Stat 330 (Spring 2020): Homework 3

Due: February 5, 2020

Show all of your work, and *please* staple your assignment if you use more than one sheet. Write your name, the course number and the section on every sheet. Problems marked with * will be graded and one additional randomly chosen problem will be graded.

1. Proof Questions:

   (a) With conditional probability, $\mathbb{P}(A|B)$, the axioms of probability hold for the event on the left side of the bar. A useful consequence is applying the complement rule to conditional probability. We have that $\mathbb{P}(A|B) = 1 - \mathbb{P}(\overline{A}|B)$.

   Prove this by showing that $\mathbb{P}(A|B) + \mathbb{P}(\overline{A}|B) = 1$ (Hint: just use the definition of conditional probability, a proof should be very short).

   Answer: $\mathbb{P}(A|B) + \mathbb{P}(\overline{A}|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} + \frac{\mathbb{P}(\overline{A}\cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$.

   Recall from our table we made or a Venn diagram that $B = (A \cap B) \cup (\overline{A} \cap B)$

   (b) If two events A and B are independent, then we know $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. A fact is that if A and B are independent, then so are all combinations of $A, \overline{B}, \ldots$ etc.

   Show that if events A and B are independent, then $\mathbb{P}(\overline{A} \cap \overline{B}) = \mathbb{P}(\overline{A})\mathbb{P}(\overline{B})$, and thus $\overline{A}$ and $\overline{B}$ are independent. (Hint: $\mathbb{P}(\overline{A} \cap \overline{B}) = 1 - \mathbb{P}(A \cup B)$. Then use addition rule and simplify.)

   Answer:

$$\begin{aligned}
\mathbb{P}(\overline{A} \cap \overline{B}) &= 1 - \mathbb{P}(A \cup B) \\
&= 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A \cap B) \\
&= [1 - \mathbb{P}(A)] - \mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(B) \quad \text{[b/c A and B independent]} \\
&= [1 - \mathbb{P}(A)] - \mathbb{P}(B)[1 - \mathbb{P}(A)] \\
&= [1 - \mathbb{P}(A)][1 - \mathbb{P}(B)] \\
&= \mathbb{P}(\overline{A})\mathbb{P}(\overline{B})
\end{aligned}$$

2. A computer has a dual-core processor. At any time, the probability that each of the processors are active is

		Processor 2		
		In Use	Not In Use	
Processor 1	In Use	0.28	0.12	0.40
	Not In Use	0.42	0.18	0.60
		0.70	0.30	

Let A be the event that processor 1 is in use and B be the event that processor 2 is in use.

   (a) From the table, give $\mathbb{P}(A), \mathbb{P}(B)$, and $\mathbb{P}(A \cap B)$

   (b) Calculate $\mathbb{P}(A|B)$.

   (c) Calculate $\mathbb{P}(B|A)$

   (d) Are the events A and B independent? Why or why not?

Answer:

   (a) $\mathbb{P}(A) = 0.40, \mathbb{P}(B) = 0.70, \mathbb{P}(A \cap B) = 0.28$

   (b) $\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} = \frac{0.28}{0.70} = 0.40$

   (c) $\mathbb{P}(B|A) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)} = \frac{.28}{.4} = 0.70$

(d) Does $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$?
Since $P(A \cap B) = 0.28 = 0.40 \times 0.70 = P(A)P(B)$, A and B are independent.
Note: Alternatively, we could also check whether $P(A|B) = P(A)$ or $P(B|A) = P(B)$.

3. Suppose you have two urns with poker chips in them. Urn I contains two red chips and four white chips. Urn II contains three red chips and one white chip. You randomly select one chip from urn I and put it into urn II. Then you randomly select a chip from urn II.

(a) What is the probability that the chip you select from urn II is white?
Answer: Let W_i = white chip chosen from urn i, R_i = red chip chosen from urn i

$$\begin{aligned}
\mathbb{P}(W_2) &= \mathbb{P}(W_1 \cap W_2) + \mathbb{P}(R_1 \cap W_2) \\
&= \mathbb{P}(W_1)\mathbb{P}(W_2|W_1) + \mathbb{P}(R_1)\mathbb{P}(W_2|R_1) \\
&= \frac{4}{6} \times \frac{2}{5} + \frac{2}{6} \times \frac{1}{5} \\
&= \frac{1}{3}
\end{aligned}$$

(b) Is selecting a white chip from urn I and selecting a white chip from urn II independent? Justify your answer numerically.
Answer: Since $P(W_2) = \frac{1}{3} \neq \frac{2}{5} = P(W_2|W_1)$, W_1 and W_2 are not independent.
Note: Alternatively, we could have also checked whether $P(W_1 \cap W_2) = P(W_1)P(W_2)$.

4. * A diagnostic test has a 98% probability of giving a positive result when given to a person who has a certain disease. It has a 10% probability of giving a (false) positive result when given to a person who does not have the disease. It is estimated that 15% of the population suffers from this disease.

Answer: Let P = positive test result, $\overline{P}$ = negative test result, D = has disease, $\overline{D}$ = does not have disease

Given:

$$\begin{aligned}
\mathbb{P}(P|D) = 0.98 &\rightarrow \mathbb{P}(\overline{P}|D) = 1 - 0.98 = 0.02 \\
\mathbb{P}(P|\overline{D}) = 0.10 &\rightarrow \mathbb{P}(\overline{P}|\overline{D}) = 1 - 0.10 = 0.90 \\
\mathbb{P}(D) = 0.15 &\rightarrow \mathbb{P}(\overline{D}) = 1 - 0.15 = 0.85
\end{aligned}$$

(a) What is the probability that a test result is positive?
Answer:

$$\begin{aligned}
\mathbb{P}(P) &= \mathbb{P}(P \cap D) + \mathbb{P}(P \cap \overline{D}) \\
&= \mathbb{P}(P|D)\mathbb{P}(D) + \mathbb{P}(P|\overline{D})\mathbb{P}(\overline{D}) \\
&= (0.98)(0.15) + (0.10)(0.85) \\
&= 0.232
\end{aligned}$$

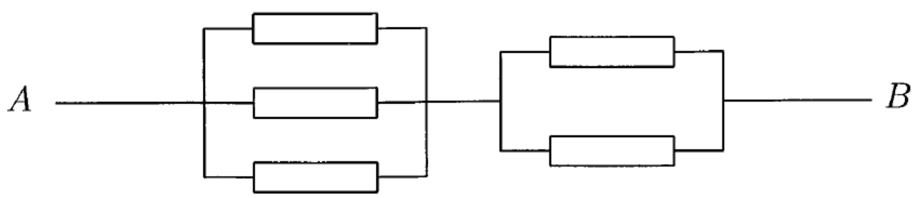
(b) A person receives a positive test result. What is the probability that this person actually has the disease?
Answer:

$$\begin{aligned}
\mathbb{P}(D|P) &= \frac{\mathbb{P}(P \cap D)}{\mathbb{P}(P)} \\
&= \frac{\mathbb{P}(P|D)\mathbb{P}(D)}{\mathbb{P}(P)} \\
&= \frac{(0.98)(0.15)}{(0.98)(0.15) + (0.10)(0.85)} \\
&= \frac{0.147}{0.232} \\
&= 0.6336
\end{aligned}$$

(c) A person receives a positive test result. What is the probability that this person does not actually have the disease? **Answer:**

$$\begin{aligned}\mathbb{P}(\overline{D}|P) &= \frac{\mathbb{P}(P \cap \overline{D})}{\mathbb{P}(P)} \\ &= \frac{\mathbb{P}(P|\overline{D})\mathbb{P}(\overline{D})}{\mathbb{P}(P)} \\ &= \frac{(0.10)(0.85)}{(0.98)(0.15) + (0.10)(0.85)} \\ &= \frac{0.085}{0.232} \\ &= 0.3664\end{aligned}$$

5. In the following system, each component *fails* with probability 0.3 independently of other components. Compute the systems reliability.



Answer:

2.21 At least one of the first three components works with probability

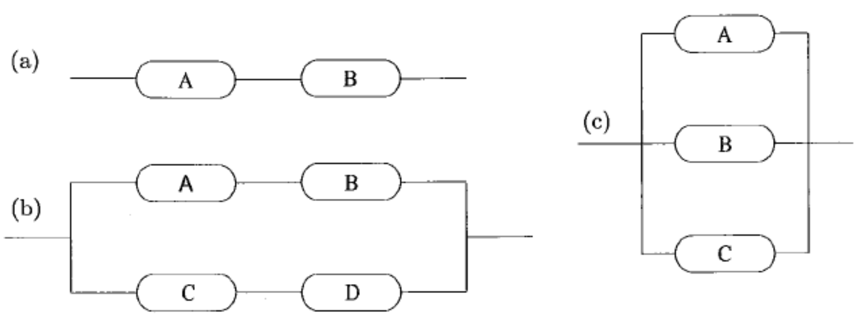
$$1 - \boldsymbol{P}\{\text{all three fail}\} = 1 - (0.3)^3 = 0.973.$$

At least one of the last two components works with probability

$$1 - \boldsymbol{P}\{\text{both fail}\} = 1 - (0.3)^2 = 0.91.$$

Hence, the system operates with probability $(0.973)(0.91) = \boxed{0.8854}$

6. * Calculate the reliability of each system show below, if components A, B, C, and D function properly (independently of each other) with probabilities 0.95, 0.9, 0.8, and 0.7 respectively.



Answer:

(a) $(0.95)(0.9) = 0.855$

(b) $1 - [(1 - (0.95)(0.9))(1 - (0.8)(0.7))] = 1 - [(1 - 0.855)(1 - 0.56)] = 0.9362$

(c) $1 - [(1 - 0.95)(1 - 0.9)(1 - 0.8)] = 1 - 0.001 = 0.999$