Recitation 8 Solutions

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1. In a congregation of 97 people, a pastor instructs everyone to stand up and shake hands with exactly 3 other people (the pastor doesn't participate in this activity). Use graph theory to explain why this cannot be done.

   Solution

   Using the First Degree theorem, $\sum_{v \in V} deg(v) = 97 \times 3 = 2|E|$. Since 97×3 is an odd number, there exists no $|E|$ satisfying the First Degree theorem. Therefore, this cannot be done.

2. You are tasked with painting the centerlines of the streets in downtown. The map of downtown consists of blocks in a regular 3x3 grid, as shown in the graph below. (Each edge represents a street.)

(a) Is it possible to paint all the centerlines without traversing a street in the above map more than once? Assume that all streets are two-way streets.

(b) Justify your answer using graph theory.

Solution

In order to satisfy Euler's path, there should be zero or two odd degree of nodes. Since this graph has 8 odd degree nodes, it is not possible.

3. A complete graph with n nodes (denoted by the symbol K_n) is a simple, undirected graph with precisely one edge joining every pair of distinct nodes. Use the First Degree theorem to deduce the number of edges in this graph in terms of n.

   Solution

   Using the First Degree theorem, $\sum_{v \in V} deg(v) = 2|E|$, where v stands for vertices and E stands for total number of edges, the total number of degrees is equal to $n(n-1)$. Therefore, the total number of edges in the graph will be $\frac{n(n-1)}{2}$.

4. A simple graph is called *cubic* if *every* node has degree 3.

(a) Draw examples of cubic graphs with $n = 4, 6, 8$ nodes.

(b) Argue why you cannot draw cubic graphs with an odd number of nodes.

Solution

Using the First Degree theorem, $\sum_{v \in V} deg(v) = 3*(2k+1) = 2(3k+1)+1 = 2|E|$ where k is an integer. Since $2(3k+1)+1$ is an odd number, there exists no

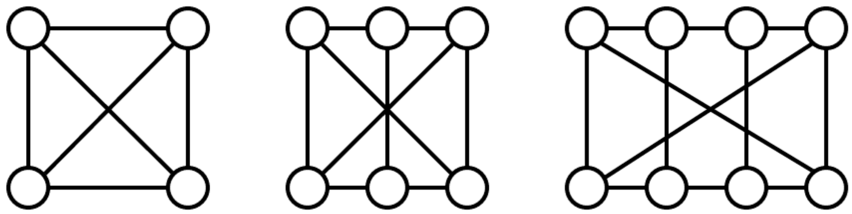

Figure 1: Cubic with n=4, 6, 8

integer satisfying $|E|$. Therefore, the cubic graphs cannot have an odd number of nodes.

5. Define the *distance* between two nodes in a graph as the number of edges along the shortest path between the nodes. Then, the *diameter* of a connected graph is the largest distance between any pair of nodes in the graph.

   (a) What is the biggest possible diameter for any connected graph with n nodes? Draw (or describe in words) a graph with this maximum diameter.

   (b) What is the smallest possible diameter for any connected graph with n nodes? Draw (or describe in words) a graph with this minimum diameter.

   Solution

   (a) Line graph.

   (b) Complete graph.