Recitation 3 Solution

---

1. Prove the statements that are true, and give a counterexample to disprove those that are false:

   (a) The product of any two odd integers is odd.
   (b) The sum of any even and any odd integer is odd.
   (c) The difference of any two odd integers is odd.
   (d) The product of any even integer and any integer is even.

   Solution

   (a) Let there is two odd integers m, n such that $m = 2k - 1$ and $n = 2l - 1$ where k and l are integers.

   $$\begin{aligned} mn &= (2k-1)(2l-1) = 4kl - 2k - 2l + 1 \\ &= 2(2kl - k - l) + 1 \end{aligned}$$

   Since $(2kl - k - l)$ is an integer, we can conclude that any product of two odd integer is an odd integer.

   (b) Let there is an odd integer and even integer respect to m and n such that $m = 2k - 1$ and $n = 2l$ where k and l are integers.

   $$m + n = 2k - 1 + 2l = 2(k + l) - 1$$

   Since $(k + l)$ is an integer, we can conclude that the summation of odd and even number is an odd integer.

   (c) This statement is false because 3 and 1 are odd integers but their difference is $3 - 1 = 2$ which is even.

   (d) Let there is an even integer and any integer respect to m and n such that $m = 2k$ where k and n are integers.

   $$mn = 2kn$$

   Therefore, mn is an even integer.

2. Recall: a *rational number* is defined as a real number r that can be written as the ratio of two integers $\frac{p}{q}$ with q not equal to 0. An irrational number is defined as a real number that is not rational. Prove that the sum of a rational number with an irrational number is always irrational. Clearly state your method of proof in the beginning.

   Solution

   Suppose, to the contrary, that the sum of a rational number with an irrational number is always rational number. By assumption, we define the following terms m, n for rational numbers and x for irrational number.

We can express m and n in following ways: $m = \frac{a}{b}$ and $n = \frac{c}{d}$ where a, b, c, d are all integers, and $b \neq 0$ and $d \neq 0$.

According to the assumption, following equations needs to be satisfied:

$$\begin{aligned} m + x &= n \\ \frac{a}{b} + x &= \frac{c}{d} \\ x &= \frac{c}{d} - \frac{a}{b} \\ x &= \frac{bc - ad}{bd} \end{aligned}$$

Since $(bc - ad)$ and bd are both integers and $bd \neq 0$, x needs to be an rational number which contradicts our assumption.

Therefore, the sum of a rational number with an irrational number is always irrational due to the proof by contradiction.

3. Identify precisely the conceptual bug(s) in the following "proof" -

- **Claim**: $1/8 > 1/4$.
- **Proof**: The proof proceeds as follows.
  - We know that $3 > 2$;
  - therefore, $3 \log_{10}(1/2) > 2 \log_{10}(1/2)$;
  - therefore, using the properties of logarithms, $\log_{10}(1/2)^3 > \log_{10}(1/2)^2$;
  - therefore, $(1/2)^3 > (1/2)^2$;
  - therefore, $1/8 > 1/4$.

Solution

- We know that 3>2; **True**
- therefore, $3 \log_{10}(1/2) > 2 \log_{10}(1/2)$; **False** $\log_{10}(1/2) < 0$

if $a > b$, then $ac > bc$ is true **only if** $c > 0$ but here, $\log_{10}(1/2) < 0$ (from the property of the log. $\log_{10} x < 0$ if $x < 1$). Thus, this is conceptual bug in the proof.

- therefore, using the properties of logarithms, $\log_{10}(1/2)^3 > \log_{10}(1/2)^2$;
- therefore, $(1/2)^3 > (1/2)^2$;
- therefore, $1/8 > 1/4$.

4. Find a counterexample for each of the following statements:
   (a) If n is prime, then $2^n - 1$ is prime.
   (b) Every triangle has an obtuse angle.
   (c) For all real numbers x, $x^2 \geq x$.
   (d) For every nonprime positive integer n, if some prime p divides n then some other prime q (where $q \neq p$) also divides n.

Solution

(a) Counterexample: n = 11. 23 divides $2^n - 1 = 2047$
(b) Counterexample: Equilateral Triangle
(c) Counterexample: $x = \frac{1}{2}$
(d) Counterexample: n = 25. p = 5 divides n. However, there does not exist q to divide n.