Lecture 16

Stochastic Process & Markov Chain

STAT 330 - Iowa State University

Stochastic Process

Stochastic Processes

Definitions

A *stochastic process* is a random variable that also depends on time. It is written as

$$X_t(\omega) = X(t, \omega) \text{ for } t \in \mathcal{T}, \omega \in \Omega$$

where $\mathcal{T}$ is a set of possible times. e.g. $[0, \infty), \{0, 1, 2, \ldots\}$ and Ω is the whole sample space.

- $X_t = X_t(\omega)$ is the random variable
- t is time
- ω is the "state"
- The *state space* is the collection of values the R.V X_t can take on: $\cup_{t \in \mathcal{T}} Im(X_t)$

Types of Stochastic Processes

Types of Stochastic Processes: $X_t(\omega)$ can be

- Continuous-time (t) continuous-state (ω)
- Discrete-time (t) continuous-state (ω)
- Continuous-time (t), discrete-state (ω)
- Discrete-time (t), discrete-state (ω)

Types of Stochastic Processes

Examples:

1. Let X_t be the result of tossing a fair coin ($0 =$ tails, $1 =$ heads) in the t^{th} trial.
   - The time (trial) $t \in \mathcal{T}$ where $\mathcal{T} = \{1, 2, 3, \ldots\}$
   - $Im(X_t) = \{0, 1\}$
   - This is an example of ____________ time, ____________ state stochastic process.
2. Let X_t be the number of customers in a store at time t.
   - The time $t \in \mathcal{T}$ where $\mathcal{T} = (0, \infty)$
   - $Im(X_t) = \{0,1,2,3, \ldots\}$
   - This is an example of ____________ time, ____________ state stochastic process.

Markov Chain

Markov Chain (MC) and Markov Property

Markov Property

A stochastic process X_t satisfies the *Markov property* if for any $t_1 < t_2 < \ldots < t_n < t$ and any sets A; $A_1, ..., A_n$:

$$P\{X_t \in A | X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n\} = P\{X_t \in A | X_{t_n} \in A_n\}.$$

- The probability distribution of X_t at time t only depends on its previous state.
- If the above is satisfied, then X_t is called a Markov Chain.

Markov Property Examples

1. A (fair) coin in flipped over and over: If coin lands on "heads", you win \$1. If coin lands on "tails", you lose \$1. Let X_t be your profit after t flips.
   - $P(X_5 = 3 | X_4 = 2) =$
   - $P(X_5 = 3 | X_4 = 2, X_3 = 1, X_2 = 2, X_1 = 1) =$

2. An urn contains 2 red balls, and 1 green ball. A ball is drawn (without replacement) from the urn yesterday and today. Another ball will be drawn tomorrow. Suppose you drew a red ball yesterday, and a red ball today.
   - $P(\text{Red tomorrow} | \text{Red today}) =$
   - $P(\text{Red tomorrow} | \text{Red today, Red yesterday}) =$

Discrete-Time Discrete-State MC

Discrete-Time Discrete-State Markov Chain (MC)

Suppose we have a Markov chain with time set $\mathcal{T} = \{0, 1, 2, \ldots\}$ and state space $\{0, 1, 2, \ldots\}$ Two things we need to know about X_t:

1. Initial distribution (P_0): $P_0(x) = P(X_0 = x)$ usually given as a vector of probabilities for the initial states of X_t.
   Ex: State space $= \{0, 1, 2\}$; $P_0 = \{0.3, 0.4, 0.3\}$
2. Transition probabilities:
   1-step transition probability: probability of moving from state i to state j in 1 step.
   $p_{ij} = P(X_{t+1} = j | X_t = i)$
   h-step transition probability: probability of moving from state i to state j in h steps.
   $p_{ij}^{(h)} = P(X_{t+h} = j | X_t = i)$

Discrete-Time Discrete-State Markov Chain (MC)

- We assume that the Markov Chain (MC) is *homogeneous.* (ie) transition probabilities p_{ij} are independent of t.
  $\rightarrow$ For all times $t_1, t_2 \in \mathcal{T}$, $p_{ij}(t_1) = p_{ij}(t_2)$.
- Then, the distribution of a homogeneous MC is completely determined by the initial distribution (P_0) and one-step transition probability (p_{ij}).

Main Idea: Start with an initial distribution P_0. Then use the one-step transition probability p_{ij} to "jump" forward to the next step. Then, we can keep going forward one step at a time.

Examples

Example

Example 1: In the summer, each day in Ames is either sunny or rainy. A sunny day is followed by another sunny day with probability 0.7, whereas a rainy day is followed by a sunny day with probability 0.4. It rains on Monday. Make weather forecasts for Tuesday and Wednesday.
Let $1 =$ "Sunny" and $2 =$ "Rainy".

To simplify and solve these types of problems, use transition matrices and matrix multiplication.

1-Step Transition Probability Matrix

For a homogeneous MC with state space $\{1, 2, ..., n\}$, the *1-step transition probability matrix* is:

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}.$$

The element from the i-th row and j-th column is p_{ij}, which is the transition probability from state i to state j.

h-Step Transition Probability Matrix

Similarly, one can define a *h*-step transition probability matrix

$$P^{(h)} = \begin{pmatrix} p_{11}^{(h)} & p_{12}^{(h)} & \cdots & p_{1n}^{(h)} \\ p_{21}^{(h)} & p_{22}^{(h)} & \cdots & p_{2n}^{(h)} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}^{(h)} & p_{n2}^{(h)} & \cdots & p_{nn}^{(h)} \end{pmatrix}.$$

Using the matrix notation the following results follow:

- 2-step transition matrix $P^{(2)} = P \cdot P = P^2$
- *h*-step transition matrix $P^{(h)} = P^h$
- The initial distribution of X_0 is written as row vector P_0.
  The distribution of X_h (h-steps in the future) is $P_h = P_0 P^h$

Example

Back to Example 1: We can solve the problem much more easily by using transition matrices . . . .

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$$

$$P^{(2)} = P \cdot P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} \cdot \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix}$$

$$P^{(3)} = P \cdot P \cdot P = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix} \cdot \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = \begin{pmatrix} 0.583 & 0.417 \\ 0.556 & 0.444 \end{pmatrix}$$