Lecture 23: Quantitative Thinking

Basics of Counting

Enumerating the elements of a set is central to several problems in computer engineering (and several other technical disciplines.)

However, counting can be hard. Consider, for example, writing an AI software program that can play a game of chess versus a human. However, even if the program were to think four or five moves "ahead", the set of possible moves that the program must consider can become incredibly complex. Often, it will help if we can establish a collection of simple principles/tools for counting, and this will also help us start reasoning about problems in a quantitative manner.

What does it even mean to *count* the elements of a set S? We briefly touched upon this when we discussed sequences. The idea is to construct a bijective mapping between the elements of S and a subset of the integers (say A), and sets of integers are easy to count. We know due to properties of bijective functions, the cardinality of S and A must be the same. This is the basic principle that we use in all of counting.

However, the above principle is somewhat abstract-sounding. From a practical standpoint, counting problems can be solved using a few simple rules involving addition and multiplication. The hard part is to know *when* to add and *when* to multiply. The best way to understand this is to give some examples.

Addition rule

The first rule of counting is called the *addition rule*, or the *sum rule*. Suppose that there are two finite sets (say A and B) that are mutually exclusive, i.e., $A \cap B$ is empty. Then, the number of ways to choose an element of $A \cup B$ is given by $|A| + |B|$.

First example:

> Bart owns five bicycles and three cars. How many different ways can he get to work?

Let B denote the set of bicycles owned by Bart, and C denote the set of cars. The total set of travel options he has is given by $B \cup C$. By the sum rule, the number of ways for Bart to choose a method to go to school is given by $|B \cup C| = |B| + |C| = 5 + 3 = 8$.

Second example:

> Lisa has decided to shop at exactly one store today, either in the north part of town or the south part of town. If she visits the north part of town, she will shop at either a mall, a furniture store, or a jewelry store. If she visits the south part of town then she will shop at either a clothing store or a shoe store. How many stores could she end up shopping in?

We use an identical argument as above: let S be the set of shops in the south part of town, and N be the set of shops at the north part of town. Then, the set of shopping choices for Lisa is given by $N \cup S$. By the sum rule, $|N \cup S| = |N| + |S| = 3 + 2 = 5$.

Third example:

> The Frying Dutchman (tFD) in Springfield has 25 customers for lunch and 37 for dinner. How many unique people visited tFD?

Here is an attempt to answer this question. Let L be the set of customers for lunch, and D be the set of customers for dinner. Then by the sum rule, $|L \cup D| = |L| + |D| = 25 + 37 = 62 \ldots$

However, the above answer is **incorrect**. We did not specify whether L and D were disjoint or not; it could be that any subset of the 25 lunch customers also returned for dinner, in which case the number of *unique* visitors will vary. Be careful while applying the sum rule.

The sum rule is stated above for 2 sets, but there is a straightforward generalization to n sets. Here is the *generalized addition rule*, stated in full:

> Suppose $A_1, A_2, \ldots, A_n$ are mutually exclusive sets. Then $|A_1 \cup A_2 \ldots A_n| = |A_1| + |A_2| + \ldots |A_n|$.

Principle of Inclusion-Exclusion

We have (briefly) seen the Principle of Inclusion-Exclusion (PIE) before; this rule is also called "subtraction" rule, and is an extension of the sum rule in the case of non-disjoint sets.

The PIE states that for any two sets A and B (disjoint or not), the following relation holds:

> $|A \cup B| = |A| + |B| - |A \cap B|$.

Here is an example. Suppose there are 350 applicants for a job at Compu-Global-Hyper-Mega-Net. You are given the following information:

> 150 are CPRE majors.
>
> 80 are business majors.
>
> 30 are double-majors in CPRE and business.

How many applicants to the job *did not* major in either CPRE or business?

This is a a straightforward application of PIE. Let C be the set of CPRE majors and B be the set of business majors applying for the job. Then $|B| = 80$, $|C| = 150$, and $|B \cap C| = 30$. Therefore, by the principle of exclusion, the number of applicants that are either CPRE or business majors is given by $|B \cup C| = |B| + |C| - |B \cap C| = 150 + 80 - 30 = 200$.

Therefore, the number of applicants that are neither CPRE nor business majors is $\overline{B \cup C} = 350 - 200 = 150$.

Product rule

The next rule of counting is called the *product rule*. If A and B are finite sets then recall that $A \times B$ is the Cartesian product of A and B, containing all ordered pairs (a, b) where $a \in A$ and $b \in B$. Then, the number of elements in $A \times B$ is $|A| \times |B|$.

> Bart owns five bicycles and three cars. He plans to drive up to the base of a trailhead and then bike up a mountain. How many different ways can he do this?

As above, let $B = \{b_1, b_2, b_3, b_4, b_5\}$ be the set of bikes, and $C = \{c_1, c_2, c_3\}$ be the set of cars, owned by Bart. Any combination that Bart uses to go to the top of the mountain can be denoted as (b, c) where $b \in B$ and $c \in C$. Therefore, the total number of choices used by Bart is given by the number of possible ordered pairs (b, c) – and by the product rule, this is equal to $|B \times C| = |B| \times |C| = 15$.

As with the sum rule, one can generalize the product rule to n sets. Here is the *generalized product rule* stated formally:

> Suppose $A_1, A_2, \ldots, A_n$ are finite sets. Then $|A_1 \times ...A_n| = |A_1| \times \ldots |A_n|$

Let's do a second example application of the product rule.

> Within the Springfield area code (636), each phone number is 7 digits, with the first digit being any number except 0 or 1. How many distinct phone numbers are possible?

Each phone number can be written as (636) $a_1a_2a_3a_4a_5a_6a_7$, where $a_1 \in A_1 = \{2, 3, \ldots, 9\}$ and $a_i \in A = \{0, 1, \ldots, 9\}$ for $2 \leq i \geq 7$. Therefore, by the product rule, the total number of distinct phone numbers is given by $|A_1| \times |A|^6 = 8 \times 10^6 = 8,000,000$.

Third example.

> Lisa sets a computer password. A valid password can contain between six and eight symbols. The first symbol must be a letter (which can be lowercase or uppercase), and the remaining symbols must be either letters or digits. How many different passwords are possible?

The solution to this problem uses *both* the sum and product rules. In fact, this is how most counting exercises will be structured. First, we observe that there are three mutually exclusive cases: Lisa's password either contains 6 symbols, or 7 symbols, or 8 symbols.

Let F denote the set of valid first symbols. Then, $F = \{a, b, \ldots, z, A, \ldots, Z\}$, and $|F| = 26 + 26 = 52$.

Let S denote the set of valid symbols for subsequent positions. Then, $S = \{a, \ldots, z, A, \ldots, Z, 1, \ldots, 9\}$ and $|S| = 26 + 26 + 10 = 62$.

Let the notation S^n denote $S \times S \times \ldots S$ (n times). If the password contains 6 symbols, the set of possible choices is $F \times S^5$. Therefore, by the product rule $|F \times S^5| = |F| \times |S| \times |S| \times |S| \times |S| = 52 \times 62^5$.

If the password contains 7 symbols, by an identical argument as above we get $|F \times S^6| = |F| \times |S|^6 = 52 \times 62^6$.

If the password contains 8 symbols, we get $|F \times S^7| = |F| \times |S|^7 = 52 \times 62^7$.

Therefore, by the sum rule, the *total* number of possible passwords is $52 \cdot 62^5 + 52 \cdot 62^6 + 52 \cdot 62^7 = 1.86 \times 10^{14}$.

One point about the product rule is that the sets A_1 and A_2 are allowed to be dependent on each other. Here is an example: suppose we modify the above problem where passwords are only allowed to contain 6 characters, all characters being the *same* digit, i.e., the only allowable passwords are of the form 000000, 111111, and so on. (Admittedly these are not very strong passwords.)

Here is how we apply the product rule for this problem. Again, let F be the first set of valid first symbols. Then, $F = \{0, 1, \ldots, 9\}$, and $|F| = 10$. However, once we choose the first digit, the set S

is a *singleton* set since the second digit has to be equal to the first digit. For example, if the first digit was 7, then $S = \{7\}$. In general, S depends on how we choose the first digit but in all cases, $|S| = 1$.

Therefore, by the product rule, the number of allowable passwords = $10 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 = 10$.

Permutations and the Arrangement Principle

Consider any set S. A *permutation* of S is a sequence that uses contains every element of S exactly once. For example, the set $S = \{a, b, c\}$ has the following permutations:

- (a, b, c)
- (a, c, b)
- (b, a, c)
- (b, c, a)
- (c, a, b)
- (c, b, a)

One can check by hand that the six sequences above represent all the possible permutations of $\{a, b, c\}$. But in fact, we can *prove* that there are 6 permutations of a 3-element set, using a version of the Product rule. Observe that there are exactly 3 choices for the first element in the sequence; once we fix the first element, there are exactly 2 choices for the second element; and once the first two elements are fixed, there is only 1 choice for the third element. Therefore, the total number of permutations is $3 \cdot 2 \cdot 1 = 6$.

Here is a slightly different problem.

> Suppose I have fridge magnets in the shapes of the letters A through Z. How many different 3-letter strings can I make?

Since I have only one magnet per letter, I cannot repeat letters. Therefore, the first character in the string has 26 choices; the second only has 25; and the third has 24. Therefore, the total number of choices is $26 \cdot 25 \cdot 24$.

This argument can be generalized into something that we call the *Arrangement Principle*. This principle is stated as follows:

> The number of ways to form a sequence of r distinct elements that are drawn from a set of n elements is given by:
>
> $$P(n, r) = n \cdot (n - 1) \cdot (n - 2) \cdots (n - r + 1)$$
>
> .

In particular, the number of ways to arrange all n elements of the set into a sequence is given by $P(n, n) = n \cdot (n - 1) \cdot \cdots \cdot 2 \cdot 1 = n!$, where ! denotes the factorial function. (By convention, we denote $1! = 1$ and $0! = 1$.)

Notice that the above expression for $P(n, r)$ can be simplified as:

$$P(n, r) = \frac{n!}{(n - r)!}.$$

Let us do some more examples of the Arrangement Principle.

> From Opening Day to September 1, every MLB team can carry up to 25 players on their roster. How many different ways are there to choose a 9-man batting order?

The answer to this is simply $P(n, r)$, where $n = 25$ and $r = 9$. This equals $25!/(25-9)! = 25 \cdot 24 \cdots 17 \cdot 16 \approx 4.7 \times 10^{11}$.

We should be a bit careful while applying the Arrangement principle; people often confuse it with the multiplication rule. Here is an example that explains the difference:

> A jar contains 10 ping-pong balls of distinct colors. Four balls are drawn in sequence. How many ways are there to do this, if: (i) the balls are replaced before the next one is drawn? (ii) the balls are drawn and not replaced?

Let D_1, D_2, D_3, D_4 be the set of choices for each of the 4 draws. In the first case, there are always 10 balls in the urn, so there are always 10 choices for each draw. By the product rule, the number of ways to draw 4 balls equals $10^4 = 10,000$.

However, n the second case, the balls are *not* replaced, therefore the number of choices decreases by one each time a ball is drawn. Therefore, the number of possible ways = $P(10, 4) = 10 \cdot 9 \cdot 8 \cdot 7 = 5,040$.

Division rule

Suppose we want to count the number of people in a room. We could do it the standard way; however, an equally valid way would be to count the number of *ears* in the room, and divide by two! A third way would be to count the number of *toes* in the room, and dividing by 10. (Assume for simplicity that everyone in the room has two ears, and 10 toes.)

This may sound like a dumb way of counting the number of people in the room, but in fact is an instance of a more general counting principle known as the *division rule*.

Define a *d-to-1* mapping as an onto function $f : A \to B$ which maps exactly d elements in the domain to every element in the co-domain. In the above example, the function that maps ears to people is a 2-to-1 mapping, and the function that maps toes to people is a 10-to-1 mapping.

Then, the division rule states that:

> If $f : A \to B$ is a $d - to - 1$ function, then $|A| = d \cdot |B|$.

Again, this should make intuitive sense. Here is an application of the division rule in action:

> King Arthur wants to seat his Knights at the Round Table. The seats are not numbered. How many different ways of arranging n Knights is possible? Two seatings are considered the same if the sequence of Knights in clockwise order starting from Knight 1 is the same.

One can imagine seating the Knights as follows: generate a random permutation of the n Knights, and seat them in that order. Let A denote the set of all possible such permutations. Therefore, at first glance, it seems that the number of arrangements in this case is precisely the number of permutations of n elements, i.e., $|A| = P(n, n) = n!$.

However, this would be a case of *overcounting*. For example, if $n = 4$, denote the Knights by $\{k_1, k_2, k_3, k_4\}$. Then, (k_1, k_2, k_3, k_4) is the same permutation as (k_2, k_3, k_4, k_1). (Indeed, these are also equivalent to (k_3, k_4, k_1, k_2) and (k_4, k_1, k_2, k_3)).

In particular, every cyclic shift of the above sequence is the same seating. Therefore, we can define an $n - to - 1$ function $f : A \to B$, where A denotes the set of all permutations of the n Knights, and B denotes the set of permissible arrangements around the round table. By the division rule, the number of circular arrangements is given by $|A|/n = n!/n = (n-1)!$.

[As an **exercise**, convince yourself that this is the case by explicitly enumerating all possible circular arrangements with $n = 4$ Knights.]