Lecture 11

Exponential Distribution

STAT 330 - Iowa State University

Exponential Distribution

Exponential Distribution

Setup: The exponential distribution is commonly used to model waiting times between single occurrences, and lifetimes of electrical/mechanical devices.

Define the random variable

$$X = \text{“time between occurrences (rare events)”}$$

This random variable X follows an *exponential distribution*

$$X \sim Exp(\lambda)$$

where $\lambda > 0$ is the rate parameter.

Exponential PDF

- Probability Density Function (pdf)
  - $\mathrm{Im}(X) = (0, \infty)$
  - $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$

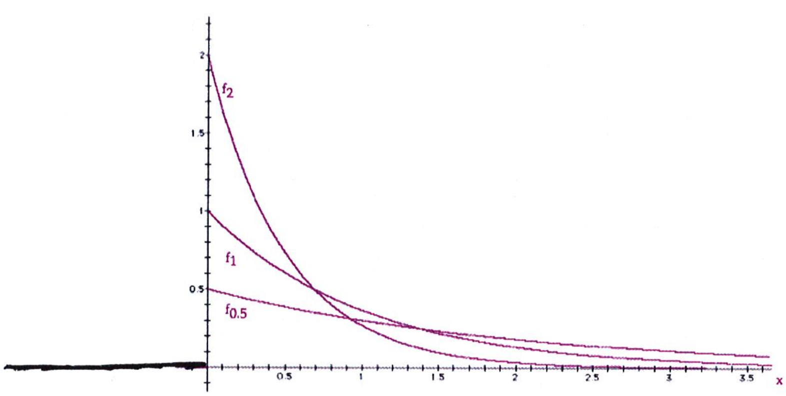

Figure 1: PDFs for exponential distributions with $\lambda = 0.5, 1, 2$

Exponential Distribution Cont.

- Cumulative distribution function (cdf)

$$F_X(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 - e^{-\lambda t} & \text{for } t > 0 \end{cases}$$

- Expected Value: $E(X) = \frac{1}{\lambda}$

- Variance: $Var(X) = \frac{1}{\lambda^2}$

Examples

Exponential Distribution Example

Example 1: Suppose you create a website and are interested in modeling the time between hits to your website. On average, you receive 5 hits per minute.

- Hits to a website is a "rare" occurrence. Time between hits can be modeled using an exponential distribution

  Define the R.V: X = time between hits to your website

  Distribution of X: $X \sim Exp(\lambda) \equiv Exp(?)$

- What value should we use for the rate parameter λ?

  $E(X) = \frac{1}{\lambda}$ is the average time between hits.
  We know there is an average of 5 hits/min. This means, on average, there is $\frac{1}{5}$ min/hit.

  $E(X) = \frac{1}{\lambda} = \frac{1}{5} \rightarrow \underline{\lambda = 5}$

5 hits/min
$\Rightarrow$ 1 min/5 hits
$\Rightarrow$ 1/5 min/hit

rate is always the # of occurances in the interval we're interested in.

Exponential Distribution Example

X = time between hits to your website

$X \sim Exp(5)$

$\lambda = 5$

PDF (general form for exponential) $\boxed{\text{PDF}}$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases} \Longrightarrow f(x) = \begin{cases} 5e^{-5x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

CDF (general form for exponential) $\boxed{\text{CDF}}$

$$F_X(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 - e^{-\lambda t} & \text{for } t > 0 \end{cases} \Longrightarrow F_X(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 - e^{-5t} & \text{for } t > 0 \end{cases}$$

Exponential distribution Example

$X \sim Exp(\lambda = 5)$

1. What is the expected time between hits?

$EX = \frac{1}{\lambda} = \frac{1}{5}$

2. What is the variance of time between hits?

$Var(X) = \frac{1}{\lambda^2} = \frac{1}{5^2} = \frac{1}{25}$

2/3 min

3. What is the probability that we wait at most 40 seconds before someone visits your website? $P(X \le 2/3)$

$P(X \le 2/3) = F_X(2/3) = 1 - e^{-(5)(2/3)} = 0.9643$

Exponential Distribution Example

4. How long do we have to wait to observe a first hit with probability 0.9?

Find the "t" that makes $P(X \le t) = 0.9$

$P(X \le t) = 0.9$

$\Rightarrow \quad F_X(t) = 0.9$

$\Rightarrow \quad 1 - e^{-5t} = 0.9$

$\Rightarrow \quad e^{-5t} = 0.1$

$\Rightarrow \quad \ln(e^{-5t}) = \ln(0.1)$

$-5t = \ln(0.1)$

$t = \frac{-\ln(0.1)}{5} = 0.4605 \text{ min}$

Memoryless Property

Memoryless Property of Exponential Distribution

- In the web page example, we said that we start to observe the web page at time point 0.
- Does the choice of this time point affect our analysis?
- If there is no hit in 1^{st} min, what is the probability that we get a hit in the next 40 seconds?

This is a conditional probability:

$$P(\text{obtain hit in 1 min and 40 sec}|\text{no hit in } 1^{st} \text{ min})$$
$$= P\left(X \leq \frac{5}{3} \middle| X > 1\right)$$

Memoryless Property of Exponential Distribution

$P(\text{obtain hit in 1 min and 40 sec}|\text{no hit in } 1^{st} \text{ min})$

$$= P\left(X \leq \frac{5}{3}\middle| X > 1\right)$$

$$= \frac{P(X \leq \frac{5}{3} \cap X > 1)}{P(X > 1)}$$

$$= \frac{P(1 \leq X \leq \frac{5}{3})}{P(X > 1)}$$

$$= \frac{F_X(\frac{5}{3}) - F_X(1)}{1 - F_X(1)} = \frac{(1 - e^{-5(5/3)}) - (1 - e^{-5(1)})}{1 - (1 - e^{-5(1)})}$$

$$= 0.9643$$

This is exactly the same as $P(X \leq \frac{2}{3})$ which we calculated in Example 1 #3:

Memoryless Property of Exponential Dist. Cont.

$$* \quad P(Y \leq t + s | Y \geq s) = 1 - e^{-\lambda t} = P(Y \leq t) \quad *$$

- In other words, a random variable with an exponential distribution "forgets" about its past.
- This phenomena is called the *memoryless property*.
- Common examples of this is in modeling the lifetime of electrical/mechanical devices that breaks down due to random error, and modeling radioactive decay.
- Exponential distribution is the only continuous distribution that has this property.
- Recall that the Geometric distribution (discrete distribution) also has this property.