Lecture 4: Rules of inference

In Lecture 1, we mentioned that a major objective of 310 is to develop a systematic framework that lets us prove mathematical statements. Recall the mathematical definition of a "proof":

> *A proof of a proposition is a chain of logical deductions leading to the proposition from a set of axioms.*

The operating terms in this definition are "proposition", "axioms", and "chain of logical deductions". We understand what propositions are. We have seen certain types of logical "deductions" but will dig a bit deeper into this; specifically, we will define what types of deductions are kosher within our framework. We also need to properly define what an "axiom" is. But first, we need some more vocabulary.

Arguments

A *axiom* is a proposition that is assumed to be (unquestionably, universally) true.

A *theorem* is a proposition that can be proved to be true.

An *argument* is a sequence of propositions that starts from an proposition (or set of propositions) and ends in a final proposition.

In any argument, all propositions (except the last one) are called *premises*; the final proposition is called the *conclusion*. A *hypothesis* is a proposition that is assumed to be true in the context of a particular argument.

A proof is a *valid* (i.e., logically sound) argument that begins with propositions that are *true* (i.e., hypotheses or axioms) and leads to a *true* conclusion.

As a simple example, consider the following "proof" that John is eligible for CprE 310.

> If John has taken ComS 228, then John is eligible to enroll in CprE 310. (**premise**)
>
> John has taken ComS 228. (**hypothesis**)
>
> Therefore, John is eligible to enroll in CprE 310. (**conclusion**)

This flow of logic seems fairly obvious and intuitive. However, here is a way from first principles (without relying on intuition) to check whether this is logically sound.

Let p denote "John has taken ComS 228" and q denote "John is eligible for CprE 310". Then, the above reasoning can be represented symbolically using the following notation:

$$\begin{array}{l} p \implies q \\ \underline{p \qquad\qquad} \\ \therefore\ q \end{array}$$

This is an example of an *argument*. In order to elevate this into a *proof* (or valid argument) of John's eligibility, we need to assign a value of "T" to *each* of the premises (everything above the horizontal line) and check that this leads to a value of "T" to the conclusion (the statement below the horizontal line). However, this is easy to see: assigning a value of "T" to both p as well as $p \implies q$ forces q to be also "T", recalling the property of " $\implies$ ".

Note: We could have expressed the above argument purely using propositional logic:

$$((p \implies q) \wedge p) \implies q$$

One can check that this logical statement is a tautology; in particular, if when both $p \implies q$ as well as p are true, then q has to be true. But the 3-line notation as used above makes the flow of logic very clear.

Truth tables

How do we check whether or not an argument is valid? As we discussed before in the case of propositions and predicates, the surest way is via truth tables. We construct a truth table of all possible assignments to internal variables, and try to look for *critical rows* where every hypothesis, premise, and conclusion is assigned a value of "T". If such a row exists, then the argument is valid; if no such row exists, then the argument is invalid.

Here is an example. Consider the argument:

If I am sleepy, then I want to drink coffee.

It is true that I am sleepy.

Therefore, I feel sleepy and I want to drink coffee.

Symbolically, we can write this as:

$$\begin{array}{l} \quad p \implies q \\ \quad p \\ \hline \therefore\ p \wedge q \end{array}$$

Again, this argument seems intuitively obvious. But how can we check this intuition? Let us build the truth table for all the 3 lines of the argument:

p	q	$p \implies q$	$p \wedge q$
T	T	**T**	**T**
T	F	F	F
F	T	T	F
F	F	T	F

The first row shows the existence of a truth assignment where each of the 3 lines is assigned a value of "T". Therefore, the argument is valid!

Rules of inference

In order to avoid the brute-force approach of truth tables, we can alternatively employ some standard logical deductions, or *rules of inference*. These are all straightforward adaptations of propositional logic that we have already seen before, but written in a somewhat different notation.

There are several rules of inference that we can use in logical arguments. Here, we list a few commonly used ones. The validity of the rules can be checked via truth tables and we leave this as an easy exercise. In each of the rules below, think of p, q, r etc. as being propositional variables.

Modus ponens. We have seen this one earlier in this lecture:

$$\begin{array}{l} p \implies q \\ \underline{p \qquad\qquad} \\ \therefore\ q \end{array}$$

An example is: "If it is snowing, I drive to work. It is snowing. Therefore, I drive to work."

By the way, the phrase "modus ponens" stands for "method of affirmation", and is a great example of:

> *"omnia dicta fortiora si dicta Latina"*

which roughly translates to

> *Everything sounds cooler when said in Latin.*

Modus tollens. Also seen previously in propositional logic, closely related to the contrapositive:

$$\begin{array}{l} p \implies q \\ \underline{\neg q \qquad\qquad} \\ \therefore\ \neg p \end{array}$$

Example: "If the sun is out, then it is daytime. It is not daytime. Therefore, the sun is not out."

Rule of addition. In general symbolic form, we write this as:

$$\begin{array}{l} \underline{\quad p \qquad} \\ \therefore\ p \vee q \end{array}$$

An example of this rule is the following sentence:

> "It is snowing. Therefore, it is either raining or snowing".

Rule of simplification. We write this as:

$$\begin{array}{l} \underline{\quad p \wedge q} \\ \therefore\ p \end{array}$$

An example is the following:

> "31 is prime and odd. Therefore, 31 is prime."

Rule of conjunction. Exactly as it sounds:

$$\begin{array}{l} p \\ \underline{q \qquad\quad} \\ \therefore\ p \wedge q \end{array}$$

In words:

> "This cup of coffee is free. This cup of coffee tastes good. Therefore, this cup of coffee is free and tastes good."

This is an incomplete set of inference rules, and you can use others. But these rules will be enough for 310. Memorize them!

We end this lecture with four **exercises**. First, is this a valid argument?

> "If today is Thursday, then John will go to the gym." "Today is Thursday." "Therefore, John will go to the gym."

Yes! The argument is valid, and follows the *modus ponens* rule.

Next example. How about:

> "If I play Overwatch all day, I won't finish Homework 1. If I don't finish Homework 1, I won't ace the exam. Therefore, if I play Overwatch all day, I won't ace the exam."

Valid argument! This is an example of the following sequence of implications:

$$\begin{array}{r} p \implies q \\ \underline{q \implies r} \\ \therefore\ p \implies r \end{array}$$

Third example. What about:

> "If I do every homework problem in 310, I will become an expert in logic. I am an expert in logic. Therefore, I did every homework problem in 310."

This is a *fallacy*. It resembles modus ponens, but really, the argument is logically represented as:

$$\begin{array}{l} p \implies q \\ \underline{q \qquad\qquad} \\ \therefore\ p \end{array}$$

which is fallacious, since the premises being true need not imply that the conclusion is true. (**Exercise**: Can you check this using a truth table?) This type of incorrect reasoning is unfortunately all too common, and is called the *fallacy of affirming the conclusion.*

Last one. What about:

> "If I do every problem in the book, I will become an expert in logic. I didn't do every problem in the book. Hence, I am not an expert in logic."

Another *fallacy*. This one looks like modus tollens, but in reality, the representation of this argument is:

$$\begin{array}{l} \quad p \implies q \\ \underline{\neg p \qquad\qquad} \\ \therefore\ \neg q \end{array}$$

which is also fallacious (Again, as an **exercise** verify this claim using a truth table.). Such an incorrect argument is called the *fallacy of denying the hypothesis*.