Solving a CSP

Outline

I. Binary CSP

II. Constraint Propagation

III. Backtracking search

* Figures/images are from the textbook site (or by the instructor).

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Alldiff$(v_1, \ldots, v_k)$: variables $v_1, \ldots, v_k$ must have different values.

e.g., Sudoku (all variables in a row, column, or 3×3 box).

Cryptarithmetic Puzzle

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Addition constraints on the four columns:

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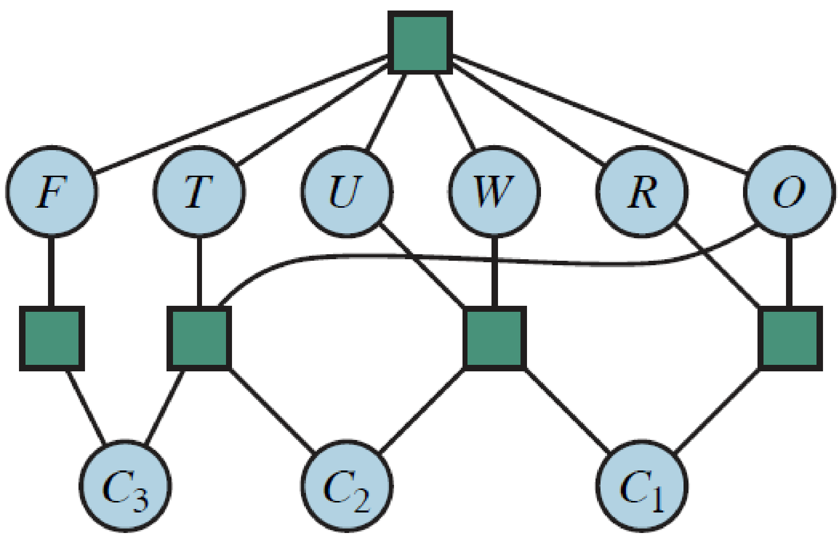

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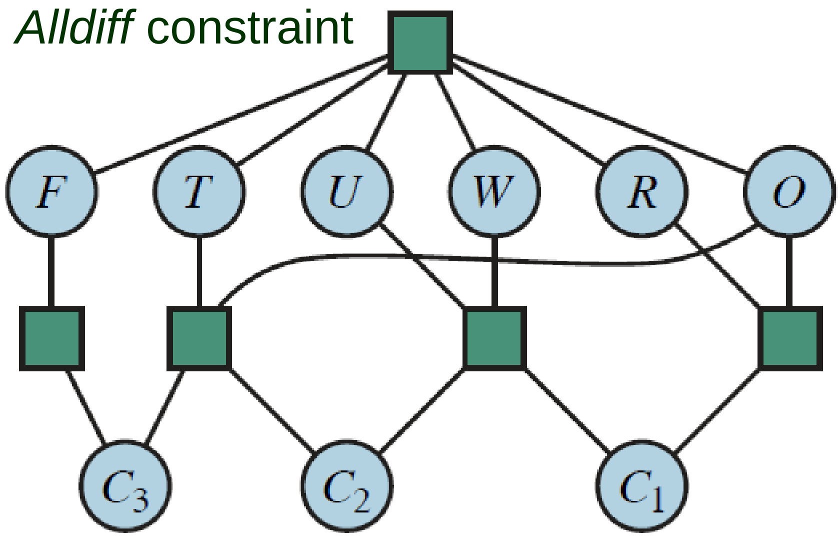
Alldiff constraint



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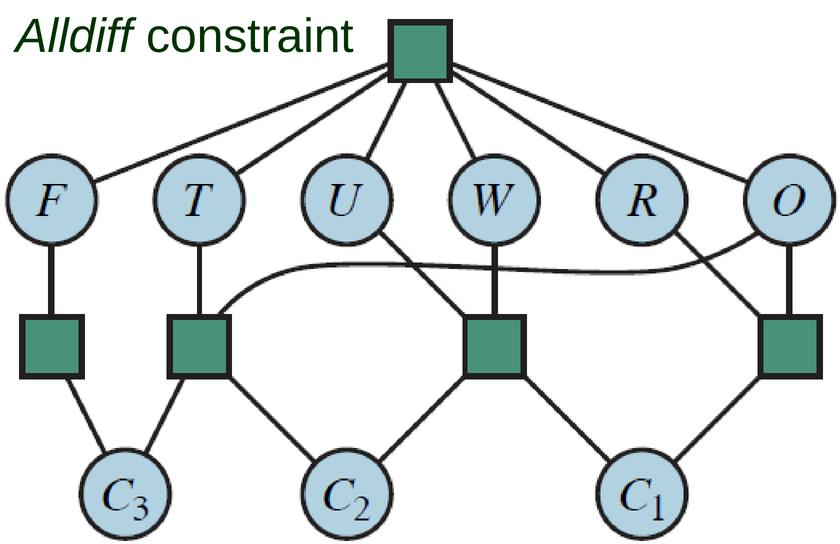

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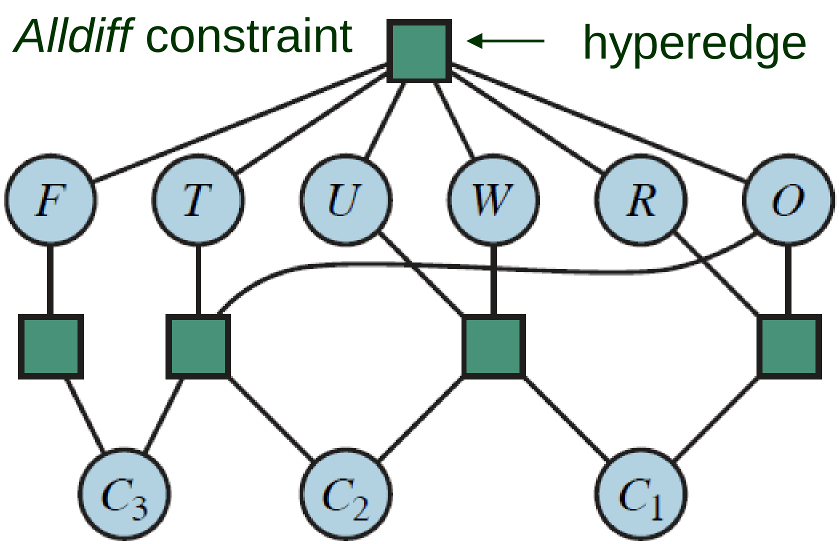

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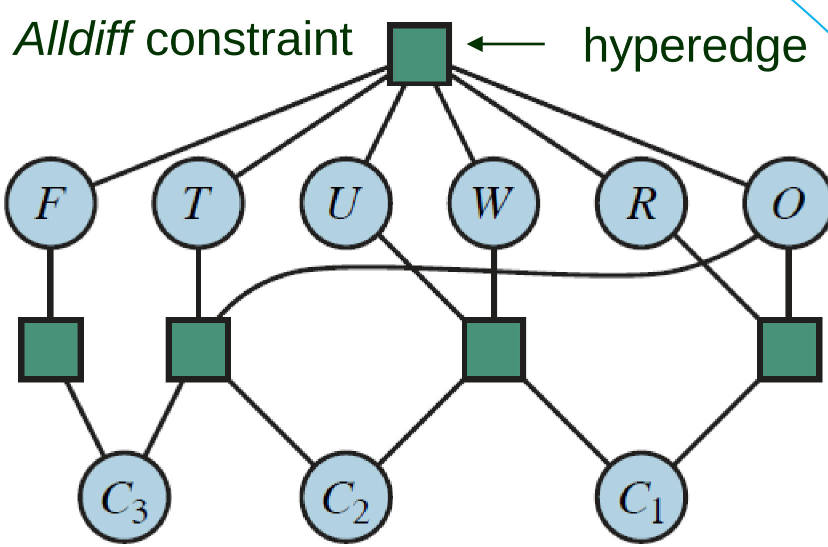

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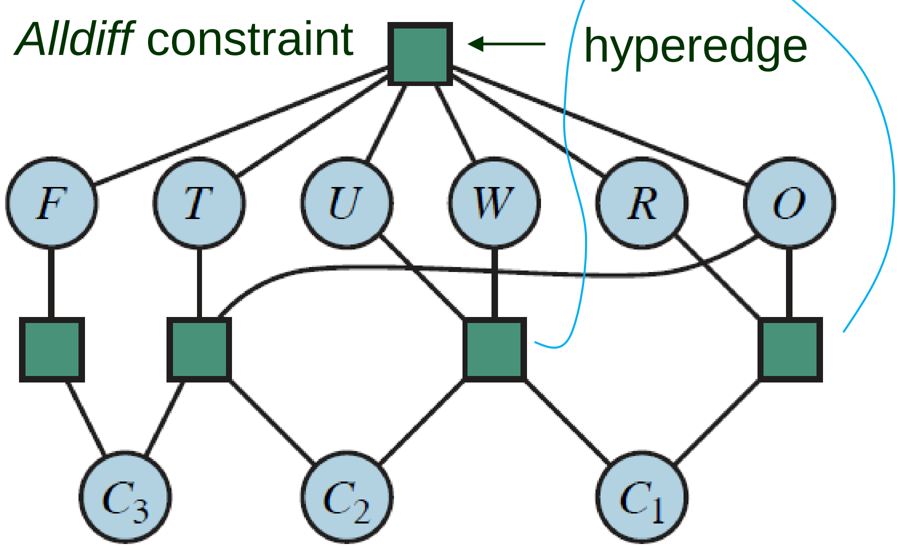

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Transformation to Binary Constraints

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Any CSP can be transformed into a binary CSP.

Advantages of a global constraint such as *Alldiff*:

- Easier and less error-prone to write.
- Allowing efficient special-purpose inference algorithms.

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Use constraints to reduce the number of legal values for a variable, which in turn reduce those for another variable, and so on.

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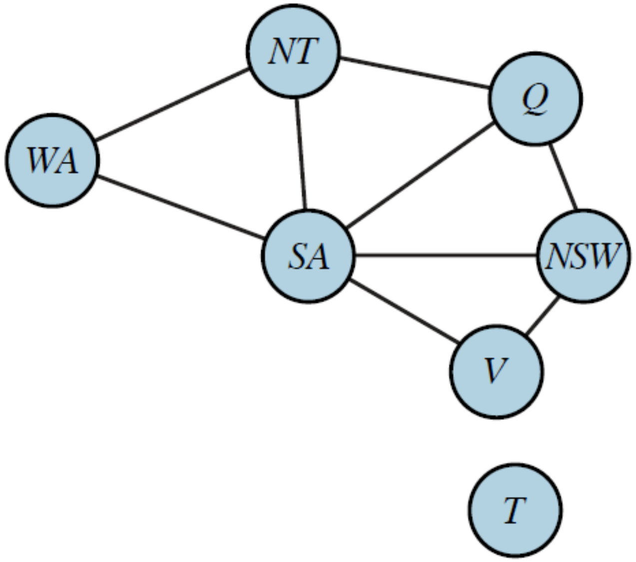

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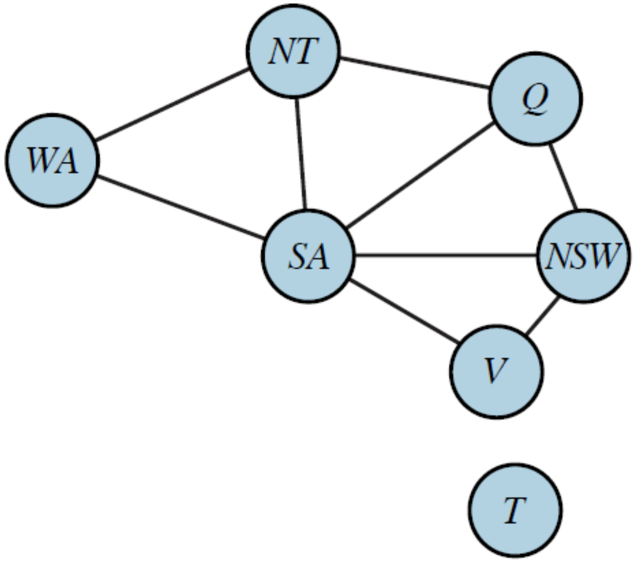

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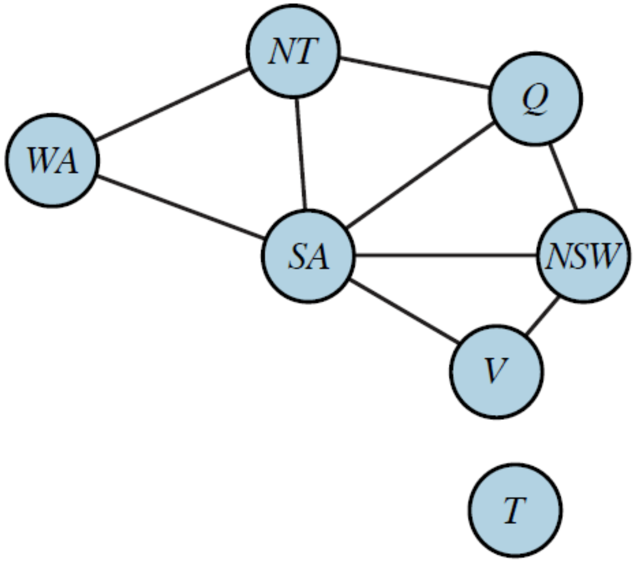

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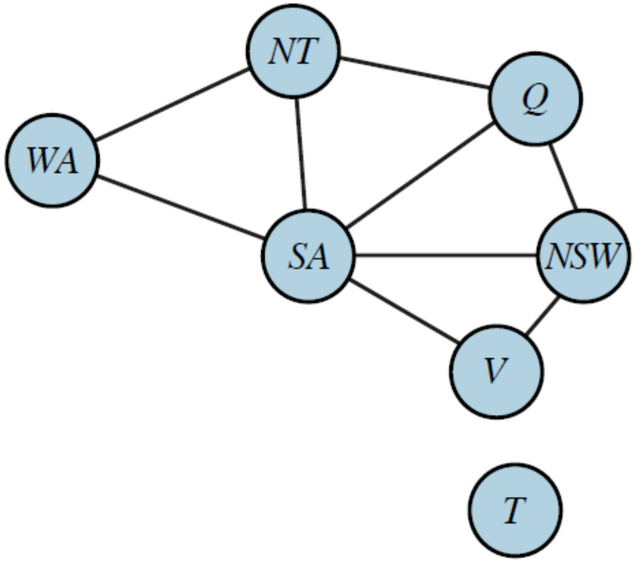

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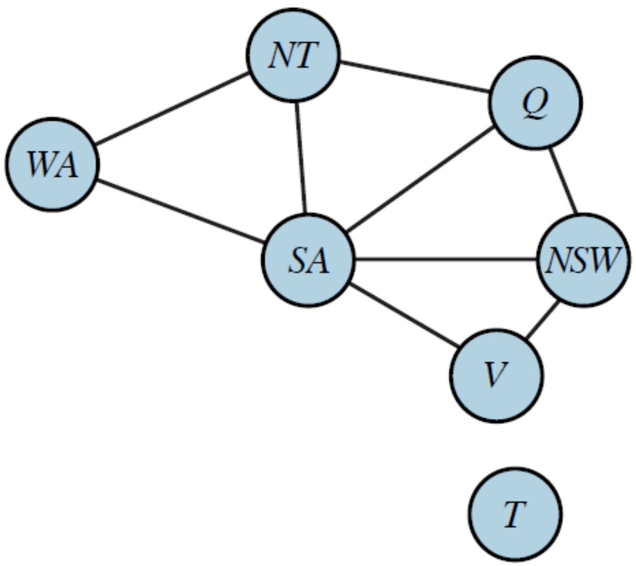

- Node consistency

Suppose South Australians dislike green.

$\Downarrow$

Domain for *SA*: {red, blue}

Eliminate all the unary constraints by reducing the corresponding variables at the start.

Arc Consistency

A variable X_i is *arc-consistent* with respect to (w.r.t.) another variable X_j if for every value in D_i there exists a value in D_j that satisfies the binary constraint on the arc (i.e., edge (X_i, X_j)).

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The constraint graph is *arc-consistent* if every variable is arc consistent with every other variable.

Application of Arc Consistency

Example 1

$$Y = X^2 \quad \text{where } X, Y \in \{0, 1, \ldots, 9\}$$

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Example 2 Australia map-coloring

$SA \neq WA$

No effect on the domain {*red*, *green*, *blue*} of each variable.

Arc-Consistency Algorithm AC-3

function AC-3(*csp*) **returns** false if an inconsistency is found and true otherwise
  queue ← a queue of arcs, initially all the arcs in *csp*

  while *queue* is not empty **do**
    $(X_i,\ X_j) \leftarrow$ POP(*queue*)
    if REVISE(*csp*, $X_i,\ X_j$) **then**
      if size of $D_i\ =\ 0$ **then return** *false*
      for each X_k **in** X_i.NEIGHBORS - $\{X_j\}$ **do**
        add $(X_k,\ X_i)$ to *queue*
  return *true*

// propagate to other variables sharing a
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Domain size $\leq d$, c binary constraints.

- Each arc inserted into the queue d times.
- Each checking takes $O(d^2)$ time.

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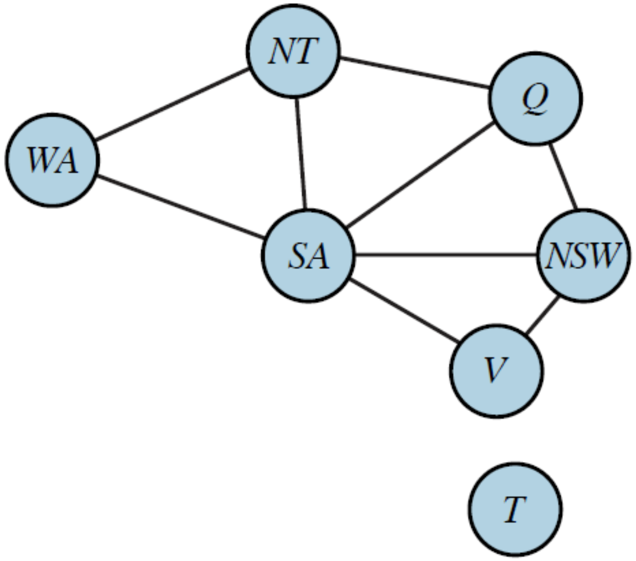

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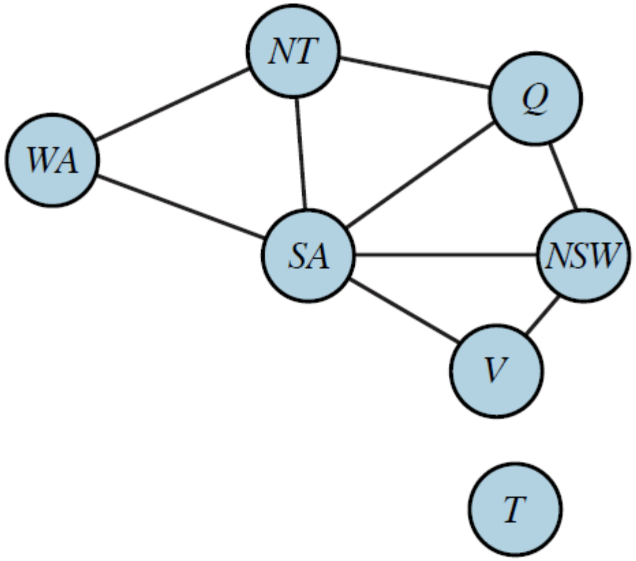

No solution if using two colors even though every constraint can be satisfied individually.

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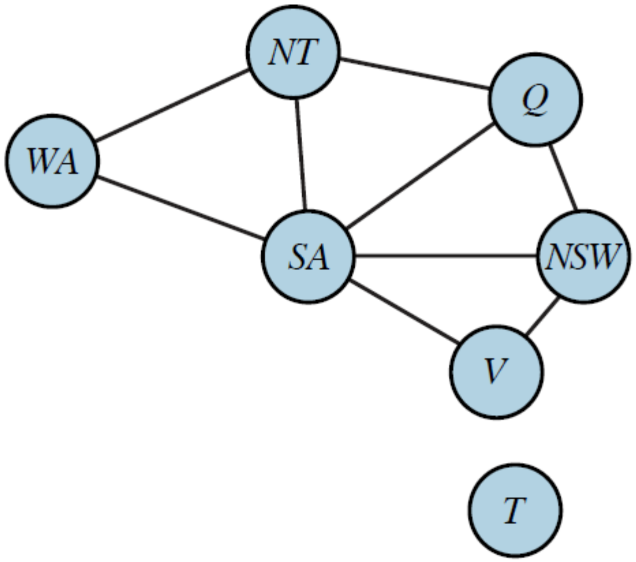

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Path consistency checks implicit constraints inferable across triples of variables along a path.

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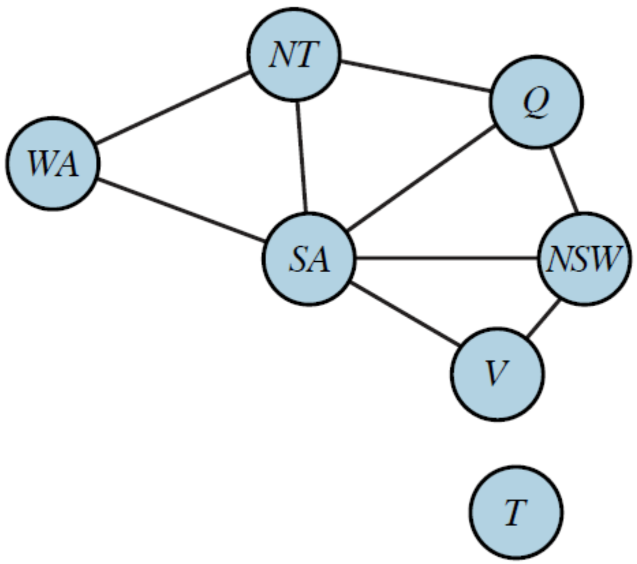

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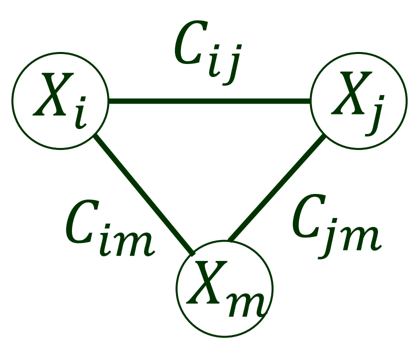
Path consistency checks implicit constraints inferable across triples of variables along a path.

$\{X_i, X_j\}$ is *path-consistent* w.r.t. X_m if for every assignment to X_i, X_j consistent with their constraint C_{ij} (if exists), there exists an assignment to X_m that satisfies the constraints C_{im} and C_{jm} between them and X_m.



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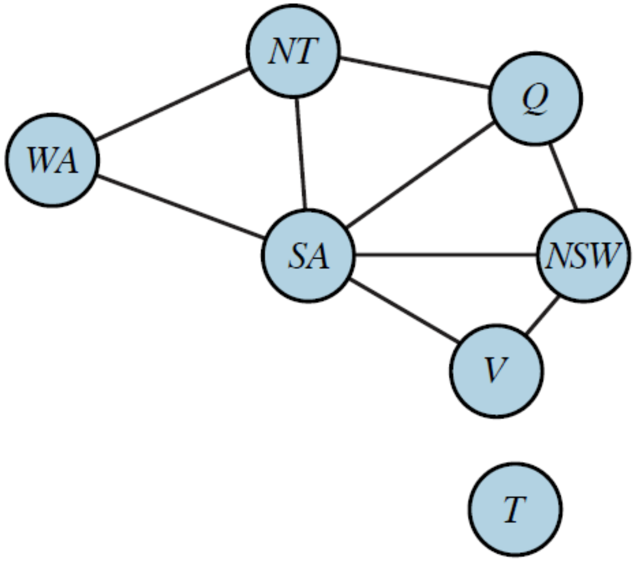

Make {*WA*, *SA*} path-consistent w.r.t. *NT*.

Assume two colors only: *red*, *blue*.

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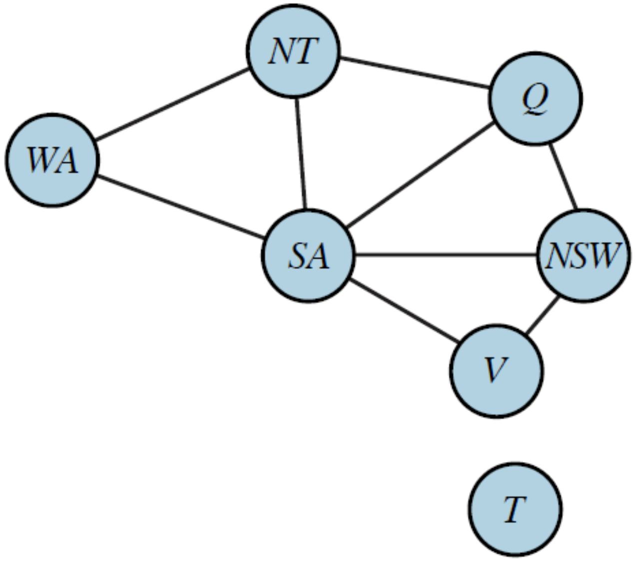

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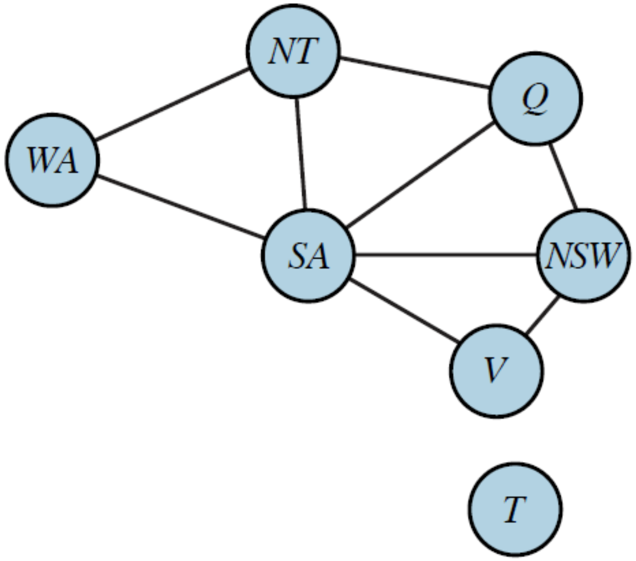

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No assignment for *NT* in either case!

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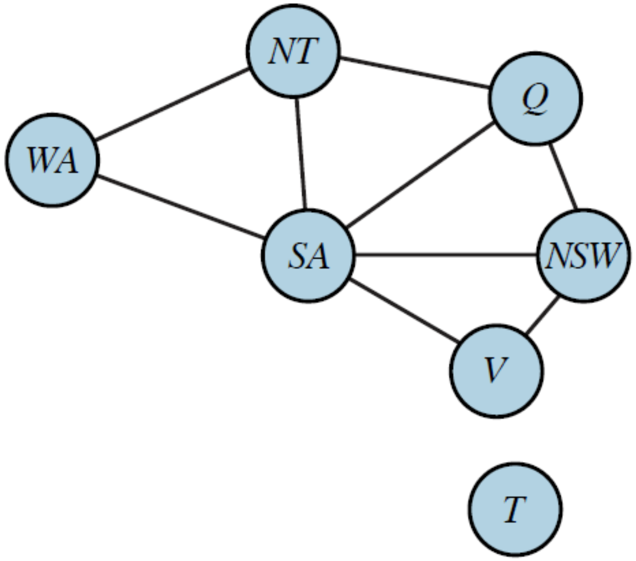

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Eliminate both assignments to *WA* and *SA*.

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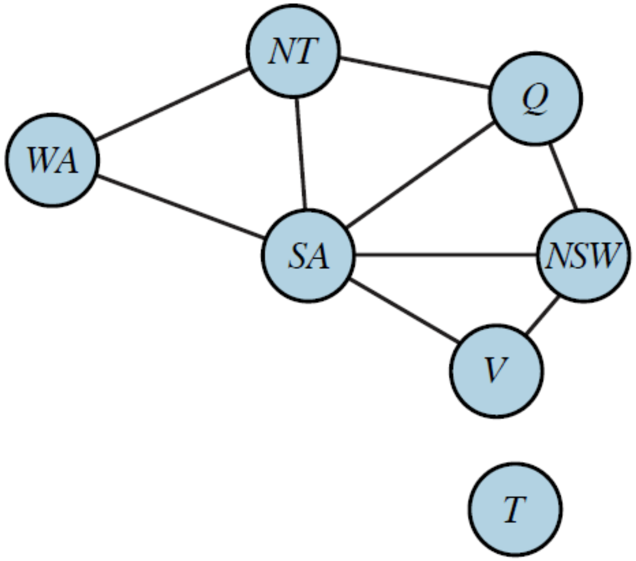

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No solution to the problem.

Bounds Propagation

Two flights F_1 and F_2 have domains:

$$D_1 = [0,165] \text{ and } D_2 = [0,385]$$

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Constraint: the two flights together carry 420 people.

$\Downarrow$ domain reduction

$$D_1 = [420 - 385,165] = [35,165]$$
$$D_2 = [420 - 165,385] = [255,385]$$

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$$D_1 = [0{,}165] \text{ and } D_2 = [0{,}385]$$

max capacity

Constraint: the two flights together carry 420 people.

$\Downarrow$ domain reduction

$$D_1 = [420 - 385{,}165] = [35{,}165]$$
$$D_2 = [420 - 165{,}385] = [255{,}385]$$

Bounds-consistent if for any two variables X, Y, whether X takes on its lower- or upper-bound values, there exists some value of Y that satisfies the constraint between X and Y.

Bounds Propagation

Two flights F_1 and F_2 have domains:

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* By the above definition, for some value of X between its lower and upper bounds, there may not exist a value of Y to meet the constraint (e.g., a nonlinear constraint).

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Sudoku

Fill the digits 1 to 9 in a 9×9 grid such that no digit appears twice in any row, column, or 3×3 box.

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
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F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

	1	2	3	4	5	6	7	8	9
A	4	8	3	9	2	1	6	5	7
B	9	6	7	3	4	5	8	2	1
C	2	5	1	8	7	6	4	9	3
D	5	4	8	1	3	2	9	7	6
E	7	2	9	5	6	4	1	3	8
F	1	3	6	7	9	8	2	4	5
G	3	7	2	6	8	9	5	1	4
H	8	1	4	2	5	3	7	6	9
I	6	9	5	4	1	7	3	8	2

Sudoku as a CSP

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
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G			2	6		9	5		
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Variables: $A1, \dots, A9, B1, \dots I9$

Domain: $D = \{1,2,\dots,9\}$

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	1	2	3	4	5	6	7	8	9
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E	7								8
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27 *Alldiff* constraints:

Sudoku as a CSP

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G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

Variables: $A1, \dots, A9, B1, \dots I9$

Domain: $D = \{1,2, \dots, 9\}$

27 *Alldiff* constraints:

Alldiff$(A1, A2, A3, A4, A5, A6, A7, A8, A9)$

$\vdots$

Alldiff$(I1, I2, I3, I4, I5, I6, I7, I8, I9)$

Sudoku as a CSP

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
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I			5		1		3		

Variables: $A1, \dots, A9, B1, \dots I9$

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Alldiff$(A1, A2, A3, A4, A5, A6, A7, A8, A9)$

$\vdots$

Alldiff$(I1, I2, I3, I4, I5, I6, I7, I8, I9)$

Alldiff$(A1, B1, C1, D1, E1, F1, G1, H1, I1)$

$\vdots$

Alldiff$(A9, B9, C9, D9, E9, F9, G9, H9, I9)$

Sudoku as a CSP

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

Variables: $A1, \ldots, A9, B1, \ldots I9$

Domain: $D = \{1,2, \ldots, 9\}$

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Alldiff$(A1, A2, A3, A4, A5, A6, A7, A8, A9)$

$\vdots$

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Alldiff$(A1, B1, C1, D1, E1, F1, G1, H1, I1)$

$\vdots$

Alldiff$(A9, B9, C9, D9, E9, F9, G9, H9, I9)$

Alldiff$(A1, A2, A3, B1, B2, B3, C1, C2, C3)$

$\vdots$

Alldiff$(A7, A8, A9, B7, B8, B9, C7, C8, C9)$

Sudoku as a CSP

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

Variables: $A1, \ldots, A9, B1, \ldots I9$

Domain: $D = \{1,2, \ldots, 9\}$

27 *Alldiff* constraints:

Alldiff$(A1, A2, A3, A4, A5, A6, A7, A8, A9)$

$\vdots$

Alldiff$(I1, I2, I3, I4, I5, I6, I7, I8, I9)$

Alldiff$(A1, B1, C1, D1, E1, F1, G1, H1, I1)$

$\vdots$

Alldiff$(A9, B9, C9, D9, E9, F9, G9, H9, I9)$

Alldiff$(A1, A2, A3, B1, B2, B3, C1, C2, C3)$

$\vdots$

Alldiff$(A7, A8, A9, B7, B8, B9, C7, C8, C9)$

A CSP solver can handle thousands of puzzles per second!

Sudoku as a CSP

	1	2	3	4	5	6	7	8	9
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Variables: $A1, \ldots, A9, B1, \ldots I9$

Domain: $D = \{1,2, \ldots, 9\}$

27 *Alldiff* constraints:

Alldiff$(A1, A2, A3, A4, A5, A6, A7, A8, A9)$

$\vdots$

Alldiff$(I1, I2, I3, I4, I5, I6, I7, I8, I9)$

Alldiff$(A1, B1, C1, D1, E1, F1, G1, H1, I1)$

$\vdots$

Alldiff$(A9, B9, C9, D9, E9, F9, G9, H9, I9)$

Alldiff$(A1, A2, A3, B1, B2, B3, C1, C2, C3)$

$\vdots$

Alldiff$(A7, A8, A9, B7, B8, B9, C7, C8, C9)$

A CSP solver can handle thousands of puzzles per second!
Only the simplest ones can be solved by AC-3.

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
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	1	2	3	4	5	6	7	8	9
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F			6	7		8	2		
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Consider E6:

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Consider E6:

$$D_{E6} \leftarrow \{1,2, \ldots, 9\}$$

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	1	2	3	4	5	6	7	8	9
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Consider E6:

$$D_{E6} \leftarrow \{1,2,\ldots,9\}$$

$\Downarrow$ constraints in the box

$$D_{E6} \leftarrow D_{E6} \setminus \{1,2,7,8\}$$

$$= \{3,4,5,6,9\}$$

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$$= \{4\}$$

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	1	2	3	4	5	6	7	8	9
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A			3		2		6		
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Consider E6:

$$D_{E6} \leftarrow \{1,2,\dots,9\}$$

constraints in the box

$$D_{E6} \leftarrow D_{E6} \setminus \{1,2,7,8\}$$

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Consider I6: $D_{I6} \leftarrow \{1,2,\dots,9\}$

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2		6		
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$\Downarrow$ constraints in the column

$$D_{I6} \leftarrow D_{I6} \setminus \{2,3,4,5,6,8,9\} = \{1,7\}$$

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B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
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Consider A6:

Consider I6:

$$D_{I6} \leftarrow \{1,2,\dots,9\}$$

$\Downarrow$ constraints in the column

$$D_{I6} \leftarrow D_{I6} \setminus \{2,3,4,5,6,8,9\} = \{1,7\}$$

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H	8			2		3			9
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Consider E6:

$$D_{E6} \leftarrow \{1,2,\ldots,9\}$$

$\Downarrow$ constraints in the box

$$D_{E6} \leftarrow D_{E6} \setminus \{1,2,7,8\}$$
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$$D_{E6} \leftarrow D_{E6} \setminus \{2,3,5,6,8,9\}$$
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H	8			2		3			9
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Consider E6:

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$$D_{E6} \leftarrow D_{E6} \setminus \{1,2,7,8\}$$
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Consider I6:

$$D_{I6} \leftarrow \{1,2,\dots,9\}$$

$\Downarrow$ constraints in the column

$$D_{I6} \leftarrow D_{I6} \setminus \{2,3,4,5,6,8,9\} = \{1,7\}$$

$\Downarrow$ constraints in the box

$$D_{I6} \leftarrow D_{I6} \setminus \{1,2,3,6,9\} = \{7\}$$

Consider A6:

$$D_{A6} \leftarrow \{1,2,\dots,9\}$$

$\Downarrow$ constraints in the column

$$D_{A6} \leftarrow \{1\}$$

Example of CSP Solution

	1	2	3	4	5	6	7	8	9
A			3		2	1	6		
B	9			3		5			1
C			1	8		6	4		
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F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1	7	3		

Consider E6:

$$D_{E6} \leftarrow \{1,2,\dots,9\}$$

$\Downarrow$ constraints in the box

$$D_{E6} \leftarrow D_{E6} \setminus \{1,2,7,8\}$$

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$$D_{E6} \leftarrow D_{E6} \setminus \{2,3,5,6,8,9\}$$

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Consider I6:

$$D_{I6} \leftarrow \{1,2,\dots,9\}$$

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$$D_{I6} \leftarrow D_{I6} \setminus \{2,3,4,5,6,8,9\} = \{1,7\}$$

$\Downarrow$ constraints in the box

$$D_{I6} \leftarrow D_{I6} \setminus \{1,2,3,6,9\} = \{7\}$$

Consider A6:

$$D_{A6} \leftarrow \{1,2,\dots,9\}$$

$\Downarrow$ constraints in the column

$$D_{A6} \leftarrow \{1\}$$

III. Backtracking Search

- Constraint propagation often ends with partial solutions.
- Backtracking search can be employed to extend them to full solutions.

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Solve a CSP using depth-limited search.

n variables of domain size d

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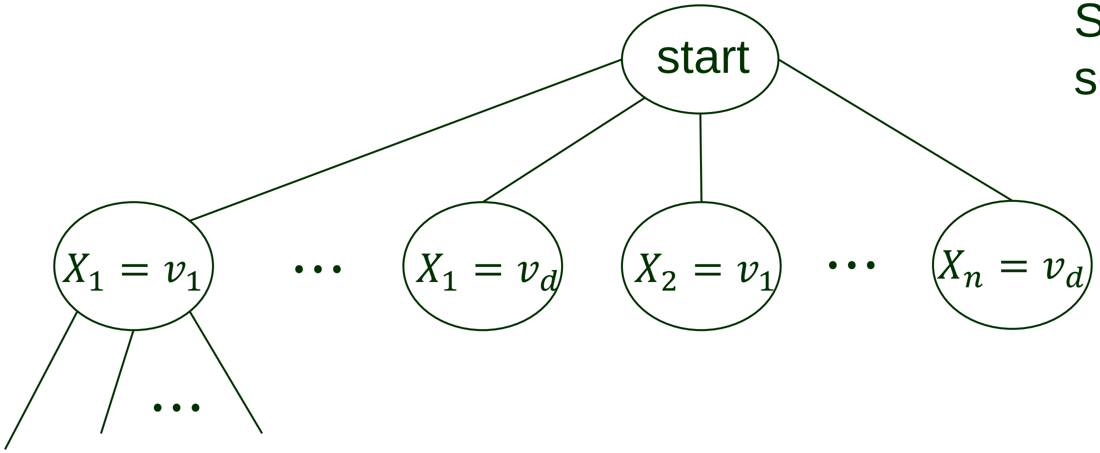

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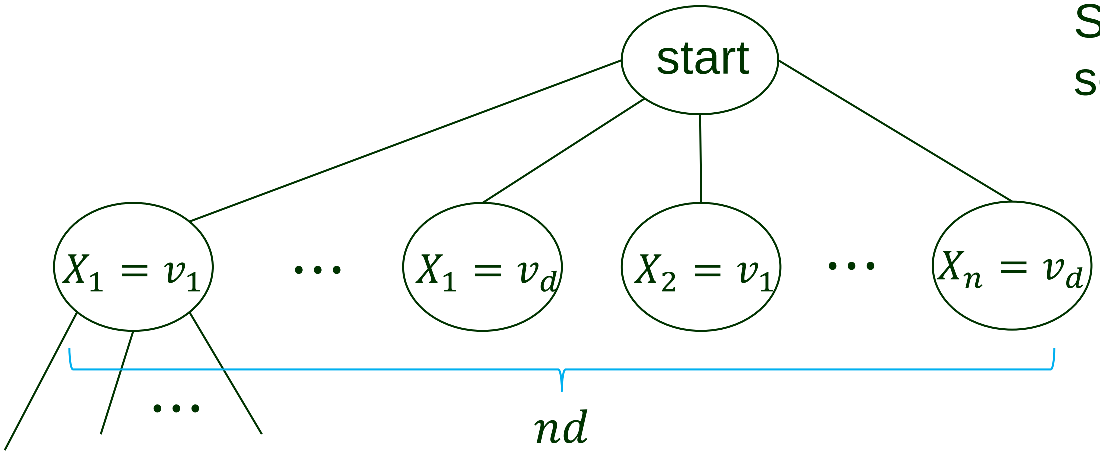

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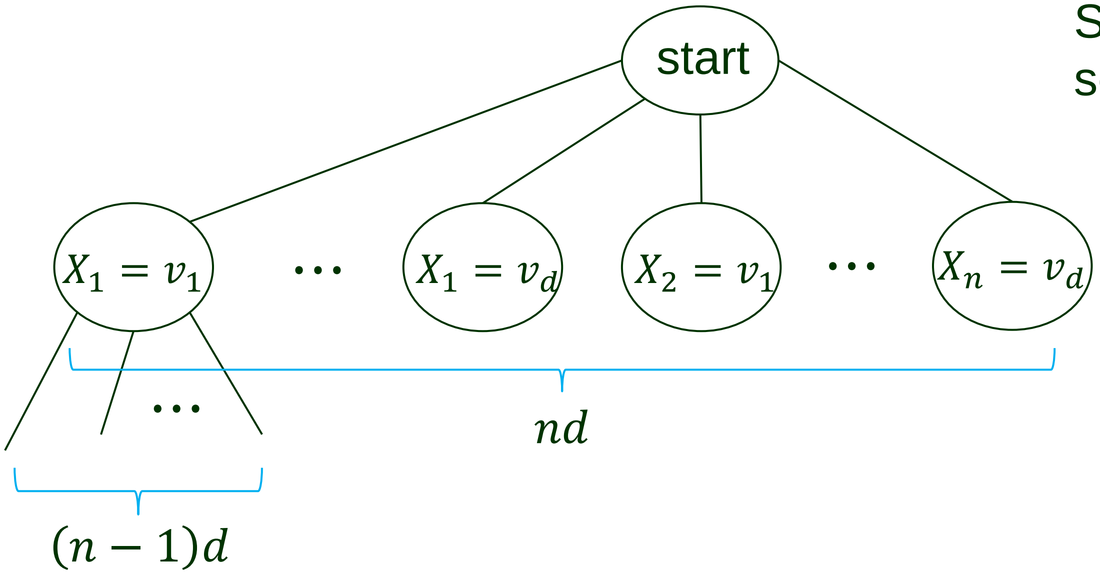

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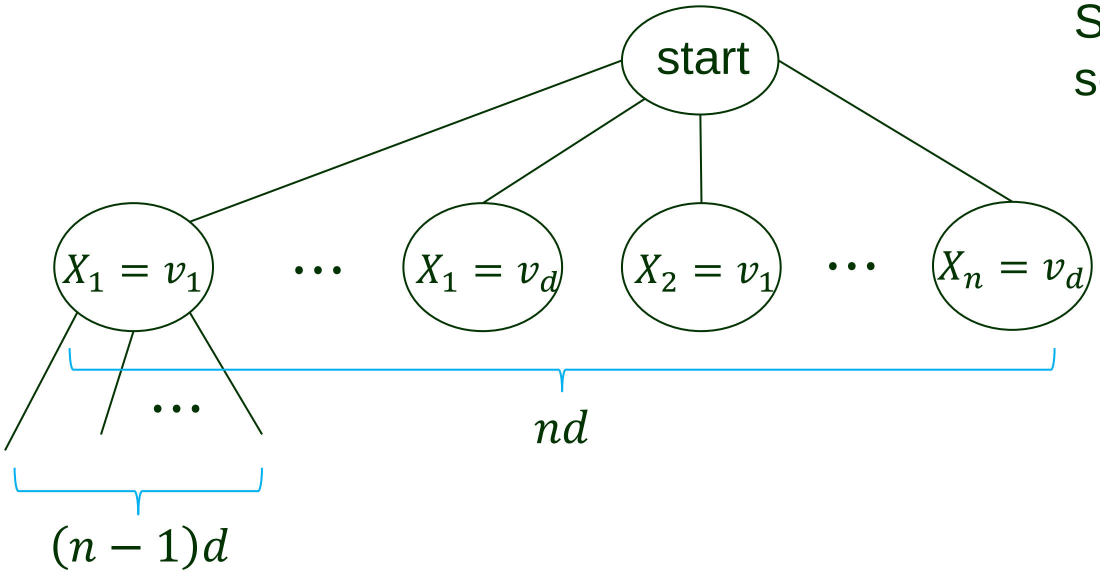

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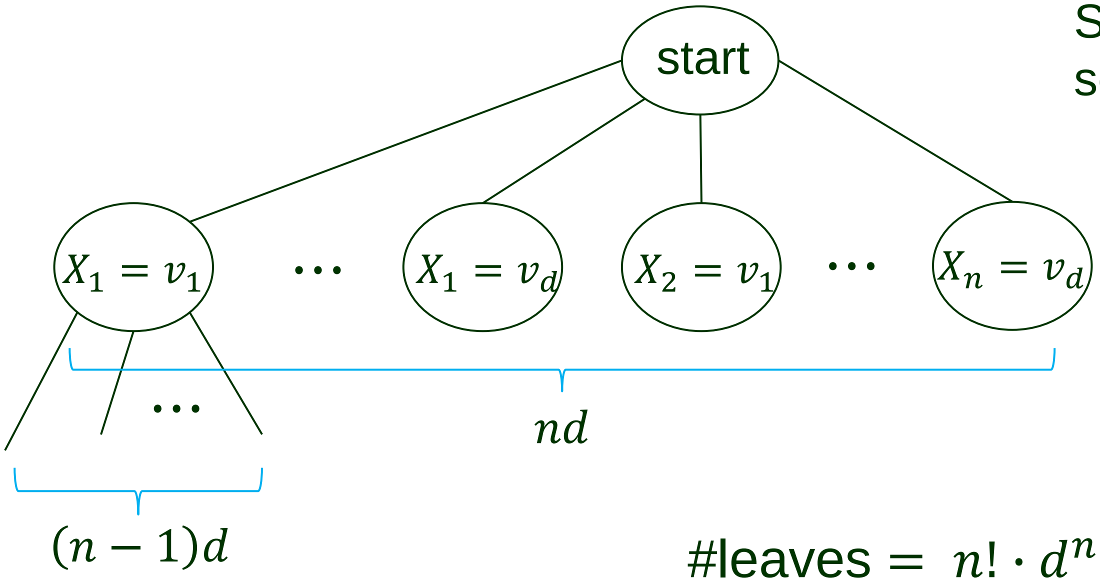

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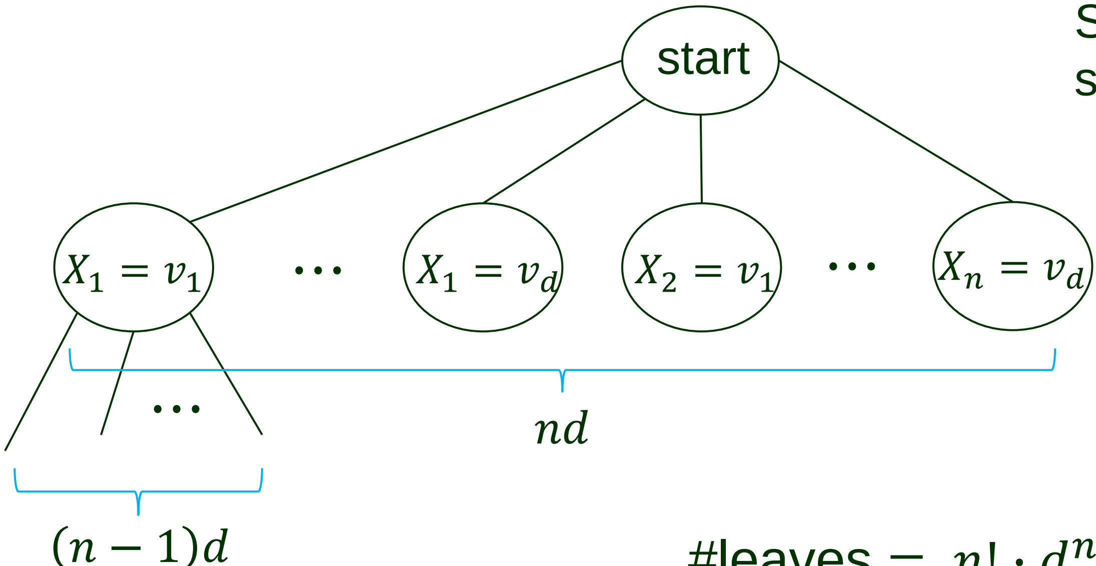

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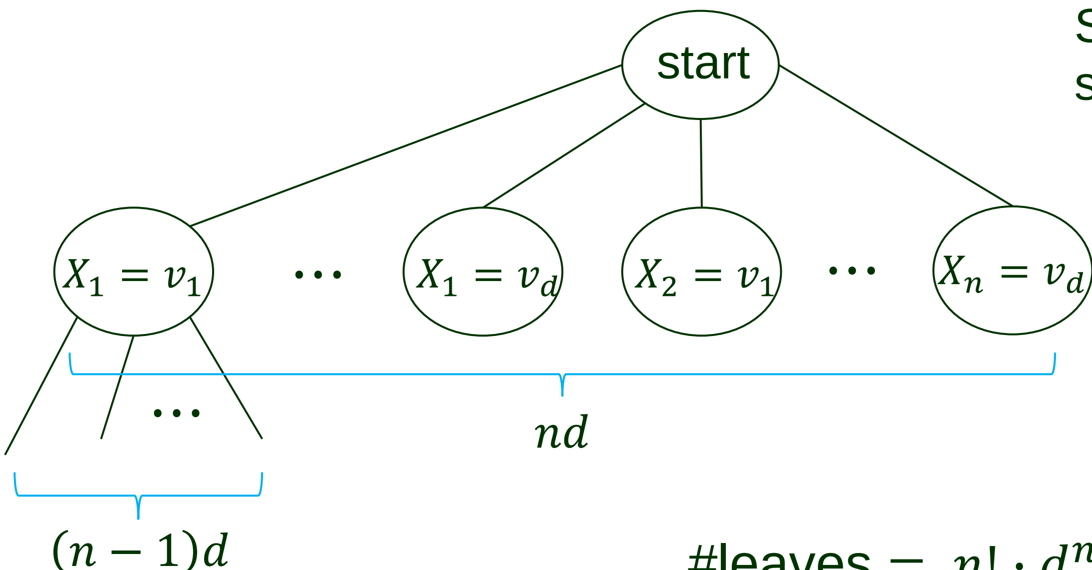

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How to get back to d^n?

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At the root choose between

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but not between

SA = *blue* and *NSW* = *red*

Backtracking Algorithm

- Repeatedly chooses an unassigned variable X_i.
- Tries all values $v_j \in D_i$ (its domain).

  - Add $X_i = v_j$ to the partial solution (after consistency checking).

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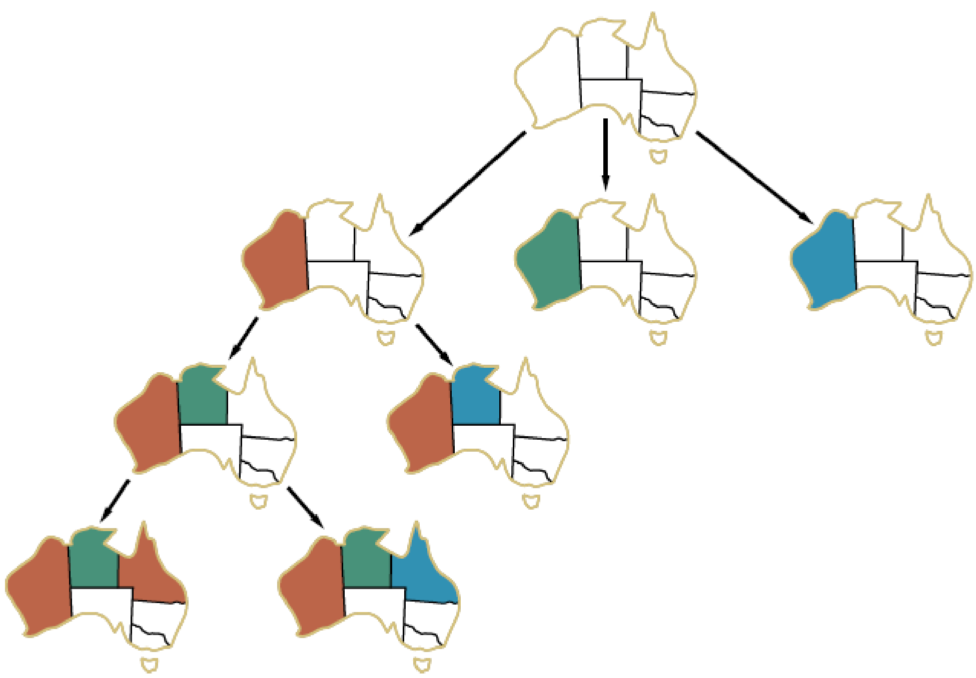

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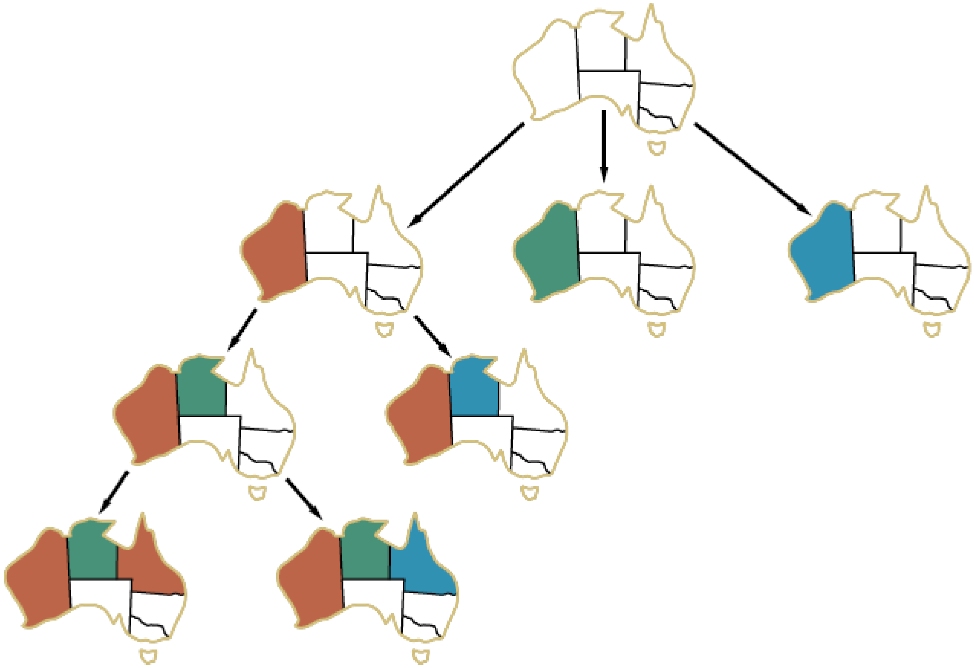

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