4.3 Homogeneous Linear Equations with Constant Coefficients

General form: $a_n\, y^{(n)} + a_{n-1}\, y^{(n-1)} + \cdots + a_1\, y' + a_0\, y = 0$

We will first focus on the 2^{nd} order case: $a\, y'' + b\, y' + c\, y = 0$

Consider the following easy cases: $y' = y$, we can see that it has solution $y = e^x$, also $y'' = y'$ would have solution $y = e^x$; $y'' = y$ has sol! $y = e^{-x}$

How about, $y'' = -y' + 2y$..? Let's try $y = e^{mx}$ and plug in $y' = me^{mx}$

and $y'' = m^2 e^{mx}$: $m^2 e^{mx} = -me^{mx} + 2e^{mx}$

$\Leftrightarrow (m^2 + m - 2)e^{mx} \stackrel{\text{we want}}{=} 0$. Since $e^{mx} \neq 0$ for all x

$\Leftrightarrow m^2 + m - 2 = 0 \Leftrightarrow (m+2)(m-1) = 0$ So e^{-2x} and e^x

are 2 l.i. (can check with $W(e^{-2x}, e^x)$) So $\boxed{y = c_1 e^{-2x} + c_2 e^x}$ is the general solution.

In general the equation $a\, y'' + b\, y' + c\, y = 0$ has solutions of the form $y = e^{mx}$ for some values of m.

Plug in: $am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$

$(am^2 + bm + c)e^{mx} = 0$, since $e^{mx} \neq 0$ for all x

all we need is to solve $am^2 + bm + c = 0$

Definition

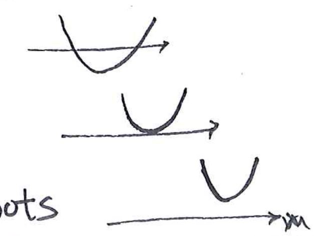
The equation $am^2 + bm + c = 0$ obtained above is called auxiliary equation to the DE $a\, y'' + b\, y' + c\, y = 0$.

To solve the DE $a\, y'' + b\, y' + c\, y = 0$, we need to solve its auxiliary equation and there will be three cases according to the discriminant $b^2 - 4ac$

Recall Quadratic Formula

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- $b^2 - 4ac > 0 \Rightarrow$ two distinct real roots
- $b^2 - 4ac = 0 \Rightarrow$ one repeated real root
- $b^2 - 4ac < 0 \Rightarrow$ two complex conjugate roots



- Case $b^2 - 4ac > 0$. We get two distinct real roots m_1 and m_2. So

$y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are two l.i. solutions, indeed

$$w(e^{m_1 x}, e^{m_2 x}) = \det \begin{bmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{bmatrix} = m_2 e^{m_3 x} - m_1 e^{m_3 x} \quad (m_3 = m_1 + m_2)$$

$= (m_2 - m_1) e^{m_3 x} \neq 0$ since $e^{m_3 x} \neq 0$ for all x and $m_1 \neq m_2$.

$\Rightarrow$ General Sol $\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$

- Case $b^2 - 4ac = 0$. We get one repeated real root m, thus only one solution $y_1 = e^{mx}$

Note that $m = -\frac{b}{2a}$ so $y_1 = e^{-\frac{b}{2a}x}$.

We'll use reduction of order to find a second l.i. solution

$$y_2 = y_1 \int \frac{e^{-\int P dx}}{y_1^2} dx. \quad \text{(see section 4.2)}$$

The equation in standard form $y'' + \frac{b}{a} y'' + \frac{c}{a} y = 0$

$-\int P dx = -\int \frac{b}{a} dx = -\frac{b}{a} x \Rightarrow e^{-\int P dx} = e^{-\frac{b}{a} x}$

and $y_1^2 = \left(e^{-\frac{b x}{2a}}\right)^2 = e^{-\frac{b}{a} x}$

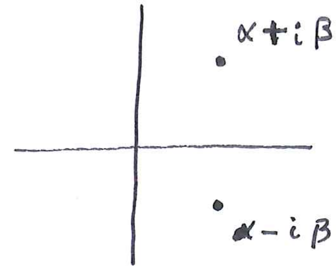
$\Rightarrow \int \frac{e^{-\int P dx}}{y_1^2} dx = \int \frac{e^{-\frac{b}{a} x}}{e^{-\frac{b}{a} x}} dx = \int dx = x$

$\Rightarrow y_2 = y_1 x = x e^{mx}$

$\therefore$ General Solution is $\boxed{y = c_1 e^{mx} + c_2 x e^{mx}}$

- Case $b^2 - 4ac < 0$. We get two conjugate complex roots:

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \underbrace{\frac{-b}{2a}}_{\alpha} \pm i \underbrace{\frac{\sqrt{|b^2-4ac|}}{2a}}_{\beta}$$



α is called the real part (α is real)
β is called the imaginary (β is real).

Recall two complex numbers $a_1 + i b_1 = a_2 + i b_2$ iff. $a_1 = a_2$ AND $b_1 = b_2$ *

$\Rightarrow$ $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ and $e^{(\alpha+i\beta)x}$ & $e^{(\alpha-i\beta)x}$ are complex solutions, but we want real solutions.

First we show that if $y = u(x) + i\, v(x)$ (with $u(x)$ & $v(x)$ real functions) is a solution to $ay'' + by' + cy = 0$ then also $u(x)$ & $v(x)$ are solutions.

We assume $a(u''(x) + i\, v''(x)) + b(u'(x) + i\, v'(x)) + c\,(u(x) + i\, v(x)) = 0$

5 / 9

$\Leftrightarrow$ $(au'' + bu' + cu) + i\,(av'' + bv' + cv) = 0 = 0 + i\,0$

(By *) $\Leftrightarrow$ $au'' + bu' + cu = 0$ AND $av'' + bv' + cv = 0$

that means both $u(x)$ and $v(x)$ are (real) solutions of $ay'' + by' + cy = 0$.

(So we'll see $y_1 = u(x)$ and $y_2 = v(x)$)

Our goal now is to write $e^{(\alpha+i\beta)x}$ in the form $u(x) + i\ v(x)$. For this we will need Euler's Formula:

$$e^{i\theta} = \cos\theta + i\ \sin\theta$$

$$e^{(\alpha+i\beta)x} = e^{\alpha x} e^{i\beta x} = e^{\alpha x}(\cos\beta x + i \sin\beta x)$$

$$= \underbrace{e^{\alpha x}\cos\beta x}_{u(x)} + i\ \underbrace{e^{\alpha x}\sin\beta x}_{v(x)}$$

So that $y_1 = e^{\alpha x}\cos\beta x$ and $y_2 = e^{\alpha x}\sin\beta x$, are 2 l.i. real sols.

Exercise Verify that $W(e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x) \neq 0$ for all x.

$\therefore$ General Solution $\boxed{y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x}$

Example

Find the solution to the IVP: $y'' + 16y = 0$, $y(0) = 2$, $y'(0) = -2$.

Aux Eqn. $m^2 + 16 = 0 \Rightarrow m = \pm 4i \Rightarrow \alpha = 0, \beta = 4$

$\therefore$ $y = c_1\cos 4x + c_2\sin 4x \Rightarrow y(0) = c_1 = 2$

$y' = -4c_1\sin 4x + 4c_2\cos 4x \qquad y'(0) = 4c_2 = -2 \Rightarrow c_2 = -\frac{1}{2}$

$y = 2\cos 4x - \frac{1}{2}\sin 4x.$

Higher Order Equations

The method works for higher order equations, the problem reduces to finding roots of a polynomial (auxiliary equation would be a higher degree algebraic equation).

Example. Find the general solution of

$$y^{(4)} + y''' + y'' = 0$$

$$m^4 + m^3 + m^2 = 0$$

$$m^2(m^2 + m + 1) = 0$$

$m^2 = 0$ repeated

$y_1 = e^{0x} = 1$

$y_2 = xe^{0x} = x$

$m = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ $\quad \alpha = -1/2$, $\beta = \sqrt{3}/2$

$y_3 = e^{-1/2 x}\cos(\sqrt{3}/2\, x)$

$y_4 = e^{-1/2 x}\sin(\sqrt{3}/2\, x)$

General Solution: $y = c_1 + c_2 x + c_3 e^{-1/2 x}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_4 e^{-1/2 x}\sin\left(\frac{\sqrt{3}}{2}x\right)$