Recitation 14 Solutions

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1. *Craps* is a game played in casinos, where players roll a pair of ordinary (6-sided) dice. If the total of the numbers on the rolled dice on the first roll is 2, 3, or 12, the person "craps out", i.e., they lose the bet. If the total is 7 or 11, they win the bet. In other cases, the game continues (and other rules kick in).

   Solution

   a. What is the probability that you roll a 7 on the first roll? $\frac{6}{36}$

   b. What is the probability that you **win** the bet on the first roll? $\frac{8}{36}$

   c. What is the probability that you **lose** the bet on the first roll? $\frac{4}{36}$

   d. What is the probability that you roll a 7 on the second roll? $\frac{24}{36} \times \frac{6}{36} = \frac{1}{9}$

2. If A_1 and A_2 are arbitrary events of any sample space, then prove that

$$P(A_1 \cup A_1) \leq P(A_1) + P(A_2).$$

   In words, the probability of the union of two events can never be greater than the sum of the probabilities of the individual events. This is called the *union bound*, and is one of the foundational results of probability theory. (Hint: use the Principle of Inclusion-Exclusion.)

   Solution

   Let S is the sample space.

   From the principle of inclusion and exclusion,

$$\begin{aligned}
|A_1 \cup A_2| &= |A_1| + |A_2| - |A \cap B| \\
\frac{|A_1 \cup A_2|}{|S|} &= \frac{|A_1|}{|S|} + \frac{|A_2|}{|S|} - \frac{|A_1 \cap A_2|}{S} \\
P(A_1 \cup A_2) &= P(A_1) + P(A_2) - P(A_1 \cap A_2) \\
P(A_1 \cup A_2) &\leq P(A_1) + P(A_2)
\end{aligned}$$

3. We discuss a curious phenomenon in probability where a certain trend seems to appear in two groups of data but reverses course when the data is combined. Bart and Lisa take a series of 5 courses spread over 2 semesters. They are all CPRE courses (and hence very difficult.) In Semester 1, Lisa does not study at all, and has a 0% probability of passing any given course, while Bart has a marginally higher (25%) probability of passing any course.

   a. If Bart takes 4 courses and Lisa takes 1 course in Semester 1, how many courses is each student expected to pass?

b. In Semester 2, they both improve; Lisa has a 75% probability of passing the course, while Bart has a 100% probability of passing any course. If Bart takes 1 course in Semester 2 and Lisa takes 4 courses, how many courses is each student expected to pass?

c. Calculate the total number of courses that each student is expected to pass, and argue that Lisa is the better student (even though Bart shows a higher probability of doing well in *both* semesters.)

d. What is the source of this paradox?

Solution

a. Bart: $4 \times 1/4 = 1$, Lisa: $1 \times 0 = 0$

b. Bart: $1 \times 1 = 1$, Lisa: $4 \times \frac{3}{4} = 3$

c. Bart: 2 courses; Lisa: 3 courses.

d. The source of the paradox is from the number of courses which Bart and Lisa took during the first and second semester. (Bart took four courses when his probability of passing course is only 25% whereas Lisa took four courses when her passing probability is 75%).