3.2 Non - Linear Models

Population Dynamics   let $P(t)$ = population at time t.

Model Equation : $\frac{dP}{dt} = KP \Rightarrow P(t) = P_0 e^{Kt}$

Exponential Growth in long term : as $t \to \infty \Rightarrow P(t) \longrightarrow \infty$

True cases that fit this growth model over long periods of time are hard to find due to limited resources. More realistic models would satisfy that

relative rate of growth, $\dfrac{dP/dt}{P}$, decreases as P increases.

We'd like a function $f(P)$ such that: $\frac{dP/dt}{P} = f(P)$ decreases as P increases.

Then work with the model:

$$\frac{dP}{dt} = P f(P)$$

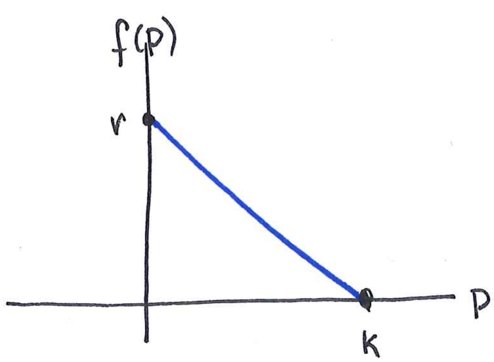
Logistic Equation

Definition

The maximum number of individuals that an environment is capable of sustaining is called the carrying capacity.

Say $k :=$ carrying capacity, let's figure out a function f that works.

Let $f(0) = r$ (some initial rate of growth), we'd like $f(k) = 0$ and we want f to be decreasing:



we use the simplest curve joining the intercepts (a straight line).

$f(P) = C_1 P + C_2$ (We know $r = C_2$ $f(0) = C_2 = r$)

We need $f(K) = C_1 K + r = 0 \Rightarrow C_1 = -r/k$

Let $a = r$ and $b = r/k$ then $f(P) = a - bP$

$$\Rightarrow \boxed{\frac{dP}{dt} = P(a - bP)} \leftarrow \text{Logistic Equation}$$

Solution to the Logistic Equation

$$\frac{dP}{dt} = P(a - bP) \qquad \text{(Seperable).}$$

Separate: & Integrate $\displaystyle\int \frac{1}{P(a-bP)}\,dP = \int dt \Rightarrow \int \frac{1/a}{P} + \frac{b/a}{a-bP}\,dP = \int dt$

$\Rightarrow \frac{1}{a}\ln|P| - \frac{1}{a}\ln|a-bP| = t + C \Leftrightarrow \cancel{\frac{1}{a}}\ln\left|\frac{P}{a-bP}\right| = at + Ca$

$\dfrac{P}{a-bP} = e^{at}e^{ac} = e^{at}C_1 \Leftrightarrow P = (a-bP)e^{at}C_1$

$P(1 + C_1 b e^{at}) = C_1 a e^{at} \Rightarrow P = \dfrac{C_1 a e^{at}}{1 + C_1 b e^{at}}$, if we know

$P(0) = P_0$ we can find $C_1 = \dfrac{P_0}{a - bP_0}$ then the solution is

$$P = \frac{aP_0 e^{at}}{bP_0e^{at} + a - bP_0}$$ which can be written

$$\boxed{P = \frac{aP_0}{bP_0 + (a-bP_0)e^{-at}}}$$

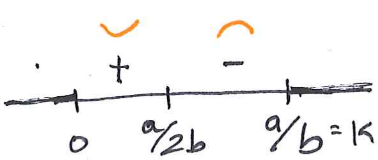
Note that $\lim\limits_{t\to\infty} P = \dfrac{a}{b} = \dfrac{r}{r/K} = K$ and $\lim\limits_{t\to-\infty} P = 0$

We will find the inflection pts / concavity. We need $P''(t)$

$P''(t) = \frac{d}{dt}\left(\frac{dP}{dt}\right) = \frac{d}{dt}\left(P(a-bP)\right) = \underbrace{\frac{dP}{dt}}(a-bP) + P\left(-b\underbrace{\frac{dP}{dt}}\right)$

$= \underbrace{[P(a-bP)]}(a-bP) - bP\overbrace{P(a-bP)} = P(a-bP)[a-bP-bP]$

$P''(t) = P(a-bP)(a-2bP) \underset{\uparrow_{\text{set}}}{=} 0 \qquad P'' = 0$ at $P = 0, a/b, a/2b$.



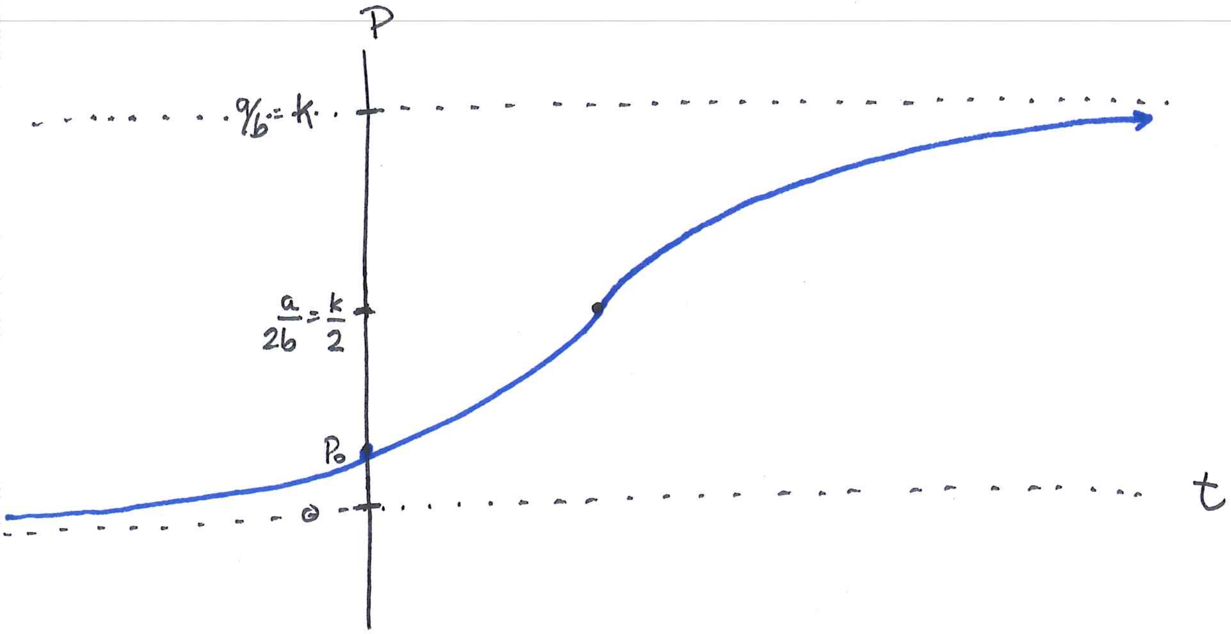
There is one inflection point at $a/2b$

$0 < P < a/2b < a/b$

$0 < a - 2bP$

if $a/2b < P < a/b$

$a - 2bP < 0$



Example A student with flu virus comes to a 1000-student campus. Assume the rate at which the virus is spread is proportional to the number x of infected students and to the number of students not infected. Determine the number of students infected after 6 days, if we also know that after 4 days there were 50 infected students.

$$\frac{dx}{dt} = cx(1000 - x) \iff \frac{dx}{dt} = x(1000c - cx) \qquad a = 1000c, \quad b = c$$

$$x(t) = \frac{1000c}{c + (1000c - c)e^{-1000ct}} = \frac{1000}{1 + 999e^{-1000ct}}$$

Plug in $t=4$ & solve for $1000c$ $\left(x(4) = 50\right) \Rightarrow 1000c = \frac{1}{4}\ln\left|\frac{50(999)}{950}\right|$

$\Rightarrow 1000c \simeq 0.9906$

$$\Rightarrow x(t) = \frac{1000}{1 + 999e^{-0.9906t}}$$ ← Solution to the D.E.

Then we can plug in $t = 6$

$X(6) \simeq 276$ students

Note: $X(10) = 952$, $x(20) = 999.\dots$, $x(30) = 999.\dots$

Other variations:

$$\frac{dP}{dt} = P(a - bP) \pm h(t)$$

Restock

Harvesting.

$$\frac{dP}{dt} = P(a - bP) + ce^{kt}$$

immigration