Decision Trees

Outline

I. Learning decision trees from examples

II. Entropy and attribute choice

** Figures are from the textbook site or plotted by the instructor.

Machine Learning Models

- Supervised learning
  - Naïve Bayes classifier
  - Nearest neighbor methods
  - Linear models
  - Decision trees
  - Neural networks
  - Support vector machines
  - Ensemble learning
- Probabilistic graphical models
  - Bayesian networks
  - Markov random fields
- Unsupervised learning
  - Clustering: mixture models, K-means, hierarchical clustering
  - Principal component analysis
  - Independent component analysis
- Sequential data
  - HMMs
  - Recurrent neural networks
- Markov decision process
- Reinforcement Learning

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I. What is a Decision Tree?

A *decision tree* maps a set of attribute values to a “decision”

- It performs a sequence of tests, starting at the root and descending down to a leaf (which specifies the output value).
- Each internal node corresponds to a test of the value of an attribute.

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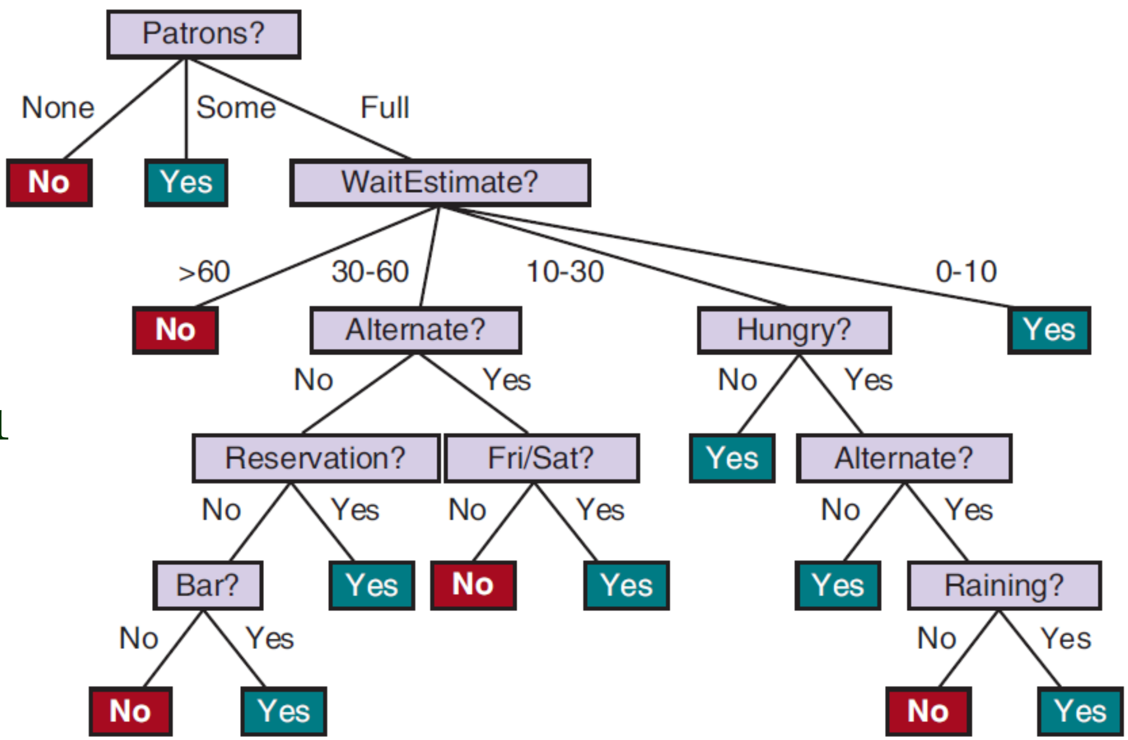
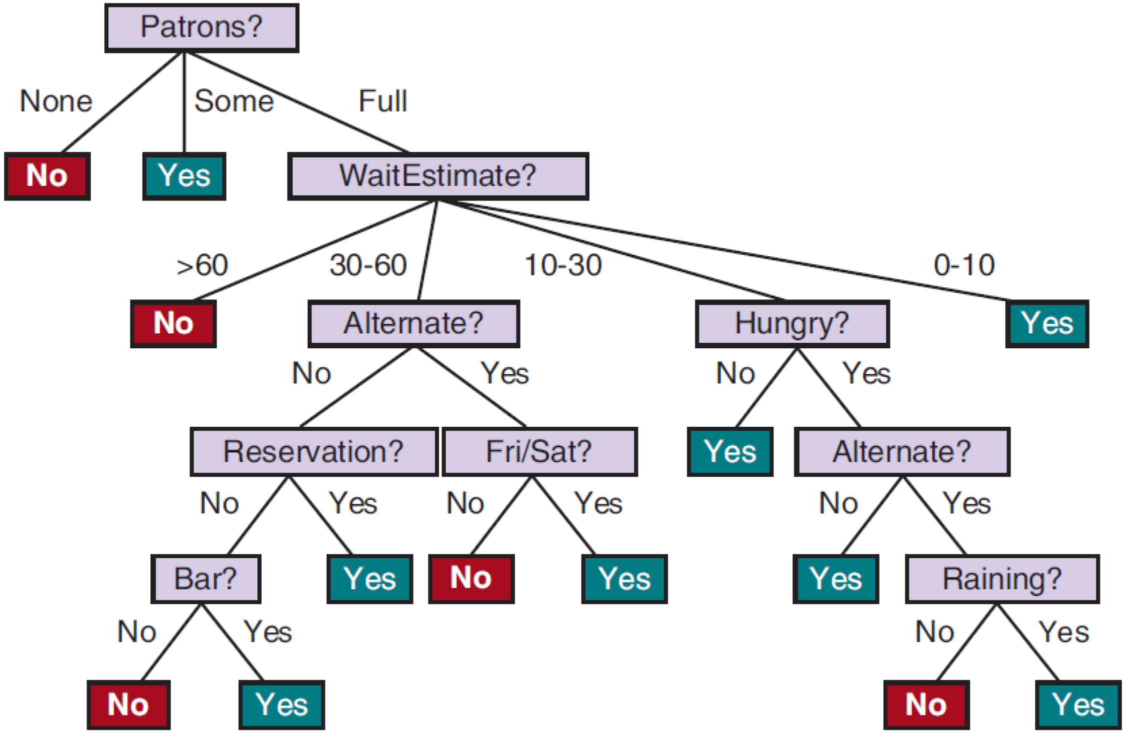
T_1



A decision tree for the restaurant waiting problem

Boolean Classification

- Inputs are discrete values.
- Outputs are either *true* (a positive example) or *false* (a negative one).



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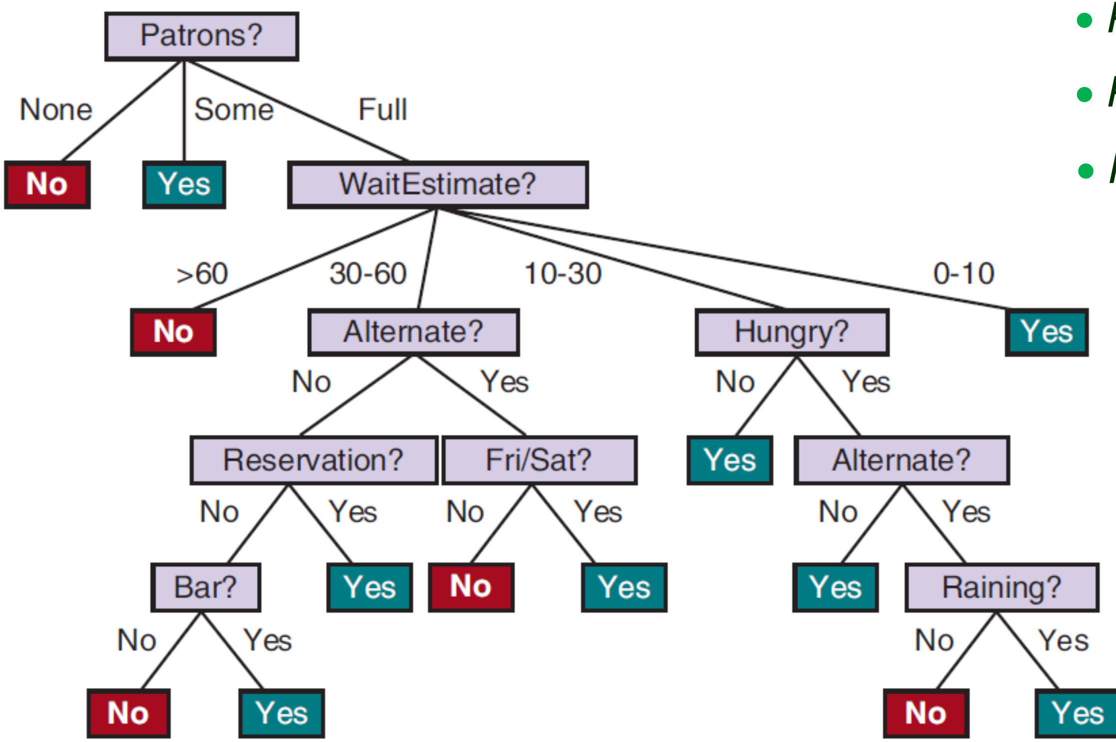

Positive examples:

- $Patrons = Yes$
- $Patrons = Full \wedge WaitEstimate = 0\text{−}10$
- $Patrons = Full$
  $\wedge\ WaitEstimate = 10\text{−}30$
  $\wedge\ Hungry = Yes$
  $\wedge\ Alternate = No$

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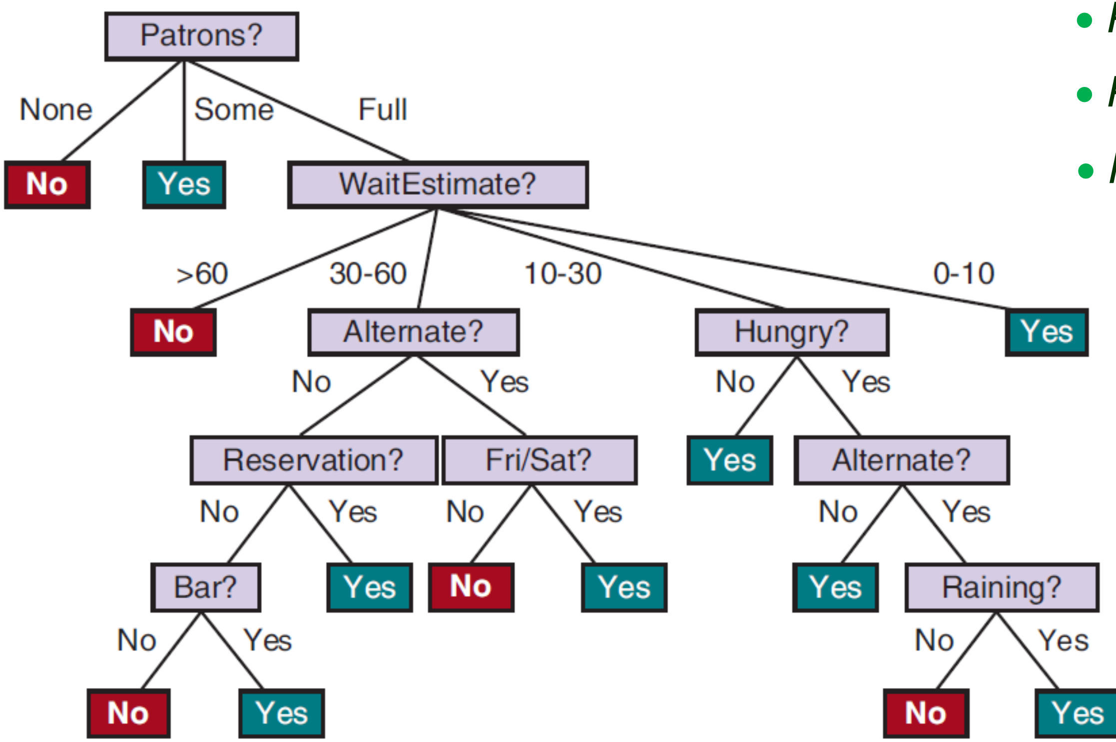

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Negative examples:

- $Patrons = No$
- $Patrons = Full$
  $\wedge\ WaitEstimate > 60$

Why Decision Trees?

- They yield nice, concise, and understandable results.
- They represent simple rules for classifying instances that are described by discrete attribute values.
- Decision tree learning algorithms are relatively efficient – linear in the size of the decision tree and the size of the data set.
- Decision trees are often among the first to be tried on a new data set.
- They are not good with real-valued attributes.

Expressiveness of a Decision Tree

Equivalent logical statement of a decision tree:

$$Output \Leftrightarrow Path_1 \vee Path_2 \vee \cdots$$

where

$$Path_i = (A_{i1} = v_{i1} \wedge A_{i2} = v_{i2} \wedge \cdots)$$

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Disjunctive normal form (DNF)

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Any sentence in propositional logic can be converted into a DNF.

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$\Downarrow$

Any sentence in propositional logic can expressed as a decision tree.

Learning DTs from Examples

Problem Find a tree that is consistent with the training examples (in, e.g, restaurant waiting) and is as small as possible.

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- Recursively solve the smaller subproblems.

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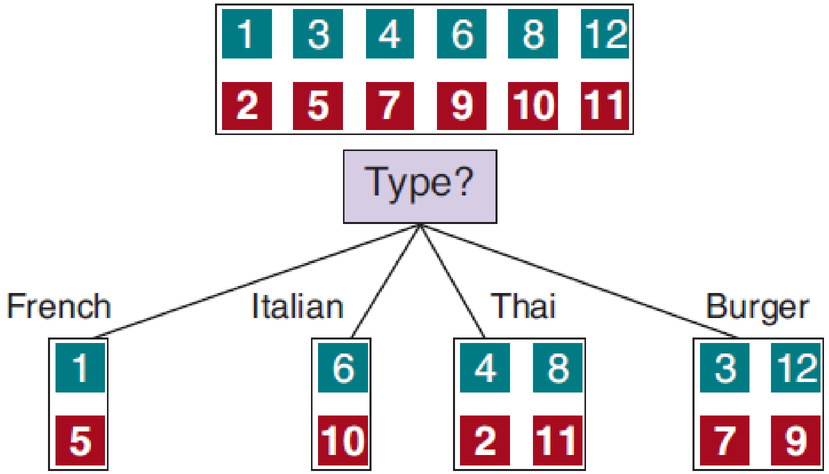
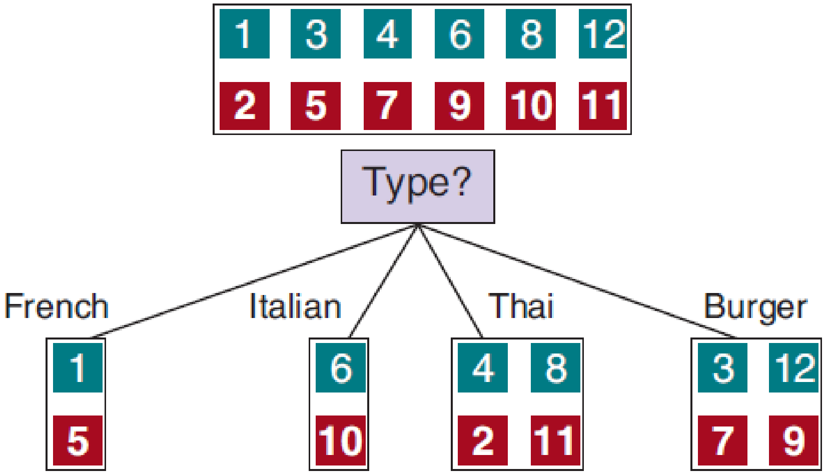
Poor and Good Attributes



Poor attribute *Type*:

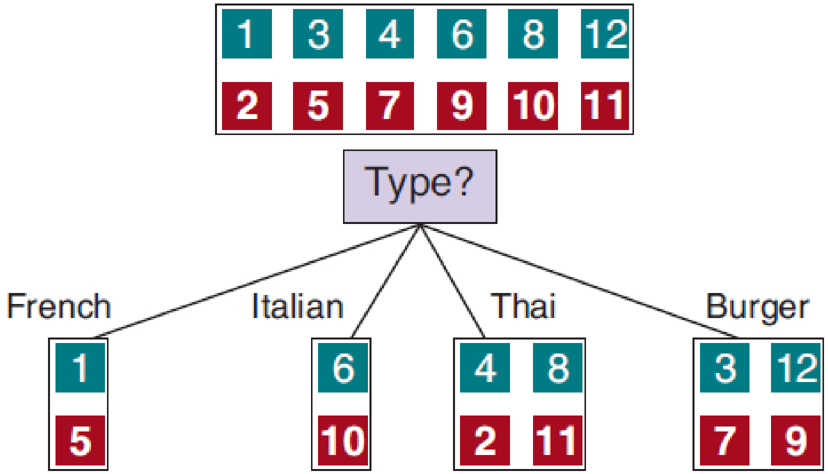
Poor and Good Attributes



Poor attribute *Type*:

♠ four outcomes

Poor and Good Attributes



Poor attribute *Type*:

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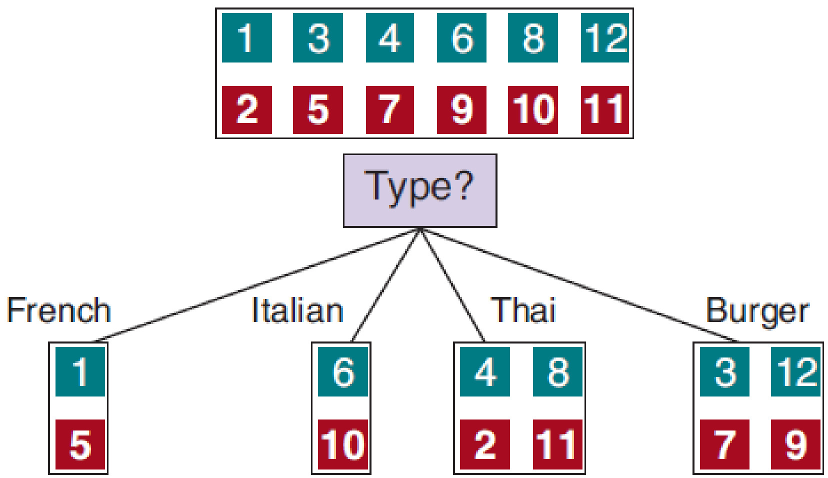
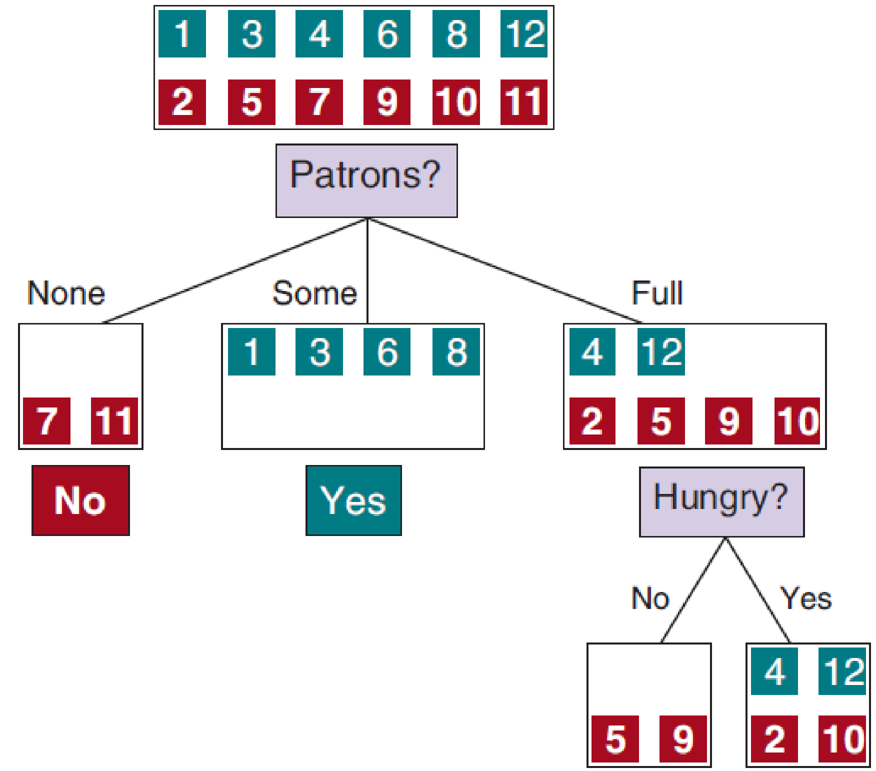
- each with the same number of positive as negative examples

Poor and Good Attributes





Poor attribute *Type*:

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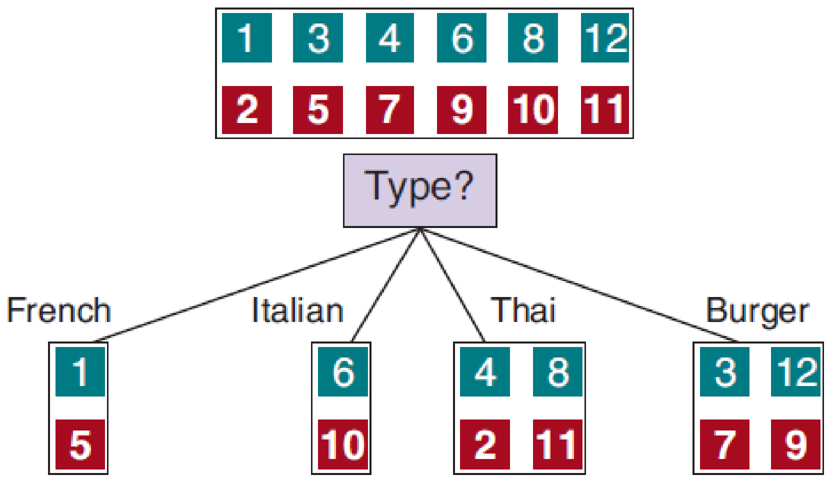
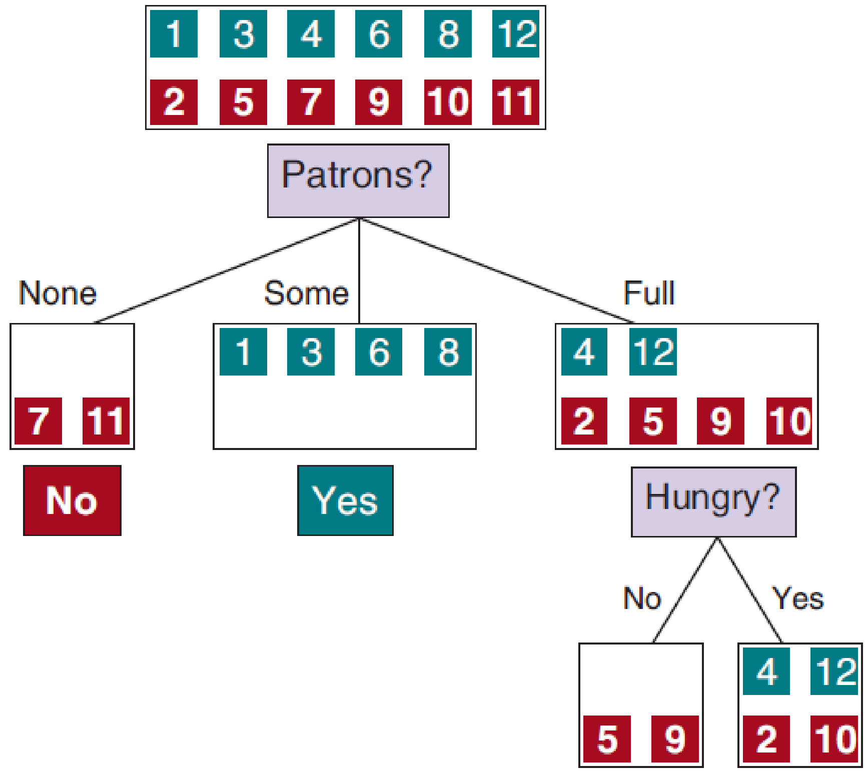
Good attributes *Patrons* and *Hungry*:

Poor and Good Attributes





Poor attribute *Type*:

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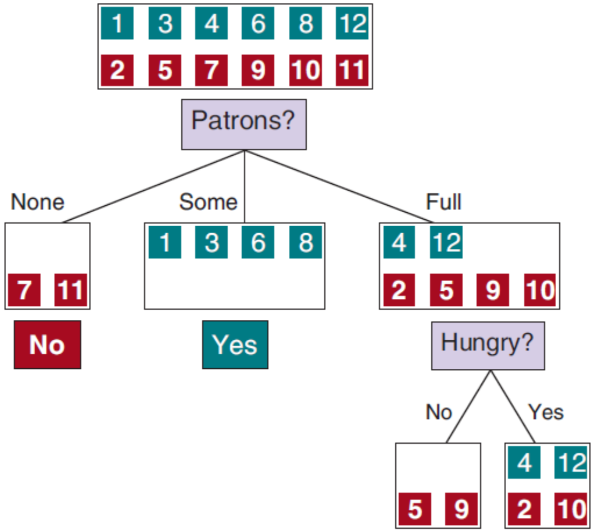
Good attributes *Patrons* and *Hungry*:

- effective separations of positive and negative examples

Four Cases

1. All positive (or all negative) remaining examples.

Done.



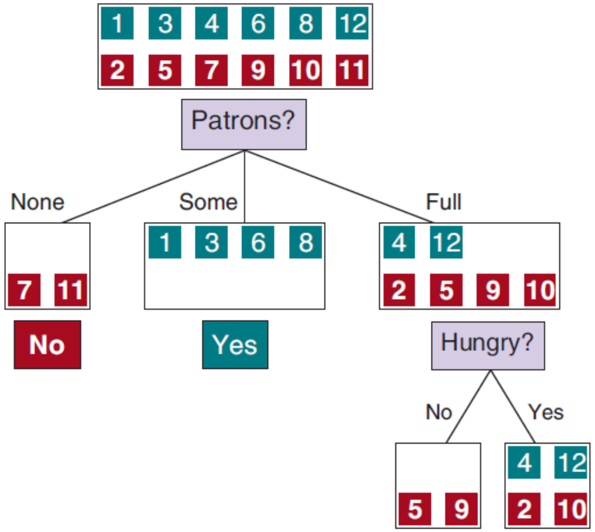
Four Cases

1. All positive (or all negative) remaining examples.

   Done.

2. Mixed positive and negative remaining examples.

   Choose the best attribute to split them.



Four Cases

1. All positive (or all negative) remaining examples.

   Done.

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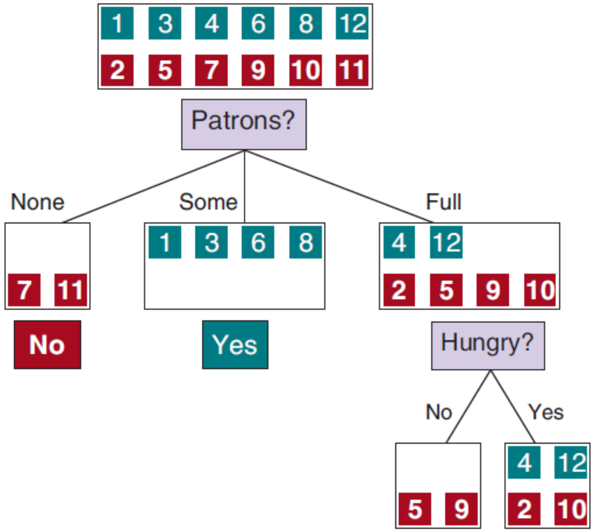
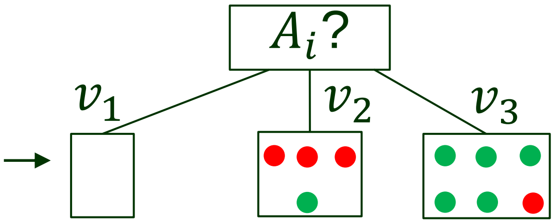


3. No examples left.



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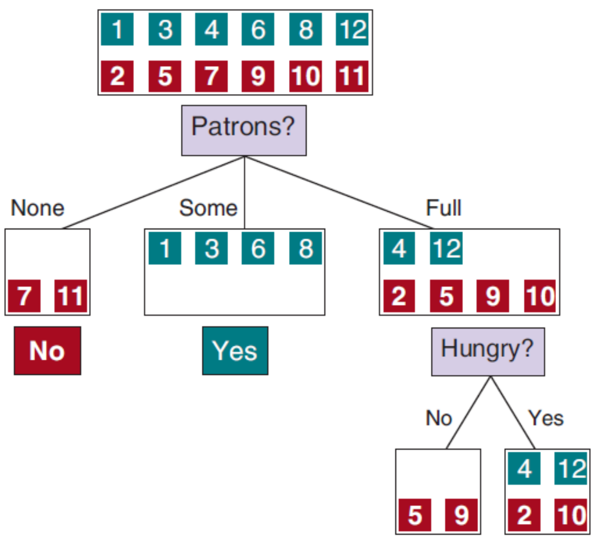

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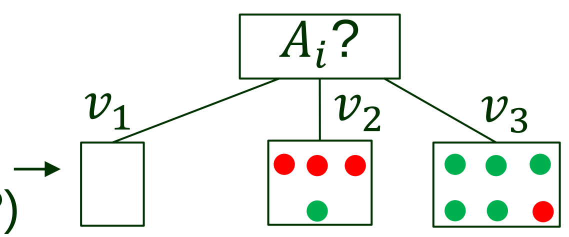
   Return the most common output value (e.g., •) for the example set used in constructing the parent (A_i?)



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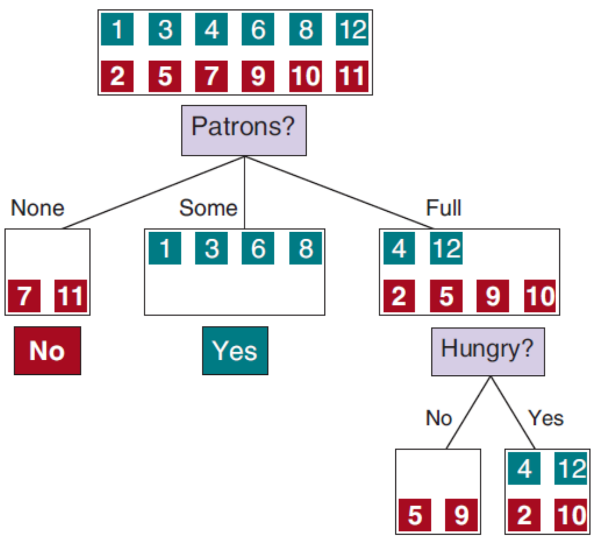

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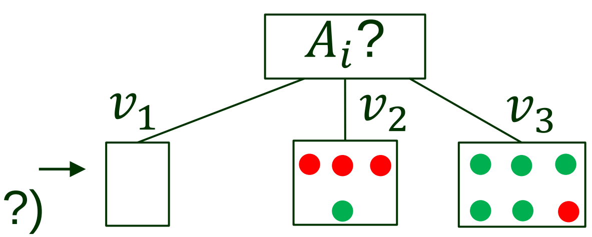

4. No attributes left but both positive and negative examples.

   Return the most common output value of these examples (by a *majority vote*)

Algorithm for Learning DTs

```
function LEARN-DECISION-TREE(examples, attributes, parent_examples) returns a tree

  if examples is empty then return PLURALITY-VALUE(parent_examples)        // case 3
  else if all examples have the same classification then return the classification  // case 1
  else if attributes is empty then return PLURALITY-VALUE(examples)        // case 4
  else  // case 2
     A ← argmax_{a ∈ attributes} IMPORTANCE(a, examples)
     tree ← a new decision tree with root test A
     for each value v of A do
        exs ← {e : e ∈ examples and e.A = v}
        subtree ← LEARN-DECISION-TREE(exs, attributes − A, examples)
        add a branch to tree with label (A = v) and subtree subtree
     return tree
```

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function LEARN-DECISION-TREE($examples$, $attributes$, $parent\_examples$) **returns** a tree

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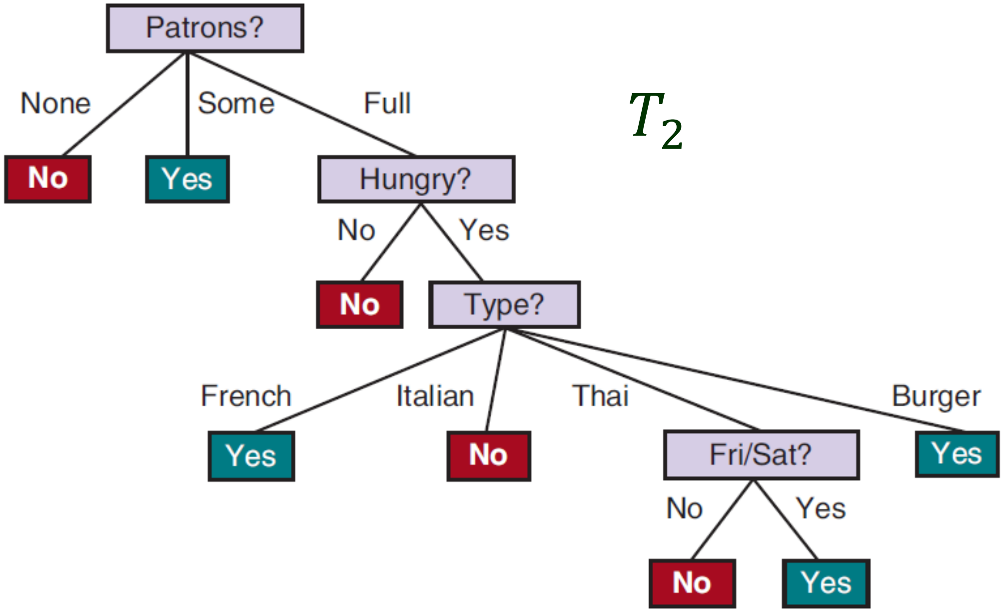
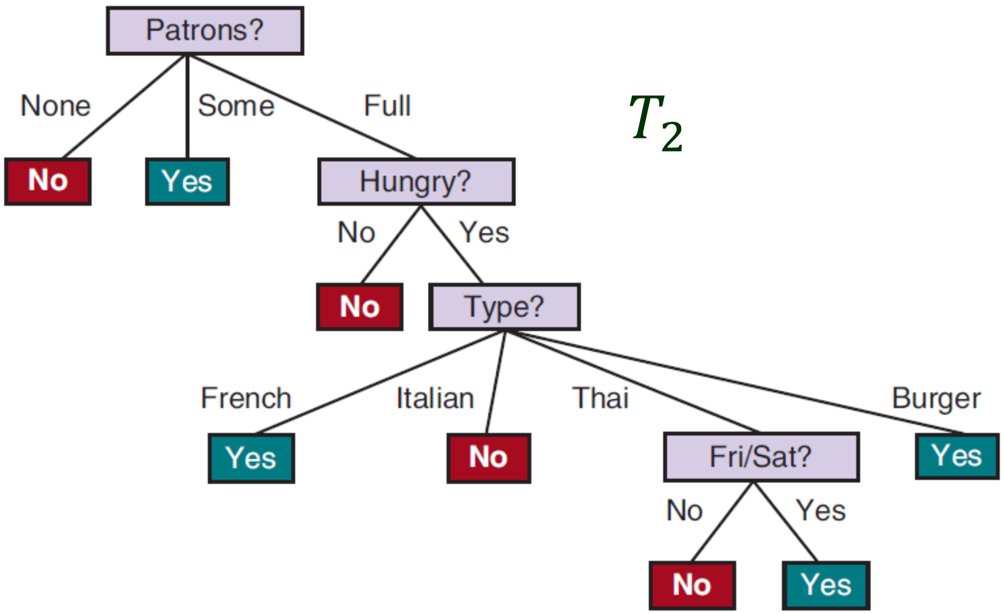
Output Decision Tree



T_2

Output Decision Tree



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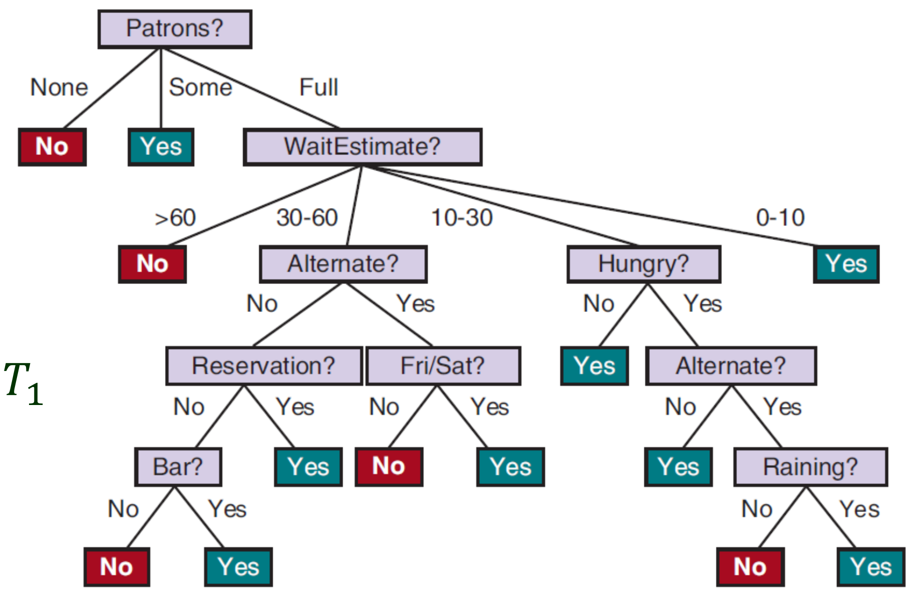

Original decision tree T_1

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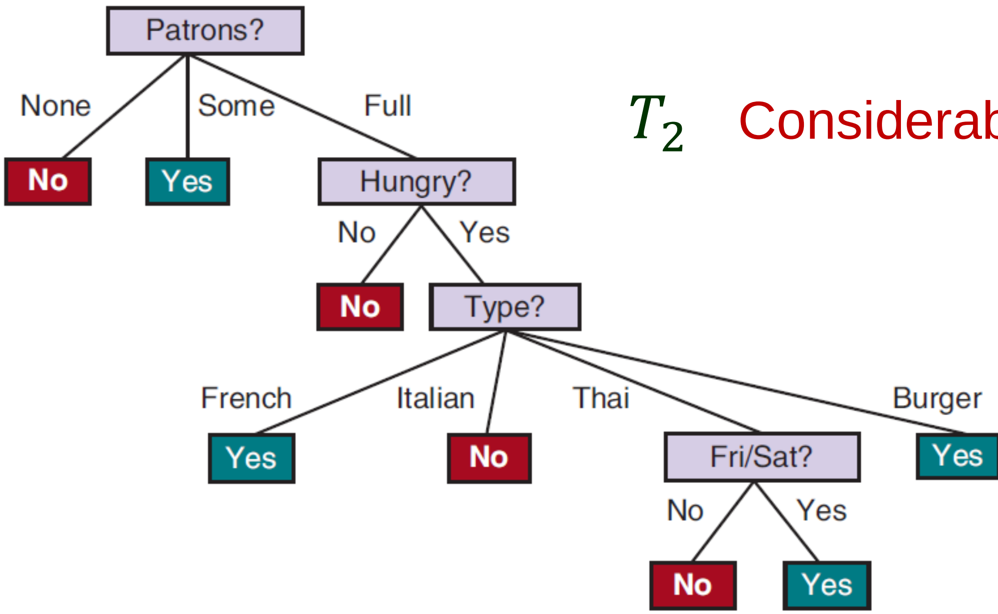
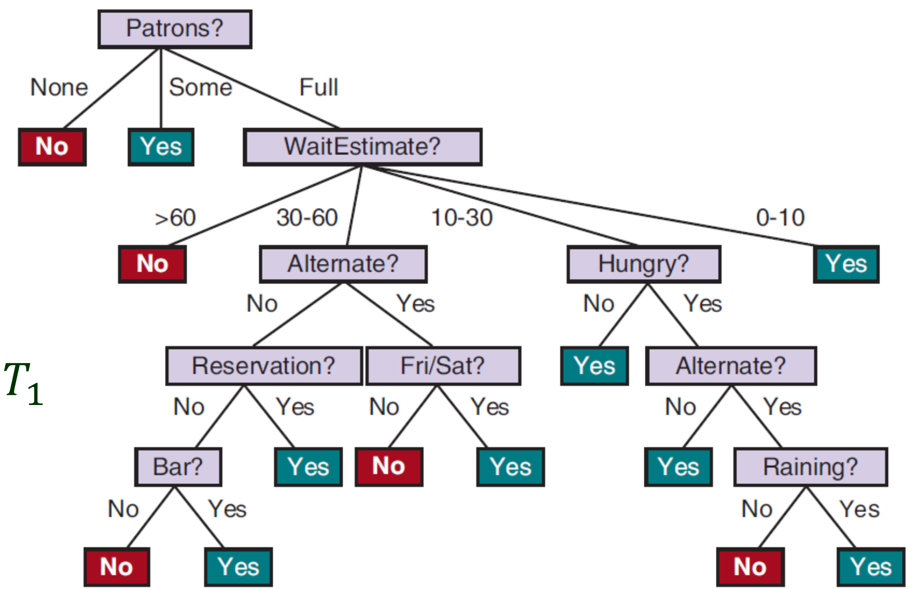


T_2 Considerably simpler!



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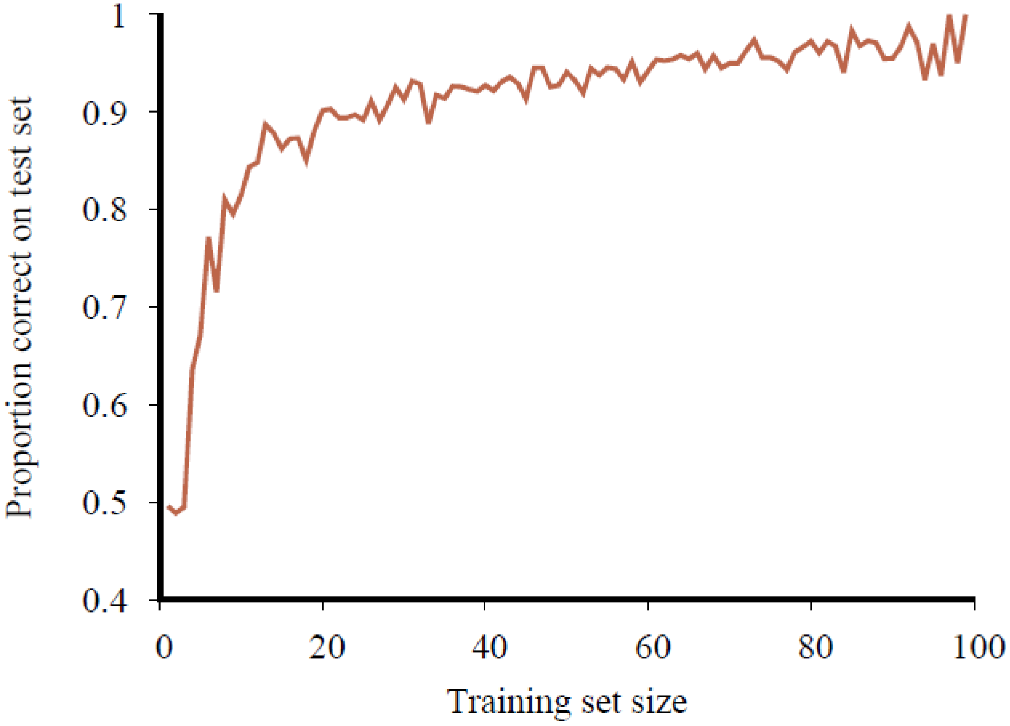
Learning Curve



"happy graph"

- 100 randomly generated examples in the restaurant domain.
- Split them randomly into a training set and a test set.
  - Split 20 times for each size (1 – 99) of the training set.
  - Average the results of the 20 trials.

II. Measure of Importance

Importance of an attribute is measured using the notion of information gain defined in terms of *entropy*.

measure of the uncertainty of a random variable:
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- A four-sided die has 2 bits of entropy since there are 2^2 equally likely choices.
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Entropy

Random variable V with values v_k having probability $P(v_k)$.

The variable has entropy:

$$H(V) = \sum_k P(v_k) \log_2 \frac{1}{P(v_k)}$$
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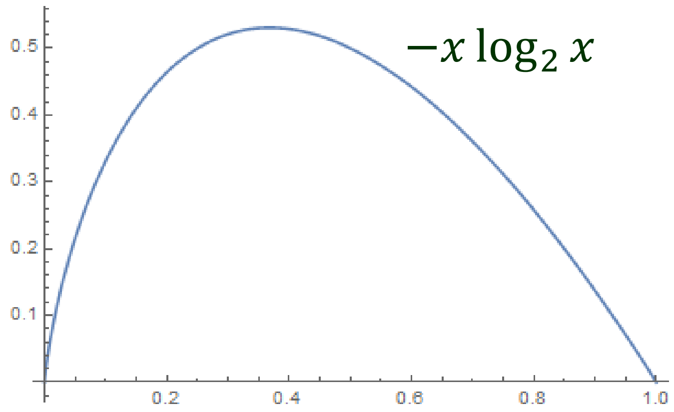

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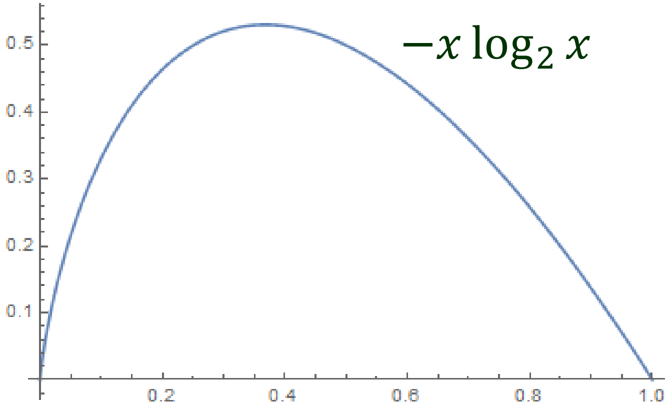

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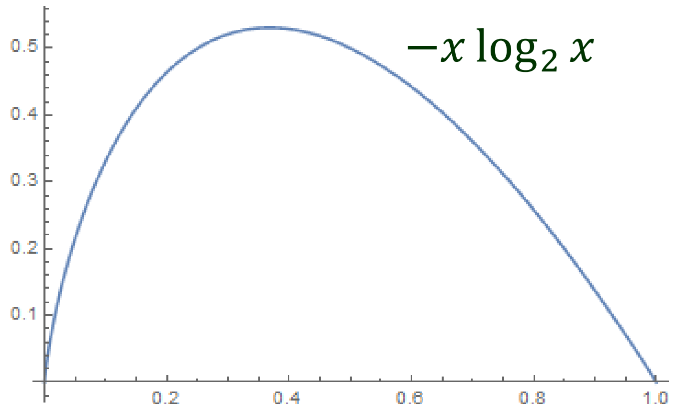

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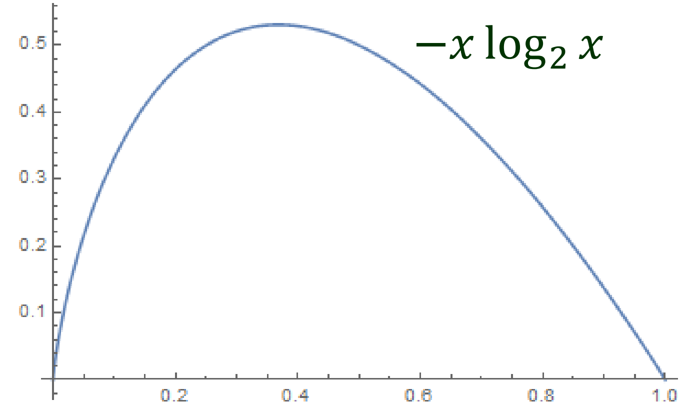

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$$H(\mathit{Die4}) = -(4 \cdot 0.25 \log_2 0.25) = 2 \text{ (bits)}$$

- A loaded coin with 99% heads

$$H(\mathit{Loaded}) = -(0.99 \log_2 0.99 + 0.01 \log_2 0.01) \approx 0.08 \text{ (bits)}$$

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A Boolean random variable that is true with probability q.

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Example	Input Attributes										Output
	Alt	*Bar*	*Fri*	*Hun*	*Pat*	*Price*	*Rain*	*Res*	*Type*	*Est*	*WillWait*
$\mathbf{x}_1$	*Yes*	*No*	*No*	*Yes*	*Some*	*\$\$\$*	*No*	*Yes*	*French*	*0–10*	$y_1 =$ *Yes*
$\mathbf{x}_2$	*Yes*	*No*	*No*	*Yes*	*Full*	*\$*	*No*	*No*	*Thai*	*30–60*	$y_2 =$ *No*
$\mathbf{x}_3$	*No*	*Yes*	*No*	*No*	*Some*	*\$*	*No*	*No*	*Burger*	*0–10*	$y_3 =$ *Yes*
$\mathbf{x}_4$	*Yes*	*No*	*Yes*	*Yes*	*Full*	*\$*	*Yes*	*No*	*Thai*	*10–30*	$y_4 =$ *Yes*
$\mathbf{x}_5$	*Yes*	*No*	*Yes*	*No*	*Full*	*\$\$\$*	*No*	*Yes*	*French*	*>60*	$y_5 =$ *No*
$\mathbf{x}_6$	*No*	*Yes*	*No*	*Yes*	*Some*	*\$\$*	*Yes*	*Yes*	*Italian*	*0–10*	$y_6 =$ *Yes*
$\mathbf{x}_7$	*No*	*Yes*	*No*	*No*	*None*	*\$*	*Yes*	*No*	*Burger*	*0–10*	$y_7 =$ *No*
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$$\Downarrow$$

$$B(0.5) = 1$$

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$\mathbf{x}_{12}$	*Yes*	*Yes*	*Yes*	*Yes*	*Full*	*$*	*No*	*No*	*Burger*	*30–60*	$y_{12} = \mathit{Yes}$

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  - Consider a randomly chosen example from the training set.
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- Expected entropy remaining after testing attribute A is

$$Remainder(A) = \sum_{k=1}^{d} \frac{p_k+n_k}{p+n} B\left(\frac{p_k}{p_k+n_k}\right)$$

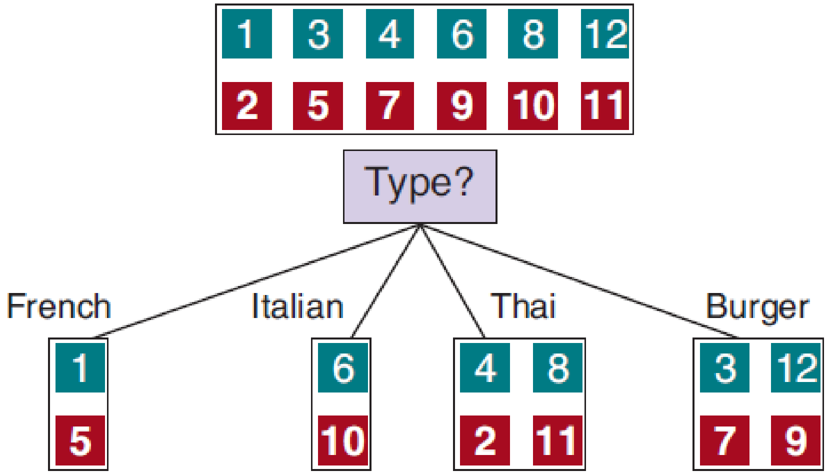
Information Gain

Expected reduction in entropy from the attribute test on A:

$$\mathit{Gain}(A) = B\left(\frac{p}{p+n}\right) - \mathit{Remainder}(A)$$

Choose the attribute A among the remaining attributes to maximize $\mathit{Gain}(A)$.

Information Gain



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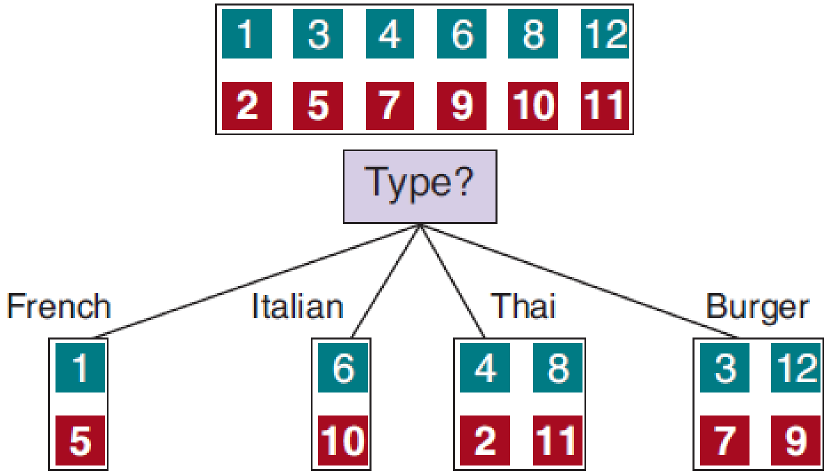

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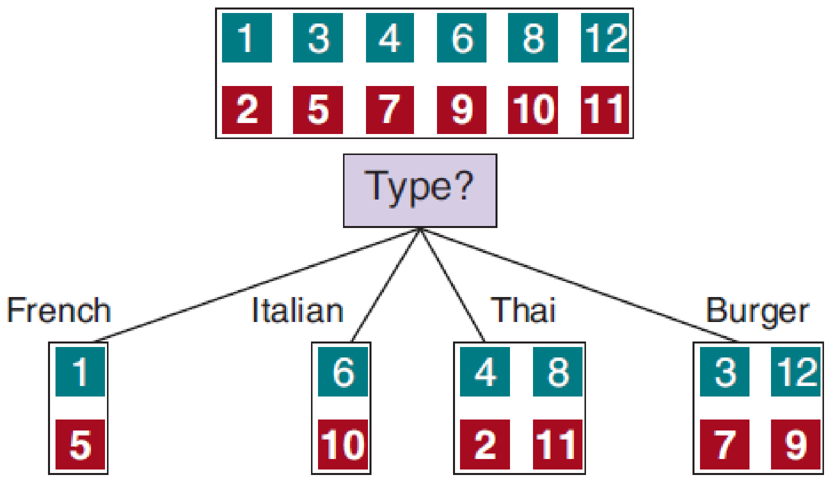
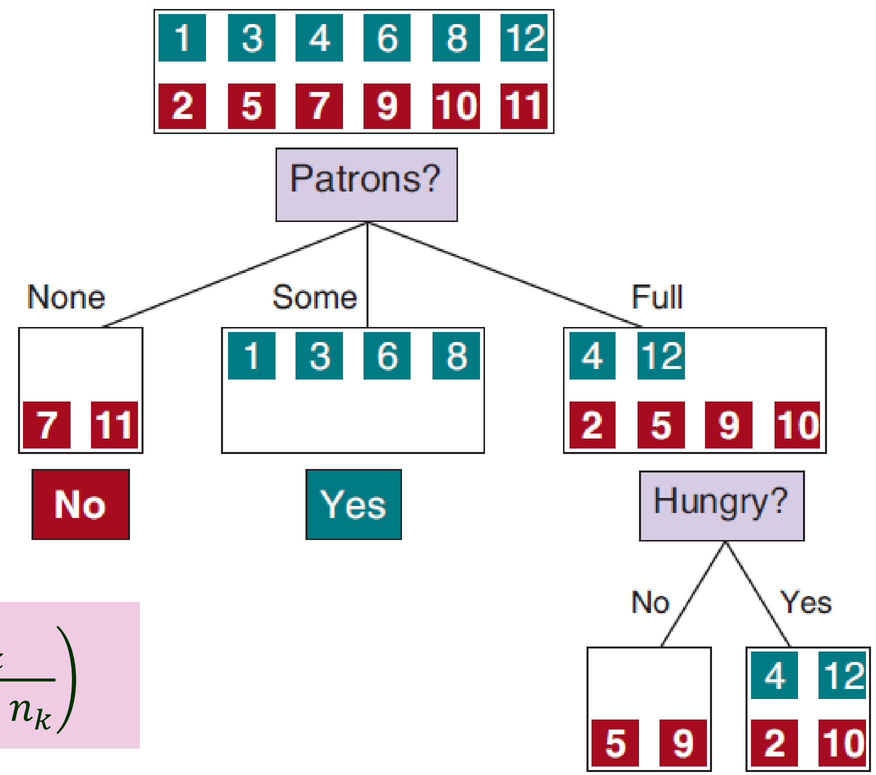
$$Gain(Type) = B\left(\frac{1}{2}\right) - \left(\frac{2}{12} B\left(\frac{1}{2}\right) + \frac{2}{12} B\left(\frac{1}{2}\right) + \frac{4}{12} B\left(\frac{2}{4}\right) + \frac{4}{12} B\left(\frac{2}{4}\right)\right) = 0 \text{ bits}$$

Information Gain

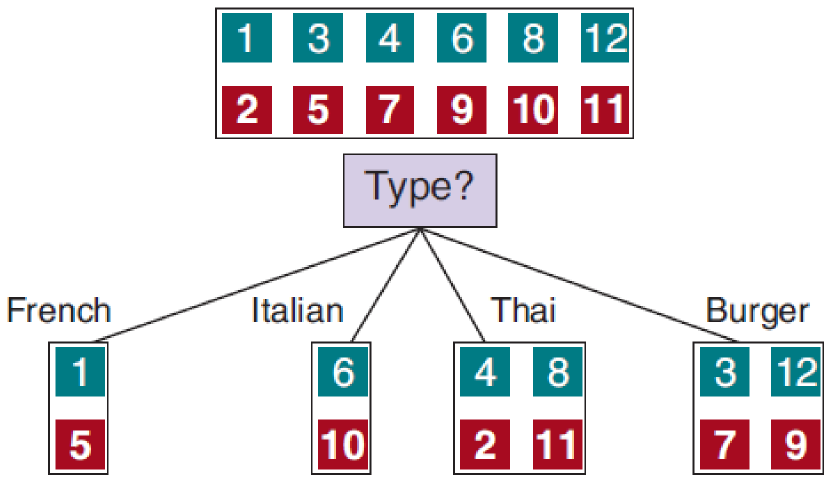


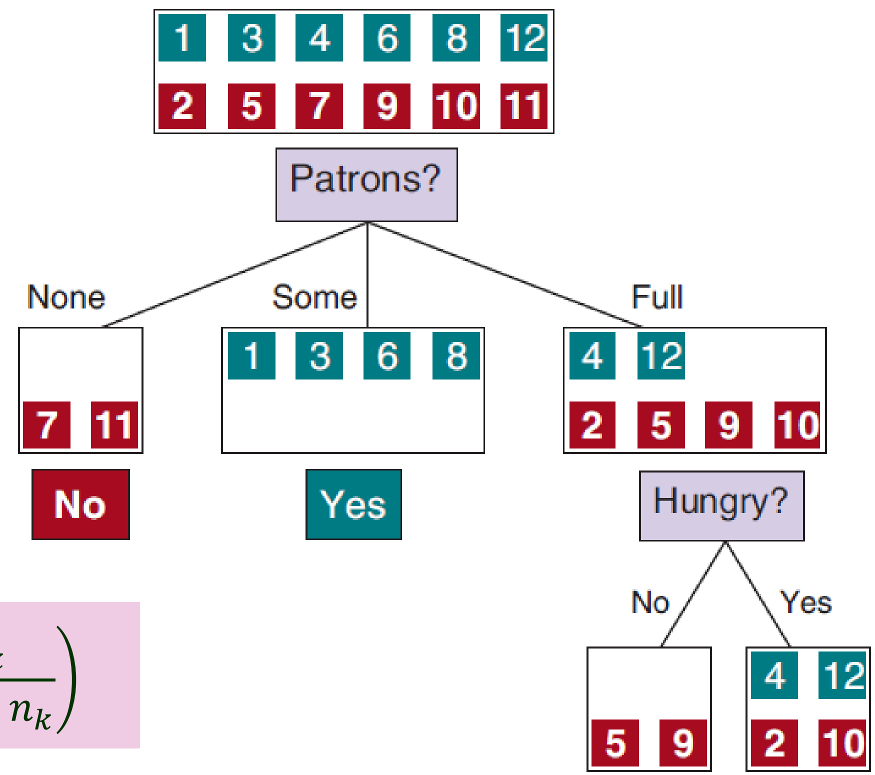


$$Gain(A) = B\left(\frac{p}{p+n}\right) - \sum_{k=1}^{d} \frac{p_k + n_k}{p+n} B\left(\frac{p_k}{p_k + n_k}\right)$$

$$Gain(Type) = B\left(\frac{1}{2}\right) - \left(\frac{2}{12} B\left(\frac{1}{2}\right) + \frac{2}{12} B\left(\frac{1}{2}\right) + \frac{4}{12} B\left(\frac{2}{4}\right) + \frac{4}{12} B\left(\frac{2}{4}\right)\right) = 0 \text{ bits}$$

Information Gain



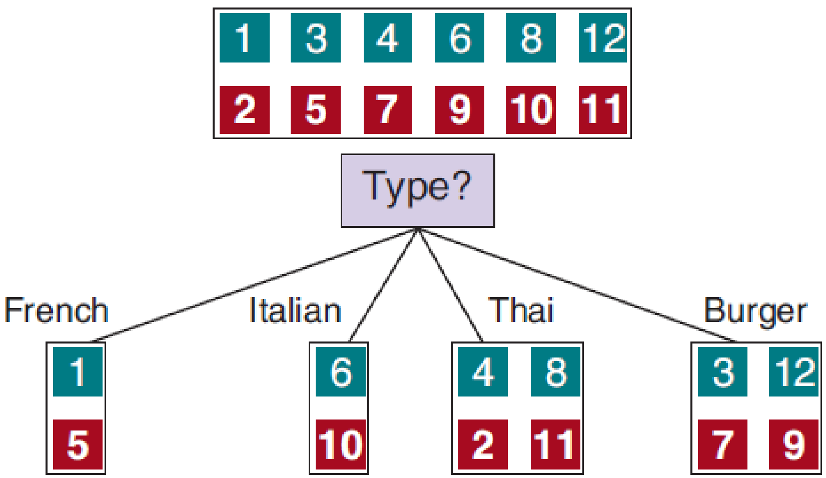


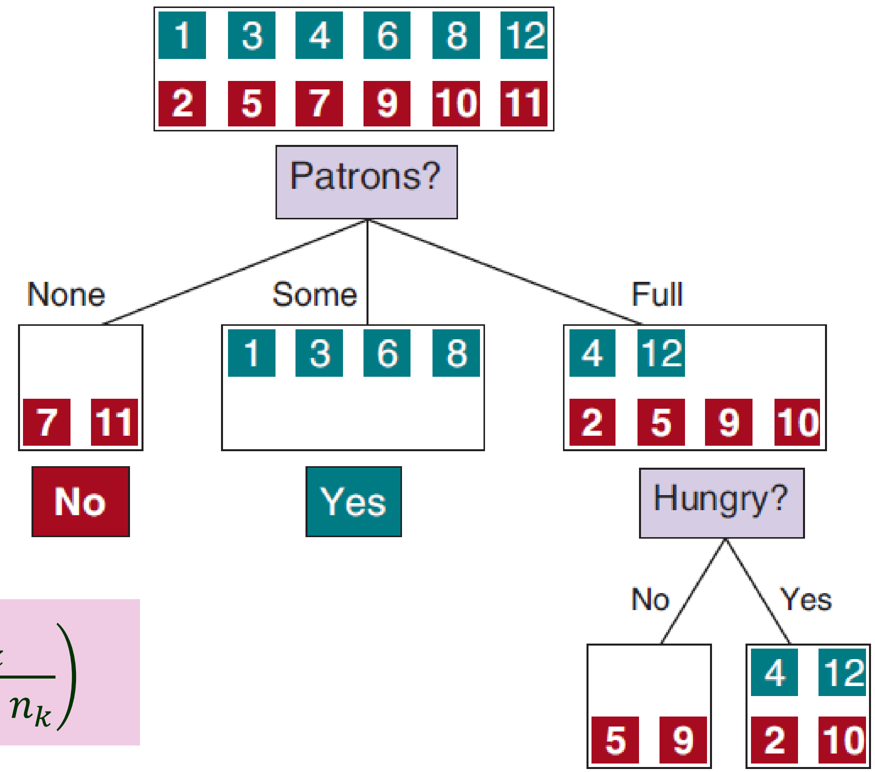
$$Gain(A) = B\left(\frac{p}{p+n}\right) - \sum_{k=1}^{d} \frac{p_k + n_k}{p+n} B\left(\frac{p_k}{p_k + n_k}\right)$$

$$Gain(Type) = B\left(\frac{1}{2}\right) - \left(\frac{2}{12}B\left(\frac{1}{2}\right) + \frac{2}{12}B\left(\frac{1}{2}\right) + \frac{4}{12}B\left(\frac{2}{4}\right) + \frac{4}{12}B\left(\frac{2}{4}\right)\right) = 0 \text{ bits}$$

$$Gain(Patrons) = B\left(\frac{1}{2}\right) - \left(\frac{2}{12}B\left(\frac{0}{2}\right) + \frac{4}{12}B\left(\frac{4}{4}\right) + \frac{6}{12}B\left(\frac{2}{6}\right)\right) \approx 0.541 \text{ bits}$$

Information Gain





$$Gain(A) = B\left(\frac{p}{p+n}\right) - \sum_{k=1}^{d} \frac{p_k + n_k}{p+n} B\left(\frac{p_k}{p_k + n_k}\right)$$

$$Gain(Type) = B\left(\frac{1}{2}\right) - \left(\frac{2}{12} B\left(\frac{1}{2}\right) + \frac{2}{12} B\left(\frac{1}{2}\right) + \frac{4}{12} B\left(\frac{2}{4}\right) + \frac{4}{12} B\left(\frac{2}{4}\right)\right) = 0 \text{ bits}$$

$$Gain(Patrons) = B\left(\frac{1}{2}\right) - \left(\frac{2}{12} B\left(\frac{0}{2}\right) + \frac{4}{12} B\left(\frac{4}{4}\right) + \frac{6}{12} B\left(\frac{2}{6}\right)\right) \approx 0.541 \text{ bits}$$