Inference in Temporal Models

Outline

I. States and observations

II. Transitions and sensor models

III. Filtering

IV. Prediction

V. Smoothing

* Figures/images are from the [textbook site] or the instructor unless the source is cited specifically.

I. Time and Uncertainty

- Probabilistic reasoning in *static worlds* discussed so far.

  Every random variable
  has a single fixed value.

- Real situations are *dynamic* with evidence evolving with time and thus actions predicted (and chosen) based on the history of evidence:

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How to model dynamic situations?

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high speed camera

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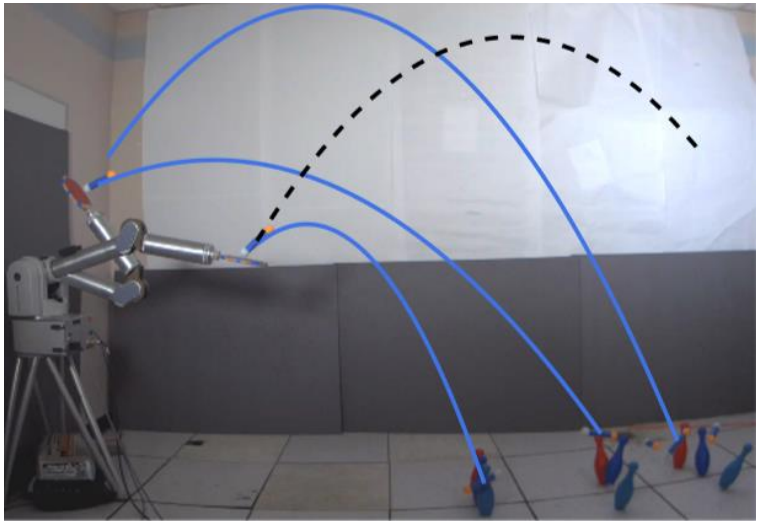
Resolution: 2048×1088 pixel



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Duration: 0.6 second with 90 frames
Motion of the object estimated by an extended Kalman filter (EKF).

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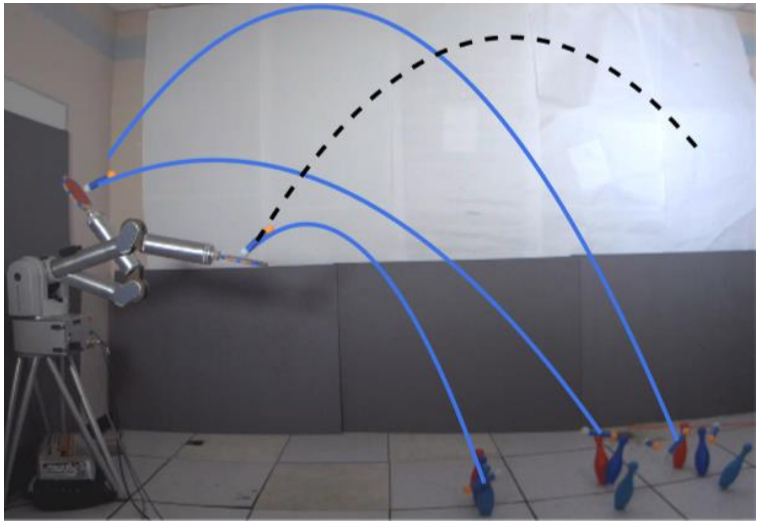

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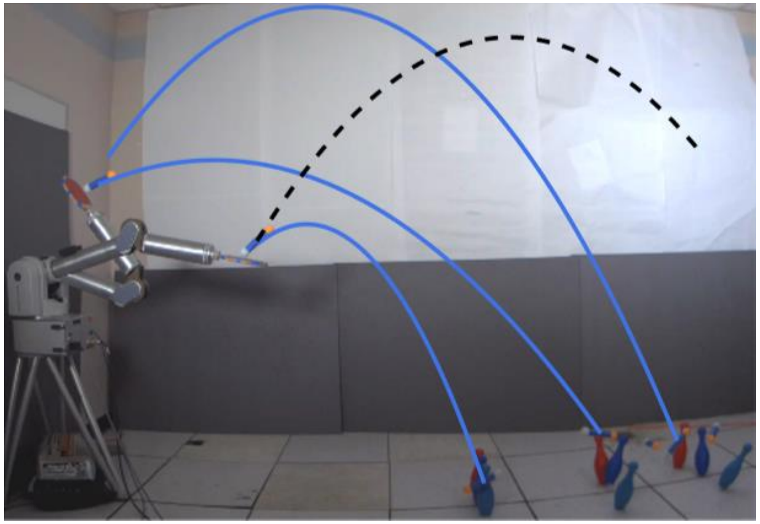

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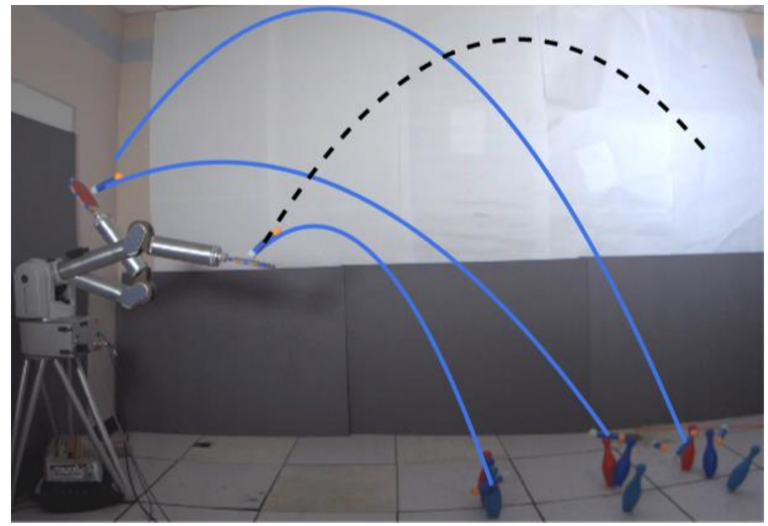

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$$\boldsymbol{E}_t = \boldsymbol{e}_t \text{ for some observed values } \boldsymbol{e}_t$$

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The umbrella world is represented by

state variable sequence: $R_0, R_1, R_2, \dots$

evidence variable sequence: $U_0, U_1, U_2, \dots$

$U_{3:5}$ corresponds to U_3, U_4, U_5.

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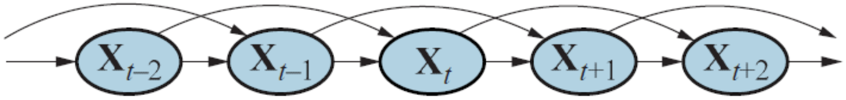

2nd order Markov process
(transition model $\boldsymbol{P}(\boldsymbol{X}_t \mid \boldsymbol{X}_{t-1}, \boldsymbol{X}_{t-2})$)

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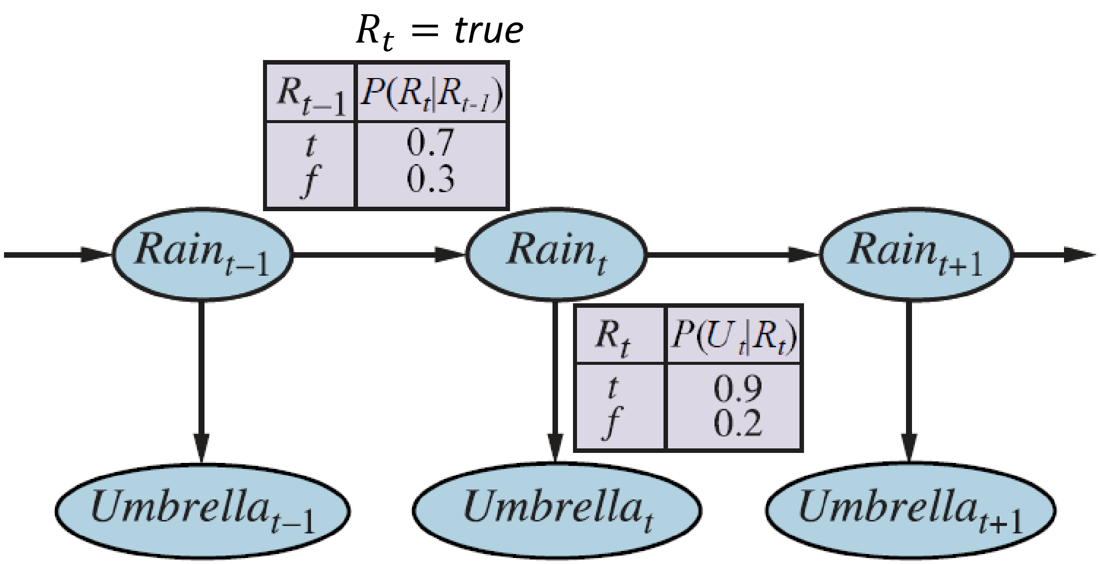
$$\boldsymbol{P}(\boldsymbol{E}_t \mid \boldsymbol{X}_{0:t}, \boldsymbol{E}_{1:t-1}) = \boldsymbol{P}(\boldsymbol{E}_t \mid \boldsymbol{X}_t)$$

sensor/observation model

Models for the Umbrella World



1st order Markov Process

Transition model:

$P(Rain_t \mid Rain_{t-1})$

Sensor model:

$P(Umbrella_t \mid Rain_t)$

The state (*Rain*) causes the sensor to take on a particular value (*Umbrella*).

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♦ Discrete time models can handle an infinite set of variables due to

- ♣ use of integer indices
- ♣ use of implicit universal quantification to define sensor and transition models

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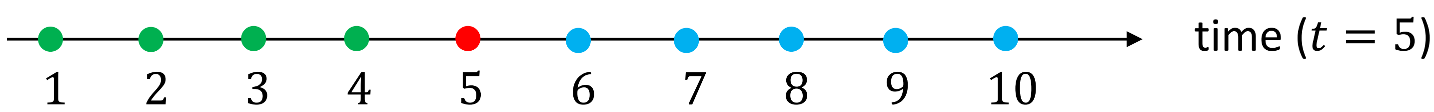

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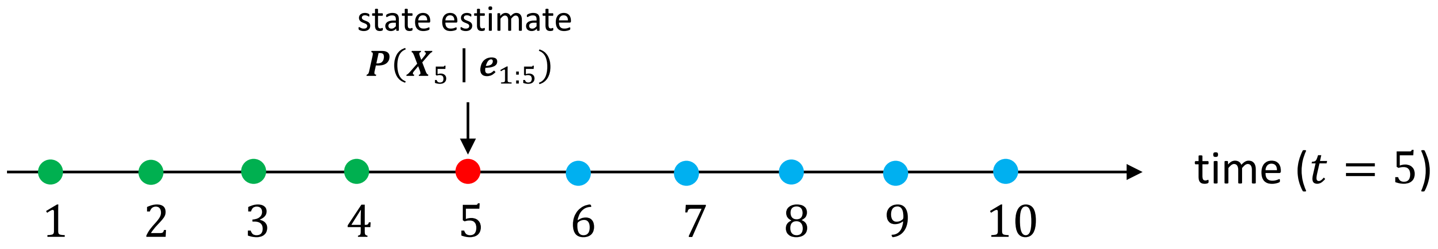

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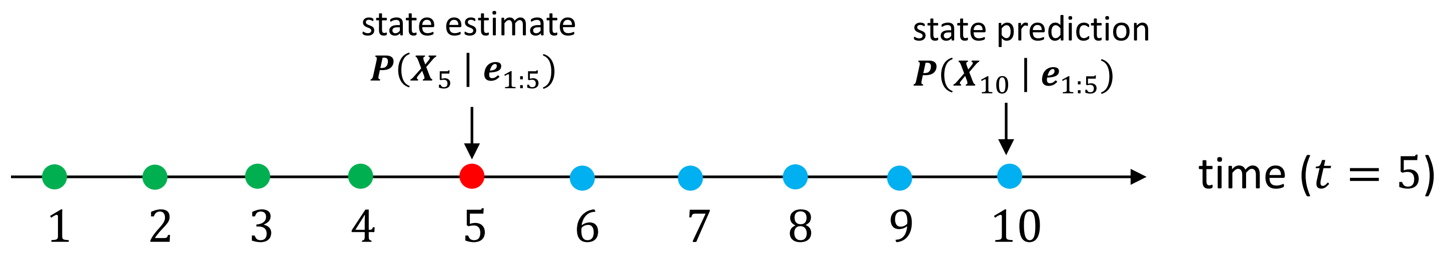

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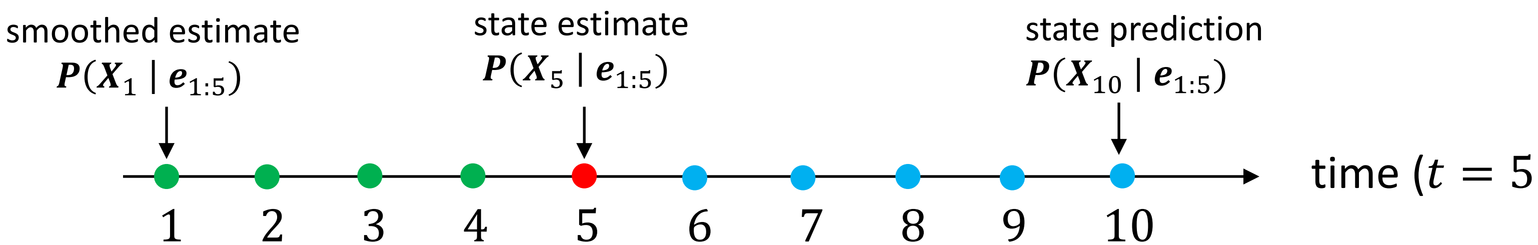

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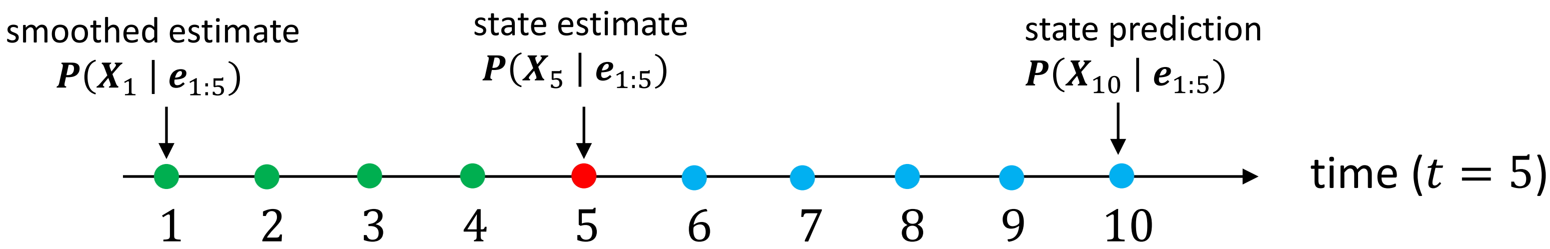

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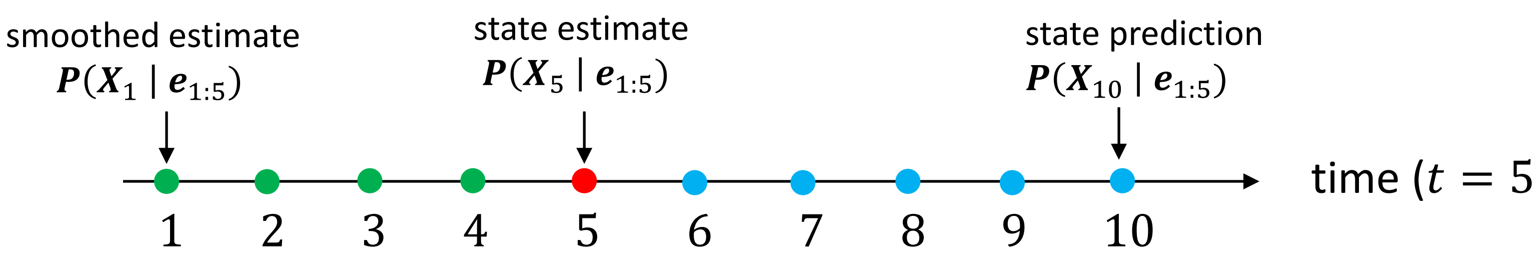

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  E.g., If the umbrella appears on each of the first three days and is absent on the fourth, then the most likely explanation is that it rained on the first three days and did not rain on the fourth.

III. Filtering

To be efficient (so usable in a real time scenario), a filtering algorithm

- ♦ needs to maintain current state estimate and update it on the fly, and
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Recursive estimation:

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$$\begin{aligned}\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1}) &= \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}, \mathbf{e}_{t+1}) \\ &= \alpha \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}, \mathbf{e}_{1:t})\, \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t})\end{aligned}$$

(Bayes' rule, given $\mathbf{e}$ $1{:}\,t$)
$1{:}\,t$ t $1{:}\,t$)

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state estimate at $t+1$ state estimate at t

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normalizing constant

(Bayes' rule, given $\boldsymbol{e}$ $1{:}\,t$)
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$$= \alpha \boldsymbol{P}(\boldsymbol{e}_{t+1} \mid \boldsymbol{X}_{t+1}, \boldsymbol{e}_{1:t})\, \boldsymbol{P}(\boldsymbol{X}_{t+1} \mid \boldsymbol{e}_{1:t})$$ (Bayes' rule, given $\boldsymbol{e}$ $1{:}\,t$)

$1{:}\,t\ t\ 1{:}\,t$)

normalizing constant

$$= \alpha \boldsymbol{P}(\boldsymbol{e}_{t+1} \mid \boldsymbol{X}_{t+1})\, \boldsymbol{P}(\boldsymbol{X}_{t+1} \mid \boldsymbol{e}_{1:t})$$ (by the sensor Markov assumption)

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(Bayes' rule, given $\boldsymbol{e}\ 1:t$)
$1:t\ t\ 1:t$)

normalizing constant

$$= \alpha \underbrace{\boldsymbol{P}(\boldsymbol{e}_{t+1} \mid \boldsymbol{X}_{t+1})}_{\text{update}}\, \underbrace{\boldsymbol{P}(\boldsymbol{X}_{t+1} \mid \boldsymbol{e}_{1:t})}_{\text{prediction}}$$

(by the sensor Markov assumption)

One-Step Prediction

$$\boldsymbol{P}(\boldsymbol{X}_{t+1} \mid \boldsymbol{e}_{1:t+1}) = \alpha \boldsymbol{P}(\boldsymbol{e}_{t+1} \mid \boldsymbol{X}_{t+1}) \sum_{\boldsymbol{x}_t} \boldsymbol{P}(\boldsymbol{X}_{t+1} \mid \boldsymbol{x}_t, \boldsymbol{e}_{1:t}) P(\boldsymbol{x}_t \mid \boldsymbol{e}_{1:t})$$

One-Step Prediction

$$\boldsymbol{P}(\boldsymbol{X}_{t+1} \mid \boldsymbol{e}_{1:t+1}) = \alpha \boldsymbol{P}(\boldsymbol{e}_{t+1} \mid \boldsymbol{X}_{t+1}) \sum_{\boldsymbol{x}_t} \boldsymbol{P}(\boldsymbol{X}_{t+1} \mid \boldsymbol{x}_t, \boldsymbol{e}_{1:t}) P(\boldsymbol{x}_t \mid \boldsymbol{e}_{1:t})$$

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$$= \alpha \underbrace{\mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1})}_{\text{sensor model}} \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t) \mathbf{P}(\mathbf{x}_t \mid \mathbf{e}_{1:t})$$

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$$\begin{cases} \boldsymbol{P}(\boldsymbol{X}_{t+1} \mid \boldsymbol{e}_{1:t+1}) = \text{FORWARD}(\underbrace{\boldsymbol{P}(\boldsymbol{X}_t \mid \boldsymbol{e}_{1:t})}_{\text{“forward” message}}, \boldsymbol{e}_{t+1}) \\ \boldsymbol{P}(\boldsymbol{X}_0 \mid \boldsymbol{e}_{1:0}) = \boldsymbol{P}(\boldsymbol{X}_0) \end{cases}$$

One-Step Prediction

$$\boldsymbol{P}(\boldsymbol{X}_{t+1} \mid \boldsymbol{e}_{1:t+1}) = \alpha \boldsymbol{P}(\boldsymbol{e}_{t+1} \mid \boldsymbol{X}_{t+1}) \sum_{\boldsymbol{x}_t} \boldsymbol{P}(\boldsymbol{X}_{t+1} \mid \boldsymbol{x}_t, \boldsymbol{e}_{1:t}) P(\boldsymbol{x}_t \mid \boldsymbol{e}_{1:t})$$

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Time and space for the update at t must be *constant* in order to keep track of the current state distribution indefinitely.

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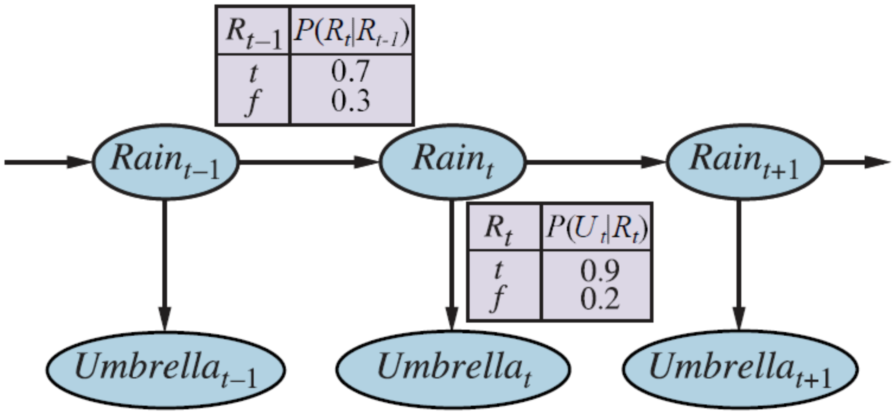
For more on Kalman filtering and its subroutine recursive least-squares (used in update), we refer to
http://web.cs.iastate.edu/~cs577/handouts/kalman-filter.pdf
http://web.cs.iastate.edu/~cs577/handouts/recursive-least-squares.pdf

Filtering in the Umbrella World



Compute $\boldsymbol{P}(R_2 \mid u_{1:2})$ as follows:

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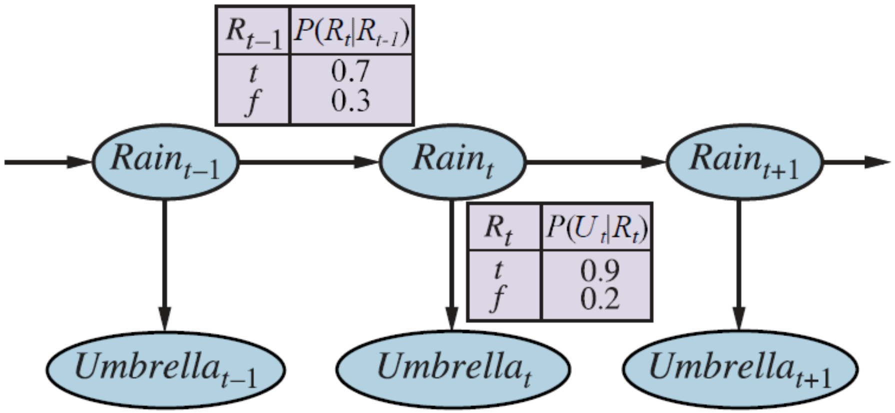

Compute $\boldsymbol{P}(R_2 \mid u_{1:2})$ as follows:

- On day 0, no observation.

$$\boldsymbol{P}(R_0) = \langle 0.5, 0.5 \rangle$$

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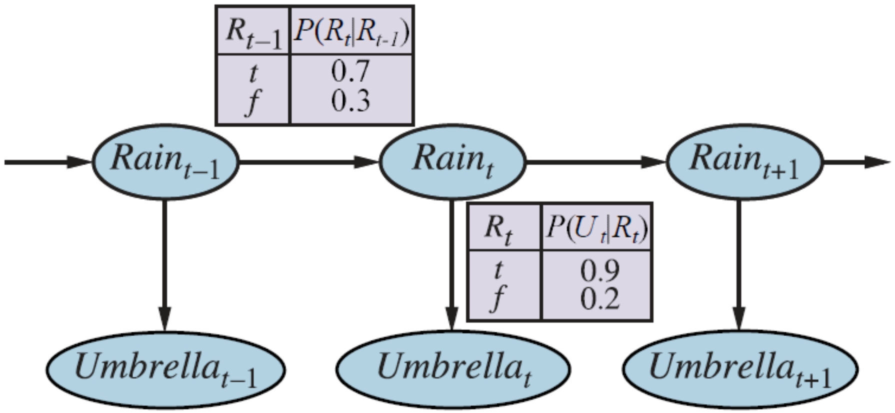

Compute $\boldsymbol{P}(R_2 \mid u_{1:2})$ as follows:

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- On day 1, the umbrella appears.

  $$U_1 = true$$

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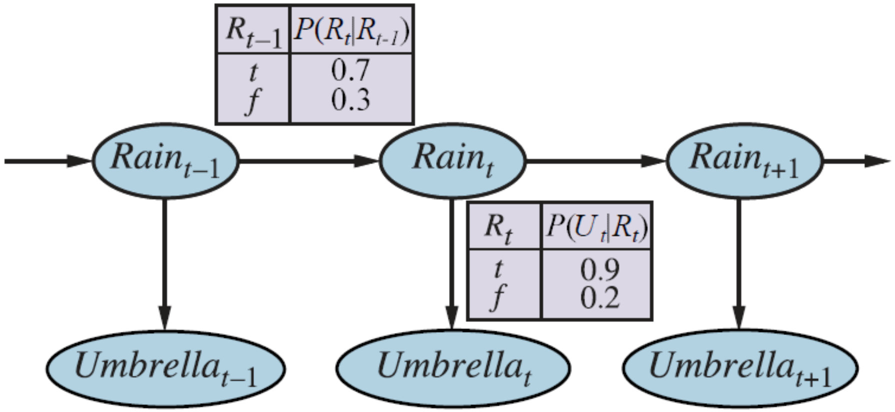

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a) Prediction:

$$\boldsymbol{P}(R_1) = \sum_{r_0 \in \{true, false\}} \boldsymbol{P}(R_1 \mid r_0) P(r_o)$$

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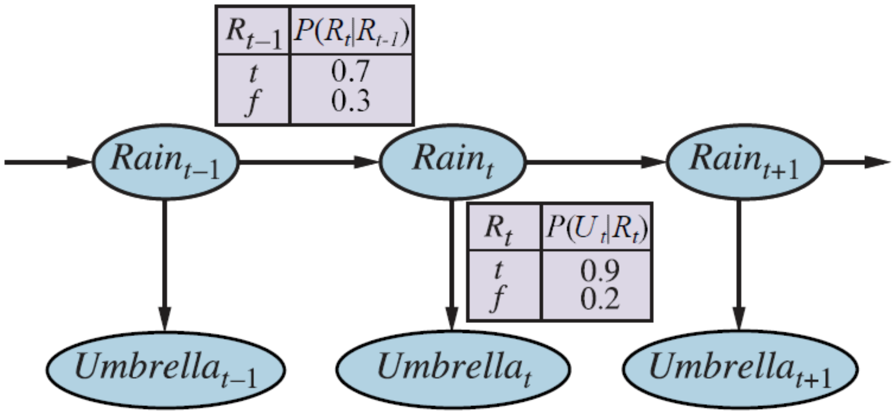

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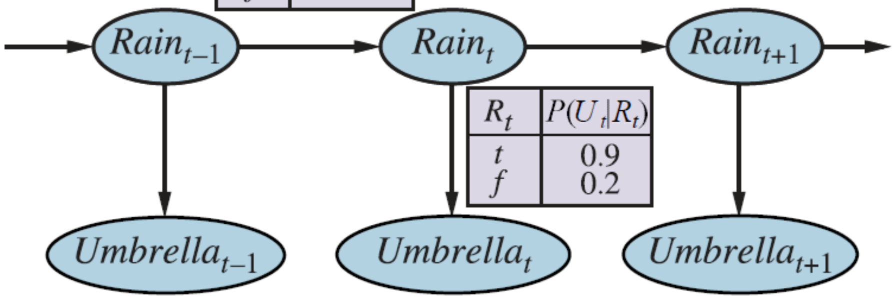
a) Prediction:

$$\boldsymbol{P}(R_1) = \sum_{r_0 \in \{true, false\}} \boldsymbol{P}(R_1 \mid r_0) P(r_o)$$

$$= \langle 0.7, 0.3 \rangle \times 0.5 + \langle 0.3, 0.7 \rangle \times 0.5 = \langle 0.5, 0.5 \rangle$$

Filtering in the Umbrella World



R_{t-1}	$P(R_t \mid R_{t-1})$
t	0.7
f	0.3

R_t	$P(U_t \mid R_t)$
t	0.9
f	0.2

Compute $\boldsymbol{P}(R_2 \mid u_{1:2})$ as follows:

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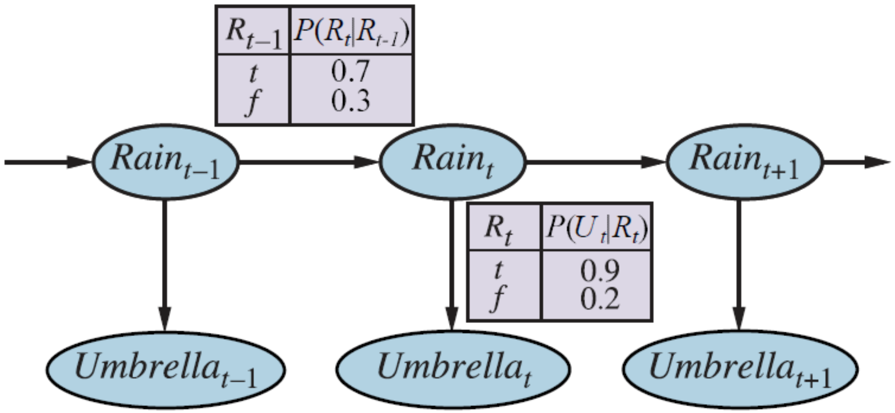
b) Update with evidence for $t = 1$ followed by normalization:

$$\boldsymbol{P}(R_1 \mid u_1) = \alpha \boldsymbol{P}(\underset{true}{u_1} \mid R_1) \boldsymbol{P}(R_1)$$

Filtering in the Umbrella World



Compute $\boldsymbol{P}(R_2 \mid u_{1:2})$ as follows:

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- On day 1, the umbrella appears.

$$U_1 = \textit{true}$$

a) Prediction:

$$\boldsymbol{P}(R_1) = \sum_{r_0 \in \{\textit{true}, \textit{false}\}} \boldsymbol{P}(R_1 \mid r_0) P(r_o)$$

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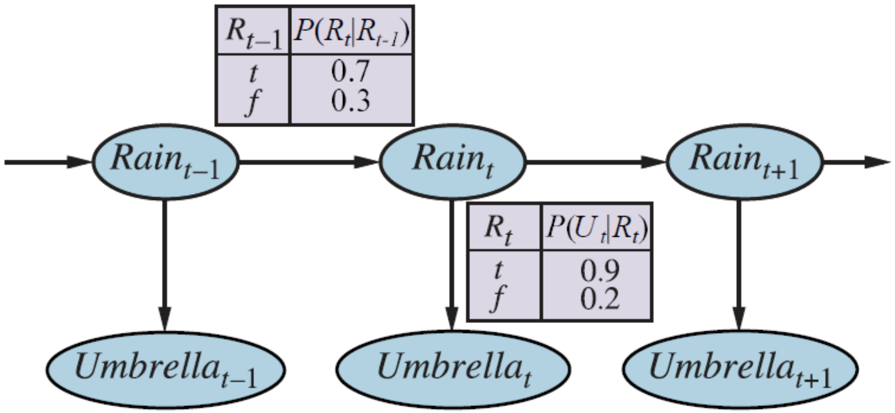
$$\boldsymbol{P}(R_1 | u_1) = \alpha \boldsymbol{P}(\underset{\textit{true}}{u_1} \mid R_1) \boldsymbol{P}(R_1)$$

$$= \alpha \langle 0.9, 0.2 \rangle \langle 0.5, 0.5 \rangle$$

Filtering in the Umbrella World



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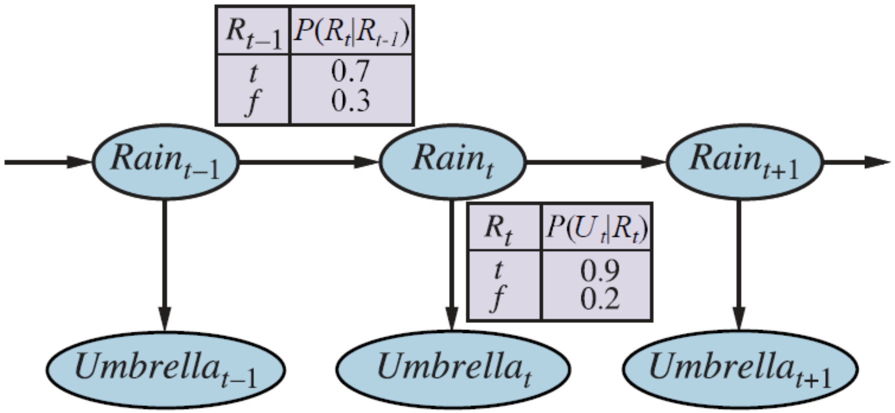
$$= \alpha \langle 0.9, 0.2 \rangle \langle 0.5, 0.5 \rangle$$

$$= \alpha \langle 0.45, 0.1 \rangle$$

Filtering in the Umbrella World



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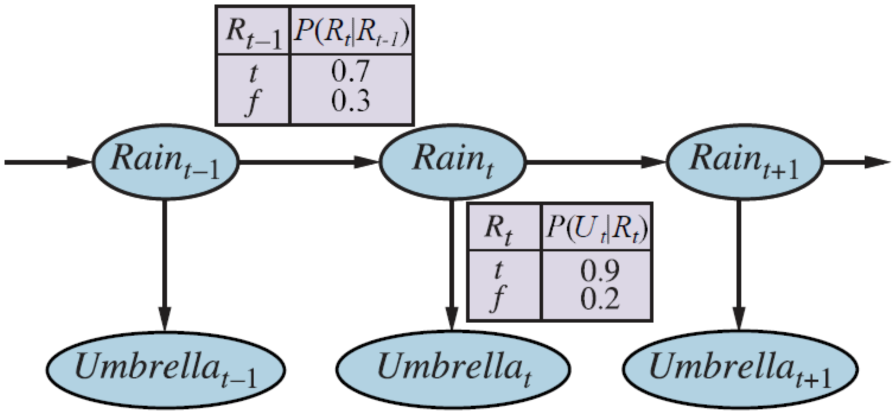
$$= \alpha \langle 0.45, 0.1 \rangle$$

$$\approx \langle 0.818, 0.182 \rangle$$

Filtering in UW (cont'd)

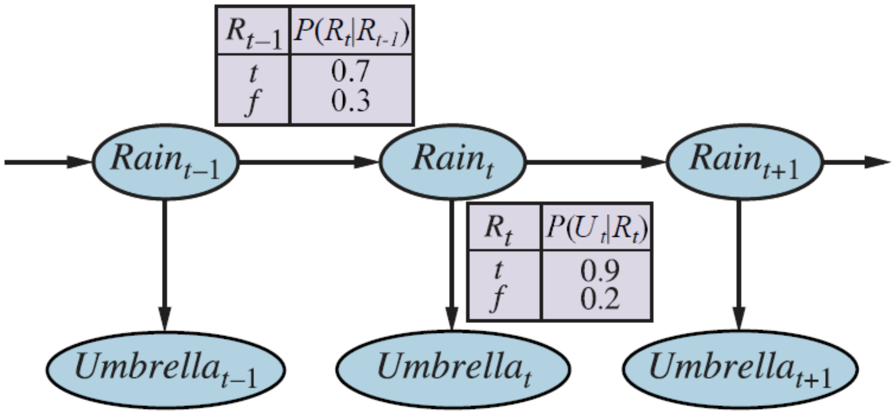


$$\boldsymbol{P}(R_1 \mid u_1) \approx \langle 0.818, 0.182 \rangle$$

- On day 2, the umbrella appears.

$$U_2 = true$$

Filtering in UW (cont'd)



$$\boldsymbol{P}(R_1 \mid u_1) \approx \langle 0.818, 0.182 \rangle$$

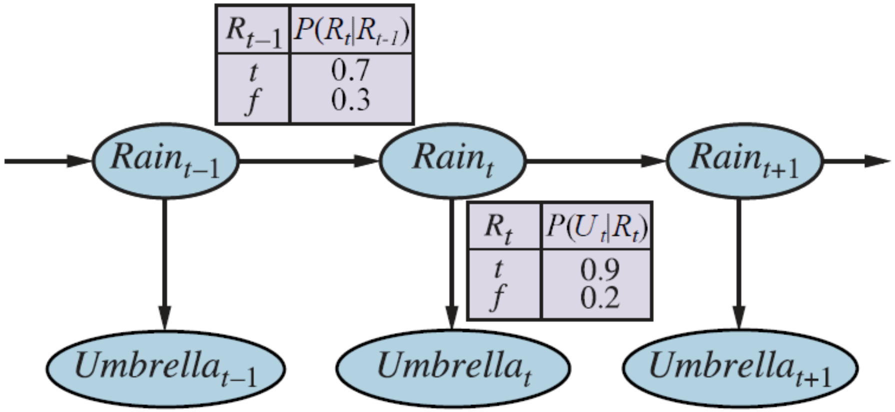
- On day 2, the umbrella appears.

$$U_2 = true$$

a) Prediction:

$$\boldsymbol{P}(R_2 \mid u_1) = \sum_{r_1} \boldsymbol{P}(R_2 \mid r_1) P(r_1 \mid u_1)$$

Filtering in UW (cont'd)



$$\boldsymbol{P}(R_1 \mid u_1) \approx \langle 0.818, 0.182 \rangle$$

- On day 2, the umbrella appears.

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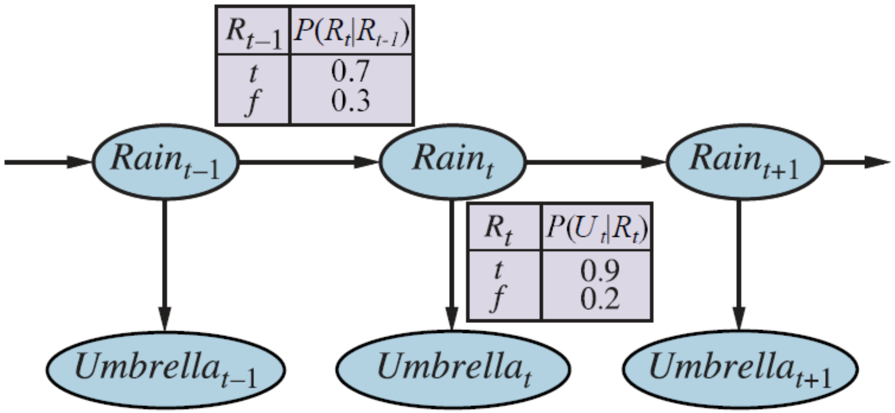
a) Prediction:

$$\boldsymbol{P}(R_2 \mid u_1) = \sum_{r_1} \boldsymbol{P}(R_2 \mid r_1) P(r_1 \mid u_1)$$

$$= \langle 0.7, 0.3 \rangle \times 0.818 + \langle 0.3, 0.7 \rangle \times 0.182 \approx \langle 0.627, 0.373 \rangle$$

Filtering in UW (cont'd)



$$\mathbf{P}(R_1 \mid u_1) \approx \langle 0.818, 0.182 \rangle$$

- On day 2, the umbrella appears.

$$U_2 = \mathit{true}$$

a) Prediction:

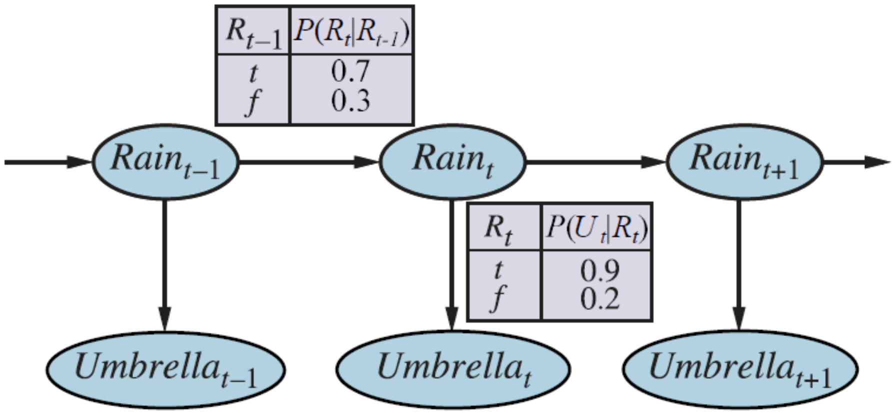
$$\mathbf{P}(R_2 \mid u_1) = \sum_{r_1} \mathbf{P}(R_2 \mid r_1) P(r_1 \mid u_1)$$

$$= \langle 0.7, 0.3 \rangle \times 0.818 + \langle 0.3, 0.7 \rangle \times 0.182 \approx \langle 0.627, 0.373 \rangle$$

b) Update with evidence for $t = 2$:

$$\mathbf{P}(R_2 \mid u_1, u_2) = \alpha \mathbf{P}(\underset{\mathit{true}}{u_2} \mid R_2) \mathbf{P}(R_2 \mid u_1)$$

Filtering in UW (cont'd)



$$\boldsymbol{P}(R_1 \mid u_1) \approx \langle 0.818, 0.182 \rangle$$

- On day 2, the umbrella appears.

$$U_2 = true$$

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$$\boldsymbol{P}(R_2 \mid u_1) = \sum_{r_1} \boldsymbol{P}(R_2 \mid r_1) P(r_1 \mid u_1)$$

$$= \langle 0.7, 0.3 \rangle \times 0.818 + \langle 0.3, 0.7 \rangle \times 0.182 \approx \langle 0.627, 0.373 \rangle$$

b) Update with evidence for $t = 2$:

$$\boldsymbol{P}(R_2 \mid u_1, u_2) = \alpha \boldsymbol{P}(\underset{true}{u_2} \mid R_2) \boldsymbol{P}(R_2 \mid u_1)$$

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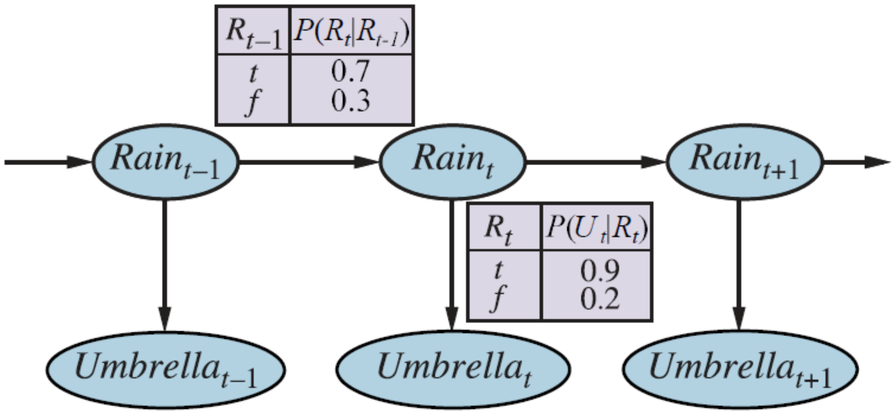
Filtering in UW (cont'd)



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$$\approx \alpha \langle 0.565, 0.075 \rangle$$

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IV. Prediction

Prediction is essentially filtering without the addition of new evidence.

Only prediction and no update at every time step.

For $k = 0,1,\ldots$

$$\boldsymbol{P}(\boldsymbol{X}_{t+k+1} \mid \boldsymbol{e}_{1:t}) = \sum_{\boldsymbol{x}_{t+k}} \underbrace{\boldsymbol{P}(\boldsymbol{X}_{t+k+1} \mid \boldsymbol{x}_{t+k})}_{\text{transition model}} \underbrace{\boldsymbol{P}(\boldsymbol{x}_{t+k} \mid \boldsymbol{e}_{1:t})}_{\text{recursion}}$$

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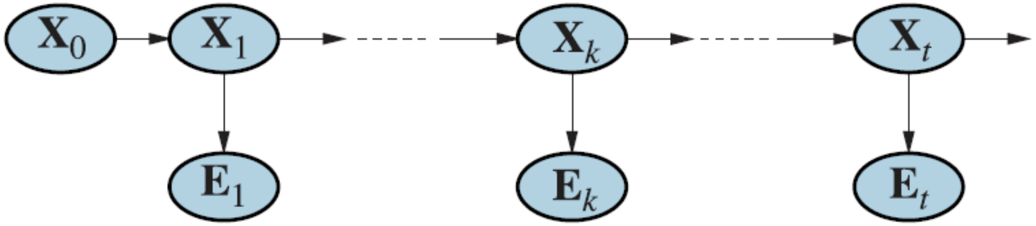
- The value of k at which convergence happens is called the *mixing time*, which has been well studied.
- The more uncertainty, the shorter will be the mixing time.

V. Smoothing

Compute $\boldsymbol{P}(\boldsymbol{X}_k \mid \boldsymbol{e}_{1:t})$ for some $0 \le k < t$.

current time



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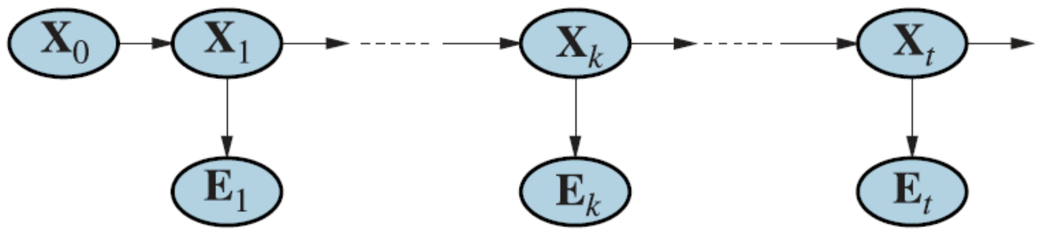

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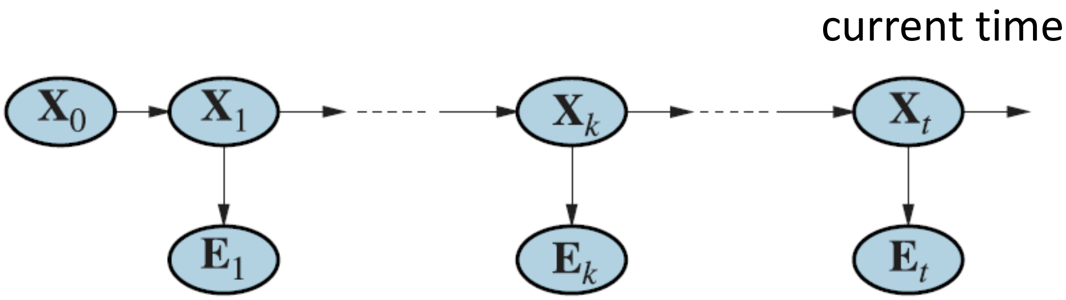

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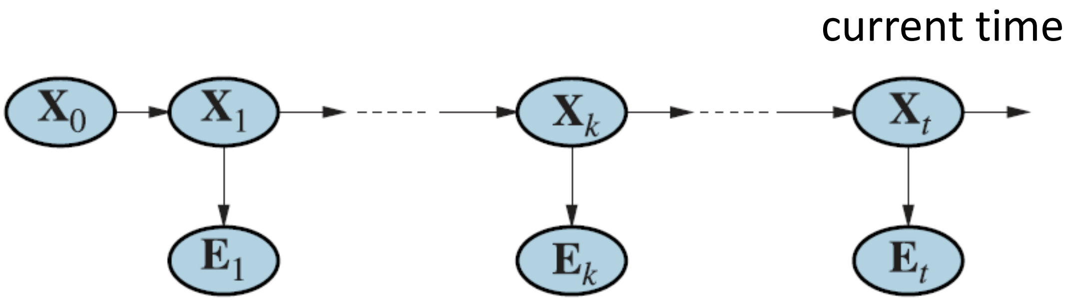

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(Bayes' rule, given $\boldsymbol{e}$ $1{:}\,k$)
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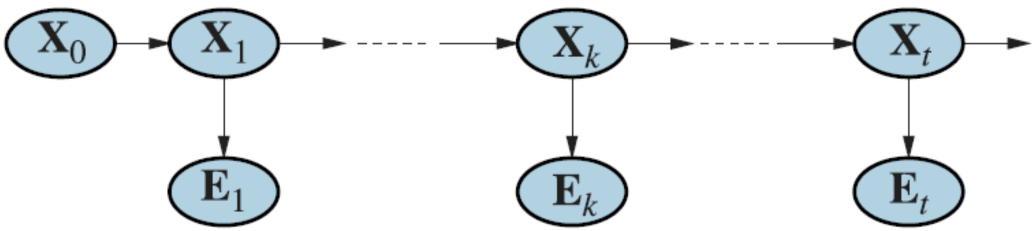

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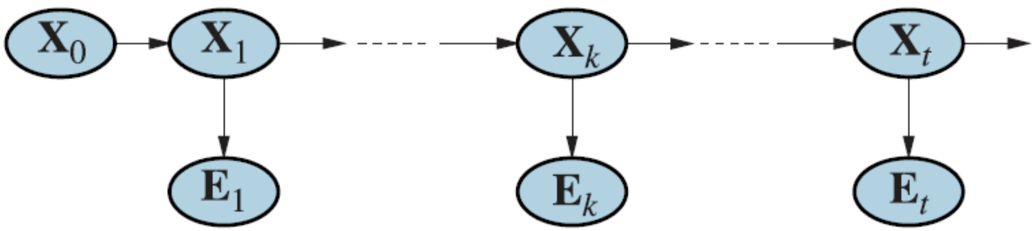

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$$= \alpha \boldsymbol{P}(\boldsymbol{X}_k \mid \boldsymbol{e}_{1:k}) \underbrace{\boldsymbol{P}(\boldsymbol{e}_{k+1:t} \mid \boldsymbol{X}_k)}_{\text{“backward” message}}$$ (conditional independence)

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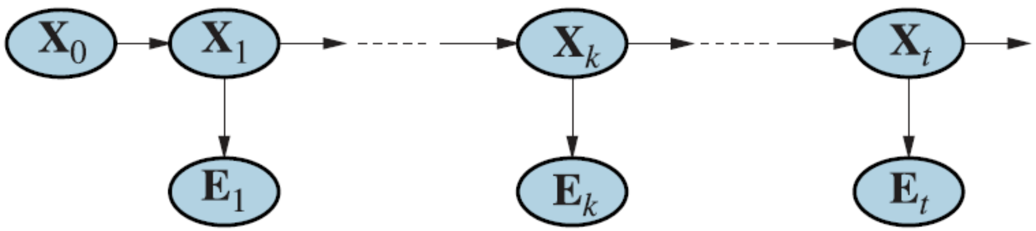

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- The backward message can also be computed recursively.

$$\boldsymbol{P}(\boldsymbol{e}_{k+1:t} \mid \boldsymbol{X}_k) = \sum_{\boldsymbol{x}_{k+1}} \underbrace{P(\boldsymbol{e}_{k+1} \mid \boldsymbol{x}_{k+1})}_{\text{sensor model}} \underbrace{P(\boldsymbol{e}_{k+2:t} \mid \boldsymbol{x}_{k+1})}_{\text{recursion}} \underbrace{\boldsymbol{P}(\boldsymbol{x}_{k+1} \mid \boldsymbol{X}_k)}_{\text{transition model}}$$

Computation

Equation for smoothing ($0 \leq k < t$):

$$\boldsymbol{P}(\boldsymbol{X}_k \mid \boldsymbol{e}_{1:t}) = \alpha \boldsymbol{P}(\boldsymbol{X}_k \mid \boldsymbol{e}_{1:k}) \boldsymbol{P}(\boldsymbol{e}_{k+1:t} \mid \boldsymbol{X}_k)$$

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- Forward computation (filtering)

$$\boldsymbol{P}(\boldsymbol{X}_0) \rightarrow \boldsymbol{P}(\boldsymbol{X}_1 \mid \boldsymbol{e}_{1:1}) \rightarrow \cdots \rightarrow \boldsymbol{P}(\boldsymbol{X}_k \mid \boldsymbol{e}_{1:k})$$

where, for $0 \leq i \leq k-1$,

$$\boldsymbol{P}(\boldsymbol{X}_{i+1} \mid \boldsymbol{e}_{1:i+1}) = \alpha \boldsymbol{P}(\boldsymbol{e}_{i+1} \mid \boldsymbol{X}_{i+1}) \sum_{\boldsymbol{x}_i} \boldsymbol{P}(\boldsymbol{X}_{i+1} \mid \boldsymbol{x}_i) \boldsymbol{P}(\boldsymbol{x}_i \mid \boldsymbol{e}_{1:i})$$

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$$\underbrace{\boldsymbol{P}(\boldsymbol{e}_{t+1:t} \mid \boldsymbol{X}_t)}_{= \boldsymbol{P}(\,\mid \boldsymbol{X}_t) = \mathbf{1} \text{ (vector of 1s)}} \to \boldsymbol{P}(\boldsymbol{e}_{t:t} \mid \boldsymbol{X}_{t-1}) \to \cdots \to \boldsymbol{P}(\boldsymbol{e}_{k+1:t} \mid \boldsymbol{X}_k)$$

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$$\boldsymbol{P}(\boldsymbol{e}_{j+1:t} \mid \boldsymbol{X}_j) = \sum_{\boldsymbol{x}_{j+1}} P(\boldsymbol{e}_{j+1} \mid \boldsymbol{x}_{j+1}) \, P(\boldsymbol{e}_{j+2:t} \mid \boldsymbol{x}_{j+1}) \boldsymbol{P}(\boldsymbol{x}_{j+1} \mid \boldsymbol{X}_j)$$

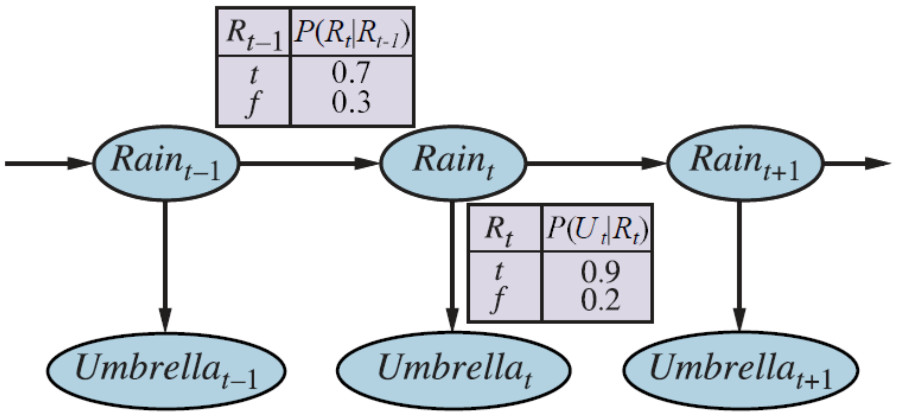
Smoothing in the Umbrella World

Compute $\underbrace{\boldsymbol{P}(R_1 \mid u_1, u_2)}$ as follows:

probability of rain on day 1, given that umbrellas were observed on days 1 and 2.

$u_1 = true$

$u_2 = true$



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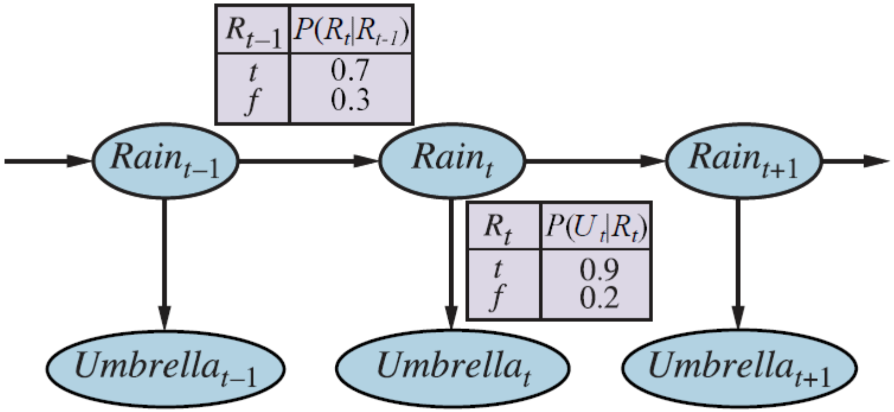

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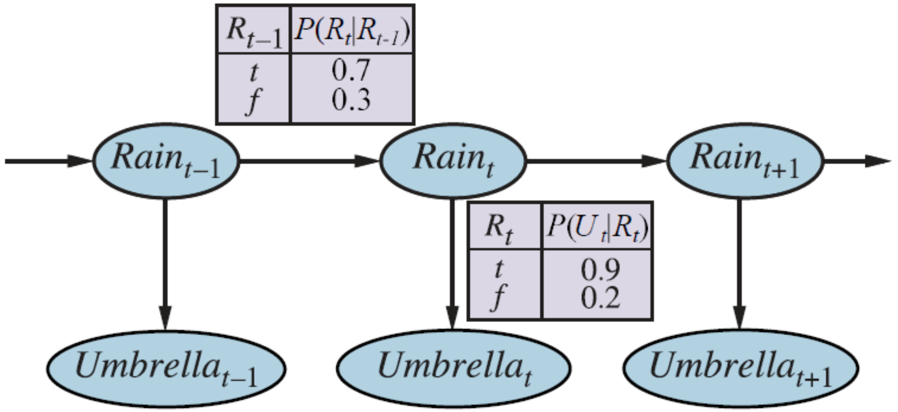

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$$\Big\Downarrow \quad \begin{matrix} t = 2 \\ k = 1 \end{matrix}$$

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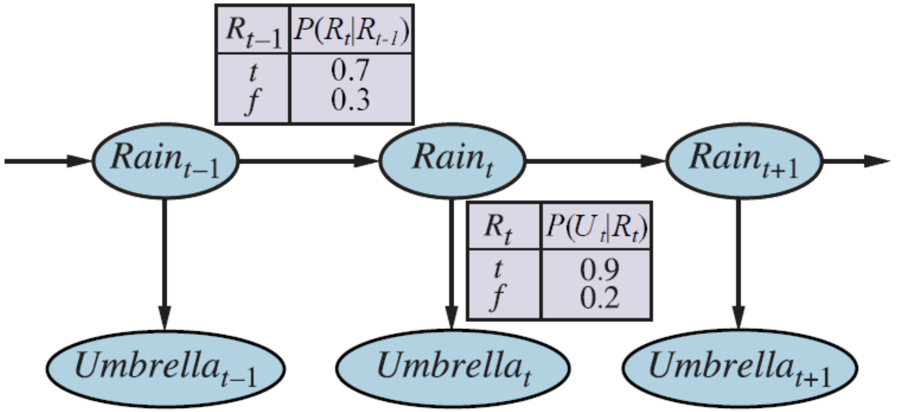

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as computed earlier

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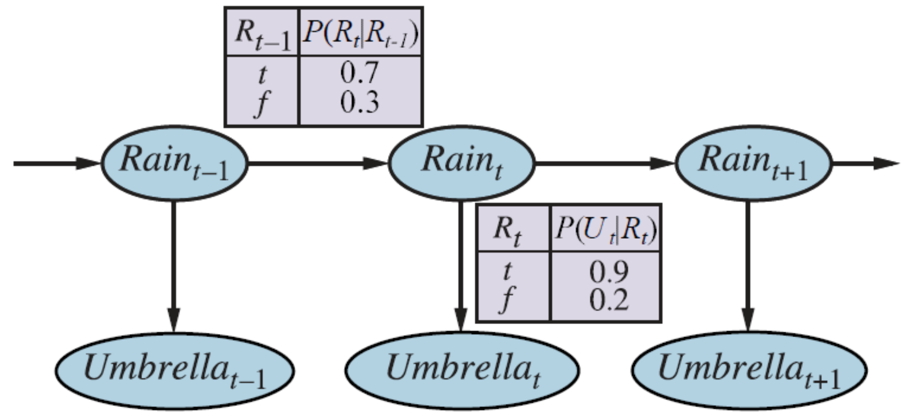

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$$= (0.9 \cdot 1 \cdot \langle 0.7, 0.3 \rangle) + (0.2 \cdot 1 \cdot \langle 0.3, 0.7 \rangle)$$

$$= \langle 0.69, 0.41 \rangle$$

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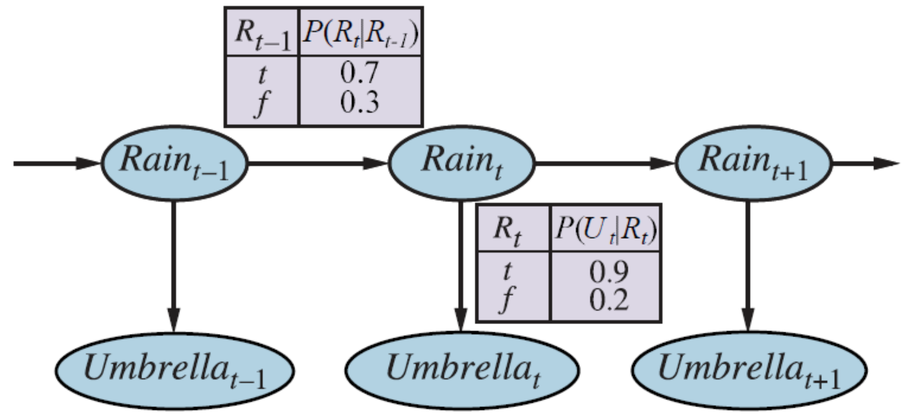

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$$\approx \alpha \langle 0.818, 0.182 \rangle \times \langle 0.69, 0.41 \rangle$$

$$\approx \langle 0.883, 0.117 \rangle$$

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