Spring 2020 **Statistics 330 - Section 2** **Exam I**

(75 points)

Please write your first and last name here:

Name ____

Instructions:

- Partial credit will be given only if you show your work.
- Reason out your answers. In many cases, a line or two of justification is enough.
- The questions are roughly in the order in which the material is presented in class, so they are not necessarily ordered easiest to hardest.
- If you get stuck on one, it may be a good idea to move on and come back to that question at the end.
- You may use your prepared notes (1 page, both sides) and a calculator only.

1. Suppose I construct two-digit numbers in the following way. First, I randomly select the tens place to be either 1, 2, or 3. Then, I randomly select the ones place to be either 1 or 5. For example, if I select my tens place to be 1 and ones place to be 5, then my two-digit number is 15. For the computation of probabilities, assume each outcome in your sample space is equally likely.

   (a) Give the sample space for this experiment. (5 points)

   Answer:

   $$\Omega = \{11, 15, 21, 25, 31, 35\}$$

   (b) Let A denote the event that my two-digit number is divisible by 5. List the outcomes in A and give $\mathbb{P}(A)$. (4 points)

   Answer:

   $$A = \{15, 25, 35\}$$

   $$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{3}{6} = 0.5$$

   (c) Let B denote the event that the sum of the digits in my two-digit number is even. For example, if your two-digit number is 51, then the sum of each digit is $5 + 1 = 6$. List the outcomes in B and give $\mathbb{P}(B)$. (4 points)

   Answer:

   $$B = \{11, 15, 31, 35\}$$

   $$\mathbb{P}(B) = \frac{|B|}{|\Omega|} = \frac{4}{6} = 0.67$$

   (d) Suppose now I construct three-digit numbers. I randomly select my hundreds place to be either 1,2,3,4 or 5. My tens and ones place follows the same rules as before (tens place can be 1,2, or 3; ones place can be 1 or 5). How many three-digit numbers can I make in such a way? (4 points)

   Answer: $5 \cdot 3 \cdot 2 = 30$

2. In a survey of 500 college students, it was found that 49% likes watching football, 62% likes watching basketball, and 17% likes watching basketball but not football.
   Answer: Let F be event "likes watching football", and B be event "likes watching basketball".

   (a) What is the probability that a randomly chosen student likes watching both football and basketball? (4 points)
   Answer: $\mathbb{P}(F \cap B) = 0.45$

   (b) What is the probability that a randomly chosen student likes watching football or basketball? (4 points)
   Answer: $\mathbb{P}(F \cup B) = \mathbb{P}(F) + \mathbb{P}(B) - \mathbb{P}(F \cap B) = 0.49 + 0.62 - 0.45 = 0.66$

   (c) What is the probability that a randomly chosen student likes watching basketball given they like watching football? (4 points)
   Answer: $\mathbb{P}(B|F) = \frac{\mathbb{P}(B \cap F)}{\mathbb{P}(F)} = \frac{0.45}{0.49} = 0.9184$

   (d) Are the events "likes watching football" and "likes watching basketball" independent? Justify your answer. (4 points)
   Answer: No, F and B are not independent since $\mathbb{P}(B|F) = 0.9184 \neq \mathbb{P}(B) = 0.62$.

   (e) Are the events "likes watching football" and "likes watching basketball" mutually exclusive/disjoint? Justify your answer. (4 points)
   Answer: No, F and B are not mutually exclusive/disjoint because $\mathbb{P}(F \cap B) = 0.45 \neq \emptyset$.

3. Suppose we take the Aces, Kings, and Queens from a standard deck of cards. There are four cards of each type (Ace of clubs, Ace of spades, Ace of diamonds, Ace of hearts. Same for Kings and Queens, giving twelve cards in total[1]). Clubs and spades are 'black' cards and diamonds and hearts are 'red' cards. We shuffle the cards and lay them face down on a table and randomly choose two cards.

   (a) What is the probability that the two cards are the same color? (5 points)
   Answer: First, $|\Omega| = \binom{12}{2} = 66$. Let $A =$ same color

$$|A| = \binom{6}{2} + \binom{6}{2} \Rightarrow \mathbb{P}(A) = \frac{\binom{6}{2} + \binom{6}{2}}{\binom{12}{2}} = \frac{30}{66} = .45$$

   (b) What is the probability that the two cards are the same type? (5 points)
   Answer: Let $B =$ same type

$$|B| = \binom{4}{2} + \binom{4}{2} + \binom{4}{2} \Rightarrow \mathbb{P}(B) = \frac{\binom{4}{2} + \binom{4}{2} + \binom{4}{2}}{\binom{12}{2}} = \frac{18}{66} = .27$$

   (c) Let $A =$ the event that both cards are the same color, and $B =$ the event that both cards are the same type. What is $\mathbb{P}(A \cup B)$? (5 points)
   Answer:
   $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

   There are 6 outcome in $(A \cap B)$: $\{(A_s, A_c), (A_h, A_d), (K_s, K_c), (K_h, K_d), (Q_s, Q_c), (Q_h, Q_d)\}$, thus $\mathbb{P}(A \cap B) = \frac{6}{66}$

   So, $\mathbb{P}(A \cup B) = \frac{30}{66} + \frac{18}{66} - \frac{6}{66} = \frac{42}{66} = .63$

[1] The twelve cards are $A_s, A_c, A_h, A_d, K_s, K_c, K_h, K_d, Q_s, Q_c, Q_h, Q_d$. The A's, K's and Q's are the types and the s and c subscript cards are black, h and d subscript cards are red

4. The longest consecutive game hitting streak in Major League Baseball is 56 games by Joe DiMaggio (He got at least one hit in 56 straight games). We will look at how impressive that record is with a few assumptions. Joe DiMaggio had a career batting average of .325, so we will assume that every time he came to bat, he had a 32.5% chance of getting a hit. Also assume that he got exactly 4 at bats every game, and each at-bat/game was *independent* of each other.

   Lets assume a baseball player is going to play in the next 56 games with the assumptions above (4 at bats per game, 32.5% chance of a hit in every at-bat etc), what is the probability that he gets at least one hit in *all* 56 games? (3 points)

   Answer: Let H_i = gets at least one hit in game i. The probability of a 56 game hit streak is $\mathbb{P}(H_1 \cap \ldots \cap H_{56}) = (\mathbb{P}(H_i))^{56}$ (b/c independence between games). $\mathbb{P}(H_i) = 1 - \mathbb{P}($no hits in game i$) = 1 - (1 - .325)^4 = .792$. Thus the probability of a 56 game hit streak is $(.792)^{56} = .0000021$. (Fun Fact: Most agree this record will never be broken. Only 6 players have ever had a streak over 40 games with 45 being the next best)

5. In statistical education measurement, researchers are interested in relating student's exam answers to whether or not they truly understood the material. Just because a question is answered right, does not mean the student fully understood. They could have guessed, cheated, simply memorized a formula etc. Suppose a student is taking a multiple choice exam where each question has five possible choices (e.g. a - e). Suppose the student has studied enough to where they have a 70% chance of knowing the answer to a random question, therefore getting it correct. If they don't know the answer, they will simply randomly guess.

   (a) For a random question on this exam, what is the probability the student gets the answer correct? (5 points)

   Answer: Let K = knows answer, and C = gets answer correct.

   We have $\mathbb{P}(K) = .7, \mathbb{P}(\overline{K}) = .3, \mathbb{P}(C|K) = 1, \mathbb{P}(C|\overline{K}) = \frac{1}{5} = .20$

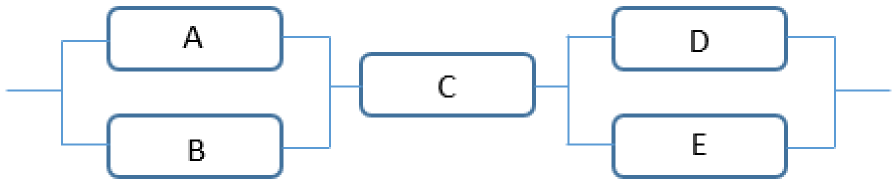
$$\begin{aligned}\mathbb{P}(C) &= \mathbb{P}(K \cap C) + \mathbb{P}(\overline{K} \cap C)\\ &= \mathbb{P}(K)\mathbb{P}(C|K) + \mathbb{P}(\overline{K})\mathbb{P}(C|\overline{K})\\ &= (.7)(1) + (.3)(.2)\\ &= .76\end{aligned}$$

   (b) The student answers a random question and gets it correct, what is the probability they truly knew the answer? (5 points)

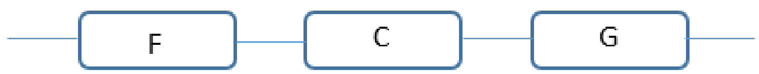
   Answer: Using Baye's rule we have: $\mathbb{P}(K|C) = \frac{\mathbb{P}(K)\mathbb{P}(C|K)}{\mathbb{P}(C)} = \frac{(.7)(1)}{.76} = .921$

6. In the following system, the probability of the individual components A, B, C, D, E working are 0.92, 0.92, 0.90, 0.92, and 0.92 respectively.

   (a) Compute the system's reliability. (Round all intermediate steps to three decimals) (8 points)



Answer: Combine A and B, which has reliability $1-[1-(0.92)]^2 = 0.994$. Combine D and E, which has reliability $1-[1-(0.92)]^2 = 0.994$. Now we have a series system as show below.



Thus the reliability is now:
$P(\text{F works and C works and G works}) = (0.994)(0.92)(0.994) = 0.889$

   (b) What would component C's probability of working need to be for the entire system to have a reliability of .96? (Round to three decimals) (2 points)

Answer: After getting the system in series, we have $(.994)(\mathbb{P}(C))(.994) = .96 \Rightarrow \mathbb{P}(C) = \frac{.96}{.994^2} = .972$

Scratch Paper