Lecture 13

Normal Distribution

STAT 330 - Iowa State University

Normal Distribution

Normal Distribution

Setup: The normal distribution is commonly used to model a wide variety of variables (weight, height, temperature, voltage, etc) due to its *"bell-shaped"* and *symmetric* shape.

If a random variable X follows a *normal distribution*,

$$X \sim N(\mu, \sigma^2)$$

where μ is the mean, and $\sigma^2 > 0$ is the variance

- Probability Density Function (pdf)

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad \text{for } -\infty < x < \infty$$

- Expected Value: $E(X) = \mu$
- Variance: $Var(X) = \sigma^2$

Normal PDF

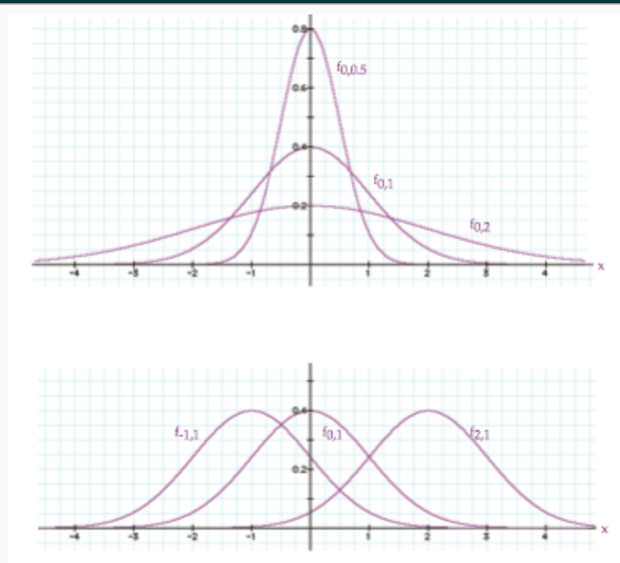


Figure 1: PDFs for normal distribution with various μ and σ^2

μ determines the location of the peak in the $x-$axis,
σ^2 determines the "width" of the bell shape.

Normal CDF

- Cumulative Distribution Function (cdf)

$$F_X(t) = \int_{-\infty}^{t} f(x)dx \qquad \text{(no closed form)}$$

→ Use cdf table ($z-$table) of *standard normal distribution* $N(\mu = 0, \sigma^2 = 1)$ to obtain probabilities.

→ Need to *standardize* any normal random variable, X, into standard normal random variable, Z.

Standardization of Normal Distribution

Let $X \sim N(\mu, \sigma^2)$. Then,

1. $Z = \frac{X-\mu}{\sigma}$ is a standard normal random variable
2. $Z \sim N(0, 1)$ (normal distribution with $\mu = 0$, $\sigma^2 = 1$)

Standardization

Example 1: Suppose $X \sim N(20, 100)$. What is the probability that X is less than 23.5?

To find this probability, we usually ...

- Integrate the PDF (too difficult)
- Plug into CDF (impossible - no closed form for CDF of X)

Instead we standardize X, and obtain probabilities using the standard normal cdf table ($z-$table)

- The standardized R.V is $Z = \frac{X-\mu}{\sigma} = \frac{X-20}{\sqrt{100}} \sim N(0, 1)$
- The standardized observation is $z = \frac{x-\mu}{\sigma} = \frac{23.5-20}{\sqrt{100}} = 0.23$
- $P(X < 10) = P(Z < 0.35)$ (obtain this from $z-$table)

Standard Normal Distribution

Standard Normal Distribution

Suppose a random variable, X, follows a $N(\mu, \sigma^2)$ distribution.
Then, $Z = \frac{X-\mu}{\sigma^2}$ follows a *standard normal distribution*

$$Z \sim N(0,1)$$

- Probability Density Function (pdf)

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \qquad \text{for } -\infty < z < \infty$$

- Expected Value:

$$E(Z) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{E(X)-\mu}{\sigma} = \frac{\mu-\mu}{\sigma} = 0$$

- Variance:

$$Var(Z) = Var\left(\frac{X-\mu}{\sigma}\right) = Var(\frac{X}{\sigma}) = \frac{1}{\sigma^2} Var(X) = \frac{\sigma^2}{\sigma^2} = 1$$

Standard Normal CDF

- Cumulative Distribution Function (cdf)

$$F_Z(t) = \int_{-\infty}^{t} f(z)dz = \Phi(t) \qquad \text{(no closed form)}$$

→ Just like the normal cdf, the standard normal cdf does not have a closed form expression.

→ The cdf of N(0,1) random variable is denoted by $\Phi(t)$ (or more commonly $\Phi(z)$)

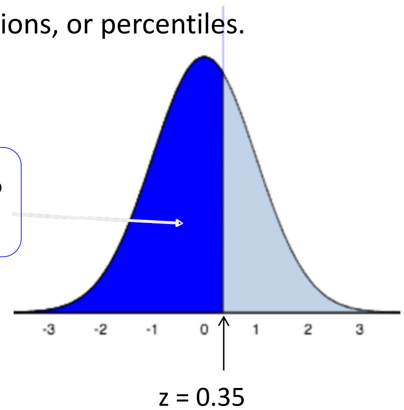
→ The values of the cdf, $\Phi(z)$, are found in the standard normal table (*z-table*)

Z-table (Standard Normal Table)

Z-table

- Z-Table gives proportion of normal curve less than a particular z score
  - Gives left-hand area (dark blue shaded region)
  - This is same as the *percentile* value for z
  - Can be referred to as areas, proportions, or percentiles.
  - Denoted P(Z< z)

Proportion of area less than z=0.35
Denoted as "P(Z<0.35)"



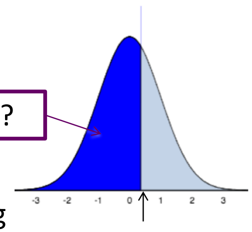
How to read the Z-table

- z values range from –3.99 to 3.99 on the z-table
- Row – ones and tenths place for z
- Column – hundredths place for z
- P(Z < z) found *inside* z-table

	Second decimal place in z						
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06
0.0	0.5000	0.5040	0.5080	0.5120	[illegible]	[illegible]	0.5239
0.1	0.5398	0.5438	0.5478	[illegible]	[illegible]	[illegible]	0.5636
0.2	0.5793	0.5832	[illegible]	[illegible]	[illegible]	[illegible]	0.6026
0.3	0.6179	[illegible]	[illegible]	[illegible]	0.6331	0.6368	0.6406
0.4	0.6554	[illegible]	[illegible]	0.6664	0.6700	0.6736	0.6772

Left-hand area/probability
P(Z < z)

How to read the Z-table



- Look up z = 0.35 in the margins on z-table
- Percentile/left-hand area is corresponding value inside z-table

z	Second decimal place in z						
	0.00	0.01	0.02	0.03	0.04	0.05	0.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	$\boxed{0.6368}$	0.6406
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772

P(Z < 0.35)

Examples

Z-table Practice

Suppose $Z \sim N(0, 1)$

1. $P(Z < 1)$

2. $P(Z > -2.31)$

Z-table Practice

Suppose $Z \sim N(0, 1)$

3. $P(0 < Z < 1)$

4. $P(|Z| > 2)$

Normal Distribution Example

Suppose $X \sim N(1, 2)$, and we want to find $P(1 < X < 2)$.

Normal Distribution Example