Approximate Inference in Bayesian Networks

Outline

I. Direct sampling methods

II. Rejection sampling

III. Importance sampling

* Figures are either from the textbook site.

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- Two families of algorithms: direct sampling and Markov chain sampling.

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inputs: bn, a Bayesian network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$

$\mathbf{x} \leftarrow$ an event with n elements
for each variable X_i **in** $X_1, \ldots, X_n$ **do**
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from the domain of
X_i, e.g., true or false

The Sprinkler Network

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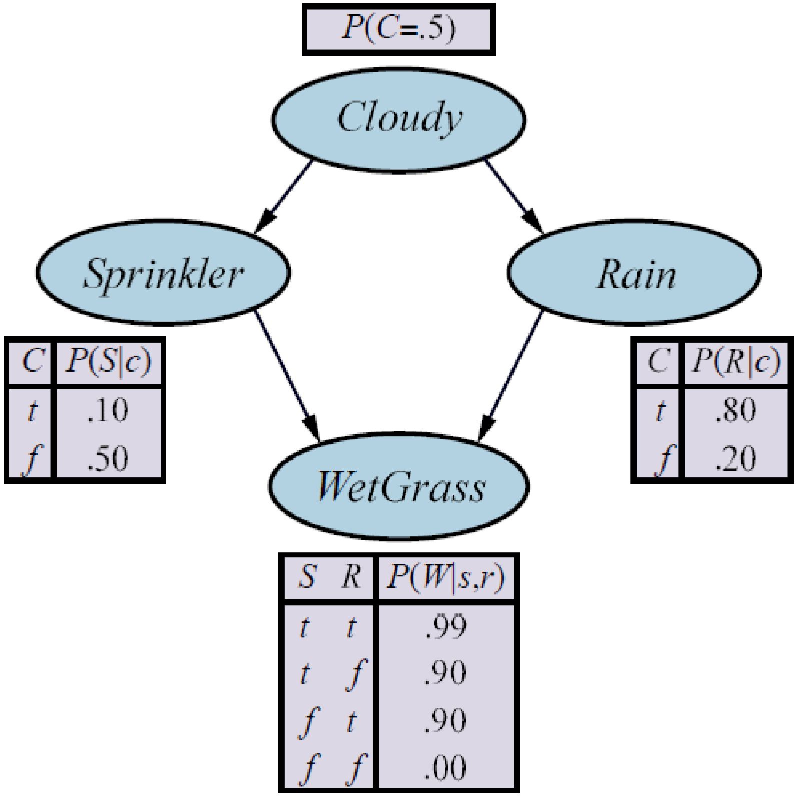
- If it’s cloudy, she usually does not turn on the sprinkler.
- The grass will be wet if the sprinkler is on, or if it rains during the day.



$P(C{=}.5)$

C	$P(S \mid c)$
t	.10
f	.50

C	$P(R \mid c)$
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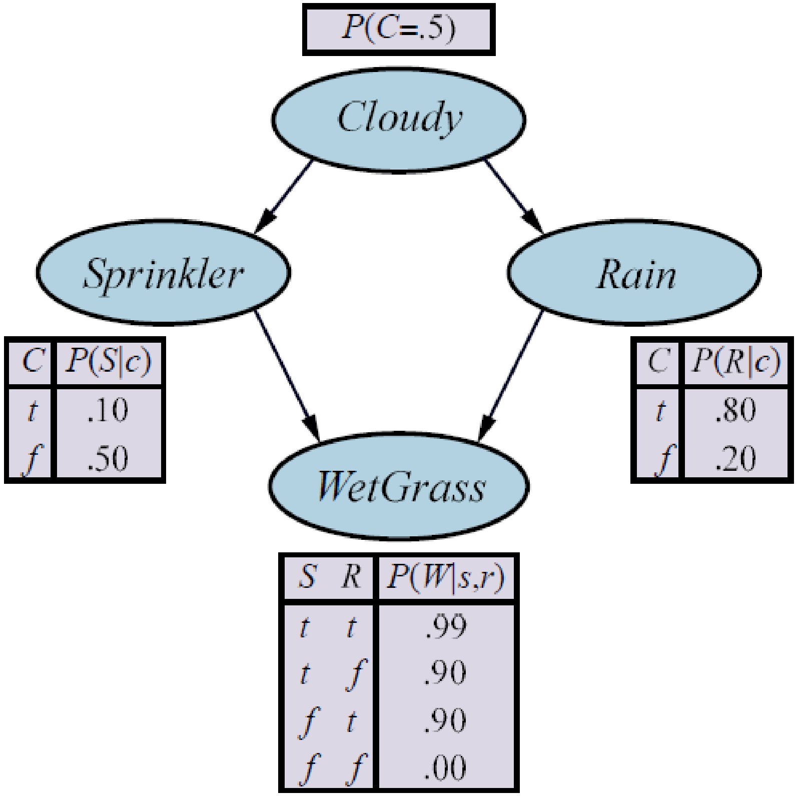

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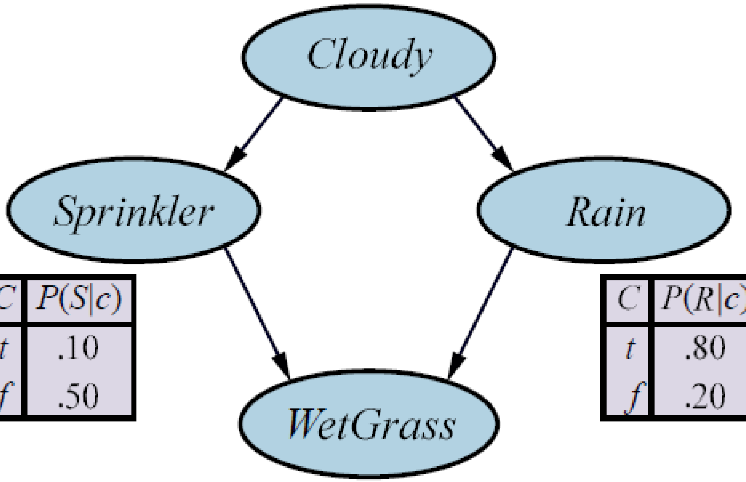

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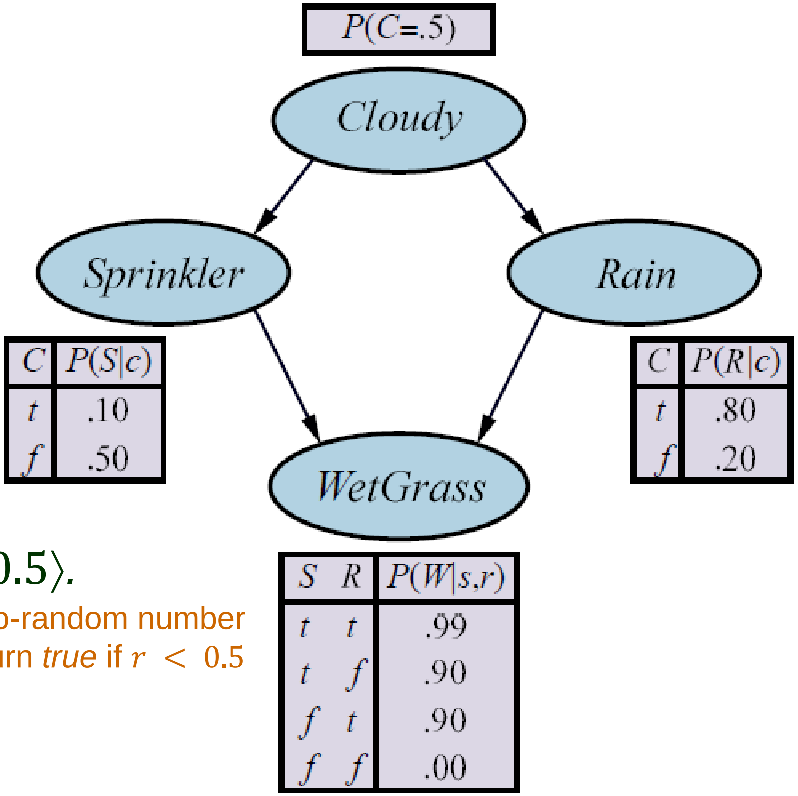

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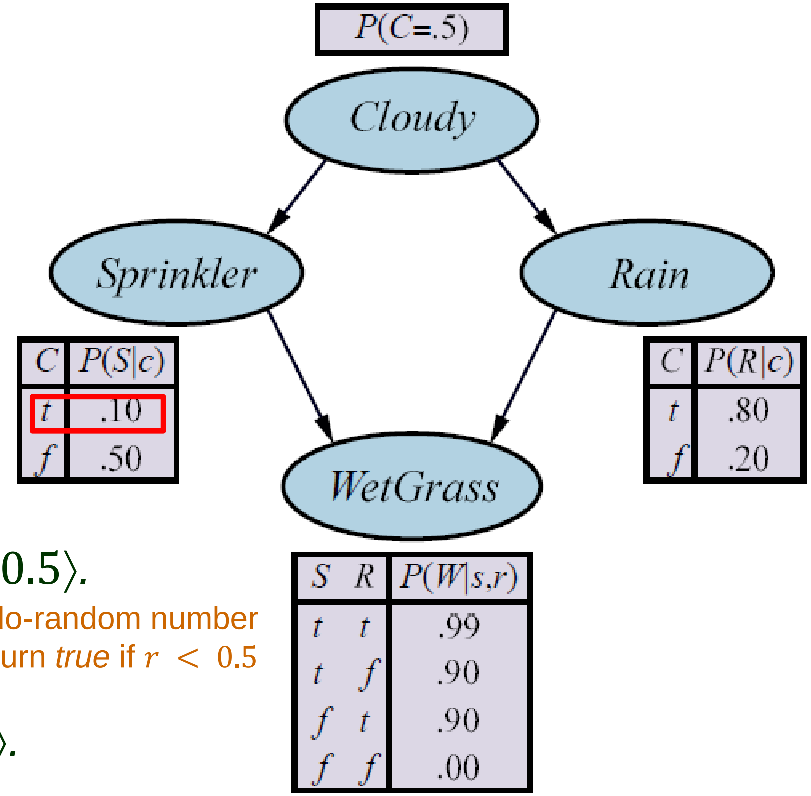

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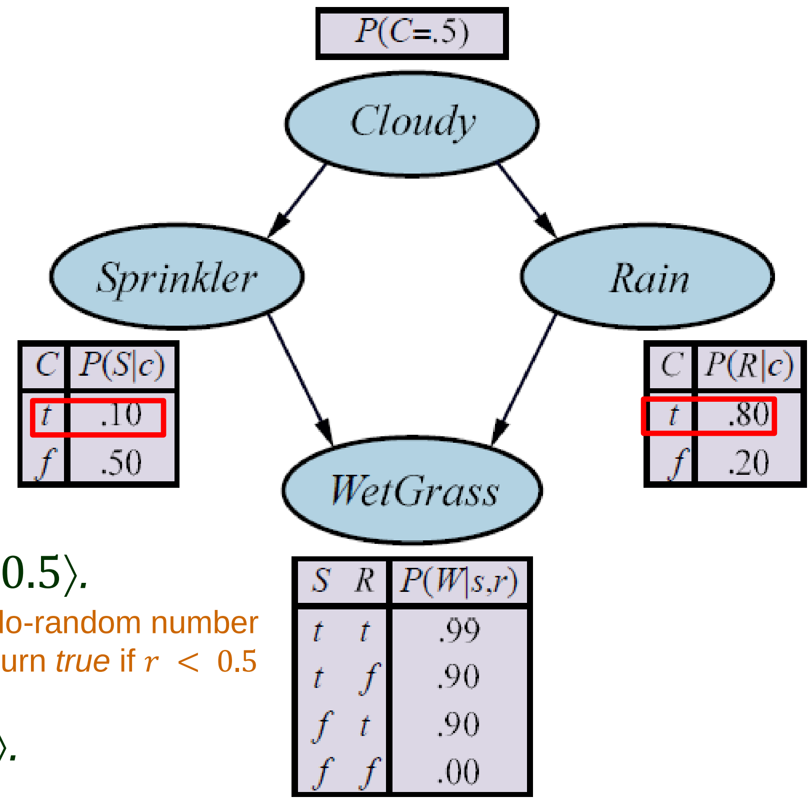

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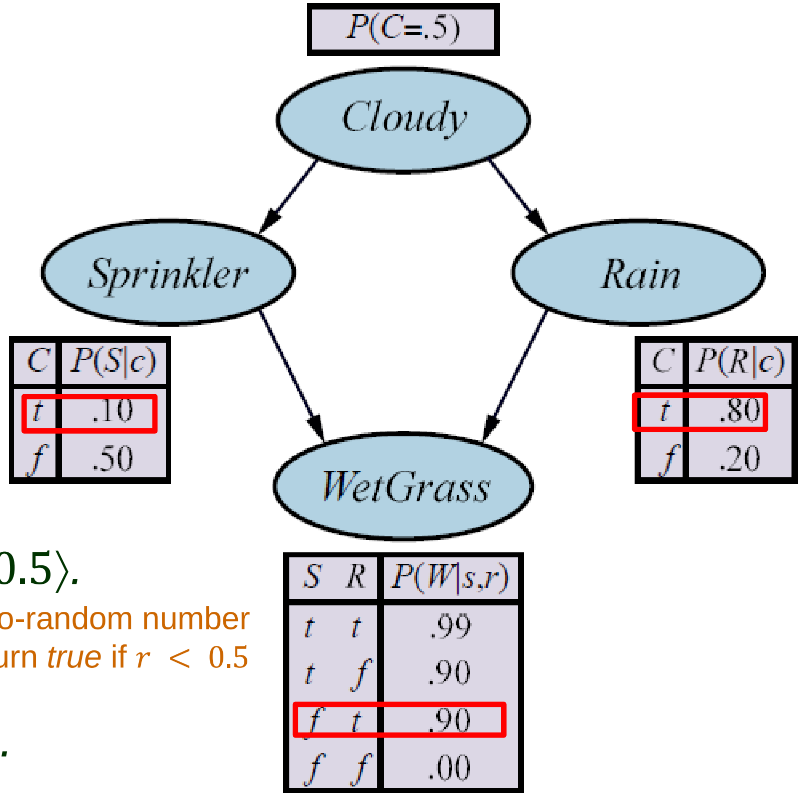

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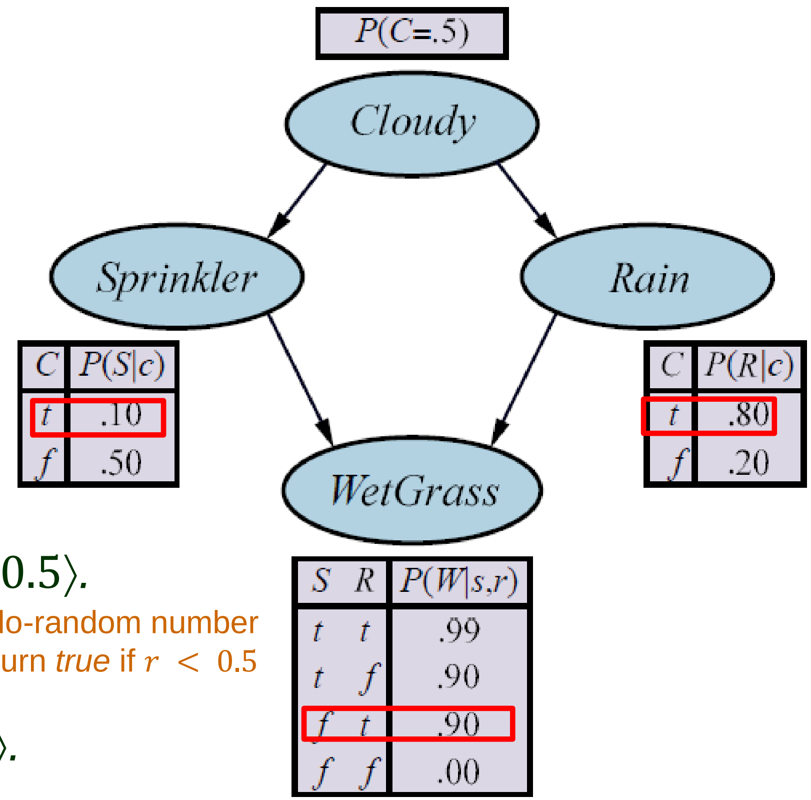

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PRIOR-SAMPLE() returns the event [*true*, *false*, *true*, *true*].

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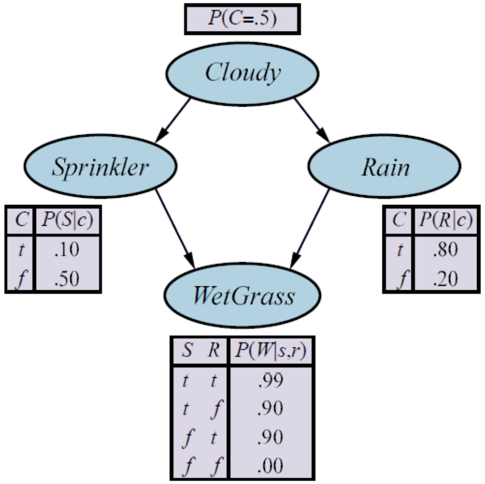

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$$S_{PS}(true, false, true, true) = 0.5 \cdot 0.9 \cdot 0.8 \cdot 0.9 = 0.324$$

Partially Specified Event

Estimate the probability of the partial event $X_1 = x_1 \wedge \cdots \wedge X_m = x_m$, $m \le n$:

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Example *Rain* = *true* holds for 511 of 1,000 samples generated from the sprinkler network.

$$\hat{P}(\mathit{Rain} = \mathit{true}) = 0.511$$

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$$\widehat{\boldsymbol{P}}(X \mid \boldsymbol{e}) = \alpha \boldsymbol{N}_{PS}(X, \boldsymbol{e}) = \frac{\boldsymbol{N}_{PS}(X, \boldsymbol{e})}{N_{PS}(\boldsymbol{e})}$$

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Example Estimate $\mathbf{P}(\mathit{Rain} \mid \mathit{Sprinkler} = \mathit{true})$ using 100 samples.

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- Of the 27 samples with *Sprinkler* = *true*, only 8 have *Rain* = *true*.

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$$\boldsymbol{P}(Rain \mid Sprinkler = true) \approx \alpha\langle 8{,}19\rangle = (0.296{,}0.704)$$

How Fast Does RS Converge?

- How many samples are needed before the resulting estimates are close to the correct answers with high probability?

- The complexity of rejection sampling depends primarily on the $\underbrace{\text{fraction of samples}}_{=\ \text{prior probability of the evidence}\ P(\boldsymbol{e})}$ that are accepted.

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- $P(\boldsymbol{e})$ is vanishingly small for complex networks with many evidence variables.
  - The fraction of samples consistent with $\boldsymbol{e}$ drops exponentially as the number of evidence variables grows.
  - Rejection sampling is unusable for complex problems.

III. Importance Sampling (IS)

- Emulate the effect of sampling from one distribution P using samples from another distribution Q.
  - It is too hard to sample from the true posterior distribution on all the evidence.
  - So we sample from an easy distribution.
- To ensure correctness in the limit, we use a correction factor $P(\boldsymbol{x})/Q(\boldsymbol{x})$, to each sample $\boldsymbol{x}$ when it is counted.

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How does this work?

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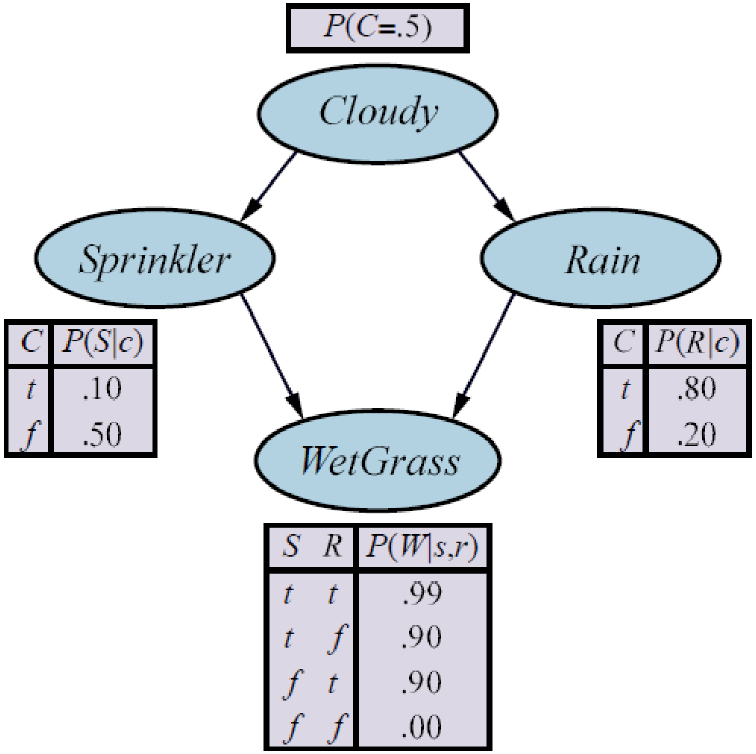

Query $\mathbf{P}(Rain \mid Cloudy = true, WetGrass = true)$

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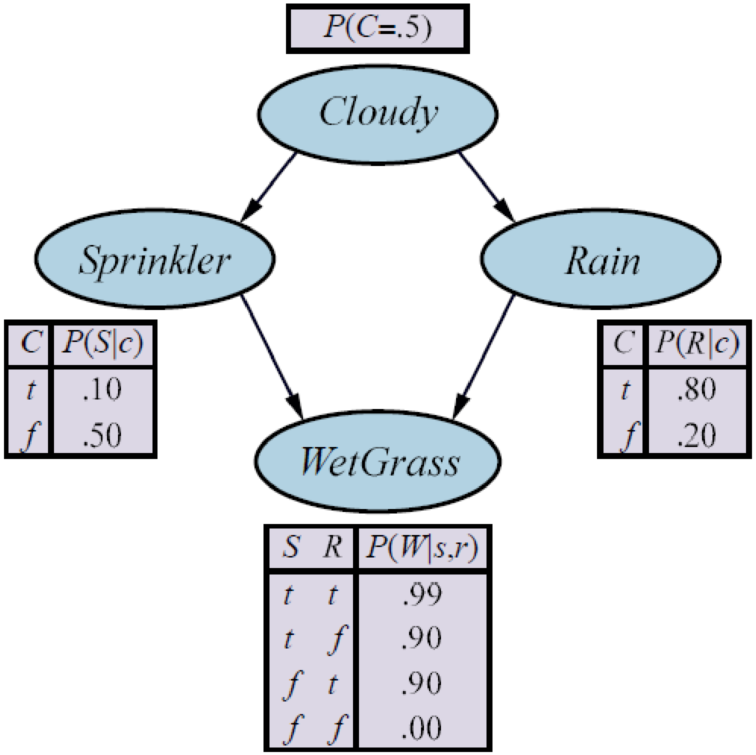

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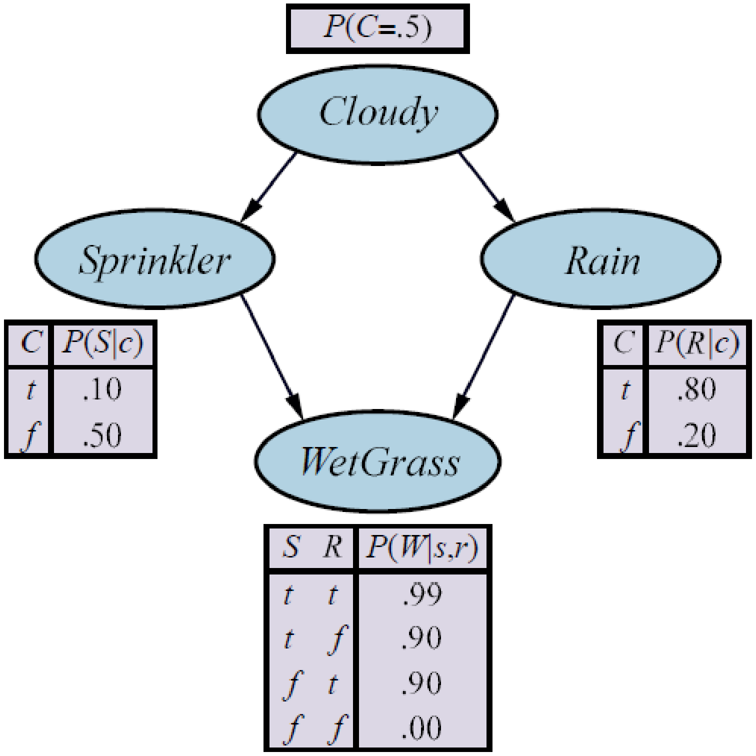

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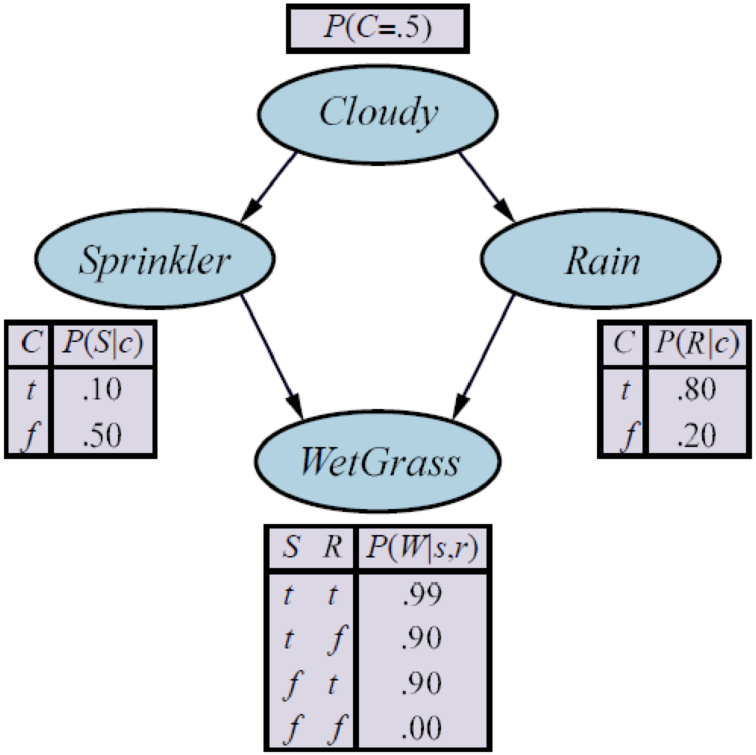

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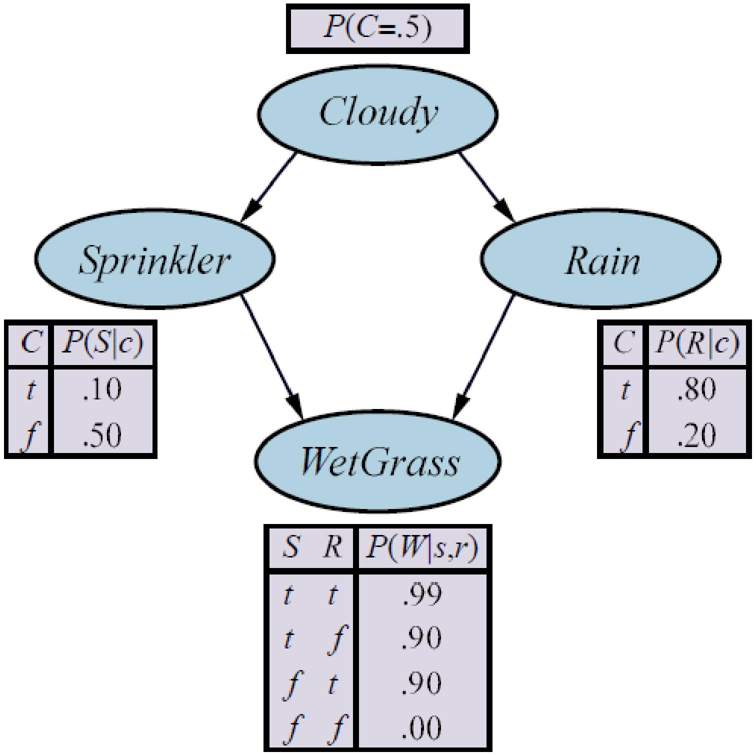
evidence variables $\boldsymbol{E} = \boldsymbol{e}$

hidden variable $\boldsymbol{Z}$

Weight of a Sample



Query $\boldsymbol{P}(Rain \mid Cloudy = true, WetGrass = true)$

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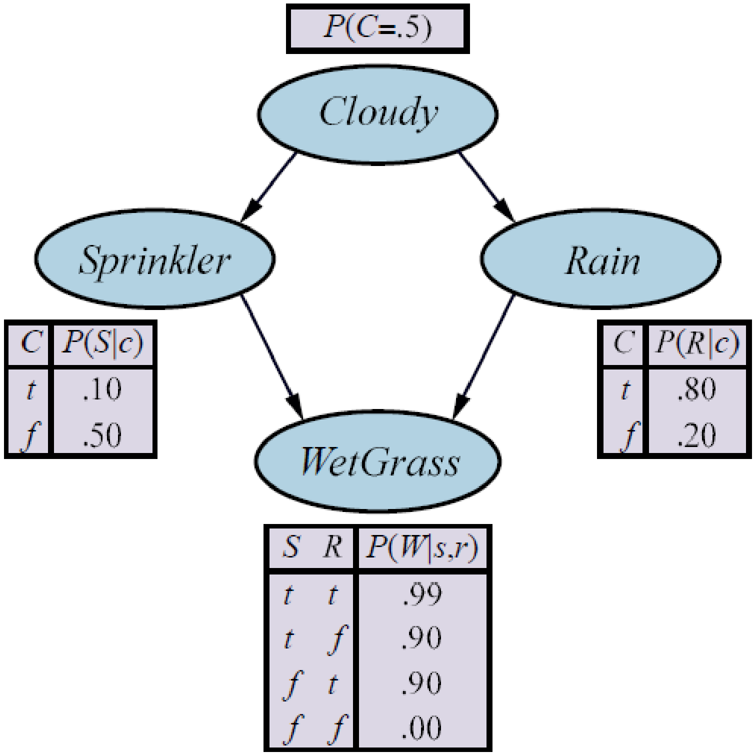

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- Set the weight $w = 1$.

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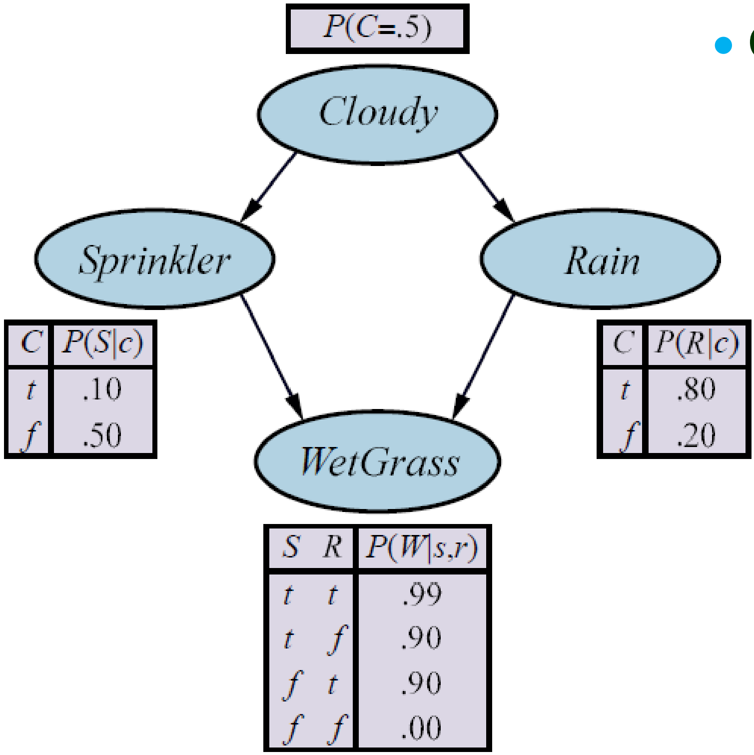
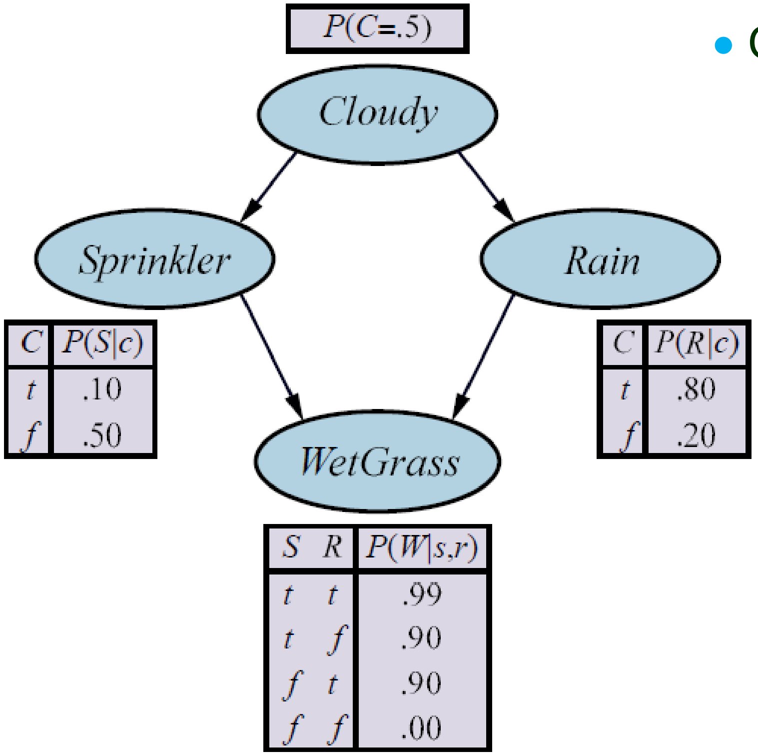


- Generate an event in the chosen topological order.

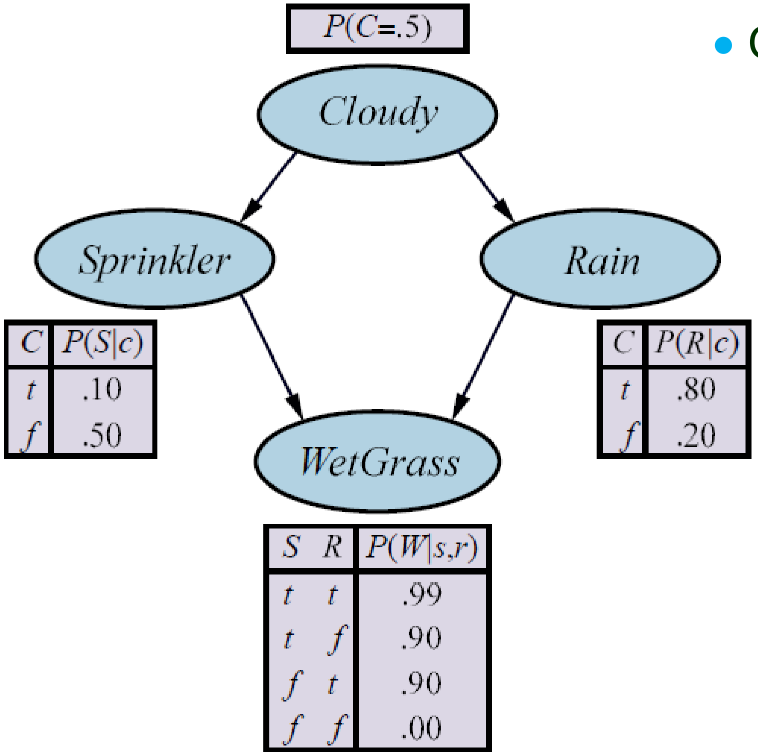
(cont'd)



- Generate an event in the chosen topological order.
    1. *Cloudy* is an evidence variable with value *true*.

$$w \leftarrow w \times P(\mathit{Cloudy} = \mathit{true}) = 0.5$$

(cont'd)



- Generate an event in the chosen topological order.

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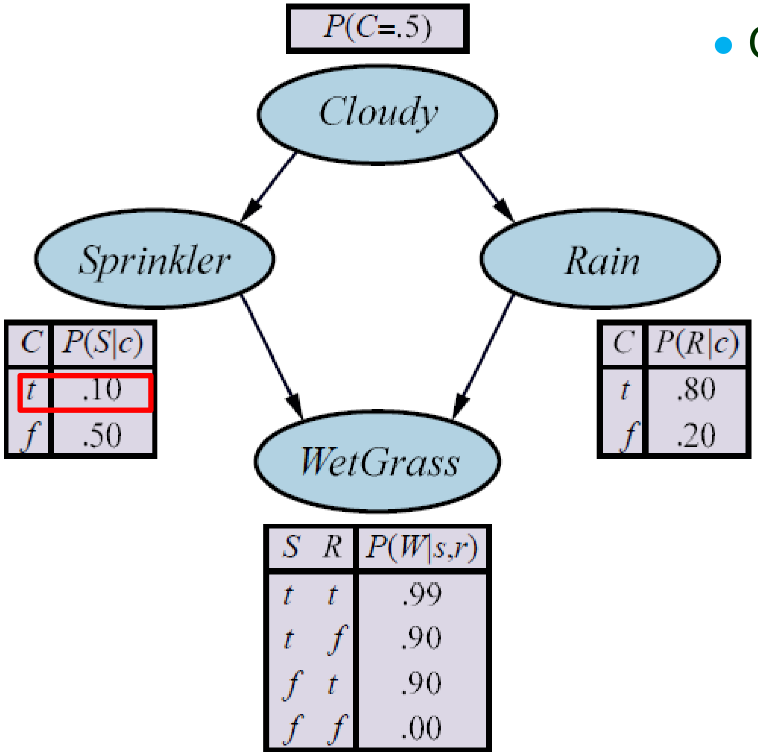
(cont'd)



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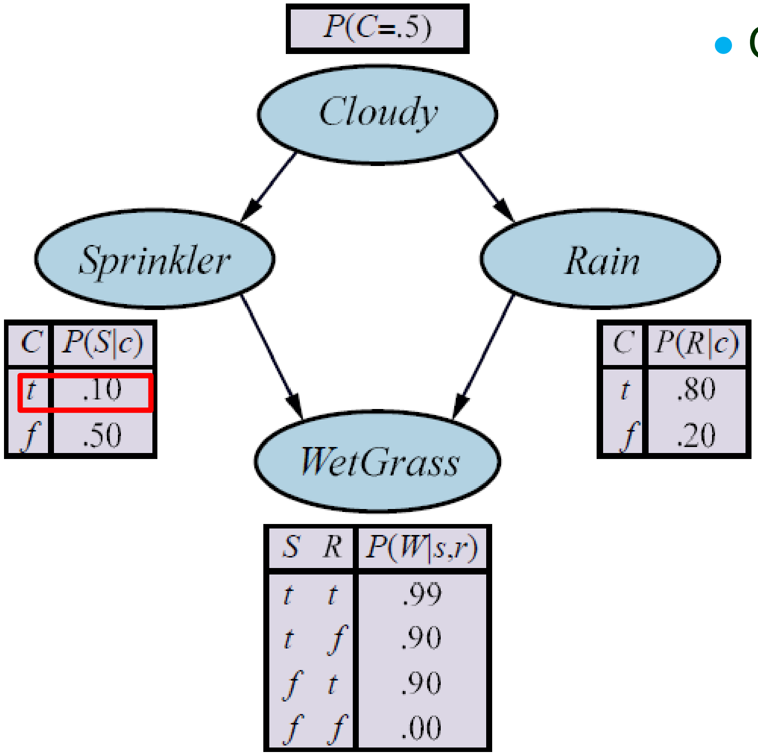

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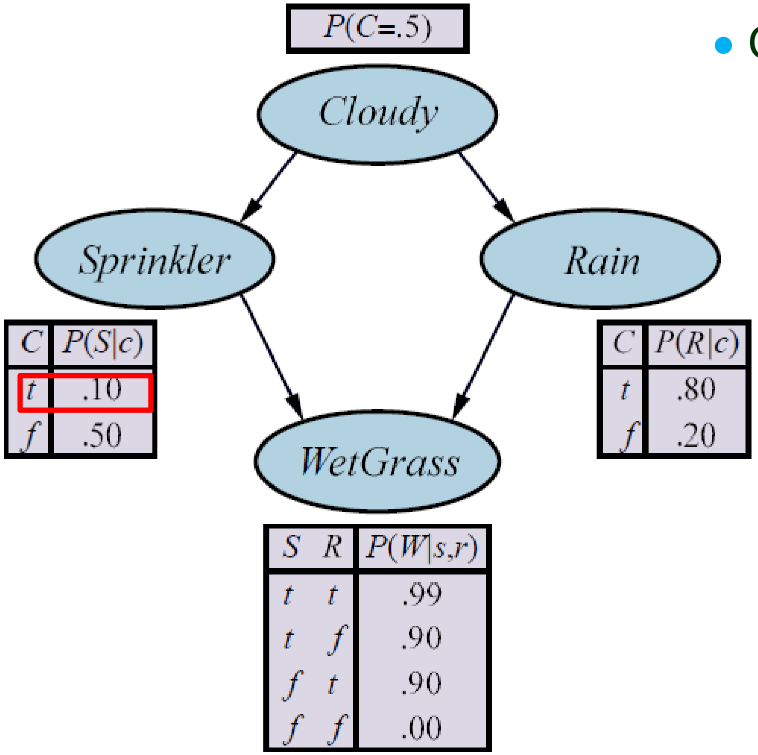
     Suppose this returns *false*.

(cont'd)



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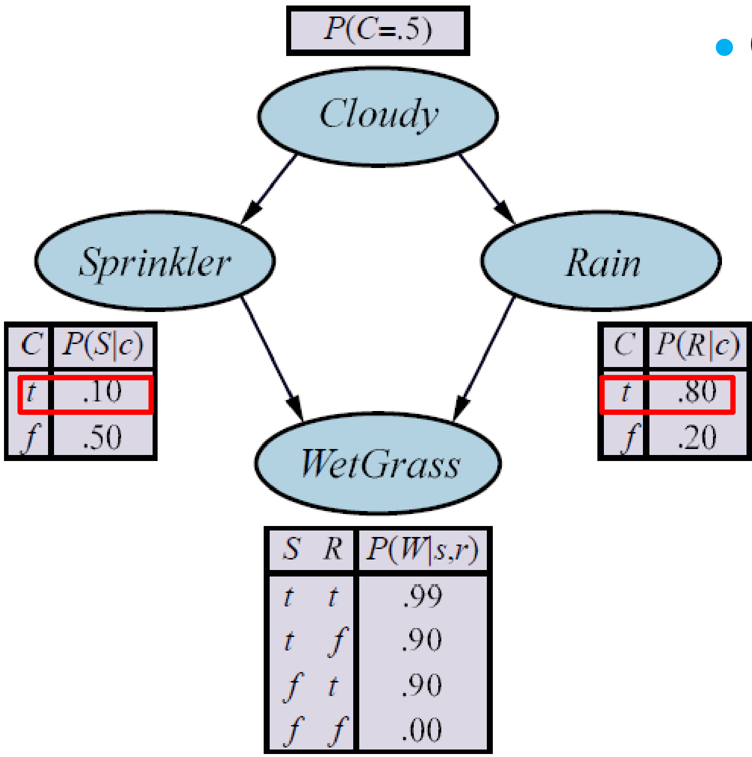
     Suppose this returns *false*.

  3. *Rain* is not an evidence variable. Sample from $\mathbf{P}(\mathit{Rain} \mid \mathit{Cloudy} = \mathit{true}) = \langle 0.8, 0.2 \rangle$.

(cont'd)



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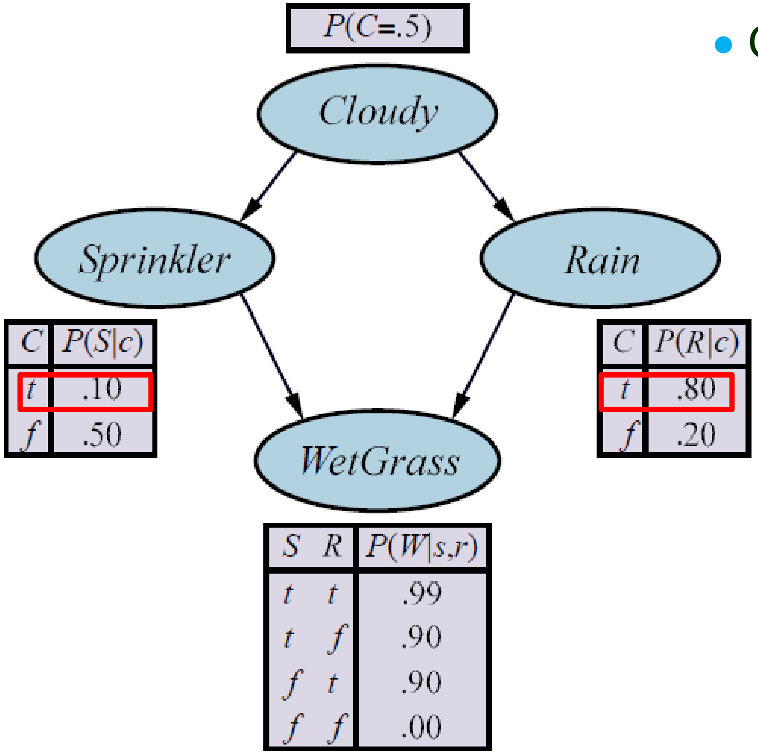
(cont'd)



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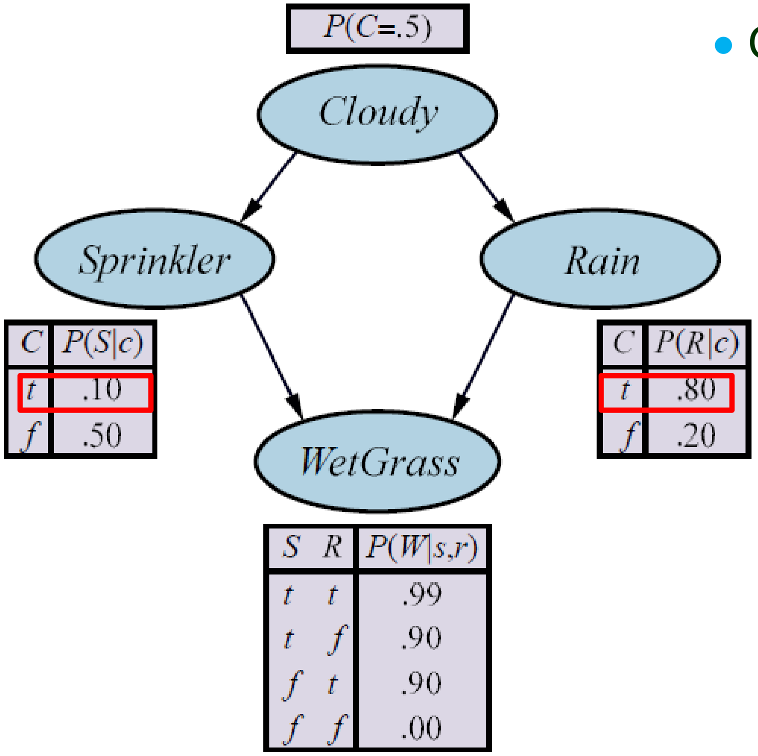
(cont'd)



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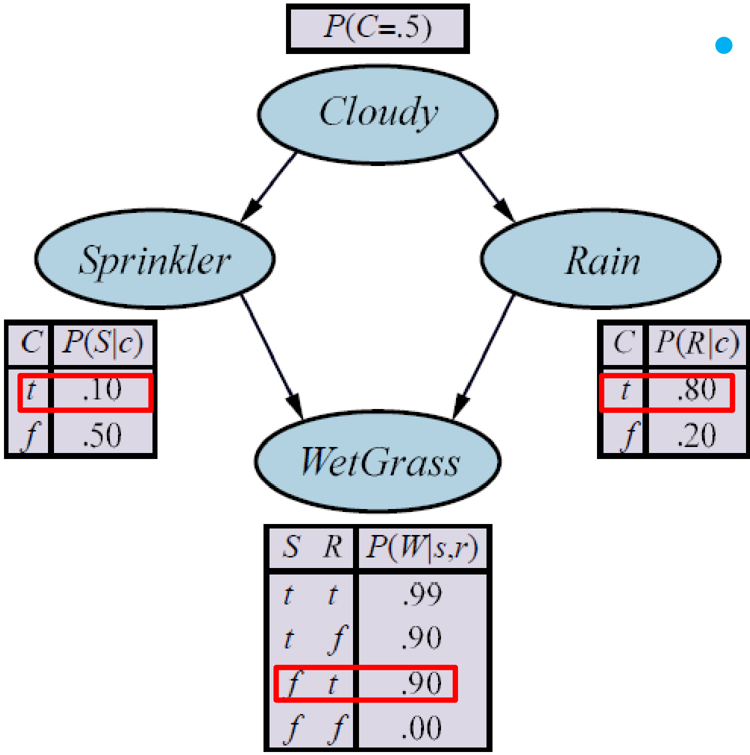
   Suppose this returns *true*.

4. *WetGrass* is an evidence variable with value *true*.

(cont'd)



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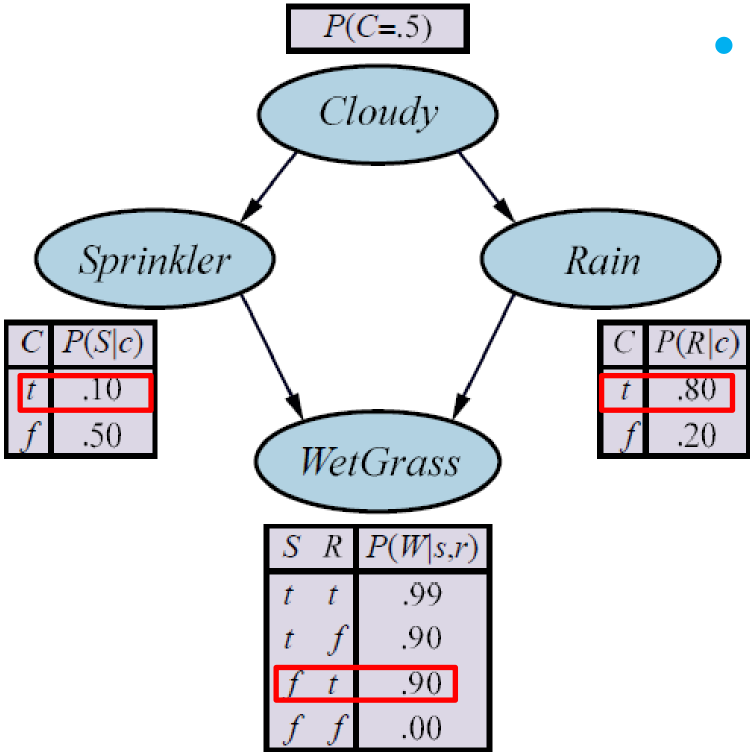
(cont'd)



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     Suppose this returns *false*.

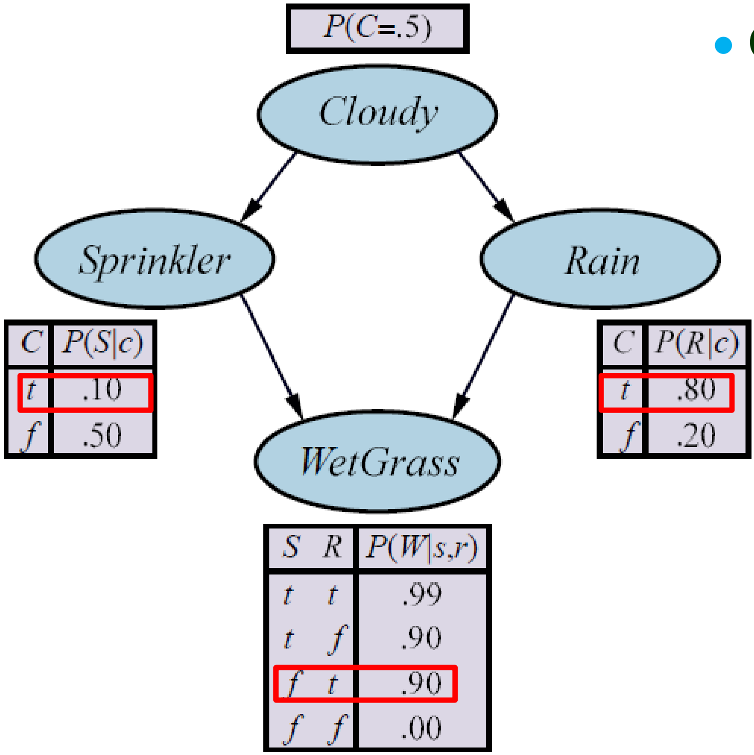
  3. *Rain* is not an evidence variable. Sample from $\mathbf{P}(\mathit{Rain} \mid \mathit{Cloudy} = \mathit{true}) = \langle 0.8, 0.2 \rangle$.

     Suppose this returns *true*.

  4. *WetGrass* is an evidence variable with value *true*.

$$w \leftarrow w \times P(\mathit{WetGrass} = \mathit{true} \mid \mathit{Sprinkler} = \mathit{true}, \mathit{Rain} = \mathit{true}) \\ = 0.5 \times 0.9 = 0.45$$

(cont'd)



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   Suppose this returns *false*.

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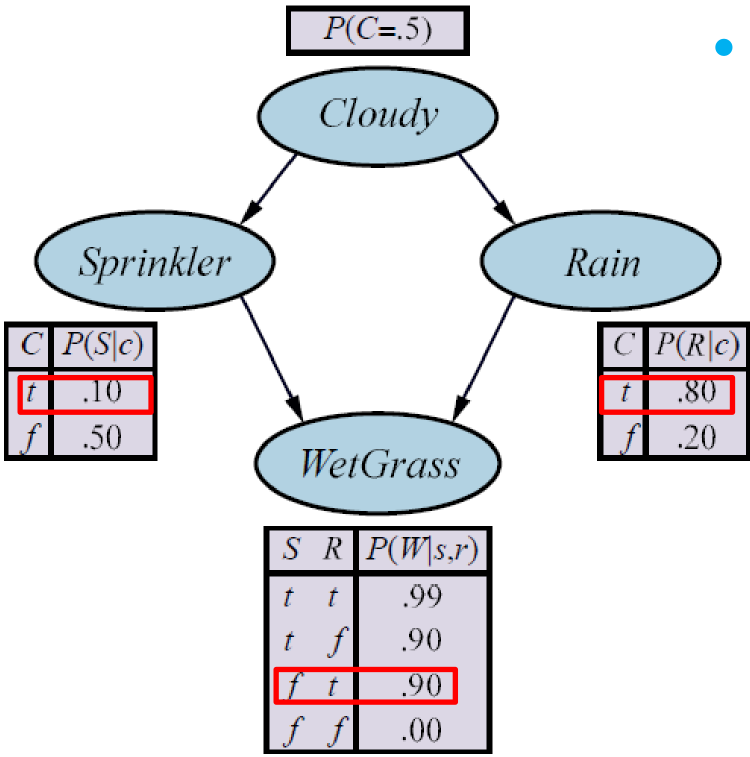
   Suppose this returns *true*.

4. *WetGrass* is an evidence variable with value *true*.

$$w \leftarrow w \times P(\mathit{WetGrass} = \mathit{true} \mid \mathit{Sprinkler} = \mathit{true}, \mathit{Rain} = \mathit{true}) = 0.5 \times 0.9 = 0.45$$

- This round of sampling returns the event [*true*, *false*, *true*, *true*] with weight 0.45.

(cont'd)



- Generate an event in the chosen topological order.

  1. *Cloudy* is an evidence variable with value *true*.

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  2. *Sprinkler* is not an evidence variable. Sample from $\mathbf{P}(\textit{Sprinkler} \mid \textit{Cloudy} = \textit{true}) = \langle 0.1, 0.9 \rangle$.

     Suppose this returns *false*.

  3. *Rain* is not an evidence variable. Sample from $\mathbf{P}(\textit{Rain} \mid \textit{Cloudy} = \textit{true}) = \langle 0.8, 0.2 \rangle$.

     Suppose this returns *true*.

  4. *WetGrass* is an evidence variable with value *true*.

$$w \leftarrow w \times P(\textit{WetGrass} = \textit{true} \mid \textit{Sprinkler} = \textit{true}, \textit{Rain} = \textit{true}) = 0.5 \times 0.9 = 0.45$$

- This round of sampling returns the event [*true*, *false*, *true*, *true*] with weight 0.45.
- This event is tallied under *Rain* = *true* in generating the distribution estimate

$$\hat{\mathbf{P}}(\textit{Rain} \mid \textit{Cloudy} = \textit{true}, \textit{WetGrass} = \textit{true})$$

Likelihood Weighting

Fix the values for the evidence variables $\boldsymbol{E}$ and sample all the nonevidence variables $\{X\} \cup \boldsymbol{Z}$ in topological order, each conditioned on its parents.

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The weight $w(\boldsymbol{z})$ must satisfy

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$$\Downarrow$$

$$w(\boldsymbol{z}) = \frac{1}{P(\boldsymbol{e}\,)} \cdot \frac{P(\boldsymbol{z}, \boldsymbol{e})}{Q_{WS}(\boldsymbol{z})} = \alpha \frac{P(\boldsymbol{z}, \boldsymbol{e})}{Q_{WS}(\boldsymbol{z})}$$

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$$= \alpha \frac{\prod_{i=1}^{l} P(z_i) \mid parents(Z_i)) \cdot \prod_{i=1}^{m} P(e_i) \mid parents(E_i))}{\prod_{i=1}^{l} P(z_i) \mid parents(Z_i))}$$

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$$= \alpha \prod_{i=1}^{m} P(e_i) \mid parents(E_i))$$

Weighted Sampling

$$w(\mathbf{z}) = \alpha \prod_{i=1}^{m} P(e_i) \mid parents(E_i))$$

The weight is the product of the conditional probabilities for the evidence variables given their parents.

function WEIGHTED-SAMPLE(bn, **e**) **returns** an event and a weight

  $w \leftarrow 1$; $\mathbf{x} \leftarrow$ an event with n elements, with values fixed from **e**
  for i = 1 **to** n **do**
    if X_i is an evidence variable with value x_{ij} in **e**
      then $w \leftarrow w \times\ P(X_i = x_{ij} \mid parents(X_i))$
      else $\mathbf{x}[i] \leftarrow$ a random sample from $\mathbf{P}(X_i \mid parents(X_i))$
  return $\mathbf{x}$, w

The Likelihood Weighting Algorithm

function LIKELIHOOD-WEIGHTING(X, $\mathbf{e}$, bn, N) **returns** an estimate of $\mathbf{P}(X \mid \mathbf{e})$
  inputs: X, the query variable
    $\mathbf{e}$, observed values for variables $\mathbf{E}$
    bn, a Bayesian network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$
    N, the total number of samples to be generated
  local variables: $\mathbf{W}$, a vector of weighted counts for each value of X, initially zero

  for j = 1 **to** N **do**
    $\mathbf{x}$, $w \leftarrow$ WEIGHTED-SAMPLE(bn, $\mathbf{e}$)
    $\mathbf{W}[j] \leftarrow \mathbf{W}[j] + w$ where x_j is the value of X in $\mathbf{x}$
  return NORMALIZE($\mathbf{W}$)

function WEIGHTED-SAMPLE(bn, $\mathbf{e}$) **returns** an event and a weight

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  for i = 1 **to** n **do**
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  return $\mathbf{x}$, w