Lecture 25: Counting (fin)

A very good way to practice counting rules is to perform *card counting*. Recall that a standard card deck consists of 52 cards, with 13 ranks (Ace through King) and 4 suits (clubs, hearts, spades, diamonds).

Five-card draw is a variant of poker where each player is dealt a hand of 5 cards from the deck (the order of the cards in the hand doesn't matter). Different hands have different values, and as a rule, the rarer hands have higher value. Let's do some card counting practice in five-card draw.

> What is the total number of possible hands?

This one is easy! We have 52 cards and we select 5 cards to form a hand, so by the Selection Principle, the answer is given by $\binom{52}{5}$, which is roughly 2.5 million.

In poker, some hands are more valuable than others. A *very* valuable hand is known as a *straight flush*, where all five cards in the hand are the same suit (i.e., the hand is a "flush"), and their ranks are in order (i.e., the hand is a "straight"). So next question:

> How many hands are a straight flush?

This one is a direct application of the product rule. There are 4 suits. Moreover, since the hand is a straight, it is uniquely determined by the lowest-rank card, which can range from an A (i.e, an A-2-3-4-5) all the way to a 10 (i.e., a 10-J-Q-K-A). So there are 10 choices for the straight. By the product rule, we get $4 \times 10 = 40$ possible straight flushes.

Next: A *four-of-a-kind* is a 5-card hand where 4 cards have the same rank. (e.g., 8Spades-AceHearts-8Clubs-8Diamonds-8Hearts is a four-of-a-kind since we have 4 eights).

> How many hands contain a four-of-a-kind?

Now it becomes interesting, but really just the product rule again. Here is the reasoning. There are 13 ways to choose the rank of the four-of-a-kind. Once you do so, you have to pick all 4 suits of that rank. The remaining card can be any other card from the other 12 ranks and the 4 suits. Therefore, the total number is given by:

$$13 \times 12 \times 4 = 624.$$

Next: A *full house* is a 5-card hand where 3 cards share one rank and the remaining two share the other hand. (e.g., 3Spades- 3Diamonds- KingDiamonds- KingClubs- KingHearts) is a full house.

> How many hands are a full house?

Here we have to use both the product rule and the selection principle. Each full house can be specified using:

- the rank of the triple, which can be chosen in 13 ways.
- the 3 suits of the triple, which can be chosen in $\binom{4}{3}$ ways.
- the rank of the pair, which can be chosen in 12 ways (*not 13* – you have already used up one rank for the triple)

- the 2 suits of the pair, which can be chosen in $\binom{4}{2}$ ways.

Therefore, the number of full houses, via the product rule, is

$$13 \times \binom{4}{3} \times 12 \times \binom{4}{2}.$$

Counting in two ways

Combinatorial proofs usually involve counting the elements of some set in two different ways, and equating the results. This notion of "counting-by-two-ways" is a general technique used in counting problems.

For example, let us try to prove the following identity:

$2^n = \sum_{i=0}^{n} \binom{n}{i}$.

As with all counting proofs, we need to construct a counting problem, calculate the answer in two different ways, and equate the results.

The left hand side of the identity should be familiar by now: 2^n is the number of subsets of a set of size n. This should give us some idea of where to start.

So here is the proof: consider the problem of counting the number of subsets of a given set X containing n elements. We know that this number equals 2^n.

However, we can also count the number of subsets in a different way. Any given subset can have size 0, or 1, or 2, . . . or n. Moreover,

- The number of subsets of size 0 is $\binom{n}{0}$.
- The number of subsets of size 1 is $\binom{n}{1}$.
- The number of subsets of size 2 is $\binom{n}{2}$.
- . . .
- The number of subsets of size n is $\binom{n}{n}$.

Therefore, by the sum rule (since the above cases are mutually exclusive), we get that the total number is given by $\sum_{i=0}^{n} \binom{n}{i}$. Equating this to our first answer (2^n), we complete the proof.

A different, *algebraic* proof of the above identity can be derived as follows. From elementary algebra, we know the following facts about polynomials:

- $(1 + x)^2 = 1 + 2x + x^2$
- $(1 + x)^3 = 1 + 3x + 3x^2 + x^3$
- $(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$
- . . .

In fact, these are all special cases of a more general result called the *Binomial theorem*:

$$(1 + x)^n = \sum_{i=0} \binom{n}{i} x^i.$$

which itself can be derived using induction or other means. Plugging in $x = 1$ in the above formula, we get the desired identity.

Here is a (somewhat) more challenging **exercise**: can you come up with a combinatorial proof of the following equation?

$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$.

(A hint to get you started: imagine selecting a subset of size $r + 1$ given a set of n numbers, and focus on the *biggest* number that you select.)