Lecture 18: Mathematical Induction

We now present a powerful proof technique, known as *mathematical induction*. Indeed, this will be perhaps the single most important concept that you would need to pick up in CPRE 310.

Induction, simply put, is a technique that will be used to prove that a given property is true for all nonnegative integers. We will see that induction provides an elegant mechanism to prove seemingly-difficult statements involving sequences, summations, and the like.

The best way to understand the method of induction is using the following example: suppose (after a particularly good showing by the class in the midterm) a hypothetical CPRE 310 professor wants to distribute candy among all the students. The rule by which he distributes them:

- Line up all the students in a sequence.
- The first student gets a piece of candy.
- If a student gets candy, then the next student in line also gets candy.

Simple enough, and it is clear that eventually everyone in line will get candy. Now, suppose we wish to **prove** a statement of the form:

Student #116 will get the candy bar.

Obviously, the statement is true. But how do we (formally) prove it? The idea is to use the following sequence of arguments:

- It is true that Student 0 gets candy.
- Student 0 gets candy $\implies$ Student 1 gets candy.
- Student 1 gets candy $\implies$ Student 2 gets candy.
- ...
- Student 115 gets candy $\implies$ Student 116 gets candy.

If we observe the above reasoning carefully, we realize this is nothing but *modus ponens* applied a bunch of times. In fact, for any n (up to the size of the class), it is true that Student n will receive that candy. Induction is basically an abstraction of this idea.

A good mental picture to have is imagine a sequence of dominoes stacked up one next to the other. If the first domino falls, then eventually the n^{th} domino also will fall for any $n \geq 1$.

The principle of induction

Let us be a bit abstract in the beginning. Let $P(n)$ be a predicate defined on the nonnegative integers. If

- (Base case) $P(0)$ is true, and
- (Inductive step) $P(n) \implies P(n+1) \quad \forall n \geq 0$

then

- $P(m)$ is true for all nonnegative integers.

In terms of predicate logic, we can write the induction principle as the following *rule of inference*:

$$\begin{array}{l} P(0) \\ \forall k \geq 0,\ P(k) \implies P(k+1) \\ \hline \therefore\ \forall n,\ P(n) \end{array}$$

To be more precise, the above rule of inference is called "ordinary induction." Later we will talk about a different flavor of induction called "strong induction" which is a variant of the above idea.

The validity of the above rule may seem fairly obvious. However, this is essentially the structure of all induction proofs;

- establish the *base case* ($P(0)$) and
- establish the *induction step* $P(k) \implies P(k+1)$ for any arbitrary k.

A few examples will show how truly powerful induction is.

Induction: An example

Recall that the summation of the first n nonnegative integers is given by the following closed expression:

$$0 + 1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

Let $P(n)$ be the predicate that the summation of the first n numbers is $n(n+1)/2$. We need to prove that $P(n)$ is true for all values of n.

To do this, we need to establish two simple arguments:

- (Base case) Show that the statement is true for $n = 0$. To prove this, we observe for $n = 0$, the left hand side of the summation is zero, while the right hand side is $0(0+1)/2 = 0$. Therefore, the statement $P(n)$ is true for $n = 0$.

- (Induction step) Now consider $P(k)$ for **some generic** $k \geq 0$. We need to show that if $P(k)$ is true then necessarily $P(k+1)$ also has to be true. Suppose $P(k)$ is true, i.e.:

  $$0 + 1 + 2 + \ldots + k = \frac{k(k+1)}{2}$$

  Starting from here, we need to somehow deduce that $P(k+1)$ is true, i.e.,

  $$0 + 1 + 2 + \ldots + k + k + 1 = \frac{(k+1)(k+2)}{2}$$

In order to do this, we start from the left hand side of the first expression for $0 + 1 + \ldots k$, add $k + 1$ to both sides, and simplify:

$$\begin{aligned}0 + 1 + \ldots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}\end{aligned}$$

which is precisely the right hand side of the second expression. Therefore, if $P(k)$ is true, then so is $P(k+1)$.

Combining the above arguments, the principle of induction tells us that $P(n)$ is true for all nonnegative integers n.

Template for induction proofs

A (well-written) proof of induction will have the following essential components.

- Begin by saying "We prove this statement by induction."
- Define an appropriate **predicate** $P(n)$. This is called the *induction hypothesis.* Identifying the correct induction hypothesis is the most important part of any induction proof; a well-modeled induction hypothesis can greatly simplify the problem, while a poorly modeled hypothesis can lead you down a rather deep rabbit hole. Typically, the induction hypothesis pops out of the statement, but some other times you need to carefully model it based on the problem that you are trying to solve.
- Prove that the **base case** $P(0)$ is true. Clearly mark it. End this part by stating "This establishes the base case."
- Show that the **induction step** is true, i.e., use your assumption that $P(n)$ is true to show that $P(n+1)$ is also true. Usually this involves the most work, but End this part by stating "this completes the induction step."
- Conclude by invoking the induction principle. "By induction, $P(n)$ is true for all n."

More proofs by induction

Let us prove 3 different examples of inductive proofs. They are all pretty similar.

1. Suppose we are given a recurrence:

$$f(n) = \begin{cases} 1, & n = 0 \\ 2f(n-1), & n > 0. \end{cases}$$

We wish to prove that for all $n \geq 0$, we have $f(n) = 2^n$. Let $P(n)$ denote the assertion (predicate) that $f(n) = 2^n$. Clearly, $P(0)$ is true since $2^0 = 1$. Suppose we assume that $P(k)$ is true for some k, i.e., $f(k) = 2^k$. Then, multiplying by 2 on both sides, we get

$$2f(k) = 2^{k+1}.$$

However, by definition of the recursion, we have $f(k+1) = 2^{k+1}$. This means that $P(k+1)$ is also true. Therefore, by induction, $P(n)$ is true for all $n \geq 0$.

2. Suppose we wish to prove that $n^2 + n + 42$ is even for all $n \geq 0$. Let $P(n)$ be the predicate that $n^2 + n + 42$ is zero. Clearly, $P(0)$ is true since $0^2 + 0 + 42 = 42$, which is even. Suppose that $P(k)$ is true for some k, i.e., $k^2 + k + 42$ is even. We need to show that $P(k+1)$ is true, i.e., $(k+1)^2 + (k+1) + 42$ is even. Simplifying,

$$(k+1)^2 + (k+1) + 42 = k^2 + 2k + 1 + k + 1 + 42 = k^2 + k + 42 + 2k + 2.$$

   But we already know that $k^2 + k + 42$ is even (by the induction hypothesis) and $2k + 2$ is even (by definition of "even"). Combining these two facts, we get that $(k+1)^2 + k + 1 + 42$ is even. Therefore, via induction, $P(n)$ is true for all n.

3. Suppose we wish to prove that the sum of the first n *odd* positive integers is given by n^2.

$$1 + 3 + 5 + \ldots + 2n - 1 = n^2.$$

   Try proving this by a straightforward (but interesting) **exercise**.

Faulty induction proof

Using induction, let us now prove a statement that is obviously false:

All horses in the world have the same color.

Here is the "proof".

Let $P(n)$ be the predicate that the number of distinct colors in any set of $n \geq 1$ horses in the world is exactly 1. Then $P(n)$ is true for all n.

- (Base case) $n = 1$ is trivially true since any set containing 1 horse contains exactly 1 color.
- (Induction hypothesis) Consider any arbitrary set of k horses, and assume that they are all of the same color.
- (Induction step) We have to show that any arbitrary set of $k+1$ horses, $\{H_1, H_2, \ldots, H_k, H_{k+1}\}$ have the same color. Consider the first k horses $H_1, \ldots, H_k$. By the induction hypothesis, they are of the same color. Therefore, H_1 is of the same color as $H_2, \ldots, H_k$. Consider the last k horses $H_2, \ldots, H_{k+1}$. Again, by hypothesis they are of the same color. Therefore,, H_{k+1} is the same color as $H_2, \ldots, H_k$.
- Therefore the set of horses $\{H_1, H_2, \ldots, H_{k+1}\}$ are all of 1 color. Done!

Where is the fallacy? Clearly the statement is false since we have black, brown, and white horses in the world.

The fallacy lies in the following (very subtle) flaw in the induction step. The base case is for $n = 1$. However, the induction step *only* works for $n \geq 2$; otherwise, there is no meaning to the statement "H_1 is of the same color as $H_2, \ldots, H_n$".

In terms of symbolic logic, we have proved that $P(1)$ is true, and we have proved that $P(n) \implies P(n+1)$ for $n \geq 2$. But the crucial link $P(1) \implies P(2)$ is missing, and therefore the entire argument falls apart.

Moral of the story: it is easy to go wrong while doing induction!