Homework 5 Solutions

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1. **(10 points)** Suppose you want to write a program that collects information on a customer's tastes, and customize web content. By monitoring online shopping habits, you are able to collect pairwise preferences of a customer on some given set of products X. If x and y are two products in X, we say that $x \preceq y$ if the customer prefers y over x. (For simplicity, also assume by convention that $x \preceq x$ for all $x \in X$, and also that $\preceq$ is a transitive relation.) Suppose you know that your customer prefers:

   - lettuce over broccoli
   - cabbage over broccoli
   - tomatoes over cabbage
   - carrots over cabbage
   - carrots over lettuce
   - asparagus over lettuce
   - mushrooms over tomatoes
   - corn over tomatoes
   - corn over carrots
   - eggplant over carrots
   - eggplant over asparagus
   - onions over mushrooms
   - onions over corn

   Answer the following questions:

   a. Draw the Hasse diagram for $(X, \preceq)$.
   b. What is/are the customer's favorite vegetable(s)? (i.e., what is/are the maximal elements?) What is/are the least favorite vegetables?
   c. Use topological sorting to produce an ordering the vegetables according to the customer's preferences.
   d. Is this ordering unique?

   Solution

   (a)

   (b) The minimal elements are those appearing on the top of the graph, and they don't have any further connection. According to the diagram, Onions and Eggplant are customer's favorite. Also, we can see that the least favorite element would be the one lying at the bottom of the graph, which is broccoli in this case.

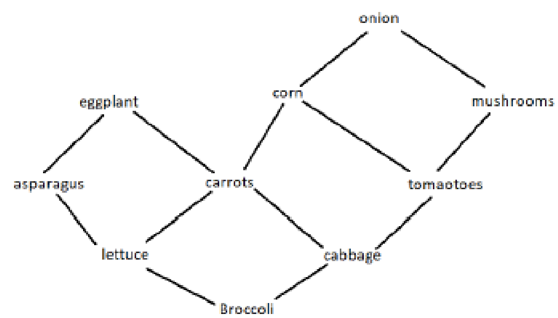

Figure 1: hasse diagram

(c) For topological ordering, we start with the minimal element, add it to our list, then remove it from the Hasse diagram. We will then repeat the same process with the remaining part of the diagram. We will proceed till all the elements are listed. One such ordering is : *broccoli, cabbage, lettuce, tomatoes, carrots, asparagus, mushroom, corn, eggplant, onions.*

(d) Here, at many points, the diagram is having more than one minimal element. In that case, we can chose any one of them, and proceed further. This suggests that the list is not unique. For example, after removing broccoli from the diagram and adding it up to the list, the remaining diagram hase 2 minimal elements (cabbage and lettuce). We can chose either of them as the next element, thus we will get at least two different kind of lists.

2. **(10 points)** The National Quidditch League contains 32 teams.

   a. Suppose each team plays every other team exactly once in a given season. Model the season as a relation R in terms of its graph representation. Clearly say what the nodes and edges represent in this graph.
   b. Calculate how many games in total need to be played.
   c. How would your answer to part (b) change if each team now plays every other team *twice* every season (one at home and one away)?
   d. Generalize your answers to (b) and (c) if the League contains n teams, where n is any integer greater than or equal to 2.

Solution

(a) Here, the relation aRb would mean that the team a played team b. The nodes of the graph would be the 32 teams, so there would be 32 nodes in total. An edge between any two nodes suggests that both these teams have played a game with each other. As each team plays every other team exactly once in the given season, any two nodes of the graph will have an edge. Thus, every node would have 31 edges corresponding to 31 other teams.

(b) We can select any two team out of 32 teams, and they would have exactly 1 match with each other, i.e. total number of matches = number of ways of selecting 2 teams out of 32 = $\binom{32}{2}$ = 496 games.

(c) If every pair of teams plays twice instead of once, the total number of games would double to 992 (each undirected edge on the graph would be replaced with two directed edges distinguishing home and away games).

(d) We can select any two team out of total n teams, and they would have exactly 1 match with each other, i.e. total number of matches = number of ways of selecting 2 teams out of n teams = $\binom{n}{2}$. Similarly, for part (b), the total number of matches will get doubled, i.e., $2 \times \binom{n}{2}$.

3. **(10 points)** Consider the set $A = \{1, 2, 3\}$.

   a. Enumerate all elements of its power set, pow(A).
   b. Clearly express a bijective function between pow(A) and the set of all possible 3-bit strings. (E.g. 000, 001, 010, ...are 3-bit strings.)
   c. Suppose we construct an (undirected) graph where nodes correspond to 3-bit strings, and an edge between nodes indicates that the two corresponding strings differ in exactly 1 location. Draw this graph, and clearly label each vertex.
   d. What is the degree of each node in this graph?

   Solution

   (a) pow(A) = {$\emptyset$, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}. {1,2,3}}

   (b) $\emptyset$ to 000; {1} to 001; {2} to 010; {3} to 011; {1,1} to 100; {1,2} to 101; {1,3} to 110; {1,2,3} to 111

   (c)

   (d) 3

4. **(10 points)** Recall that a *complete binary tree* is a binary tree where every non-leaf node has two children, while every leaf node has no children.

   a. Draw complete binary trees with 1, 2, 3, and 4 levels.
   b. If a complete binary tree has k internal (non-leaf) nodes, then verify that it has $k + 1$ leaf nodes (no need for a proof; you can just check your answers.)
   c. *By applying the First Degree theorem*, prove that a complete binary tree with k internal nodes has $2k + 1$ total nodes.

   Solution

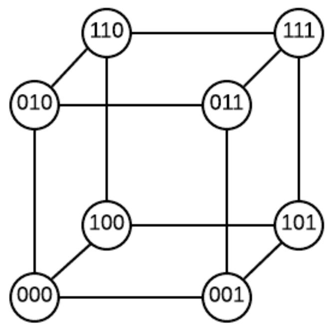

Figure 2: Problem 3c

(c) We know that a tree with n nodes will have n-1 edges.
Let the number of leaf nodes = m.
First degree theorem states that $\sum \deg|n| = 2\ (n\text{-}1)\ -(1)$
Since we have k internal nodes (which includes the root node), the total number of nodes in this tree, n = k + m
Since it is a complete graph, the root will have a degree of 2, the other internal nodes have a degree of 3 and the leaf nodes have a degree of 1.
So the total degree for n nodes = 2 + 3(k-1) + 1(m)
Plugging this back into eqn (1), we get:
$2 + 3(k\text{-}1) + m = 2(n - 1)$
$\therefore 2 + 3(k\text{-}1) + m = 2(k + m - 1)$
Solving the above, we get: $m = k + 1$
We have, $n = m + k = k + 1 + k = 2k + 1$
$\therefore$ the total number of nodes in a complete graph is equal to 2k + 1.

5. **(10 points)** Recall that an *Euler* path is one that traverses all edges of a (connected) graph without repeating any edge. Using properties of Euler paths, figure out whether which of the following figures can be drawn without lifting your pen from the paper and without repeating any line.

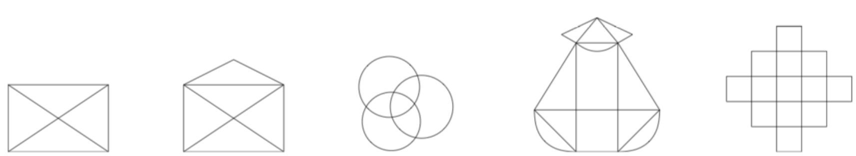

Figure 3: Can you draw these shapes without lifting your pen and repeating lines?

Solution Figures 2,3,4 and 5 can be drawn without lifting the pen. They have number of nodes with odd degree to be either 0 or 2.