Final exam

- Please **write your name and netid** on the top of this page.
- There are 11 questions in this exam, totaling 105 points.
- The maximum score will be capped to 100 points. Any score above 100 will be rounded to 100.
- Total duration: 120 minutes.
- You **can** use three pages as cheat sheets.
- You **cannot** consult your notes, textbook, your neighbor, or Google.
- Use the back of the page if needed for any rough work.
- Give sufficient explanation behind your thinking. You will get partial credit even if your final answer is wrong but if you have thought about it correctly.

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1. **(5 points)** Let S be the set of countries in mainland Europe. For $a, b \in S$, define a relation R such that aRb iff a and b border each other. Which properties of an equivalence relation (reflexive, symmetric, transitive) does R satisfy? Explain your reasoning for each of these properties.

2. **(10 points)** Consider a complete bipartite graph K_n with $2n$ nodes, where each of the first n nodes are connected via undirected edges to each of the last n nodes.

(a) Let e_n be the sequence denoting the number of edges in K_n. Evaluate the sequence $e_1, e_2, e_3, e_4, e_5, \ldots$.

(b) Guess a closed form expression for e_n.

(c) Construct a recurrence relation for the number of edges, $e(n)$, in this graph. (Hint: how do you construct K_n given K_{n-1}?)

(d) Verify that your recurrence relation in part b is correct by plugging in the derived expressions for e_n and e_{n-1} from part b.

3. **(10 points)** The power set $A = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$ is partially ordered with respect to the "subset of" relation.

(a) Draw the Hasse diagram representation of the above relation.

(b) List all minimal and maximal elements.

(c) Run topological sort on the Hasse diagram to obtain a compatible total ordering of the elements.

4. **(10 points)** On any given game night in Hilton Coliseum, there are at least 700 spectators. Your goal is to prove that irrespective of who attends the game, there *always* will exist two people in the crowd with the same initials of their first and last names.

(a) First, calculate the maximum number of possibilities of (first-initial, last-initial) pairs.

(b) Use the above calculation to (clearly) state your proof using a simple combinatorial principle discussed in class.

5. **(10 points)** Find a counterexample for each of the following statements. You can simply write down, or draw, or clearly describe your counterexample.

(a) Every triangle has at least one angle greater than $60°$.

(b) For all real numbers x, $x^3 \geq x$.

(c) If $a_1, a_2, a_3, \ldots$ is an increasing sequence of real numbers, i.e., $a_1 < a_2 < a_3 < \ldots$, then there exists some a_i in this sequence that is bigger than 10.

6. **(10 points)** King Arthur wants to seat his Knights at the Round Table. The seats are not numbered, and therefore, two seating arrangements are considered identical if the sequence of Knights in clockwise order starting from Knight 1 is the same.

(a) For $n = 4$, enumerate all possible seatings.

(b) How many different ways of arranging n Knights is possible?

(c) How would your answer change if the seats *were* numbered?

7. **(10 points)** This is a 2-part question.

(a) Show that the following 3 statements are logically equivalent:

$p \implies q \vee r, \qquad p \wedge \neg q \implies r, \qquad p \wedge \neg r \implies q$

(b) Let n denote a positive integer. Using the logical equivalences above, rewrite the following assertion in two other ways.

"If *n* is a prime number not equal to 2, then *n* is odd".

8. **(10 points)** Prove that every amount of postage that is at least 12c can be made from some combination of 4c and 5c stamps. (Hint: (i) strong induction. (ii) you need to check multiple base cases.)

9. **(10 points)** You are the CEO of a startup. You want to choose a team of m people from a pool of n applicants, and from these m people you want to choose k to be the team managers. Since you took CPRE 310, you think about this a bit, and conclude that you can do so in:

$$\binom{n}{m}\binom{m}{k}$$

ways. However, your CFO, who is a Hawkeye, comes up with the strange looking formula:

$$\binom{n}{k}\binom{n-k}{m-k}$$

ways. Before doing the obvious thing – dumping on all UofI alums – you decide to check your answer against theirs.

(a) Explain your calculations, clearly stating which counting rule(s) you used.

(b) By direct algebra and simplification, show that your answer is mathematically equivalent to that obtained by the CFO.

(c) Explain how your CFO may have arrived at their answer by guessing a different way to count the same quantity.

10. **(10 points)** Several departments across campus have wireless access points. However, interference problems can arise if two access points are within 200 feet of each other and operating on the same frequency. Your goal is to assign different frequencies to different access points such that there is no interference.

Here is some information about the geographic locations of the departments.

Department	is within 200 feet of
MATH	PHY, CHEM, SOCIO
SOCIO	ECON, MATH, PSYCH
PHY	MATH, CHEM
PSYCH	CHEM, SOCIO, ECON
ECON	SOCIO, PSYCH
CHEM	MATH, PSYCH, PHY

(a) Model the above table using an undirected graph, where nodes denote departments and the "is within 200 feet of" information denotes edges. Draw this graph, and clearly mark the nodes.

(b) Assign labels (or colors) to the nodes such that no two nodes connected by an edge are assigned the same label/color.

(c) Use your answer above to argue that only 3 frequencies are required for the overall wireless system to function properly.

11. **(10 points)** Suppose you pick a positive integer n (where $1 \leq n \leq 100$) uniformly at random.

(a) What is the probability that n is divisible by 5?

(b) What is the probability that n is divisible by 20?

(c) Given that n is divisible by 5, what is the probability that n is divisible by 20?