Basics of Probability

Outline

I. Acting under uncertainty

II. Probability model

III. Probability distribution

* Figures are from the textbook site or by the instructor.

** A small part of the notes are adapted from those by Dr. Jin Tian.

I. Acting Under Uncertainty

Keeping track of a belief state has several drawbacks:

- ♣ Large belief state full of unlikely possibilities because every possible explanation of the percept needs to be considered.
- ♣ Correct contingent plan growing arbitrarily large to handle every eventuality.
- ♣ No successful plan guaranteed sometimes yet action required.

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Make a rational decision depending on

♦ relative importance of various goals, and

♦ likelihood that, and degree to which, they will be achieved.

From Categorical to Uncertain Beliefs

- ♦ Intelligent behavior requires knowledge about the world.

- ♦ Propositional and first-order logics are effective for representing and reasoning with categorical beliefs about the world.

- ♣ Due to uncertainty, any logical sentence could be true, false or unknown.

- ♣ Logical approach can break down.

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Fix the rule by augment the LHS with all qualifications required for a cavity to cause a toothache!

- ♠ Too much work.
- ♠ Medical science has no complete domain theory.
- ♠ Not all the necessary tests have been or can be run on a patient.

Example of Reasoning Under Uncertainty

Beliefs

- If a patient has lung cancer, there is a 60% chance that an X-ray test will come back positive ; and a 40% percent chance negative.
- If a patient does not have lung cancer, there is 2% percent chance that an X-ray test will come back positive ; and a 98% percent chance negative.
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Probability theory provides a framework for representing and reasoning under uncertainty

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$\omega \in \Omega$ is a sample point, possible world, or atomic event.

♣ mutually exclusive

♣ exhaustive

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- b is the event that the results of rolling two dice sum to 10.

  $P(\text{Total} = 10) = P(b) = P((4,6)) + P((5,5)) + P((6,4)) = 1/12$

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The prior of a is 1/6, and the posterior of a given b is 1/3.

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Use of Proportional Logic

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$$P(cavity \mid \neg toothache, teen) = 0.1$$

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Example 4 distinct atomic events (or 4 possible worlds):

$\textit{cavity} \wedge \textit{toothache}$

$\neg \textit{cavity} \wedge \textit{toothache}$

$\textit{cavity} \wedge \neg \textit{toothache}$

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III. Probability Distribution

- Probabilities of all the possible values of a random variable.

$$P(Weather = sun) = 0.6$$
$$P(Weather = rain) = 0.1$$
$$P(Weather = cloud) = 0.29$$
$$P(Weather = snow) = 0.01$$

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$\mathbf{P}(X \mid Y)$ gives the values of $P(X = x_i \mid Y = y_j\}$ for all i, j.

Joint Probability Distribution

Joint probability distribution gives the probability of every atomic event on a set of r.v.s (i.e., every combination of their values).

$\mathbf{P}(\mathit{Weather}, \mathit{Cavity})$ is a 4×2 matrix.

Cavity \ Weather	*sun*	*rain*	*cloud*	*snow*
true	0.144	0.02	0.016	0.02
false	0.576	0.08	0.064	0.08

Concise **P** Notation

$$\boldsymbol{P}(Weather, Cavity) = \boldsymbol{P}(Weather \mid Cavity)\, \boldsymbol{P}(Cavity)$$

{ *sun*, *rain*, *cloud*, *snow* }   { *true*, *false* }

replaces 8 equations (corresponding to 8 possible worlds)

$$P(W = sun \wedge C = true) = P(W = sun \mid C = true)P(C = true)$$
$$P(W = rain \wedge C = true) = P(W = rain \mid C = true)P(C = true)$$
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Kolmogorov's Axioms

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2. $\sum_{\omega \in \Omega} P(\omega) = 1$

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The rest of probability theory are built up from these axioms.

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Inconsistent Beliefs

Proposition	Agent 1's belief	Agent 2 bets	Agent 1 bets	Agent 1 payoffs for each outcome a, b	$a, \neg b$	$\neg a, b$	$\neg a, \neg b$
a	0.4	\$4 on a	\$6 on $\neg a$	–\$6	–\$6	\$4	\$4
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$a \vee b$	0.8	\$2 on $\neg(a \vee b)$	\$8 on $a \vee b$	\$2	\$2	\$2	–\$8
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- Agent 2 can devise a set of bets to guarantee a win.

Inconsistent Beliefs

Agent 2 will lose \$4 if $\neg a$

Agent 1 will lose \$6 if $\neg\neg a = a$

Proposition	Agent 1's belief	Agent 2 bets	Agent 1 bets	Agent 1 payoffs for each outcome a, b	$a, \neg b$	$\neg a, b$	$\neg a, \neg b$
a	0.4	$\boxed{\$4 \text{ on } a}$	$\boxed{\$6 \text{ on } \neg a}$	–\$6	–\$6	\$4	\$4
b	0.3	\$3 on b	\$7 on $\neg b$	–\$7	\$3	–\$7	\$3
$a \vee b$	0.8	\$2 on $\neg(a \vee b)$	\$8 on $a \vee b$	\$2	\$2	\$2	–\$8
				–\$11	–\$1	–\$1	–\$1

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