Show all of your work, and *please* staple your assignment if you use more than one sheet. Write your name, student ID, the course number, and the section on every sheet. Problems marked with * will be graded and one additional randomly chosen problem will be graded.

1. * Suppose that your bus arrives at your bus stop uniformly between 9:05am and 9:10am. Let X = time you wait for the bus. Thus we have that $X \sim \text{Unif}(5, 10)$.

   (a) Give the PDF and CDF for X.

   (b) What is the expected time that the bus will arrive?

   (c) Suppose you slept in a little and can make it to the bus stop at 9:07. What is the probability that you will have missed the bus?

   Answer:

   (a) The PDF is

   $$f_X(x) = \begin{cases} \frac{1}{10-5} = \frac{1}{5} & 5 < x < 10 \\ 0 & \text{otherwise} \end{cases}$$

   The CDF is

   $$F_X(t) = \begin{cases} 0 & t \leq 0 \\ \frac{t-5}{10-5} = \frac{t-5}{5} & 0 < t < 5 \\ 1 & t \geq 10 \end{cases}$$

   (b) $\mathbb{E}(X) = \frac{b+a}{2} = \frac{5+10}{2} = 7.5$. Thus we expect the bus to arrive at 9:07 and 30 seconds.

   (c) We want $\mathbb{P}(X \leq 7) = F_X(7) = \frac{7-5}{5} = 0.4$. Thus if you arrive at 9:07, there is an 40% chance that the bus has already come and gone.

2. * A web page is accessed at an average of 20 times an hour. Assume that waiting time until the next hit has an exponential distribution.

   (a.) Determine the rate parameter λ of the distribution of the time until the first hit?

   (b.) What is the expected time between hits?

   (c.) What is the distribution of the time until the second hit? (Give the name of the distribution and the value(s) of parameter(s).)

   (d.) What is the probability that the next hit is within 20 minutes?

   (e.) Describe the distribution of the total waiting time for 5 hits? (Give the name of the distribution and the value(s) of parameter(s).)

   (f.) What is the expected total waiting time for 5 hits on the web page?

   (g.) What is the probability that there will be less than 5 hits in the first hour? (Hint: Consider Poisson distribution instead.)

   Answer: Let X be the time until the next hit. It is the same as the time between hits.

   (a) Let X be the time until the next hit. By the the description, the rate parameter (number of hits per hour) $\lambda = 20$. Alternatively, the expected waiting time between hits, $E[X] = 1/20 = 1/\lambda$ giving $\lambda = 20$.

   (b) Since $X \sim Exp(20)$, we have $E[X] = 1/\lambda = 1/20 = .05$ (hours)

   (c) The time until the second hit is $Y = X_1 + X_2$ where $X_1 \sim Exp(20)$ and $X_2 \sim Exp(20)$ and X_1 and X_2 are independent. Thus $Y \sim Gamma(2, 20)$.

   (d) Need $P(X \leq 20/60)$ where $X \sim Exp(20)$. Using the cdf for the exponential distribution $F_X(0.3333) = 1 - e^{-0.3333 \times 20} = 1 - 0.00127 = 0.9987$

   (e) The waiting time for 5 hits is $W = \sum_{i=1}^{5} = X_i$; so $W \sim Gamma(5, 20)$.

   (f) We need $E[W]$ which is given by $k/\lambda = 5/20 = 0.25$ hour, as one might expect!

   (g) Let N be the number of hits in the first hour and assume $N \sim Poi(20 \times 1)$. The we need $P(N < 5) = P(N \leq 4) = 0.000017$.

3. The amount of time a postal clerk spends with his customer can be modeled using an exponential distribution. On average, the clerk spends 5 minutes with a customer. Let X = the amount of time (in minutes) a postal clerk spends with his customer.

   (a) Give the distribution of X and the value for its parameter.

   (b) Give the probability density function (PDF) and the cumulative distribution function (CDF) of X.

   (c) What is the probability that the clerk spends less than 5 minutes with a customer?

   (d) If the clerk hasn't finished assisting the customer in 2 minutes, what is the probability that he spends less than 5 minutes with the customer?

   Answer:

   (a) $E(X) = \frac{1}{\lambda} = 5 \text{ min} \rightarrow \lambda = \frac{1}{5} = 0.2$
   $X \sim Exp(0.2)$

   (b)

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 0.2e^{-0.2x} & \text{for } x > 0 \end{cases}$$

$$F_X(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 - e^{-0.2t} & \text{for } x > 0 \end{cases}$$

   (c) $P(X < 5) = F_X(5) = 1 - e^{-0.2(5)} = 0.6321$

   (d) $P(X < 5|X > 2) = P(X < 3)$ (by the memoryless property of exponential distribution)

$$P(X < 5|X > 2) = P(X < 3) = F_X(3) = 1 - e^{-0.2(3)} = 0.4512$$

   Or we can calculate the conditional probability $P(X < 5|X > 2)$ directly

$$\begin{aligned} P(X < 5|X > 2) &= \frac{P((X < 5) \cap (X > 2))}{P(X > 2)} \\ &= \frac{P(2 < X < 5)}{P(X > 2)} \\ &= \frac{F_X(5) - F_X(2)}{1 - F_X(2)} \\ &= \frac{[1 - e^{-0.2(5)}] - [1 - e^{-0.2(2)}]}{1 - [1 - e^{-0.2(2)}]} \\ &= 0.4512 \end{aligned}$$

4. On the average, a computer experiences breakdowns every 5 months. The time until the first breakdown and the times between any two consecutive breakdowns are independent Exponential random variables. After the third breakdown, a computer requires a special maintenance.

   (a) Compute the probability that a special maintenance is required within the next 9 months.

   (b) Given that a special maintenance was not required during the first 12 months, what is the probability that it will not be required within the next 4 months?

Answer:

4.9 The time T until the third breakdown has Gamma distribution with parameters $\alpha = 3$ and $\lambda = 1/5$ months^{-1}.

(a) By the Gamma-Poisson formula with a Poisson($\lambda t = 1/5 \cdot 9 = 1.8$) variable X and Table A3,

$$\boldsymbol{P}\{T \le 9\} = \boldsymbol{P}\{X \ge 3\} = 1 - F_X(2) = 1 - 0.731 = \boxed{0.269}$$

(b) $$\boldsymbol{P}\{T > 16 \mid T > 12\} = \frac{\boldsymbol{P}\{T > 16 \cap T > 12\}}{\boldsymbol{P}\{T > 12\}} = \frac{\boldsymbol{P}\{T > 16\}}{\boldsymbol{P}\{T > 12\}}$$

$$= \frac{\boldsymbol{P}\{X_1 < 3\}}{\boldsymbol{P}\{X_2 < 3\}} = \frac{e^{-3.2}(1 + 3.2 + 3.2^2/2)}{0.570} = \boxed{0.666}$$

by the Gamma-Poisson formula, the formula of Poisson pmf, and Table A3, where X_1 has Poisson distribution with parameter $(1/5)(16) = 3.2$ and X_2 has Poisson distribution with parameter $(1/5)(12) = 2.4$.

5. Suppose a phone call on average lasts 5 minutes at a phone booth.

(a) If a person arrives at a public telephone booth just before you, calculate the probability that you have to wait more than 15 minutes to make your call. (Hint: Use exponential distribution to model waiting time)

(b) You get tired of waiting, and decide to do a little shopping and come back. When you come back there are now 2 people ahead of you. What is the probability that that you have to wait more than 20 minutes?

Answer:

(a) X = time to complete a phone call where $X \sim Exp(\lambda = \frac{1}{5})$

$$\begin{aligned}\mathbb{P}(X > 15) = 1 - \mathbb{P}(X \le 15) &= 1 - F_X(15)\\ &= 1 - (1 - e^{-(1/5)(15)})\\ &= e^{-3} = 0.0498\end{aligned}$$

(b) T = time to complete 2 phone calls where $T \sim Gamma(\alpha = 2, \lambda = \frac{1}{5})$. We want $\mathbb{P}(T > 20)$ Define a new random variable $Y \sim Pois(\lambda t) \equiv Pois((1/5)(20)) \equiv Pois(4)$. Using the Gamma-Poisson formula,

$$P(T > 20) = P(Y < \alpha) = P(Y < 2) = P(Y \le 1) = F_Y(1) = 0.0916$$