

# First-Order Logic

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## Outline

I. Syntax of FOL

II. Quantifiers

III. Model for FOL

# I. Propositional Logic: Strength and Weakness

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- ♠ Programming languages lack a general mechanism for deriving facts from other facts.
- ♠ They lack the expressiveness required to handle partial information.

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$$B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$$

$$B_{1,2} \Leftrightarrow (P_{1,1} \vee P_{1,3} \vee P_{2,2})$$

$\vdots$

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- ♠ Propositional logic assumes the world contains facts only.

# Combining Formal and Natural Languages

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## First-order logic

- ◆ built around objects and relations
  - **Objects:** people, houses, cars, trees, colors, days, ...
  - **Relations:**
    - ◆ unary properties such as `big`, `windy`, ...
    - ◆ *n*-ary properties such as `bigger than`, `parent of`, `on`, `owns`, ...
  - **Functions:** `square of`, `best friend`, `age`, ...
- ◆ capable of expressing facts about some or all objects



# Formal Languages

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Language	Ontological Commitment (What exists in the world)	Epistemological Commitment (What an agent believes about facts)
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief $\in [0, 1]$
Fuzzy logic	facts with degree of truth $\in [0, 1]$	known interval value

# Alphabet of First-Order Logic

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## ◆ Logical symbols

- connectives:  $\wedge, \vee, \Rightarrow, \Leftrightarrow, \neg$
- parenthesis:  $(, )$  and punctuation  $,$
- equality:  $=$
- quantifiers:  $\forall$  (universal quantification),  $\exists$  (existential quantification)
- variables:  $x, y, z, \dots; x_1, x_2, \dots$

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- constants: Socrates, Turing, 1, earth, ...

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*Father*( $x, y$ )      //  $x$  is father of  $y$   
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- function symbols:
  - gcd*( $x, y$ )      // greatest common divisor of  $x$  and  $y$
  - FatherOf*( $x$ )      // father of  $x$

# Terms and Atomic Sentences

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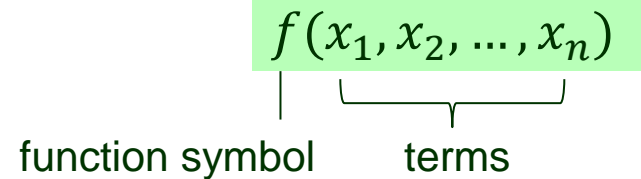
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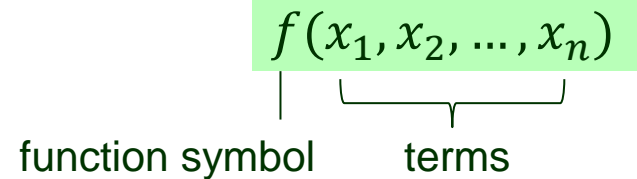


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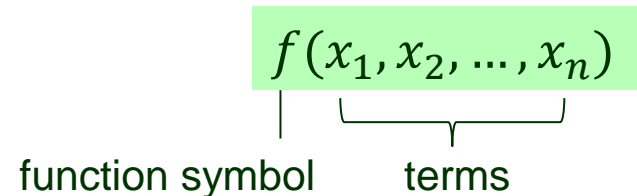
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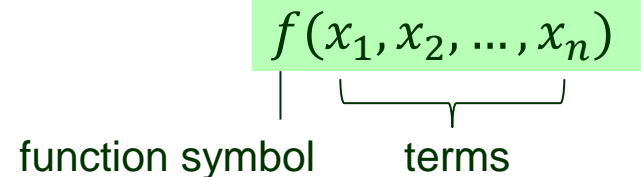
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*Mother*(Aphrodite, Harmonia)  
*Male*(John)

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## ◆ Atomic sentences

- predicates: *true*, *false*  
 $\text{Mother}(\text{Aphrodite}, \text{Harmonia})$   
 $\text{Male}(\text{John})$
- term equalities

$$\text{FatherOf}(\text{Apollo}) = \text{Zeus}$$

# Complex Sentences

---

- made of atomic sentences using logical connectives

*Father(x, y)  $\Rightarrow$  Male(x)*

*Female(x)  $\vee \neg$  Mother(x, y)*

*Likes(Mary, John)  $\Leftrightarrow$  Likes(John, Mary)*

*(Parent(x, y)  $\wedge$  Parent(y, z))  $\Rightarrow$  GrandParent(x, z)*

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$Father(x, y) \Rightarrow Male(x)$

$Female(x) \vee \neg Mother(x, y)$

$Likes(Mary, John) \Leftrightarrow Likes(John, Mary)$

$(Parent(x, y) \wedge Parent(y, z)) \Rightarrow GrandParent(x, z)$

- universal quantification

$\forall x \ Circle(x) \Rightarrow Ellipse(x)$       // Every circle is an ellipse.

$\neg \forall x \ Likes(x, sushi)$       // Not everyone likes sushi.

$\forall x \ Integer(x) \Rightarrow (Even(x) \vee Odd(x))$       // Every integer is either even or odd.

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- existential quantification

$\exists x \ Star(x) \wedge \neg (x = Sun)$       // There are stars other than the sun.

$\exists x \ Whale(x) \wedge (Age(x) = 200)$       // Some whales live to 200 years.



# Syntax of First-Order Logic

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*Sentence*  $\rightarrow$  *AtomicSentence* | *ComplexSentence*

*AtomicSentence*  $\rightarrow$  *Predicate* | *Predicate*(*Term*,...) | *Term* = *Term*

*ComplexSentence*  $\rightarrow$  ( *Sentence* )

|  $\neg$  *Sentence*

| *Sentence*  $\wedge$  *Sentence*

| *Sentence*  $\vee$  *Sentence*

| *Sentence*  $\Rightarrow$  *Sentence*

| *Sentence*  $\Leftrightarrow$  *Sentence*

| *Quantifier* *Variable*,... *Sentence*

*Term*  $\rightarrow$  *Function*(*Term*,...)

| *Constant*

| *Variable*

*Quantifier*  $\rightarrow$   $\forall$  |  $\exists$

*Constant*  $\rightarrow$  *A* | *X*<sub>1</sub> | *John* | ...

*Variable*  $\rightarrow$  *a* | *x* | *s* | ...

*Predicate*  $\rightarrow$  *True* | *False* | *After* | *Loves* | *Raining* | ...

*Function*  $\rightarrow$  *Mother* | *LeftLeg* | ...

OPERATOR PRECEDENCE :  $\neg, =, \wedge, \vee, \Rightarrow, \Leftrightarrow$

## II. Nested Quantifiers

---

$\forall x \exists y \text{ Student}(x) \wedge \text{Course}(y) \wedge \text{Enrolled}(x, y)$

$\forall x \exists y \text{ Brother}(x, y) \Rightarrow \text{Sibiling}(x, y)$

Order matters for quantifiers of different types:

$\forall x \exists y \text{ Loves}(x, y)$  // Everybody loves somebody

$\exists x \forall y \text{ Loves}(y, x)$  // There is someone whom everyone loves.

$\exists x \exists y \text{ Loves}(x, y) \quad \equiv \quad \exists y \exists x \text{ Loves}(x, y)$

$\forall x \forall y (\text{Brother}(x, y) \Rightarrow \text{Sibiling}(x, y))$   
 $\equiv \quad \forall y \forall x (\text{Brother}(x, y) \Rightarrow \text{Sibiling}(x, y))$

# Connections Between $\forall$ and $\exists$ Through $\neg$

---

$$\forall x \neg Likes(x, Parsnips) \equiv \neg \exists x Likes(x, Parsnips)$$

$$\forall x Likes(x, Icecream) \equiv \neg \exists x \neg Likes(x, Icecream)$$

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De Morgan's rules still apply:

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

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Move negation inward, flipping the quantifiers:

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free

bound

$$P(x) \Rightarrow (\exists x Q(x)) \wedge R(x)$$

different bound  
variable

same free variable

# Equality

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- It states that two terms refer to the same object.

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 $\wedge (\forall z \text{ Brother}(z, \text{Zeus}) \Rightarrow (z = x) \vee (z = y))$



# Equality

---

- It states that two terms refer to the same object.

*Father*(Zeus) = Cronus

*Father*(Cronus) = Uranus

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// ( $x \equiv \text{Poseidon}$  and  $y \equiv \text{Hades}$ , or  $x \equiv \text{Hades}$  and  $y \equiv \text{Poseidon}$ )

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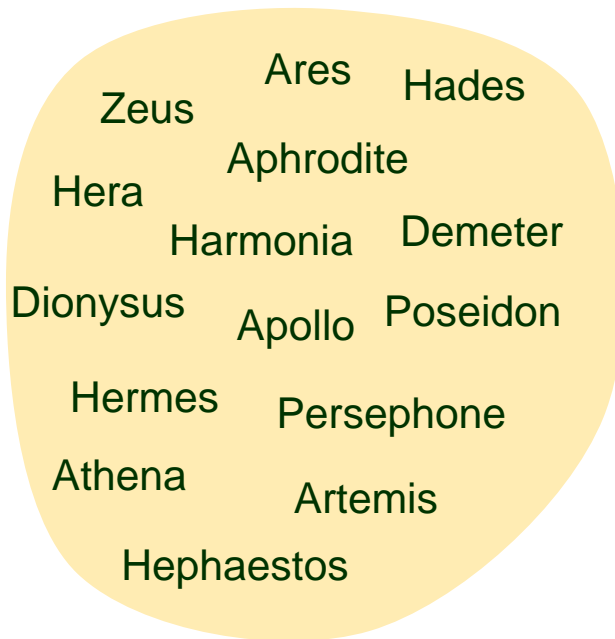
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# Model Example

Model for the family relationships of the Greek gods (incomplete).



domain  $D$

*Father*(Zeus, Hermes)  
*Mother*(Hera, Ares)  
*Mother*(Aphrodite, Harmonia)  
*Father*(Zeus, Athena)

⋮

predicates

*Weapon*(Zeus) //  $\equiv$  Thunderbolt  
*Weapon*(Apollo) //  $\equiv$  BowAndArrows  
*Carry*(Hermes) //  $\equiv$  Flute  
*Carry*(Aphrodite) //  $\equiv$  Apple

⋮

functions

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Quantifiers allow us to express properties of a collection of objects instead of enumerating them by name.

$\forall$  (universal): “for all”

$\exists$  (existential): “there exists”

# Truths with Quantifications

---

- ♦  $\forall x P(x)$  is true in a model  $M$  iff  $P(x)$  is true with  $x$  assuming every object in the model

$\forall x \text{ Father}(x, y) \Rightarrow \text{Male}(x)$

*true* (in every model)

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