4.1 Higher Order DEs- Linear Equations

Homogeneous Equations. Recall a linear n^{th} order DE has general form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0 y = g(x) \qquad (*)$$

We say this equation is homogeneous if $g(x) \equiv 0$, and non-homogeneous when $g(x) \not\equiv 0$.

Differential Operators (Notation). Define $D := \frac{d}{dx}$

- $D(y) = dy/dx = y'$
- $D^2(y) = D(D(y)) = y''$
- $(D^2 + D)(y) = D^2y + Dy = y'' + y'$
- $(\sin x D^2 + x^2 + 1)y = (\sin x)\,y'' + (x^2+1)y$
- $(xD - \sin x)(\ln x) = x(\ln x)' - (\sin x)(\ln x) = x(\frac{1}{x}) - \sin x \ln x = 1 - (\sin x)(\ln x)$

We can use then a more compact notation (using operator notation) to write differential equations. For instance we can write equation $(*)$ as:

$$[a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)](y) = g(x)$$

More over we can define/call the left hand side L, thus the equation can be abbreviated as

$$L(y) = g(x).$$

Example

The equation $y'' + 5y' + 6y = 5x - 3$ can be written $L(y) = 5x - 3$

where $L = D^2 + 5D + 6$

Since differentiation is linear, so is L (a linear combination of differential operators). That is $L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$

(α, β = constants
f, g = functions of x.)

Theorem (The Superposition Principle)

Let $y_1, y_2, \ldots, y_k$ be solutions of the homogeneous equation $(*)$ on an interval I. Then the linear combination

$$y = c_1y_1 + c_2y_2 + \cdots + c_ky_k$$

where $c_1, c_2, \ldots, c_k$ are arbitrary constants, is also a solution on I.

Proof.

We show only second order case with $k = 2$. (order also 2).

$a_2(x)y'' + a_1(x)y' + a_0(x) = 0$. Let $L = a_2(x)D^2 + a_1(x)D + a_0(x)$

then $\Leftrightarrow L(y) = 0$. We assume y_1 and y_2 are solutions, that mean $L(y_1) = 0$ and $L(y_2) = 0$. Consider $c_1y_1 + c_2y_2$ and evaluate $L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2) = c_1 \cdot 0 + c_2 \cdot 0 = 0$

$\therefore c_1y_1 + c_2y_2$ is a solution.

Definition

A set of functions $y_1, y_2, \ldots, y_n$ is said to be linearly dependent on an interval I if there exist constants $c_1, c_2, \ldots, c_n$ (not all zero) such that:

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0 \quad \text{for all } x \text{ on } I.$$

In other words we can write any y_i in terms of the other y_j's

If no such constants exist, then $y_1, y_2, \ldots, y_n$ are linearly independent

*FACT: An n^{th} order linear DE has exactly n linearly independent (l.i.) solutions.

Example: Let $y_1 = x$ and $y_2 = x^2$, are linearly independent because $c_1x + c_2x^2 = 0$ if and only if $c_1 = c_2 = 0$.

$c_1x + c_2x^2 = 0 \cdot x + 0 \cdot x^2 \Leftrightarrow c_1 = c_2 = 0$

Example: (# 5 pg. 123) Let $y_1 = \sin^2 x$, $y_2 = \cos^2 x$, $y_3 = \tan^2 x$ and $y_4 = \sec^2 x$. These are linearly dependent because we can write $c_1(\sin^2 x) + c_2(\cos^2 x) + c_3(\tan^2 x) + c_4(\sec^2 x) = 0$, with

$c_1 = c_2 = c_3 = 1$ & $c_4 = -1 \Rightarrow \underbrace{\sin^2 x + \cos^2 x}_{1} + \underbrace{\tan^2 x - \sec^2 x}_{=-1} = 1 - 1 = 0$

Definition

The Wronskian of $y_1, y_2, \ldots, y_n$ is defined as

$$W(y_1, \ldots, y_n) = det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

Recall $\det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_2 y_1'$

Theorem

Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous n^{th} order DE $(*)$ on I $(g(x) \equiv 0)$. Then the set of solutions is linearly independent if and only if $W(y_1, \ldots, y_n) \neq 0$ for all x on I.

Examples

1) (Ex7. p126) The equation $y'' - 9y = 0$ has solutions $y_1 = e^{3x}, y_2 = e^{-3x}$

$$W(y_1, y_2) = \det\begin{bmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{bmatrix} = -3\underbrace{e^{3x}e^{-3x}}_{=1} - 3\underbrace{e^{-3x}e^{3x}}_{=1} = -6 \neq 0 \text{ for all } x \text{ on } I$$

$(I = (-\infty, \infty)$ in this problem$)$. $\therefore y_1$ & y_2 are l.i.

2) The equation $y'' + 9y = 0$ has solution $y_1 = \sin(3x)$ and $y_2 = \cos(3x)$ $(I = (-\infty, \infty))$.

$$W(y_1, y_2) = \det\begin{bmatrix} \sin 3x & \cos 3x \\ 3\cos 3x & -3\sin 3x \end{bmatrix} = -3\sin^2(3x) - 3\cos^2(3x) = -3(\sin^2(3x) + \cos^2(3x))$$

$= -3 \neq 0$ for all x on I.

$\therefore y_1 = \sin 3x$ & $y_2 = \cos 3x$ are l.i.

Definition

Any set $y_1, y_2, \ldots, y_n$ of l.i. solutions of the equation $(*)$ (homogeneous case, $g = 0$) is called a fundamental set (of solutions).

Theorem

Let $\{y_1, y_2, \ldots, y_n\}$ be a fundamental set for the homogeneous equation $(*)$, then its general solution is

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

where the c_is are arbitrary constants.

Theorem

Let y_p be any particular solution of the non-homogeneous equation $(*)$ on I. And let $\{y_1, y_2, \ldots, y_n\}$ be a fundamental set to the associated homogeneous equation. Then the general solution to $(*)$ is:

$$y = \underbrace{c_1 y_1 + c_2 y_2 + \cdots + c_n y_n}_{y_c} + y_p$$

where the c_is are arbitrary constants. We call y_c, complementary function.

Example: To solve the non-homogeneous equation $y'' + 9y = 27$, we need the solutions y_1, y_2 of $y'' + 9y = 0$. In this case $y_1 = \sin 3x, y_2 = \cos 3x$. Also note that a particular solution is: $y_p = 3$

$y_c = C_1 \sin 3x + C_2 \cos 3x$

$\Rightarrow$ General Sol is $y = C_1 \sin 3x + C_2 \cos 3x + 3$.

In general for $ay'' + by' + cy = g(x)$, to find a particular solution y_p by inspection, it is recommendable to look at functions resembling $g(x)$.

Theorem (Superposition principle (Non-homogeneous equations))

Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be k particular solutions (respectively) to

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y1 + a_0 y = g_i(x), \quad i = 1, \ldots, k$$

Then $y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$ is a solution to:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y1 + a_0 y = g_1(x) + g_2(x) + \cdots + g_k(x)$$