: $\mathbb{Q} = \{x | x = \frac{a}{b}, a, b \in \mathbb{Z}, b \neq 0\}$

Sometimes, we talk about:

- The *universal* set, which consists of all elements in a given domain of discourse (or universe). This is not an *a priori* fixed set like the above definitions, but depends on what we are talking about.
- The empty set (or the *null* set) $\phi = \{ \}$

Equality of sets

Two sets are called *equal* if and only if they contain the same elements.

In symbols: let A and B be sets. Stating that $A = B$ is logically equivalent to the proposition:

$$\forall x(x \in A \iff x \in B)$$

This proposition can be used as an algorithm for checking the equality of two given sets via enumeration: cycle through each element of the first set, and check if it belongs to the second; cycle through each element in the second set and check that this belongs to the first.

For example, one can check (via exhaustive enumeration) that the set $E = \{4, 6, 7\}$ is equal to $F = \{7, 6, 4\}$.

Subsets

A set S is called a *subset* of another set T if **every** element of S also belongs to T.

If S is a subset of T, we use the notation $S \subseteq T$. Think of $\subseteq$ as pointing to a smaller set, just as how the symbol $\leq$ points to a smaller number.

In symbols: let A and B be sets. Then, stating that $A \subseteq B$ is logically equivalent to the proposition:

$$\forall x(x \in A \implies x \in B)$$

Notice the (subtle) difference from the definition of set equality defined above. To check that some set A is a subset of some bigger set B, cycle through each element of A and verify that it is a member of B.

As an example: if $U = \{2, 4, 6\}$, $V = \{1, 2, 3, 4, 5, 6\}$ and $W = \{2, 4, 5\}$, we observe that the following are true:

$U \subseteq V$

$W \subseteq V$

On the other hand, consider:

$U \subseteq W$

This statement is **false** since $6 \in U$ but $6 \notin W$. That is, U is not a subset of W since there at least one element in U that is not an element in W. Symbolically, we denote this as $U \nsubseteq W$.

Some properties of subsets

0. We say that A is a *proper* subset of B if $A \subseteq B$ but $A \neq B$, i.e., there is at least one element of B that is not in A. We denote this by the symbol $\subset$.

1. By definition, any set is a subset of itself, i.e., $A \subseteq A$.

2. Also, by definition, the null set ϕ is a subset of any arbitrary set S, i.e., $\phi \subset S$. This can be proved as follows. Invoking the definition of $\subseteq$, we need to show that for any arbitrary set S:

$\forall x(x \in \phi \implies x \in S)$

3. However, $x \in \phi$ is *false* for any x since ϕ is empty. Therefore, the implication_is true (recall the property of $\implies$). This is an example of a *vacuous proof.*

4. Note that $\subseteq$ is **distinct** from $\in$. This is a common type error that people make often. Writing down $1 \in \{1, 2, 3\}$ is *not* the same as $\{1\} \subseteq \{1, 2, 3\}$. In the former, the number 1 denotes an *element* of the *set* $\{1, 2, 3\}$, while in the latter, the set $\{1\}$ denotes a *subset* of the *set* $\{1, 2, 3\}$. Watch out for the curly braces!

5. The *power* set is the set of all subsets of some given set S. For example, if $S = \{6, 8\}$, then the power set of S, denoted by $\mathbb{P}(S)$ is the set:

   $\{\{6\}, \{8\}, \{6, 8\}, \phi\}$

Two things to note here. First, each of the elements of $\mathbb{P}(S)$ is itself a set (and therefore is written with curly braces). Next, by the argument we made above, both ϕ and S itself are subsets of S, and therefore are included in the power set.

Let's do an **exercise.** Can you identify which two of the following statements are incorrect?

- $3 \in \{1, 2, 3\}$
- $\{2\} \in \{1, 2, 3\}$
- $2 \subseteq \{1, 2, 3\}$
- $\{3\} \subset \{1, 2, 3\}$
- $\{1\} \subseteq \{1, 2, 3\}$
- $\{3\} \in \{\{2\}, \{3\}\}$

The incorrect ones are the second and third statements. There is a type error in each of these statements; convince yourself why that is the case! Also, the last one is correct: the right hand side is a set of sets; therefore *its* elements are the sets $\{2\}$ and $\{3\}$ and it is accurate to claim that $\{3\}$ is a member of $\{\{2\}, \{3\}\}$.

Finally: If two sets A and B are equal then, $A \subseteq B$, and also, $B \subseteq A$. In fact, the reverse is also true:

If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

(As an **exercise**, try proving this.)

We now consider some basic set operations. Specifically, given two or more sets, we will see how to manipulate their elements in non-trivial ways using unions, intersections, complements, products, and so on.

Basic operations on sets

In what follows, let A and B be two generic sets, and U denote the universal set. Let us start with three familiar operations:

- *Union* : The set of elements that belong to either A or B:

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

- *Intersection* : The set of elements that are common to both sets:

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

Two sets are called *disjoint* iff their intersection is empty, i.e., $A \cap B = \phi$.

- *Complement* : The set of all elements (in the universal set) that do *not* belong to a given set:

$$\overline{A} = \{x \mid x \notin A\}$$

Sometimes the symbol used for set complement is superscript-c, i.e, A^c.

You have probably seen/heard of these operations before in an earlier course. The typical way to illustrate these operations is via Venn Diagrams.

Here is an example. Suppose the domain of discourse under consideration is the set of positive integers up to 10, i.e., $U = \{1, 2, 3, 4, \ldots, 10\}$. Consider sets $A = \{2, 3, 5, 7\}$ and $B = \{2, 4, 5\}$. Then, it is easy to see that:

- $A \cup B = \{2, 3, 4, 5, 7\}$
- $A \cap B = \{2, 5\}$
- $\overline{A} = \{1, 6, 8, 9, 10\}$

A property (easily proved) of unions and intersections is that they are *commutative*; the order in which you specify the constituent sets doesn't matter. So, $A \cup B$ is the same as $B \cup A$, and $A \cap B$ is the same as $B \cap A$. We will discuss commutativity and several other properties below.

The above definitions of union and intersection can be extended to any number of sets:

$$S_1 \cup S_2 \ldots \cup S_n = \{x \mid (x \in S_1) \vee (x \in S_2) \ldots \vee (x \in S_n)\}$$

$$S_1 \cap S_2 \ldots \cap S_n = \{x \mid (x \in S_1) \wedge (x \in S_2) \ldots \wedge (x \in S_n)\}$$

There are two operations that are somewhat less commonly encountered in introductory courses:

- *Set difference*: The set of elements that belong to one set but not the second:

$$A - B = \{x \mid (x \in A) \wedge (x \notin B)\}$$

In the above example, $A - B$ is the set $\{3, 7\}$ since 3 and 7 are members of A but not B. Note that this is **different** from $B - A$, which is the set $\{4\}$. In that sense, set difference is *not commutative*.

One can combine the definitions of complement and intersection to arrive at the identity:

$$A - B = A \cap \overline{B}$$

Set differences satisfy the property that $A - B$ and $B - A$ are disjoint. As an **exercise**, try proving this by contradiction.

- *Symmetric difference*: The set of elements that belong to either one set of the other, but not both:

$$A \oplus B = \{x \mid (x \in A) \oplus (x \in B)\}$$

In contrast to ordinary set difference, symmetric difference *is* commutative. Try proving the following identity:

$$A \oplus B = (A - B) \cup (B - A)$$

Cartesian products

A somewhat different type of set operation is the *Cartesian product*. This is named after Descartes, the famous French mathematician from the 17th century. Here, the word "product" is different from what we typically call a product between numbers, although we use the same symbol $\times$. (If you are familiar with SQL, this is the same as doing a *cross-join*.)

To define Cartesian products, we first introduce the concept of an *ordered pair*. An ordered pair, or a *tuple*, is a collection of two objects (a_1, a_2) such that a_1 is designated as the first element and a_2 is designated as the second element.

Note that this is different from the *set* $\{a_1, a_2\}$. Ordering of elements does not matter while talking about sets, i.e., $\{a_1, a_2\} = \{a_2, a_1\}$. However, order matters while talking about tuples; $(a_1, a_2) \neq (a_2, a_1)$. We will use the regular parentheses $(\cdot, \cdot)$ to denote tuples.

The notion of ordered pairs can be generalized to more than 2 elements. An ordered n*-tuple* is simply the ordered collection of elements $(a_1, a_2, a_3, \ldots, a_n)$.

We now introduce Cartesian products. Let A and B be two sets. Then, the Cartesian product of A and B is the set of all ordered pairs (a, b) such that a belongs to A and b belongs to B, i.e,.

$$A \times B = \{(a, b) \mid (a \in A) \wedge (b \in B)\}$$

For example, if S denotes the set of ISU students, and C denotes the set of computer engineering courses, then:

$$S \times C = \{(s, c) \mid s \in S, c \in C\}$$

denotes the set of all possible enrollments of students in computer engineering courses.