Example 9: Flip a coin 3 times. Let $X = \#$ of heads obtained in 3 flips. The probability mass function (pmf) of X is

x	0	1	2	3
$p_X(x)$	1/8	3/8	3/8	1/8

Calculate the variance and standard deviation of X.

use shortcut formula $Var(X) = E(X^2) - (E(X))^2$

- $E(X) = \sum_x x p_X(x) = 1.5$ (example 8)
- $E(X^2) = \sum_x x^2 p_X(x) = (0)^2(1/8) + (1)^2(3/8) + (2)^2(3/8) + (3)^2(1/8) = 3$

dont forget to square

- $Var(X) = E(X^2) - [E(X)]^2 = 3 - (1.5)^2 = 0.75$
- $\sigma = \sqrt{Var(X)} = \sqrt{0.75} = 0.866$ (Standard Deviation)

Example 2: Consider random variable X and Y where $Y = X^2$

Y \ X	-1	0	1	
0	0.0	0.6	0.0	0.6
1	0.2	0.0	0.2	0.4
	0.2	0.6	0.2	

Are X and Y independent?

$\boxed{\text{Get the Marginal Probabilities}}$

x	-1	0	1
$P_X(x)$	0.2	0.6	0.2

y	0	1
$P_Y(y)$	0.6	0.4

$\boxed{\text{Calculate Covariance}}$ $Cov(X,Y) = E(XY) - E(X)E(Y)$

$E(X) = \sum_x x P_X(x) = (-1)(0.2) + (0)(0.6) + (1)(0.2) = 0$

$E(Y) = \sum_y y P_Y(y) = (0)(0.6) + (1)(0.4) = 0.4$

$E(XY) = \sum_{x,y} x \cdot y \cdot P_{X,Y}(x,y) = \cancel{(-1)(0)(0)} + \cancel{(0)(0)(0.6)} + \cancel{(1)(0)(0)}$
$+ (-1)(1)(0.2) + \cancel{(0)(1)(0)} + (1)(1)(0.2)$
$= -0.2 + 0.2 = 0$

$\rightarrow Cov(X,Y) = E(XY) - E(X)E(Y) = 0 - (0)(0.4) = 0$

Check whether $P_{X,Y}(x,y) = P_X(x) P_Y(y)$ for every (x,y) pair

$P_{X,Y}(-1,0) = 0 \neq (0.2)(0.6) = P_X(-1)P_Y(0)$

Hence X & Y are dependent (not independent) even though $Cov(X,Y) = 0$.

15 / 18

A quality control engineer tests the quality of produced computers in a shipment of 6 computers. Suppose that 5% of computers have defects, and defects occur independently of each other.

(a) Find the probability of exactly 2 defective computers in the shipment.

(b) Find the probability of at most 2 defective computers in the shipment.

(a) $P(X=2) = \binom{6}{2}(0.05)^2(0.95)^{6-2}$
$15*(0.05)^2(0.95)^4 = .0305$

(b) $P(X \le 2) = P(X=0) + P(X=1) + P(X=2) \Rightarrow$
$\binom{6}{0}(0.05)^0(0.95)^{6-0} + \binom{6}{1}(0.05)^1(0.95)^{6-1} + \binom{6}{2}(0.05)^2(0.95)^{6-2} =$
$0.735 + 0.232 + 0.031 = 0.998$

Before a computer is assembled, its motherboard goes through a special inspection. Assume only 85% of motherboards pass this inspection.

(a) What is the probability that at least 13 of the next 15 motherboards pass inspection?

(b) On the average, how many motherboards should be inspected until a motherboard that passes inspection is found?

Answer:

(a) Let X be the number of motherboards that pass the inspection. It is the number of successes in 15 Bernoulli trials, thus it has Binomial distribution with n = 15 and p = 0.85.

$$\begin{aligned}\mathbb{P}(X \ge 13) &= \mathbb{P}(X=13) + \mathbb{P}(X=14) + \mathbb{P}(X=15)\\ &= \binom{15}{13}.85^{13}(.15)^2 + \binom{15}{14}.85^{14}(.15)^1 + \binom{15}{15}.85^{15}(.15)^0\\ &= 0.6042\end{aligned}$$

(b) Let Y be the number of motherboards that should be inspected until a motherboard that passes inspection is found. It is the number of trials needed to see the first success, thus it has Geometric distribution with p = 0.85.
Thus, we want $\mathbb{E}(Y) = \frac{1}{p} = \frac{1}{.85} \approx 1.18$

An insurance company divides its customers into 2 groups. Twenty percent are in the high-risk group, and eighty percent are in the low-risk group. The high-risk customers make an average of 1 accident per year while the low-risk customers make an average of 0.1 accidents per year. Eric had no accidents last year. What is the probability that he is a high-risk driver?

Answer:

3.29 Denote the events: H = {high risk}, L = {low risk}, N = {no accidents}. The number of accidents is the number of "rare events", discrete, ranging from 0 to infinity, thus it has a Poisson distribution. We have:

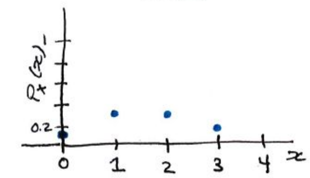
$$P\{H\} = 0.2,\ P\{L\} = 0.8,\ P\{N|H\} = 0.368,\ P\{N|L\} = 0.905$$

(from Table A3, with $\lambda = 1$ and $\lambda = 0.1$).

$$P\{H|N\} = \frac{P\{N|H\}P\{H\}}{P\{N|H\}P\{H\} + P\{N|L\}P\{L\}} = \frac{(0.368)(0.2)}{(0.368)(0.2) + (0.905)(0.8)} = \boxed{0.0923}$$

Example 6: Suppose you toss 3 coins, and observe the face up for each flip. We are interested in the number of heads we obtain in 3 coin tosses.

PMF

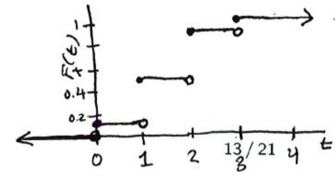


From Example 4, the pmf is

x	0	1	2	3
(pmf) $p_X(x)$	1/8	3/8	3/8	1/8
(cdf) $F_X(x)$	1/8	4/8	7/8	1

What is the cdf of X?

CDF



Operations with $E(\cdot)$

- $E(aX) = aE(X)$
- $E(aX + b) = aE(X) + b$
- $E(aX + bY) = aE(X) + bE(Y)$

Operations with $Var(\cdot)$

- $Var(aX) = a^2 Var(X)$
- $Var(aX + b) = a^2 Var(X)$
- $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + \boxed{2abCov(X,Y)}$

(when X,Y are independent, Cov(X,Y) = 0. We'll discuss more about independence and define covariance later)

· square constants when you pull it out of variance
· ignore any constants that are not attached to a R.V

A box contains seven marbles. Four of them are red and three of them are green. You reach in and choose three at random without replacement. Define a random variable X as: X = the number of red marbles selected.

(a) What are the possible values X can take on? (i.e. give $Im(X)$)
Answer: $Im(X) = \{0, 1, 2, 3\}$

(b) Find $\mathbb{P}(X = x)$ for all x in $Im(X)$.
Answer:

$\mathbb{P}(X=0) = \frac{\binom{4}{0}\cdot\binom{3}{3}}{\binom{7}{3}} = \frac{1}{35}$

$\mathbb{P}(X=1) = \frac{\binom{4}{1}\cdot\binom{3}{2}}{\binom{7}{3}} = \frac{12}{35}$

$\mathbb{P}(X=2) = \frac{\binom{4}{2}\cdot\binom{3}{1}}{\binom{7}{3}} = \frac{18}{35}$

$\mathbb{P}(X=3) = \frac{\binom{4}{3}\cdot\binom{3}{0}}{\binom{7}{3}} = \frac{4}{35}$

Two variables are independent if, for all values of X and Y:
P(x | y) = P(x)
P(x ∩ y) = P(x)*P(y)

1. * Consider the following joint distribution for the weather in two consecutive days. Let X and Y be the random variables for the weather in the first and the second days, with the weather coded as 0 for sunny, 1 for cloudy, and 2 for rainy.

X \ Y	0	1	2
0	0.3	0.1	0.1
1	0.2	0.1	0
2	0.1	0.1	0

(a) Find the marginal probability mass functions for X and Y.
(b) Calculate the expectation and variance for X and Y.
(c) Calculate the covariance and correlation between X and Y. Are they correlated?
(d) Are the weather in two consecutive days independent?

Answer:

(a) The marginal distributions for X and Y are

x	0	1	2
$p_X(x)$	0.5	0.3	0.2

y	0	1	2
$p_Y(y)$	0.6	0.3	0.1

(b) The expectation and variance are

$$\begin{aligned}E(X) &= (0)(0.5) + (1)(0.3) + (2)(0.2) = 0.7\\ E(X^2) &= (0)^2(0.5) + (1)^2(0.3) + (2)^2(0.2) = 1.1\\ Var(X) &= E(X^2) - [E(X)^2] = 1.1 - 0.7^2 = 0.61\\ E(Y) &= (0)(0.6) + (1)(0.3) + (2)(0.1) = 0.5\\ E(Y^2) &= (0)^2(0.6) + (1)^2(0.3) + (2)^2(0.1) = 0.7\\ Var(Y) &= E(Y^2) - [E(Y)]^2 = 0.7 - 0.5^2 = 0.45\end{aligned}$$

(c) We have:

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$\begin{aligned}E(XY) &= (0)(0)(0.3) + (0)(1)(0.5) + (0)(2)(0.1)\\ &+ (1)(0)(0.2) + (1)(1)(0.1) + (1)(2)(0)\\ &+ (2)(0)(0.1) + (2)(1)(0.1) + (2)(2)(0)\\ &= 0.3\end{aligned}$$

$$Cov(X,Y) = .3 - (.7)(.5) = -0.05$$

$$\rho = Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{-0.05}{\sqrt{(0.61)(0.45)}} = -0.095$$

(d) X and Y are not independent since $Cov(X,Y) = -0.05 \neq 0$,
or
X and Y are not independent since

$$0 = \mathbb{P}(X=2, Y=2) \neq \mathbb{P}(X=2)\mathbb{P}(Y=2) = 0.2 \times 0.1 = 0.02$$

Bernoulli Dist: X ~ Bern (p) X = “Outcome of a single trial”	Geometric Dist: X ~ Geo (p) X = “# of trials until first success”
Binomial Dist: X ~ Bin (n, p) X = “# of successes in n trials”	Poisson Dist: X ~ Pois (λ) X = “# of events during an interval”

Example: Flip Coin
S=Heads, F=Tails, P=.5
Bernoulli: Flip once, record heads.
Binomial: Flip 10 times, record heads.

Binomial: A box contains 15 components that each have a defective rate of 5%.

1. What is the probability that exactly 2 out of 15 components are defective?

$$P(X=2) = P_X(2) = \binom{15}{2}(0.05)^2(0.95)^{15-2} = \binom{15}{2}(0.05)^2(0.95)^{13} = \frac{15!}{13!\,2!}(0.05)^2(0.95)^{13} = (105)(0.05)^2(0.95)^{13} = 0.1348$$

2. What is the probability that at most 2 components are defective?

(using PMF) $P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$
$= P_X(0) + P_X(1) + P_X(2)$
$= \binom{15}{0}(0.05)^0(0.95)^{15} + \binom{15}{1}(0.05)^1(0.95)^{15-1} + \binom{15}{2}(0.05)^2(0.95)^{15-2}$
$= 0.9638$

(using CDF) $P(X \le 2) = F_X(2) = 0.9638$ (using Appendix A Binomial CDF table)

3. What is the probability that more than 3 components are defective? $P(X>3) = ? = 1 - P(X \le 3)$

$P(X \le 3) = P_X(0) + P_X(1) + P_X(2) + P_X(3)$

using PMF $= \binom{15}{0}(0.05)^0(0.95)^{15} + \binom{15}{1}(0.05)^1(0.95)^{14} + \binom{15}{2}(0.05)^2(0.95)^{13} + \binom{15}{3}(0.05)^3(0.95)^{12}$
$= 0.9945$

→ $P(X>3) = 1 - P(X \le 3) = 1 - 0.9945 = 0.0055$

using CDF $P(X \le 3) = F_X(3) = 0.9945$ (Appendix A Binom. Table)

→ $P(X>3) = 1 - P(X \le 3) = 1 - 0.9945 = 0.0055$

4. What is the probability that more than 1 but less than 4 components are defective?

using PMF $P(1 < X < 4) = P(X=2) + P(X=3)$
$= P_X(2) + P_X(3)$
$= \binom{15}{2}0.05^2\,0.95^{13} + \binom{15}{3}0.05^3\,0.95^{12}$
$= 0.1655$

using CDF $P(1 < X < 4) = P(X<4) - P(X \le 1)$
$= P(X \le 3) - P(X \le 1)$
$= F_X(3) - F_X(1)$
$= 0.9945 - 0.8290$
$= 0.1655$

To find CDF, we need probabilities with "≤" sign
CDF table only given $P(X \le t)$

18/18

Geometric Example: Flip an unfair coin until we get our first head. P(Head) = 0.3

F F S

1. What is the probability that the first head occurs on the third flip?

$$P(Y=3) = (1-0.3)^{3-1}(0.3) = 0.7^2 \cdot 0.3 = 0.147$$

2. What is the probability that we get the first head before the third flip?

(using PMF) $P(Y<3) = P(Y \le 2) = P_Y(1) + P_Y(2)$
$= P(Y=1) + P(Y=2)$
$= 0.3 + (0.7)(0.3) = 0.51$

(using CDF) $P(Y<3) = P(Y \le 2) = F_X(2) = 1-(1-0.3)^2$
$= 1 - 0.7^2$
$= 0.51$

6/14

3. What is the probability that we have to flip the coin at least 3 times, but at most 7 times?

$P(3 \le Y \le 7) = P_Y(3) + P_Y(4) + P_Y(5) + P_Y(6) + P_Y(7)$
$= 0.7^2 0.3 + 0.7^3 0.3 + 0.7^4 0.3 + \cdots + 0.7^6 0.3$
$= 0.4076$

$P(3 \le Y \le 7) = P(Y \le 7) - P(Y<3)$
$= P(Y \le 7) - P(Y \le 2)$
$= F_Y(7) - F_Y(2)$
$= [1-(1-0.3)^7] - [1-(1-0.3)^2] = 0.9176 - 0.51$
$= 0.4076$

4. What is the expected value?

$$EY = \frac{1}{P} = \frac{1}{0.3} = 3.33$$

5. What is the variance?

$$\mathrm{Var}\,Y = \frac{1-P}{P^2} = \frac{1-0.3}{0.3^2} = 7.78$$

7/14

Poisson

Suppose the number of customers entering West Street Deli can be modeled using a Poisson distribution. Customers enter the deli at an average rate of 10 customers every 15 minutes during the lunch rush.

$\lambda = 10$ interval

Between 12pm and 12:15pm today, what is the probability that

1. What is the probability that exactly 3 customers enter?

$$P(X=3) = \frac{e^{-10}(10)^3}{3!} = 0.00757$$

2. What is the probability that at most 3 customers enter

(using PMF) $P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$
$= P_X(0) + P_X(1) + P_X(2) + P_X(3)$
$= \frac{e^{-10}10^0}{0!} + \frac{e^{-10}10^1}{1!} + \frac{e^{-10}10^2}{2!} + \frac{e^{-10}10^3}{3!}$
$= 0.0103$

$P(X \le 3) = F_X(3) = 0.0103$ (Appendix A Poisson Table)

11/14

3. What is the probability that at least 4 customers enter?

$$P(X \ge 4) = 1 - P(X<4)$$

4. What is the probability that between 8 and 10 customers enter (inclusive)

$P(8 \le X \le 10) = P(X=8) + P(X=9) + P(X=10)$
$= \frac{e^{-10}10^8}{8!} + \frac{e^{-10}10^9}{9!} + \frac{e^{-10}10^{10}}{10!} = 0.3628$

$P(8 \le X \le 10) = P(X \le 10) - P(X<8)$
$= P(X \le 10) - P(X \le 7)$
$= F_X(10) - F_X(7)$
$= 0.5830 - 0.2202 = 0.3628$

5. What is the expected value of the random variable?

$EX = \lambda = 10$

6. What is the variance of the random variable?

$\mathrm{Var}\,X = \lambda = 10$