Stat 330 (Spring 2020): Homework 9

Due: April 8, 2020

Show all of your work, and *please* staple your assignment if you use more than one sheet. Write your name, the course number and the section on every sheet. Problems marked with * will be graded and one additional randomly chosen problem will be graded.

1. Every day, Eric takes the same street from his home to the university. There are 4 street lights along his way, and Eric has noticed the following Markov dependence. If he sees a green light at an intersection, then 60% of time the next light is also green, and 40% of time the next light is red. However, if he sees a red light, then 75% of time the next light is also red, and 25% of time the next light is green. Let 1 = "green light" and 2 = "red light" with the state space $\{1, 2\}$.

   (a) Construct the 1-step transition probability matrix for the street lights.

   (b) If the first light is red, what is the probability that the third light is red?

   (c) Eric's classmate Jacob has *many* street lights between his home and the university. If the *first* street light is red, what is the probability that the *last* street light is red? (Use the steady-state distribution.)

   Answer:

   (a) Let $X(n) = 1$ is the n^{th} light is green; $X(n) = 2$ if it is red.

   $$P = \begin{bmatrix} 0.60 & 0.40 \\ 0.25 & 0.75 \end{bmatrix}$$

   (b) Since the first light is red, the initial state is $P_0 = [0 \;\; 1]$.
   To predict the third light, we predict 2-steps ahead. The 2-step transition probability matrix is

   $$P^{(2)} = P \cdot P \begin{bmatrix} 0.60 & 0.40 \\ 0.25 & 0.75 \end{bmatrix} \begin{bmatrix} 0.60 & 0.40 \\ 0.25 & 0.75 \end{bmatrix} = \begin{bmatrix} 0.46 & 0.54 \\ 0.3375 & 0.6625 \end{bmatrix}$$

   Prediction: $P_2 = P_0 P^{(2)} = [0 \;\; 1] \begin{bmatrix} 0.46 & 0.54 \\ 0.3375 & 0.6625 \end{bmatrix} = [0.3375 \;\; 0.6625]$
   The probability that the third light is red is 0.6625.

   (c) We need to find $\pi_2 = \lim_{n\to\infty}\{X(n) = 2\}$. The steady state distribution $\boldsymbol{\pi} = [\pi_1 \;\; \pi_2]$ is found by solving the system of equations given by

   (i) $\boldsymbol{\pi} P = \boldsymbol{\pi} \to [\pi_1 \;\; \pi_2] \begin{bmatrix} 0.60 & 0.40 \\ 0.25 & 0.75 \end{bmatrix} = [\pi_1 \;\; \pi_2]$

   (ii) $\sum \pi_x = 1 \to \pi_1 + \pi_2 = 1$

   (i) gives the relation $(0.6\pi_1 + 0.25\pi_2 = \pi_1)$ and $(0.4\pi_1 + 0.75\pi_2 = \pi_2) \to \pi_2 = 1.6\pi_1$
   (ii) gives the relation $\pi_1 + \pi_2 = 1 \to \pi_2 = 1 - \pi_1$

   Combining the above into a system of equations, we solve for π_1 and π_2 to obtain

   $\pi_1 = 0.3846$ is the long run probability that the light is green
   $\pi_2 = 0.6154$ is the long run probability that the light is red.

   (Alternatively, we can multiply $P \cdot P \cdot P \cdots P$ until convergence and arrive at the same answer.)

   So, $\pi_2 = 0.6154$ is the probability that the last light is red.

2. * We want to model the daily movement of a particular stock (say Amazon, ticker = AMZN) using a homogenous markov-chain. Suppose at the close of the market each day, the stock can end up higher or lower than the previous day's close. Assume that if the stock closes higher on a day, the probability that it closes higher the next day is 0.58. If the stock closes lower on a day, the probability that it closes higher the next day is 0.46.

(a) What is the 1-step transition matrix? (Let 1 = higher, 2 = lower)

Answer:

$$P = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0.58 & 0.42 \\ 0.46 & 0.54 \end{pmatrix} \end{array}$$

(b) On Monday, the stock closed higher. What is the probability that it will close higher on Thursday (three days later)

Answer:

$$P^{(3)} = P \cdot P \cdot P = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0.5236 & 0.4764 \\ 0.5218 & 0.4782 \end{pmatrix} \end{array}$$

The distribution of X three steps ahead is:

$P_3 = P_0 \cdot P^{(3)} = (1, 0) \cdot \begin{pmatrix} 0.5236 & 0.4764 \\ 0.5218 & 0.4782 \end{pmatrix} = (0.5236, 0.4764)$

So, the probability it closes higher Thursday (three steps in the future) is 0.5236.

3. * A Markov chain has 3 possible states: A, B, and C. Every hour, it makes a transition to a *different* state. From state A, transitions to states B and C are equally likely. From state B, transitions to states A and C are equally likely. From state C, it always makes a transition to state A.

(a) Write down the transition probability matrix.

(b) If the initial distribution for states A, B, and C is $P_0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, find the distribution of state after 2 transitions, i.e., the distribution of X_2.

(c) Show that this is a regular Markov Chain.

(d) Find the steady-state distribution of states.

Answer:

(a)

$$P = \begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 0 & .5 & .5 \\ .5 & 0 & .5 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

(b) $P_2 = P_0 \times P^2$ where

$$P = \begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 0 & .5 & .5 \\ .5 & 0 & .5 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

and

$$P^2 = \begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} .75 & 0 & .25 \\ .5 & .25 & .25 \\ 0 & 0.5 & 0.5 \end{pmatrix} \end{array}$$

Thus $P_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \times P^2 = (.4167, .25, .3333)$

(c) We have

$$P^3 = \begin{pmatrix} 0.250 & 0.375 & 0.375 \\ 0.375 & 0.250 & 0.375 \\ 0.750 & 0.000 & 0.250 \end{pmatrix}, P^4 = \begin{pmatrix} 0.5625 & 0.1250 & 0.3125 \\ 0.5000 & 0.1875 & 0.3125 \\ 0.2500 & 0.3750 & 0.3750 \end{pmatrix}$$

Since all of the entries in P^4 are positive, this is a regular markov chain

(d)

$$\begin{cases} .5\pi_B + \pi_C = \pi_A \\ .5\pi_A = \pi_B \\ .5\pi_A + .5\pi_B = \pi_C \\ \pi_A + \pi_B + \pi_C = 1 \end{cases} \implies \begin{cases} \pi_A = 2\pi_B \\ \pi_C = 1.5\pi_B \\ 2\pi_B + \pi_B + 1.5\pi_B = 1 \end{cases} \implies \pi_B = 2/9, \quad \pi_A = 4/9, \quad \pi_C = 3/9$$

4. Every second in a hockey game, we recorded the possession status of a hockey puck where the possibilities are that team A possesses the puck, team B possesses the puck, or nobody possesses the puck (this is called a loose puck). Then a Markov chain model of the possession status of the puck is

$$P = \begin{array}{c} \\ A \\ B \\ L \end{array} \begin{array}{c} \begin{array}{ccc} A & B & L \end{array} \\ \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.3 \\ 0.5 & 0.4 & 0.1 \end{pmatrix} \end{array}$$

(a) What is the probability that team A retains possession of the puck in 1 second?

(b) What is the probability that team B losses the puck to team A in 1 second?

(c) Which team is better at picking up loose pucks? Why?

(d) What is the probability that a loose puck is **stays** loose for 2 seconds, i.e. it was loose at 1 second and again at 2 seconds?

(e) What is the probability that if a puck is loose now, that it will be loose after 2 seconds? (This probability is different than the previous.)

(f) Find the steady-state distribution of this Markov chain. Use a computer program that can do matrix multiplication to make things easier.

(g) At the end of the game, what is the expected proportion of time that team A will possess the puck? (Note: A hockey game has 3 20-minute periods for a total of 3600 seconds in the game).

Answer:

(a) 0.8

(b) 0.1

(c) Team A is better at picking up loose pucks since the probability of a loose puck being picked up by A (0.5) is greater than the probability of a loose puck being picked up by B (0.4) .

(d) In order for the puck to *still* be loose, the Markov chain must be $L \to L \to L$. Since each of these transitions has probability 0.1 and they are independent of each other, then the probability of these set of transitions is $0.1^2 = 0.01$.

(e) This is the two step transition probability ($P^2 = P \cdot P$) starting from L and ending at L, i.e. the bottom right element. Note that it is different than the previous question because the transitions $L \to A \to L$ and $L \to B \to L$ are also possibilities. Since

$$P^2 = \begin{array}{c} \\ A \\ B \\ L \end{array} \begin{array}{c} \begin{array}{ccc} A & B & L \end{array} \\ \begin{pmatrix} 0.70 & 0.18 & 0.12 \\ 0.29 & 0.49 & 0.22 \\ 0.49 & 0.33 & 0.18 \end{pmatrix} \end{array},$$

the probability is 0.18.

(f) The steady state distribution of the chain can be found by solving $\pi = \pi P$ or, by multiplying P by itself until all rows are the same. Using the second strategy, it turns out the steady-state is $\pi = (0.5454545, 0.2954545, 0.1590909)$.

(g) Team A will be expected to control the puck for about 55% of the game.