Lecture 5

Discrete Random Variables

STAT 330 - Iowa State University

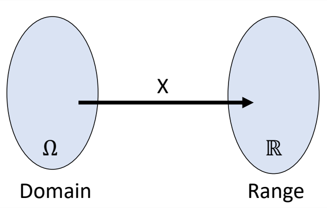
Random Variable

Random Variable

Definition

A *random variable (R.V.)* is a function that maps the sample space (Ω) to real numbers ($\Re$)

$$X : \Omega \rightarrow \Re$$



- Random variables (R.V.) connect random experiment to data
- Denote random variables with capital letters (X, Y, Z, etc)
- The values of a R.V. are determined by the outcome of a random experiment.

Random Variable Cont.

Example 1: Suppose you toss 3 coins, and observe the face up for each flip. $\Omega = \{HHH, HHT, \ldots, TTT\}; |\Omega| = 8$
We are interested in the number of heads we obtain in 3 coin tosses.

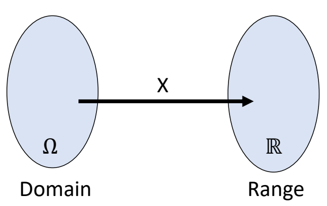
What is the random variable X?
$X = \#$ of heads in 3 coin tosses

Notation:
$X \equiv$ Random variable
$x \equiv$ Realized value
$X = x \rightarrow$ "random variable X takes on the value x".
$\{X = x\}$ is just an event

Consider the event 1 or 2 heads. This is $\{X = 1\} \cup \{X = 2\}$

Types of Random Variables

Types of Random Variables



Two types of random variables:

Discrete Random Variable

Sample space (Ω) maps to finite or countably infinite set in $\Re$

Ex: $\{1, 2, 3\}$, $\{1, 2, 3, 4, \ldots\}$

Continuous Random Variable

Sample space (Ω) maps to an uncountable set in $\Re$.

Ex: $(0, \infty)$, $(10, 20)$

Image of a Random Variable

Definition

The *image* of a random variable is defined as the values the random variable can take on.

$$Im(X) = \{x : x = X(\omega) \text{ for some } \omega \in \Omega\}$$

Example 2:

1. Put a disk drive into service. Let Y = time till the first major failure. $Im(Y) = (0, \infty)$.
   Image of Y is an interval (uncountable)
   $\rightarrow$ Y is a continuous random variable.
2. Flip a coin 3 times. Let $X = \#$ of heads obtained.
   $Im(X) = \{0, 1, 2, 3\}$. Image of X is a finite set
   $\rightarrow$ X is a discrete random variable.

Probability Mass Function (PMF)

Probability Mass Function

Two things to know about a random variable X:

(1) What are the values X can take on? (what is its image?)

(2) What is the probability that X takes on each value?

(1) and (2) together gives the *probability distribution* of X.

Definition

Let X be a discrete random variable.
The *probability mass function (pmf)* of X is $p_X(x) = P(X = x)$.

Properties of pmf:

1. $0 \leq p_X(x) \leq 1$
2. $\sum_x p_X(x) = 1$

Probability Mass Function Cont.

Example 3: Which of the following are *valid* probability mass functions (pmfs)?

1.

x	-3	-1	0	5	7
$p_X(x)$	0.1	0.45	0.15	0.25	0.05

2.

y	-1	0	1.5	3	4.5
$p_Y(y)$	0.1	0.45	0.25	-0.05	0.25

3.

z	0	1	3	5	7
$p_Z(z)$	0.22	0.18	0.24	0.17	0.18

Probability Mass Function Cont.

Example 4: Suppose you toss 3 coins, and observe the face up for each flip. We are interested in the number of heads we obtain in 3 coin tosses.

1. Define the random variable X.

2. What is the image of X?

3. What is the pmf of X? (find $p_X(x)$ for all x)

Probability Mass Function Cont.

Cumulative Distribution Function (CDF)

Cumulative Distribution Function

Definition

The *cumulative distribution function (cdf)* of X is

$$F_X(t) = P(X \leq t)$$

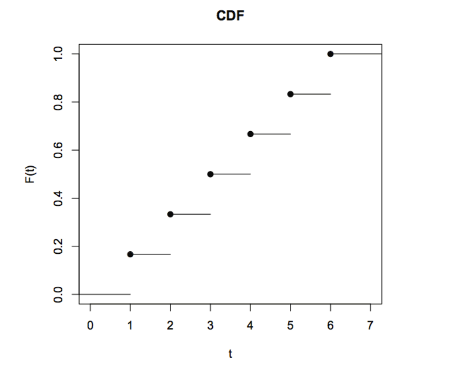
- The pmf is $Px(x) = P(X = x)$, the probability that R.V. X is equal to value x.
- The cdf is $F_X(t) = P(X \leq t)$, the probability that R.V. X is less than or equal to t.

Relationship between pmf and cdf

- $F_X(t) = P(X \leq t) = \sum_{x \leq t} p_X(x) = \sum_{x \leq t} P(X = x)$

Properties of CDFs

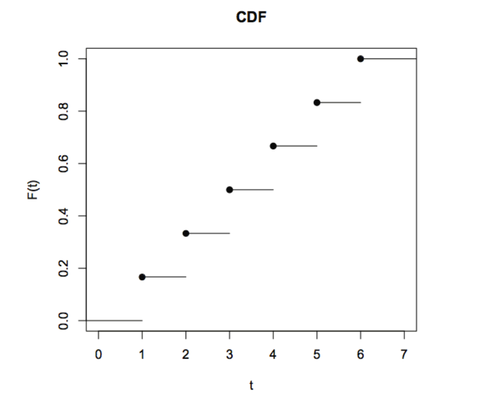
1. $0 \leq F_X(t) \leq 1$
2. F_X is non-decreasing (if $a \leq b$, then $F(a) \leq F(b)$.
3. $\lim_{t \to -\infty} F_X(t) = 0$ and $\lim_{t \to \infty} F_X(t) = 1$
4. F_X is right-continuous with respect to t



Cumulative Distribution Function Cont.

Example 5: Roll a fair die. Let X = the number of dots on face up

x	1	2	3	4	5	6
(pmf) $p_X(x)$	1/6	1/6	1/6	1/6	1/6	1/6
(cdf) $F_X(x)$	1/6	2/6	3/6	4/6	5/6	1



Cumulative Distribution Function Cont.

Example 6: Suppose you toss 3 coins, and observe the face up for each flip. We are interested in the number of heads we obtain in 3 coin tosses.

From Example 4, the pmf is

x	0	1	2	3
(pmf) $p_X(x)$	1/8	3/8	3/8	1/8
(cdf) $F_X(x)$				

What is the cdf of X?

Expected Value

Expected Value

Example 7: Flip a coin 3 times. Let $X = \#$ of heads obtained in 3 flips. The probability mass function (pmf) of X is

x	0	1	2	3
$p_X(x)$	1/8	3/8	3/8	1/8

What number of heads do we "expect" to get?

0 obtained $\frac{1}{8}$ of the time

1 obtained $\frac{3}{8}$ of the time

2 obtained $\frac{3}{8}$ of the time

3 obtained $\frac{1}{8}$ of the time

Intuitively, we can think about taking $0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right)$ as the "expected" number of heads

Expected Value

Definition

Let X be a discrete random variable. The *expected value* or *expectation* of $h(X)$ is

$$E\left[h(X)\right] = \sum_x h(x)p_X(x) = \sum_x h(x)P(X = x)$$

- The **MOST IMPORTANT** version of this is when h(x) = x

$$E(X) = \sum_x xp_X(x) = \sum_x xP(X = x)$$

- $E(X)$ is usually denoted by μ
- $E(X)$ is the weighted average of the x's, where the weights are the probabilities of the x's.

Expected Value Cont.

Example 8: Flip a coin 3 times. Let $X = \#$ of heads obtained in 3 flips. The probability mass function (pmf) of X is

x	0	1	2	3
$p_X(x)$	1/8	3/8	3/8	1/8

Calculate the expected value of X.

$$\begin{aligned} E(X) &= \sum_x x p_X(x) \\ &= 0P(X=0) + 1P(X=1) + 2P(X=2) + 3P(X=3) \\ &= \end{aligned}$$

Variance

Variance & Standard Deviation

Definition

The *variance* (σ^2) of a random variable X is

$$Var(X) = E\left[(X - E(X))^2\right] = \sum_x (x - E(X))^2 \cdot p_X(x)$$

The *standard deviation* (σ) of a random variable X is

$$\sigma = \sqrt{Var(X)}$$

- Units for variance is squared units of X.
- Units for standard deviation is same as units of X.

SHORT CUT (usually more convenient)

$$\begin{aligned} Var(X) &= E(X^2) - \left[E(X)\right]^2 \\ &= \sum_x x^2 P(X = x) - \left[\sum_x xP(X = x)\right]^2 \end{aligned}$$

Variance Cont.

Example 9: Flip a coin 3 times. Let $X = \#$ of heads obtained in 3 flips. The probability mass function (pmf) of X is

x	0	1	2	3
$p_X(x)$	1/8	3/8	3/8	1/8

Calculate the variance and standard deviation of X.

- $E(X) = \sum_x x p_X(x) =$

- $E(X^2) = \sum_x x^2 p_X(x) =$

- $Var(X) = E(X^2) - \left[E(X)\right]^2 =$

- $\sigma = \sqrt{Var(X)} =$

Operations involving E(X) & Var(X)

Operations

X, Y are random variables; a, b are constants.

Operations with $E(\cdot)$

- $E(aX) = aE(X)$
- $E(aX + b) = aE(X) + b$
- $E(aX + bY) = aE(X) + bE(Y)$

Operations Cont.

X, Y are random variables; a, b, c are constants.

Operations with $Var(\cdot)$

- $Var(aX) = a^2 Var(X)$
- $Var(aX + b) = a^2 Var(X)$
- $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2abCov(X, Y)$
  (when X,Y are independent, Cov(X,Y) = 0. We'll discuss more about independence and define covariance later)

Chebyshev's Inequality

Chebyshev's Inequality:

For any positive real number k, and a random variable X with variance σ^2:

$$P(|X - E(X)| \leq k\sigma) \geq 1 - \frac{1}{k^2}$$

- bounds the probability that X lies within a certain number of standard deviations from $E(X)$