Maximum Likelihood Estimation

Suppose $X_1 \overset{iid}{\sim} Exp(\lambda)$ for $i = 1, \cdots, n$.

1. Find the maximum likelihood estimator for λ
2. If we observe data $x = 2, 4, 7, 10$, give the value for the maximum likelihood estimate.

Answer:

1. Maximum Likelihood
   The Likelihood function is

$$L(\lambda) = f(x_1, \cdots, x_n) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \sum x_i}$$

   The log-Likelihood functions is

$$\ell(\lambda) = logL(\lambda) = log\left(\lambda^n e^{-\lambda \sum x_i}\right) = nlog(\lambda) - \lambda \sum_{i=1}^{n} x_i$$

   We maximize the above log-Likelihood function by setting it's first derivative with respect to λ equal to 0, and solving for λ.

$$\frac{d}{d\lambda}\ell(\lambda) = \frac{d}{d\lambda}\left[nlog(\lambda) - \lambda \sum_{i=1}^{n} x_i\right] = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i$$

   Setting $\frac{d}{d\lambda}\ell(\lambda)$ equal to 0, we get,

$$\begin{aligned}
\frac{d}{d\lambda}\ell(\lambda) = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i &\overset{set}{=} 0 \\
\implies \frac{n}{\lambda} &= \sum_{i=1}^{n} x_i \\
\implies \lambda &= \frac{n}{\sum_{i=1}^{n} x_i} \\
\implies \hat{\lambda}_{MLE} &= \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}
\end{aligned}$$

   2^{nd} derivative test to check if we have maximum:

$$\frac{d^2}{d\lambda^2}\ell(\lambda) = \frac{d}{d\lambda}\left[\frac{d}{d\lambda}\ell(\lambda)\right] = \frac{d}{d\lambda}\left[\frac{n}{\lambda} - \sum_{i=1}^{n} x_i\right] = \frac{-n}{\lambda^2} < 0$$

   So, we have a maximum at $\hat{\lambda}_{MLE}$.
   Maximum likelihood estimator is $\hat{\lambda}_{MLE} = \frac{1}{\bar{X}}$

2. Plugging our data into the estimator,
   The maximum likelihood estimate for λ is $\hat{\lambda}_{MLE} = \frac{1}{\bar{x}} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{4}{2+4+7+10} = 0.1739$