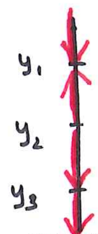
Review for Exam 1

- Types: Know how to classify the differential equaitons into: order, separable, linear, exact, homogeneous, bernoulli, of the type $\frac{dy}{dx} = f(Ax+By+C)$, autonomous.
- Autonomous DEs: Find critical points (equilibrium/constant solutions) and know if they are stable, unstable or semistable.

$\frac{dy}{dx} = f(y)$; Know how to find the critical pts.

Set $f(y) = 0$ & find y_1, y_2, y_3 (etc) that solve the equality.

Phase Portrait



← A. stable

← unstable

← semistable

- Applications (Models)
  - Population Dynamics $\frac{dP}{dt} = KP \quad (K>0)$
  - Radioactive Decay $\frac{dA}{dt} = -KA \quad (K>0)$
  - Newton's Law of Cooling $\frac{dT}{dt} = -K(T-T_m) \quad (K>0)$
  - Mixing Problems $\frac{dA}{dt} = \begin{pmatrix}\text{input}\\ \text{rate}\end{pmatrix} - \begin{pmatrix}\text{output}\\ \text{rate}\end{pmatrix}$ A = Amount of salt in a tank.

    ↑ $R_{flow in}$ * Concentration

- IVPs (Initial Value Problems) 1st find the general solution which includes an undetermined constant.

2nd Plug initial condition into general sol. to find the constant.

- Methods to solve 1st order DEs

▸ Separable Equations: A DE is separable if $\frac{dy}{dx} = h(y)\,g(x)$

To solve, separate & integrate $\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$

▸ Linear Equations: Have the form (Standard form): $\frac{dy}{dx} + P(x)y = f(x)$

To solve find $\mu(x) = e^{\int P dx}$ then the eqn becomes.

$(\mu \cdot y)' = f(x)\mu(x) \Rightarrow y = \frac{1}{\mu}\left(\int f(x)\mu(x)\,dx + C\right)$

▸ Exact Equations: Given $M(x,y)\,dx + N(x,y)\,dy = 0$, the eqn. is exact $\Leftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then find $f(x,y)$ such that

$\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$ and the sol is $f(x,y) = C$

▸ Homogeneous Equations: $\frac{dy}{dx} = G(y/x)$. We can test for homogeneous, if $\frac{dy}{dx} = f(x,y)$; $f(tx,ty) = t^{\alpha} f(x,y)$. The substitution:

$u = \frac{y}{x}$ $(y = ux)$ and $\frac{dy}{dx} = x\frac{du}{dx} + u$, & get a separable eqn.

▸ Bernoulli Equations: $\frac{dy}{dx} + P(x)y = f(x)y^n$ $(n>1)$. We let

$u = y^{1-n}$; $\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx}$ & substitute to get a linear eqn.

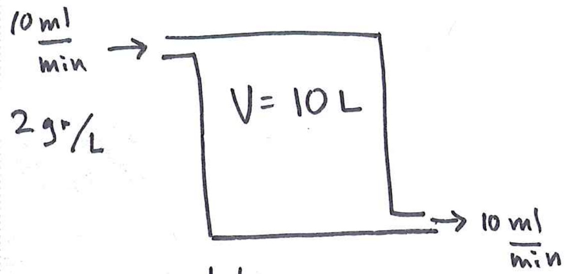
▸ Equations of the form $\frac{dy}{dx} = f(Ax + By + C)$

let $u = Ax + By + C \Rightarrow \frac{du}{dx} = A + B\frac{dy}{dx}$ & substitute

to get a separable equation.

Examples

Example 1. Suppose that a large mixing tank initially holds 10 liters of **pure water**. Brine at a concentration of 2 gr/L is pumped into the tank at a rate of 10 ml/min, and when the solution is well stirred, it is then pumped out at the same rate. Determine a differential equation for the amount of salt $A(t)$ in the tank at time t, and find the concentration of salt in the tank after 1 hour of pumping.



$$10\,ml \times \frac{1\,L}{1000\,ml} = \frac{1}{100}\,L$$

$$\frac{dA}{dt} = \begin{pmatrix}\text{input}\\ \text{rate}\end{pmatrix} - \begin{pmatrix}\text{output}\\ \text{rate}\end{pmatrix}$$

$$\frac{dA}{dt} = \frac{1}{100}\frac{\cancel{L}}{min} \cdot \frac{2\,gr}{\cancel{L}} - \frac{1}{100}\frac{\cancel{L}}{min} \cdot \frac{A(t)}{10}\frac{gr}{\cancel{L}}$$

$$\frac{dA}{dt} = \frac{2}{100} - \frac{A}{1000} = -\frac{1}{1000}(-20 + A) \quad \begin{pmatrix}\text{Solve as}\\ \text{separable}\end{pmatrix}$$

$$\int \frac{1}{A-20}\,dA = \int -\frac{1}{1000}\,dt \Rightarrow \ln|A-20| = -\frac{t}{1000} + C_1$$

$$A(t) = 20 + Ce^{-t/1000} \Rightarrow C(t) = \frac{A(t)}{Vol} = \frac{20 + Ce^{-t/100}}{10}$$

We need $C(60)$. (Concentration).

Note we have an initial condition $A(0) = 0$ (pure water).

$$\Rightarrow \quad A(0) = 20 + C = 0 \quad \Rightarrow \quad C = -20$$

$$\therefore \quad A(t) = 20 - 20e^{-t/100}$$

$$\text{and} \quad C(60) = \frac{20 - 20e^{-60/1000}}{10} =$$

Example 2. Find the value of b so that the following equation is exact and find its general solution.

$$\underbrace{(2y\cos(2xy) - 3e^{3\cos x}\sin x)}_{M(x,y)}\,dx + \underbrace{(bx\cos(2xy))}_{N(x,y)}\,dy = 0$$

We need :

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(2y\cos(2xy) - 3e^{3\cos x}\sin x\right) = 2\cos(2xy) + 2y(-\sin(2xy))2x$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(bx\cos(2xy)) = b\cos(2xy) + bx(-\sin(2xy))2y$$

These are equal if $b = 2$.

To find the solution we need $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = M \quad \& \quad \frac{\partial f}{\partial y} = N$$

$$\frac{\partial f}{\partial y} = N \Rightarrow f = \int N\,dy = \int 2x\cos(2xy)\,dy = \sin(2xy) + g(x)$$

$$\frac{\partial f}{\partial x} = M \Leftrightarrow \frac{\partial}{\partial x}(\sin(2xy) + g(x)) = 2y\cos(2xy) - 3e^{3\cos x}\sin x$$

$$\cancel{2y\cos 2xy} + g'(x) = \cancel{2y\cos(2xy)} - 3e^{3\cos x}\sin x$$

$$\int g'(x)\,dx = \int -3e^{3\cos x}\sin x\,dx \Rightarrow g(x) = e^{3\cos x}$$

update $f(x,y) = \sin(2xy) + e^{3\cos x}$

Sol : $\sin(2xy) + e^{3\cos x} = C$

Example 3. Find the general solution of the following IVP

$$x^2 \frac{dy}{dx} - 2xy = 3y^4 \; ; \quad y(1) = 1/2$$

$$\frac{dy}{dx} - \frac{2}{x} y = \frac{3}{x^2} y^4$$

(Bernoulli) let $u = y^{-3}$

$$\boxed{y^{-4}\frac{dy}{dx}} - \frac{2}{x} y^{-3} = \frac{3}{x^3}$$

$$\Rightarrow \frac{du}{dx} = -3\,\boxed{y^{-4}\frac{dy}{dx}}$$

$$-\frac{1}{3}\frac{du}{dx} - \frac{2}{x} u = \frac{3}{x^3}$$ (linear)

$$\frac{du}{dx} + \frac{6}{x} u = -\frac{9}{x^3}$$ ← In standard form.

Solve for u (linear equ).

Find $\mu = e^{\int 6/x \, dx} = e^{6\ln|x|} = |x|^6 = x^6$

After multiplying by μ the equ. becomes:

$$(x^6 u)' = -\frac{9}{x^3} x^6 = -9x^4.$$

$$\Rightarrow \quad x^6 u = \int -9x^4 \, dx = -\frac{9}{5} x^5 + C$$

$$u = -\frac{9}{5x} + Cx^{-6} \quad \Rightarrow \quad y^{-3} = -\frac{9}{5x} + Cx^{-6}$$

So $y(x) = \left(-\frac{9}{5x} + Cx^{-6}\right)^{-1/3}$ ← General Sol.

IVP sol: Plug initial condition $y(1) = \left(-\frac{9}{5} + C\right)^{-1/3} = \frac{1}{2}$

$$\Rightarrow -\frac{9}{5} + C = 8 \quad \Rightarrow C = 8 + \frac{9}{5} = \frac{49}{5}$$

∴ Sol. $\boxed{y = \left(\frac{-9}{5x} + \frac{49}{5} x^{-6}\right)^{-1/3}}$

Example 4. Find the general solution of

$$\frac{dy}{dx} = \frac{\sec(x-y+1)+y-x}{y-x}$$

$$\Longleftrightarrow \quad \frac{dy}{dx} = \frac{\sec(x-y+1) - (x-y)}{-(x-y)}$$

Note
$x-y = u-1$

Let $u = x-y+1 \Rightarrow \frac{du}{dx} = 1 - \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 1 - \frac{du}{dx}$

Sub: $\cancel{1} - \frac{du}{dx} = \frac{\sec u - (u-1)}{-(u-1)} = -\frac{\sec u}{u-1} - \cancel{1}$

$$\Rightarrow \quad \frac{du}{dx} = \frac{\sec u}{u-1} \qquad \text{(separable)}$$

Separate: $\frac{u-1}{\sec u} \cdot du = dx$

Integrate: $\int \frac{u}{\sec u} - \frac{1}{\sec u}\, du = \int dx$

$$\Longleftrightarrow \int \underbrace{u\cos u}_{\text{I.B. Parts}} - \cos u \; du = x + C$$

$u\sin u + \cos u - \sin u = x + C$

$(u-1)\sin u + \cos u = x + C$

Back Sub: $\underbrace{(x-y)\sin(x-y+1) + \cos(x-y+1) = x + C}$

(Implicit Sol)