Bayesian Networks

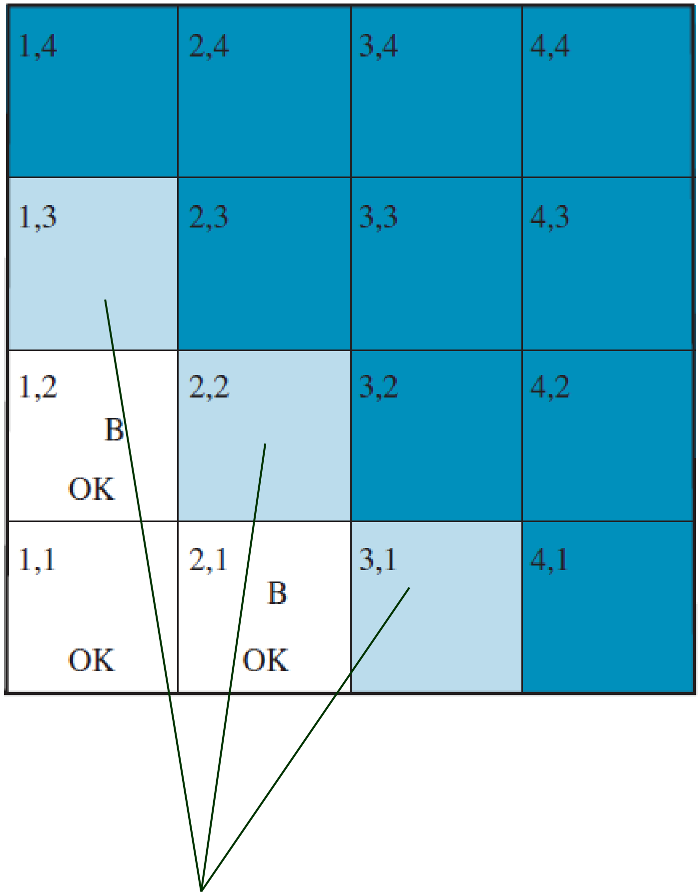
Outline

I. Revisiting the wumpus world

II. Bayesian networks: semantics

* Figures are either from the textbook site or by the instructor.

I. The Wumpus World Revisited



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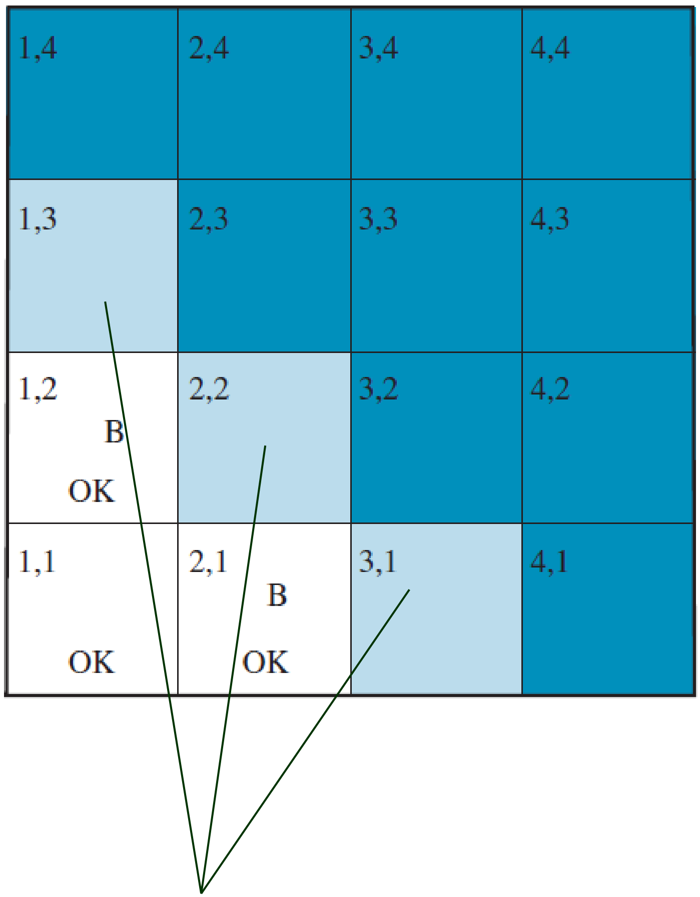

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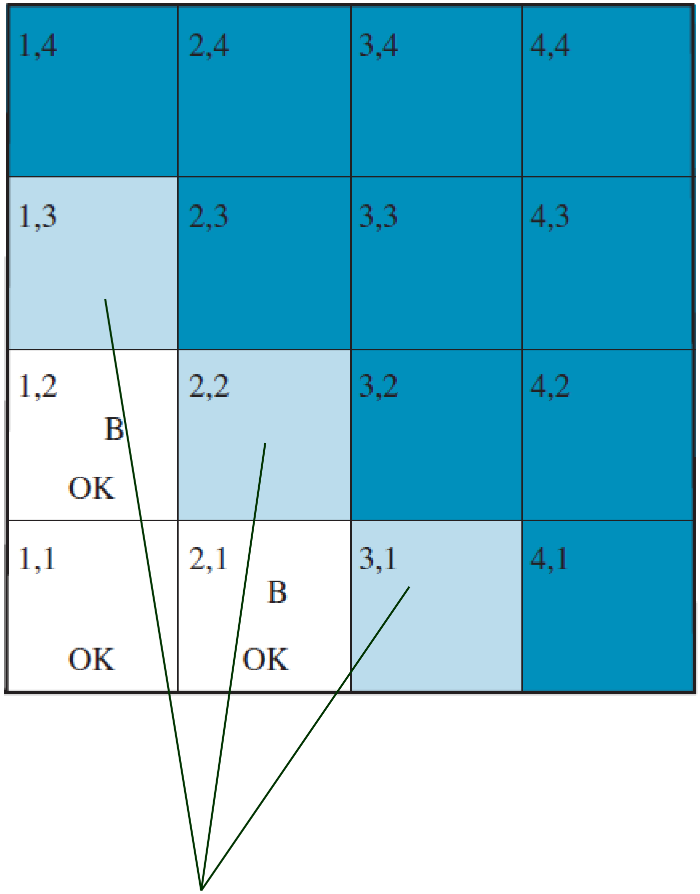

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  - Each square other than [1, 1] contains a pit with probability 0.2.

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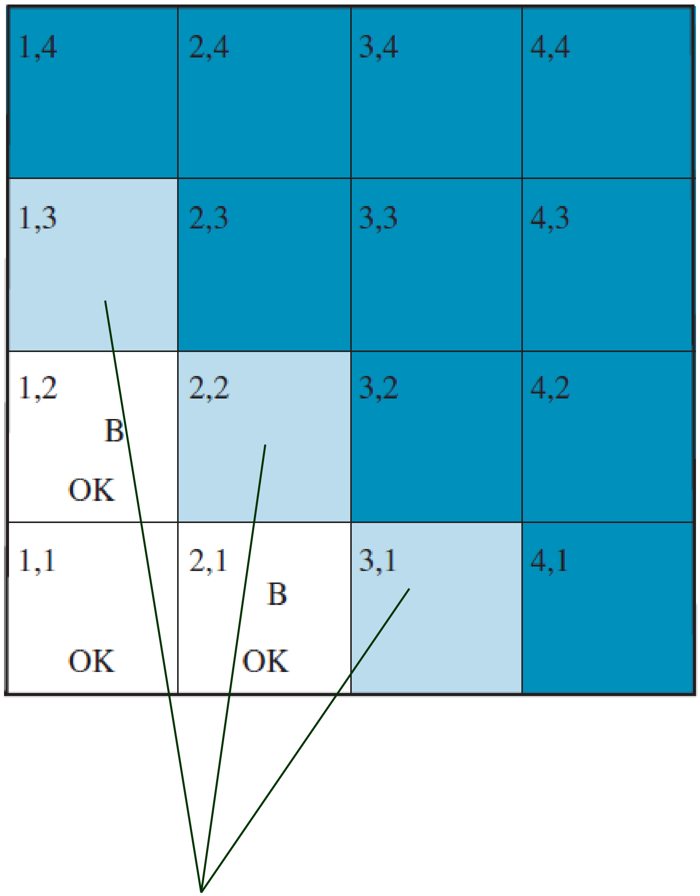

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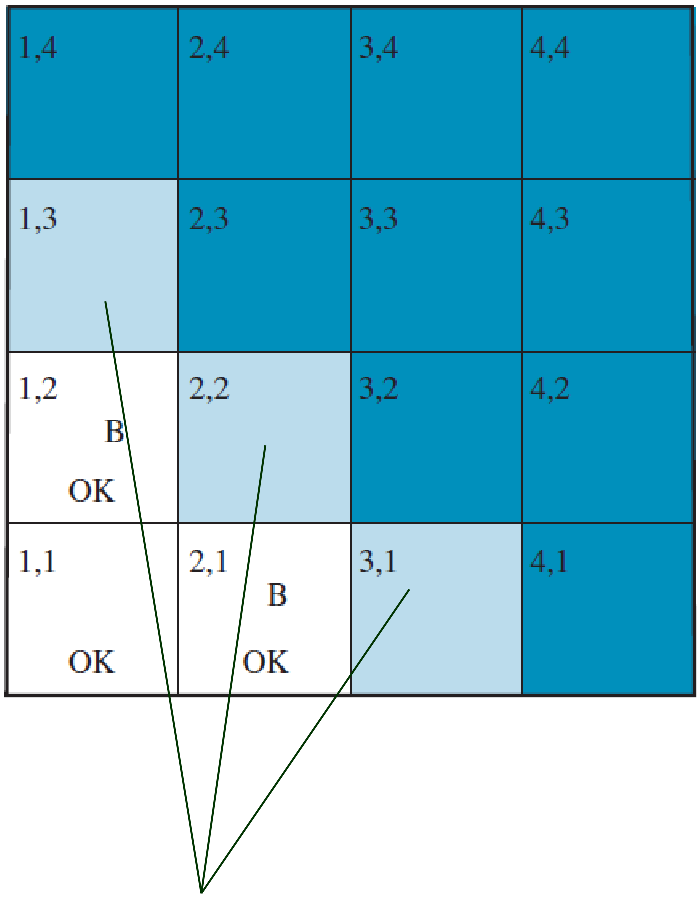

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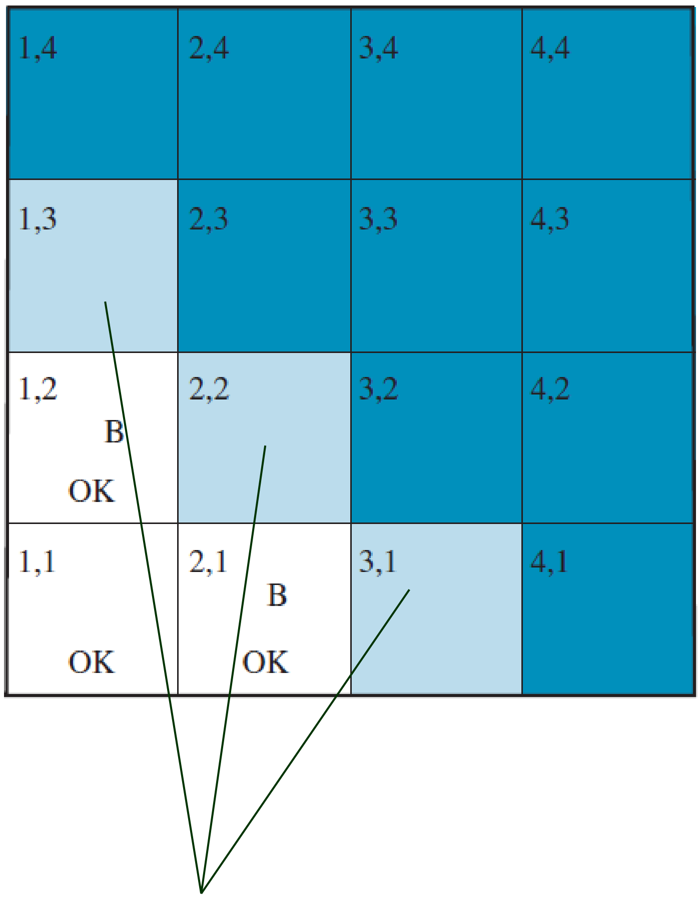

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  - A pit causes breeze in all neighboring squares.
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- ♦ Identify the set of random variables.
  - $P_{i,j}$: true if square $[i,j]$ contains a pit.
  - $B_{i,j}$: true if square $[i,j]$ is breezy – included only for the observed squares, [1,1], $[1,2]$, [2,1].

Full Joint Distribution

$$\boldsymbol{P}\left(P_{1,1}, \ldots, P_{4,4}, B_{1,1}, B_{1,2}, B_{2,1}\right) =$$

$$\boldsymbol{P}\left(B_{1,1}, B_{1,2}, B_{2,1} \mid P_{1,1}, \ldots, P_{4,4}\right) \boldsymbol{P}\left(P_{1,1}, \ldots, P_{4,4}\right)$$

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$$\underbrace{\mathbf{P}(B_{1,1}, B_{1,2}, B_{2,1} \mid P_{1,1}, \ldots, P_{4,4})}_{\text{values in the distribution, for a given pit configuration, are 1 if all the breezy squares among } [1,1], [1,2], [2,1] \text{ are adjacent to pits and 0 otherwise.}} \mathbf{P}(P_{1,1}, \ldots, P_{4,4})$$

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$$\mathbf{P}(P_{1,1}, \dots, P_{4,4}) = \prod_{i,j=1,1}^{4,4} \mathbf{P}(P_{i,j})$$

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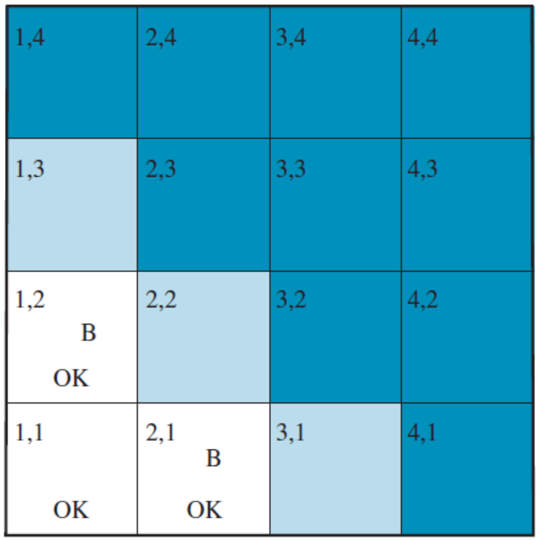
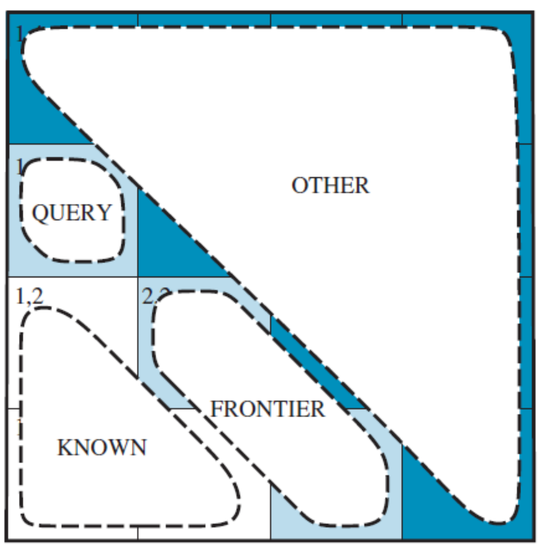
$0.2^n \times 0.8^{16-n}$ for a configuration with n pits.

Evidence





$b = \neg b_{1,1} \wedge b_{1,2} \wedge b_{2,1}$

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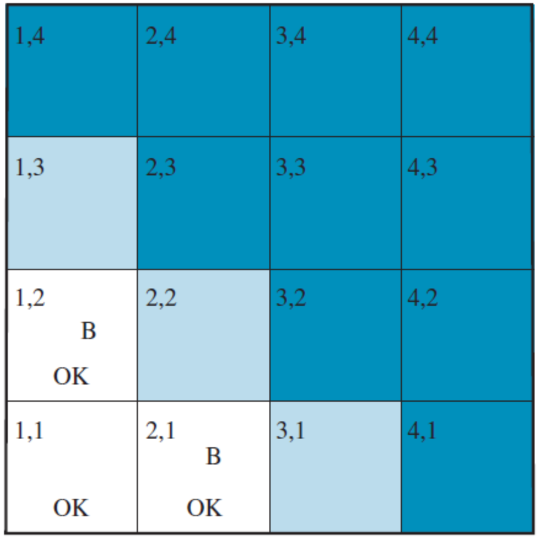

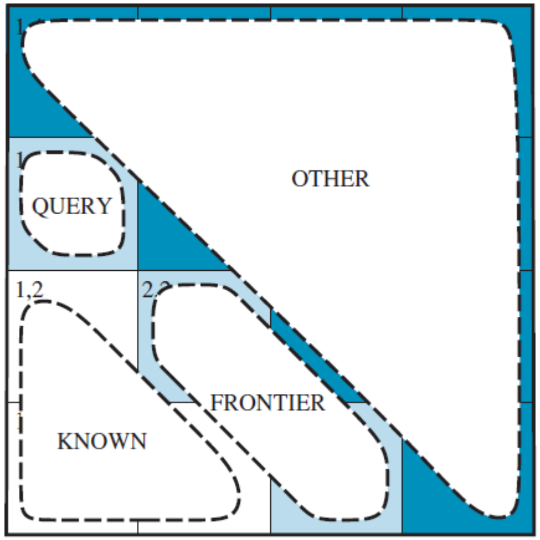

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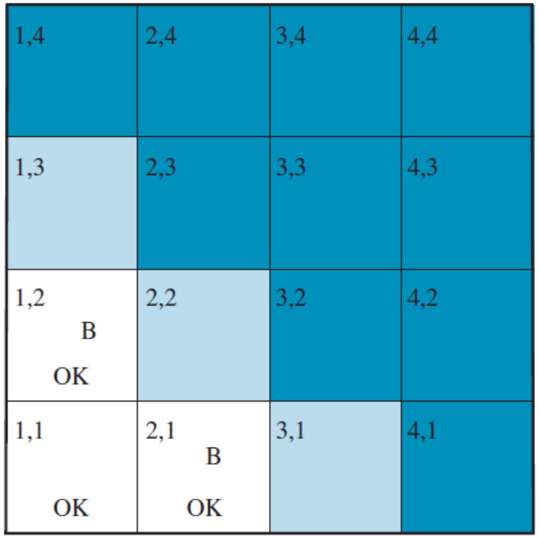

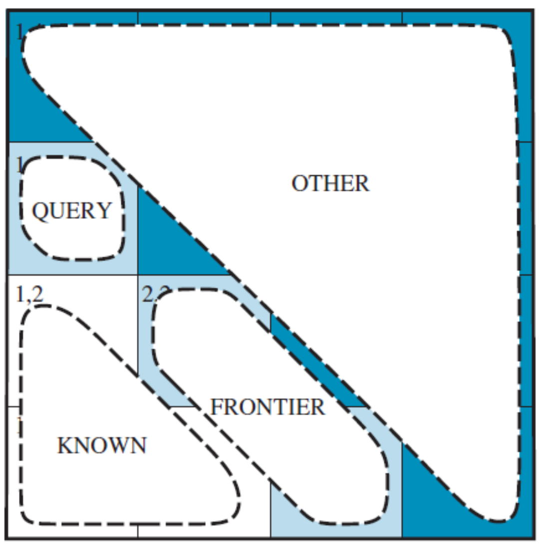

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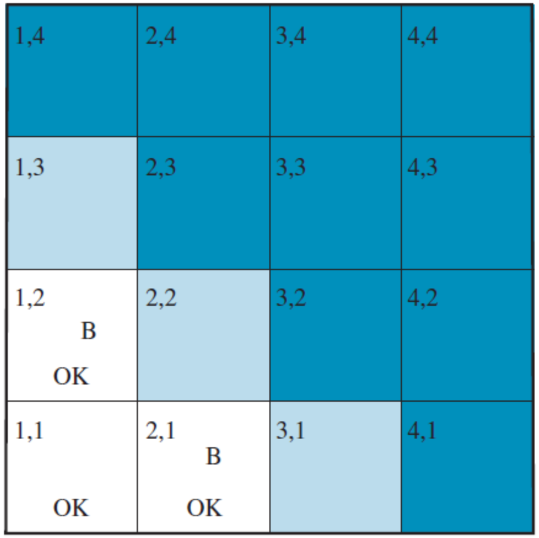

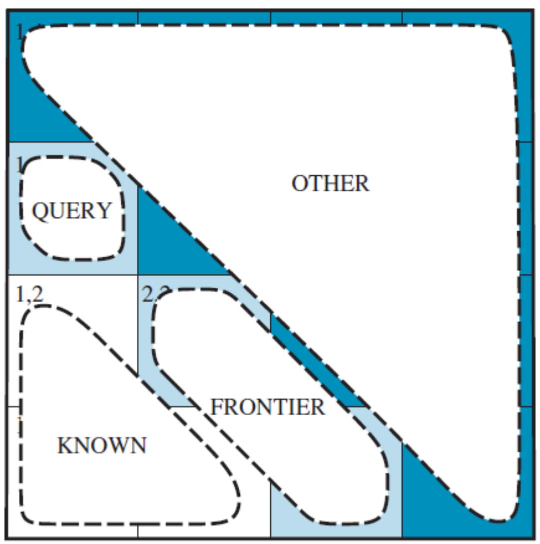

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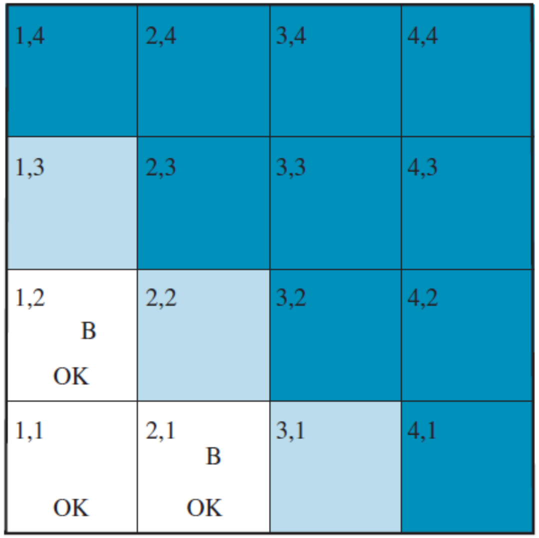

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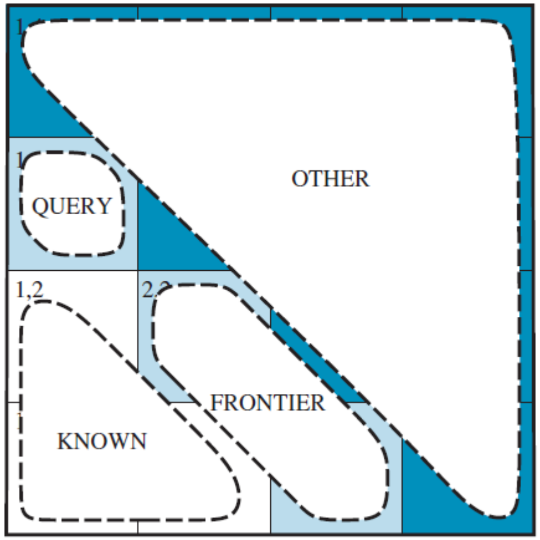

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♠ summation over $2^{12} = 4096$ terms (if the full joint probabilities are available).

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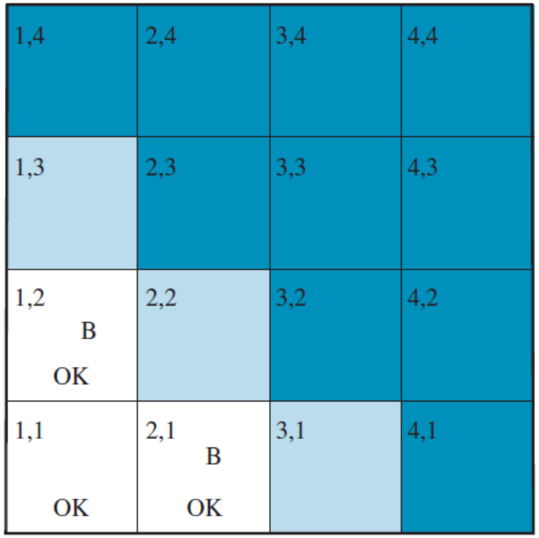

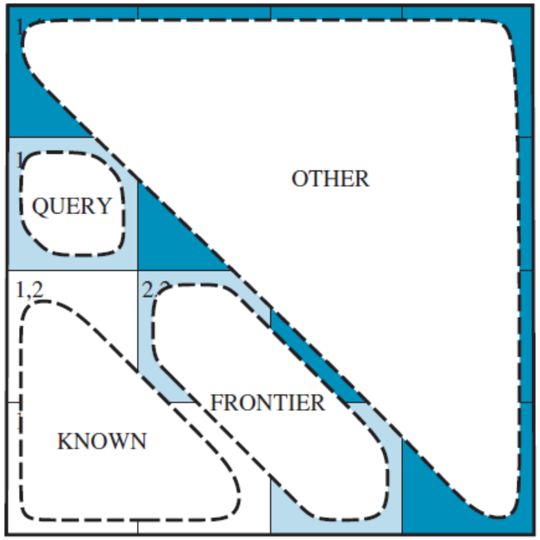

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Exponential in the number of squares!

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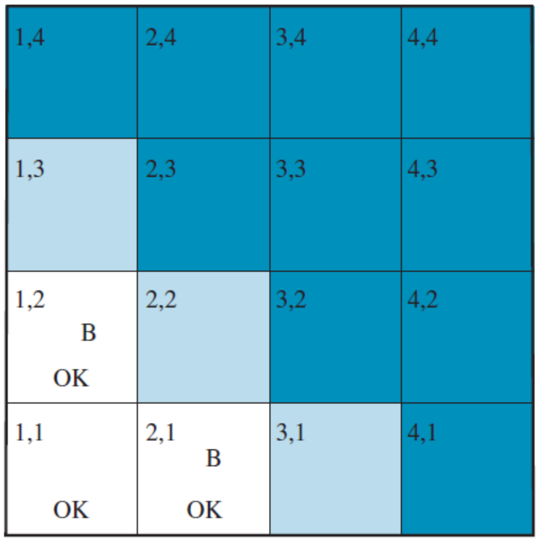

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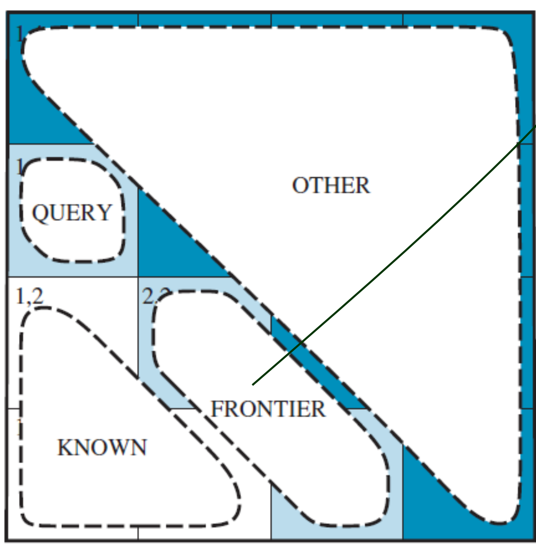

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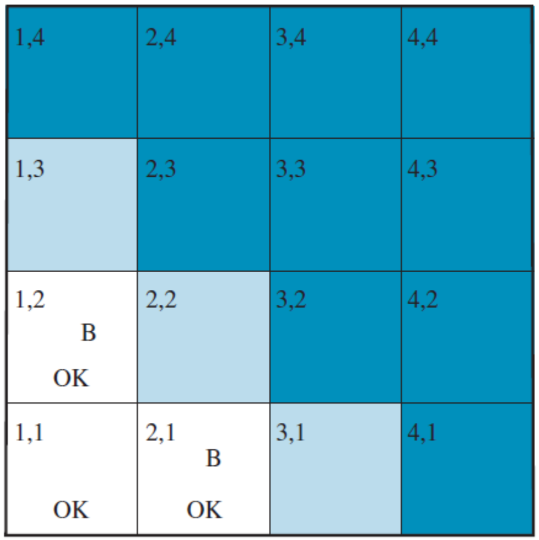

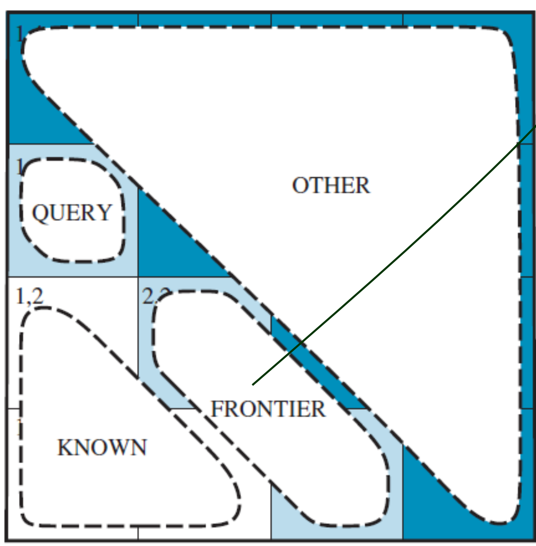

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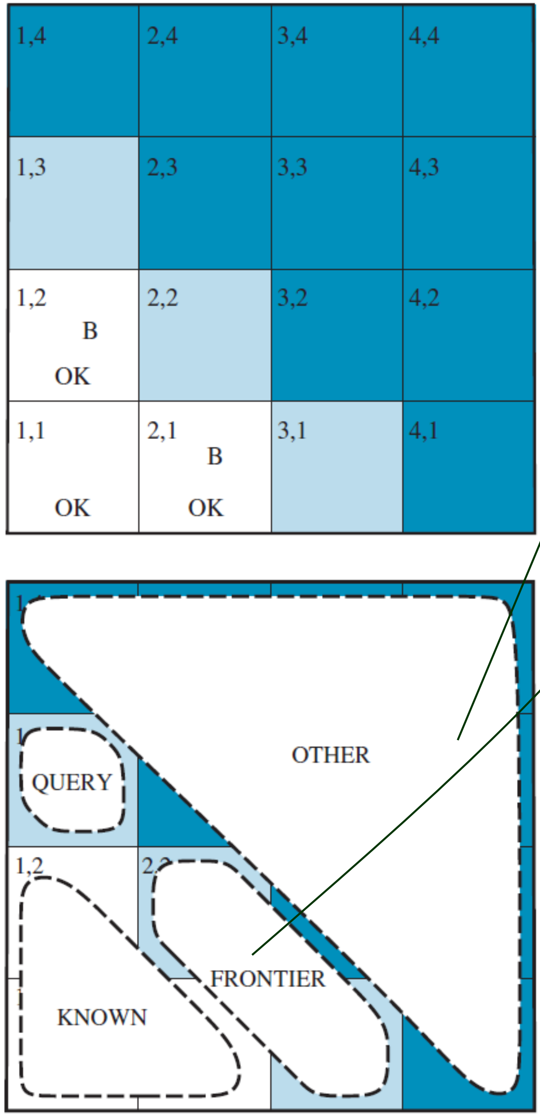

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Other: pit variables for the other unknown squares.

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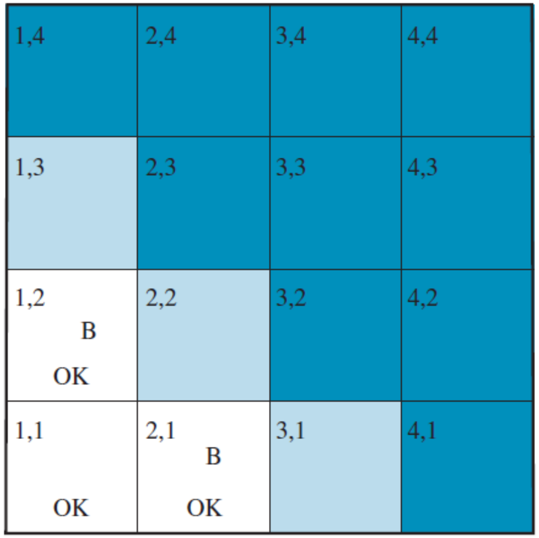

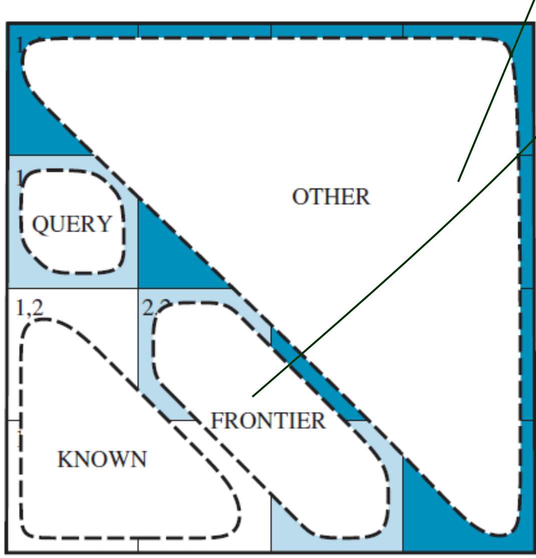

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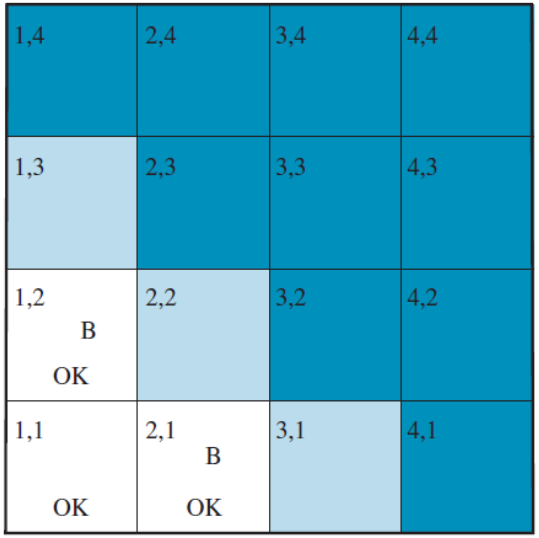

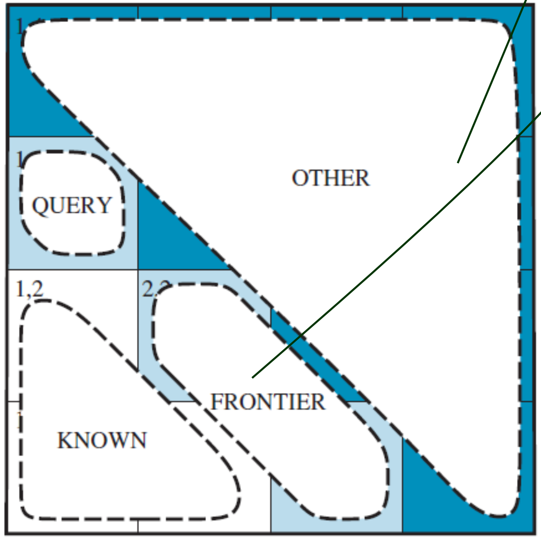

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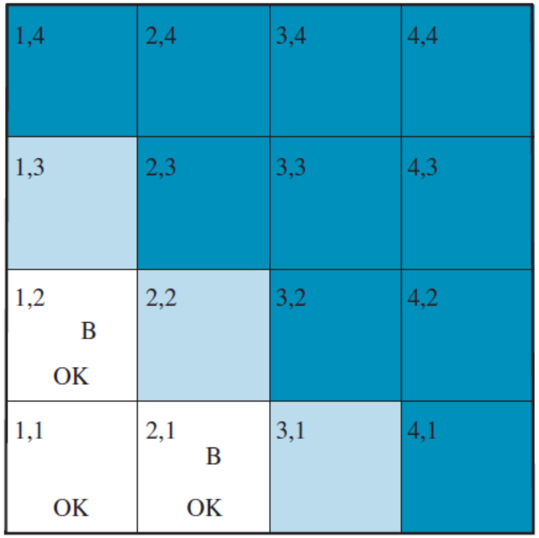

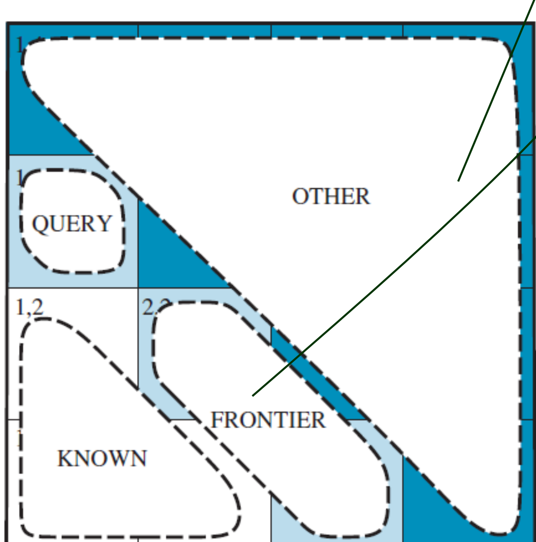

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$$Unknown = Frontier \cup Other$$

Insight: The observed breezes are conditionally *independent of Other*, given *Known*, *Frontier*, and the query variable.

Applying Conditional Independence

$$\boldsymbol{P}\left(P_{1,3} \mid known,\ b\right) = \alpha \sum_{Unknown} \boldsymbol{P}\left(P_{1,3}, known, b, Unknown\right)$$

not "unknown"
as in the text

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b is independent of *Other,* given *known*, $P_{1,3}$, and *Frontier*.

$$= \alpha \sum_{Frontier} \sum_{Other} \mathbf{P}(b \mid P_{1,3}, known, Frontier)\mathbf{P}(P_{1,3}, known, Frontier, Other)$$

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$$= \alpha \sum_{Frontier} \mathbf{P}(b \mid P_{1,3}, known, Frontier) \sum_{other} \mathbf{P}(P_{1,3}, known, Frontier, Other)$$

Elimination of Other Squares (*other*)

$\mathbf{P}(P_{1,3} \mid known, b)$

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$\alpha' = \alpha P(known)$ and

$\sum_{other} P(other) = 1$

$$= \alpha' \mathbf{P}(P_{1,3}) \sum_{Frontier} \mathbf{P}(b \mid P_{1,3}, known, Frontier) P(Frontier)$$

Probability of Containing a Pit

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$$= \{P_{2,2}, P_{3,1}\}$$

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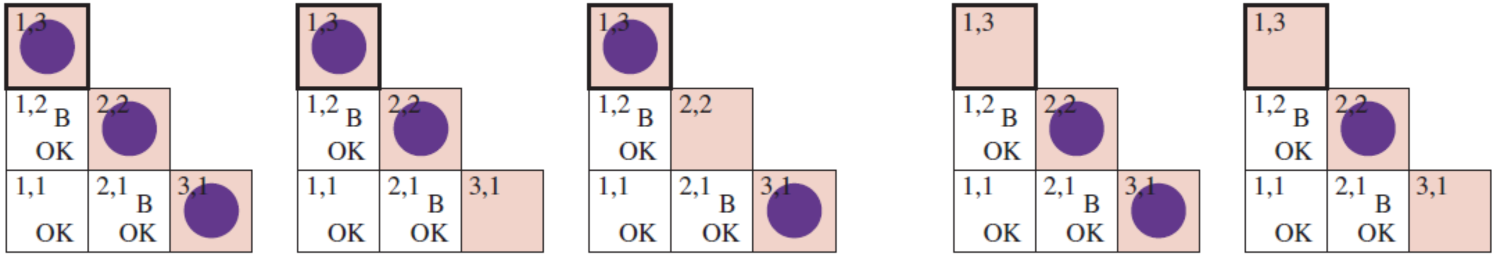

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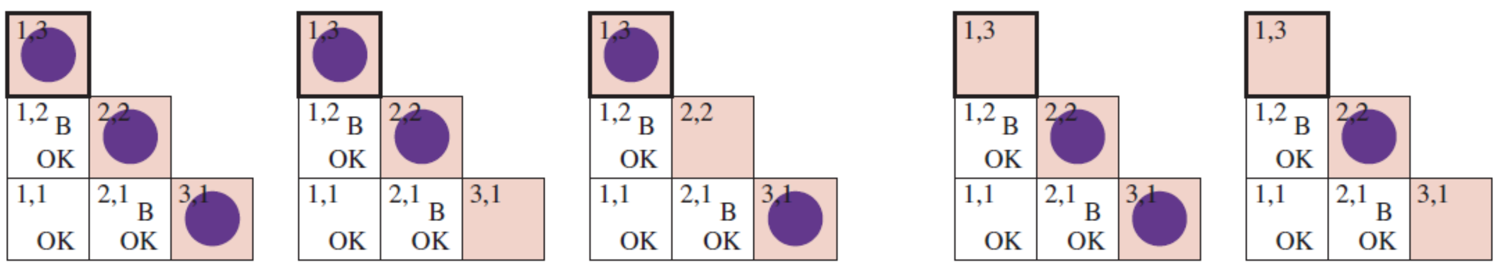

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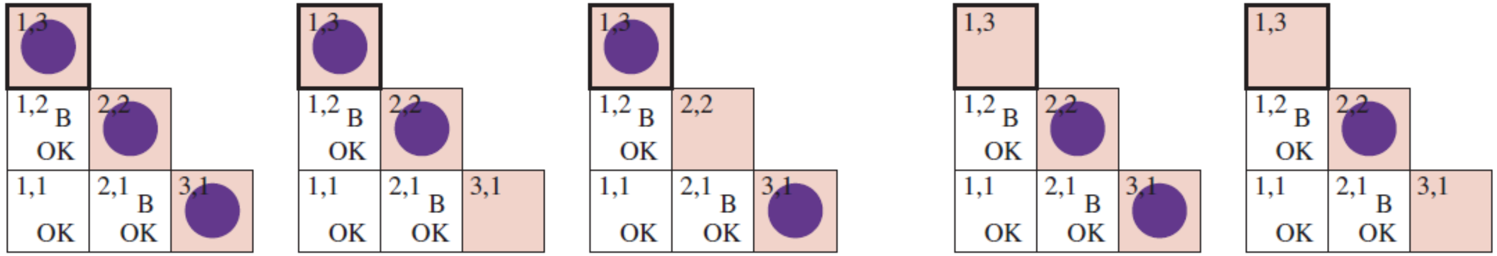

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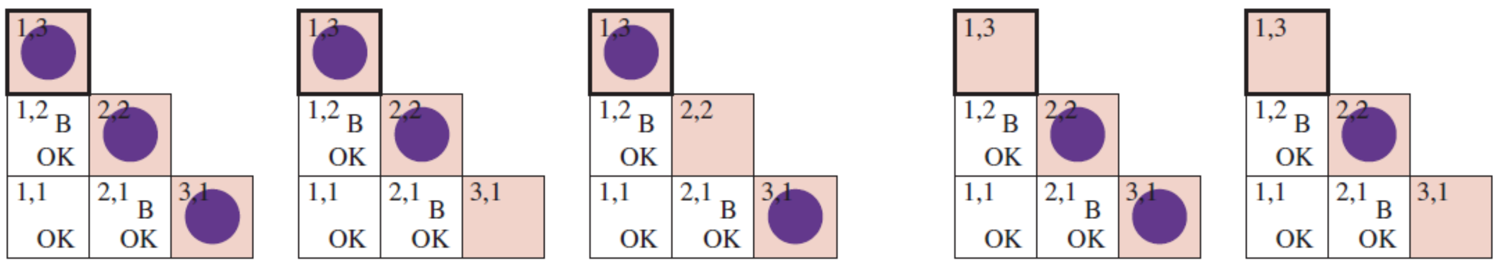

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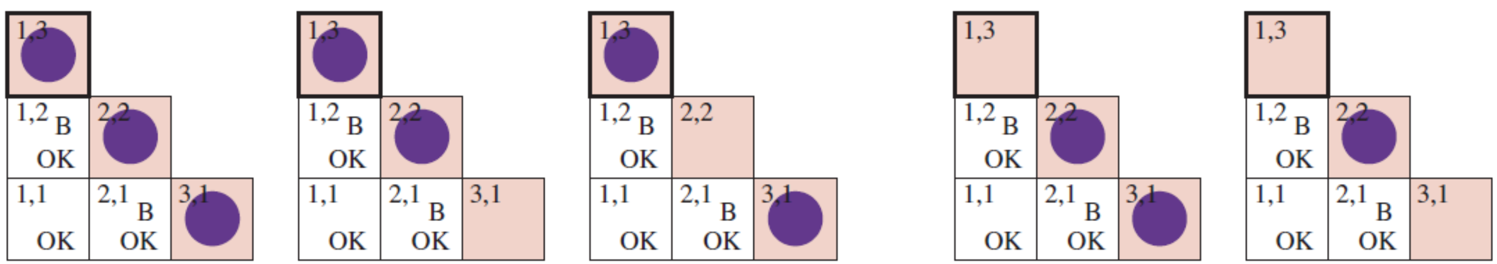

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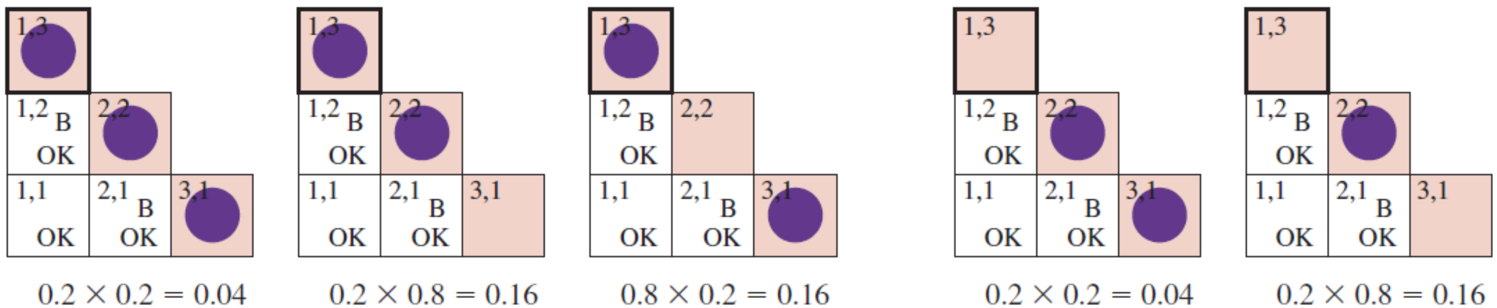

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[1,3] contains a pit with 31% probability.

Knowledge in an Uncertain Domain

- The full joint probability distribution can answer any question, but it also has several drawbacks:
  - exponential in the number n of variables and intractable as n grows very large
  - unnatural and tedious to specify probabilities of outcomes one by one
  - inadequate for representing human reasoning (good at conditional probabilities but poor at joint probabilities)
- The number of probabilities can be greatly reduced by exploring the absolute and conditional independence relationships among the variables.
- These dependencies can be *concisely represented* by a Bayesian network, which can represent any full joint probability distribution.

Bayesian Network

A *Bayesian network* (aka a *Bayes net*) is a directed acyclic graph (DAG) such that

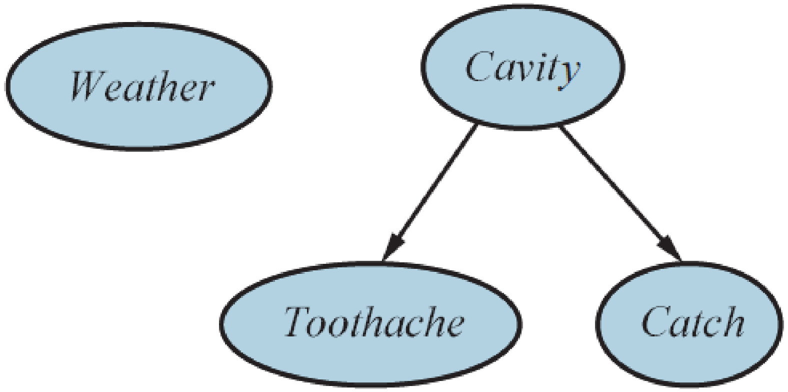
a) every node corresponds to a random variable, either discrete or continuous;

b) every edge (X, Y) specifies X (a cause) as a parent of Y (an effect);

c) every node X has associated probability information $\theta(X \mid \textit{parent}(X))$ that quantifies the effect of the parents on X.

The network topology specifies the conditional independence relationships that hold in the domain.

BN as a Modeling Tool



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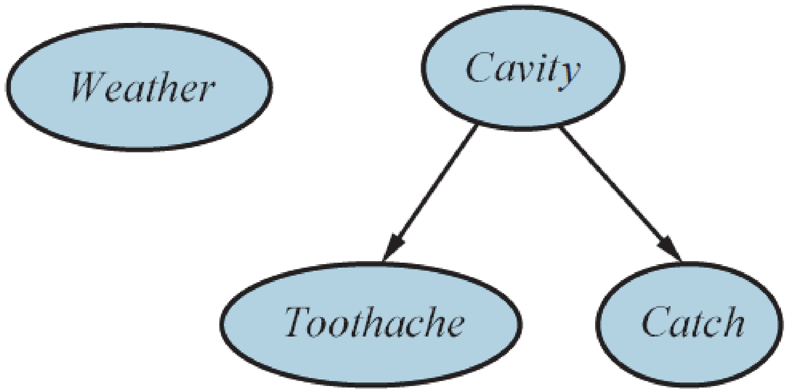

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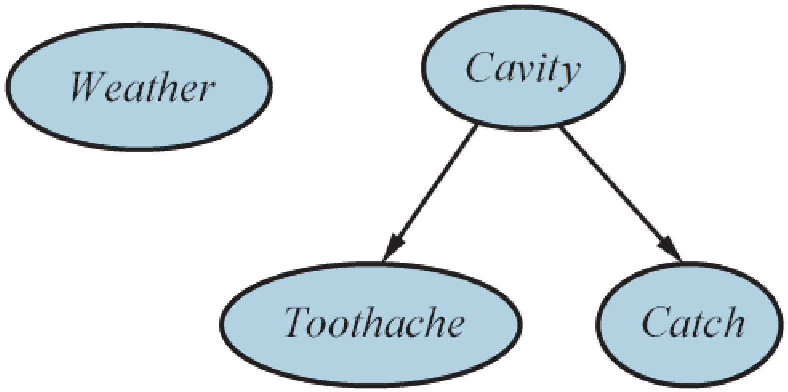

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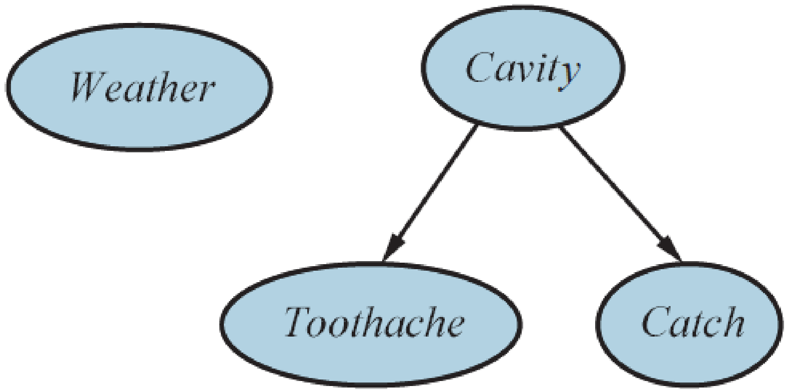

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- The parameters required for model construction are conditional probabilities that quantify cause-effect relations, which are
  - psychologically meaningful
  - often measurable

Burglar Alarm Problem

- A newly installed burglar alarm is fairly reliable at detecting a burglary.
- But it can also be occasionally set off by earthquakes.
- Neighbors John and Mary have promised a call when they hear the alarm.
  - ♣ John nearly always calls but sometimes confuses the alarm with the telephone ringing.
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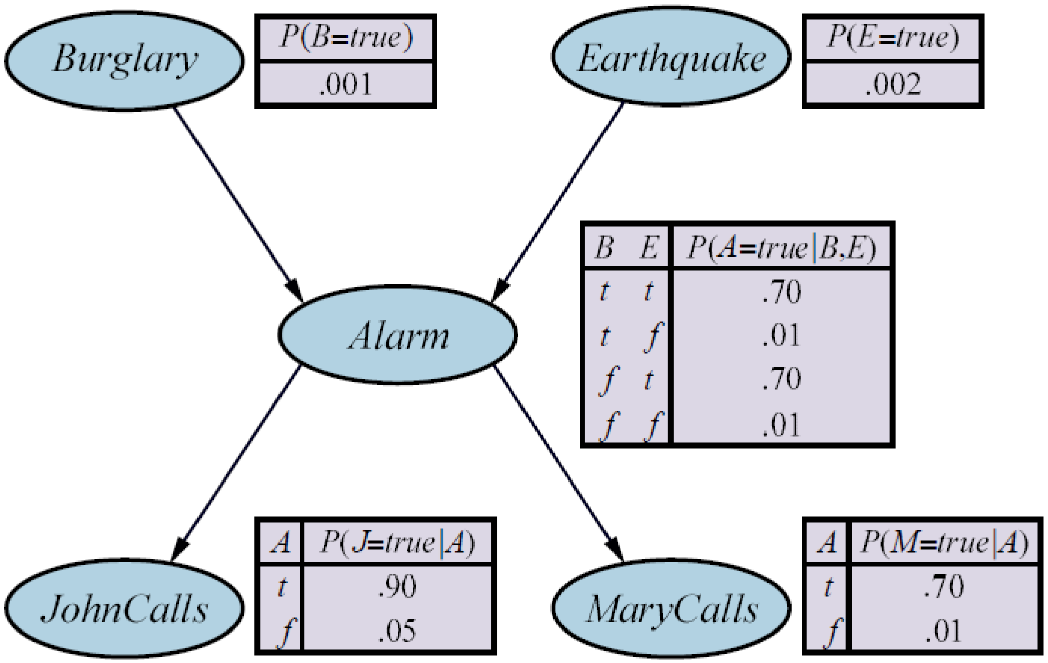

P(B=true)
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P(E=true)
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B	E	P(A=true|B,E)
t	t	.70
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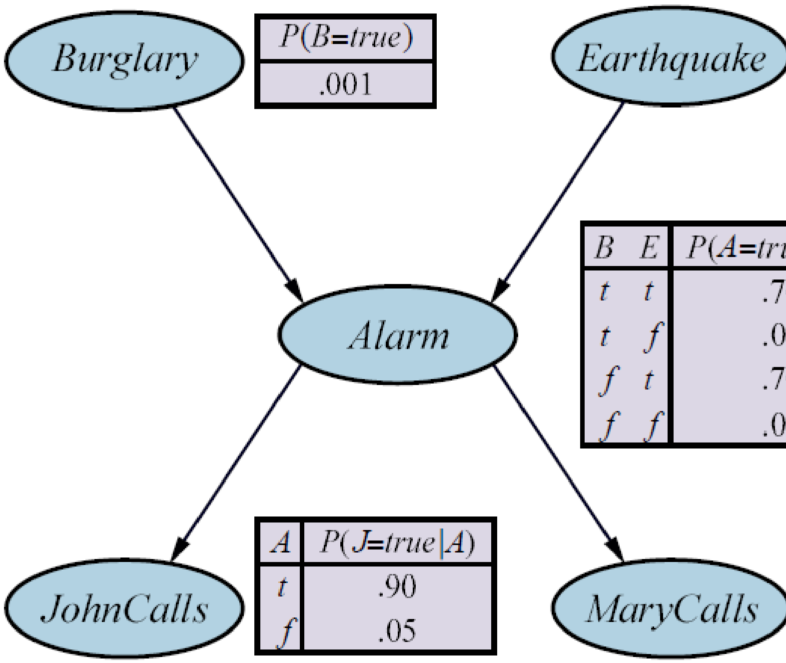

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conditional probability tables (CPTs)

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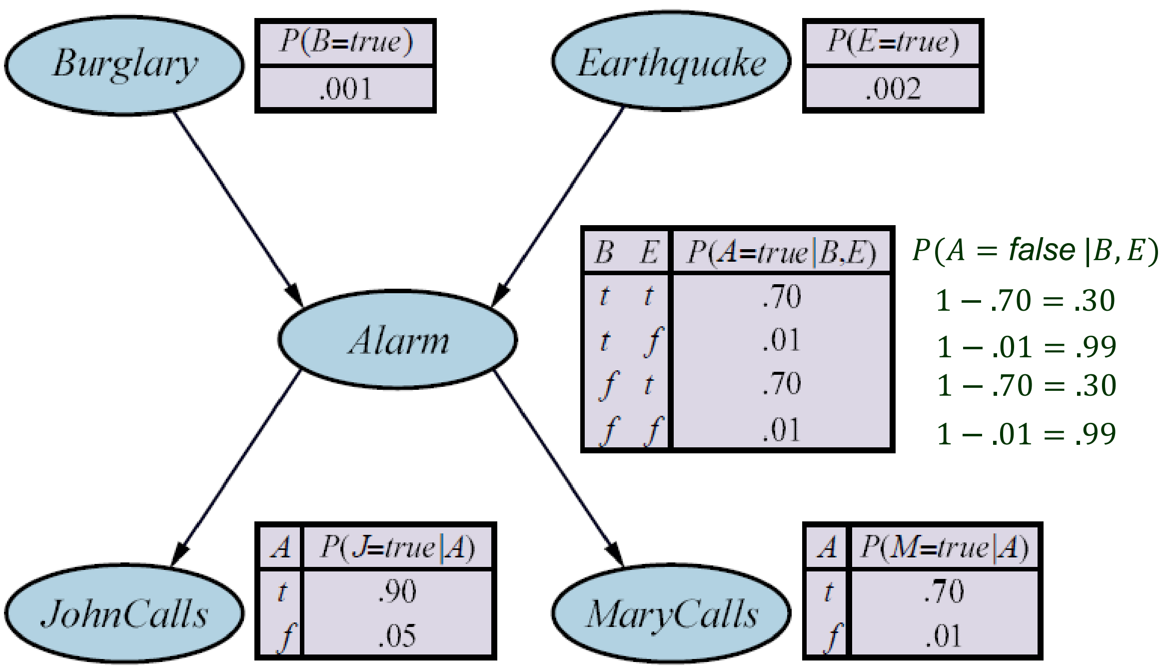

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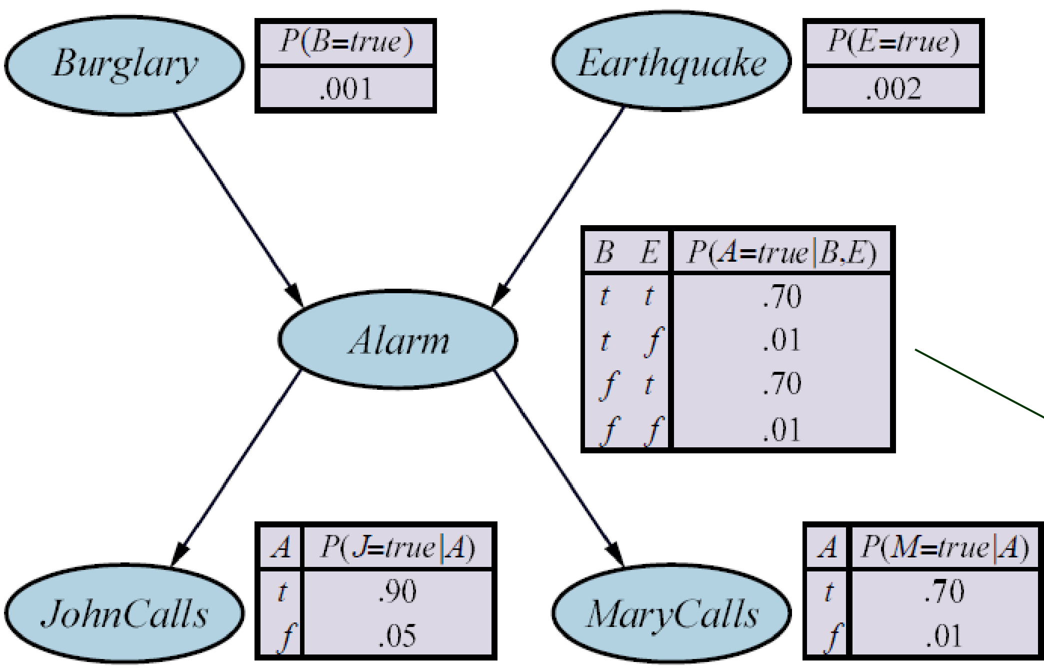

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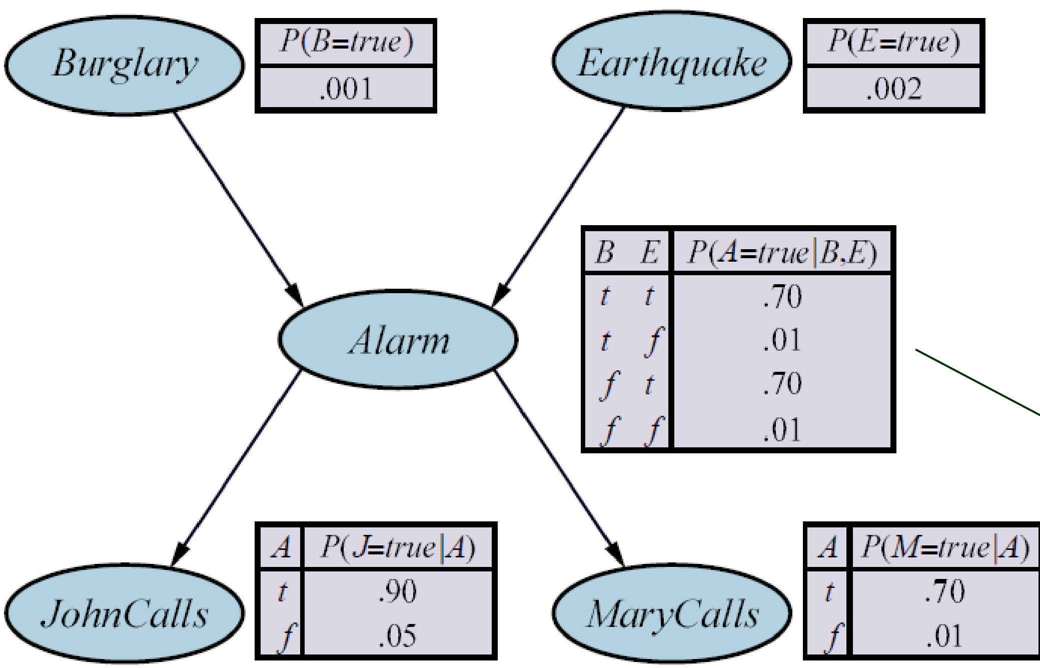

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Problem Estimate the probability of a burglary given the evidence of who has or has not called.

conditional probability tables (CPTs)

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$$\boxed{= \prod_{i=1}^{n} \theta_i(x_i \mid \mathit{parents}(X_i))}$$

where

$\mathit{parents}(X_i) = \{x_j \mid X_j \in \mathit{Parents}(X_i)\}$,

// the values of $\mathit{Parents}(X_i)$ that appear as part of $x_1, \ldots, x_n$

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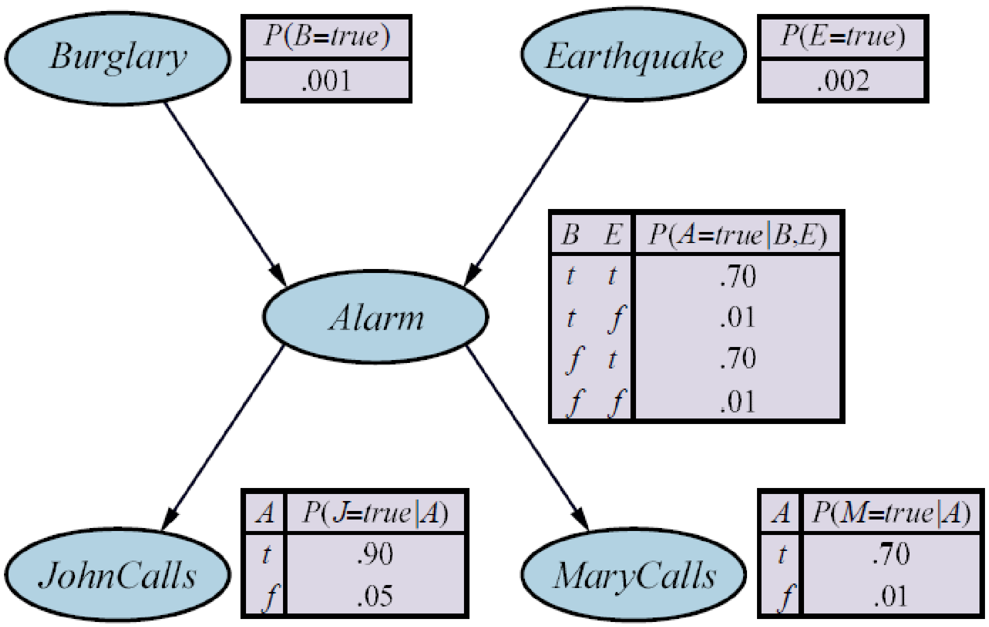

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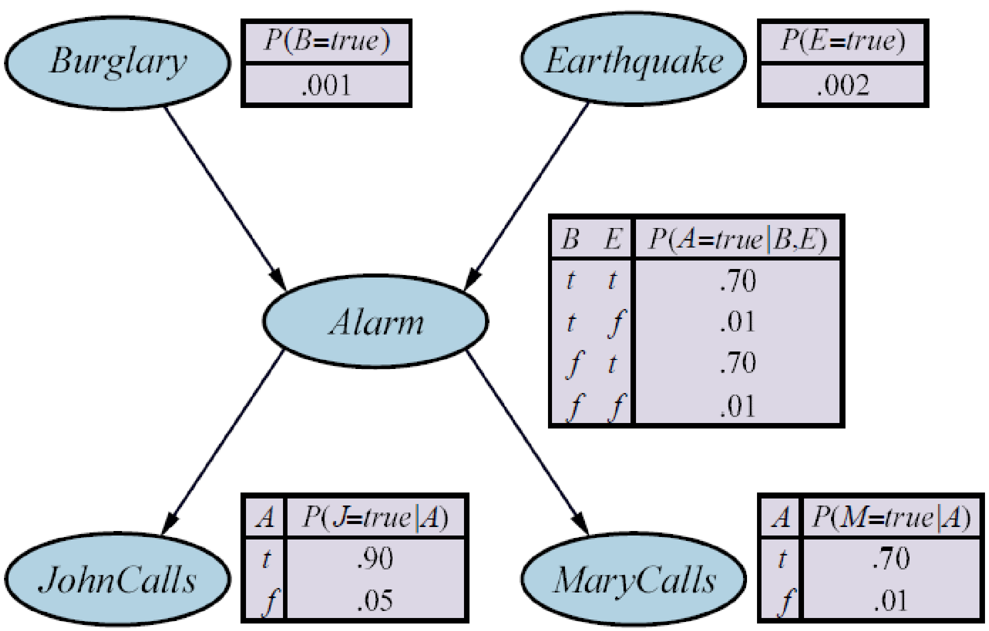

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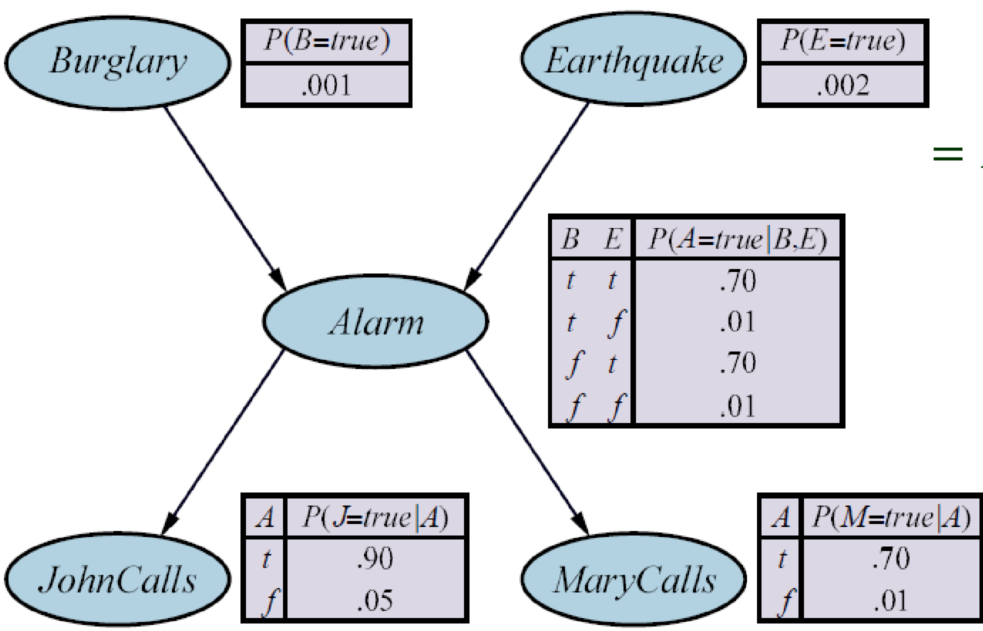

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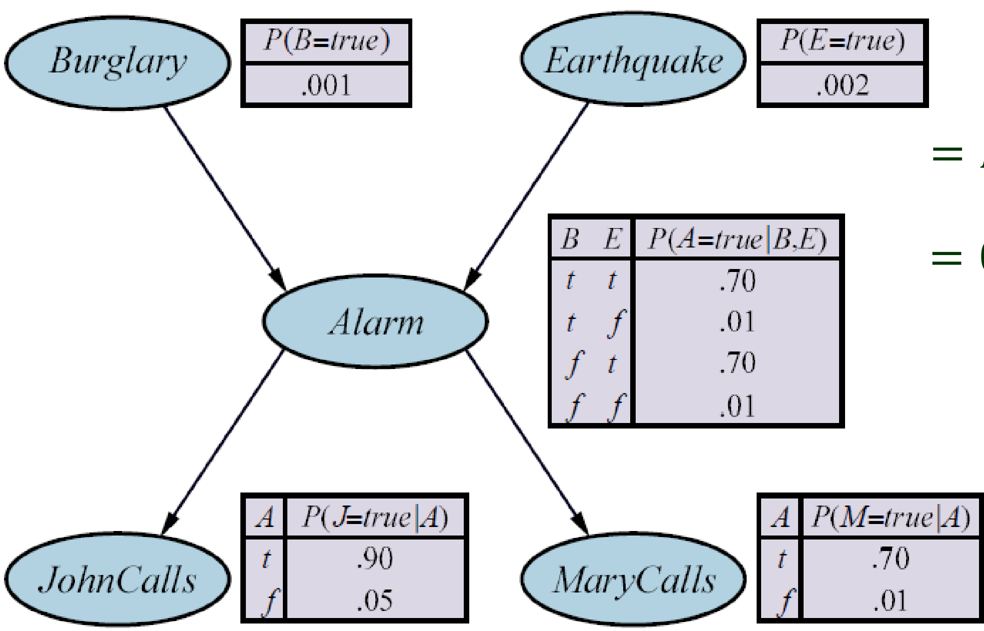

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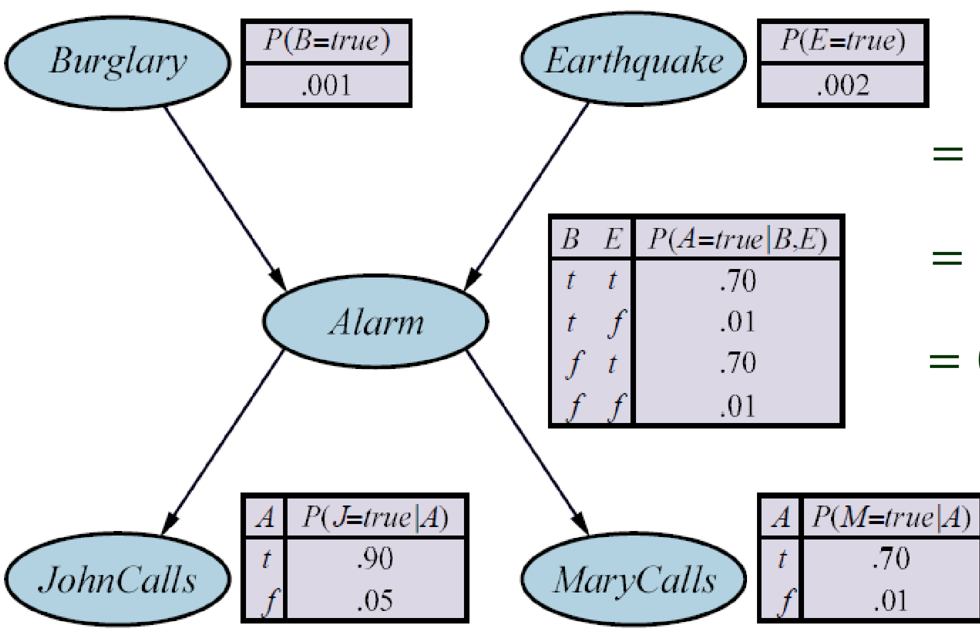

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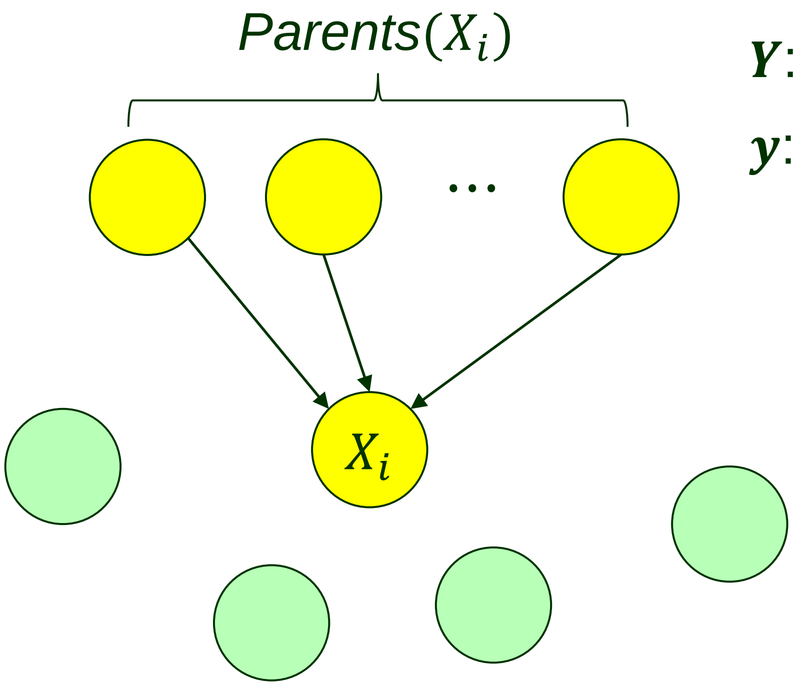

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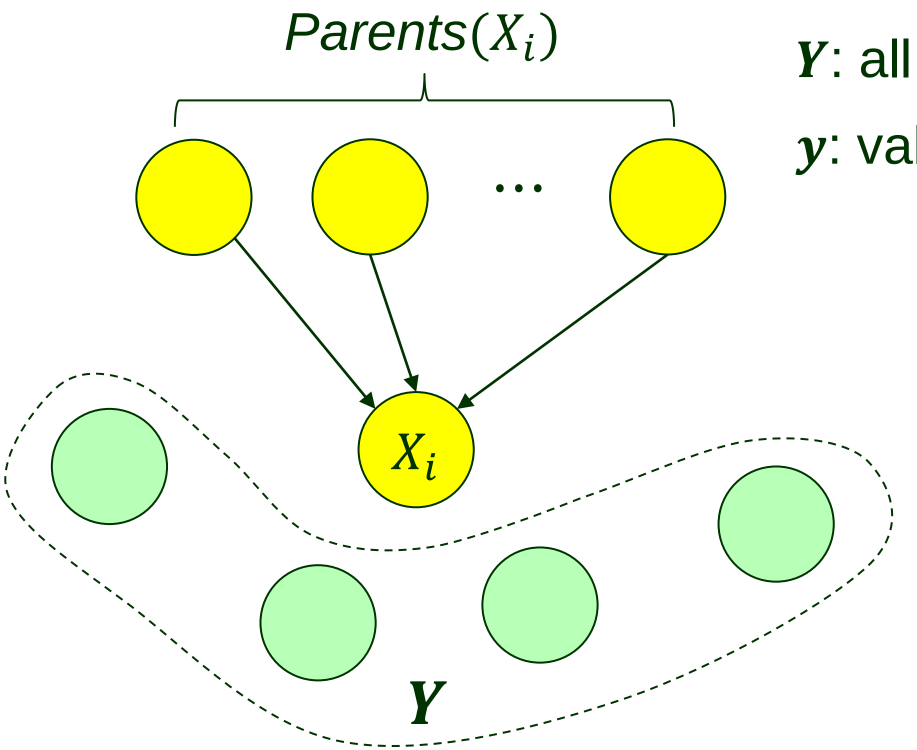

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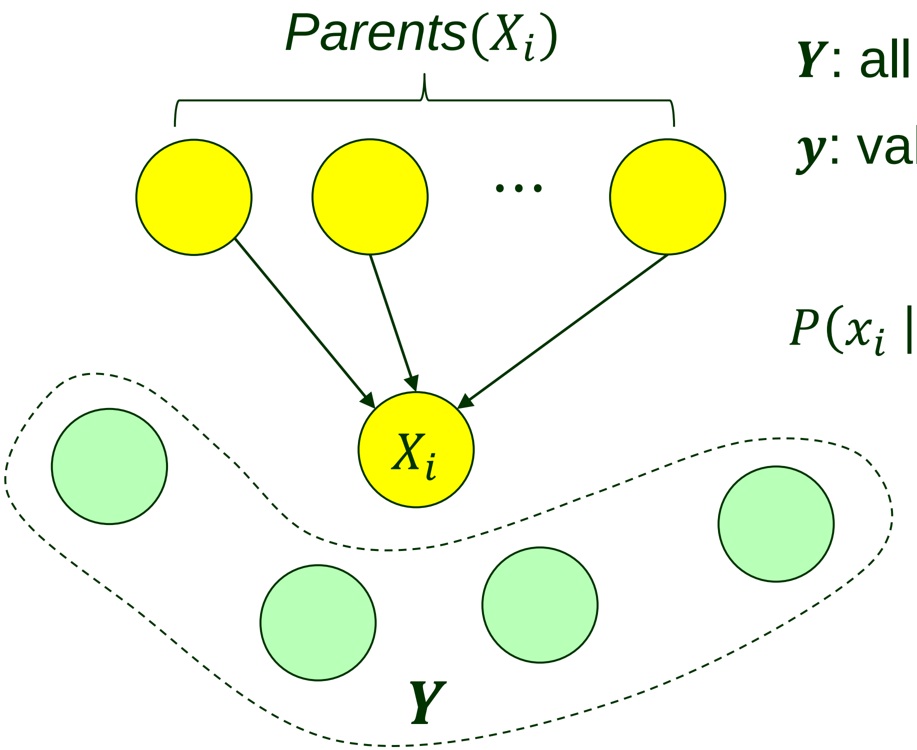

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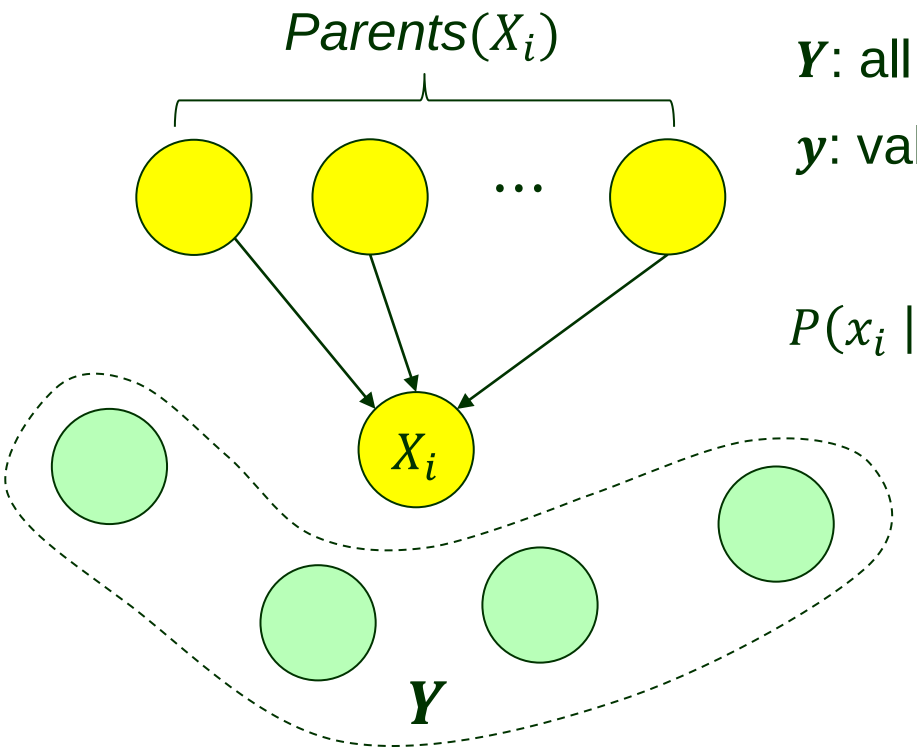
Conditional Probabilities



$\boldsymbol{Y}$: all variables other than X_i and $Parents(X_i)$

$\boldsymbol{y}$: values of $\boldsymbol{Y}$

$$P(x_i \mid parents(X_i)) \equiv \frac{P(x_i, parents(X_i))}{P(parents(X_i))}$$

$$= \frac{\sum_{\boldsymbol{y}} P(x_i, parents(X_i), \boldsymbol{y})}{\sum_{x_i', \boldsymbol{y}} P(x_i', parents(X_i), \boldsymbol{y})}$$

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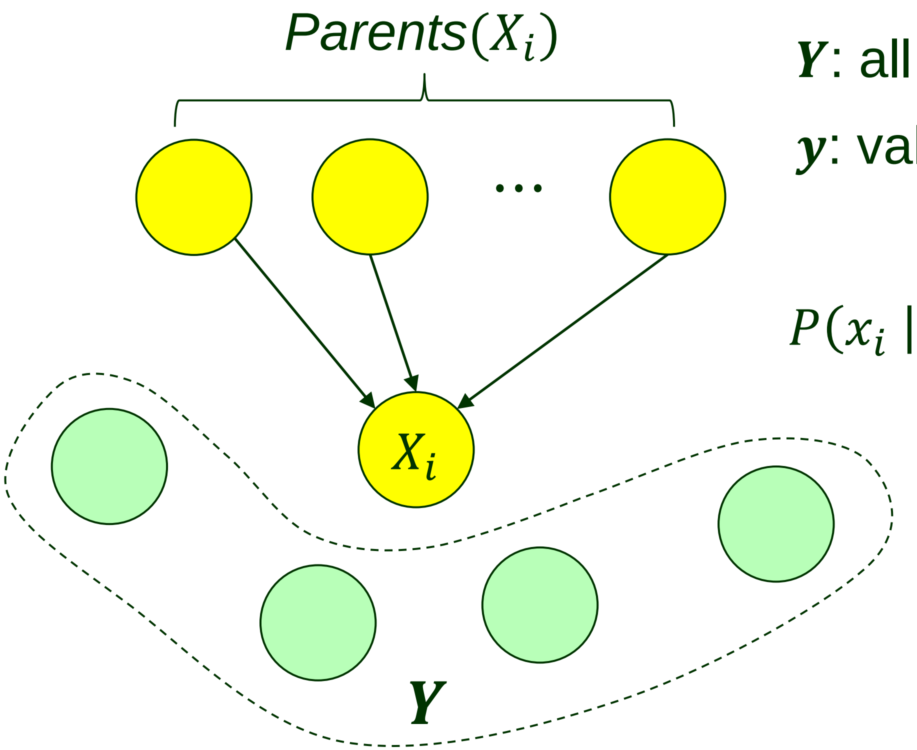

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$$= \theta_i(x_i \mid parents(X_i))$$

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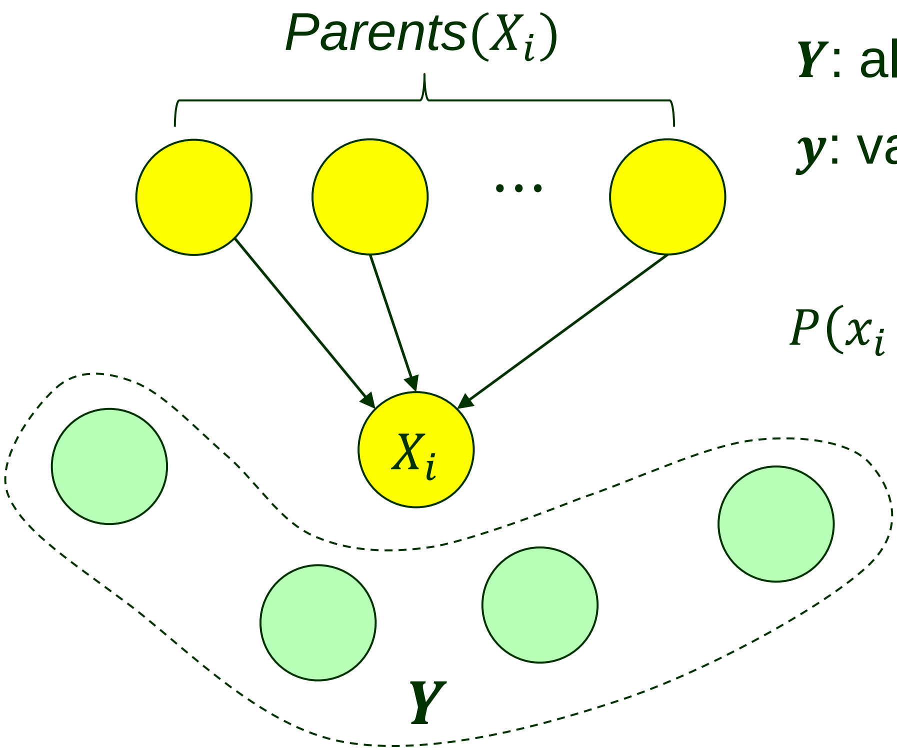

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$$= \theta_i(x_i \mid parents(X_i))$$

$$P(x_1, \ldots, x_n) = \prod_{i=1}^{n} \theta_i(x_i \mid parents(X_i))$$

$$\Downarrow$$

$$P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i \mid Parents(X_i))$$

Correct Domain Representation

Chain rule:

$$P(x_1, \dots, x_n) = P(x_n \mid x_{n-1}, \dots, x_1) P(x_{n-1}, \dots, x_1)$$

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$\boldsymbol{P}(X_i \mid X_{i-1}, \dots, X_1) = \boldsymbol{P}(X_i \mid \mathit{Parents}(X_i))$ for $i = 2, \dots, n$

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$$\textit{Parents}(X_i) \subseteq \{X_1, \dots X_{i-1}\} \text{ for } i = 2, \dots, n$$

Topological Order

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The above is guaranteed if we number the nodes in *topological order* (which exists since the Bayesian network is a DAG).

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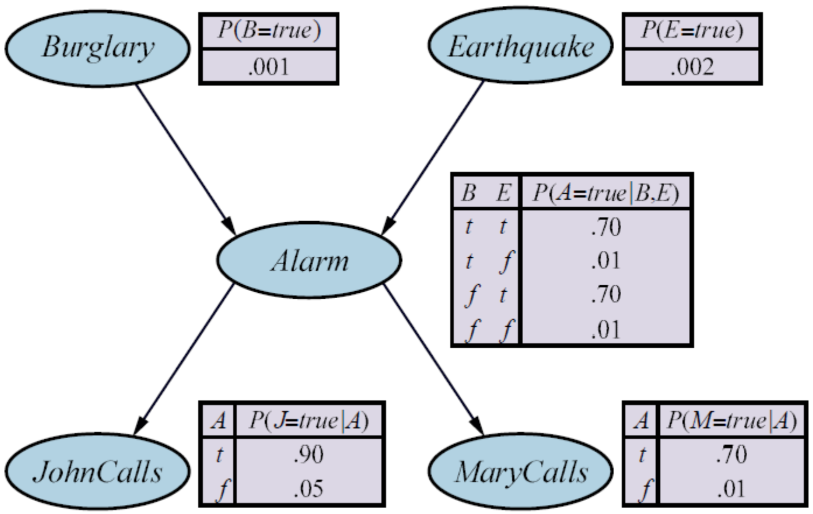

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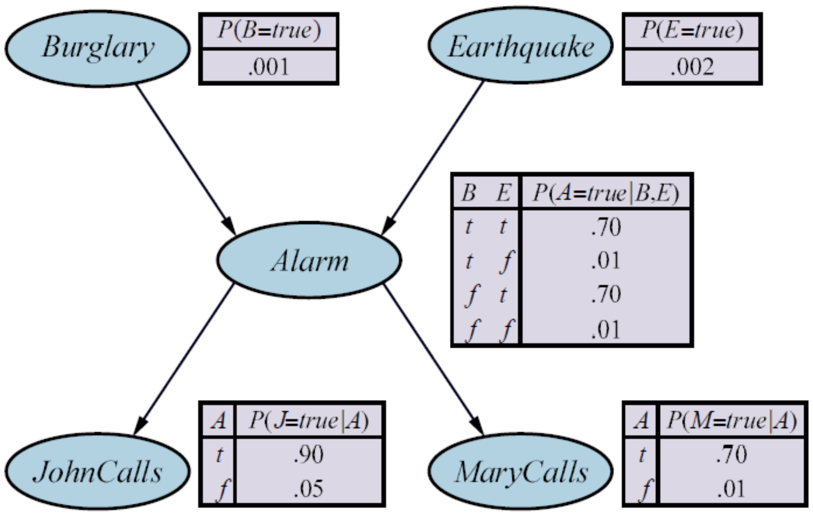

Four topological orders:

B, E, A, J, M

B, E, A, M, J

E, B, A, J, M

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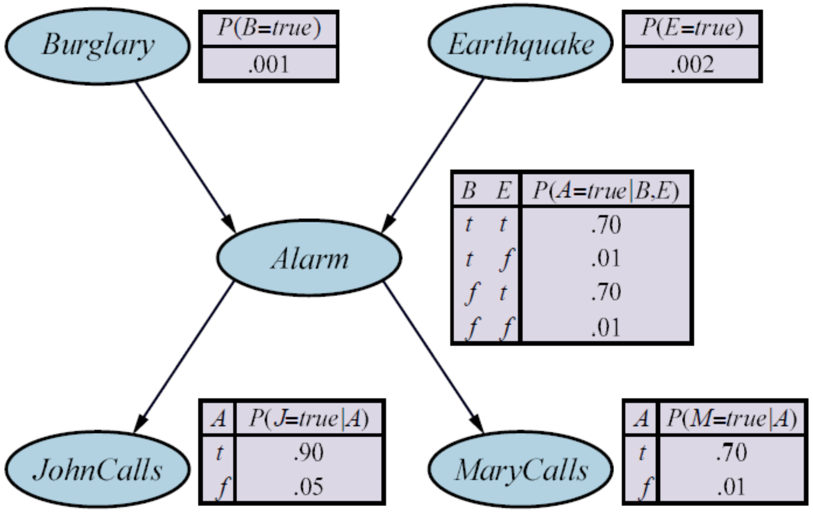

Four topological orders:

B, E, A, J, M
B, E, A, M, J
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Any one of the four suffices.