Set Operations

Recall that we have introduced the notion of a set, and some properties of sets (elements, equality, cardinality, subsets, and set operations).

Cardinality of sets

If a set contains finitely many elements, then it is called a finite set. The number of elements in a finite set is called its *size*, or *cardinality*. If a set is not finite, then its cardinality is not a finite number and the set is said to be *infinite*.

For example, the cardinality of the set of letters in the English alphabet is 26, and the cardinality of the set of living ex-Presidents is 4. The set of real numbers $\mathbb{R}$ contains infinitely many elements (since there are an infinity of real numbers), and hence $\mathbb{R}$ is infinite. By convention, the empty set is said to have cardinality 0.

A useful factoid: if a finite set S has cardinality n, then its power set $\mathbb{P}(S)$ has cardinality 2^n. We will prove this formally by induction, but as an **exercise**, convince yourself that this is the case using the set $S = \{\text{red, blue, green}\}$ as an example.

Set operations and cardinality properties

Let $|A|$ denote the cardinality of a set A. There are some interesting relationships between the cardinality of a pair of sets, their unions and intersections, etc.

First, we easily see that the following relationships are true: $|A \cap B| \leq |A|$ and $|A \cap B| \leq |B|$. This is because intersecting a set with some other set can never increase its size. Moreover, $|A \cup B| \geq |A|$ and $|A \cup B| \geq |B|$; taking the union of a set with another set cannot decrease its size.

Here is a more non-trivial relationship. In the above example involving sets of numbers up to 10, we observe that $|A| = 4, |B| = 3$. Moreover, $|A \cup B| = 5$ and $|A \cap B| = 2$.

More generally, we have the following theorem:

> For any pair of finite sets A and B, $|A| + |B| \geq |A \cup B|$.

We will give a full, rigorous proof of this theorem later. In fact, we will prove a more general principle. Observe that $|A \cup B| = 5$ and $|A \cap B| = 2$, which rather conveniently adds up to the sum of the sizes of $|A|$ and $|B|$.

This is an instance of a more general result called the *principle of inclusion-exclusion* (PIE). In its simplest form, this principle can be stated as follows:

For any pair of finite sets A and B, $|A \cup B| = |A| + |B| - |A \cap B|$.

If one takes the PIE for granted, then the theorem stated above follows quite easily, since $|A \cap B|$ cannot be negative (and therefore, $|A| + |B|$ must be at least as big as $A \cup B$.)

Cardinality and Cartesian products

We now discuss cardinalities in the context of *Cartesian products*.

Three quick points. First, observe that taking Cartesian products of sets is *not* a commutative operation; $A \times B$ is very different from $B \times A$ (unless $A = B$).

Second, the number of possible ordered pairs (a, b), by a simple counting argument is given by $|A|$ multiplied by $|B|$. In other words, the cardinality of the Cartesian product of two sets is the product of their cardinalities.

Lastly, observe that we can generalize Cartesian products to more than 2 sets using n-tuples. The Cartesian product of n sets, $A_1 \times A_2 \times \ldots \times A_n$, can be defined as the set of all n-tuples $(a_1, a_2, \ldots, a_n)$ such that $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

The power of Cartesian products is that they can be used to construct concise descriptions of large, complex sets. For example, suppose we are interested in the set of all possible license plate numbers in Iowa. For simplicity, let us assume that license plate numbers in Iowa consist of 3 letters of the alphabet (in caps) followed by 3 digits. We can concisely describe this set as follows. Let A be the set of letters in the alphabet (in caps) and D be the set of digits. Then, the set of possible license plate numbers can be written as:

$$A \times A \times A \times D \times D \times D,$$

or $A^3 \times D^3$ for short.

On the other hand, suppose that license plates in Texas consist of 6 alphanumeric symbols in any arbitrary order. Therefore, the set of license plate numbers in Texas can be denoted as:

$$(A \cup L) \times (A \cup L) \times (A \cup L) \times (A \cup L) \times (A \cup L) \times (A \cup L),$$

or $(A \cup L)^6$ for short.

Functions

In several applications, we want to assign, to each element of a given set, a specific element of a second set. Such assignments are instances of *functions*. The notion of a "function" is fundamental to all of math and computer science.

Let X and Y be nonempty sets. A function (also called a *mapping* or *transformation*) from X to Y maps an element in X to exactly one (unique) element in Y. We denote this mapping using the notation $f : X \to Y$. Sometimes, x is called the *argument* of the function, and y is called the *value*.

The set X is called the *domain* of f, while the set Y is called the *co-domain* of f. For short, we use the notation $\text{Dom}(f)$ and $\text{Co-dom}(f)$.

The unique element in Y that $x \in X$ gets mapped to is called the *image* of x under f. If $f(x) = y$, then x is called a *pre-image* of y.

It is important to note that while *not* every element in Y necessarily gets a pre-image. (In other words, there could be elements in Y that do not have any corresponding x.) The subset of Y that consist of all elements y that are images is called the *range* of f, denoted by $\text{Range}(f)$. In set-builder notation, we can express this as:

$$\text{Range}(f) = \{y \in Y \mid \exists x \in X,\ f(x) = y\}.$$

For example, if at the end of the class, each student in CPRE 310 will be assigned a letter grade. Grades are assigned to each student. Moreover, this grade is unique (i.e., students get exactly one grade in the end.). Therefore, we can model this assignment as a function f.

Let's use a simpler example. Suppose there are 5 people in a class - Ava, Bob, Chuck, Don, and Emma. Their grade assignments (in order) are B, B, A, C, A.

We model this as follows. Define $X = \{\text{Ava}, \text{Bob}, \text{Chuck}, \text{Don}, \text{Emma}\}$ be the set of students in the class, $Y = \{A, B, C, D, F\}$ be the set of letter grades. Let f denote the grade assignment function. Then, we have:

- $f(\text{Ava}) = B$
- $f(\text{Bob}) = B$
- $f(\text{Chuck}) = A$
- $f(\text{Don}) = C$
- $f(\text{Emma}) = A$

Moreover, $\text{Dom}(f)$ is X, $\text{Co-dom}(f)$ is Y, and $\text{Ran}(f) = \{A, B, C\}$ is the set of all grades that the students in the class end up getting. (If everyone got an A, then $\text{Ran}(f)$ would be $\{A\}$.)

A convention: we will always assume that any function f that we discuss is *everywhere well-defined*, i.e., each element in the domain X will have a well-defined image under f. In terms of an example, under the "grade-assignment function" that we defined above, the everywhere-well-defined property implies that *every* student in 310 gets mapped to a letter grade.

Inverse image

Consider some function f, and let y be some element in $\mathrm{Ran}(f)$. The *inverse image* of y, denoted as $f^{-1}(y)$, is the set of all elements $x \in X$ such that $f(x) = y$. As opposed to $f(x)$, which is an *element* of Y, keep in mind that $f^{-1}(y)$ is a *subset* of elements of X. In set-builder notation, we have:

$$f^{-1}(y) = \{x \in X \mid f(x) = y\}$$

In the above example with grades, we have:

- $f^{-1}(A) = \{\text{Chuck}, \text{Emma}\}$
- $f^{-1}(B) = \{\text{Ava}, \text{Bob}\}$
- $f^{-1}(C) = \{\text{Don}\}$

Note that $f^{-1}(D)$ or $f^{-1}(E)$ are **not defined** since D and E do not belong to $\mathrm{Ran}(f)$.

Let us do a few more examples.

1. Let $X = Y = \mathbb{R}$. Suppose that $f(x) = x + 1$. Then, $\mathrm{Dom}(f) = \mathbb{R}$, $\text{Co-dom}(f) = \mathbb{R}$ (since the function maps real numbers to real numbers), and $\mathrm{Ran}(f) = \mathbb{R}$ as well, since every real number y is the image of some x under f.

2. On the other hand, $x^2 + y^2 = 1$ is **not** a function the way we have defined it. This is because if we solve for x, we can see that $x = \pm\sqrt{1 - y^2}$, i.e., x is assigned two possible values. This is not allowed under our definition above since functions map elements to *uniquely* defined values.

3. Let $X = Y = \mathbb{R}$. Suppose that $f(x) = 0$. Then, $\mathrm{Dom}(f) = \mathbb{R}$, $\text{Co-dom}(f) = \mathbb{R}$, and $\mathrm{Ran}(f) = \{0\}$.