Proof Using Resolution

Outline

I. Rule of resolution

II. Resolution refutation

* Figures are from the textbook site unless the source is specifically cited.

I. Resolution

An inference algorithm i is

sound if *KB* $\models \alpha$ whenever *KB* $\vdash_i \alpha$

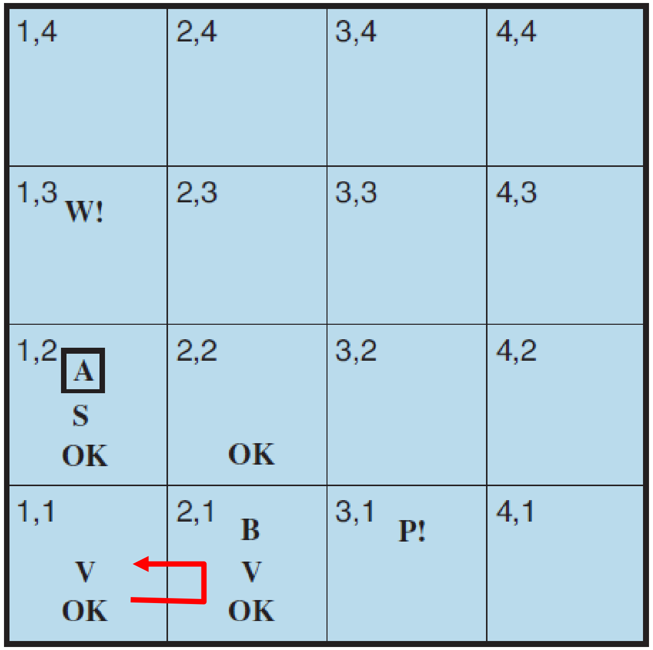
complete if *KB* $\vdash_i \alpha$ whenever *KB* $\models \alpha$

- Inference rules covered so far are sound.
- The inference algorithms using them may not be complete.

resolution + a complete search algorithm $=$ a complete inference algorithm

single inference rule

Wumpus World Revisited



Agent: [1,1] → [2,1] → [1,1]

KB:

R_1: $\neg P_{1,1}$

R_2: $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$

R_3: $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$

R_4: $\neg B_{1,1}$

R_5: $B_{2,1}$

Rules

R_6: $\left(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})\right) \wedge \left((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1}\right)$

R_7: $(P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1}$

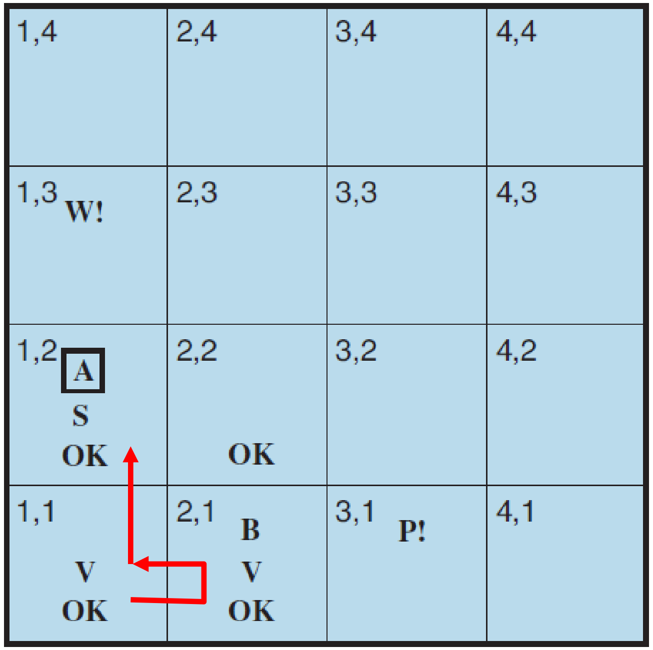
R_8: $\neg B_{1,1} \Rightarrow \neg(P_{1,2} \vee P_{2,1})$

R_9: $\neg(P_{1,2} \vee P_{2,1})$ // R_4, R_8

R_{10}: $\neg P_{1,2} \wedge \neg P_{2,1}$

Added to KB via inferences

(cont'd)



[1,1] → [1,2]: stench but no breeze

Add to KB:

R_{11}: $\neg B_{1,2}$

R_{12}: $B_{1,2} \Leftrightarrow \left(P_{1,1} \vee P_{2,2} \vee P_{1,3}\right)$

$\Downarrow$ Similarly, as in deriving R_{10}

R_{13}: $\neg P_{2,2}$

R_{14}: $\neg P_{1,3}$

R_3: $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$

$\Downarrow$ biconditional elimination
R_5: $B_{2,1}$

R_{15}: $P_{1,1} \vee P_{2,2} \vee P_{3,1}$

Resolvent

R_{13}: $\neg P_{2,2}$ $\qquad$ R_{15}: $P_{1,1} \lor P_{2,2} \lor P_{3,1}$

resolving the two literals that are negations of each other

R_{16}: $P_{1,1} \lor P_{3,1}$ (*resolvent*)

If there's a pit in one of [1,1], [2,2], and [3,1] and it's not in [2,2],then it's in [1,1] or [3,1].

R_1: $\neg P_{1,1}$ $\qquad$ R_{16}: $P_{1,1} \lor P_{3,1}$

R_{17}: $P_{3,1}$

Simple Resolution Rule

$$\frac{l_1 \vee \cdots \vee l_i \vee \cdots \vee l_k, \qquad m}{l_1 \vee \cdots \vee l_{i-1} \vee l_{i+1} \vee \cdots \vee l_k}$$

(l_i and m are complementary literals, i.e.,$l_i = \neg m$ or $m = \neg l_i$.)

Since m is true, then l_i must be false. But one of $l_1, \ldots, l_k$ must be true. Therefore, we can exclude l_i and assert that one of the remaining $k-1$ literals must be true.

Clause: a disjunction of literals.

R_{15}: $P_{1,1} \vee P_{2,2} \vee P_{3,1}$

$$\frac{P_{1,1} \vee P_{2,2} \vee P_{3,1}, \qquad \neg P_{2,2}}{P_{1,1} \vee P_{3,1}}$$

Unit clause: a single literal.

R_1: $\neg P_{2,2}$ $\qquad$ R_5: $B_{2,1}$

Full Resolution Rule

l_i and m_j are complementary literals:

$$\frac{l_1 \vee \cdots \vee l_i \vee \cdots \vee l_k, \qquad m_1 \vee \cdots \vee m_j \vee \cdots \vee m_k}{l_1 \vee \cdots \vee l_{i-1} \vee l_{i+1} \vee \cdots \vee l_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n}$$

If l_i is true, then m_j is false. Hence $m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n$ must be true.
If l_i is false, then $l_1 \vee \cdots \vee l_{i-1} \vee l_{i+1} \vee \cdots \vee l_k$ must be true.

$$\frac{P_{1,1} \vee P_{3,1}, \qquad \neg P_{1,1} \vee \neg P_{2,2}}{P_{3,1} \vee \neg P_{2,2}}$$

One Pair at a Time

Only one pair of complementary literals can be resolved at each step.

$$\frac{P \lor \neg Q \lor R, \qquad\qquad \neg P \lor Q}{\neg Q \lor R \lor Q \quad \equiv \textit{true}}$$

$$\frac{P \lor \neg Q \lor R, \qquad\qquad \neg P \lor Q}{R}$$

Incorrect conclusion!

Conjunctive Normal Form

The resolution rule applies to clauses only.

Conjunctive normal form (CNF): a conjunction of clauses

$$
\begin{aligned}
CNFSentence &\rightarrow Clause_1 \wedge \cdots \wedge Clause_n \\
Clause &\rightarrow Literal_1 \vee \cdots \vee Literal_m \\
Fact &\rightarrow Symbol \\
Literal &\rightarrow Symbol \mid \neg Symbol \\
Symbol &\rightarrow P \mid Q \mid R \mid \ldots
\end{aligned}
$$

Every sentence of propositional logic is equivalent to a CNF.

Converting to CNF

1. Eliminate $\Leftrightarrow$.

$$\alpha \Leftrightarrow \beta$$

replaced with

$$(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$$

2. Eliminate $\Longrightarrow$.

$$\alpha \Rightarrow \beta$$

$$\neg\alpha \vee \beta$$

3. Move $\neg$ inwards, repeatedly applying

$$\neg(\neg\alpha) \equiv \alpha$$
$$\neg(\alpha \vee \beta) \equiv (\neg\alpha \wedge \neg\beta)$$
$$\neg(\alpha \wedge \beta) \equiv (\neg\alpha \vee \neg\beta)$$

4. Apply the distributivity law

$$B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$$

$$(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Longrightarrow B_{1,1})$$

$$(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg(P_{1,2} \vee P_{2,1}) \vee B_{1,1})$$

$(\neg B$ 1,1 $\vee \neg$ P 1,2 $\vee$ P 2,1 $) \wedge ((\neg$ P 1,2 $\wedge \neg,$
1,1 1 1,1 , 1 1 2,1 2 1,2 1 2,1 , 2 2 1,2 1,1 1,1 1,

$(\neg B$ 1,1 $\vee \neg$ P 1,2 $\vee$ P 2,1 $) \wedge (\neg$ P 1,2 $\vee$ B 1,1 $) \wedge (\neg$ P
1,1 1 1,1 , 1 1 2,1 1 1,1 , 2 2 1,2 1 2,1 , 2 2 1,2 1,1 1,1 1,1

II. Proof by Resolution – An Example

KB:

P

$P \rightarrow (Q \vee R)$ ----→ $\neg P \vee Q \vee R$

$Q \rightarrow S$ ----→ $\neg Q \vee S$

$R \rightarrow (S \wedge T)$ ----→ $\neg R \vee (S \wedge T)$

----→ $(\neg R \vee S) \wedge (\neg R \vee T)$

Q: *KB* $\vdash S$?

1. Converting sentences to CNF

KB:

P

$\neg P \vee Q \vee R$

$\neg Q \vee S$

$(\neg R \vee S) \wedge (\neg R \vee T)$

2. Spilt each conjunction into clauses.

KB:

P

$\neg P \vee Q \vee R$

$\neg Q \vee S$

$\neg R \vee S$

$\neg R \vee T$

Proof by Resolution

KB (updated):

(1) P

(2) $\neg P \vee Q \vee R$

(3) $\neg Q \vee S$

(4) $\neg R \vee S$

(5) $\neg R \vee T$

(1) P (2) $\neg P \vee Q \vee R$

resolve

(3) $\neg Q \vee S$ (6) $Q \vee R$

(4) $\neg R \vee S$ (7) $S \vee R$

(8) $S \vee S \equiv S$

Resolution tree

Resolution Refutation

(Proof by contradiction)
To show that $KB \vDash \alpha$, we show that $KB \wedge \neg\alpha$ is unsatisfiable. .

KB (about a summer day):

```
(1) If it is raining and you are outside then you will get wet.
(2) If it is warm and there is no rain then it is a pleasant day.
(3) You are not wet.
(4) You are outside.
(5) It is a warm day.
```

Prove

```
It is a pleasant day.
```

* Example taken from http://watson.latech.edu/book/intelligence/intelligenceApproaches2b2.html

KB in Propositional Sentences

KB (rewritten):

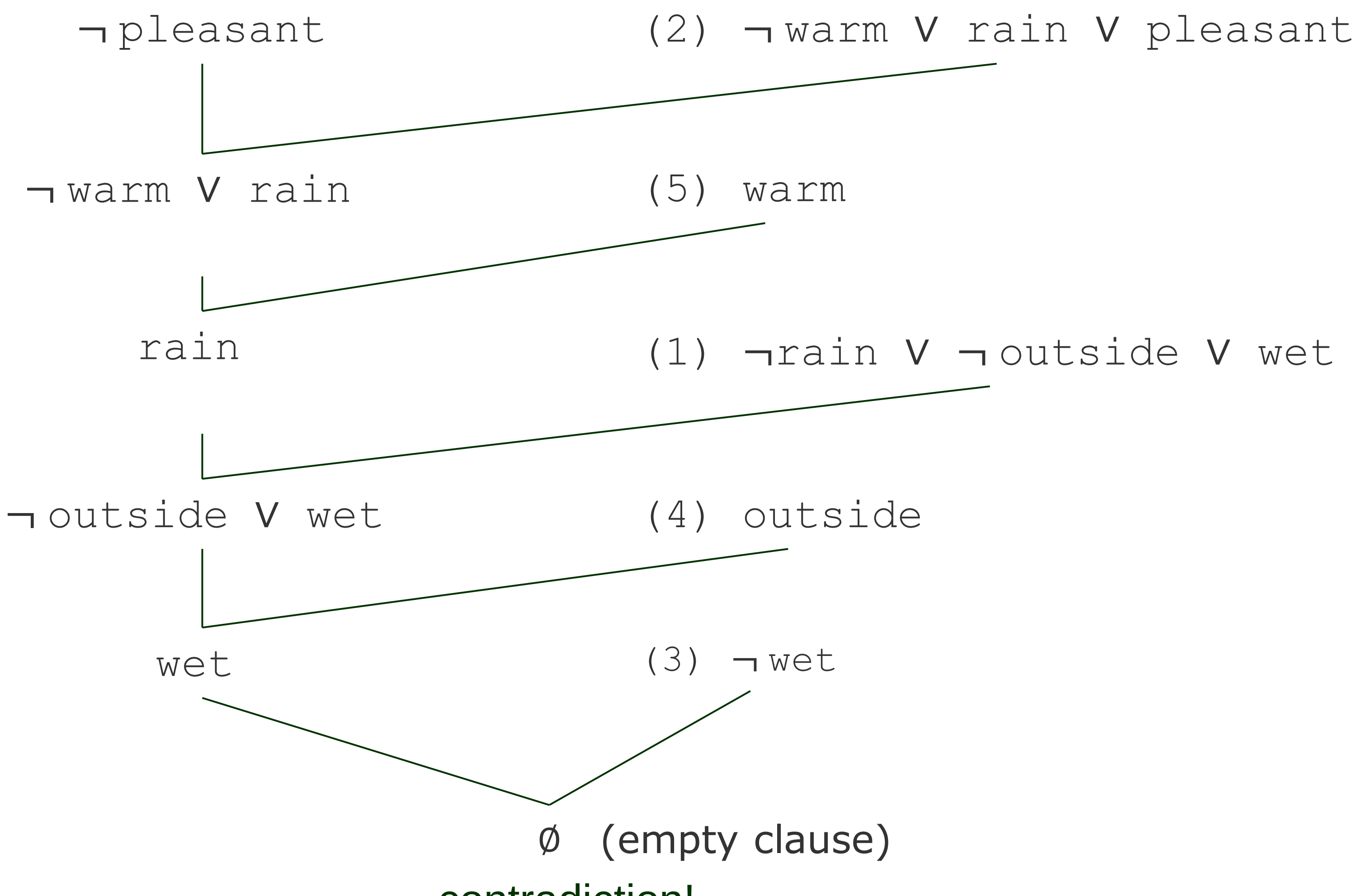
```
(1)  ( rain ∧ outside ) ⇒ wet
(2)  ( warm ∧ ¬rain ) ⇒ pleasant
(3)  ¬wet
(4)  outside
(5)  warm
```

⇓ converted into clauses

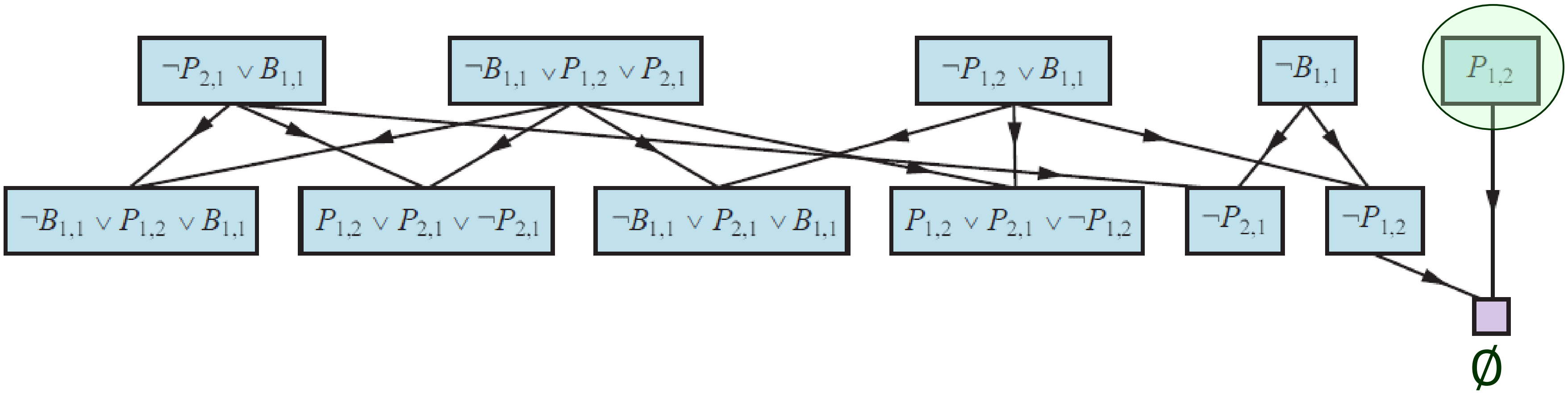
```
(1)  ¬rain ∨ ¬outside ∨ wet
(2)  ¬warm ∨ rain ∨ pleasant
(3)  ¬wet
(4)  outside
(5)  warm
```

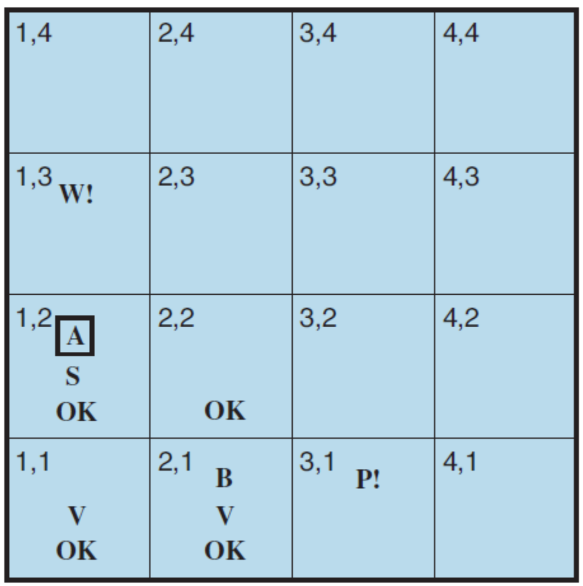
We add ¬`pleasant` to KB and try to derive `false`.

Resolution Refutation Tree



Proving $\neg P_{1,2}$ in the Wumpus World





Resolution Algorithm

function PL-RESOLUTION(KB, α) **returns** *true* or *false*
  inputs: KB, the knowledge base, a sentence in propositional logic
    α, the query, a sentence in propositional logic

  $clauses \leftarrow$ the set of clauses in the CNF representation of $KB \wedge \neg\alpha$
  $new \leftarrow \{\ \}$
  while $true$ **do**
    for each pair of clauses C_i, C_j **in** $clauses$ **do**
      $resolvents \leftarrow$ PL-RESOLVE(C_i, C_j)
      if $resolvents$ contains the empty clause **then return** $true$
      $new \leftarrow new \cup resolvents$
    if $new \subseteq clauses$ **then return** $false$ // no new clauses can be added.
    $clauses \leftarrow clauses \cup new$

The process ends in one of two situations below:

- No new clauses can be added, in which case *KB* does not entail α;
- Two clauses resolve to yield the empty clause, in which case *KB* entails α.

Completeness of Resolution

Given a set of clauses *S*, its *resolution closure* *RC*(S) includes all the clauses in S as well as all the resolvents from repeated applications of the resolution rule.

RC(S) is finite because only 3^n distinct clauses can be constructed out of n propositional symbols appearing in *S*.

Ground Resolution Theorem: If S is unsatisfiable, then *RC*(S) contains the empty clause ∅.

Constructive proof by explicitly generating an assignment for S if $\emptyset \notin$ *RC*(S).