Review for Exam 2

Topics for Exam 2 (All sections covered in class except for 3.1 and 3.2):

From Chapter 4 you should know:

- How to find the Wronskian • Know meaning: l.i., fundamental set.
- How to find y_c the general sol to $ay'' + by' + cy = 0$
- How to find y_p a particular solution of $ay'' + by' + cy = g(x)$
  - Undetermined Coefficients
  - Variation of Parameters
  - Superposition Principle
- How to find the general solution of $ay'' + by' + c = g(x)$
- How to solve a Cauchy-Euler Equation (only homogeneous)
- How to solve Initial Value Problems (IVPs)

5.1 Application problems (**free** undamped and damped motion cases).

One of the most comprehensive types of problems (2^{nd} order DE) is solving a nonhomogeneous IVP:

$$ay'' + by' + cy = g(x), \quad y(0) = y_0, \; y'(0) = y_1.$$

1st We find y_c, that is, find the general solution of $ay'' + by' + cy = 0$

auxiliary Equation: $am^2 + bm + c = 0$ yields three cases:

Case 1 Two distinct (real) roots $m_1 \neq m_2 \Rightarrow y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

Case 2 One repeated root $m_1 = m_2 \Rightarrow y_c = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$

Case 3 Two complex conjugate roots $m_{1,2} = \alpha \pm i\beta$

$$y_c = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

2nd We find a particular solution y_p, if $g(x)$ allows us to use undetermined coefficients remember the form of y_p:

	Form of $g(x)$	Form of y_p
i	$c_0 + c_1x + \cdots + c_nx^n$	$A_0 + A_1x + \cdots + A_nx^n$
ii	$ce^{\alpha x}$	$A\,x^t e^{\alpha x}$
iii	$c_1 \sin\beta x + c_2\cos\beta x$	$x^t(A\cos\beta x + B\sin\beta x)$
iv	$e^{\alpha x}(c_1\sin\beta x + c_2\cos\beta x)$	$x^t e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

Where t will be $0, 1$ or 2 depending on the roots of the auxiliary equation:

- In case *ii*: $t=0$ if $\alpha \neq$ root of aux. equ, $t=1$ if $\alpha =$ root; $t=2$ if $\alpha =$ repeated root.
- In case *iii*: $t=1$ if root of aux equ. $m = \pm\beta i$; $t=0$ otherwise.
- In case *iv*: $t=1$ if $m = \alpha \pm \beta i$; $t=0$ otherwise.

For the given DE and $g(x)$ determine the form of y_p required for undetermined coefficients method:

I. $y'' + 9y = g(x)$ $\Rightarrow$ $m = \pm 3i$ $(y_1 = \cos 3x,\ y_2 = \sin 3x)$
- $g(x) = 2\cos 3x$ $\Rightarrow$ $y_p = x(A\cos 3x + B\sin 3x)$
- $g(x) = 5e^{2x}\sin 3x$ $\Rightarrow$ $y_p = e^{2x}(A\cos 3x + B\sin 3x)$
- $g(x) = 3e^{-x}$ $\Rightarrow$ $y_p = Ae^{-x}$

II. $y'' - 4y' + 9y = g(x)$ $\rightarrow$ $m = 2 \pm \sqrt{5}i$
- $g(x) = e^{2x}$ $\rightarrow$ $y_p = Ae^{2x}$
- $g(x) = \cos(\sqrt{5}x)$ $\rightarrow$ $y_p = A\cos(\sqrt{5}x) + B\sin(\sqrt{5}x)$
- $g(x) = e^{2x}\sin(\sqrt{5}x)$ $y_p = x\,e^{2x}(A\cos(\sqrt{5}x) + B\sin(\sqrt{5}x))$
- $g(x) = 3x^2$ $\rightarrow$ $y_p = A_0 + A_1x + A_2x^2$

III. $y'' = g(x)$ $\rightarrow$ $m = 0$ repeated $y_1 = 1$ $y_2 = x$
- $g(x) = e^{-x/2}$ $\leftarrow$ $y_p = Ae^{-x/2}$
- $g(x) = x$ $\leftarrow$ $y_p = x^2(A_0 + A_1x)$

$\uparrow x^2$, because $m=0$ is repeated root!

When $g(x)$ doesn't allow us to use undetermined coefficients we can use variation of parameters. Recall that given y_1 and y_2 the l.i. solutions to the homogeneous case, we can find u_1 and u_2 such that $y_p = u_1 y_1 + u_2 y_2$ where

$$u_1 = -\int \frac{f \cdot y_2}{W}\,dx \quad \text{and} \quad u_2 = \int \frac{f\, y_1}{W}\,dx, \quad \text{and} \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

* Careful: $f(x)$ corresponds to the standard form of the D.E.

$$y'' + Py' + Qy = f(x)$$

Recall if $g(x) = g_1(x) + g_2(x)$ we can use superposition principle to find y_p, sometimes we might mix and match methods to find y_{p_1} and y_{p_2}:

Find y_{p_1} for $ay'' + by' + cy = g_1(x)$

y_{p_2} for $ay'' + by' + cy = g_2(x)$

and then $y_p = y_{p_1} + y_{p_2}$

3rd Once we have y_c and y_p we form the general solution:

$$y = y_c + y_p$$

$$y = c_1 y_1 + c_2 y_2 + y_p \quad \leftarrow \text{General Sol.}$$

4th Finally we use the general solution above to find c_1 and c_2 using the initial conditions.

Example

Solve the IVP: $9y'' - y = xe^{x/3} + 3x^2, \;\; y(0) = 1, \; y'(0) = 0.$

1st Find y_c. Aux. Equation: $9m^2 - 1 = 0 \Rightarrow m = \pm 1/3$

$y_1 = e^{x/3}$ and $y_2 = e^{-x/3}$ then $y_c = c_1 e^{x/3} + c_2 e^{-x/3}$

2nd Find y_{P_1} for $9y'' - y = xe^{x/3}$ using variation of parameters.

$$W = \begin{vmatrix} e^{x/3} & e^{-x/3} \\ \frac{1}{3}e^{x/3} & -\frac{1}{3}e^{-x/3} \end{vmatrix} = -\frac{1}{3} - \frac{1}{3} = -2/3$$ Note: $f(x) = \dfrac{xe^{x/3}}{9}$

$$u_1 = -\int \frac{f\, y_2}{W}\,dx = -\int \frac{xe^{x/3}}{9} \cdot \frac{e^{-x/3}}{-2/3}\,dx = \frac{1}{6}\int x\,dx = \frac{x^2}{12}$$

$$u_2 = \int \frac{f\, y_1}{W}\,dx = \int \frac{xe^{x/3}}{9}\,\frac{e^{x/3}}{-2/3} = -\frac{1}{6}\int xe^{2x/3}\,dx = -\frac{1}{6}\left[x\,\frac{3}{2}e^{2x/3} - \frac{9}{4}e^{2x/3}\right]$$

$$y_{P_1} = u_1 y_1 + u_2 y_2 = \frac{x^2}{12}e^{x/3} - \frac{x}{4}e^{x/3} + \frac{3}{8}e^{x/3}$$

Next find y_{P_2} for $9y'' - y = 3x^2$ with undetermined coefficients. $y_{P_2} = A + Bx + Cx^2$; $y_{P_2}' = B + 2Cx$; $y_{P_2}'' = 2C$

Plug in: $9(2C) - A - Bx - Cx^2 = 3x^2 \Rightarrow C = -3$; $B = 0$; $18C - A = 0$

$\Rightarrow A = 18C = -54 \quad \therefore \; y_{P_2} = -54 - 3x^2$

$$y_P = y_{P_1} + y_{P_2} = \frac{x^2}{12}e^{x/3} - \frac{x}{4}e^{x/3} + \frac{3}{8}e^{x/3} - 54 - 3x^2$$

Gen. Sol: $y = y_c + y_p = \underline{c_1 e^{x/3} + c_2 e^{-x/3} + \frac{x^2}{12}e^{x/3} - \frac{x}{4}e^{x/3} + \frac{3}{8}e^{x/3} - 54 - 3x^2}$. Gral Sol.

Plug initial Conditions:

$$\left.\begin{aligned} y(0) &= c_1 + c_2 - 54 = 1 - 3/8 \\ y'(0) &= \tfrac{1}{3}c_1 - \tfrac{1}{3}c_2 - \tfrac{1}{4} + \tfrac{1}{8} = 0 \end{aligned}\right\} \text{2x2 System}$$

$c_1 = 55/2$

$c_2 = 217/8.$

Cauchy-Euler Equation

Recall the solutions to $a\,x^2\,y'' + b\,x\,y' + cy = 0$ are of the form $y = x^m$ where m is a root of the auxiliary equation: $am^2 + (b-a)m + c = 0$.

Cases

$m_1 \neq m_2$ (real) $\Rightarrow y = c_1 x^{m_1} + c_2 x^{m_2}$

$m_1 = m_2$ $\Rightarrow y = c_1 x^{m_1} + c_2 x^{m_1} \ln x$

$m_{1,2} = \alpha \pm i\beta$ $\Rightarrow y = x^{\alpha}\left(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right)$

Spring-Mass Systems

Free Undamped Motion: $x'' + \underbrace{\frac{k}{m}}_{\omega^2} x = 0 \Rightarrow x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$

k = spring constant

m = mass (kg, slug)

or $x(t) = A \sin(\omega t + \phi)$

$A = \sqrt{c_1^2 + c_2^2}$ $\phi = \tan^{-1}\left(\frac{c_1}{c_2}\right)$

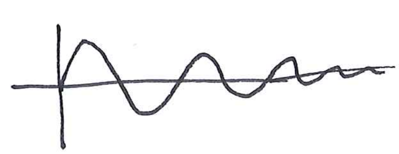
If $x(0) > 0$ (below equilibrium), $x'(0) < 0$ (upward velocity).

Free Damped Motion: $x'' + \frac{\beta}{m} x' + \frac{k}{m} x = 0$ $\beta \leftarrow$ damping constant

According to the discriminant $\lambda^2 - \omega^2$ we have three cases.

$\lambda^2 - \omega^2 > 0 \Rightarrow$ overdamped motion 

$\lambda^2 - \omega^2 = 0 \Rightarrow$ critically damped motion 

$\lambda^2 - \omega^2 < 0 \Rightarrow$ underdamped motion 

Example

A mass weighing 16 pounds is attached to a 5-ft long spring. At equilibrium the spring measures 8.2 ft. If the mass is initially released from rest at a point 2 ft above the equilibrium position, find the equation of motion of the mass if it is also known that the surrounding medium offers a resistance numerically equal to the instantaneous velocity.

$\Rightarrow \beta = 1$

Hooke's Law: $F = k \cdot s \Rightarrow 16 = k\, 3.2 \Rightarrow k = 5$

Mass: $W = m \cdot g \Rightarrow m = \frac{16}{32} = 1/2$

DE: $\frac{1}{2} x'' + 1 \cdot x' + 5x = 0$

$x'' + 2x' + 10x = 0$

Aux. Equ: $m^2 + 2m + 10 = 0 \Leftrightarrow$

$m^2 + 2m + 1 = -9$

$(m+1)^2 = -9$

$m + 1 = \pm 3i$

$m = -1 \pm 3i$

$x(t) = e^{-t}\left(c_1 \cos(3t) + c_2 \sin(3t)\right)$

Initial Conditions: $x(0) = -2$, $x'(0) = 0$.

$x(0) = c_1 = -2$

$x'(0) = -c_1 + 3c_2 = 0 \Rightarrow c_2 = c_1/3 = -2/3$

Equation of motion: $x(t) = e^{-t}\left(-2\cos(3t) - 2/3 \sin(3t)\right)$

We have underdamped motion.

If there's time...

A 20-kg mass is attached to a spring with stiffness 16 N/m. If the mass is initially at equilibrium position and given an upward velocity of 2 m/s, find the amplitude and period of the equation of motion of the mass.

$K = 16, \quad m = 20 \quad \Rightarrow DE: \quad x'' + \frac{16}{20} x = 0 \Rightarrow m = \frac{\pm 4i}{\sqrt{20}} = \frac{\pm 2}{\sqrt{5}}$

$x(t) = C_1 \cos\left(\frac{2}{\sqrt{5}} t\right) + C_2 \sin\left(\frac{2}{\sqrt{5}} t\right)$

$x(0) = C_1 = 0$

$\text{Period} = \frac{2\pi}{\omega} = \frac{2\pi}{2/\sqrt{5}} = \sqrt{5}\pi$

$x'(0) = \frac{2}{\sqrt{5}} C_2 = -2 \Rightarrow C_2 = -\sqrt{5}$

$A = \sqrt{C_1^2 + C_2^2} = \sqrt{5}.$