FINAL EXAM FALL '17 (Solutions)

1. Find the general solution of $x^2y' + 2y = xe^{2/x}$, $x > 0$. (Linear)

$$y' + \frac{2}{x^2}y = \frac{e^{2/x}}{x} \quad \Rightarrow P(x) = 2/x^2$$

$$\mu(x) = e^{\int P dx} = e^{\int 2/x^2 dx} = e^{-2/x}$$

Equation becomes: $(e^{-2/x}y)' = \frac{1}{x} \Rightarrow e^{-2/x}y = \ln|x| + C$

$$\Rightarrow \quad y = e^{2/x}(\ln x + C)$$

2. Solve IVP: $y'' - 2y' + 2y = 0$, $y(0) = 2$, $y'(0) = 1$

Aux. Eqn: $m^2 - 2m + 2 = 0 \Leftrightarrow m^2 - 2m + 1 = -1 \Leftrightarrow (m-1)^2 = -1$

$\Rightarrow m = 1 \pm i$ So $\alpha = 1$ & $\beta = 1$

General Sol: $y = c_1 e^x \cos x + c_2 e^x \sin x$

$y' = c_1(e^x \cos x - e^x \sin x) + c_2(e^x \sin x + e^x \cos x)$

Plug in initial conditions: $y(0) = c_1 e^0 = 2 \Rightarrow c_1 = 2$

$y'(0) = c_1 + c_2 = 1 \Rightarrow c_2 = -1$

Sol: $y = 2e^x \cos x - e^x \sin x$

3. Find $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2(s-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s^2} + \frac{D}{s-1}\right\} = \mathcal{L}^{-1}\left\{\frac{-2}{s} - \frac{1}{s^2} + \frac{2}{s-1}\right\}$

$$= -2 - t + 2e^t$$

4. Only b is exact. Find $f(x,y)$; using $\frac{\partial f}{\partial x} = M$ & $\frac{\partial f}{\partial y} = N$

$M = \cos y - 1/x$ & $N = -x\sin y - 1/y$

$f = \int \cos y - 1/x \, dx = x\cos y - \ln|x| + g(y)$

$\frac{\partial f}{\partial y} = \cancel{-x\sin y} + g'(y) = \cancel{-x\sin y} - 1/y \Rightarrow g(y) = -\ln|y|$

$\Rightarrow f = x\cos y - \ln|x| - \ln|y|$

General solution $\quad x\cos y - \ln|x| - \ln|y| = C$

or $\quad \ln|x| + \ln|y| - x\cos y = C$

or $\quad \ln|xy| - x\cos y = C \quad \ldots$

5. $y'' - y = e^x - 2\sin x$

aux Eqn. $\quad m^2 - 1 = 0 \Rightarrow m = \pm 1 \Rightarrow y_c = C_1 e^x + C_2 e^{-x}$

Part. Sol.

let $y_{p_1} = xAe^x$, plug in: $(2Ae^x + \cancel{xAe^x}) - \cancel{xAe^x} = e^x \Rightarrow 2A = 1 \Rightarrow A = 1/2$

$y_{p_1}' = Ae^x + xAe^x \qquad \therefore \; y_{p_1} = \frac{1}{2}xe^x$

$y_{p_1}'' = Ae^x + Ae^x + xAe^x$

let $y_{p_2} = A\cos x + B\sin x$, $y_{p_2}' = B\cos x - A\sin x$, $y_{p_2}'' = -A\cos x - B\sin x$

plug in: $-A\cos x - B\sin x - A\cos x - B\sin x = -2\sin x \Rightarrow A = 0, \; B = 1$

$y_{p_2} = \sin x \quad \Rightarrow \quad y_p = \frac{1}{2}xe^x + \sin x$

General Sol: $y = C_1 e^x + C_2 e^{-x} + \frac{1}{2}xe^x + \sin x$

6. $\vec{x}' = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\vec{x}$, $\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Char. Equ: $\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 4 = 0$

$1-\lambda = \pm 2i \Rightarrow \lambda = 1 \pm 2i$

Find Eigenvector: Solve for $\vec{k}$

$(A-\lambda I)\vec{k} = \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -2ik_1 + 2k_2 = 0 \Rightarrow ik_1 = k_2$ let $\vec{k} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\Rightarrow \vec{k} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{a}} + i\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{b}}$ $\quad \vec{x}_1 = e^t(\vec{a}\cos 2t - \vec{b}\sin 2t)$

$\vec{x}_2 = e^t(\vec{b}\cos 2t + \vec{a}\sin 2t)$

$\Rightarrow$ General Sol: $\vec{X} = c_1 e^t \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}$

plug initial conditions $\vec{x}(0) = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow c_1 = 0$

$c_2 = 1$ $\quad$ Sol: $\underline{\vec{X} = e^t\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}}$

7. Solve IVP use <u>variation of parameters</u>

A fund. matrix is $\Phi = \begin{pmatrix} e^{2t} & 3e^{-5t} \\ e^{2t} & -4e^{-5t} \end{pmatrix}$ $\quad \det\Phi = (-4-3)e^{-3t} = -7e^{-3t}$

$\Phi^{-1} = -\frac{e^{3t}}{7}\begin{pmatrix} -4e^{-5t} & -3e^{-5t} \\ -e^{2t} & e^{2t} \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 4e^{-2t} & 3e^{-2t} \\ e^{5t} & -e^{5t} \end{pmatrix}$

$\int \Phi^{-1} f = \int \frac{1}{7}\begin{pmatrix} 4e^{-2t} & 3e^{-2t} \\ e^{5t} & -e^{5t} \end{pmatrix}\begin{pmatrix} 7e^{2t} \\ 14e^{-5t} \end{pmatrix} = \int \begin{pmatrix} 4 + 6e^{-7t} \\ e^{7t} - 2 \end{pmatrix} dt = \begin{pmatrix} 4t - 6/7\, e^{-7t} \\ 1/7\, e^{7t} - 2t \end{pmatrix} = \vec{u}$

$\vec{X}_p = \Phi\vec{u} = \begin{pmatrix} 4te^{2t} - 6/7\, e^{-5t} + 3/7\, e^{2t} - 6te^{-5t} \\ 4te^{2t} - 6/7\, e^{-5t} - 4/7\, e^{2t} + 8te^{-5t} \end{pmatrix}$

plug initial conditions into general sol:

$$\vec{x}(0) = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}3\\-4\end{pmatrix} + \begin{pmatrix}-6/7+3/7\\-10/7\end{pmatrix} = \begin{pmatrix}4/7\\-3/7\end{pmatrix}$$

$$\Rightarrow c_1 + 3c_2 = 1 \qquad c_1 = 1 + 4c_2 = 1 + 0$$

$$-(\ c_1 - 4c_2 = 1\ )$$

$$7c_2 = 0 \Rightarrow c_2 = 0 \qquad \therefore \boxed{\begin{matrix}c_1 = 1\\ c_2 = 0\end{matrix}}$$

8. Apply $\mathcal{L}$ 1st: $\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\delta(t-2)\} + \mathcal{L}\{f\}$

$\mathcal{L}\{y''\} = s^2Y - s\cdot 1 - 0$

$\mathcal{L}\{y\} = Y$

$\mathcal{L}\{\delta(t-2)\} = e^{-2s}$

$\mathcal{L}\{f\} = \frac{1}{s} - \frac{e^{-s}}{s}$

$\left\{ \begin{matrix} f = u(t-0) - u(t-1) \\ f = 1 - u(t-1) \end{matrix} \right.$

$$\Rightarrow s^2Y - s + Y = e^{-2s} + \frac{1}{s} - \frac{e^{-s}}{s}$$

$$\Rightarrow Y = \frac{e^{-2s}}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{e^{-s}}{s(s^2+1)} + \frac{s}{s^2+1}$$

P.F. decomp.

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$y = \mathcal{L}^{-1}(Y) = u(t-2)\sin(t-2) + 1 - \cos t - (1 - \cos(t-1))u(t-1) + \cos t$$

9. $y'' + x^2 y = 0$, $y(0) = 1$, $y'(0) = 0$

$\Downarrow$ $C_0 = 1$ $\Downarrow$ $C_1 = 0$

let $y = \sum_{n=0}^{\infty} C_n x^n$

$y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$

Plug in: $\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + x^2 \sum_{n=0}^{\infty} C_n x^n = 0$

$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+2} = 0$

Reindex: $\sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n + \sum_{n=2}^{\infty} C_{n-2} x^n = 0$

$2C_2 + 6C_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) C_{n+2} + C_{n-2}\right] x^n = 0$

$\Rightarrow$ $2C_2 = 0$ & $6C_3 = 0$ $\Rightarrow$ $C_2 = 0$ and $C_3 = 0$

Recurrence Relation: $C_{n+2} = \dfrac{-C_{n-2}}{(n+2)(n+1)}$, $n \geq 2$

when $n=2$ $C_4 = \dfrac{-C_0}{4 \cdot 3} = -\dfrac{1}{12}$; $n = 3$ $C_5 = -\dfrac{C_1}{7 \cdot 6} = 0$

$\therefore$ First 6 terms of $y = 1 + 0x + 0x^2 + 0x^3 - \frac{1}{12}x^4 + 0x^5$

$y = 1 - \frac{1}{12}x^4$