Lecture 12: Relations

Having defined sets, functions, and so on, we now introduce *relations*. Relations capture *pairwise interactions* between objects, and are a powerful generalization of the concept of a function.

First, a quick note about *ordered pairs* (also known as *tuples*). Note that if we use the notation $\{a, b\}$ then we are talking about the *set* containing a and b, and in sets, order doesn't matter and this is the same as writing $\{b, a\}$. However, if we *do* want to preserve order, we use parentheses instead of curly braces: (a, b). Note that this is different from (b, a) since the order is reversed. The same idea can be extended to more than 2 elements – whenever we wish to preserve order we use parentheses.

We now define relations. Let A and B be sets. Then, a *binary relation* R, defined from A to B, is a subset of $A \times B$, i.e.,

$$R \subseteq A \times B$$

In other words, R is a set of ordered pairs (a, b) where $a \in A$ and $b \in B$

The set A is called the *domain* of the relation, The set B is called the *co-domain* of the relation.

To denote that some object $a \in A$ is related to some other object $b \in B$, we will use the notation:

$$aRb, \text{ or, } (a, b) \in R$$

The above definition is a bit abstract, but some examples might help to clarify. Suppose that A denotes the set of all men in the world, and B is the set of all women. Suppose R denotes the relation "is the husband of", then some subset of ordered pairs in $A \times B$ belong to R.

Another example. Suppose that P denotes the set of all webpages and L is a relation between two different webpages a and b if a links to b. Then, L can be written as:

$$L = \{(a, b) \in P \times P \mid a \text{ has a link to } b\}$$

Relations are even easier to understand if we are talking about numerical objects. Let

$$A = \{0, 1, 2\}, \ B = \{1, 2, 3\}$$

be two sets of numbers. Let R denote the "is less than or equal to" relation, i.e,

$$xRy \iff x \leq y.$$

Then, we can assert that

$$0R1, 0R2, 0R3, 1R2, 1R3, 2R3$$

i.e., R is the set of of ordered pairs:

$$\{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3)\}$$

On the other hand, if R denotes the "equal to" relation, then we can assert that:

$$1R1, \ 2R2$$

Not surprisingly, such a relation is called the *equality* relation and is commonly encountered in several applications.

Functions as relations

If we stare at the definition of a relation a bit carefully, then we realize that *every function can be interpreted as a relation*. Let

$$f : A \to B$$

be an arbitrary function from A to B. We can always define a relation $F \subseteq A \times B$ which consists of ordered pairs (a, b) such that $b = f(a)$ for a given a.

Example. Define $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) = n^2$. Then, the corresponding relation $F \subset \mathbb{N} \times \mathbb{N}$ consists of all pairs of natural numbers $\{(1, 1), (2, 4), (3, 9), (4, 16) \ldots, (n, n^2), \ldots\}$.

However, there is an important distinction between relations and functions: relations can include objects of the form (a, b) and (a, c). This is an instance of a one-to-many mapping, where a given object a in A can be related to more than one object in B. So *not every relation is a function*.

A good example of this effect is the "Circle" relation. Consider the graph of the algebraic equation corresponding to the unit circle:

$$x^2 + y^2 = 1.$$

We have previously seen that this is *not* a function in the strict sense, since the value $x = 0$ can be mapped to both $y = 1$ and $y = -1$. However, it *is* a relation, constructed as follows. Define $A = B = \mathbb{R}$ and define a relation C as follows: $(x, y) \in C$ iff the Cartesian coordinates (x, y) lie on the unit circle. Therefore, C is a relation but not a function.

Graphs provide a particularly intuitive way to represent relations. Given two sets A and B and a binary relation R, we can represent every ordered pair $(a, b) \in R$ by drawing a directed edge (arrow) from a to b. We already drew such "arrow diagrams" to represent functions; relations are represented basically the same way. An important distinction is that relations can potentially have several "outgoing" arrows, while functions can have only 1 outgoing arrow for each element in A.

(Both relations and functions can have elements in B with several "incoming" arrows, however. This is what differentiates onto functions, one-to-one functions, etc.)

Properties of relations

Binary relations are important concepts, and some of them satisfy some interesting properties. There are three main properties that we will need to understand. Let us state them first (assume here that $A = B$ in all the examples below):

1. Reflexivity. A relation R is *reflexive* if aRa.
2. Symmetry. A relation R is *symmetric* if aRb iff bRa.
3. Transitivity. A relation R is *transitive* if for any triple $a, b, c \in A$, aRb and bRc then necessarily aRc.

Some examples for each of these properties. If $A = B = \mathbb{N}$, then the equality relation is reflexive since any number a is equal to itself. Similarly, the "$\leq$" relation is reflexive. On the other hand, the "exists a direct flight" relation between airports is *not reflexive* (since having a direct flight from DSM to itself is meaningless.) As an **exercise**, convince yourself that "<" is not a symmetric relation.

Now let A denote the set of all users on Facebook. The "friend" relation is a *symmetric* relation (if Person i is a friend of j, then j is automatically a friend of i). On the other hand, the "follow" relation is not symmetric; I follow LeBron James, but that does not mean that LeBron James follows me (as of today, at least).

The $=$, $\leq$ and $<$ relations are all transitive with respect to the set of natural numbers (or real numbers). The logic being: if $a = b$ and $b = c$, then necessarily $a = c$. Likewise for $<$ and $\leq$. On the other hand, let A be the set of sports teams in a league, and R denote the "has defeated" relationship. If Team a has defeated Team b, and Team b has defeated Team c, that does not automatically imply that Team a has defeated Team c. The "has defeated" relation is *not transitive*.

Consider a "ping" relation in a computer network, i.e., Computer i is related to Computer j if i can ping j. Now we know that:

- every computer can ping itself.
- if Computer 1 can ping Computer 2, then Computer 2 can ping Computer 1
- if 1 can ping 2 and 2 can ping 3, then 1 can ping 3.

Therefore, the "ping" relation satisfies all three relations. Any relation that is reflexive, symmetric, transitive is called an *equivalence* relation, which we will study in more detail.

Here are some examples of different relations, and some of their properties. As an **exercise**, convince yourself that the stated properties hold:

- $A = B = \mathbb{R}$, R = "<". This relation is not reflexive; no symmetric; and transitive.
- $A = B$ = set of positive integers, $R = \{(a, b) | a + b = 4\}$. This relation is not reflexive; symmetric; not transitive.
- $A = B$ = set of all people who ever lived, R: "has the same blood group as". This relation is reflexive; symmetric; and transitive.
- $A = B$ = set of professional tennis players, R: "has a ranking greater than or equal to". This relation is reflexive; not symmetric; and transitive.
- $A = B$ = set of subsets of some universal set U; R: "is a subset of". This relation is reflexive; not symmetric; and transitive.

Interpreting the properties via graphs

The above properties may be a bit tricky to digest, but luckily there are some systematic ways to check whether a given relation satisfies a given property. The most intuitive way to check for a property is to look at the graph representation of the relation.

Recall that a graph has nodes, representing objects, and (directed) edges, representing relationships between objects. A *path* is a sequence of contiguous edges along this graph. A *cycle* is a path that starts and ends at the same node.

Given these definitions, we can identify relational properties as follows:

- A reflexive relation has the distinct property that *every node has a self-loop*, i.e., each node a has a cycle of length 1 from a to itself.

- A symmetric relation has the property that *every edge must be two-sided*, i.e., if there exists a "forward" edge from a to b, then necessarily there has to be a "reverse" edge from b to a as well.
- A transitive relation has the "complete-the triangle" property, i.e., if there is an edge from a to b and an edge from b to c, then necessarily there is an edge from a to c.

Equivalence relations

How do we mathematically state whether two objects "behave" the same vis-a-vis some given characteristic? This is captured via the notion of *equivalence*. Formally, any relation that is

- reflexive
- symmetric
- transitive

is called an *equivalence relation*. If R is an equivalence relation and aRb, then a is said to be equivalent to b.

Some examples:

1. Define the relation $R = \{(a, b)|a - b \text{ is even}\}$ over the integers. Then, R is reflexive (since for any integer a, we always have $a - a = 0$, which is even). R is symmetric (since $a - b$ is even iff $b - a$ is even.). Moreover, R is transitive (since $a - b$ is even *and* $b - c$ is even implies that $a - c$ is even.) Therefore, R is an equivalence relation.
2. Define the *Congruence* relation over the integers as follows: for a fixed integer m, a is related to b iff $a \mod m = b \mod m$. As an **exercise**, convince yourself that congruence is an equivalence relation.
3. A couple of non-numeric examples. Let A be the set of all cars, and R be the relation defined over A as follows. Let a and b denote cars; aRb iff a and b have the same make+model+year. Then, R is an equivalence relation (prove it as an **exercise**).
4. Let A be the set of all lines on a plane. Let R be the relation defined over A as follows: line a is related to line b iff a and b are parallel. As an **exercise**, prove that R is an equivalence relation.

Every equivalence relation induces one or more *equivalence classes*. An equivalence class is defined as follows: let R be an equivalence relation on A. Then, the set of all elements related to $a \in A$ is called the equivalence class of a, denoted by $[a]$.

$$[a] = \{b|b \in A,\ (b, a) \in R\}$$

Let's go back to the above list of examples for equivalence classes. In the first example, we see that the following elements are all related to each other:

$$\{\ldots, -4, -2, 0, 2, \ldots\}$$

Similarly, the following elements are also related:

$$\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$$

In other words, we see there are precisely *two* equivalence classes: the set of all integers related to 0, and the set of all elements related to 1. (These are precisely the even and the odd integers).