Recitation 6 Solutions

---

1. For each of the following functions defined from the reals to the reals, indicate whether it is an injection and/or a surjection and/or a bijection.
   (a) $f(x) = x + 2$
   (b) $f(x) = 7x$
   (c) $f(x) = x^3$
   (d) $f(x) = \sin x$
   (e) $f(x) = e^x$

Solution

(a) $f(x) = x + 2$

Injection

Let $a, b \in \mathbb{R}$.

$f(a) = f(b)$

$a + 2 = b + 2$

$a = b$

$\therefore f$ is an injection.

Surjection

Let $e \in \mathbb{R}$.

$e - 2 \in \mathbb{R}$

$f(e - 2) = e$

$\therefore f$ is a surjection.

Bijection

Yes, f is an injection and surjection.

(b) $f(x) = 7x$

Injection

Let $a, b \in \mathbb{R}$.

$f(a) = f(b)$

$7a = 7b$

$a = b$

$\therefore f$ is an injection.

Surjection

Let $e \in \mathbb{R}$.

$\frac{e}{7} \in \mathbb{R}$

$f(\frac{e}{7}) = e$

$\therefore f$ is a surjection.

Bijection

Yes, f is an injection and surjection.

(c) $f(x) = x^3$

Injection

Let $a, b \in \mathbb{R}$.

$f(a) = f(b)$

$a^3 = b^3$

$a = b$

$\therefore f$ is an injection.

Surjection

Let $e \in \mathbb{R}$.

$e^{\frac{1}{3}} \in \mathbb{R}$

$f(e^{\frac{1}{3}}) = e$

$\therefore f$ is a surjection.

Bijection

Yes, f is an injection and surjection.

(d) $f(x) = \sin x$

Injection

Counterexample: $0 = \sin(0) = \sin(\pi)$

$\therefore f$ is not a injection.

Surjection

$\forall x \in \mathbb{R}, -1 \leq \sin(x) \leq 1$

$\therefore f$ is not a surjection.

Bijection

NO!

(e) $f(x) = e^x$

Injection

Let $a, b \in \mathbb{R}$

$f(a) = f(b)$

$e^a = e^b$

$\ln(e^a) = \ln(e^b)$

$a = b$

$\therefore f$ is an injection.

Surjection

$\forall x \in \mathbb{R}, e^x > 0$

$\therefore f$ is not a surjection.

Bijection

NO!

2. Let k be a positive integer, and define $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = x^k$. For what values of k is $f(x)$ an onto function? Provide a brief explanation (no proof necessary.)

   Solution

   Onto: $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}$ s.t. $f(x) = y$

   k is a positive integer

   $y = x^k$

   $y^{\frac{1}{k}} = x$

   x must be in $\mathbb{R}$. If y is positive or 0 x is always in $\mathbb{R}$. If y is negative and k is even x will be imaginary but if k is odd then x will be in $\mathbb{R}$.

   If k is odd, f is onto.

3. Let $f : \{1, 2, 3\} : \mathbb{N}$ denoted by $f(1) = 3, f(2) = 5, f(3) = 1$.

(i) Is f one-to-one?

(ii) Is f onto?

(iii) What is the range of f?

   Solution

   (i) yes. For each element in the domain f has a unique value.

   (ii) no. The co-domain is $\mathbb{N}$ but for every element in the domain f can only return an element from $\{1, 3, 5\}$.

   (iii) Ran(f)=$\{1, 3, 5\}$

4. Let $f : A \to B$ and $g : B \to C$ be functions. Let $h : A \to C$ be their composition, i.e., $h(a) = g(f(a))$.

(a) Prove that if f and g are surjections, then so is h.

(b) Prove that if f and g are bijections then so is h.

   Solution

   (a)

   We can say:

   $\forall b \in B, \exists a \in A$ such that $f(a) = b$

   $\forall c \in C, \exists b \in B$ such that $g(b) = c$

   For an element $e \in C$ there exists an element $b' \in B$ s.t. $g(b') = e$.

   For an element $b' \in B$ there exists an element $a' \in A$ s.t. $f(a') = b'$.

   $h(a') = g(f(a')) = g(b') = e$

$h(a') = e$, this is true for any element $e \in C$.

$\therefore h$ is a surjection.

(b)

From above we know if f and g are surjections then so is h. We must prove that if f and g are injections then so is h.

We can say:

$\forall a, a' \in A, f(a) = f(a') \implies a = a'$

$\forall b, b' \in B, g(b) = g(b') \implies b = b'$

$h(a) = h(a')$

$g(f(a)) = g(f(a'))$

g is an injection so $f(a) = f(a')$

f is an injection so $a = a'$

If $h(a) = h(a') \implies a = a'$

$\therefore h$ is an injection.

If f and g are bijections h is a bijection.