Lecture 11: Functions

Types of functions

Injective function

Injective functions are also known as *one-to-one* functions. Injective functions are functions where distinct elements in the domain get mapped to distinct elements in the co-domain. Formally, the injective property can be captured in terms of predicate logic:

$$\forall x_1, x_2 \in X,\ f(x_1) = f(x_2) \implies x_1 = x_2$$

The easiest way to imagine a one-to-one function is by drawing arrows from the domain X to the co-domain Y. If each element in the co-domain has **at most** 1 "incoming" arrow, then the function is injective.

As a simple example, if $X = 1, 2, 3, 4$ and $Y = A, B, C, D$, consider the function f defined as:

$$f(1) = A, f(2) = D, f(3) = C, f(4) = B$$

This function would be a one-to-one function. On the other hand, if the function were defined as:

$$f(1) = f(2) = f(3) = f(4) = B$$

then the function would *not* be a one-to-one function.

Here are some other examples. The function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = 2x + 3$ is injective. We prove this by contradiction. Suppose, to the contrary, that the function is not injective, i.e., there exist two distinct numbers x_1, x_2 such that $f(x_1) = f(x_2)$. By definition of f, we have:

$$2x_1 + 3 = 2x_2 + 3$$

Solving for x_1, we get $x_1 = x_2$, which contradicts the assumption that x_1 and x_2 are distinct. Therefore, the function is injective.

On the other hand, the function $f(x) = x^2$ is not injective. A direct proof would simply follow by observing that for any $x \neq 0$, $f(x) = f(-x) = x^2$.

In general, when discussing functions defined over real numbers, try to find out if the function is *strictly* increasing or *strictly* decreasing. (For instance, linear functions that are not constant are either increasing or decreasing; so are exponential functions; so are log functions.) In all these cases, the function can be proved to be one-to-one.

Surjective function

Surjective functions are also known as *onto* functions. Surjective functions are functions where every element in the co-domain is the image of some element in the domain. Formally, the surjective property is defined as:

$$\forall y \in Y,\ \exists x \in X,\ f(x) = y.$$

Again, the easiest way to imagine an onto function is by drawing arrows from the domain X to the co-domain Y. If each element in the co-domain has **at least** 1 incoming arrow, then the function is surjective.

In the example above where $X = 1, 2, 3, 4$ and $Y = A, B, C, D$, the function f is surjective (in addition to being injective.) Observe that each element in Y has some x that gets mapped to it.

The function $f(x) = 2x + 3$ is surjective if the domain of f is specified as $\mathbb{R}$. This is because every real number (say, y) can always be written as $y = 2x + 3$ for *some* real number x.

However, the function $f(x) = 2x + 3$ is *not* surjective if the domain of f is specified as the set of integers $\mathbb{Z}$. For example, $y = 2$ cannot be written as $2 = 2x + 3$ for *integer* x.

The function $f(x) = x^2$ is not surjective if the domain is $\mathbb{R}$ *or* $\mathbb{Z}$. Try proving this! For now, we leave this as an **exercise**.

Bijective function

Bijective functions are also known as *one-to-one correspondence* functions. Bijective functions are functions that are both one-to-one as well as onto.

Back to our mental picture of arrows from X to Y. A bijective function will have: * one arrow out of every element in X (since it is a function) * **exactly** arrow into **every** element in Y.

Examples: the linear function $f(x) = 2x + 3$, defined over the real numbers as the domain, is bijective; we proved above that it is both one-to-one as well as onto. Any linear function of the form $f(x) = ax + b$ will be a one-to-one correspondence in general, unless $a = 0$ (in which case the function f is a constant/flat function.)

A simpler example is the *identity* function $f(x) = x$. (Here, the domain could be any set A.)

On the other hand, $f(x) = x^2$ is *not* a bijection if the domain of f is $\mathbb{R}$. First of all, f is not one-to-one since $f(1) = f(-1)$. Moreover, f is not onto since negative real numbers have no pre-image under f.

Some useful "310" functions

Let us now discuss some common functions that often arise in CPRE/SE applications.

1. Consider any set S that is a subset of a given universal set U. The *characteristic* function f with respect to S is defined as:

   $$f(a) = \begin{cases} 1, & \text{for } a \in S \\ 0, & \text{for } a \notin S \end{cases}$$

   The characteristic function is onto if S is a proper subset of U, but not one-to-one.

2. The *ceiling* function $f : \mathbb{R} \to \mathbb{Z}$, denoted by $f(x) = \lceil x \rceil$ is the smallest integer that is greater than or equal to x.

3. The *floor* function $f : \mathbb{R} \to \mathbb{Z}$, denoted by $f(x) = \lfloor x \rfloor$ is the largest integer that is smaller than or equal to x. Neither the ceiling nor the floor functions are one-to-one, but both are onto.

4. The *modulo* function $f_p : \mathbb{Z} \to \mathbb{Z}$, denoted by $f_p(x) = x \mod p$, is the remainder when x is divided by p. This function is neither one-to-one nor onto.

Some rules of thumb

The best way to check whether a function is injective/surjective/bijective is to mentally imagine the "arrow diagram" and check the number of "incoming" arrows for every element in the co-domain.

However, it is often important to formally prove whether a function f is injective/surjective/bijective. Here are some rules of thumb that can be followed for constructing such proofs:

- To prove that f is one-to-one, try doing a proof-by-contradiction. Assume that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$. Somehow deduce that x_1 and x_2 must be equal, thus leading to a contradiction.
- To prove that f is onto, try doing a direct proof. Assume some generic y in the co-domain and proving that $y = f(x)$ for some x in the domain.
- To disprove that f is one-to-one, try doing a proof-by-counterexample: find some pair x_1, x_2 such that $f(x_1) = f(x_2)$. To disprove that f is onto, also try a counterexample: find y that has no inverse image in X.