Lecture 19: Induction (II)

While proving a statement by induction, it is often fruitful to ask yourself the following questions:

- Is the domain of discourse the nonnegative integers (or some subset of the nonnegative integers).
- Does the statement hold for the first few cases?
- If an oracle told us that the statement is true for k, can we use that fact to prove it to be true for $k+1$?

In order to construct a proof by induction, the answer should be yes in each of these cases.

Some more examples

Let us do a couple more induction examples.

First example. Suppose we want to prove that the summation of a geometric series (with ratio $r \neq 1$) is given by:

$$1 + r + \ldots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$

Again, previously we had given a proof by mathematical jugglery. Now we prove it by induction.

Let $P(n)$ be the assertion that the summation has the above form.

(base case) Clearly $P(0)$ is true since the left hand side is equal to 1, while the left hand side is given by $\frac{r^{0+1}-1}{r-1} = 1$.

(induction) Suppose that $P(k)$ is true for some $k \geq 0$, i.e.,

$$1 + r + \ldots + r^k = \frac{r^{k+1} - 1}{r - 1}.$$

Adding r^{k+1} on both sides, we get:

$$\begin{aligned} 1 + r + \ldots + r^k + r^{k+1} &= \frac{r^{k+1} - 1}{r - 1} + r^{k+1} \\ &= \frac{r^{k+1} - 1 + r^{k+1}(r-1)}{r-1} \\ &= \frac{r^{k+1} - 1 + r^{k+2} - r^{k+1}}{r-1} \\ &= \frac{r^{k+2} - 1}{r - 1}. \end{aligned}$$

Therefore, $P(k+1)$ is true, and we are done.

Second example. Suppose we have a robot moving about on a grid. Let its coordinates be denoted as (x, y). In each time step, the robot makes a *diagonal* move, i.e., it can move to $(x+1, y+1)$ or $(x+1, y-1)$ or $(x-1, y+1)$ or $(x-1, y-1)$. Here is a question:

Suppose the robot starts at $(0, 0)$*. Is it possible for the robot to ever reach* $(63, 56)$*?*

Reachability is an important concept in robotic path planning. We prove that $(63, 56)$ is not a reachable location. In fact, we prove the following, more general claim:

If (x_n, y_n) *is the location of the robot after* n *steps. If the robot starts at* $(0, 0)$ *then* $(x_n + y_n)$ *is an even integer for all* n*.*

We prove this by induction. Let $P(n)$ be the assertion that $x_n + y_n$ is even.

(base case) P(0) is true since $x_0 + y_0 = 0 + 0 = 0$, which is an even integer.

(induction) Assume that $P(k)$ is true, i.e., $x_k + y_k$ is even for some $k \geq 0$. We need to show that $P(k + 1)$ is true. Due to the rules of transition of the robot, we have exactly 4 cases:

1. $(x_{k+1}, y_{k+1}) = (x_k + 1, y_k + 1)$. In this case, $x_{k+1} + y_{k+1} = x_k + y_k + 2$ which is an even number plus 2. Therefore, $x_{k+1} + y_{k+1}$ is even.
2. $(x_{k+1}, y_{k+1}) = (x_k + 1, y_k - 1)$. In this case, $x_{k+1} + y_{k+1} = x_k + y_k$ which is even.
3. $(x_{k+1}, y_{k+1}) = (x_k - 1, y_k + 1)$. In this case, $x_{k+1} + y_{k+1} = x_k + y_k$ which is even.
4. $(x_{k+1}, y_{k+1}) = (x_k - 1, y_k - 1)$. In this case, $x_{k+1} + y_{k+1} = x_k + y_k + 2$ which is an even number minus 2. Therefore, $x_{k+1} + y_{k+1}$ is even.

Therefore, in any case, $P(k + 1)$ is true. Thus the claim is proved by induction.

Third example. This one is from geometry. Define a *convex* polygon as any convex planar figure that is bounded by n line segments (where $n \geq 3$). For example, a triangle is a convex polygon with $n = 3$, a quadrilateral has $n = 4$, a pentagon has $n = 5$, and so on. Prove that the sum of internal angles in any polygon is $(n - 2) \times 180°$.

As above: let $P(0)$ be the assertion that *every* convex polygon with n sides has internal angles that add up to $(n - 2) \times 180$ degrees.

(base case) When $n = 3$, the polygon is a triangle, whose angles add up to $(3 - 2) \times 180 = 180$ degrees.

(induction hypothesis) Suppose that $P(k)$ is true for some k, i.e., every convex polygon with k sides has internal angles that add up to $(k - 2) \times 180$ degrees.

(induction step) We prove $P(k + 1)$. Consider any arbitrary convex polygon with $k + 1$ sides. Label its vertices as $v_1, v_2, \ldots v_k, v_{k+1}$. Therefore, we can imagine this polygon as composed of two pieces: a *smaller* convex polygon with merely k sides, and a triangle (with vertices v_1, v_k, v_{k+1}).

Therefore, the sum of internal angles of the bigger polygon equals the sum of the angles of the smaller polygon (with k sides) and the sum of the angles of the triangle. But we know the expressions to each of these smaller sums! The former is equal to $180 \times (k - 2)$ degrees (via the induction hypothesis) and the latter is equal to 180 degrees. Adding the two, we get:

$$180(k - 2) + 180 = 180(k - 2 + 1) = 180(k - 1)$$

as the sum of internal angles of the bigger polygon (with $k + 1$ sides), which is exactly what we set out to prove. QED.