Homework 2 - Solutions

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1. **(10 points)** Translate the following arguments into symbolic logic. Give your reasoning whether each argument is valid or not. If the logical expressions corresponds to a rule of inference discussed in lecture or in notes, identify the name of that rule.

a. If I go to the movies, I won't finish my homework. If I don't finish my homework, I won't do well on the exam. Therefore, if I go to the movies, I won't do well on the exam.

Solution p: I go to the movies q: I will not finish my homework r: I don't do well on the exam

p => q and q=>r. Therefore p => r. This is true by Hypothetical Syllogism.

b. If this number is larger than 2, then its square is larger than 4. This number is not larger than 2. Therefore, the square of this number is not larger than 4.

Solution p: The number is larger than 2 q: The square is larger than 4

¬ p is true. Therefore the square of the number is not larger than 4. This is not true as it does not satisfy any of our inference rules.

c. Mary knows C and Mary knows Python. Therefore, Mary knows Python.

Solution p: Mary knows C q: Mary knows Python

Therefore p ∧ q is true via the simplification rule.

2. **(10 points)** Recall that a number is called *rational* if it can be written as the ratio of two integers. Numbers that are not rational are called *irrational*. Prove by contraposition the following statement:

- "If r is irrational, then $\sqrt{r}$ is irrational."

Solution

Let p = "r is irrational" and q = "$\sqrt{r}$ is irrational". The contraposition of this sentance will be of the form $\not q \implies \neg p$,which is in this case, "If $\sqrt{r}$ is rational, then r is rational." (because every real number can either be rational or irrational).

Now, as $\sqrt{r}$ is a rational number, it can be written as ratio of two integers m and n,

$\sqrt{r} = m/n \implies r = m^2/n^2$ (squaring both the sides)

as m and n are integers, m^2 and n^2 are also integers.

$\implies r$ can also be written as a ratio of two intergers. $\implies r$ is a rational number.

hence, it proves that "If $\sqrt{r}$ is rational, then r is rational" which is equivalent to "If r is irrational, then $\sqrt{r}$ is irrational."

3. **(10 points)** Most geometry that we learn in high-school is an instance of *Euclidean* geometry. A central axiom of Euclidean geometry is that the sum of angles in a triangle equals 180°. Prove by contradiction that in Euclidean geometry, if two lines are each perpendicular to a given line segment, then the lines have to be parallel.

Solution

For proof by contradiction, we suppose that the given statement is false, "If two lines share a common perpendicular, then they are not parallel."

Now, suppose the common perpendicular intersects one line at point A, and the other at point B. As the lines are not parallel, they intersect at some point C, creating a nonzero angle with each other. The sum of the angles in triangle ABC must be 180, as central axiom of Euclidean geometry provided in the question. Because line AB is perpendicular to line AC, the angle at vertex A must be 90 degree . Because line AB is perpendicular to line BC, the angle at vertex B must be 90 degree . The angle at vertex C must be 180 degree - 90 degree - 90 degree = 0 degree . Therefore, lines AC and BC do not intersect, creating a nonzero angle. Which contradicts with the conclusion obtained that angle at C must be 0 degree .

Hence, it is proven by contradiction that if two lines share a common perpendicular, then they are parallel.

4. **(10 points)** An integer n is called *frumpy* if $n^3 + 2n^2 + 4n + 6$ is an odd number. Prove that all frumpy numbers are themselves odd numbers. (Clearly state your method of proof in the beginning.)

   Solution

   The proof of contraposition can be used to prove this statement. The original statement says:

   $\forall n \in \mathbb{N}$ (n is a frumpy $\implies$ n is an odd number).

   The contraposition of this statement is:

   $\forall n \in \mathbb{N}$ (n is an even number $\implies$ n is not a frumpy)

   Assume $n = 2k$ for $k \in \mathbb{N}$. The next step is to show that $n^3 + 2n^2 + 4n + 6$ is not an odd number by substituting with $n = 2k$.

   $n^3 + 2n^2 + 4n + 6 = (2k)^3 + 2(2k)^2 + 4(2k) + 6 = 2(4k^3 + 4k^2 + 4k + 3)$ which shows that n is not a frumpy if n is an even number. $\square$

5. **(10 points)** Given n arbitrary real numbers $a_1, a_2, ..., a_n$, prove that at least one of these numbers is greater than or equal to their average. (Clearly state your method of proof in the beginning).

   Solution

   The proof of contradiction can be used to prove this statement.

   Let $\forall i \in 1, 2, ..., n$. Then, $a_i < \frac{1}{n} \sum_{i=1}^{n} a_i$. (Every elements are smaller than the average.)

   Then, $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n < (\frac{1}{n} \sum_{i=1}^{n} a_i) \times n = \sum_{i=1}^{n} a_i$ which leads to the contradition.

   Therefore, at least one of element of arbitrary real numbers should be greater than or equal to their average. $\square$