4.6 Variation of Parameters

We are still trying to find particular solutions to

$$a\,y'' + b\,y' + c\,y = g(x),$$

so that we can give the general solution of the nonhomogeneous equation.

$$y = y_c + y_p$$

Undetermined coefficients method works for certain kinds of $g(x)$, but we need another method for when $g(x)$ is of the form: $\tan x$, $\ln x$, $\sqrt{x+1}$, x^{-n} $(n > 0)$, etc. That is, not a polynomial, exponential or sine/cosine.

We will work with the standard form:

$$y'' + P\,y' + Q\,y = f(x)$$

In this method our "guess" solution will have the form

$$y_p = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$$

where y_1 and y_2 are solutions to the associated homogeneous equation. (the l.i. solutions forming y_c). The goal is to find functions $u_1(x)$ and $u_2(x)$ so that y_p solves the DE $y'' + P\,y' + Q\,y = f(x)$.

$$Q\cdot[y_p = u_1 y_1 + u_2 y_2]$$

$$+P[y_p' = u_1 y_1' + u_2 y_2'] + \underbrace{u_1' y_1 + u_2' y_2}$$ Condition 1: ← let $= 0$.

$$+1\cdot y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

$$\underbrace{(y_1'' + Py_1' + Qy_1)}_{=0} u_1 + \underbrace{(y_2'' + Py_2' + Qy_2)}_{=0} u_2 + \underbrace{u_1' y_1' + u_2' y_2' = f(x)}$$ Condition 2

Because y_1 and y_2 are solutions of $y'' + Py' + Qy = 0$.

Conditions 1 & 2 yield the following 2x2 system:

$$u_1' y_1 + u_2' y_2 = 0 \quad \cdots \text{①}$$

$$u_1' y_1' + u_2' y_2' = f(x) \quad \cdots\text{②}$$

Note

$$W := \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

and we know $W \neq 0$ for all x, because y_1 & y_2 are l.i.

Multiply ① by y_2' and ② by y_2:

$$\begin{array}{rl} & u_1' y_1 y_2' + u_2' y_2 y_2' = 0 \\ - & u_1' y_1' y_2 + u_2' y_2' y_2 = f(x) y_2 \\ \hline & u_1'(y_1 y_2' - y_1' y_2) = -f(x) y_2 \end{array}$$

$$u_1' = \frac{-f(x) y_2}{y_1 y_2' - y_2 y_1'} \quad \Rightarrow \quad \boxed{u_1 = -\int \frac{f y_2}{W}\, dx}$$

Similarly we can eliminate the term with u_1' and solve for u_2', to get $\boxed{u_2 = \int \frac{f y_1}{W}\, dx}$

With these $u_1(x)$, $u_2(x)$ we build our y_p:

$$y_p = u_1 y_1 + u_2 y_2$$

Example. Find a particular solution to $2y'' - 4y' + 2y = 2x^{-1}e^x$.

Standard Form: $y'' - 2y' + y = x^{-1}e^x$

Aux. Eqn: $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1$ is a repeated root

$\Rightarrow y_1 = e^x$ and $y_2 = xe^x \Rightarrow W = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = xe^{2x} + e^{2x} - xe^{2x} = e^{2x}$

$$u_1 = -\int \frac{fy_2}{W}\,dx = -\int \frac{x^{-1}e^x}{e^{2x}} \cdot xe^x\,dx = -\int dx = -x$$

$$u_2 = \int \frac{fy_1}{W}\,dx = \int \frac{x^{-1}e^x}{e^{2x}}\,e^x\,dx = \int \frac{1}{x}\,dx = \ln|x|$$

General Solution: $y = \underbrace{c_1 e^x + c_2 x e^x}_{y_c} \underbrace{- xe^x + xe^x \ln|x|}_{y_p}$

Example. Find a particular solution in the interval $(-\pi/2, \pi/2)$

$$y'' + y = \underbrace{\tan x}_{y_{p_1}} + \underbrace{3x - 1}_{y_{p_2}}. \quad \Rightarrow y_p = y_{p_1} + y_{p_2}$$

Aux. Eqn:

$m^2 + 1 = 0$

$\Rightarrow m = \pm i \begin{cases} \alpha = 0 \\ \beta = 1 \end{cases} \Rightarrow y_1 = \cos x$ and $y_2 = \sin x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$u_1 = -\int \frac{fy_2}{W}\,dx = -\int \tan x \sin x\,dx = -\int \frac{\sin^2 x}{\cos x}\,dx = -\int \frac{1-\cos^2 x}{\cos x}\,dx$$

$$= -\int \sec x - \cos x\,dx = -\ln(\sec x + \tan x) + \sin x$$

In fact $\sec x + \tan x = \dfrac{1+\sin x}{\cos x} > 0$ for x in $(-\pi/2, \pi/2)$.

$$u_2 = \int \frac{f \cdot y_1}{W} dx = \int \tan x \cos x \, dx = \int \sin x \, dx = -\cos x$$

$$\therefore \; y_{p_1} = u_1 y_1 + u_2 y_2 = -\cos x \ln(\sec x + \tan x) + \cancel{\sin x \cos x} - \cancel{\cos x \sin x}$$

We can now find y_{p_2} using undetermined coefficients

$$y'' + y = 3x - 1$$

Let $y_{p_2} = Ax + B \Rightarrow y_{p_2}' = A$; $y_{p_2}'' = 0$ and plug in:

$$0 + Ax + B = 3x - 1 \quad \Rightarrow \quad y_{p_2} = 3x - 1$$

So $\quad y_p = -\cos x \ln(\sec x + \tan x) + 3x - 1$

General Sol : $\quad y = \underbrace{c_1 \cos x + c_2 \sin x}_{y_c} \underbrace{- \cos x \ln(\sec x + \tan x) + 3x - 1}_{y_p}$

Example. Find a particular solution in the interval $(-\pi/2, \pi/2)$

$$y'' + y = \tan^2 x.$$

Like in previous example $y_1 = \cos x$, $y_2 = \sin x$ and $W = 1$.

$$u_1 = -\int \frac{f y_2}{W} dx = -\int \tan^2 x \sin x \, dx = -\int \frac{\sin^2 x}{\cos^2 x} \sin x \, dx = -\int \frac{1 - \cos^2 x}{\cos^2 x} \sin x \, dx$$

(substitution $\Rightarrow$) $= \int \frac{1 - u^2}{u^2} du = \int u^{-2} - 1 \, du = -\frac{1}{u} - u = -\sec x - \cos x$

$u = \cos x$

$du = -\sin x \, dx$

$$u_2 = \int \frac{f y_1}{W} dx = \int \tan^2 x \cdot \cos x \, dx = \int \frac{\sin^2 x}{\cos x} dx = \ln(\sec x + \tan x) - \sin x$$

$$y_p = u_1 y_1 + u_2 y_2 = (-\sec x - \cos x)\cos x + (\ln(\sec x + \tan x) - \sin x)\sin x$$

$$= -1 - \cos^2 x + \sin x \ln(\sec x + \tan x) - \sin^2 x$$

$$\boxed{y_p = -2 + \sin x \ln(\sec x + \tan x)}$$

General Solution

$$\therefore \; y = c_1 \cos x + c_2 \sin + \sin x \ln(\sec x + \tan x) - 2$$