Exam 2 - Wed, March 11 (regular class/time)
Bring calculator, 1 page (front & back) note sheet
Coverage: Discrete Dist - General Continuous Dist (Lecture 5-9)
Practice Exam Posted
In-class Review on Monday.

Lecture 12

Gamma Distribution

STAT 330 - Iowa State University

Gamma Distribution

Gamma Distribution

Setup: The gamma distribution is commonly used to model the total time for a procedure composed of α independent occurrences, where the time between each occurrence follows $Exp(\lambda)$

$$\underbrace{\boxed{Exp(\lambda)}\boxed{Exp(\lambda)}\boxed{Exp(\lambda)}\boxed{Exp(\lambda)}}_{\text{Gamma}}$$

If a random variable follows a *Gamma distribution*, alpha α

lambda λ (rate)

$$X \sim Gamma(\alpha, \lambda)$$

where $\lambda > 0$ is there rate parameter, and $\alpha > 0$ is the shape parameter

Don't need to know. We won't use this directly.

- Probability Density Function (pdf)
  - $Im(X) = (0, \infty)$
  - $f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$

  where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$ is called the *"gamma function"*.

Gamma PDF

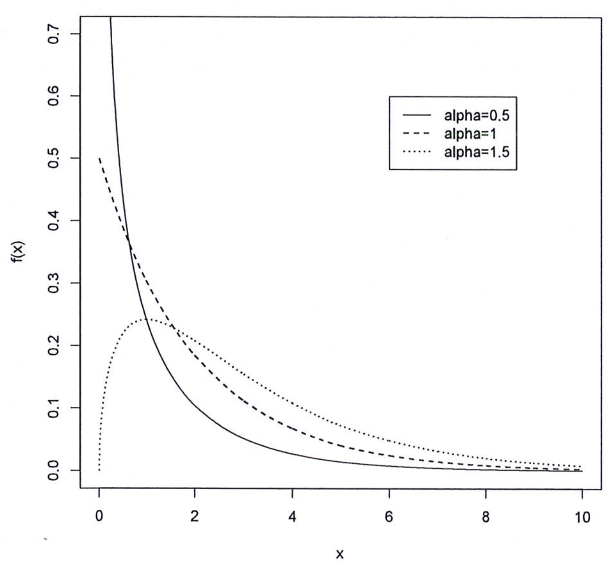

Figure 1: PDFs for gamma distribution with fixed λ and $\alpha = 0.5, 1, 1.5$

Gamma Distribution Summary

hard to integrate

- Cumulative distribution function (cdf)

Won't deal with this directly

$$F_X(t) = \int_0^t f(x)dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^t x^{\alpha-1} e^{-\lambda x} dx$$

- Expected Value: $E(X) = \frac{\alpha}{\lambda}$
- Variance: $Var(X) = \frac{\alpha}{\lambda^2}$

* Won't use the Gamma PDF/CDF directly to calculate probabilities

* Instead we will use a trick to turn a Gamma R.V into a different Poisson R.V, and then calculate probabilities using Poisson CDF.

Examples

Gamma Distribution Example

Example 1: Compilation of a computer program consists of 3 blocks that are processed sequentially, one after the other. Each block is independent of the other blocks, and takes Exponential time with mean of 5 minutes. We are interested in the total compilation time.

$\underbrace{\boxed{Exp(1/5)}}_{T_1}\underbrace{\boxed{Exp(1/5)}}_{T_2}\underbrace{\boxed{Exp(1/5)}}_{T_3}$ $\quad T = \sum_{i=1}^{3} T_i$

- Total compilation time modeled using Gamma distribution.

  Define the R.V: T = total compilation time

  Distribution of T: $T \sim Gamma(\alpha, \lambda) \equiv Gamma(?, ?)$

- What value should we use for α and λ?
  - α is the number of independent occurrences (blocks) in the full procedure: $\alpha =$ 3
  - Time for each occurrence (call this "T_i") is exponential with mean 5 min. $E(T_i) = \frac{1}{\lambda} = 5 \rightarrow \lambda =$ 1/5

$T \sim Gamma(\alpha = 3, \lambda = 1/5)$

Gamma Distribution Example

T = total compilation time

$$T \sim Gamma(3, \frac{1}{5})$$

$\alpha = 3$ $\quad \lambda = \frac{1}{5}$

1. What is the expected value of total compilation time?

   $E(T) = \frac{\alpha}{\lambda} = \frac{3}{(1/5)} = 15$ min

2. What is the variance of total compilation time?

   $Var(T) = \frac{\alpha}{\lambda^2} = \frac{3}{(1/5)^2} = 75$ min^2

3. What is the probability for the entire program to be compiled in less than 12 minutes.

$$P(T<12) = \int_{-\infty}^{12} f(x)dx = \int_{-\infty}^{12} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx$$

$$= \int_{-\infty}^{12} \frac{(1/5)^3}{\Gamma(3)} x^{3-1} e^{-x/5} dx$$

need to do repeated integration by parts

$= \ldots$

$=$ Grooss

$=$

Poisson Approximation to Gamma Distribution

Gamma Distribution Example

- Could answer the previous question by using the Gamma CDF directly (requires repeated integration by parts)
- Instead, simplify Gamma probabilities by turning it into a Poisson problem!
- Turn a Gamma random variable into Poisson random variable using the Gamma-Poisson formula.

short cut

define a new R.V

Gamma-Poisson Formula

For $T \sim \text{Gamma}(\alpha, \lambda)$ and $X \sim \text{Pois}(\underbrace{\lambda t}_{\text{parameter for Poisson}})$,

$$① \quad P(T > t) = P(X < \alpha)$$

and

$$② \quad P(T \leq t) = P(X \geq \alpha)$$

Gamma Distribution Example

3. What is the probability that total compilation is under 12 min? $P(T<12)$

- Step 1: Define our Gamma random variable:
  $T \sim Gamma(\alpha, \lambda) \equiv Gamma(\overset{\alpha}{3}, \overset{\lambda}{\tfrac{1}{5}})$
  We want to know $P(T < t) = P(T < \overset{t=12}{12}) = ?$
- Step 2: Convert the Gamma R.V (T) into a Poisson R.V (X):
  $X \sim Pois(\lambda t) \equiv Pois(\frac{1}{5} \cdot 12) \equiv Pois(\underbrace{2.4}_{\text{parameter for Poisson Distribution}})$
- Step 3: Use Gamma-Poisson formula: $P(T \leq t) = P(X \geq \alpha)$

$$
\begin{aligned}
P(T < 12) = P(T \leq 12) &= P(X \geq 3) \quad \leftarrow \text{Gamma-Poisson Formula} \\
&= 1 - P(X < 3) \\
&= 1 - P(X \leq 2) \quad \leftarrow \text{b/c } X \text{ is a (discrete) Poisson R.V} \\
&= 1 - F_X(2) \\
&= 1 - 0.5697 = 0.4303
\end{aligned}
$$

↑ b/c T is a (continuous) Gamma R.V

or $F_X(2) = p_X(0) + p_X(1) + p_X(2) = P(X=0) + P(X=1) + P(X=2)$

Gamma Distribution Example

4. What is the probability that it takes at least 5 minutes to compile the entire program? $P(T \geq 5)$

STEP 1: Define my Gamma R.V
$T \sim Gamma(\alpha = 3, \lambda = 1/5)$. Want $P(T \geq \overset{t=5}{5})$

STEP 2: Convert Gamma R.V(T) into Poisson R.V (X)
$X \sim Pois(\lambda \cdot t) \equiv Pois(\frac{1}{5} \cdot 5) \equiv Pois(1)$

STEP 3: Use Gamma-Poisson Formula: $P(T > t) = P(X < \alpha)$

$$
\begin{aligned}
P(T \geq 5) &= P(T > 5) \quad (T \text{ is continuous R.V}) \\
&= P(X < \alpha) \\
&= P(X < 3) \quad \leftarrow \text{Gamma-Poiss Formula} \\
&= P(X \leq 2) \quad (X \text{ is Discrete (Poisson) R.V}) \\
&= F_X(2) \\
&= 0.9196
\end{aligned}
$$