Final exam

- Please **write your name and netid** on the top of this page.
- There are 11 questions in this exam, totaling 105 points.
- The maximum score will be capped to 100 points. Any score above 100 will be rounded to 100.
- Total duration: 120 minutes.
- You **can** use three pages as cheat sheets.
- You **cannot** consult your notes, textbook, your neighbor, or Google.
- Use the back of the page if needed for any rough work.
- Give sufficient explanation behind your thinking. You will get partial credit even if your final answer is wrong but if you have thought about it correctly.

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1. **(5 points)** Let S be the set of countries in mainland Europe. For $a, b \in S$, define a relation R such that aRb iff a and b border each other. Which properties of an equivalence relation (reflexive, symmetric, transitive) does R satisfy? Explain your reasoning for each of these properties.

Solution

Not Reflexive - A country cannot border itself.

Symmetric - If country A borders country B, then country B must border country A.

Not Transitive - Germany borders Austria and Austria borders Hungary but Germany doe not border Hungary.

2. **(10 points)** Consider a complete bipartite graph K_n with $2n$ nodes, where each of the first n nodes are connected via undirected edges to each of the last n nodes.

(a) Let e_n be the sequence denoting the number of edges in K_n. Evaluate the sequence $e_1, e_2, e_3, e_4, e_5, ...$

Solution

$$\begin{aligned} e_1 &= 1 \\ e_2 &= 4 \\ e_3 &= 9 \\ e_4 &= 16 \end{aligned}$$

(b) Guess a closed form expression for e_n.

Solution

$e_n = n^2$

(c) Construct a recurrence relation for the number of edges, $e(n)$, in this graph. (Hint: how do you construct K_n given K_{n-1}?)

Solution

$e_n = e_{n-1} + 2n - 1$

(d) Verify that your recurrence relation in part b is correct by plugging in the derived expressions for e_n and e_{n-1} from part b.

$e_n = (n-1)^2 + (2n-1) = n^2$

3. **(10 points)** The power set $A = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$ is partially ordered with respect to the "subset of" relation.

(a) Draw the Hasse diagram representation of the above relation.

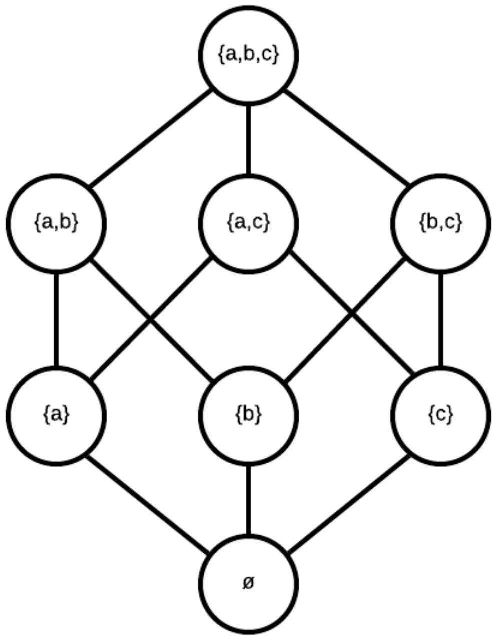

Figure 1: Hasse

(b) List all minimal and maximal elements.

Minimal: $\emptyset$

Maximal: $\{a,b,c\}$

(c) Run topological sort on the Hasse diagram to obtain a compatible total ordering of the elements.

Topological Sort: $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$. This topological sort is not unique

4. **(10 points)** On any given game night in Hilton Coliseum, there are at least 700 spectators. Your goal is to prove that irrespective of who attends the game, there *always* will exist two people in the crowd with the same initials of their first and last names.

(a) First, calculate the maximum number of possibilities of (first-initial, last-initial) pairs.

Solution

There are 26 ways to choose the first initial and 26 ways to choose the second initial. Total number of possibilities = 26*26 = 676

(b) Use the above calculation to (clearly) state your proof using a simple combinatorial principle discussed in class.

Solution

We can use the Piegon Hole Principle. We have 700 people (piegons) and 676 different pairs (holes). So since we have more people than first name initial -last name initial pairs, atleast 2 people will have to share the same initials.

5. **(10 points)** Find a counterexample for each of the following statements. You can simply write down, or draw, or clearly describe your counterexample.

(a) Every triangle has at least one angle greater than $60°$.

Solution

Equilateral Triangle.

(b) For all real numbers x, $x^3 \geq x$.

Solution

$\frac{1}{2}^3 < \frac{1}{2}$

(c) If $a_1, a_2, a_3, ...$ is an increasing sequence of real numbers, i.e., $a_1 < a_2 < a_3 < ...$, then there exists some a_i in this sequence that is bigger than 10.

Solution

We can have an infinite number of possibilities for numbers less than 10 that the sequence may never converge to 10 and hence there may never exist a number greater than 10. $9 < 9.9 < 9.99 < 9.999 < 9.9999 < ...$

6. **(10 points)** King Arthur wants to seat his Knights at the Round Table. The seats are not numbered, and therefore, two seating arrangements are considered identical if the sequence of Knights in clockwise order starting from Knight 1 is the same.

(a) For $n = 4$, enumerate all possible seatings.

Solution

$$n_1, n_2, n_3, n_4$$
$$n_1, n_4, n_3, n_2$$
$$n_1, n_3, n_4, n_2$$
$$n_1, n_2, n_4, n_3$$
$$n_1, n_4, n_2, n_3$$
$$n_1, n_3, n_2, n_4$$

(b) How many different ways of arranging n Knights is possible?

Solution

By the division rule, we can arrange them in $\frac{n!}{n}$ ways. Which will give us (n-1)! arrangements.

(c) How would your answer change if the seats *were* numbered?

Solution If they were numbered, then we would not have any duplicates so we would have n! number of orderings.

7. **(10 points)** This is a 2-part question.

(a) Show that the following 3 statements are logically equivalent:

$$p \implies q \vee r, \qquad p \wedge \neg q \implies r, \qquad p \wedge \neg r \implies q$$

Solution

We will do this using logic manupulation. You can also do it using truth tables. We simply substitute the fact that $a \implies b$ is equivalent to $\neg a \vee b$ in each of the statements

$$\begin{aligned} p &\implies q \vee r = \neg p \vee q \vee r \\ p \wedge \neg q &\implies r = \neg(p \wedge \neg q) \vee r = \neg p \vee q \vee r \\ p \wedge \neg r &\implies q = \neg(p \wedge \neg r) \vee q = \neg p \vee q \vee r \end{aligned}$$

Hence we can see that the three statements are equivilent.

(b) Let n denote a positive integer. Using the logical equivalences above, rewrite the following assertion in two other ways.

"If n is a prime number not equal to 2, then n is odd".

Solution

p: n is a prime number q: n is equal to 2 r: n is odd

1. If n is a prime number then n is equal to 2 or n is odd.
2. If n is prime and n is not equal to 2 then n is odd.
3. If n is prime and n is not odd then n is 2.

8. **(10 points)** Prove that every amount of postage that is at least 12c can be made from some combination of 4c and 5c stamps. (Hint: (i) strong induction. (ii) you need to check multiple base cases.)

Solution

Method 1: Using the ordinary induction.

This question is equivalent to the proof such that $n = 4a + 5b$ where $n, a, b \in \mathbb{N}$ and $n \geq 12$.

Base case: $n = 12$, $a = 3$, $b = 0$.

Induction Hypothesis: For some k, there exists a and b such that $k = 4a + 5b$.

Induction Step: Assuming the induction hypothesis is true, we need to think about two different cases: $a \geq 1$ and $a = 0$.

If $a \geq 1$,

$$\begin{aligned} n + 1 &= 4a + 5b + 1 \\ &= 4a + 4 - 4 + 5b + 1 \\ &= 4a - 4 + 5b + 5 \\ &= 4(a - 1) + 5(b + 1) \end{aligned}$$

Denoting $a^* = a - 1$ and $b^* = b + 1$, we get new a^* and b^* for the next sequence from previous a and b values from induction hypothesis.

If $a = 0$,

$$\begin{aligned} n + 1 &= 5b + 1 \\ &= 5b + 1 + 16 - 16 \\ &= 16 + 5b - 15 \\ &= 4 \cdot 4 + 5(b - 3) \end{aligned}$$

Denoting $a^* = 4$ and $b^* = b - 3$, we can define new a^* and b^* with the same reasoning in the first case.

Method 2: Using a strong induction,

Define the predicate $P(n)$ that there exists non-negative integers a, b, and an integer n such that $4a + 5b = n$ where $n \geq 12$.

Base case: Need to consider following 4 cases, and you can see that the pattern repeats every four steps by calculating further.

a. $n = 12$, $a = 3$, $b = 0$

b. $n = 13$, $a = 2$, $b = 1$

c. $n = 14$, $a = 1$, $b = 2$

d. $n = 15$, $a = 0$, $b = 3$

Strong Induction Hypothesis: $P(k)$ is true for all $k \in \{12, 13, \dots, n\}$.

Induction Step: Assuming that the strong induction hypothesis is true, we now show that $P(n+1)$ is true. Since we know $P(n-3)$ is true, we have $n-3 = 4a+5b$ for some non-negative integers a and b. Therefore, $n+1 = 4(a+1)+5b$, which proves $P(n+1)$.

Therefore, by induction, $P(n)$ is true for all n.

9. **(10 points)** You are the CEO of a startup. You want to choose a team of m people from a pool of n applicants, and from these m people you want to choose k to be the team managers. Since you took CPRE 310, you think about this a bit, and conclude that you can do so in:

$$\binom{n}{m}\binom{m}{k}$$

ways. However, your CFO, who is a Hawkeye, comes up with the strange looking formula:

$$\binom{n}{k}\binom{n-k}{m-k}$$

ways. Before doing the obvious thing – dumping on all UofI alums – you decide to check your answer against theirs.

(a) Explain your calculations, clearly stating which counting rule(s) you used.

Solution We choose m people out of a pool of n **and** we choose k team managers from these m people choosen. So by using the combination rule, since the order in which we pick people does not matter, it makes sense that we compute

$$\binom{n}{m}\binom{m}{k}$$

(b) By direct algebra and simplification, show that your answer is mathematically equivalent to that obtained by the CFO.

Solution

$$\binom{n}{m}\binom{m}{k} = \frac{n!}{m! * (n-m)!} * \frac{m!}{k! * (m-k)!} = \frac{n!}{k! * (n-m)! * (m-k)!}$$

$$\binom{n}{k}\binom{n-k}{m-k} = \frac{n!}{k! * (n-k)!} * \frac{(n-k)!}{(m-k)! * (n-m)!} = \frac{n!}{k! * (n-m)! * (m-k)!}$$

Hence, they are both equal!

(c) Explain how your CFO may have arrived at their answer by guessing a different way to count the same quantity.

Solution

The CFO first choose k people from the n people to be the managers and then from the remaining number of people (n-k) he choose the (m-k) people who would be non-managers. Hence, it would give us the same answer as part (a)

10. **(10 points)** Several departments across campus have wireless access points. However, interference problems can arise if two access points are within 200 feet of each other and operating on the same frequency. Your goal is to assign different frequencies to different access points such that there is no interference.

Here is some information about the geographic locations of the departments.

Department	is within 200 feet of
MATH	PHY, CHEM, SOCIO
SOCIO	ECON, MATH, PSYCH
PHY	MATH, CHEM
PSYCH	CHEM, SOCIO, ECON
ECON	SOCIO, PSYCH
CHEM	MATH, PSYCH, PHY

(a) Model the above table using an undirected graph, where nodes denote departments and the "is within 200 feet of" information denotes edges. Draw this graph, and clearly mark the nodes.

(b) Assign labels (or colors) to the nodes such that no two nodes connected by an edge are assigned the same label/color.

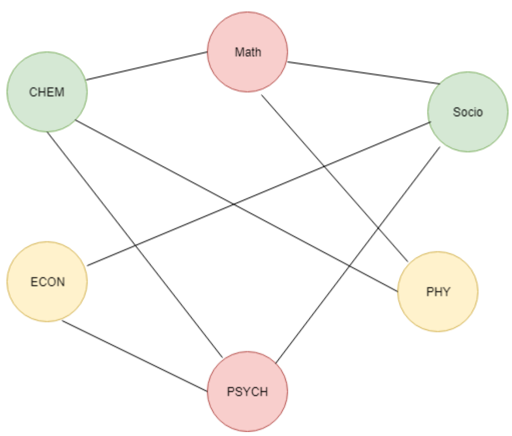

Figure 2: Graph

(c) Use your answer above to argue that only 3 frequencies are required for the overall wireless system to function properly.

Solution

As seen from the graph, we need only 3 frequencies for the system to function properly since none of the nodes that share an edge share the same color and we needed only 3 colors.

11. **(10 points)** Suppose you pick a positive integer n (where $1 \leq n \leq 100$) uniformly at random.

(a) What is the probability that n is divisible by 5?

Solution

$\frac{100}{5} = 20$. There are 20 numbers divisible by 5. Probablilty that n is divisible by 5 $= \frac{20}{100} = 0.2$

(b) What is the probability that n is divisible by 20? **Solution**

$\frac{100}{20} = 5$. There are 5 numbers divisible by 20. Probablilty that n is divisible by 20 $= \frac{5}{100} = 0.05$

(c) Given that n is divisible by 5, what is the probability that n is divisible by 20?

Solution

5 numbers are divisible by 20 and 20 numbers divisible by 5 Therefore, The probability is $\frac{5}{20} = 0.25$