Final exam

- Please **write your name and netid** on the top of this page.
- There are 11 questions in this exam, totaling 105 points.
- The maximum score will be capped to 100 points. Any score above 105 will be rounded to 100.
- Total duration: 120 minutes.
- You **can** use three pages as cheat sheets.
- You **cannot** consult your notes, textbook, your neighbor, or Google.

---

1. **(5 points)** Let S be the set of all states in the continental United States. For $a, b \in S$, define a relation R such that aRb iff a and b border each other. Which properties of an equivalence relation (reflexive, symmetric, transitive) does R satisfy? Explain your reasoning for each of these properties.

Not reflexive : State cannot border itself.

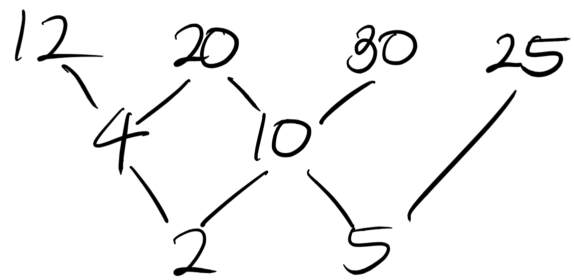
Symmetric : $\forall_{a,b \in S}$ $aRb \rightarrow bRa$

Not transitive : (Nebraska, Iowa) $\in R$, (Iowa, Illinois) $\in R$

(Nebraska, Illinois) $\notin R$.

2. **(10 points)** The set $A = \{2, 4, 5, 10, 12, 20, 25, 30\}$ is partially ordered with respect to the "divides" relation.

(a) Draw the Hasse diagram representation of the above relation.



(b) List all minimal and maximal elements.

Minimal : 2,5

Maximal : 12,20,30,25

(c) Run topological sort on the Hasse diagram to obtain a compatible total ordering of the elements.

2,5,4,10,12,20,30,25

3. **(10 points)** Prove via mathematical induction that for any natural number $n \geq 8$, we have:

$$n - 2 < \frac{n^2}{10}.$$

Base case: $(8-2) < \frac{64}{100}$

Induction Hypothesis: Assume $k-2 < \frac{k^2}{10}$ is true for some k.

Induction step:

$$(k+1)-2 = (k-2)+1 < \frac{k^2}{10} + 1$$

$$\Rightarrow \frac{k^2}{10}+1 = \frac{k^2+2k+1}{10} + \frac{-2k-1}{10} + 1$$

$$= \frac{(k+1)^2}{10} + \frac{9-2k}{10}$$

Since $\frac{9-2k}{10} < 0$ for all $k \geq 8$,

$$\frac{k^2}{10}+1 = \frac{(k+1)^2}{10} + \frac{9-2k}{10} < \frac{(k+1)^2}{10} \quad \square$$

According to the induction, $n-2 < \frac{n^2}{10}$ is true.

4. **(10 points)** Consider a 82×4 array (i.e, 82 rows, 4 columns) of tiles. Each tile is colored either red, white, or blue. (The coloring is arbitrary and repetitions of colors are allowed within each row or column).

(a) Calculate the total number of ways to color any given row.

$3 \times 3 \times 3 \times 3 = 81$

(b) Argue that some two rows must be colored identically.

According to PHP, the total number of ways to color is 81 (holes) and # of rows is 82 (pigeons). Therefore, some two rows must be colored identically.

(c) Use the above result to conclude that no matter how the 82×4 array is colored, there exist 4 tiles of the same color which form the corners of a rectangle.

From same two rows, each row has 4 slots (column). Using PHP, the number of distinct colors (holes) is 3, while slots (pigeons) is 4. Therefore, at least two slots share same colors which form the corners of a rectangle.

5. **(10 points)** Let x and y be non-negative integers. If x and y satisfy the following equation:

$$3x + 5y = 1069,$$

then give a formal proof that at least one of x and y has to be odd. (Clearly state your method of proof in the beginning.)

Using contraposition, if both x and y are even, $3x+5y \neq 1069$.

Let $x=2k$ and $y=2l$ where k and l are some integers.

$$3 \cdot 2k + 5 \cdot 2l = 6k + 10l$$

$$= \underbrace{2(3k+5l)}_{\text{even}} \neq \underbrace{1069}_{\text{odd}}.$$

6. **(10 points)** Using the English alphabet (with 26 characters, repetitions allowed), count the following. No need to provide explicit numbers, you can leave your answers in terms of factorials and/or powers.

(a) the number of strings of length 10.

26^{10}

(b) the number of strings that begin and end with the same letter.

26^{9}

(c) the number of *palindromes* of length 10. (A palindrome is a string whose reversal gives the same string; e.g. abba, cdbdc, eye, racecar, madam, etc.)

26^{5}

(d) strings of length 8 that have the letters (C,P,R,E) in any order.

4^{8}

(e) strings of length 8 that begin with C **or** end with R.

A: begin with C, B: end with R

$|A \cup B| = |A| + |B| - |A \cap B|$

$= \boxed{26^7 + 26^7 - 26^6}$

7. **(10 points)** Apply the rules of inference studied in class to deduce the conclusion from the given hypotheses:

$$\begin{array}{l} p \vee q \\ q \implies r \\ p \wedge s \implies t \\ \neg r \\ \neg q \implies u \wedge s \\ \hline \therefore t \end{array}$$

Modus Tollens :

$$\begin{array}{l} \neg r \\ q \Rightarrow r \\ \hline \therefore \neg q \end{array}$$

Disjunctive Syllogism :

$$\begin{array}{l} \neg q \\ p \vee q \\ \hline \therefore p \end{array}$$

Modus Ponens :

$$\begin{array}{l} \neg q \\ \neg q \Rightarrow u \wedge s \\ \hline \therefore u \wedge s \end{array}$$

Rule of Simplification :

$$\begin{array}{l} u \wedge s \\ \hline \therefore s \end{array}$$

Rule of conjunction

$$\begin{array}{l} p \\ s \\ \hline p \wedge s \end{array}$$

Modus ponens

$$\begin{array}{l} p \wedge s \\ p \wedge s \Rightarrow t \\ \hline \therefore t \end{array}$$

8. **(10 points)** The Astros and the Dodgers are favored to meet again in this year's World Series. The series is a best-of-seven, and ends when one team wins 4 games. Count the total number of win-loss sequences that can potentially arise.

For Astros winning case

1) 4-0 $\longrightarrow \binom{3}{3} = 1$

2) 4-1 $\longrightarrow \binom{4}{3} = 4$

3) 4-2 $\longrightarrow \binom{5}{3} = 10$

4) 4-3 $\longrightarrow \binom{6}{3} = 20$

$\therefore 20 + 10 + 4 + 1 = 35$

For overall, $35 \times 2 = 70$

9. **(10 points)** In a standard 64-square chessboard, rows are numbered 1 through 8 and columns are numbered a through h. Suppose you are programming a robot to move from the bottom-left square (a1) to the top-right square (h8) of an 8×8 chessboard. The robot is only allowed to make "up" or "right" moves.

(a) How many such possible paths are available for the robot?

$$\frac{16!}{8!\,8!}$$

(b) Now, suppose that the square e4 contains a mine, and the bot has to avoid this mine *at all costs*. How many possible paths are now available to the robot? (Hint: inclusion-exclusion).

Paths going through e4: $\frac{9!}{5!4!} \cdot \frac{9!}{5!4!}$

Answer: Total − paths going thorough e4.

$$\therefore \boxed{\frac{16!}{9!9!} - \left(\frac{9!}{5!4!}\right)^2}$$

10. **(10 points)** A *complete bipartite* graph is an undirected graph with $m + n$ nodes, where each of the first m nodes are connected with each of the last n nodes. Assume for the sake of this problem that m and n are both greater than 2.

(a) Use the First Degree theorem to count the number of edges in this graph.

$\sum \text{degs} = 2|E|.$

$mn + nm = 2 \cdot mn = 2 \cdot |E|$

$E = mn$

(b) What is the minimum number of colors needed to color the vertices of such a graph so that no adjacent vertices have the same color?

2.

(c) Under what conditions on m and n does this graph admit an Euler path?

m & n should be even

11. **(10 points)** Suppose you pick a positive integer $n \leq 100$ uniformly at random.

(a) What is the probability that n is divisible by 5?

$$\frac{20}{100} = \frac{1}{5}$$

(b) What is the probability that n is divisible by 7?

$$\frac{14}{100} = \frac{7}{50}$$

(c) $\underbrace{\text{Given that } n \text{ is divisible by 5}}_{B}$, $\underbrace{\text{what is the probability that } n \text{ is divisible by 7}}_{A}$?

35, 70.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2/100}{1/5} = \frac{10}{100}$$

$$= \frac{1}{10}$$

SCRATCH