Search for CSPs

Outline

I. Backtracking algorithm

II. Local search

III. CSP structure

* Figures/images are from the textbook site (or by the instructor).

I. Backtracking

```
function BACKTRACKING-SEARCH(csp) returns a solution or failure
   return BACKTRACK(csp, { })

function BACKTRACK(csp, assignment) returns a solution or failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(csp, assignment)
   for each value in ORDER-DOMAIN-VALUES(csp, var, assignment) do
      if value is consistent with assignment  then
         add {var = value} to assignment
         inferences ← INFERENCE(csp, var, assignment)   // arc-, path-, or k-consistency
         if inferences ≠ failure then                    // forward checking, etc.
            add inferences to csp
            result ← BACKTRACK(csp, assignment)
            if result ≠ failure then return result
            remove inferences from csp
         remove {var = value} from assignment
   return failure
```

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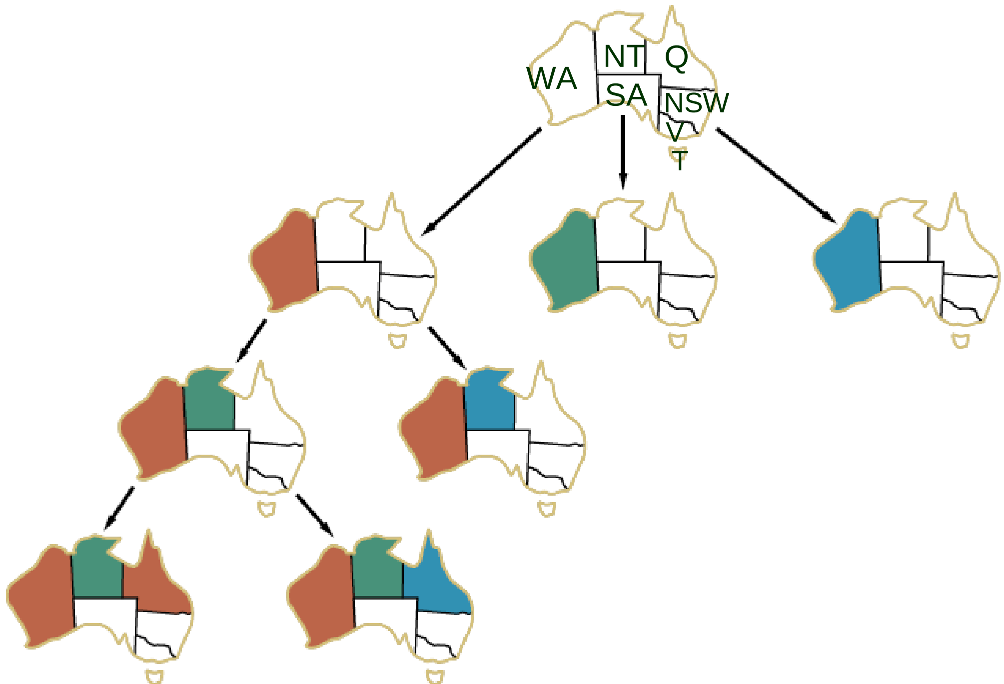

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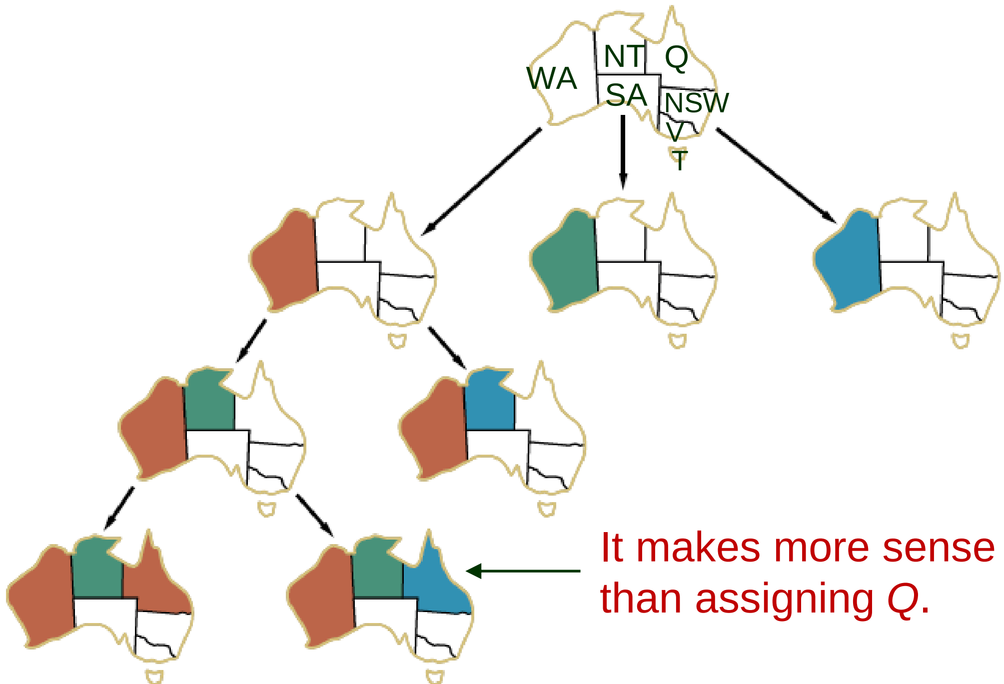

It makes more sense to assign $SA = blue$ than assigning *Q*.

MRV and Degree Heuristics

Minimum-remaining-values (MRV): Choose the variable with the *fewest "legal" values*.

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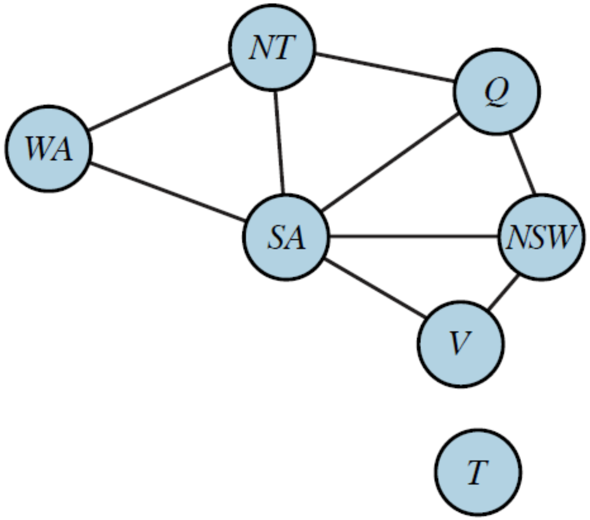

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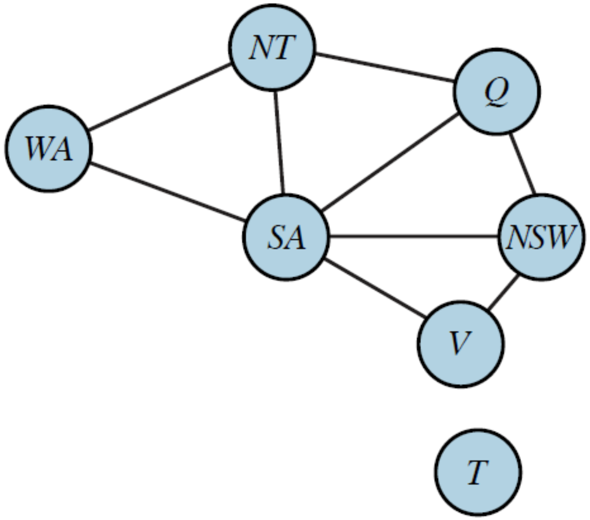

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  Others have degrees ≤ 3.

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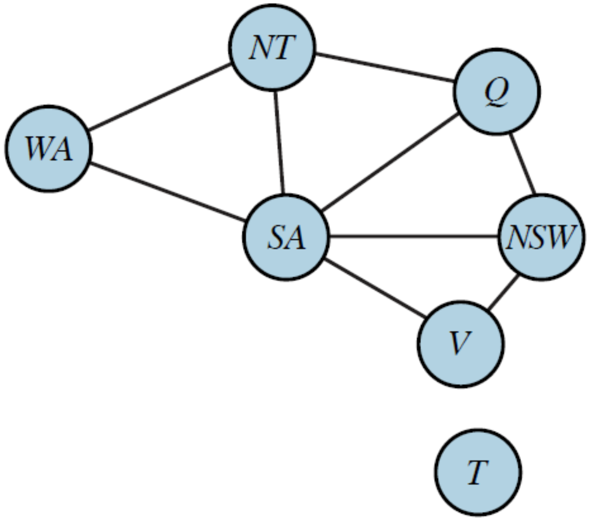

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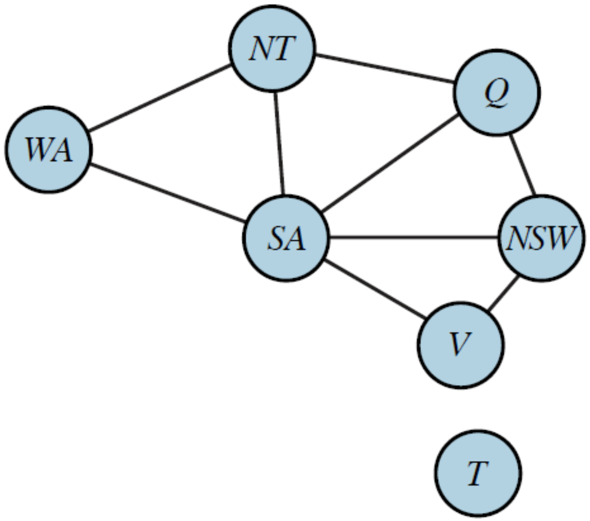


- Use the *degree heuristic* as a tie-breaker or at the start.

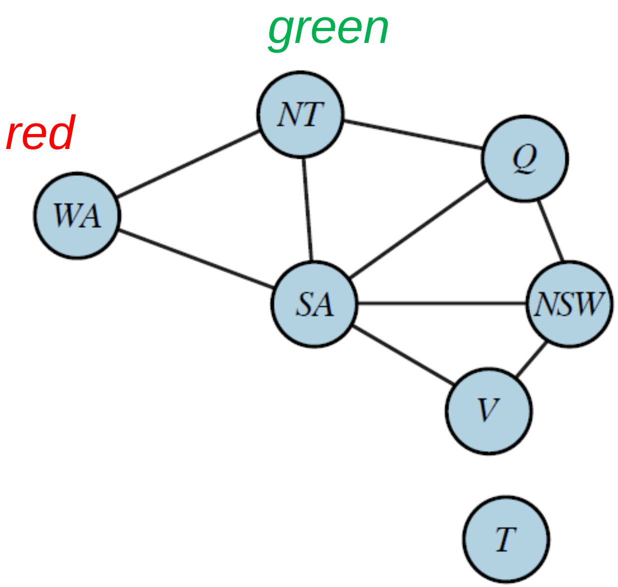
  $\deg(SA) = 5$
  Others have degrees ≤ 3.

  $\Downarrow$

  Color *SA* first.

Least Constraining Value

For the selected variable, choose its value that *rules out the fewest choices* for the neighboring variables in the constraint graph.



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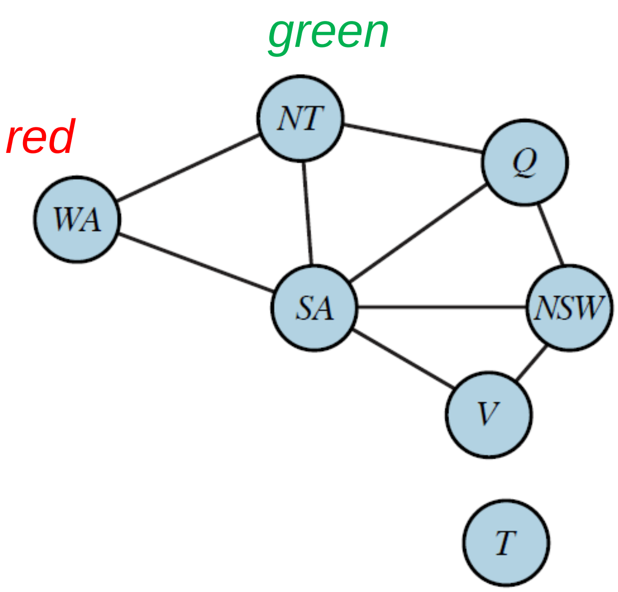

Which color to assign to *Q* next?

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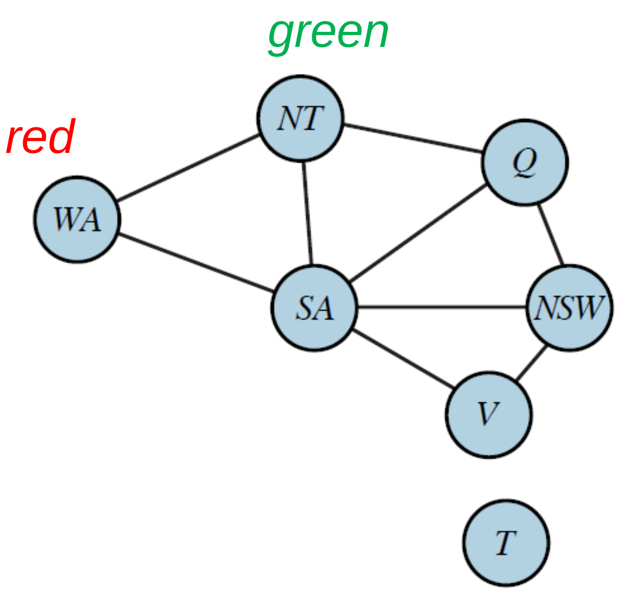

Which color to assign to *Q* next?

If *blue*, then SA would have no color left.

Choose *red*.

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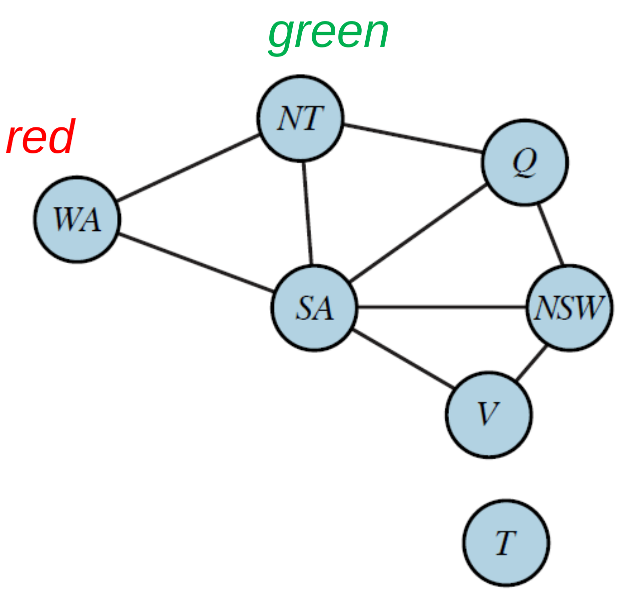

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Choose *red*.

The least-constraining-value heuristic tries to create the maximum room for subsequent variable assignments.

Variable vs. Value Selections

Variable order: *fail-first*.

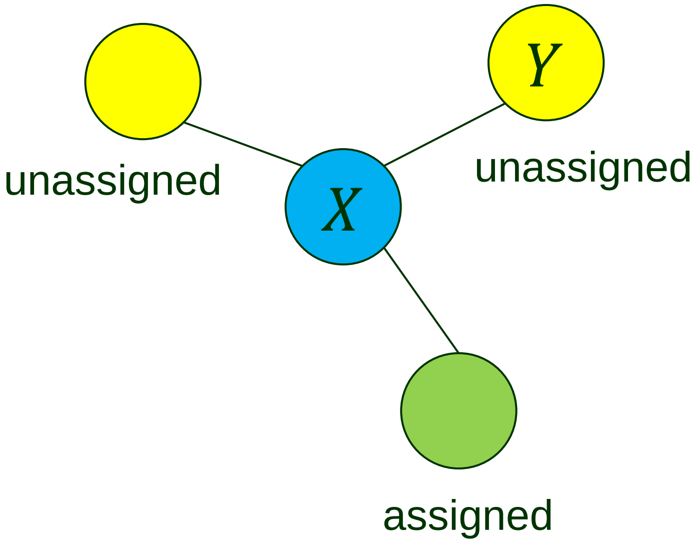
Fewer successful assignments to backtrack over.

Value order: *fail-last*.

- Only one solution needed.
- It makes sense to look for the most likely values first.

Forward Checking

Inference: Every new variable assignment opens the door for new domain reductions on neighboring variables.



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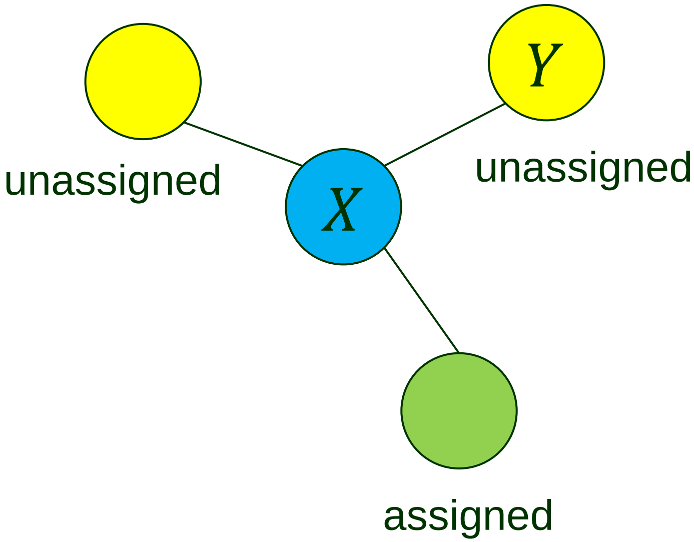

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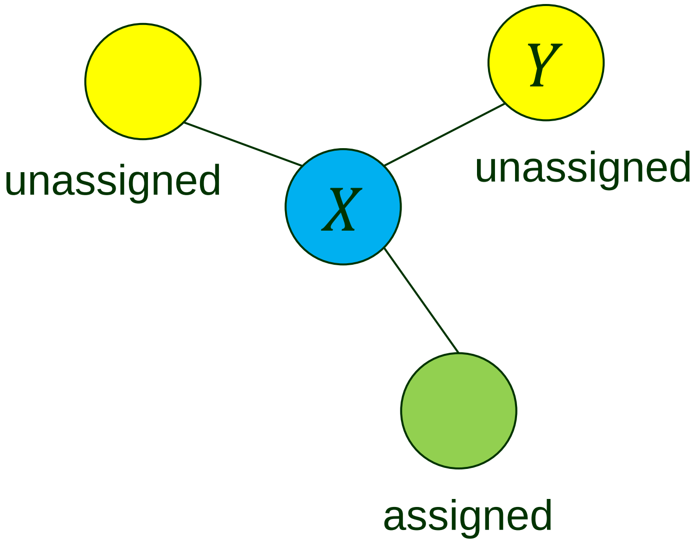
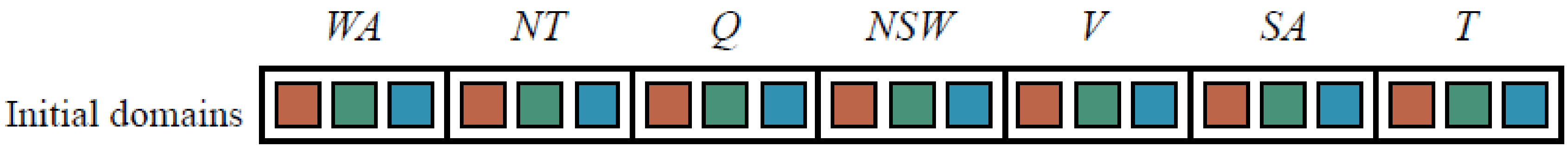


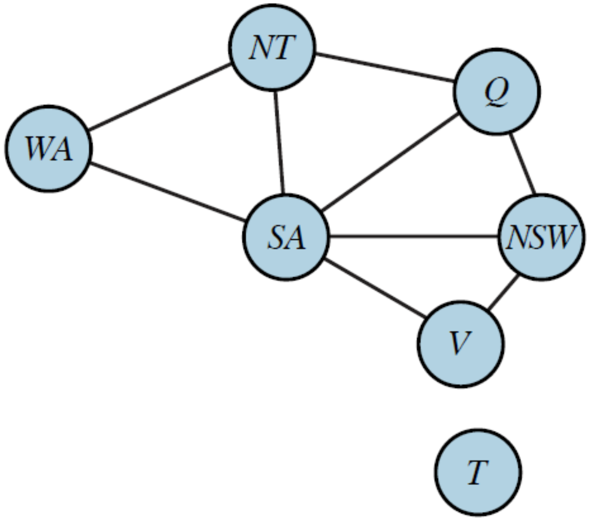
Assignment $X = v$

$\Downarrow$

For every unassigned Y connected to X , delete any value from Y's domain that is inconsistent with v.

Backtracking with Forward Checking





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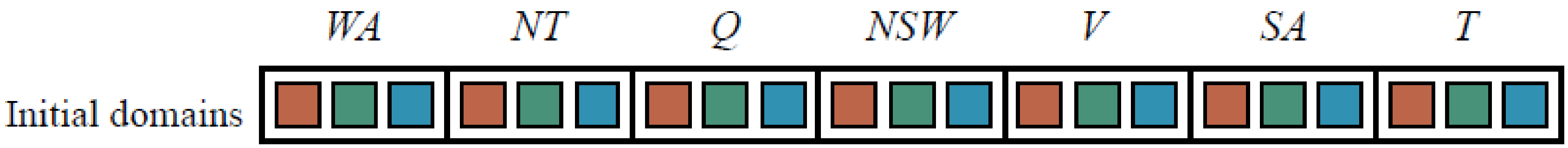

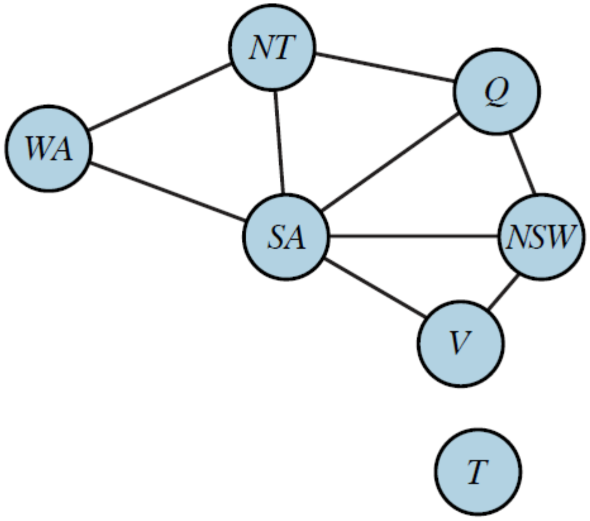

- *WA* = *red*

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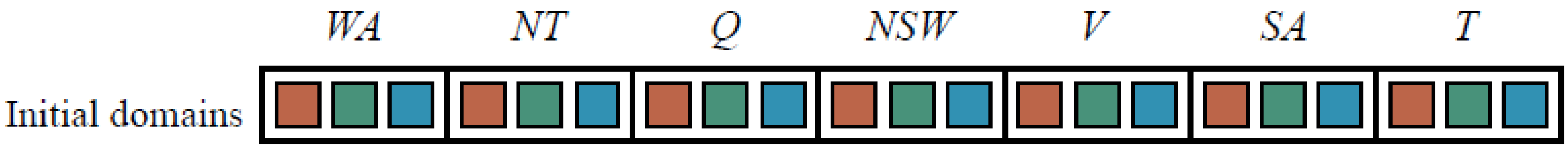

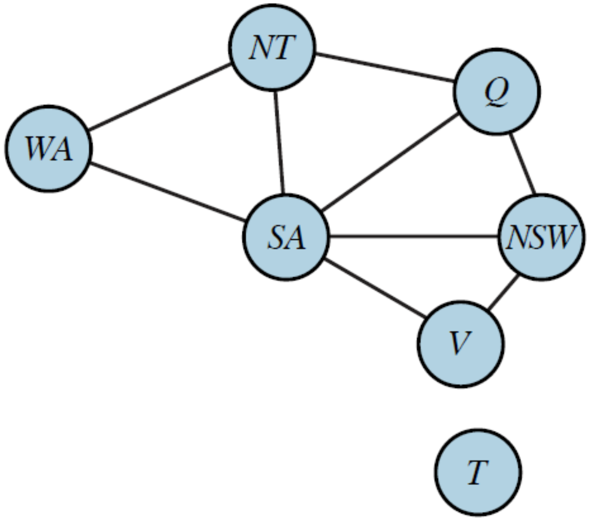

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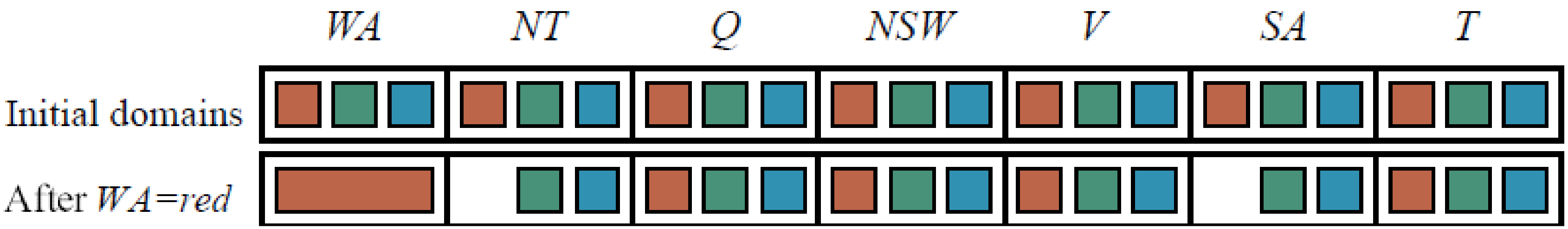

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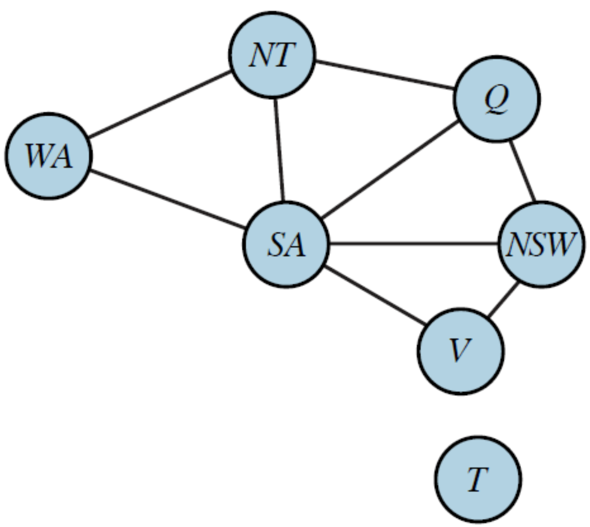

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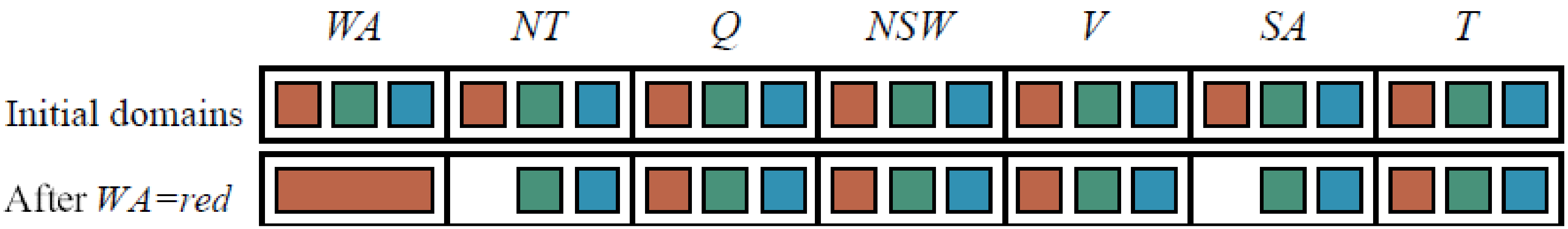

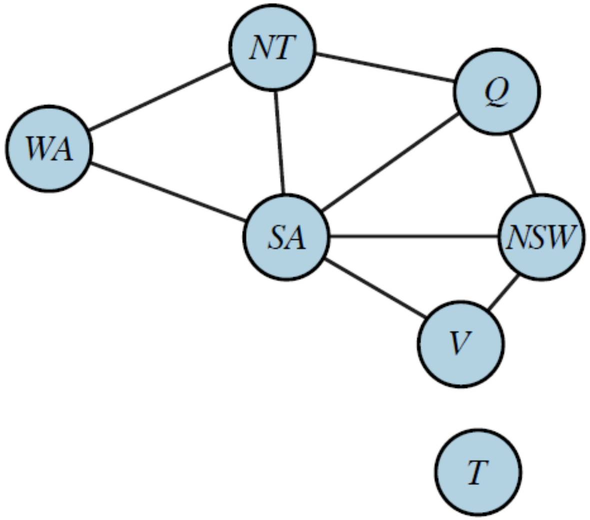

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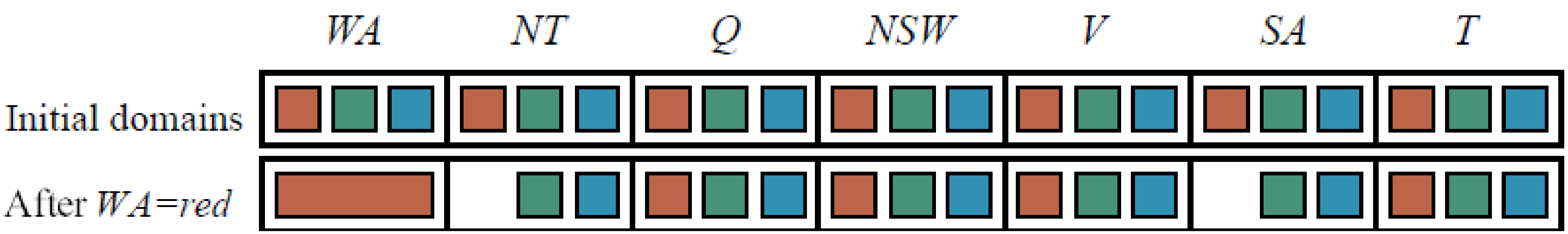

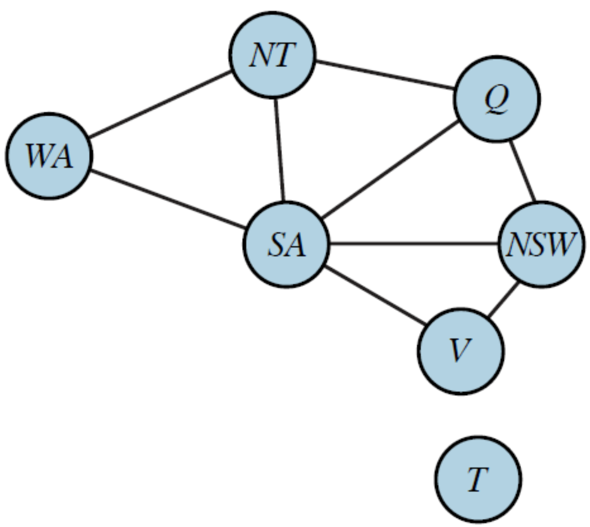

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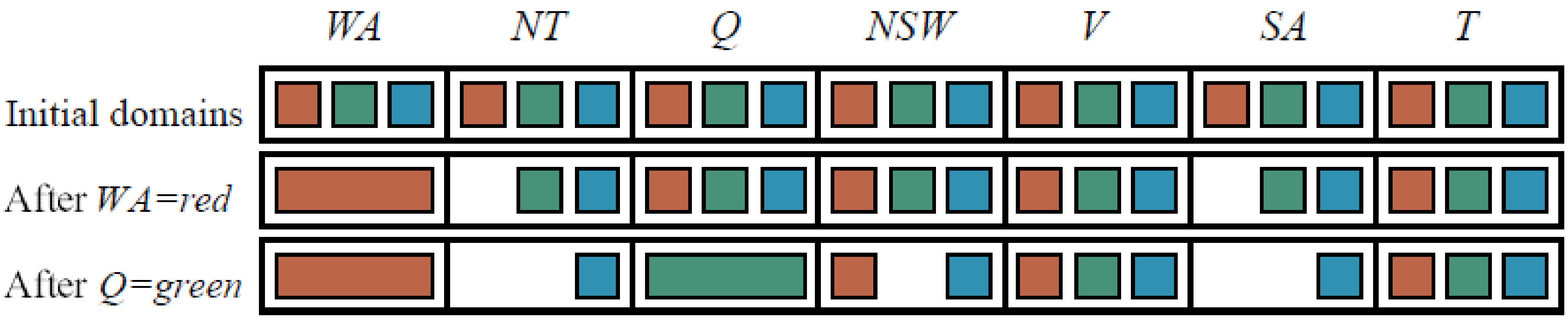

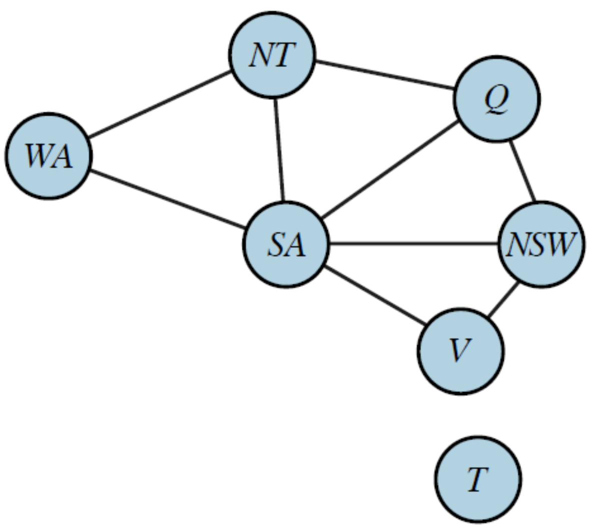

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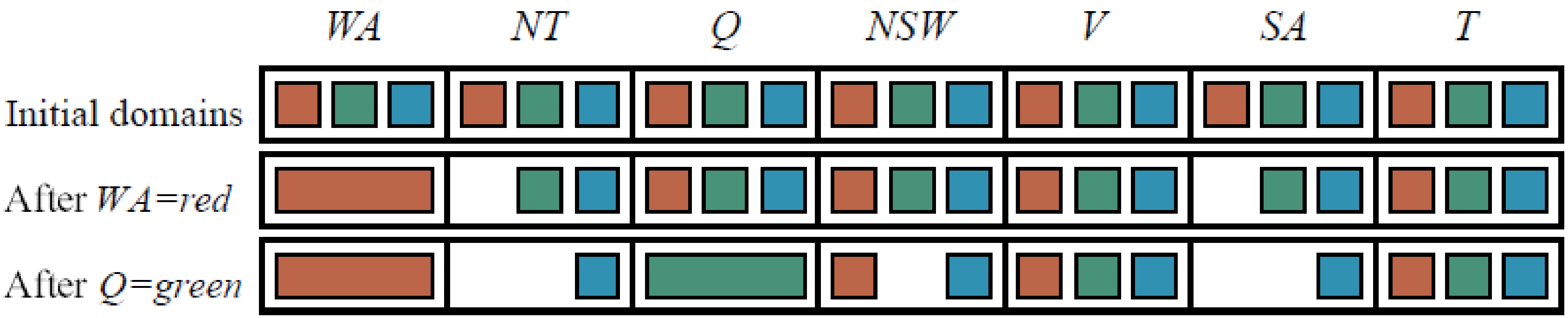

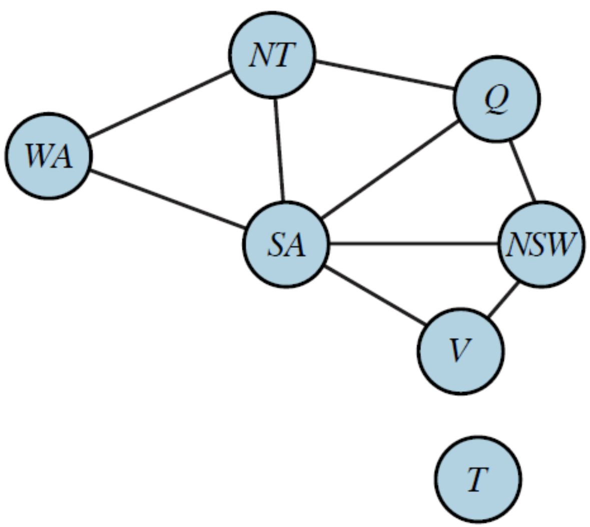

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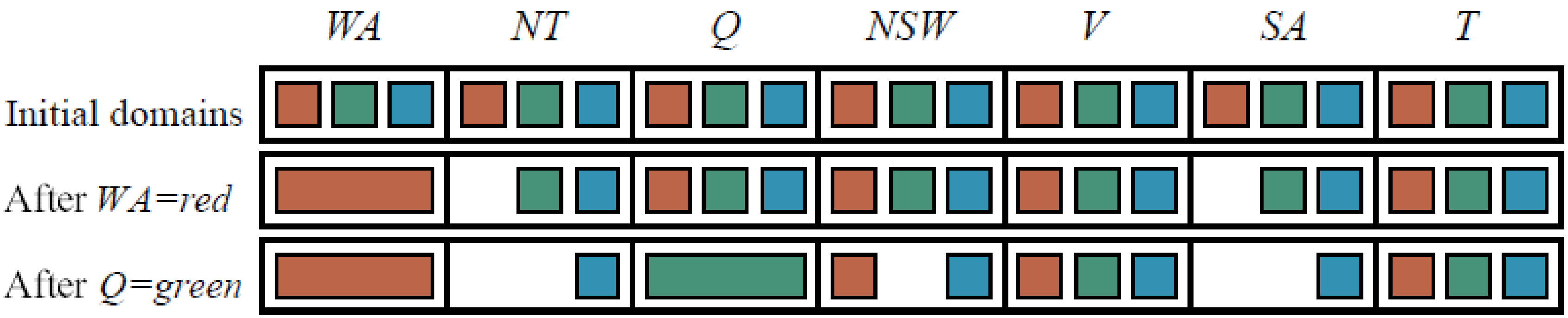

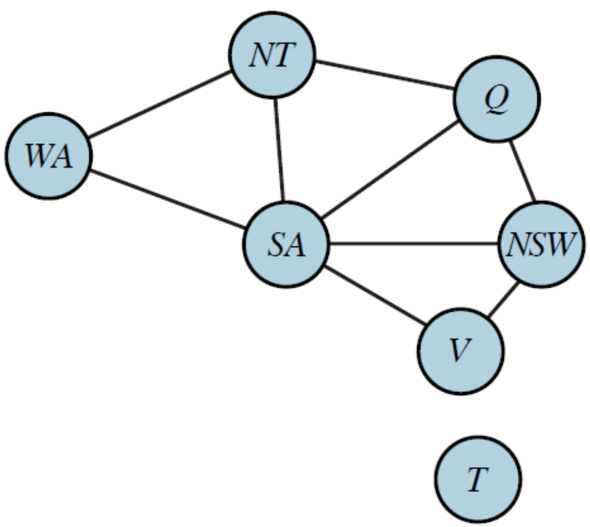

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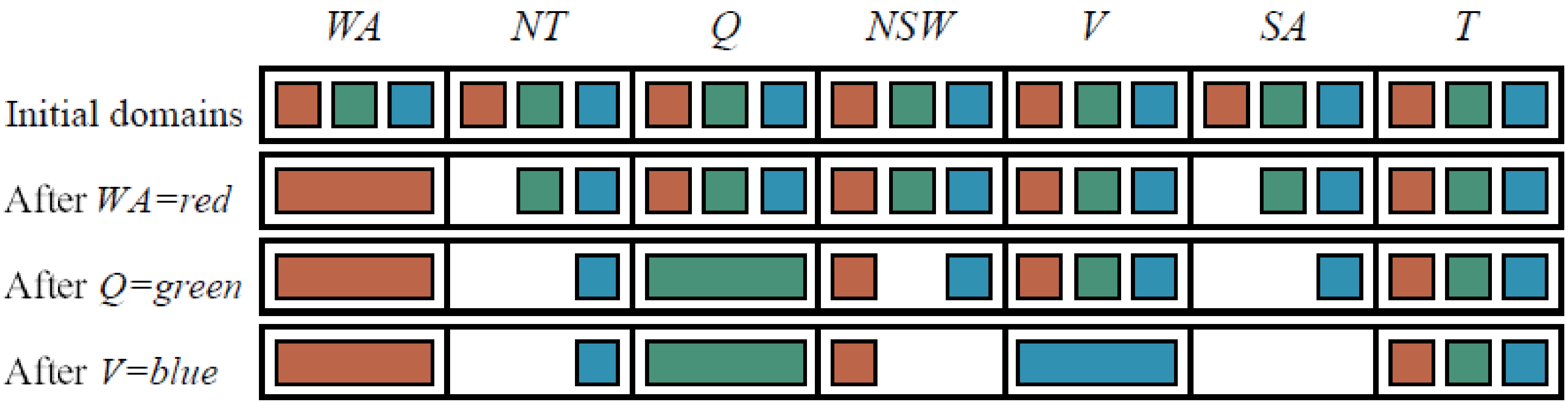

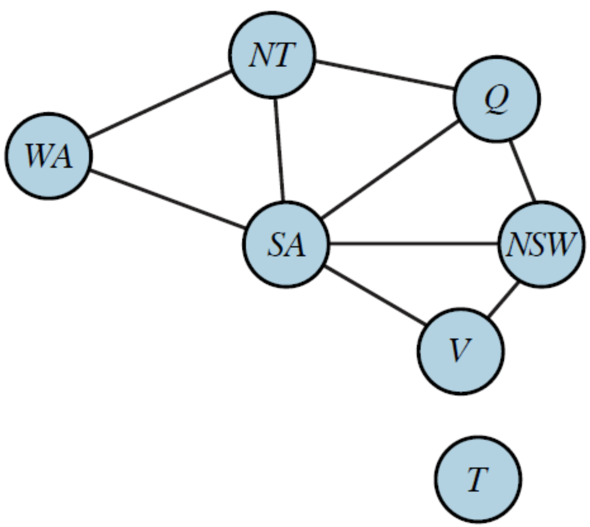

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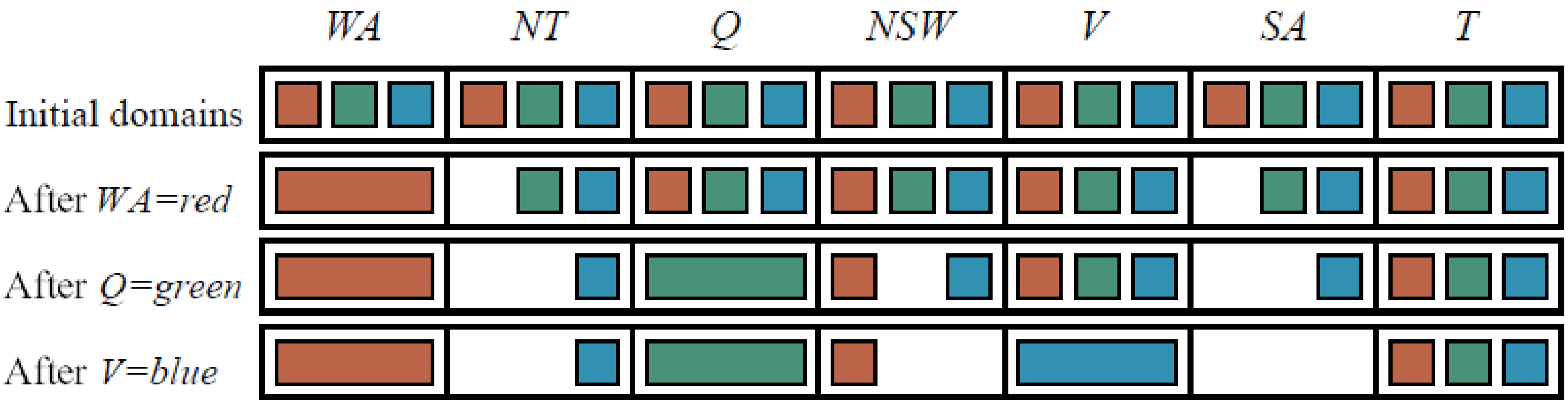

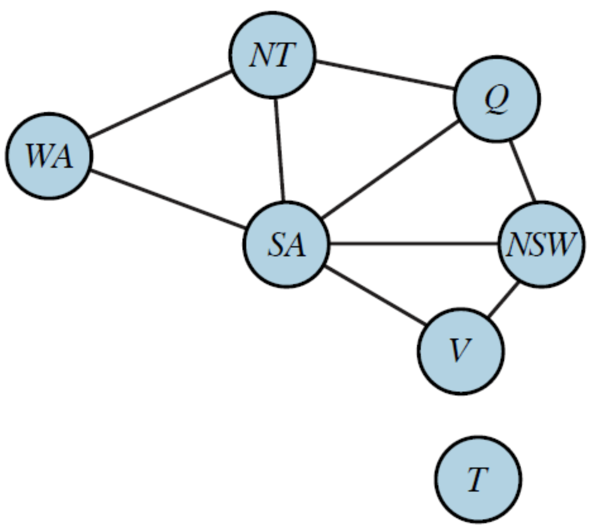

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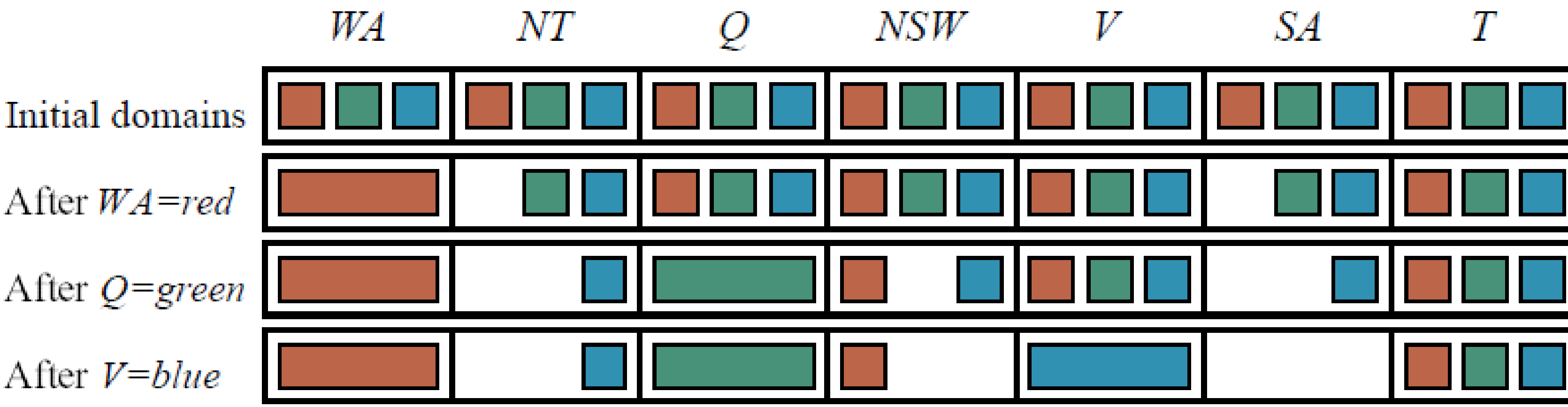

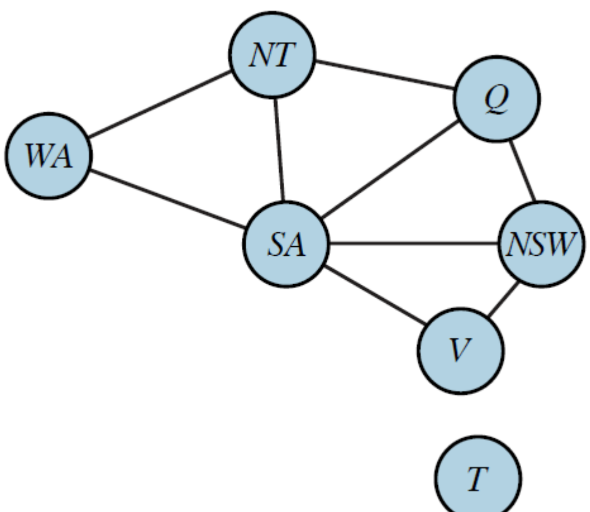

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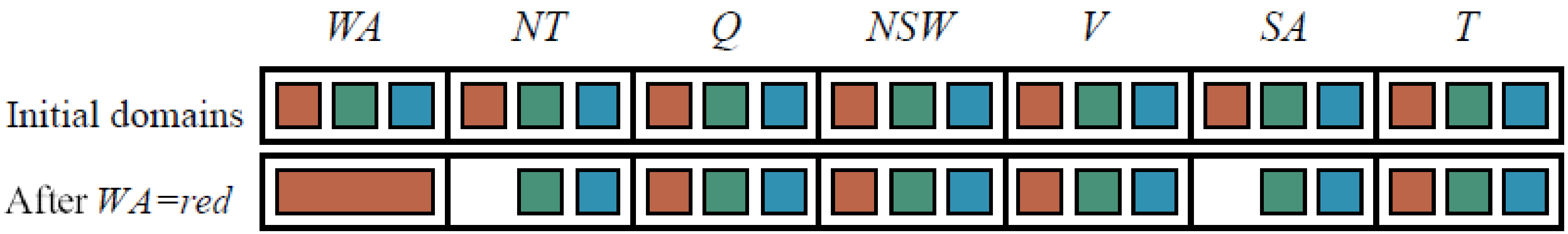
  Delete {*WA* = *red*, *Q* = *green*, *V* = *blue*}.
  Start backtracking.

Combining MRV and FC Heuristics

Search becomes more effective when they are combined.



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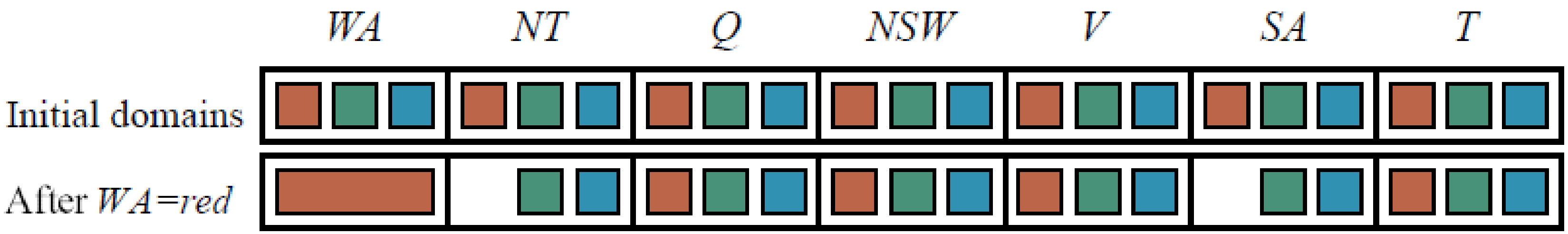

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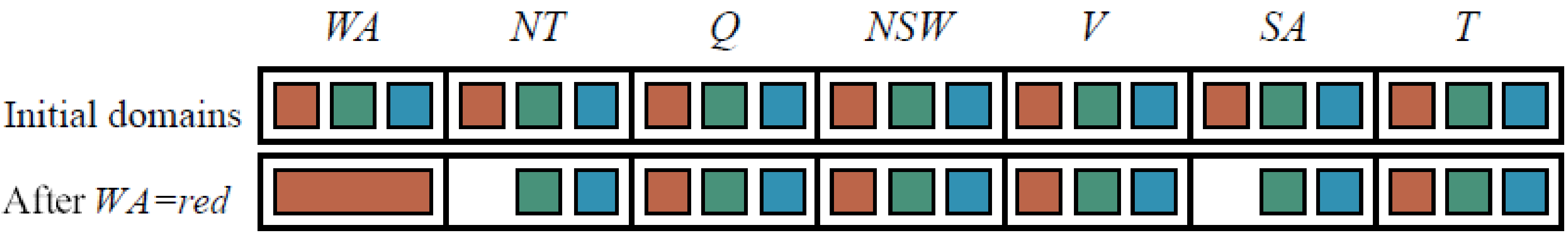

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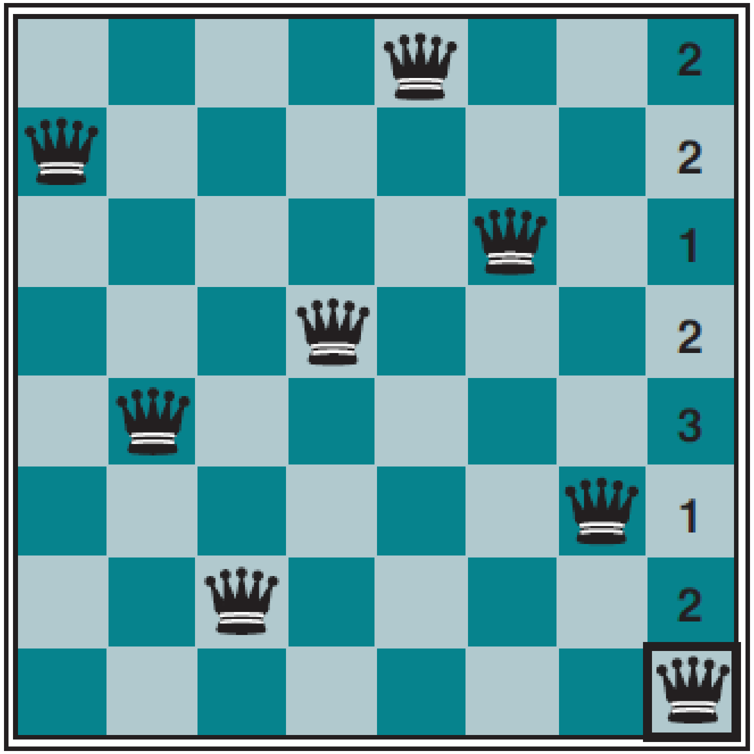
Min-conflicts heuristic:

- Start with a complete assignment.
- Randomly choose a conflicted variable.
- Select the value that results in the least conflicts with other variables.

Applying Min-conflicts to 8-Queen

Variable set: $\mathcal{X} = \{Q_1, Q_2, \dots, Q_8\}$

Q_i: the row number of the queen placed in the ith column.



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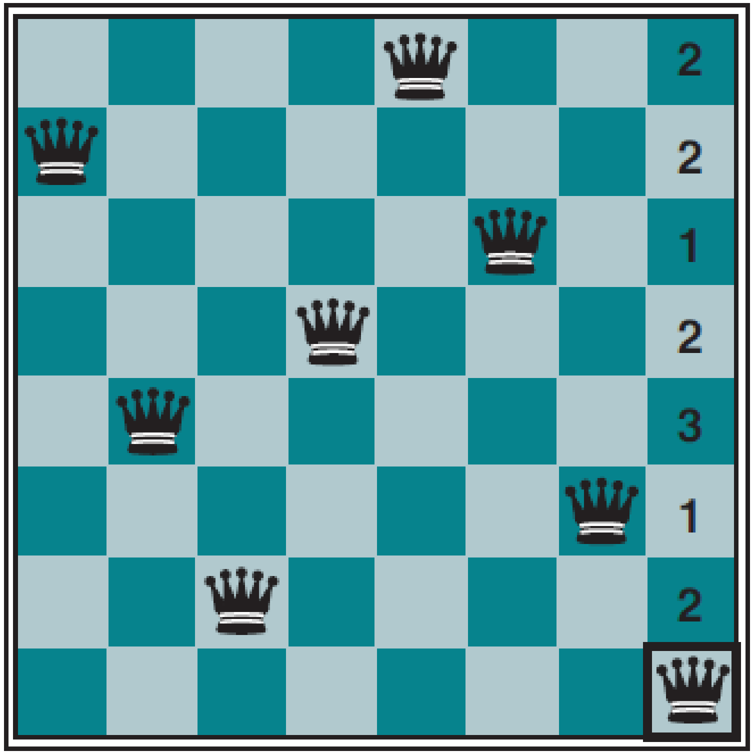

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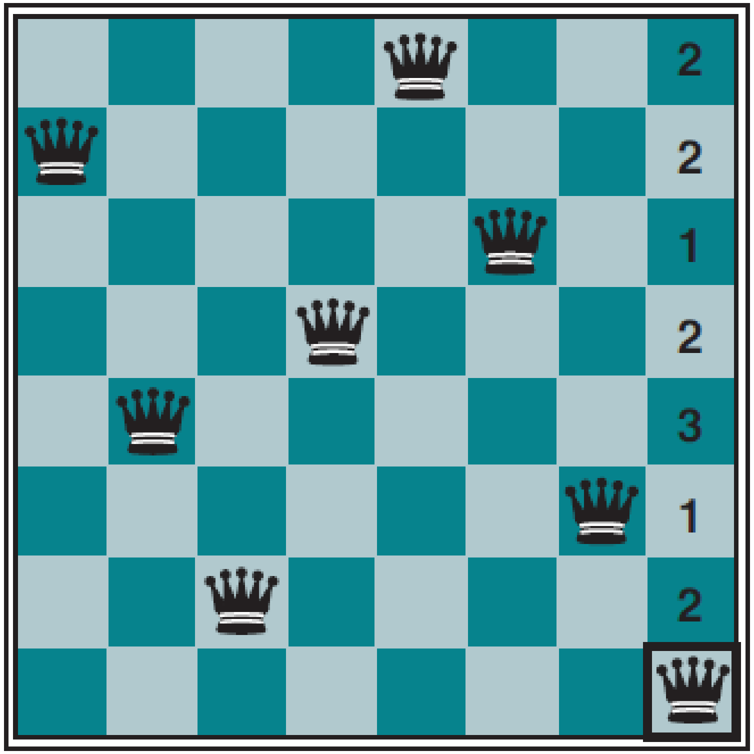

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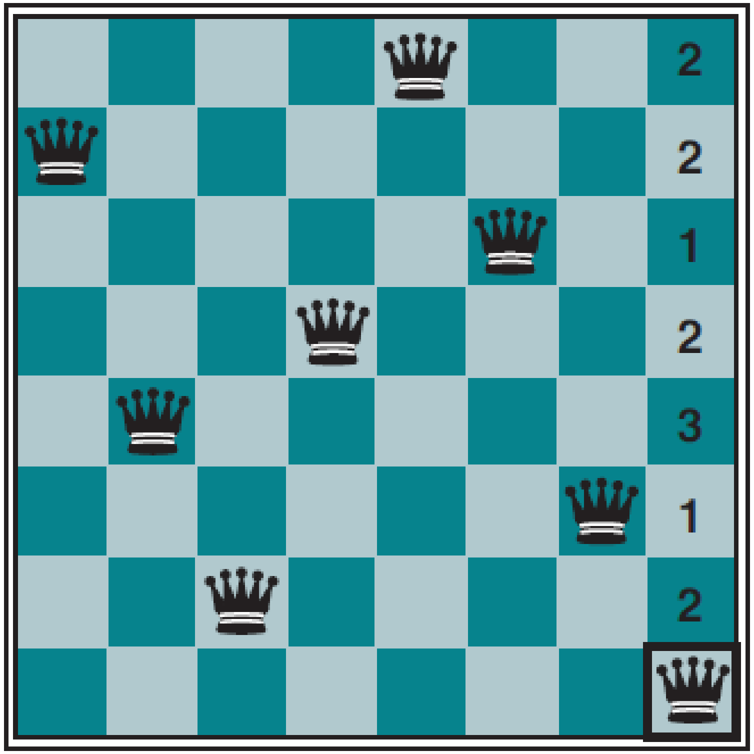

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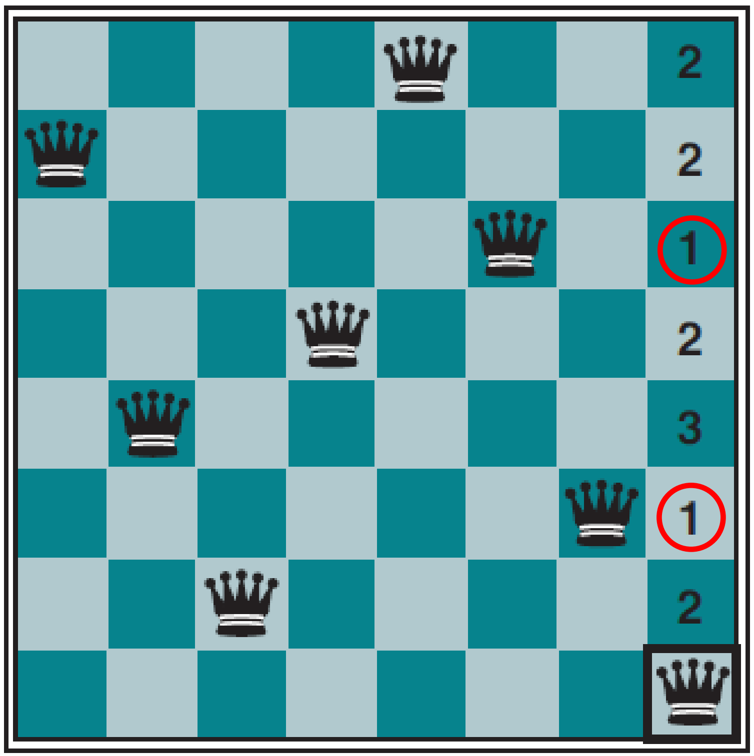
- The 8th queen in row 3 or 6 would violate only one constraint.



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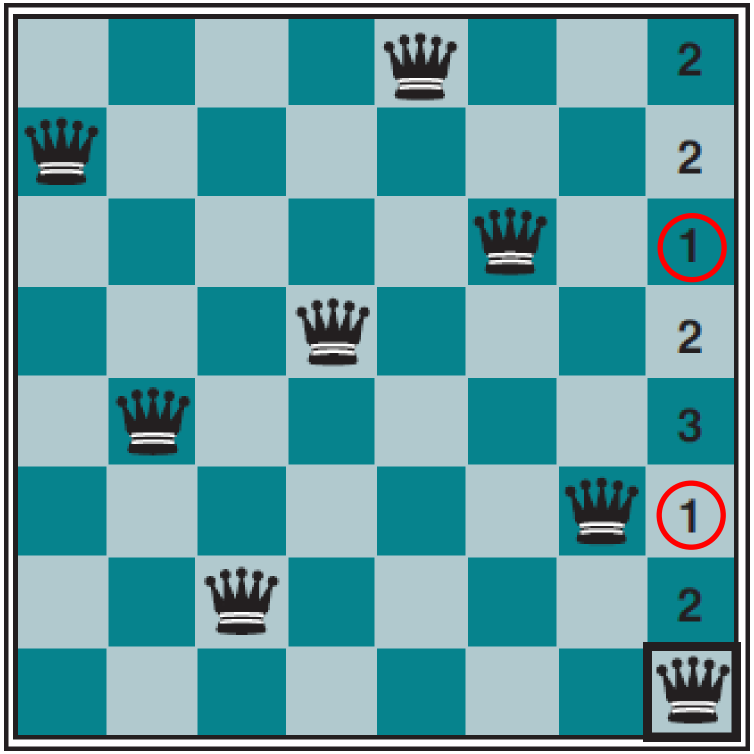
Variable set: $\mathcal{X} = \{Q_1, Q_2, \ldots, Q_8\}$

Q_i: the row number of the queen placed in the ith column.

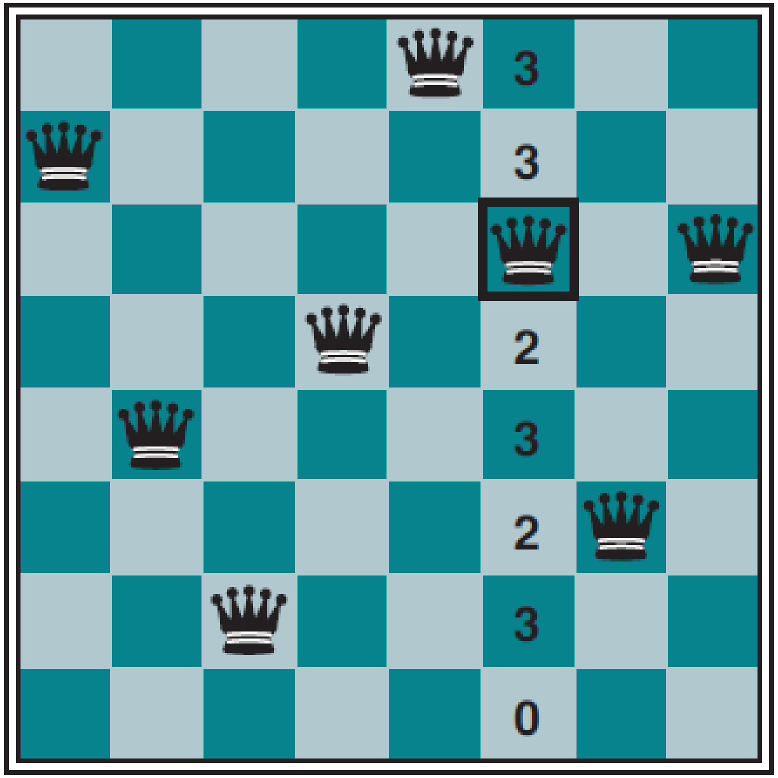


← 2 conflicts if $Q_8 = 7$

$Q_8 = 8$

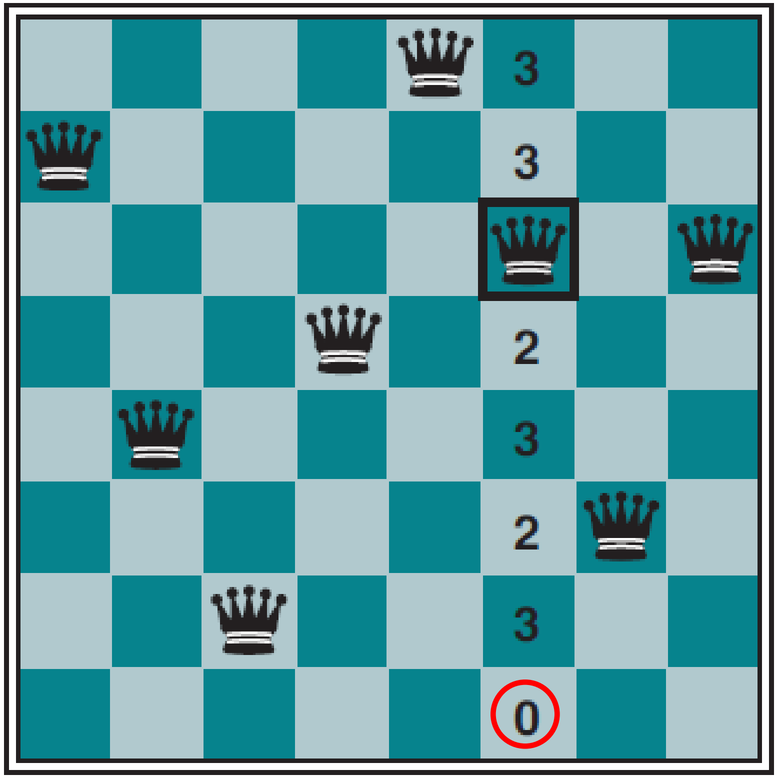
- Q_8 out of the set $\{Q_4, Q_8\}$ of conflicted variables by a random choice.
- The 8th queen in row 3 or 6 would violate only one constraint.
- Move the queen to, say, row 3.

8- and n-Queen problems



- Q_6 out of $\{Q_6, Q_8\}$.

8- and n-Queen problems



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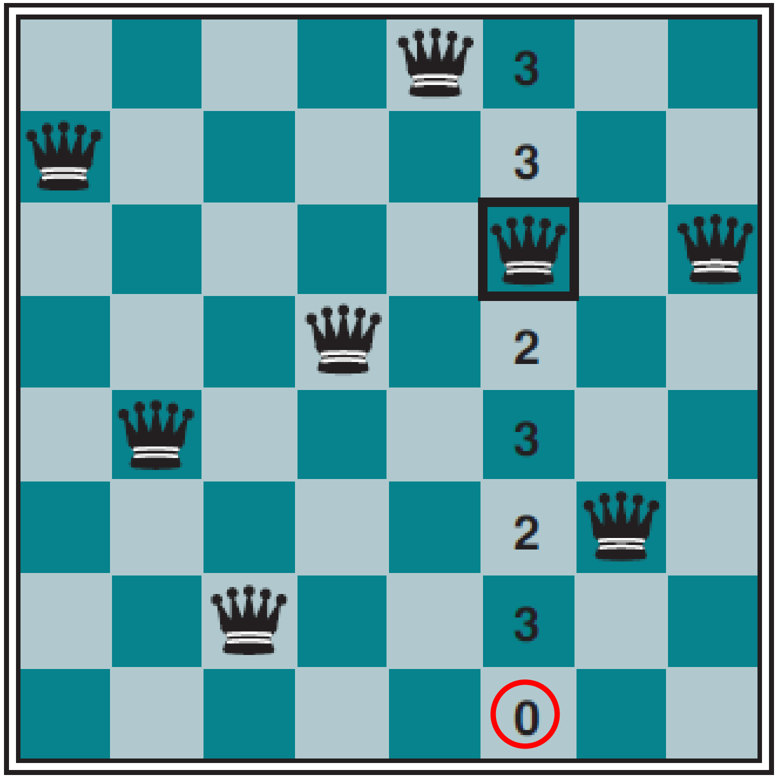

- Q_6 out of $\{Q_6, Q_8\}$.
- Reassignment: $Q_6 = 8$.

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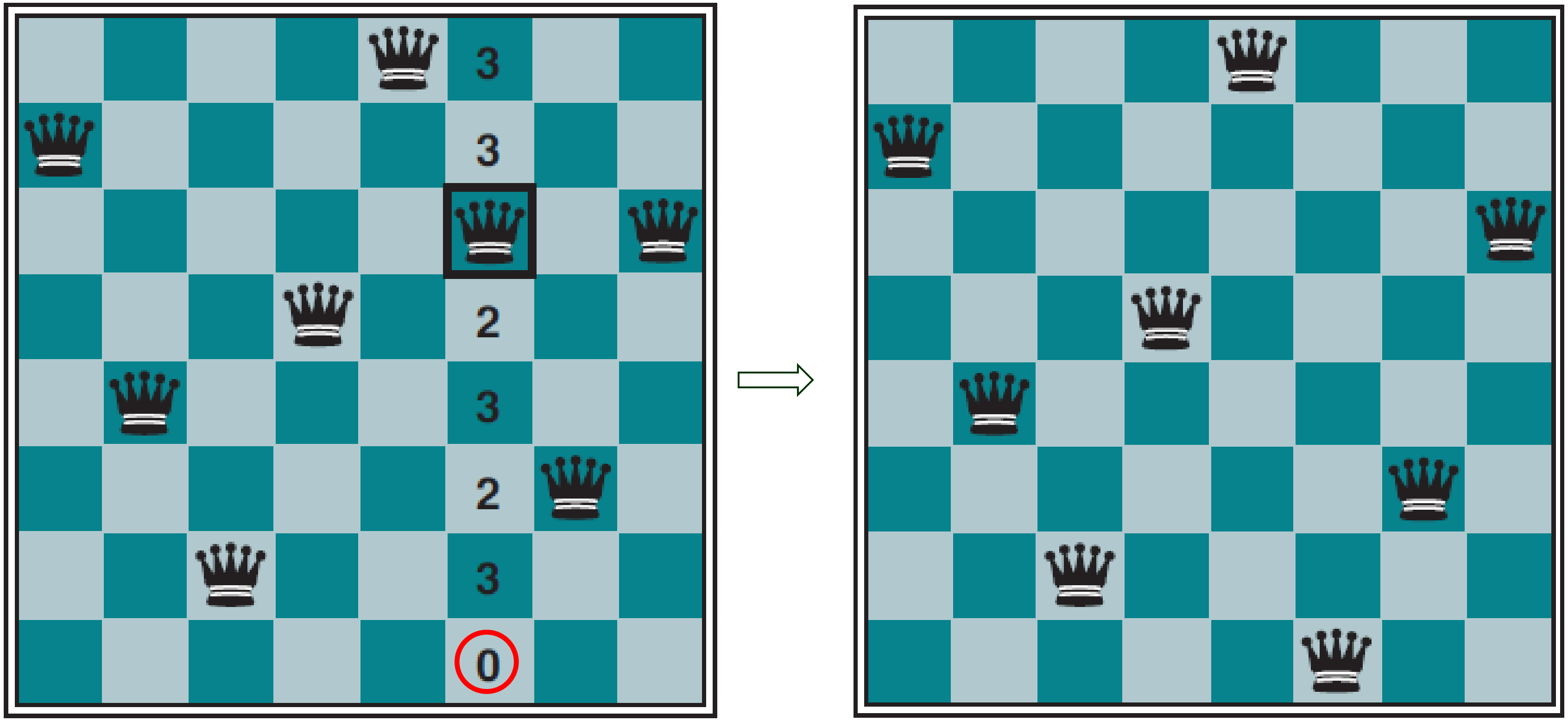

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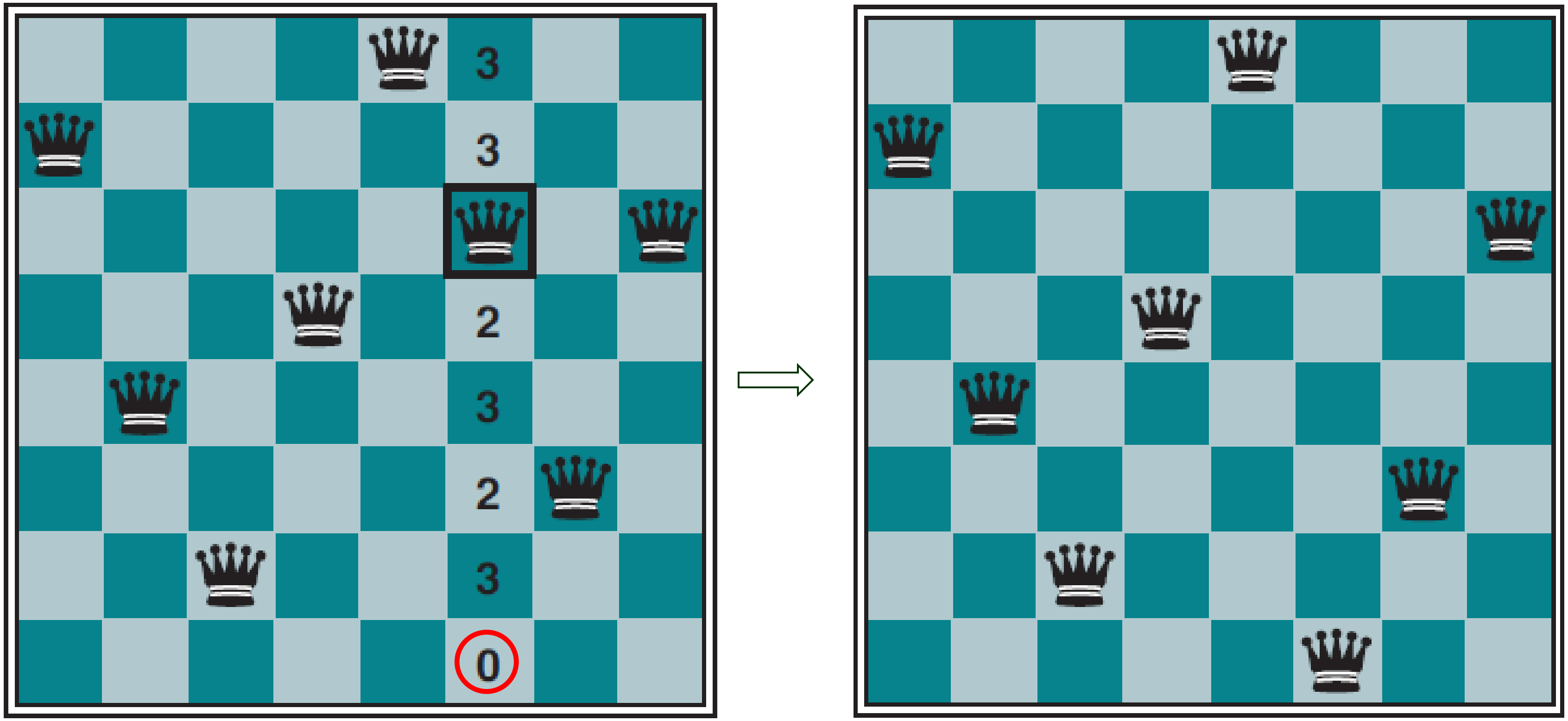

Solution

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Local Search: n-Queen and Beyond

- Run time of min-conflicts on n-queen is roughly independent of n.

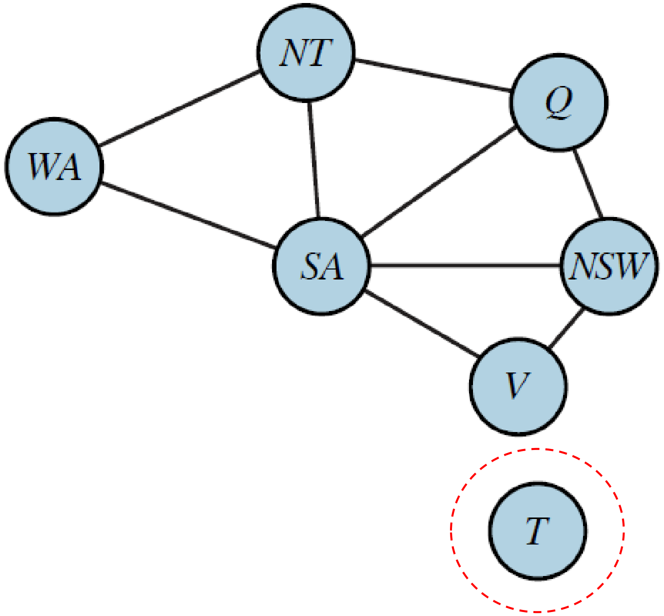
  10^6-queen problems are solved in an average of 50 steps.

- Ease of solving n-queen due to dense distribution of solutions throughout the state space.

- Min-conflicts also effective on hard problems such as observation scheduling for the Hubble Space Telescope.

- Local search is applicable in an online setting (e.g., repairing the scheduling of an airline's weekly activities --- in the advent of bad weather).

III. The Structure of CSP Problems



Independent subproblems

- Connected components in the constraint graph.
- Each subproblem can be solved independently.

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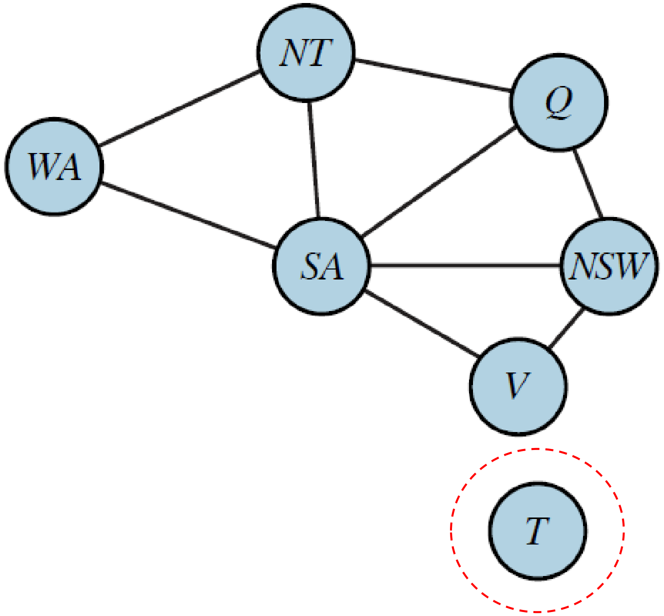

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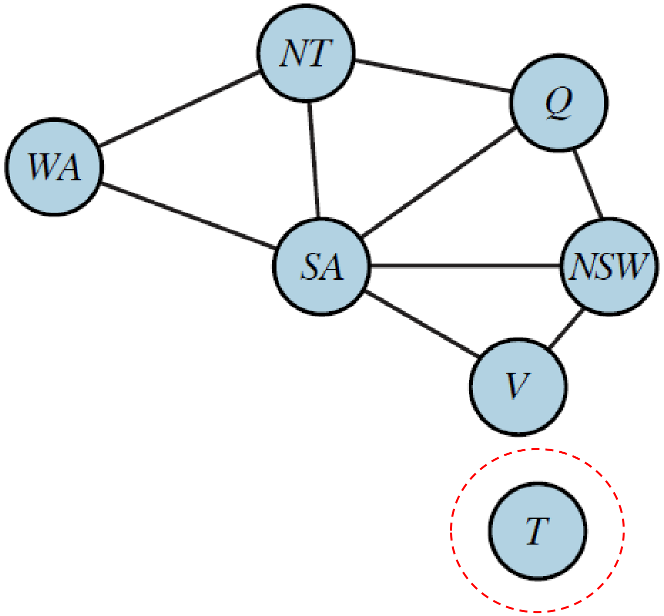

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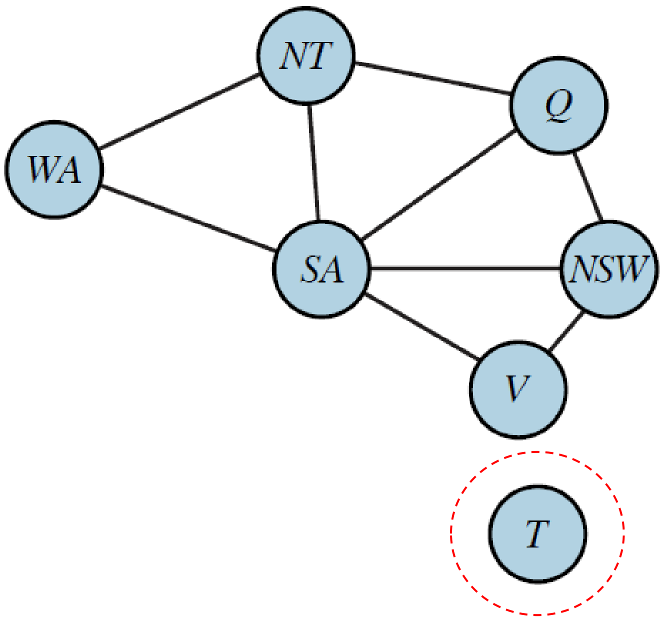

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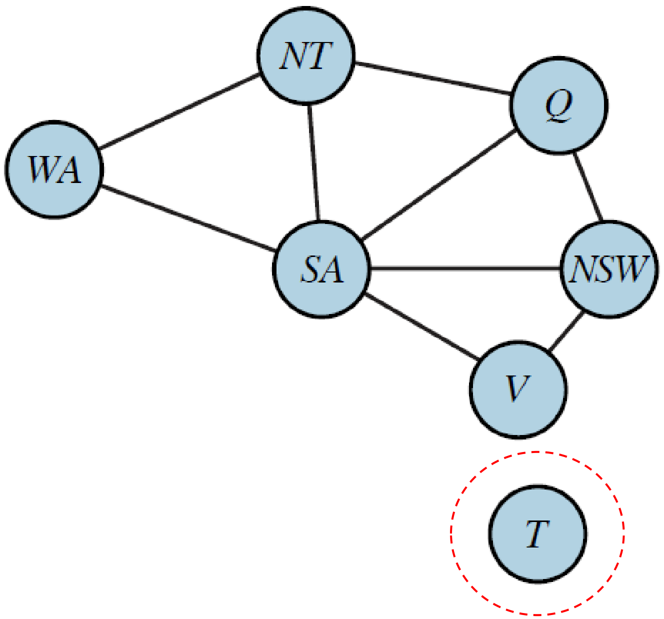

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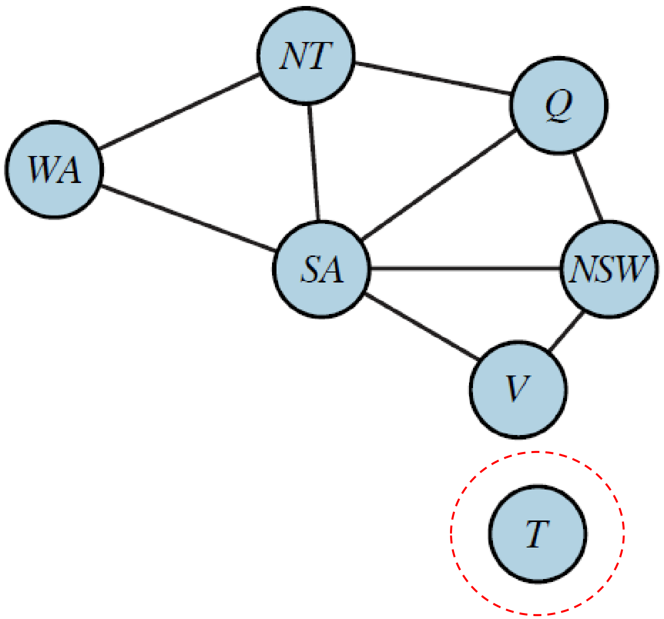

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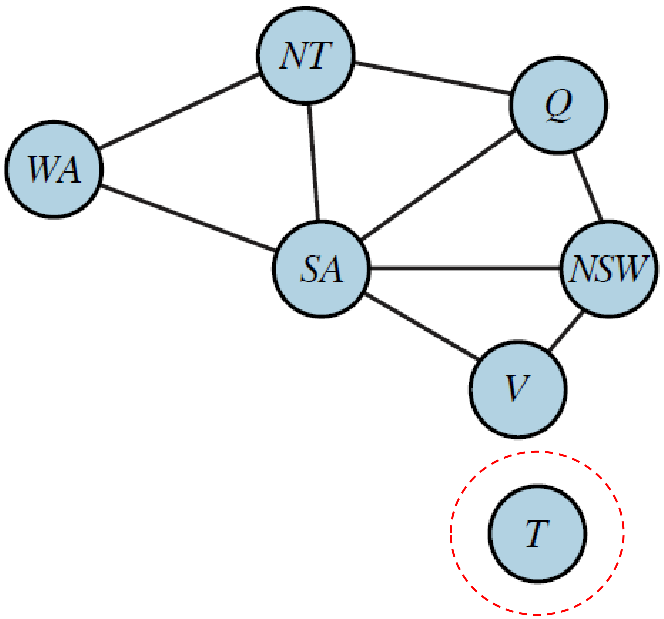

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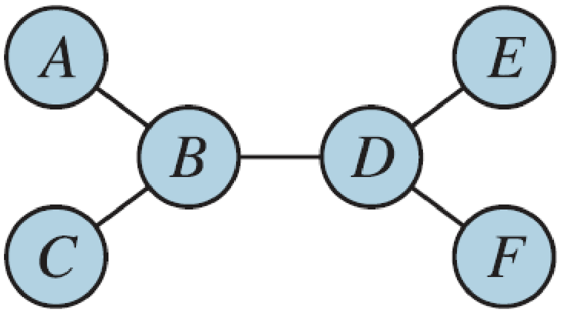
$\Longrightarrow$ Total work $O(d^c n/c)$ Linear in n.

Tree-Structured CSPs

Constraint graph is a tree.



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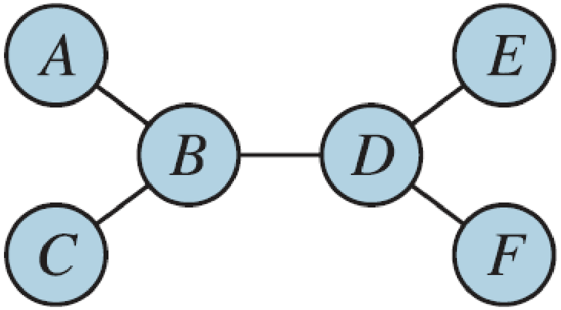

Solution:

- Generate a topological order of the variables.

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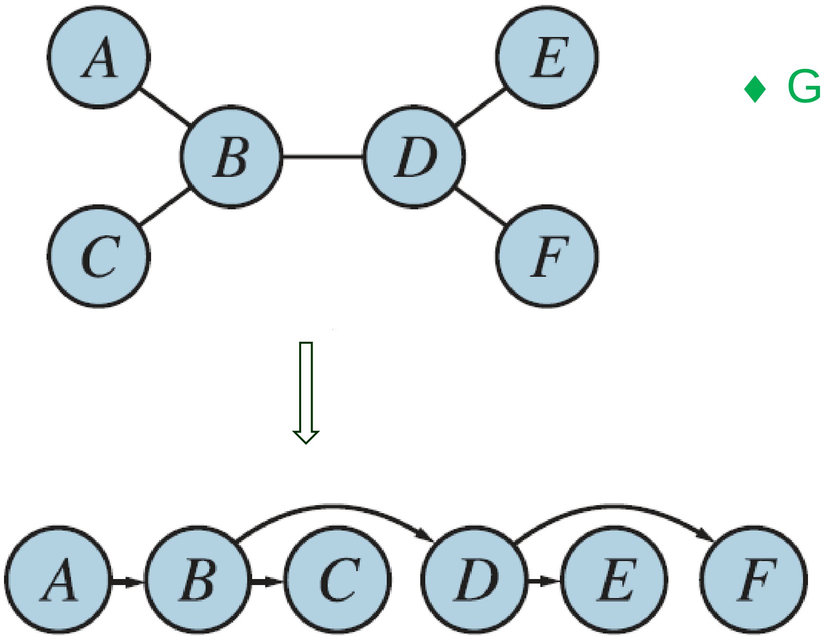

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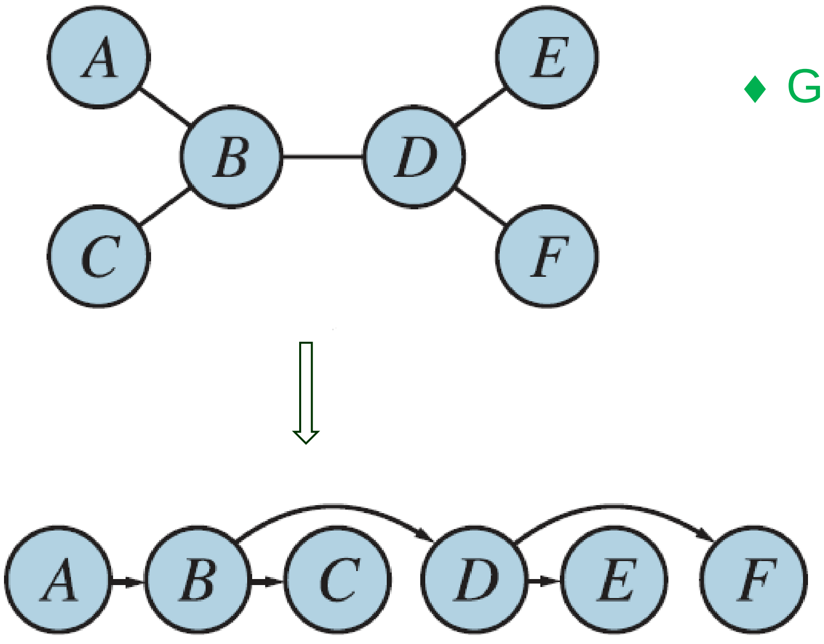

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$$O(n)$$

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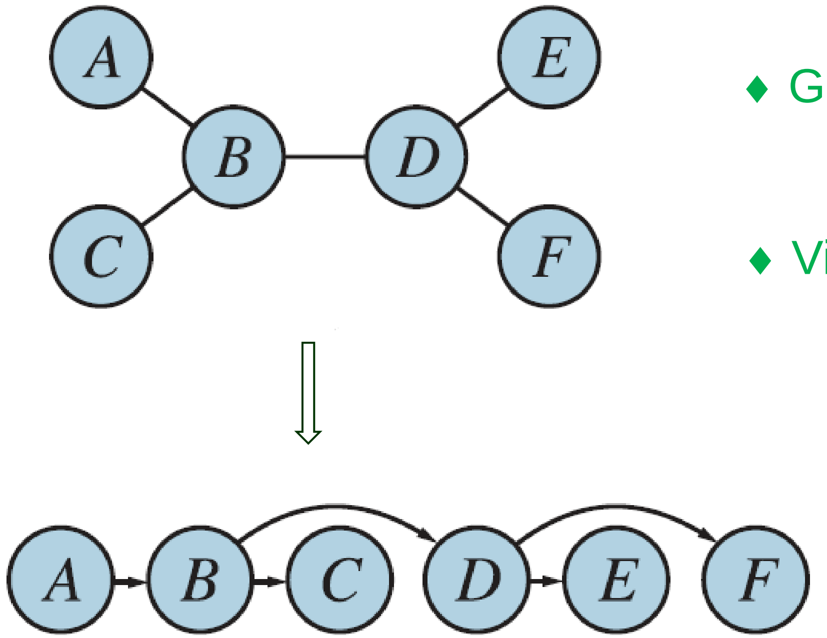

Solution:

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Tree-Structured CSPs

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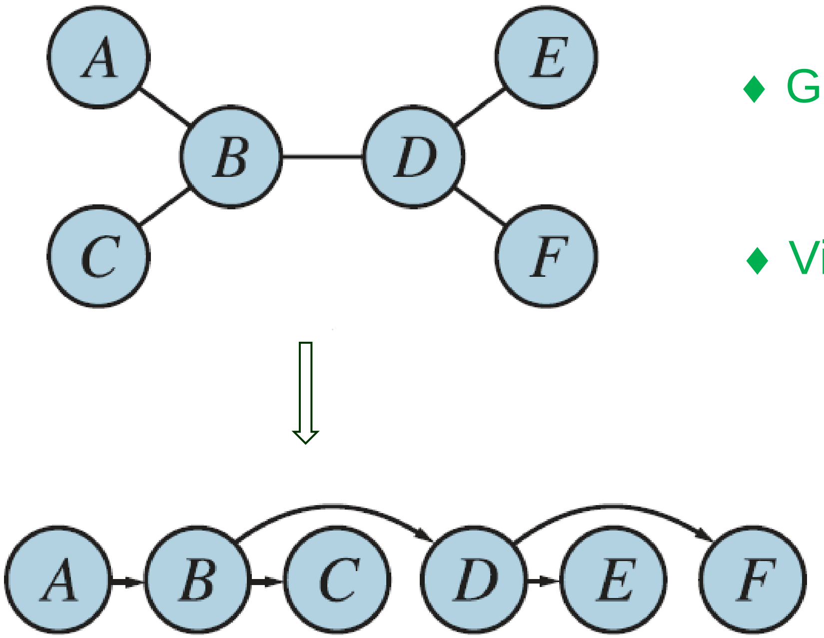

Solution:

- Generate a topological order of the variables. $O(n)$
- Visit variables in the order. $O(n)$
  - At each visited variable, make every outgoing edge arc-consistent.

Tree-Structured CSPs

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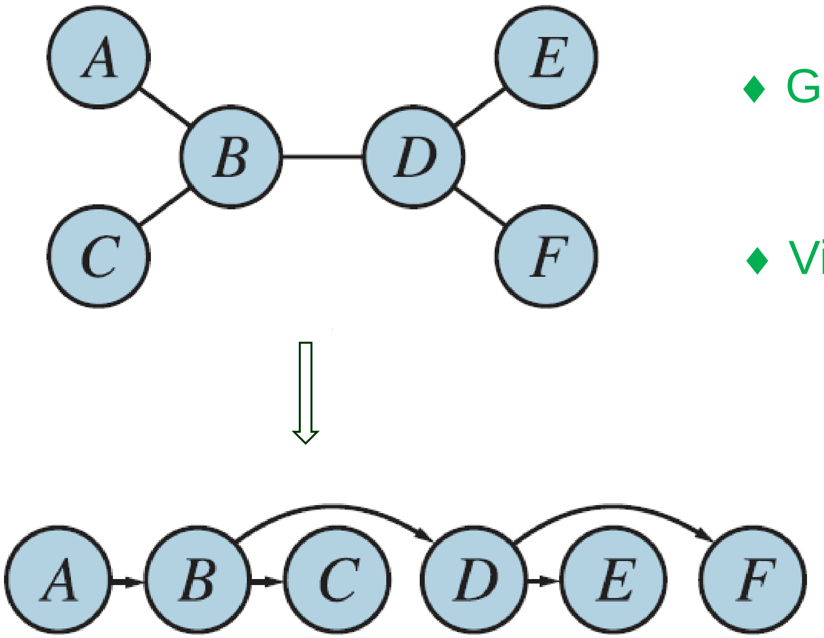

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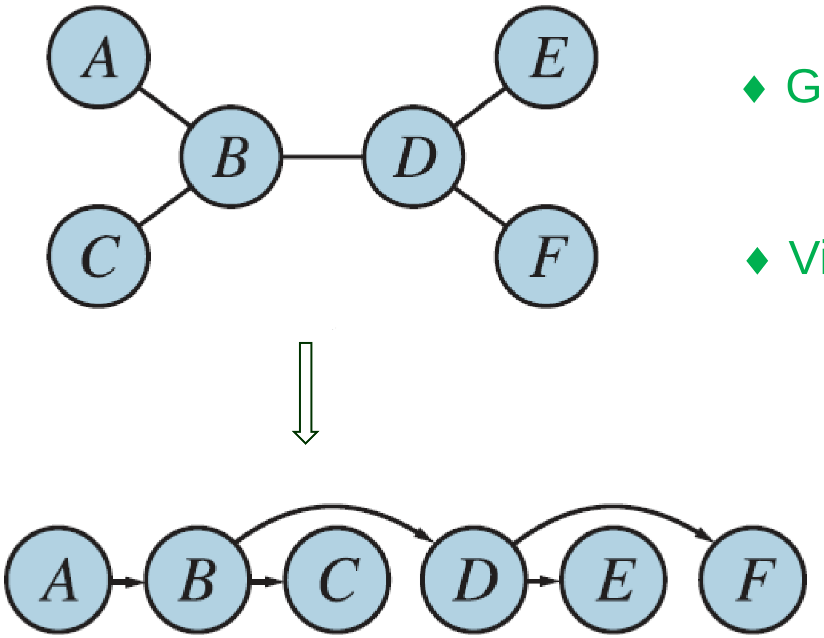

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- Finally, visit variables in the topological order again and choose any value from its reduced domain,

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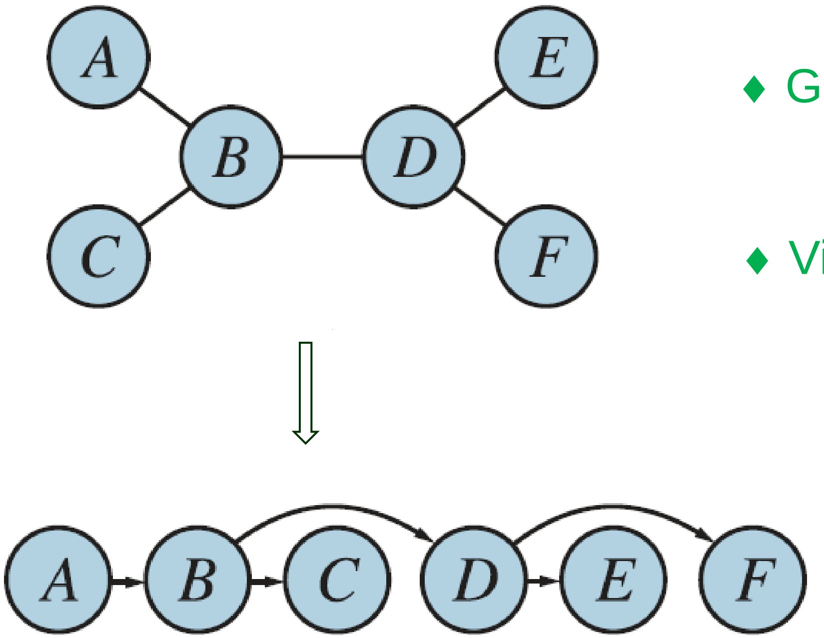

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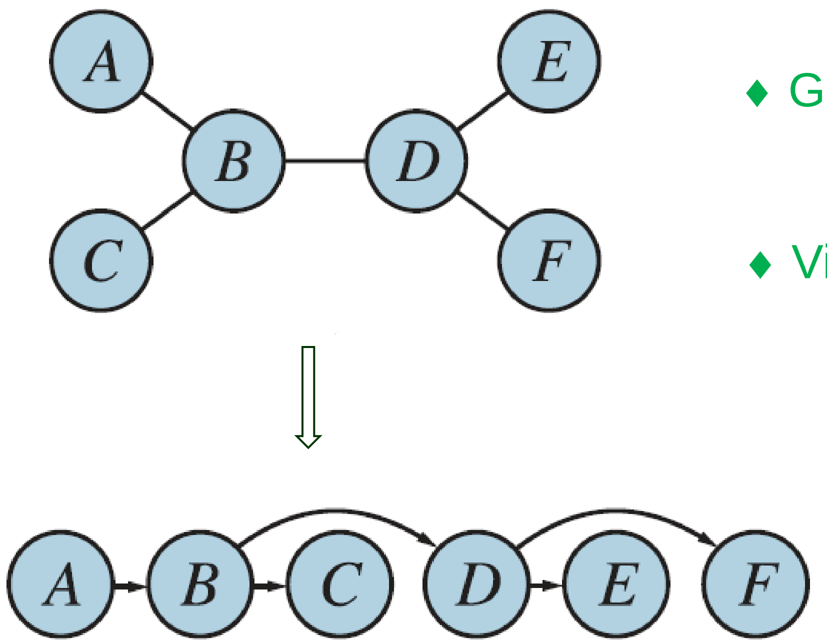

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  $O(d^2)$

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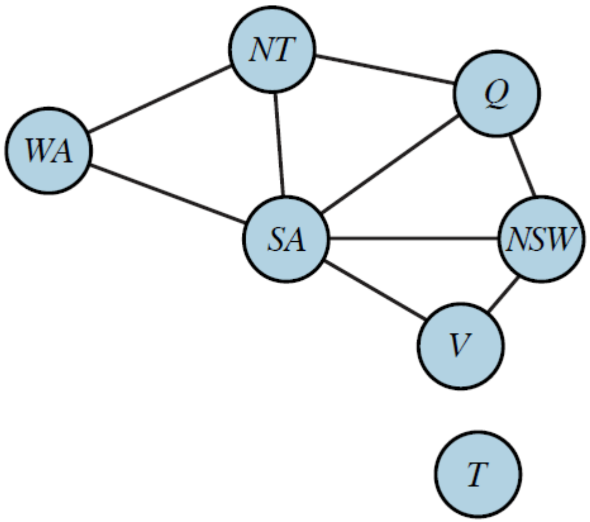
Tree CSP Solver

```
function TREE-CSP-SOLVER(csp) returns a solution, or failure
  inputs: csp, a CSP with components X, D, C

  n ← number of variables in X
  assignment ← an empty assignment
  root ← any variable in X
  X ← TOPOLOGICALSORT(X, root)
  for j = n down to 2 do
     MAKE-ARC-CONSISTENT(PARENT(X_j), X_j)
     if it cannot be made consistent then return failure
  for i = 1 to n do
     assignment[X_i] ← any consistent value from D_i
     if there is no consistent value then return failure
  return assignment
```

Cutset Conditioning

Reduce a constraint graph to a tree (or a forest) by assigning values to some variables.



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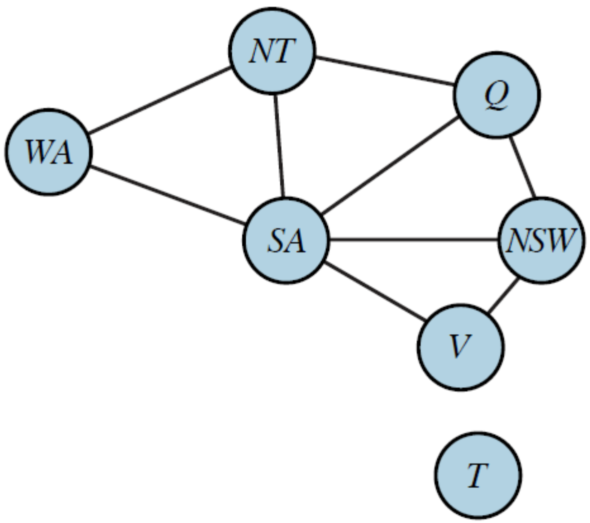
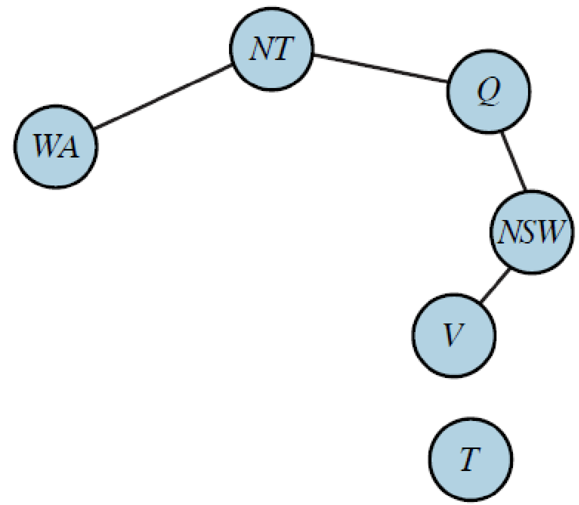


Assign a value to *SA* and remove the node.



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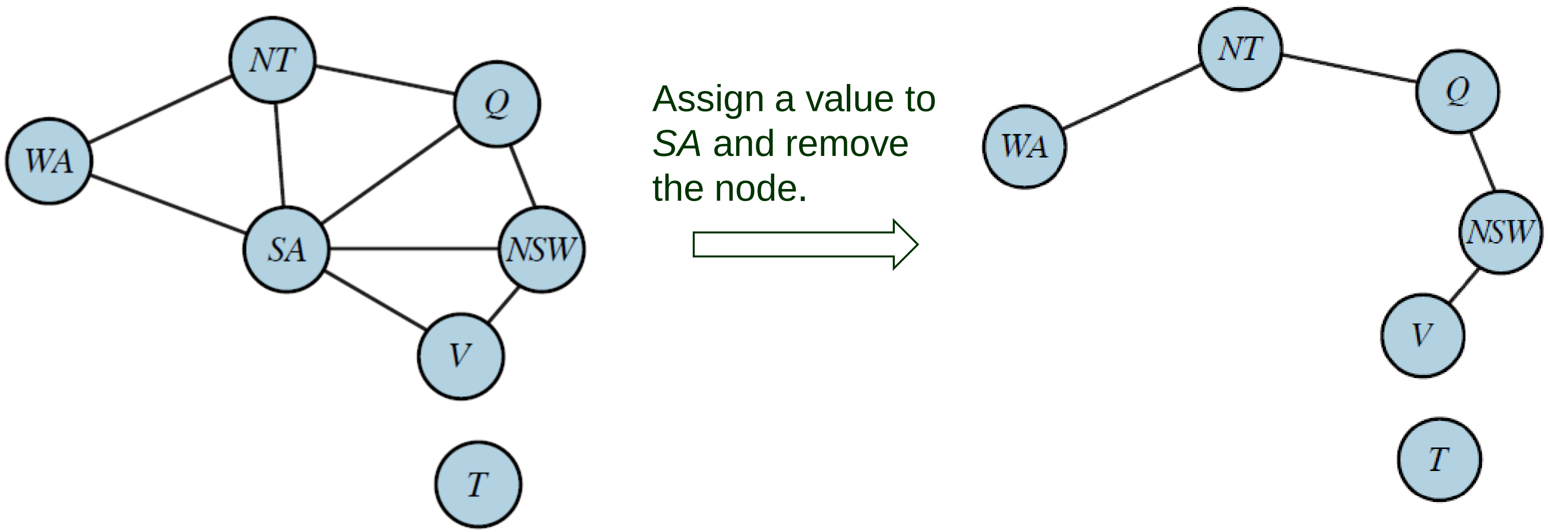

1. Choose a subset $S \subset \mathcal{X}$ of variables whose removals reduce the constraint graph to a tree (or a forest).

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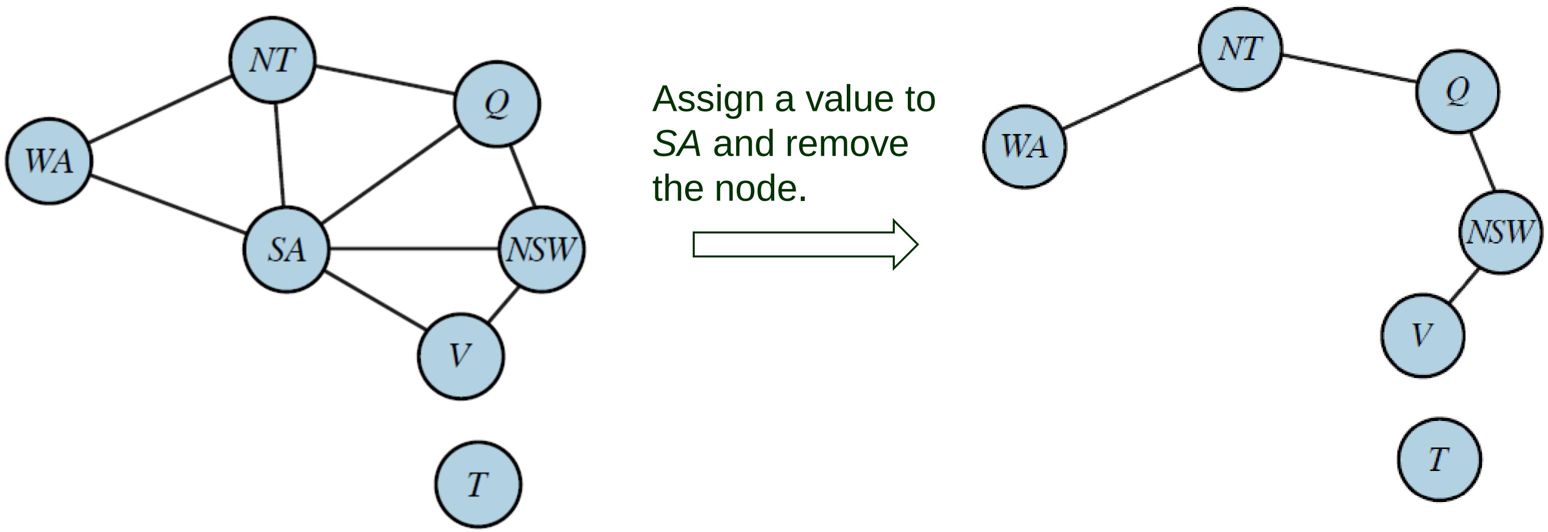

1. Choose a subset $S \subset \mathcal{X}$ of variables whose removals reduce the constraint graph to a tree (or a forest).
2. For every consistent assignment A to variables in S:
   - remove from the domain of every $X \in \mathcal{X} \setminus S$ all values that are inconsistent with A.
   - return the solution to the reduced CSP (if exists) along with A.

Tree Decomposition

Transform the constraint graph into a tree where each node consists of a set of variables such that

- Every variable X must appear in at least one tree node n.
- Two variables X, Y sharing a constraint must appear together in at least one node n.
- If X appears in two nodes n_1 and n_2, it must appear in every node on the path connecting n_1 and n_2.

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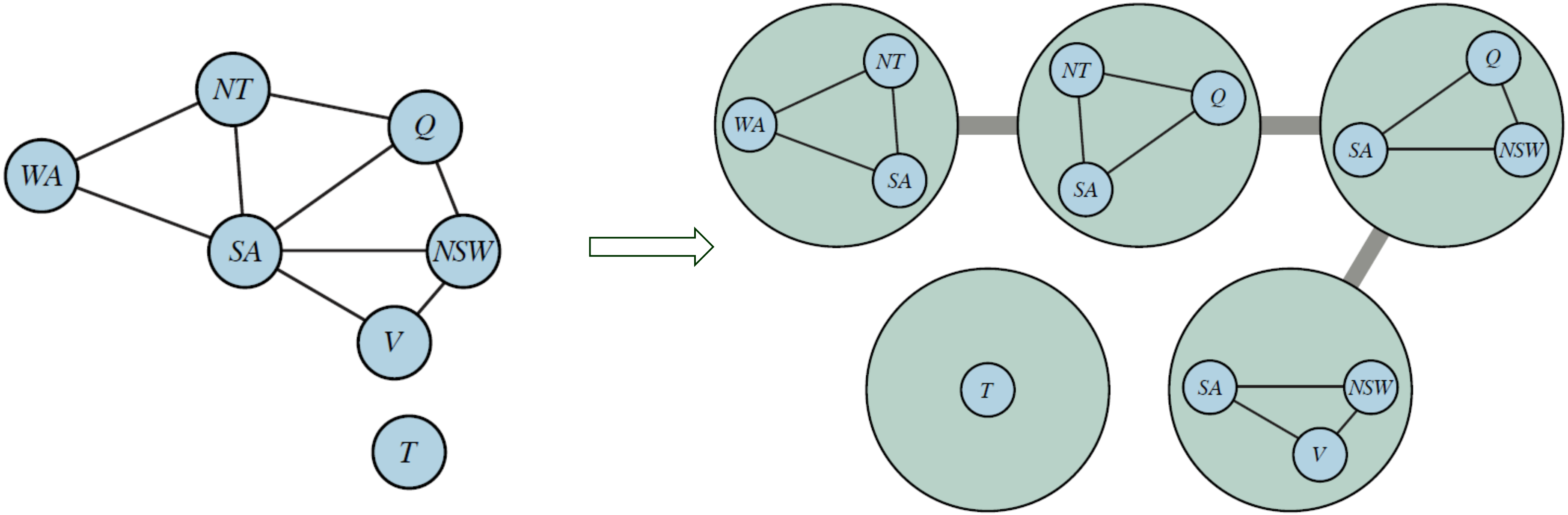

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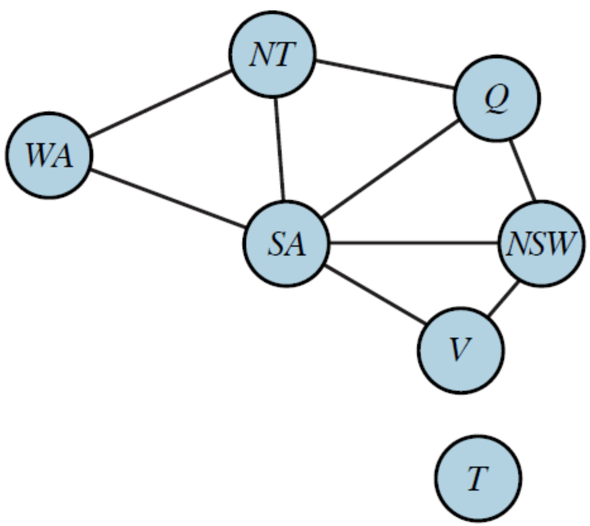

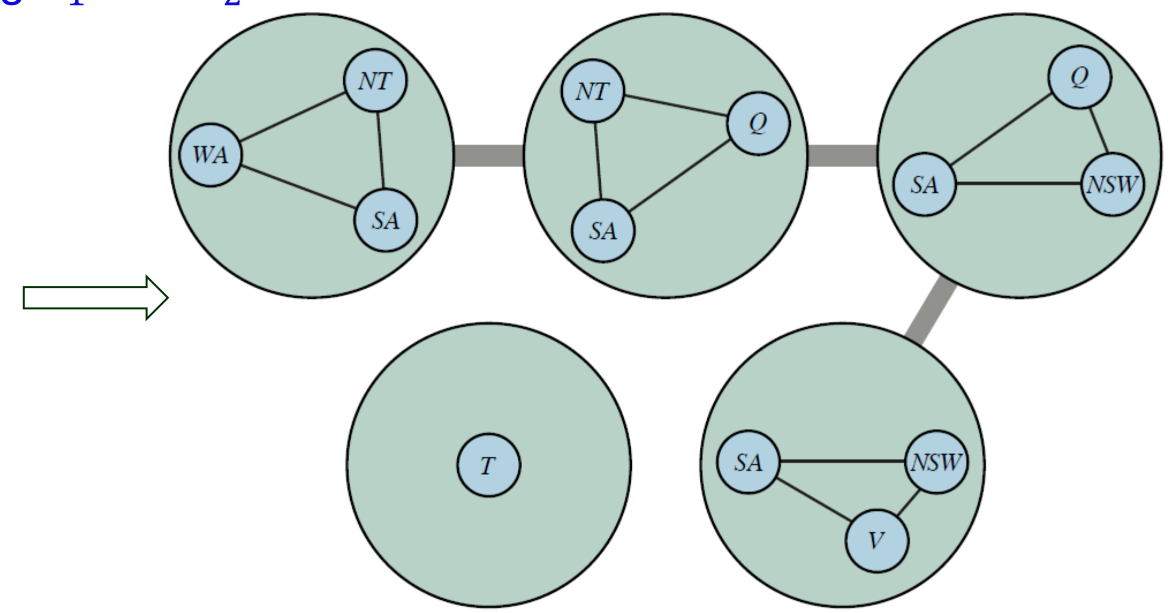

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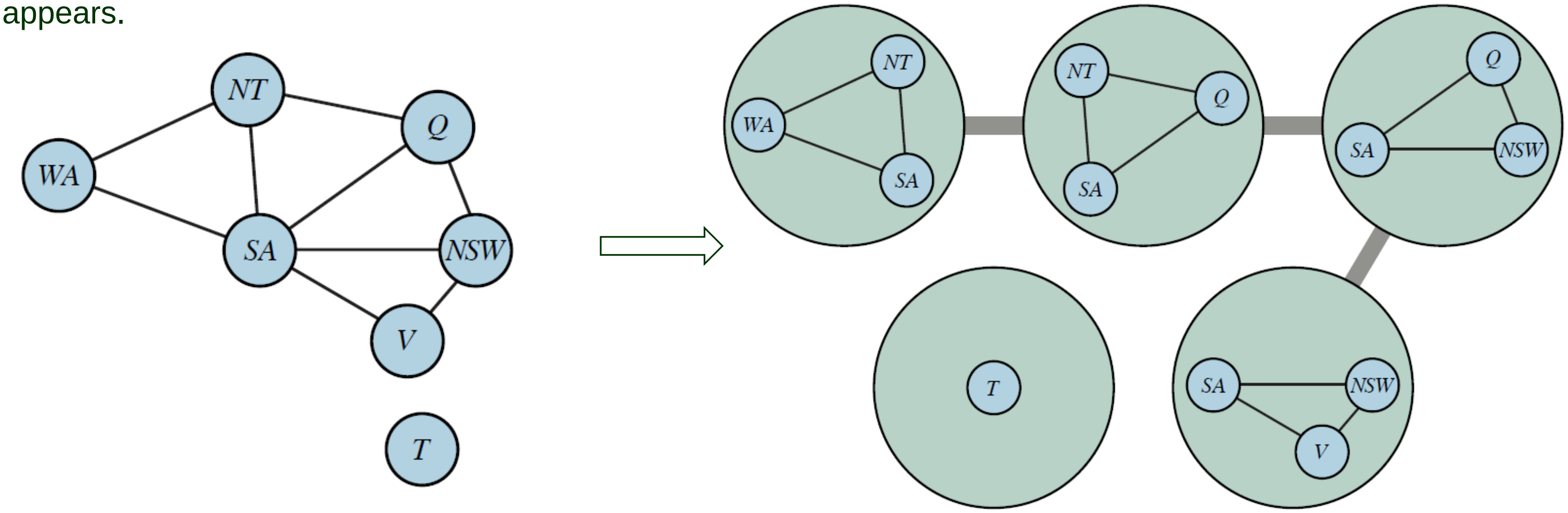
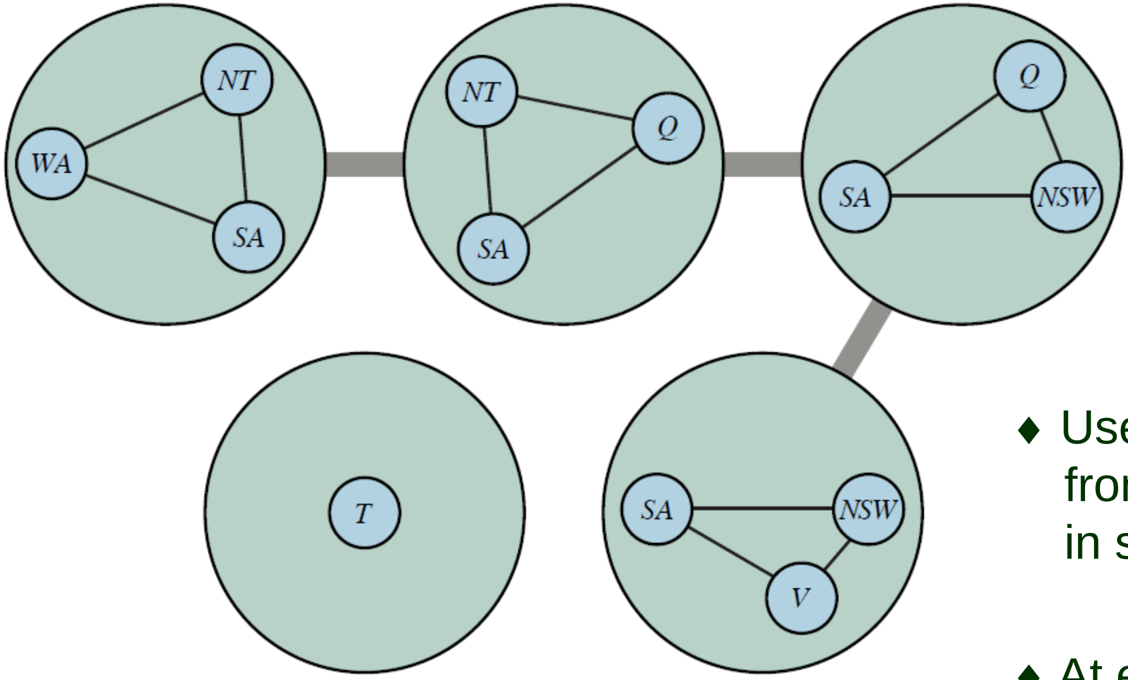


Solution After Tree Decomposition



- Use the CSP tree solver to move from one tree node to the next in some topological order.
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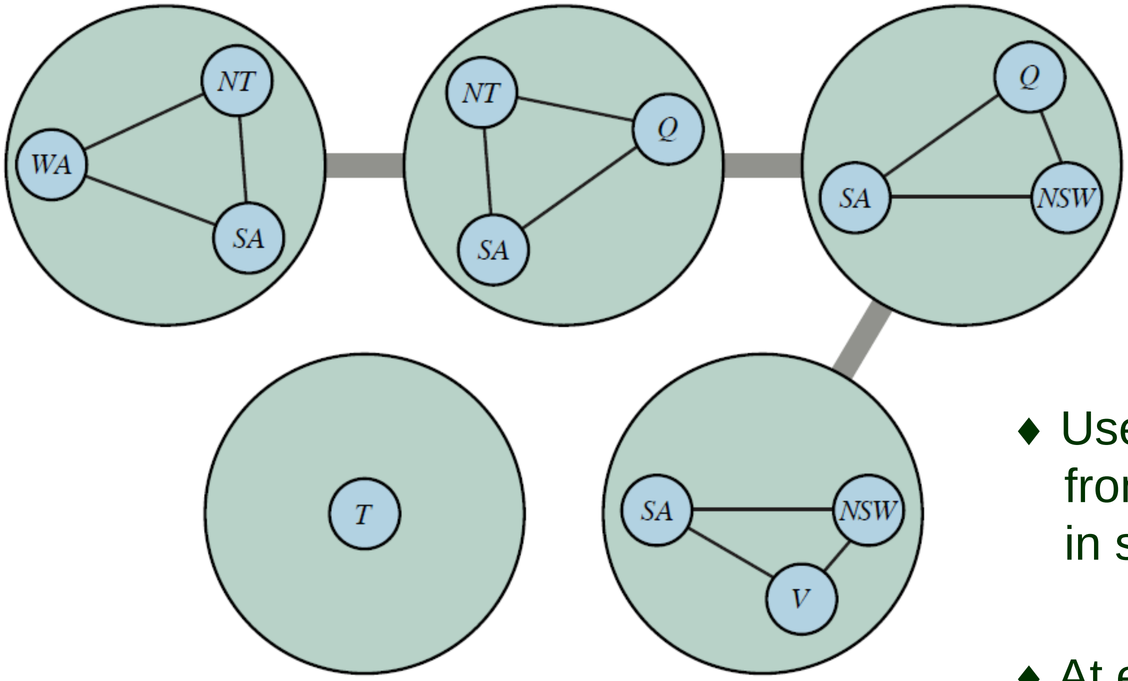

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