4.7 Cauchy-Euler Equation

An equation of the following form is called a Cauchy-Euler Equation:

$$a_n\, x^n\, \frac{d^n y}{dx^n} + a_{n-1}\, x^{n-1}\, \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1\, x\, \frac{dy}{dx} + a_0\, y = g(x)$$

We will focus on the special case (second order and homogeneous)

$$a\, x^2\, y'' + b\, x\, y' + c\, y = 0 \quad (*)$$

The solutions to this equation will have the form $y = x^m$

To figure out what power m works, we plug our proposed solution $y = x^m$ into the equation (*) ; $y' = m x^{m-1}$; $y'' = m(m-1) x^{m-2}$

$$a x^2 m(m-1) x^{m-2} + b x m x^{m-1} + c x^m = 0$$

$$(am(m-1) + bm + c)\, x^m = 0$$

$$\Leftrightarrow \quad \underbrace{am^2 + (b-a)m + c = 0}$$

← Auxiliary Equation of (2nd order) Cauchy-Euler D.E.

The nature of the roots of the auxiliary equation will again yield different cases:

- Case 1: Real distinct roots $m_1 \neq m_2$, then $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$ form a fundamental set.

  ⇒ General Solution is $y = c_1 x^{m_1} + c_2 x^{m_2}$

- Case 2. One (real) repeated root m, then we get one solution $y_1 = x^m$

We use reduction of order (4.2) to find a second l.i. solution.

$$y_2 = y_1 \int \frac{e^{-\int P dx}}{y_1^2} dx$$; where P corresponds to the standard form

of the D.E: $y'' + \frac{b}{ax} y' + \frac{c}{ax^2} y = 0 \Rightarrow P = \frac{b}{ax}$

$$\Rightarrow e^{-\int P dx} = e^{-\int \frac{b}{ax} dx} = e^{-b/a \ln|x|} = |x|^{-b/a} = x^{-b/a} \quad \text{(assume } x>0\text{)}$$

$$y_1^2 = (x^m)^2 = x^{2m} = x^{1-b/a} = x\, x^{-b/a}$$

Note:

$2m = \frac{\cancel{2}(a-b)}{\cancel{2}a}$

$2m = 1 - \frac{b}{a}$

$$\Rightarrow \int \frac{e^{-\int P dx}}{y_1^2} dx = \int \frac{\cancel{x^{-b/a}}}{x \cancel{x^{-b/a}}} dx = \ln|x| \Rightarrow y_2 = x^m \ln|x|$$

$\Rightarrow$ General Solution: $\underline{y = c_1 x^m + c_2 x^m \ln|x|}$

- Case 3. A pair of complex conjugate roots $m = \alpha \pm i\,\beta$ (Recall α, β are real)

A complex solution is $x^{\alpha + i\beta}$. We will again use Eulers formula $e^{i\theta} = \cos\theta + i \sin\theta$ and the fact that $x = e^{\ln x}$

$$x^{\alpha+i\beta} = x^\alpha x^{i\beta} = x^\alpha (e^{\ln x})^{i\beta} = x^\alpha e^{i\beta \ln x} = x^\alpha (\cos(\beta \ln x) + i \sin(\beta \ln x))$$

$\Rightarrow \{x^\alpha \cos(\beta \ln x), x^\alpha \sin(\beta \ln x)\}$ form a fundamental set.

$\therefore$ General Solution : $\underline{y = c_1 x^\alpha \cos(\beta \ln x) + c_2 x^\alpha \sin(\beta \ln x)}$

Another Approach

We could instead turn the C-E equation into a constant coefficient equation making a substitution: let $t = \ln x$ or $e^t = x$.

$\{ * = \text{chain rule} \}$

$$\frac{dy}{dx} \overset{*}{=} \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}$$

prod. rule

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt} \cdot \frac{1}{x}\right) = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) - \frac{1}{x^2}\frac{dy}{dt}$$

$$= \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\underbrace{\frac{dt}{dx}}_{=\frac{1}{x}} - \frac{1}{x^2}\frac{dy}{dt} = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}$$; now substitute

$$ax^2\left(\frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}\right) + bx\left(\frac{1}{x}\frac{dy}{dt}\right) + cy = 0$$

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0 \Rightarrow \text{aux Eq.}: am^2 + (b-a)m + c = 0.$$

$\Rightarrow$ Solutions $y_i = e^{m_i t} = e^{m_i \ln x} = x^{m_i}$

Repeated Root $y_1 = e^{m_1 t} = e^{m_1 \ln x} = x^{m_1}$ and $y_2 = t e^{m_1 t} = (\ln x) x^{m_1}$

etc...

Example. Solve the DE $4x^2 y'' + y = 0$. $a = 4, b = 0, c = 1$

aux Eqn: $4m^2 + (0-4)m + 1 = 0$

$4m^2 - 4m + 1 = 0$

$(2m-1)^2 = 0 \Rightarrow m = 1/2$ repeated root.

$\therefore \; y = c_1 x^{1/2} + c_2 (\ln x) x^{1/2}$

Example. Solve the DE $x^2\, y'' + y' = 0$. $a=1,\ b=1,\ c=0$.

$\Rightarrow$ Aux. Eqn: $1\cdot m^2 + (1-1)m + 0 = 0$

$m^2 = 0 \Rightarrow m = 0$ repeated root $\Rightarrow$ $y_1 = x^0 = 1$, $y_2 = \ln x$

$\therefore$ General Solution. $y = c_1 + c_2 \ln x$

Example. (#3 in pg 169) Solve the DE $4x^2\, y'' + 17\, y = 0$. $a=4,\ b=0,\ c=17$

(Read full example).

aux. Eqn:

$$4m^2 - 4m + 17 = 0$$
$$4m^2 - 4m + 1 = -16$$
$$(2m-1)^2 = -16$$
$$2m - 1 = \pm 4i$$
$$m = \frac{1}{2} \pm 2i$$
$$\Rightarrow \alpha = 1/2 \quad \beta = 2$$

Sol:

$$y = c_1 x^{1/2} \cos(2 \ln x) + c_2 x^{1/2} \sin(2 \ln x)$$

Example. (#4 in pg 169) Solve the DE $x^3\, y''' + 5x^2\, y'' + 7x\, y' + 8y = 0$.

The solution still has the form $y = x^m$, so we plug it into the D.E. to get the corresponding aux. eqn:

$$x^3 m(m-1)(m-2)x^{m-3} + 5x^2 m(m-1)x^{m-2} + 7x\, m\, x^{m-1} + 8x^m = 0$$

$$[m(m-1)(m-2) + 5m(m-1) + 7m + 8]\, x^m = 0$$

$\Leftrightarrow$ $\vdots$

$$(m+2)(m^2+4) = 0$$

$\nearrow m_1 = -2 \Rightarrow y_1 = x^{-2}$

$\searrow m_{2,3} = \pm 2i \Rightarrow y_2 = \cos(2 \ln x)$, $y_3 = \sin(2 \ln x)$

$$\therefore \quad y = c_1 x^{-2} + c_2 \cos(2 \ln x) + c_3 \sin(2 \ln x)$$