5.1 Continued

Spring-Mass System (Free damped motion):

Now we consider a damping force, usually modeled by a term proportional to the velocity: $F = \beta x'$ $(\beta > 0)$

Thus the resultant of the forces acting on m is as follows: $-Kx - \beta x'$

We get the differential equation: $m \cdot \frac{d^2x}{dt^2} = -Kx - \beta \frac{dx}{dt}$

$$\boxed{\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{K}{m}x = 0}$$

Free Damped motion D.E. Model.

let $2\lambda = \beta/m$ and $\omega^2 = K/m$ $\Rightarrow$ $\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$

Auxiliary Equation Roots: $\frac{-2\lambda \pm \sqrt{4\lambda^2 - 4\omega^2}}{2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$

Recall we have three cases (according to the discriminant $\lambda^2 - \omega^2$):

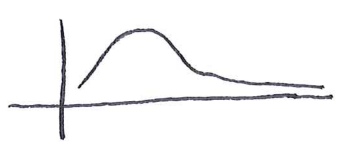
Case $\lambda^2 - \omega^2 > 0$: Two (real) distinct roots $r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$, the two l.i. solutions $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$

Note: $\sqrt{\lambda^2} > \sqrt{\lambda^2 - \omega^2}$, $\sqrt{\lambda^2} = \lambda$

$0 > -\lambda + \sqrt{\lambda^2 - \omega^2}$

General sol: $x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

Since $r_1, r_2 < 0$ we have $x(t) \xrightarrow[t \to \infty]{} 0$

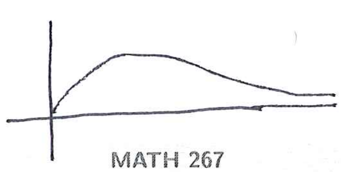


"overdamped motion"

Case $\lambda^2 - \omega^2 = 0$: One repeated root $r_1 = r_2 = -\lambda$, the general solution

$x(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t} \xrightarrow[t \to \infty]{} 0$

(*Verify with L'Hôpital's rule).

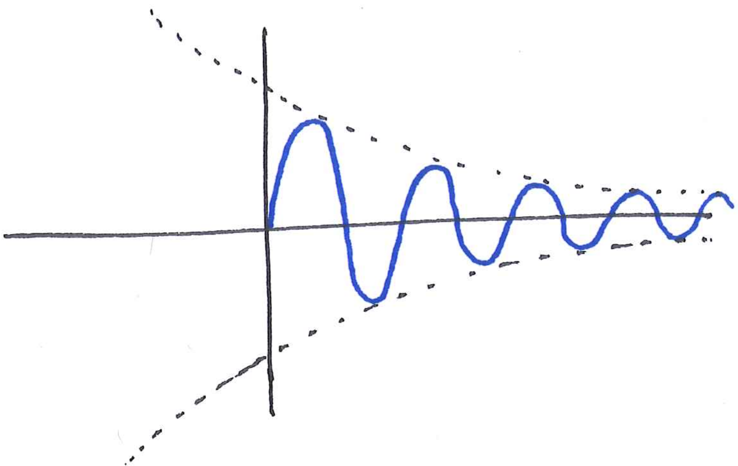


"critically damped motion"

Case $\lambda^2 - \omega^2 < 0$: Two complex conjugate roots $r_{1,2} = -\lambda \pm i\sqrt{|\lambda^2-\omega^2|}$

$\Rightarrow$ General Sol: $x(t) = c_1 e^{-\lambda t}\cos(\sqrt{|\lambda^2-\omega^2|}\,t) + c_2 e^{-\lambda t}\sin(\sqrt{|\lambda^2-\omega^2|}\,t)$

$$x(t) = e^{-\lambda t}\underbrace{\left(c_1\cos(\sqrt{|\lambda^2-\omega^2|}\,t) + c_2\sin(\sqrt{|\lambda^2-\omega^2|}\,t)\right)}_{\text{a sine graph. } A\sin(\sqrt{|\lambda^2-\omega^2|}\,t+\phi)}$$



Again $x(t) \xrightarrow[t\to\infty]{} 0$

"underdamped motion"

The nonhomogeneous case corresponds to a Driven of Forced Motion (not Free), where an external force f acts on the vibrating mass:

Equation: $$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = f$$

β could be zero or non zero (undamped or damped).

(see pg. 204-207)

Example

A mass weighing 8 pounds stretches a spring 2 feet. Assuming that a damping force numerically equal to 2 times the instantaneous velocity acts on the system, determine the equation of motion if the mass is initially released from the equilibrium position with an upward velocity of 3 ft/s. State the kind of motion the mass presents.

Hooke's Law : $F = Ks \iff 8 = K2 \Rightarrow K = 4$

Mass : $W = mg \Rightarrow 8 = m \cdot 32 \Rightarrow m = \frac{8}{32} = \frac{1}{4}$

Damping force $2x' = 2\frac{dx}{dt} \Rightarrow \beta = 2$

$$\frac{d^2x}{dt^2} + \frac{2}{1/4}\frac{dx}{dt} + \frac{4}{1/4}x = 0$$

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0 \Rightarrow \text{Aux. Eqn } r^2 + 8r + 16 = 0$$

$$(r+4)^2 = 0$$

$\Rightarrow r = -4$ repeated.

General Sol : $x(t) = C_1 e^{-4t} + C_2 t e^{-4t}$

$$x'(t) = -4C_1 e^{-4t} + C_2 e^{-4t} - 4C_2 t e^{-4t}$$

Plug in initial conditions $x(0) = 0$, $x'(0) = -3$

$x(0) = C_1 = 0$

$x'(0) = \cancel{-4C_1} + C_2 = -3$

$\therefore$ Sol. (Equation of Motion)

$$x(t) = -3te^{-4t}$$

"critically damped motion".

Review for Exam 2

Topics for Exam 2 (All sections covered in class except for 3.1 and 3.2):

From Chapter 4 you should know:

- How to find the Wronskian • Know meaning: l.i., fundamental set.
- How to find y_c the general sol to $ay'' + by' + cy = 0$
- How to find y_p a particular solution of $ay'' + by' + cy = g(x)$
  - Undetermined Coefficients
  - Variation of Parameters
  - Superposition Principle
- How to find the general solution of $ay'' + by' + c = g(x)$
- How to solve a Cauchy-Euler Equation (only homogeneous)
- How to solve Initial Value Problems (IVPs)

5.1 Application problems (**free** undamped and damped motion cases).

One of the most comprehensive types of problems (2^{nd} order DE) is solving a nonhomogeneous IVP:

$$ay'' + by' + cy = g(x), \quad y(0) = y_0, \ y'(0) = y_1.$$

1st We find y_c, that is, find the general solution of $ay'' + by' + cy = 0$

auxiliary Equation: $am^2 + bm + c = 0$ yields three cases:

Case 1 Two distinct (real) roots $m_1 \neq m_2 \Rightarrow y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

Case 2 One repeated root $m_1 = m_2 \Rightarrow y_c = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$

Case 3 Two complex conjugate roots $m_{1,2} = \alpha \pm i\beta$

$$y_c = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

2nd We find a particular solution y_p, if $g(x)$ allows us to use undetermined coefficients remember the form of y_p:

	Form of $g(x)$	Form of y_p
i	$c_0 + c_1x + \cdots + c_nx^n$	$A_0 + A_1x + \cdots + A_nx^n$
ii	$ce^{\alpha x}$	$A\,x^t e^{\alpha x}$
iii	$c_1 \sin \beta x + c_2 \cos \beta x$	$x^t(A\cos\beta x + B\sin\beta x)$
iv	$e^{\alpha x}(c_1 \sin\beta x + c_2\cos\beta x)$	$x^t e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

Where t will be $0, 1$ or 2 depending on the roots of the auxiliary equation:

- In case *ii*: $t=0$ if $\alpha \neq$ root of aux. equ, $t=1$ if $\alpha=$ root; $t=2$ if $\alpha=$ repeated root.
- In case *iii*: $t=1$ if root of aux equ. $m=\pm\beta i$; $t=0$ otherwise.
- In case *iv*: $t=1$ if $m=\alpha\pm\beta i$; $t=0$ otherwise.

For the given DE and $g(x)$ determine the form of y_p required for undetermined coefficients method:

I. $y'' + 9y = g(x)$ $\Rightarrow$ $m=\pm 3i$ $(y_1=\cos 3x, y_2=\sin 3x)$
- $g(x) = 2\cos 3x$ $\Rightarrow y_p = x(A\cos 3x + B\sin 3x)$
- $g(x) = 5e^{2x}\sin 3x \Rightarrow y_p = e^{2x}(A\cos 3x + B\sin 3x)$
- $g(x) = 3e^{-x}$ $\Rightarrow y_p = Ae^{-x}$

II. $y'' - 4y' + 9y = g(x) \rightarrow m = 2\pm\sqrt{5}i$
- $g(x) = e^{2x}$ $\rightarrow y_p = Ae^{2x}$
- $g(x) = \cos(\sqrt{5}x)$ $\rightarrow y_p = A\cos(\sqrt{5}x) + B\sin(\sqrt{5}x)$
- $g(x) = e^{2x}\sin(\sqrt{5}x)$ $y_p = xe^{2x}(A\cos(\sqrt{5}x) + B\sin(\sqrt{5}x))$
- $g(x) = 3x^2$ $\rightarrow y_p = A_0 + A_1x + A_2x^2$

III. $y'' = g(x)$ $\rightarrow m=0$ repeated $y_1 = 1$ $y_2 = x$
- $g(x) = e^{-x/2}$ $\leftarrow y_p = Ae^{-x/2}$
- $g(x) = x$ $\leftarrow y_p = x^2(A_0 + A_1x)$

$\uparrow x^2$, because $m=0$ is repeated root!