Review for Final Exam Part 3

Systems of Differential Equations

We'd like to solve the $n \times n$ system of linear first order differential equations of the form:

$$
\begin{aligned}
\frac{dx_1}{dt} &= a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1n}\,x_n \\
\frac{dx_2}{dt} &= a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2n}\,x_n \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}\,x_1 + a_{n2}\,x_2 + \cdots + a_{nn}\,x_n
\end{aligned}
$$

Which can be written in the more compact matrix/vector form:

$$\vec{X}\,'(t) = A\vec{X}(t), \quad \text{where } A \text{ is the matrix of coefficients, and } \vec{X}(t) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Steps to solve $\vec{X}\,' = A\vec{X}$ (A homogeneous system)

1. Find the eigenvalues of A: Solve for λ in $\det(A-\lambda I)=0$ (Eigenvalues are the roots λ_i)
2. Find the eigenvectors that correspond to the found eigenvalues in step 1: Find $\vec{K}_i$ such that $(A-\lambda_i I)\vec{K}_i = \vec{0}$
3. Form the linearly independent solutions, we have the following cases:

- Case $\lambda_1 \neq \lambda_2$ (real) $\vec{X}_1 = \vec{K}_1 e^{\lambda_1 t}$ and $\vec{X}_2 = \vec{K}_2 e^{\lambda_2 t}$
- Case $\lambda_1 = \lambda_2$ (real) $\vec{X}_1 = \vec{K}e^{\lambda t}$ and $\vec{X}_2 = \vec{K}te^{\lambda t} + \vec{P}e^{\lambda t}$
  $\lambda_1 = \lambda_2 = \lambda$ where $(A-\lambda I)\vec{P} = \vec{K}$
- Case $\lambda_{1,2} = \alpha \pm i\beta$ A complex sol: $(\vec{B}_1 + i\vec{B}_2)e^{\alpha t}(\cos\beta t + i\sin\beta t)$

$\Rightarrow$ $\vec{X}_1 = e^{\alpha t}(\vec{B}_1\cos\beta t - \vec{B}_2\sin\beta t)$ and $\vec{X}_2 = e^{\alpha t}(\vec{B}_1\sin\beta t + \vec{B}_2\cos\beta t)$.

Some other concepts to remember:

If $\vec{X}_1$ & $\vec{X}_2$ are l.i. solutions we say: $\left\{\vec{X}_1, \vec{X}_2\right\}$ is a fundamental set

and $\Phi = \left(\vec{X}_1 \ \vec{X}_2\right)$ is a fundamental matrix.

We can write the general solution $\vec{X}_c = C_1\vec{X}_1 + C_2\vec{X}_2$ can also be written:

$$\vec{X}_c = \Phi\vec{C} = \left(\vec{X}_1 \ \vec{X}_2\right)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

The non-homogeneous problem

$$\vec{X}\,' = A\vec{X} + \vec{f}(t),$$

has general solution $\vec{X} = \vec{X}_c + \vec{X}_p$, where $\vec{X}_p$ is a particular solution, which can be found with undetermined coefficients or variation of parameters.

Examples

1) Find the general solution of $\vec{X}\,' = \overbrace{\begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}}^{A}\vec{X} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Char. Eqn: $\det\begin{pmatrix} 5-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix} = (5-\lambda)(3-\lambda)+1 = \lambda^2 - 8\lambda + 16 = (\lambda-4)^2 = 0$

$\Rightarrow$ Eigenvalues $\lambda_1 = \lambda_2 = 4$.

Find the eigenvector $\vec{K}$, i.e. solve for $\vec{K}$ in $(A-4I)\vec{K} = \vec{0}$

$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad k_1 - k_2 = 0 \Rightarrow k_1 = k_2$ Let $\vec{K} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow \vec{X}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t}$

Next find $\vec{P}$ suchthat

$(A-4I)\vec{P} = \vec{K} \Leftrightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\Rightarrow P_1 - P_2 = 1$ let $P_1 = 1 \Rightarrow P_2 = 0$

Thus:

$\vec{X}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}te^{4t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{4t}$

Next, find a particular solution, with undetermined coefficients we let $\vec{X}_p = \begin{pmatrix} a \\ b \end{pmatrix}$ & plug into the system.

$$\vec{X}_p' = \begin{pmatrix} a \\ b \end{pmatrix}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{aligned} 5a - b &= -1 \\ a + 3b &= -2 \end{aligned} \iff \begin{aligned} 15a - 3b &= -3 \\ a + 3b &= -2 \\ \hline 16a &= -5 \end{aligned} \Rightarrow a = -5/16, \quad b = -\frac{9}{16}$$

$$b = 5a + 1 = \frac{-25}{16} + 1$$

Finally, the general sol is:

$$\vec{X}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{4t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{4t}\right] + \begin{pmatrix} -5/16 \\ -9/16 \end{pmatrix}$$

2) Find the general solution of $\vec{X}' = \begin{pmatrix} 4 & -7 \\ 14 & -10 \end{pmatrix} \vec{X}$

Char. Eqn: $\det\begin{pmatrix} 4-\lambda & -7 \\ 14 & -10-\lambda \end{pmatrix} = (4-\lambda)(-10-\lambda) + 98 = 0$

$\iff \lambda^2 + 6\lambda + 58 = 0 \iff \lambda^2 + 6\lambda + 9 = -49$

$\iff (\lambda+3)^2 = -49 \iff \lambda + 3 = \pm 7i \iff \lambda = -3 \pm 7i$ $\alpha = -3$, $\beta = 7$

Find the eigenvector $\vec{K}$:

$$\begin{pmatrix} 7-7i & -7 \\ 14 & -7-7i \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(7-7i)K_1 = 7K_2$

$(1-i)K_1 = K_2$

let $\vec{K} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{B_1} + i\underbrace{\begin{pmatrix} 0 \\ -1 \end{pmatrix}}_{B_2}$

$$\vec{X}_1 = e^{-3t}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos 7t - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin 7t\right) = e^{-3t}\begin{pmatrix} \cos 7t \\ \cos 7t + \sin 7t \end{pmatrix}$$

$$\vec{X}_2 = e^{-3t}\left(\begin{pmatrix}1\\1\end{pmatrix}\sin 7t + \begin{pmatrix}0\\-1\end{pmatrix}\cos 7t\right) = e^{-3t}\begin{pmatrix}\sin 7t\\ \sin 7t - \cos 7t\end{pmatrix}$$

Then the general Solution:

$$\vec{X} = \underbrace{\begin{pmatrix} e^{-3t}\cos 7t & e^{-3t}\sin 7t \\ e^{-3t}(\cos 7t + \sin 7t) & e^{-3t}(\sin 7t - \cos 7t)\end{pmatrix}}_{\uparrow \text{ a fundamental matrix } \phi.}\begin{pmatrix}C_1\\C_2\end{pmatrix}$$

3) Using variation of parameters find a particular solution for

$$\vec{X}' = \begin{pmatrix}4 & -7\\ 14 & -10\end{pmatrix}\vec{X} + \begin{pmatrix}14e^{-3t}\\ 14e^{-3t}\end{pmatrix}$$

From the previous example we have a fundamental matrix ϕ.

$$\vec{X}_p = \phi\int \phi^{-1}\vec{f}\,dt$$

$$\det\phi = e^{-6t}\left[\cancel{\cos 7t\sin 7t} - \cos^2 7t - \cancel{\sin 7t\cos 7t} - \sin^2 7t\right] = -e^{-6t}$$

$$\phi^{-1} = -e^{6t}e^{-3t}\begin{pmatrix}\sin 7t - \cos 7t & -\sin 7t\\ -\cos 7t - \sin 7t & \cos 7t\end{pmatrix} \quad \& \quad \vec{f} = 14e^{-3t}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\int \Phi^{-1}\vec{f}\,dt = \int -14\begin{pmatrix} -\cos 7t \\ -\sin 7t \end{pmatrix} dt = \int \begin{pmatrix} 14\cos 7t \\ 14\sin 7t \end{pmatrix} dt = \begin{pmatrix} 2\sin 7t \\ -2\cos 7t \end{pmatrix}$$

$$\vec{X}_p = \Phi \begin{pmatrix} 2\sin 7t \\ -2\cos 7t \end{pmatrix} = \begin{pmatrix} 0 \\ 2e^{-3t} \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{-3t}$$

The system $\vec{X}\,' = \begin{pmatrix} 0 & 4 \\ -1 & 5 \end{pmatrix}\vec{X} + \begin{pmatrix} 9 \\ -9 \end{pmatrix} e^t$ has a fundamental matrix $\Phi = \begin{pmatrix} 4e^t & e^{4t} \\ e^t & e^{4t} \end{pmatrix}$ and a particular solution $\vec{X}_p = \begin{pmatrix} 5e^t + 24te^t \\ 5e^t + 6te^t \end{pmatrix}$. Find the solution to the IVP when $\vec{X}(0) = \begin{pmatrix} 25 \\ 4 \end{pmatrix}$.

General Sol : $\vec{X} = \Phi\vec{C} + \vec{X}_p$ & plug initial conditions here

$$\vec{X}(0) = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 25 \\ 4 \end{pmatrix}$$

$$\begin{aligned} 4C_1 + C_2 &= 20 \\ C_1 + C_2 &= -1 \\ \hline 3C_1 &= 21 \\ \Rightarrow C_1 &= 7 \end{aligned}$$

$C_2 = -1 - C_1$

$C_2 = -1 - 7 = -8$ $\quad \therefore \vec{C} = \begin{pmatrix} 7 \\ -8 \end{pmatrix}$.