Homework 4 Solution

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1. **(10 points)** Prove the *distributive law* for any three sets A, B, C:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

You can use any method of proof. For example, for a formal logic proof you might want to consider an element $x \in A \cup (B \cap C)$ and construct a chain of logical deductions to show that x also belongs to $(A \cup B) \cap (A \cup C)$. Or you could use Venn diagrams.

Solution

For direct proof, we need to prove A ∪ (B ∩ C) ⊆ (A ∪ B) ∩ (A ∪ C) and (A ∪ B) ∩ (A ∪ C) ⊆ A ∪ (B ∩ C) .

For (i), suppose x ∈ A ∪(B ∩ C). Then either x ∈ A or x ∈ B ∩ C. If x ∈ A, then x ∈ A ∪ B and x ∈ A ∪ C, so x ∈ A $\implies$ x ∈ (A ∪ B) ∩ (A ∪ C) If c ∈ B ∩ C, then x ∈ B and x ∈ C. Therefore, x ∈ A ∪ B and x ∈ A ∪ C, so x ∈ B ∩ C $\implies$ x ∈ (A ∪ B) ∩ (A ∪ C) Combining these two cases shows that A ∪ (B ∩ C) ⊆ (A ∪ B) ∩ (A ∪ C)

For (ii), Now suppose x ∈ (A ∪ B) ∩ (A ∪ C). This means that x ∈ A ∪ B and x ∈ A ∪ C. There are two possible cases: either x ∈ A or x 6∈ A. If x ∈ A, then it obviously follows that x ∈ A ∪ (B ∩ C). If x 6∈ A, then in order for x to be in both A ∪ B and A ∪ C, we know x must be an element in B and in C, so x ∈ B ∩ C. This means that x ∈ (A ∪ B) ∩ (A ∪ C) $\implies$ x ∈ A ∪ (B ∩ C) so (A ∪ B) ∩ (A ∪ C) ⊆ A ∪ (B ∩ C)

Because each set is a subset of the other, they must be equal, so A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

2. **(10 points)** Let E be the set of even integers and O be the set of odd integers. Define a function:

$$f : E \times O \to \mathbb{Z}$$

such that $f(x, y) = xy$. Is f one-to-one? Is f onto? If yes, prove it; if not, provide a counterexample.

Solution

Consider the pairs (2, 3) and (-2, -3). We know $2 \cdot 3 = 6 = (-2) \cdot (-3)$, so f(2, 3) = f(-2, -3) = 6. These pairs aren't the same element, but f maps them to the same value in Z, so f is not one-to-one. Consider any even value in ∈ Z, such as 6,8,10... etc. There is no (x, y) ∈ E × O such that f(x, y) = even number, because the product of an even number and an odd number is always odd. Therefore, f is not onto.

3. **(10 points)** Let $f : A \to B$ and $g : B \to C$ be functions. Let $h : A \to C$ be their composition, i.e., $h(a) = g(f(a))$.
   (a) Prove that if f and g are surjections, then so is h.
   (b) Prove that if f and g are bijections then so is h.

Solution

(a)

We can say:

$\forall b \in B, \exists a \in A$ such that $f(a) = b$

$\forall c \in C, \exists b \in B$ such that $g(b) = c$

For an element $e \in C$ there exists an element $b' \in B$ s.t. $g(b') = e$.

For an element $b' \in B$ there exists an element $a' \in A$ s.t. $f(a') = b'$.

$h(a') = g(f(a')) = g(b') = e$

$h(a') = e$, this is true for any element $e \in C$.

$\therefore h$ is a surjection.

(b)

From above we know if f and g are surjections then so is h. We must prove that if f and g are injections then so is h.

We can say:

$\forall a, a' \in A, f(a) = f(a') \implies a = a'$

$\forall b, b' \in B, f(b) = f(b') \implies b = b'$

$h(a) = h(a')$

$g(f(a)) = g(f(a'))$

g is an injection so $f(a) = f(a')$

f is an injection so $a = a'$

If $h(a) = h(a') \implies a = a'$

$\therefore h$ is an injection.

If f and g are bijections h is a bijection.

4. **(10 points)** Consider an n-player round robin tournament where every pair of players play each other once. Assume that there are no ties and every game has a winner. Then, the tournament can be represented via a directed graph with n nodes where the edge $x \to y$ means that x *has beaten* y in their game.

   a. Explain why the tournament graph does not have cycles (loops) of size 1 or 2.
   b. We can interpret this graph in terms of a relation where the domain of discourse is the set of n players. Explain whether the "beats" relation for any given tournament is always/sometimes/never:
      (i) asymmetric
      (ii) reflexive
      (iii) irreflexive
      (iv) transitive.

Solution

a. Exist of a loop with size 1 means there exist a node with a self loop. Since a player cannot challenge themselves, self-loop does not exist. Having a loop size of 2 in this relation means that two player played each other two times and scores are 1:1. Due to the round robin tournament rule, it is not possible.

b. (i) Asymmetric: $\forall x, y \in S \; x\mathrm{R}y \implies \neg y\mathrm{R}x$ This quantification tells that there is no loop with size 2. Due to the reasoning above, it is asymmetric.
   (ii) Reflexive: $\forall x \in S \; x\mathrm{R}x$. Since the player cannot challenge themselves, it is not reflexive.

(iii) Irreflexive: $\forall x \in S\ \neg x \mathrm{R} x$. Since there is no self loop for all vertices, it is irreflexive.

(iv) Transitive: $\forall x, y, z \in S\ x\mathrm{R}y \wedge y\mathrm{R}z \implies x\mathrm{R}z$. It is not transitive. Counterexample: Pick a, b, and c as vertices (players) from n vertices. If the $a\mathrm{R}b$, $b\mathrm{R}c$, and $c\mathrm{R}a$, it shows that this relation is not transitive.

5. **(10 points)** Let W be the set of all words in the sentence, "The sky above the port was the color of television, tuned to a dead channel."" Define a relation R on W as follows: for any words $w_1, w_2 \in W$, $w_1 R w_2$ if the first letter of w_1 is the same as the first letter of w_2 without regard to upper or lower cases.

   (a) Prove that this is an equivalence relation.

   (b) Enumerate down all possible equivalence classes. (Recall that any equivalence class is the set of all elements in W that are related to each other via R.)

   Solution

   a. In order to satisfy an equivalence relation, it needs to satisfy three properties: reflexive, symmetric, and transitive.

   Reflexive: $\forall_{w_i \in W} w_i R w_i$ is true. (Trivial)

   Symmetric: $\forall_{w_i, w_j \in W} w_i R w_j \implies w_j R w_i$. If any pair of two words is in the relation, the ordered pair with switched elements (in position) is in the relation. (First letter should be same)

   Transitive: $forall_{w_i, w_j, w_k \in W} w_i R w_j \wedge w_j R w_k \implies w_i R w_k$. Similar reasoning as the symmetric. If the first and second words have same first letter and second and third words have same first letter, the trivially the first and third letter should have the same first letter.

   As the relation satisfies reflexive, symmetric, and transitive, it satisfies an equivalence relation.

   b. $\{the, television, tuned, to\}$

   $\{sky\}$

   $\{above, a\}$

   $\{port\}$

   $\{color, channel\}$

   $\{was\}$

   $\{of\}$

   $\{dead\}$