## 6.2 Solutions about Ordinary Points

**Definition**

A function $f$ is said to be analytic at a point $x_0$ if it can be represented by a power series in $(x - x_0)$ with $R > 0$.

**Definition**

A point $x = x_0$ is said to be an ordinary point of the differential equation $y'' + P(x)y' + Q(x)y = 0$ if both $P(x)$ and $Q(x)$ are analytic at $x_0$. A point that is not an ordinary point is called a singular point.

Examples.

1. $y'' + y' = 0$ No singular points
2. $y'' + 3y' + 2y = 0$ No singular points
3. $y'' + e^x y - (\sin x)y = 0$ No singular points
4. $y'' + xy' + (\ln x)y = 0$ $\leftarrow$ singular point at zero.
5. $xy'' + y' + xy = 0$ $\Leftrightarrow y'' + \frac{1}{x}y' + y = 0$ $\leftarrow$ singular pt at zero.



We will mostly work with differential equations with polynomial coefficients, that is the equations will have the form:

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,$$

where $a_2(x)$, $a_1(x)$ and $a_0(x)$ are polynomials. So in standard form, we would have:

$$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = 0, \quad (*)$$

that is, $P(x) = \frac{a_1(x)}{a_2(x)}$ and $Q(x) = \frac{a_0(x)}{a_2(x)}$ will be rational functions.

It is known that rational functions are analytic at all points except at the zeros of the denominator. (Analytic in their domain).

Then we can conclude that a number $x = x_0$ is an ordinary point of (*) if $a_2(x_0) \neq 0$, and singular point if $a_2(x_0) = 0$.

* The zeros of $a_2$ are singular points even if they are complex.

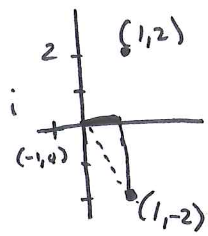
## Theorem

*If $x = x_0$ is an ordinary point of the differential equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, we can always find two linearly independent solutions of the form of a power series centered at $x_0$., that is,*

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

*A power series solution converges at least on some interval defined by $|x - x_0| < R$, where $R$ is the distance from $x_0$ to the closest singular point, also called minimum radius of convergence.*

Example. Find the minimum radius of convergence of a power series solution of the differential equation: $(x^2 - 2x + 5)y'' + xy' - y = 0$ ; about $x = 0$.




Roots of $x^2-2x+5=0 \Rightarrow x = \frac{2\pm\sqrt{4-20}}{2} = \frac{2\pm\sqrt{-16}}{2} = \frac{2\pm 4i}{2} = 1\pm 2i$

$\Rightarrow R = \sqrt{5}$

If $x_0 = -1$

$\Rightarrow R = \sqrt{(2-0)^2+(1+1)^2} = \sqrt{8}$



Example. Find a power series solution about $x_0 = 0$ of $y'' + xy = 0$.

Here $a_2 = 1 \neq 0$ (never zero) so there are no singular points

$\Rightarrow R = \infty$.

Assume $y = \sum_{n=0}^{\infty} c_n x^n$ ; $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$ ; $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

& plug into D.E:

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + x \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Leftrightarrow \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$(i = n-2)$ $\quad$ $(i = n+1)$ aux. variable for reindexing if needed.

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=1}^{\infty} c_{n-1} x^n = 0$$

(write constant term separately)

$$2 \cdot 1 \cdot C_2 X^0 + \sum_{n=1}^{\infty} (n+2)(n+1)C_{n+2} X^n + \sum_{n=1}^{\infty} C_{n-1} X^n = 0$$

$$2C_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)C_{n+2} + C_{n-1}\right] X^n = 0 \qquad (\text{for all } x)$$

Then by identity property:

$$2C_2 = 0 \quad \text{and} \quad (n+2)(n+1)C_{n+2} + C_{n-1} = 0$$

$$\Rightarrow C_2 = 0 \quad \text{and} \quad \boxed{C_{n+2} = -\frac{C_{n-1}}{(n+1)(n+2)}}$$ Recurrence Relation.

<u>Note</u>: $C_0 = y(0)$ and $C_1 = y'(0)$ ← the initial conditions.

$n=1 \quad C_3 = -\dfrac{C_0}{2 \cdot 3}$

$n=2 \quad C_4 = -\dfrac{C_1}{3 \cdot 4}$

$n=3 \quad C_5 = -\dfrac{C_2}{4 \cdot 5} = 0$

$n=4 \quad C_6 = -\dfrac{C_3}{5 \cdot 6} = -\left(-\dfrac{C_0}{2 \cdot 3}\right)\dfrac{1}{5 \cdot 6} = \dfrac{C_0}{2 \cdot 3 \cdot 5 6}$

$n=5 \quad C_7 = -\dfrac{C_4}{6 \cdot 7} = \dfrac{C_1}{3 \cdot 4 \cdot 6 \cdot 7}$

$n=6 \quad C_8 = -\dfrac{C_5}{7 \cdot 8} = 0$

We collect terms with $C_0$ & terms w $C_1$

← $y_1$

$$\Rightarrow y(x) = C_0\left[1 - \_\_ X^3 + \_\_ X^6 - \_\_ X^9 + \_\_ X^{12} - \cdots\right]$$
$$+ C_1\left[1 - \_\_ X^4 + \_\_ X^7 - \_\_ X^{10} + \_\_ X^{13} - \cdots\right].$$

↑ $y_2$

(see pg 247-8)

$y_1$ & $y_2$ the two l.i. solutions we need for a 2nd order d.e.!

Indeed recall general sol is $y = C_1 y_1 + C_2 y_2$ here $C_1, C_2$ are the $C_0, C_1$ from the p series...

Example. Find a power series solution about $x_0 = 0$ of

$$(x-1)y'' - xy' + y = 0\,; \quad y(0) = -2, \;\; y'(0) = 6.$$

Since $(x-1) = 0$ at $1$ and distance from $X_0 = 0$ to $1$ is $1$

Then $R = 1$.

Recall that $y(0) = c_0 = -2$ and $y'(0) = c_1 = 6$

Assume $y(x) = \sum_{n=0}^{\infty} c_n x^n \;;\; y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} \;;\; y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

& plug in $x y'' - y'' - x y' + y = 0$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

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$$\sum_{n=1}^{\infty} (n+1) n\, c_{n+1} x^n - \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

(write the constant terms - corresponding to $n=0$ - seperately).

$$\underbrace{\left(-2 \cdot 1 \cdot c_2 + c_0\right)}_{\text{set } = 0} + \sum_{n=1}^{\infty} \underbrace{\left[n(n+1) c_{n+1} - (n+1)(n+2) c_{n+2} - n c_n + c_n\right]}_{\text{set } = 0} x^n = 0$$

by Identity Property

$$-2c_2 + c_0 = 0 \Rightarrow c_2 = +\frac{c_0}{2} = +\left(\frac{-2}{2}\right) = -1$$

Recurrence Relation: $\left\{ c_{n+2} = \dfrac{n(n+1) c_{n+1} - (n-1) c_n}{(n+1)(n+2)} \right.$

For $n=1$ $\quad c_3 = \dfrac{2c_2}{2 \cdot 3} = \dfrac{-1}{3}$

For $n=2$ $\quad c_4 = \dfrac{2 \cdot 3 c_3 - c_2}{3 \cdot 4} = \dfrac{-2+1}{12} = \dfrac{-1}{12} \ldots$

Then the solution

$$y(x) = -2 + 6x - x^2 - \frac{1}{3}x^3 - \frac{1}{12}x^4 + \cdots$$