

1. The discrete random variable X has a PMF described by the table below.

x	2	4	6	8	10
$p_X(x)$	0.2	0.25	0.05	0.3	0.2

- (a) What is the probability that X is between 5 and 9?

$$P(5 < X < 9) = P(X = 6) + P(X = 8) = .05 + .3 = .35$$

- (b) Given that X is at least 4, what is the probability that X is at least 8?

$$P(X \geq 8 | X \geq 4) = \frac{P(X \geq 8)}{P(X \geq 4)} = \frac{.3 + .2}{.25 + .05 + .3 + .2} = .625$$

- (c) Calculate the expected value and variance of X.

$$E(X) = 2*.2 + 4*.25 + 6*.05 + 8*.3 + 10*.2 = 6.1$$

$$\text{Var}(X) = E(X^2) - E(X)^2 = [2^2*.2 + 4^2*.25 + 6^2*.05 + 8^2*.3 + 10^2*.2] - [6.1^2] = 8.59$$

- (d) Let  $Z = 10X - 5$ . Find the PMF of Z.

z	15	35	55	75	95
$p_Z(z)$	0.2	0.25	0.05	0.3	0.2

2. Let Y be a discrete random variable with PMF described by the function below

$$p_Y(y) = \frac{(y+3)^2}{174} \quad \text{if } y = 2; 3; 4; 5$$

0 otherwise

- (a) Verify that Y has a legitimate PMF.

All probabilities are between 0 and 1 and they all sum to 1

If you were to draw a PMF chart, it would look like this:

y	2	3	4	5
$p_Y(y)$	0.14	0.21	0.28	0.37

- (b) What is the probability that Y is smaller than 4?

$$P(Y < 4) = .14 + .21 = .35$$

(c) What is the probability that Y is smaller than 4 if we are told Y is not 2?

$$P(Y < 4 | Y \neq 2) = \frac{P(Y = 3)}{P(Y = 3) + P(Y = 4) + P(Y = 5)} = \frac{.21}{.86} = .24$$

(d) Find the expected value and variance of Y.

$$E(Y) = 2 \cdot .14 + 3 \cdot .21 + 4 \cdot .28 + 5 \cdot .37 = 3.88$$

$$\text{Var}(Y) = [2^2 \cdot .14 + 3^2 \cdot .21 + 4^2 \cdot .28 + 5^2 \cdot .37] - [3.88]^2 = 1.12$$

3. The random variable U follows the PMF

$$p_U(u) = k(5 - u) \quad \text{if } u \text{ equals } 1, 2, 3, 4 \text{ or } 5$$

0 otherwise

(a) Find the value of k

u	1	2	3	4	5
$p_U(u)$	4k	3k	2k	k	0

All prob must add up to 1 = 10k so k=1/10

(b) Find the probability that U is any of 2, 3, or 4.

u	1	2	3	4	5
$p_U(u)$	4/10	3/10	2/10	1/10	0

$$P(U=2) + P(U=3) + P(U=4) = .6$$

4. A special deck of cards consists of two sets of spades and one set of hearts. A total of 39 cards. The cards are randomly shuffled and the top two are taken off the top of the deck. The random variable S represents the number of spades among these two cards. Find the PMF of S.

s	$p_s(s)$
0	$\frac{\binom{26}{0}\binom{13}{2}}{\binom{39}{2}}$
1	$\frac{\binom{26}{1}\binom{13}{1}}{\binom{39}{2}}$
2	$\frac{\binom{26}{2}\binom{13}{0}}{\binom{39}{2}}$

5. Suppose X and Y are random variables with  $E(X) = 3$ ,  $E(Y) = 4$  and  $\text{Var}(X)=2$ . Find

(a)  $E(2X + 1)$

$$2 \cdot E(X) + 1 = 2 \cdot 3 + 1 = 7$$

(b)  $E(X - Y)$

$$E(X) - E(Y) = 3 - 4 = -1$$

(c)  $E(X^2)$

$$\text{Var}(X) + E(X)^2 = 2 + 3^2 = 11$$

(d)  $E(X^2 - 4)$

$$E(X^2) - 4 = 11 - 4 = 7$$

(e)  $E((X - 4)^2)$

$$\text{Var}(X - 4) = E((X - 4)^2) - [E(X - 4)]^2$$

$$E((X - 4)^2) = \text{Var}(X - 4) + [E(X - 4)]^2 = 2 + [-1]^2 = 3$$

(f)  $\text{Var}(2X - 4)$

$$2^2 \cdot \text{var}(X) = 4 \cdot 2 = 8$$

6. Samantha plans to attend a volleyball game and wants to get some of her friends to go with her. Let  $X$  represent the number of the seven friends she calls that are interested. The probability any one of them will say 'yes' is 0.8, regardless of the responses from the others. Answer the following questions.

(a) What is the distribution of  $X$  and what are its parameter(s)?

$$X \sim \text{Bin}(n=7, p=0.8)$$

(b) What is the probability that at least four of her friends say 'yes'?

$$P(X \geq 4) = P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7)$$

$$\binom{7}{4} (0.8)^4 (0.2)^3 + \binom{7}{5} (0.8)^5 (0.2)^2 + \binom{7}{6} (0.8)^6 (0.2)^1 + \binom{7}{7} (0.8)^7 (0.2)^0 \\ = 0.967$$

7. Identify the parameters  $p$  and  $n$  for each of the following Binomial distributions and find the expected value and variance of the random variable described:

(a) The number of heads in 5 tosses of a fair coin.

$$n=5, p=0.5, E(X) = 5*0.5 = 2.5, \text{Var}(X) = 5*0.5*0.5 = 1.25$$

(b) The number of correct answers on a multiple choice test if each of the 25 questions has 5 possible answers and the student guesses randomly.

$$n=25, p=0.2, E(X) = 25*0.2 = 5, \text{Var}(X) = 25*0.2*0.8 = 4$$

(c) The number of 6's in 100 tosses of a fair dice.

$$n=100, p=1/6, E(X) = 100*(1/6) = 16.67, \text{Var}(X) = 100*(1/6)*(5/6) = 13.89$$

8. A pair of fair dice is rolled 10 times. Let  $X$  be the number of (double) rolls in which we see at least one six.

(a) What is the probability distribution of  $X$  and what are its parameters?

$$\text{Prob of seeing at least one 6 is } 11/36 = 0.306$$

$$X \sim \text{Bin}(n=10, p=0.306)$$

(b) What is the probability that in exactly five (double) rolls we see at least one six?

$$P(X = 5) = \binom{10}{5} (0.306)^5 (1 - 0.306)^5 = 0.109$$

9. A store has 50 light bulbs available for sale. Of these, 5 are defective. A customer buys 8 light bulbs randomly from this store.

(a) What is the probability he finds exactly 1 defective bulb among those he purchased?

$$\frac{\binom{5}{1} \binom{45}{7}}{\binom{50}{8}}$$

(b) What distribution are you using? What are its parameters?

Hypergeometric(N=50, n=8, p=5/50)

(c) What is the expected number of defective bulbs he will purchase?

$$E(X) = np = 8 * (5/50) = 0.8$$

(d) What is the variance of the number of defective bulbs he purchased?

$$\text{Var}(X) = \frac{N-n}{N-1} np(1-p) = (42/49) * 8 * (5/50) * (45/50) = 0.617$$

(e) What is the probability at most 2 of the bulbs he purchased are defective?

$$P(X=0) + P(X=1) + P(X=2) = \frac{\binom{5}{0} \binom{45}{8} + \binom{5}{1} \binom{45}{7} + \binom{5}{2} \binom{45}{6}}{\binom{50}{8}}$$

10. A huge warehouse has 5,000,000 light bulbs available for purchase. Of these, 500,000 are defective. A customer buys 8 light bulbs randomly from this warehouse. What is the approximate probability he finds exactly 1 defective bulb among those he purchased?

You can approximate this to a Binomial distribution with n=8 and p=0.1

$$P(X=1) = \binom{8}{1} (0.1)^1 (0.9)^7 = 0.383$$

11. There are 21 students in a classroom. Fifteen of these students are in the College of Management, while the other six are not. Suppose we take a random sample of five students without replacement and let X represent the number of these students who are in the College of Management.

(a) What is the distribution of X and what are its parameter(s)?

$$X \sim \text{HG}(N=21, n=5, p=15/21)$$

(b) What are the mean and variance of X?

$$E(X) = 3.57, \text{Var}(X) = 0.816$$

(c) What is the probability that there are more non-Management students in the sample?

$$\frac{\binom{15}{0}\binom{6}{5} + \binom{15}{1}\binom{6}{4} + \binom{15}{2}\binom{6}{3}}{\binom{21}{5}}$$

12. A bag contains ten blue marbles and nine red marbles. We reach in and select four at a time. Find the probability that we get either exactly three blue or exactly three red.

$$\frac{\binom{10}{1}\binom{9}{3} + \binom{10}{3}\binom{9}{1}}{\binom{19}{4}}$$

13. NOW, suppose for the previous question we have 10,000 blue marbles and 9,000 red marbles. Find the approximate probability of getting either exactly three blue or exactly three red and compare your answer to the previous question.

We now approximate to a Binomial distribution b/c  $N > 20 \cdot n$ . We we'll just define a new random variable for how many blue we grab.

$B \sim \text{Bin}(n=4, p=0.526)$  so 3 blue or 3 red would mean 3 blue or 1 blue

$$P(B=3) + P(B=1) = \binom{4}{3} (0.526)^3 (1 - 0.526)^1 + \binom{4}{1} (0.526)^1 (1 - 0.526)^3$$

14. You decide to roll a fair six-sided die until you roll either a 5 or 6 or until you roll 5 times, whichever comes first. Let X be the number of times you roll.

(a) Construct the PMF for X.

x	P(X=x)
1	2/6
2	(4/6)(2/6) = 2/9
3	(4/6)(4/6)(2/6) = 4/27
4	(4/6)^3*(2/6) = 8/81

5	$1 - ((2/6) + (2/9) + (4/27) + (8/81)) = .2$
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(b) What is the probability you roll between 3 or 4 times inclusive.

$$(4/27) + (8/81) = .25$$

(c) Given you have rolled 2 times already and have not yet succeeded, what is the probability you roll less than 5 times.

$$\frac{.25}{(4/27 + 8/81 + .2)} = .56$$

(d) Let  $Z = -5X + 6$ . Find  $E(Z)$  and  $\text{Var}(Z)$ .

$$E(X) = 1 \cdot 2/6 + 2 \cdot 2/9 + 3 \cdot 4/27 + 4 \cdot 8/81 + 5 \cdot .2 = 2.62$$

$$E(X^2) = 1 \cdot 2/6 + 4 \cdot 2/9 + 9 \cdot 4/27 + 16 \cdot 8/81 + 25 \cdot .2 = 9.14$$

$$\text{Var}(X) = 9.14 - 2.62^2 = 2.28$$

$$E(Z) = -5 \cdot 2.62 + 6 = -7.1$$

$$\text{Var}(Z) = 25 \cdot 2.28 = 57$$

15. A barber shop averages 6.5 customers per hour between noon and 5 p.m. Let  $X$  represent the number of customers arriving between 2 and 3:30 p.m.

(a) What is the distribution of  $X$  and its parameter(s)?

An average of  $6.5 \cdot 1.5 = 9.75$  customers will arrive between 2 and 3:30

$X \sim \text{Poisson}(\lambda = 9.75 \text{ customers/hour and a half})$

(b) Find the probability of exactly six customers between 2 and 3:30 p.m.

$$P(X=6) = \frac{e^{-9.75} (9.75)^6}{6!} = 0.07$$

(c) If you know that no more than six customers arrived between 2 and 3:30 p.m., what is the probability there were exactly six?

$$P(X=6 | X \leq 6) =$$

$$\frac{P(X=6)}{P(X \leq 6)} =$$

$$\frac{e^{-9.75} (9.75)^6}{6!}}{e^{-9.75} (9.75)^0 / 0! + e^{-9.75} (9.75)^1 / 1! + e^{-9.75} (9.75)^2 / 2! + e^{-9.75} (9.75)^3 / 3! + e^{-9.75} (9.75)^4 / 4! + e^{-9.75} (9.75)^5 / 5! + e^{-9.75} (9.75)^6 / 6!}}$$

- (d) Suppose the number of customers between 2 and 3:30 p.m. is recorded over a seven-day period. Assume that the number of arrivals on different days is independent; give the distribution of Y, the number of days with exactly six customers between 2 and 3:30 p.m., and its parameter(s).

$$Y \sim \text{Binomial}(n=7, p=0.07)$$

- (e) Find the probability that exactly six customers arrive between 2 and 3:30 p.m. on at least one of the seven days.

$$P(Y \geq 1) = 1 - P(Y < 1) = 1 - P(Y=0) = 1 - \left[ \binom{7}{0} (0.07)^0 (1 - 0.07)^7 \right] = .4$$

16. To decide how many new cashiers to hire, a large retail store is investigating the flow of customers to the checkout lanes. Suppose that customers arrive at a checkout register at an average rate of 7 per hour. Let X be the number of customers arriving at the checkout lane in a 45 minute period.

- (a) Give the distribution of X and its parameter(s).

If customers arrive 7 per hour, they will arrive  $7 * .75 = 5.25$  per 45 min

$$X \sim \text{Poisson}(\lambda=5.25)$$

- (b) What is the probability that exactly 5 customers arrive during this period?

$$P(X=5) = \frac{e^{-5.25} (5.25)^5}{5!} = .17$$

- (c) What is the probability that there is at least one customer arriving at the checkout lane during this period?

$$P(X \geq 1) = 1 - P(X < 1) = 1 - \frac{e^{-5.25} (5.25)^0}{0!} = .99$$

- (d) If more than one customer arrives during this period, what is the probability that the number of customers is exactly four?

$$P(X=4 | X > 1) = \frac{P(X=4)}{P(X > 1)} = \frac{\frac{e^{-5.25} (5.25)^4}{4!}}{1 - \left[ \frac{e^{-5.25} (5.25)^0}{0!} + \frac{e^{-5.25} (5.25)^1}{1!} \right]} = .17$$



17. Assume you have applied for two scholarships, a Merit scholarship (M) and an Athletic scholarship (A). The probability that you receive an Athletic scholarship is 0.18. The probability of receiving both scholarships is 0.11. The probability of getting at least one of the scholarships is 0.3.

(a) What is the probability that you will receive a Merit scholarship?

$$P(M) = 0.23$$

(b) Are events A and M mutually exclusive? Why or why not? Explain.

$$\text{No, because } P(A \cap M) \neq 0$$

(c) Are the two events A, and M, independent? Explain, using probabilities.

$$\text{No, because } P(A \cap M) \neq P(A)P(M)$$

(d) What is the probability of receiving the Athletic scholarship given that you have been awarded the Merit scholarship?

$$P(A|M) = \frac{P(A \cap M)}{P(M)} = \frac{0.11}{0.23} = 0.4783$$

(e) What is the probability of receiving the Merit scholarship given that you have been awarded the Athletic scholarship?

$$P(M|A) = \frac{P(M \cap A)}{P(A)} = \frac{0.11}{0.18} = 0.6111$$

18. You are given the following information on Events A, B, C, and D

$$\begin{array}{ll} P(A) = .4 & P(A \cup D) = .6 \\ P(B) = .2 & P(A|B) = .3 \\ P(C) = .1 & P(A \cap C) = .04 \\ & P(A \cap D) = .03 \end{array}$$

(a) Compute  $P(D)$

$$P(D) = 0.23$$

(b) Compute  $P(A \cap B)$

$$P(A \cap B) = 0.06$$

(c) Compute  $P(A|C)$

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{0.04}{0.1} = 0.4$$

(d) Compute the probability of the complement of C

$$P(C^c) = 1 - P(C) = 0.9$$

(e) Are A and B mutually exclusive? Explain your answer.

No,  $P(A \cap B) \neq 0$

(f) Are A and B independent? Explain your answer.

No,  $P(A|B) \neq P(A)$

(g) Are A and C mutually exclusive? Explain your answer.

No,  $P(A \cap C) \neq 0$

(h) Are A and C independent? Explain your answer.

Yes,  $P(A|C) = P(A)$

19. A very short quiz has one multiple choice question with five possible choices (a, b, c, d, e) and one true or false question. Assume you are taking the quiz but do not have any idea what the correct answer is to either question, but you mark an answer anyway.

(a) What is the probability that you have given the correct answer to both questions?

$$P(\text{both}) = \frac{1}{5} * \frac{1}{2} = \frac{1}{10}$$

(b) What is the probability that only one of the two answers is correct?

$P(\text{exactly one}) = P(\text{M.C. right and T/F wrong}) + P(\text{M.C. wrong and T/F right}) =$

$$\frac{1}{5} * \frac{1}{2} + \frac{4}{5} * \frac{1}{2} = \frac{5}{10}$$

(c) What is the probability that neither answer is correct?

$$P(\text{both wrong}) = \frac{4}{5} * \frac{1}{2} = \frac{4}{10}$$

(d) What is the probability that only your answer to the multiple choice question is correct?

$$P(\text{M.C. right and T/F wrong}) = \frac{1}{5} * \frac{1}{2} = \frac{1}{10}$$

(e) What is the probability that you have only answered the true or false question correctly?

$$P(\text{T/F right and M.C. wrong}) = \frac{1}{2} * \frac{4}{5} = \frac{4}{10}$$

20. In a recent survey about appliance ownership, 58.3% of the respondents indicated that they own Maytag appliances, while 23.9% indicated they own both Maytag and GE appliances and 70.7% said they own **at least** one of the two appliances.

Define the events as

M = Owning a Maytag appliance

G = Owning a GE appliance

- (a) What is the probability that a respondent owns a GE appliance?

$$P(G) = 0.363$$

- (b) **Given** that a respondent owns a Maytag appliance, what is the probability that the respondent also owns a GE appliance?

$$P(G|M) = \frac{P(G \cap M)}{P(M)} = \frac{.239}{.583} = 0.4099$$

- (c) Are events "M" and "G" mutually exclusive? Why or why not? **Explain, using probabilities.**

No, the intersection is 0.239, not 0.

- (d) Are the two events "M" and "G" independent? **Explain, using probabilities.**

No,  $P(G|M) = 0.4099$  not  $P(G) = 0.363$

21. Let the sample space be all integers from 1 to 20.

Let A be the set of all prime numbers

Let B be the set of numbers divisible by 3

Let C be the set of numbers that are at most 10

Which events are independent? Circle all that apply

- I. A and B
- II. A and C
- III. B and C
- IV. A and  $B^c$
- V.  $A^c$  and B
- VI.  $A^c$  and C
- VII.  $B^c$  and C
- VIII. A and  $C^c$
- IX. B and  $C^c$
- X.  $B^c$  and  $C^c$

$A = \{2, 3, 5, 7, 11, 13, 17, 19\}$   
 $B = \{3, 6, 9, 12, 15, 18\}$   
 $C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A \text{ and } B = \{3\}$   
 $A \text{ and } C = \{2, 3, 5, 7\}$   
 $B \text{ and } C = \{3, 6, 9\}$

$P(A) = 8/20 = .4$   
 $P(B) = 6/20 = .3$   
 $P(C) = 10/20 = .5$   
 $P(A \text{ and } B) = 1/20 = .05$   
 $P(B \text{ and } C) = 3/20 = .15$   
 $P(A \text{ and } C) = 4/20 = .2$

To be independent  $P(A \text{ and } B) = P(A)*P(B)$  and so on. The only pairs of events in which that occurs are:

$A$  and  $C$ ;  
 $B$  and  $C$

Once you find this out you need not to any further work b/c if  $A$  and  $C$  are indep, then that implies that  $A$  and  $C^c$  are indep. As well as  $A^c$  and  $C$  and  $A^c$  and  $C^c$ . The same also follows for  $B$  and  $C$ . So any combination with  $A$  and  $C$  in it (including any compliments) and  $B$  and  $C$  (including any compliments) will be independent.

So final answer:

II, III, VI, VII, VIII, IX, X

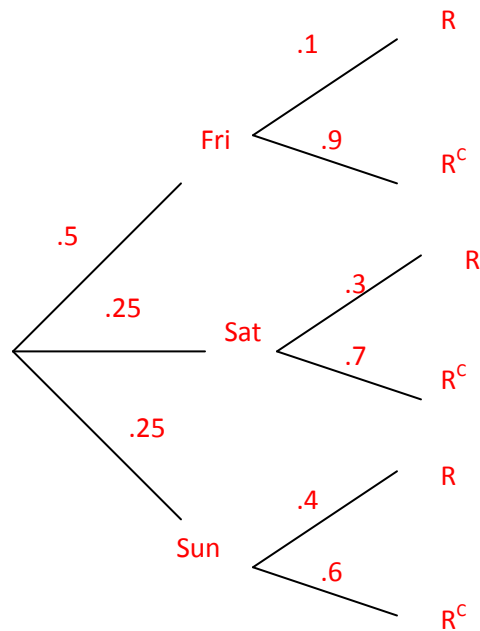
22. Your friend has 3 cards. One is black on both sides, one is red on one side black on the other, and one is red on both sides. They randomly select one card and place it on the table (so you can see only one side of the card). You see that the side facing you is Red. What is the probability that the other side is ALSO Red?

$$P(RR|R) = \frac{P(RR \cap R)}{P(R)} = \frac{P(RR) * P(R|RR)}{P(R \cap RR) + P(R \cap RB)} = \frac{\frac{1}{3} * 1}{\frac{1}{3} * 1 + \frac{1}{3} * \frac{1}{2}} = \frac{2}{3}$$

23. Kathy and her mom are planning on going to Indianapolis. She is twice as likely to go on Friday than on Saturday, but just as likely to go on Saturday as Sunday. She watches the weather forecast and sees that there is

10% chance of rain on Friday,  
 30% chance of rain on Saturday  
 40% chance of rain on Sunday

(a) Draw a tree diagram of the situation



(b) What is the probability she went to Indianapolis and it rained?

$$P(R) = .5 \cdot .1 + .25 \cdot .3 + .25 \cdot .4 = .225$$

(c) If she and her mom go to Indianapolis and it rains, what is the probability she went on the weekend (either Saturday or Sunday)?

$$P(\text{Sat or Sun} | R) = P(\text{Sat} | R) + P(\text{Sun} | R) \text{ or}$$

$$1 - P(\text{Fri} | R) = 1 - \left( \frac{P(\text{Fri} \cap R)}{P(R)} \right) = 1 - \frac{0.5(0.1)}{0.225} = 0.778$$