Probabilistic Inference

Outline

I. Probability for continuous variables

II. Inference by enumeration

* Figures are either from the textbook site or by the instructor.

I. Probability Density Function

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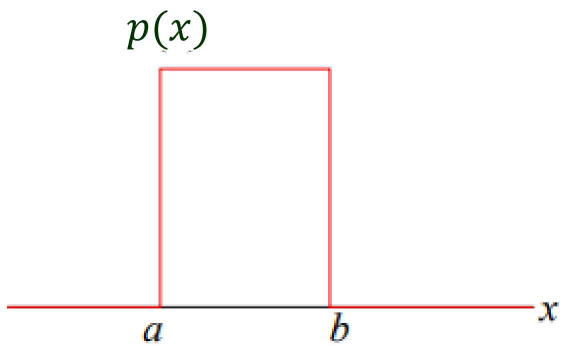
$$= \lim_{dx \to 0} \frac{P(x \le X \le x + dx)}{dx}$$

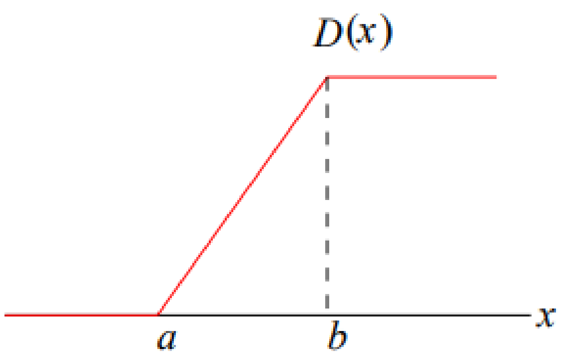
Uniform Distribution

A *uniform distribution* has constant p.d.f.

Example Continuous uniform distribution over the interval $[a, b]$.

$$p(x) = \begin{cases} 0, & \text{for } x < a, \\ \frac{1}{b-a}, & \text{for } a \le x < b, \\ 0, & \text{for } x > a, \end{cases} \qquad D(x) = \begin{cases} 0, & \text{for } x < a, \\ \frac{x-a}{b-a}, & \text{for } a \le x < b, \\ 1, & \text{for } x > b, \end{cases}$$





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Example Roll a die an infinite number of times. Each number appears 1/6 of the time.

$$E(X) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{6} i \cdot \frac{n}{6} = \frac{7}{2}$$

Variance and Standard Deviation

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The *standard deviation* of X is $\sigma = \sqrt{\text{var}(X)}$.

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Gaussian Distribution

A continuous random variable X with mean μ and variance σ^2 has *Gaussian distribution* (or *normal distribution*) if its p.d.f. is

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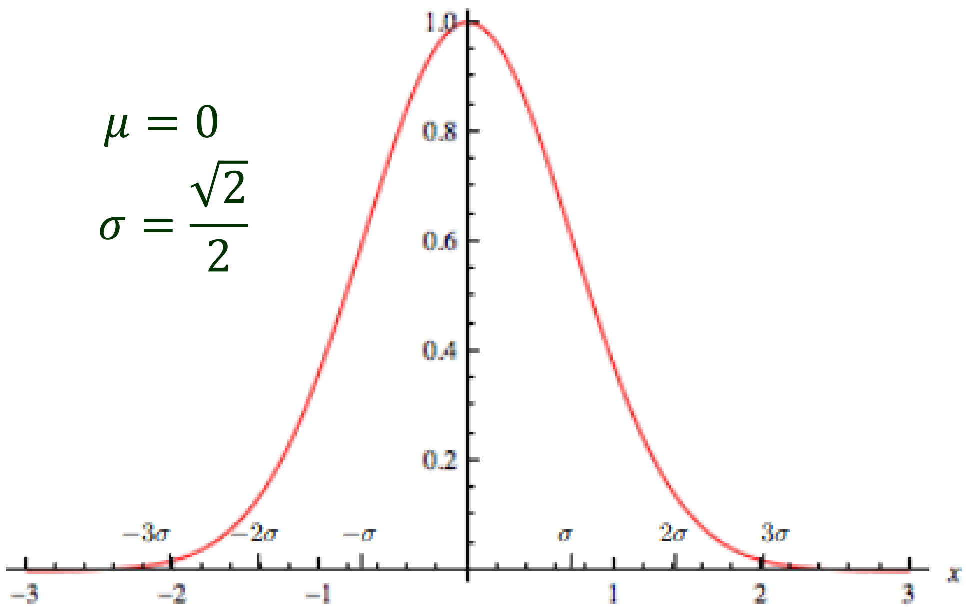

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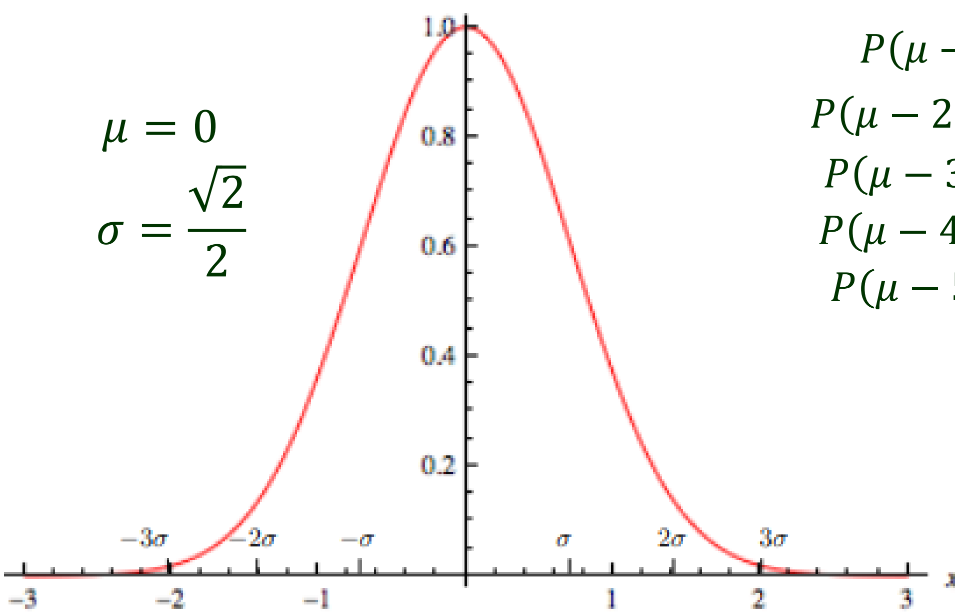

$\mu = 0$

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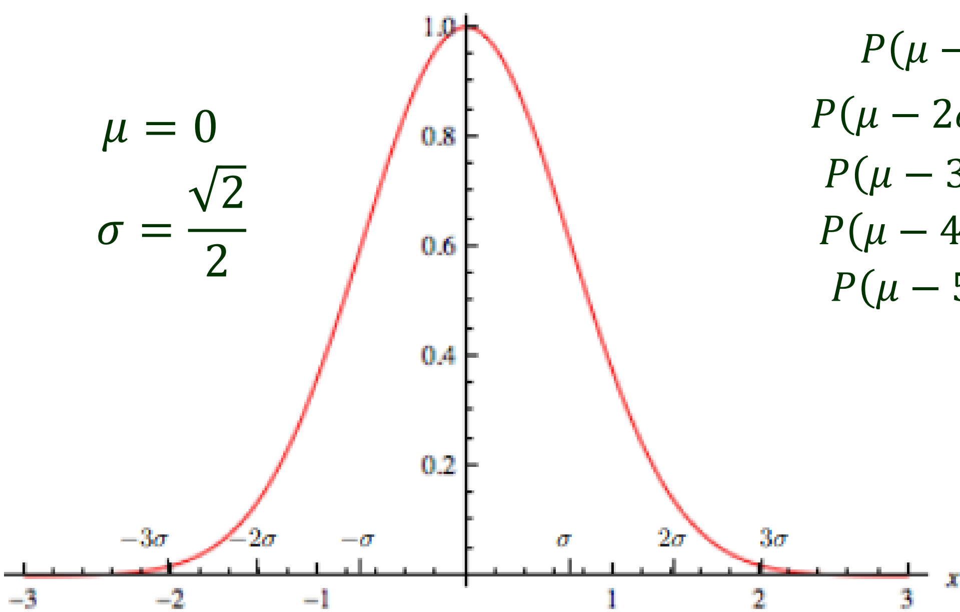

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Three-sigma rule (in practice): Consider population within $(\mu - 3\sigma, \mu + 3\sigma)$.

Why Gaussian Distribution?

- It is the most important distribution because it fits many natural phenomena (e.g., human characteristics such as weight, height, body temperature, etc.)

- It is the limiting distribution of $X_1 + \cdots + X_n$ of n independent random variables $X_1, \ldots, X_n$, as $n \to \infty$, explaining a characteristic impacted by numerous independent factors (by the central limit theorem).

- It is the foundation for important methods such as least-squares, Kalman filters that are used in statistics and engineering.

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Sums over all possible combinations of the values of the set of variables $\boldsymbol{Z}$.

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$$\boldsymbol{P}(Cavity) = \boldsymbol{P}(Cavity, toothache, catch) + \boldsymbol{P}(Cavity, toothache, \neg catch) + \boldsymbol{P}(Cavity, \neg toothache, catch) + \boldsymbol{P}(Cavity, \neg toothache, \neg catch)$$

random variable beginning with an uppercase letter.

$$= \langle 0.108, 0.016 \rangle + \langle 0.012, 0.064 \rangle + \langle 0.072, 0.144 \rangle + \langle 0.008, 0.576 \rangle$$

$$= \langle 0.2, 0.8 \rangle$$

not bold-faced

Rule of *conditioning*:

$$\boldsymbol{P}(\boldsymbol{Y}) = \sum_{\boldsymbol{z}} \underbrace{\boldsymbol{P}(\boldsymbol{Y} \mid \boldsymbol{z}) P(\boldsymbol{z})}_{= \boldsymbol{P}(\boldsymbol{Y}, \boldsymbol{z})}$$

	toothache		*¬toothache*	
	catch	*¬catch*	*catch*	*¬catch*
cavity	0.108	0.012	0.072	0.008
¬cavity	0.016	0.064	0.144	0.576

Computing Conditional Probabilities

1. $P(a \mid b) = \dfrac{P(a \wedge b)}{P(b)}$ $\Longrightarrow$ an expression in terms of unconditional probabilities.
2. Evaluate the expression from full joint distribution.

	$toothache$		$\neg toothache$	
	$catch$	$\neg catch$	$catch$	$\neg catch$
$cavity$	0.108	0.012	0.072	0.008
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$$= \frac{0.108 + 0.012}{0.108 + 0.012 + 0.016 + 0.064} = 0.6$$

	toothache		$\neg$*toothache*	
	catch	$\neg$*catch*	*catch*	$\neg$*catch*
cavity	$\boxed{0.108}$	$\boxed{0.012}$	0.072	0.008
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$$P(\neg cavity \mid toothache) = \frac{P(\neg cavity \wedge toothache)}{P(toothache)}$$

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	catch	*¬catch*	*catch*	*¬catch*
cavity	$\boxed{0.108}$	$\boxed{0.012}$	0.072	0.008
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	catch	$\neg$*catch*	*catch*	$\neg$*catch*
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$$P(\neg cavity \mid toothache) = \frac{P(\neg cavity \wedge toothache)}{P(toothache)}$$

$$= \frac{0.012 + 0.064}{0.108 + 0.012 + 0.016 + 0.064} = 0.4$$

	toothache		*¬toothache*	
	catch	*¬catch*	*catch*	*¬catch*
cavity	0.108	0.012	0.072	0.008
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Normalization

$$P(cavity \mid toothache) = \frac{P(cavity \land toothache)}{P(toothache)} = 0.6$$

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$$\boldsymbol{P}(Cavity \mid toothache) = \alpha\, \boldsymbol{P}(Cavity, toothache)$$

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$$\mathbf{P}(\mathit{Cavity} \mid \mathit{toothache}) = \alpha\, \mathbf{P}(\mathit{Cavity},\ \mathit{toothache})$$

$$= \alpha\, [\mathbf{P}(\mathit{Cavity},\ \mathit{toothache}, \mathit{catch}) + \mathbf{P}(\mathit{Cavity},\ \mathit{toothache}, \neg\mathit{catch})]$$

Normalization

$$P(cavity \mid toothache) = \frac{P(cavity \wedge toothache)}{P(toothache)} = 0.6$$

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$$= \alpha\, [\langle 0.108, 0.016\rangle + \langle 0.012, 0.064\rangle]$$

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&= \alpha\, [\langle 0.108, 0.016 \rangle + \langle 0.012, 0.064 \rangle] \\
&= \alpha\, \langle 0.12, 0.008 \rangle
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&= \alpha\, \langle 0.12, 0.008 \rangle \\
&= \langle 0.6, 0.4 \rangle
\end{aligned}$$

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$$= \langle 0.6, 0.4\rangle$$

// $\alpha = 5 = 1/P(toothache)$.

Inference Procedure

- X: single query variable (e.g., *Cavity*).
- $\boldsymbol{E}$: evidence variables (e.g., *Toothache*).
- $\boldsymbol{e}$: their observed values.
- $\boldsymbol{Y}$: unobserved variables (e.g., *Catch*).

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- $\boldsymbol{E}$: evidence variables (e.g., *Toothache*).
- $\boldsymbol{e}$: their observed values.
- $\boldsymbol{Y}$: unobserved variables (e.g., *Catch*).

$$\boldsymbol{P}(X \mid \boldsymbol{e}) \leftarrow \alpha \boldsymbol{P}(X \mid \boldsymbol{e}) = \alpha \sum_{\boldsymbol{y}} \boldsymbol{P}(X, \boldsymbol{e}, \boldsymbol{y})$$

♣ Summation is over all possible combinations $\boldsymbol{y}$ of values of variables in $\boldsymbol{Y}$.

♣ $\boldsymbol{P}(X, \boldsymbol{e}, \boldsymbol{y})$ is a subset of probabilities from the full joint distribution.

♣ The full joint distribution has size exponential in # variables and is rarely computed.

Independence

Add a fourth variable *Weather* with domain { *sun*, *rain*, *cloud*, *snow* }.

$$\mathbf{P}(Toothache, Catch, Cavity, Weather)$$

	toothache		*¬toothache*	
	catch	*¬catch*	*catch*	*¬catch*
cavity	0.108	0.012	0.072	0.008
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⇨ A 32-element table.

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$$P(toothache, catch, cavity, cloud)$$

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$$\mathbf{P}(Toothache, Catch, Cavity, Weather)$$

$$P(toothache, catch, cavity, cloud)$$

$$= P(cloud \mid toothache, catch, cavity)\, P(toothache, catch, cavity)$$

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$$P(toothache, catch, cavity, cloud)$$

$$= P(cloud \mid toothache, catch, cavity)\, P(toothache, catch, cavity)$$

$$\Downarrow \quad P(cloud \mid toothache, catch, cavity) = P(cloud)$$

$$= P(cloud)\, P(toothache, catch, cavity)$$

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$$\mathbf{P}(Toothache, Catch, Cavity, Weather) = \mathbf{P}(Toothache, Catch, Cavity)\, \mathbf{P}(Weather)$$

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$$P(toothache, catch, cavity, cloud)$$

$$= P(cloud \mid toothache, catch, cavity)\, P(toothache, catch, cavity)$$

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$$= P(cloud\,)\, P(toothache, catch, cavity)$$

$$\mathbf{P}(Toothache, Catch, Cavity, Weather) = \mathbf{P}(Toothache, Catch, Cavity)\, \mathbf{P}(Weather)$$

8-element table + 4-element table

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cavity	0.108	0.012	0.072	0.008
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$\Rightarrow$ ~~A 32-element table.~~

Independent Variables

- Two propositions a and b are *independent* if

$$P(a \mid b) = P(a) \quad \text{or} \quad P(b \mid a) = P(b) \quad \text{or} \quad P(a \wedge b) = P(a)P(b)$$

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- Two variables X and Y are *independent* if

$$\boldsymbol{P}(X \mid Y) = \boldsymbol{P}(X) \quad \text{or} \quad \boldsymbol{P}(Y \mid X) = \boldsymbol{P}(Y) \quad \text{or} \quad \boldsymbol{P}(X \wedge Y) = \boldsymbol{P}(X)P(Y)$$

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- Two continuous variables X and Y are *independent* if, for all $x, y \in \mathbb{R}$,

$$P(X \leq x \wedge Y \leq y) = P(X \leq x) \cdot P(Y \leq y)$$

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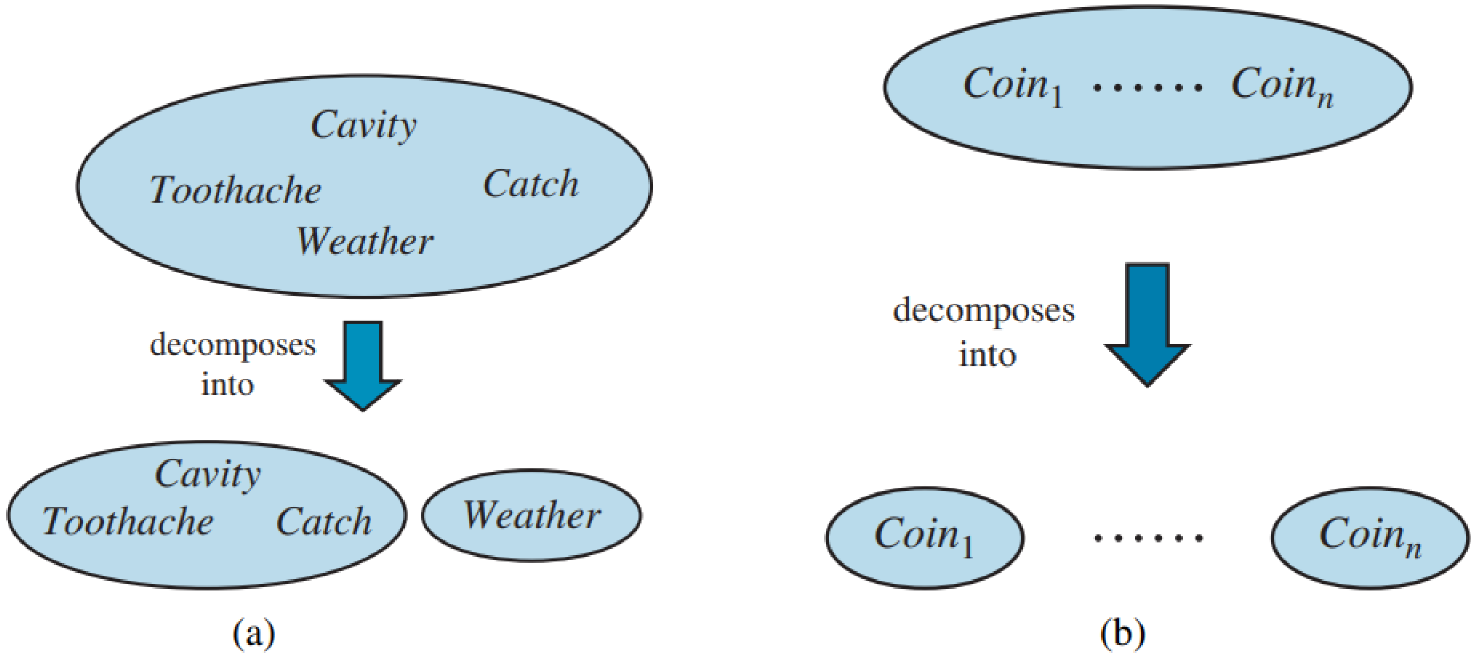
The joint probability density function $p(x, y)$ satisfies

$$p(x, y) = \int_{-\infty}^{\infty} p(x, y)\, dy \int_{-\infty}^{\infty} p(x, y)\, dx$$

Factoring a Joint Distribution

The full joint distribution can be factored into *separate* joint distributions on subsets of variables that are *independent*.



Weather and dental problems are independent.

Coin flips are independent.