Lecture 2: Propositional Logic

Thus far, we encountered:

- propositions, propositional variables, and truth values
- logical connectives (negation, conjunction, disjunction) and compound propositions
- their application for modeling semantics in spoken language.

In this lecture, we will go a bit deeper into these ideas. But first, we serve up some more vocabulary.

Special compound propositions

A compound proposition is called a *tautology* if its truth value is T under any assignment of truth values to its components.

For example,

$$p \vee (\neg p)$$

$$p \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$$

are both tautologies.

An English example: if p represents the proposition "It is sunny outside", then $p \vee (\neg p)$ is the same as saying "It is sunny outside, or it isn't sunny outside", which is obviously a tautology.

The opposite of a *tautology* is a *contradiction*; its truth value is F under any assignment of values to its components. For example,

$$p \wedge (\neg p)$$

is a contradiction. Again, an English example: if p represents the proposition "This apple is red", then $p \wedge (\neg p)$ is the same as saying "This apple is red and is not red", which is a contradiction.

Many propositions are neither tautologies nor contradictions: in such cases, their truth value depends on the states of the internal variables, and are called *contingencies*.

The surest way to check whether something is a tautology (or a contradiction) is to evaluate its *truth table*, and check whether all the output values are T (or F).

Conditionals

You have probably seen the logical construct "if `<assume something>` then `<something else happens>`" in math proofs. You have also probably used an "if-then" command when writing computer programs.

Modern logic codifies this into a special connective. A proposition of the form "if p then q", represented as "$p \implies q$" (and read aloud as "p implies q") is called a *conditional proposition* or an *implication*. Here, p is called the hypothesis (or premise, or assumption), and q is called the conclusion (or consequence.)

For example:

> *If n is divisible by 4, then n is divisible by 2.*
>
> *If John lives in Ames, then John is a resident of Iowa.*

are both conditionals.

This seems quite intuitive; however, **there is a catch** (and this trips up people all the time.) The convention is that $p \implies q$ is always assigned T, **except** when p is true and q is false. Let us write down the truth table for " $p \implies q$" :

p	q	p $\implies$ q
T	T	T
T	F	F
F	T	T
F	F	T

The above truth table indicates that the following propositions are *also* true:

1. *If time travel is possible, then 2 < 4.* (F $\implies$ T)
2. *If the sun rises in the west, then I have discovered a cure for cancer.* (F $\implies$ F).

These are somewhat bizarre statements and you will never encounter them in any reasonable setting, but they are worth taking a closer look.

In (1) above, since the conclusion "2 < 4" is clearly true, *it doesn't matter* whether the hypothesis is true or not; we are in either the 1st or 3rd line of the truth table, and the overall proposition is true!

The case (2) above is even more strange; since the hypothesis is clearly false, *it doesn't matter* whether the consequence is true or not; we are in the 3rd or 4th line of the truth table, and the overall proposition is true!

On the other hand, this is false:

> *If* $2 + 2 = 4$*, then New York is in Canada.*

The premise is true, but the consequence is not; therefore, the overall statement is false!

You may wonder why implications having false hypotheses can themselves be true. A different way to think about it is as follows. Consider the following example (adapted from Enderton's book on logic):

> *If you are telling the truth, then pigs can fly.*

Here p represents "You are telling the truth" and q represents "pigs can fly". From the truth table for $\implies$, we assign a value of T to this proposition whenever you are fibbing. This doesn't meet that you stretching the truth will suddenly *cause* pigs to sprout wings and start flying. Instead, it makes an assertion about pigs **provided** a certain hypothesis is met. If this hypothesis is false, then then the statement is said to be *vacuously true*.

Other "Common English" phrases that are the same as "$p \implies q$" are:

1. *if p then q*
2. *In order for q to be true, it is sufficient for p to occur.*
3. *p is a sufficient condition for q.*
4. *p is true only if q is true.*
5. *if p is true then it is necessary that q is also true.*
6. *q is a necessary condition for p.*

Biconditionals

The proposition $p \iff q$ (read as "p if and only if q") is a *biconditional* implication; it is true only when both *p* and *q* have the same truth values, i.e., when they are both simultaneously true or simultaneously false.

In ordinary math, this definition encodes the (commonly used phrase) "p is a necessary and sufficient condition for q".

Logical equivalence

Compound propositions can become rather complicated, and it is often prudent to replace one proposition with a simpler one. Of course, you can only do that if they are functionally the same. For example, we already saw that *p* and $\neg(\neg p)$ have the same truth assignments, regardless of what *p* is.

As a second example, you can check that $p \implies q$ and $\neg p \vee q$ have the same truth table:

p	q	$p \implies q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Compound propositions whose truth values are the same, regardless of their internal variables, are called *logically equivalent*. The symbol for logical equivalence is $\equiv$.

Exercises: check (using truth tables) that the following are logically equivalent:

- $\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$
- $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$

These are called *De Morgan's* Laws for Logic.

- $p \iff q \equiv (p \implies q) \wedge (q \implies p)$

In words, "p is a necessary and sufficient condition for q" is the same as saying "p implies q and q implies p". This is a technique that we will often use in proofs later on in the course.

Converse, contrapositive, inverse

Do these propositions say the same thing?

1. *If I am very thirsty, then I feel dizzy.*
2. *If I do not feel dizzy, then I am not very thirsty.*

Let p denote "I am thirsty" and q denote "I am tired". Then, the two propositions can be written as "$p \implies q$" and $(\neg q) \implies (\neg p)$. Writing out the truth table, we get:

p	q	$p \implies q$	$(\neg q) \implies (\neg p)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

In other words, (1) and (2) are *logically equivalent*. A statement of the form $(\neg q) \implies (\neg p)$ is called the *contrapositive* of the implication $p \implies q$. They are just two different ways of saying the same thing. Again, this is extremely useful in proofs; instead of trying to proving a given implication, we can instead prove its contrapositive, and rest assured that we are done.

On the other hand, how about:

3. *If I feel dizzy, then I am very thirsty.*

This is the proposition $q \implies p$, and is called the *converse* of $p \implies q$; even at face value, this seems like a rather different proposition than $p \implies q$ (and can be checked again using the truth table.)

Finally, consider:

4. *If I do not feel thirsty, then I do not feel dizzy*.

the proposition $\neg p \implies \neg q$ is called the *inverse* of $p \implies q$. Again, this is a *different* proposition than $p \implies q$, checked via enumerating the truth table.

Some applications

Again, let us apply some of the principles we learned.

How should we write out the natural language statement:

> You can take CprE 409 only if you have taken either CprE 301 or EE 324.

Denote r as "You can take CprE 409", p as "You have taken CprE 301", and q as "You have taken EE 324". Then, the above expression is:

> $r \implies p \vee q$.

Here is a second, real(istic) application that arises in model-based system design. Design specifications are sometimes given as a series of *rules*, say:

If system sensors encounter phenomenon C_i then perform action A_i for i=1,...,n.

Whether or not a system is acting according to the *overall* specification can be written as the logical expression:

$$(C_1 \implies A_1) \wedge (C_2 \implies A_2) \wedge \ldots \wedge (C_n \implies A_n)$$

Now say only conditions C_1 and C_7 occur. Then if the system does indeed take actions A_1 and A_7, the quantities $(C_1 \implies A_1)$ and $(C_7 \implies A_7)$ are both equal to T; the rest of the implications are also T *vacuously*. Therefore, the overall system is behaving according to specification.

On the other hand, if C_7 occurs and the system does *not* take action A_7, then the quantity $(C_7 \implies A_7)$ is false and the system is not acting according to the overall specification.