4.4 Undetermined Coefficients

Our goal now is to solve a nonhomogeneous linear differential equation with constant coefficients.

$$a\,y'' + b\,y' + c\,y = g(x) \qquad (*)$$

We saw in 4.1 that the general solution has the form:

$$y = y_c + y_p$$

Where y_c = complementary function the general sol. of the homogeneous associated equation.

And y_p is a particular solution. (of $*$)

We will focus now on a method to find y_p. In 4.1 we briefly mentioned that y_p shall have similar form to $g(x)$...

- Case $g(x) = cx^m$ (m any positive integer).

For instance, find a particular solution to $y'' + 3y' + 2y = 3x$.
We first "guess" that $y_p = Ax$ & plug into the equation $y_p' = A$; $y_p'' = 0$

$$(0) + 3(A) + 2(Ax) = 3x + 0$$

We'd need $3A = 0 \Rightarrow A = 0$
$2A = 3 \Rightarrow A = 3/2$ $\nexists$ impossible.

Instead we let $y_p = Ax + B$; $y_p' = A$; $y_p'' = 0$ and plug in:

$$0 + 3A + 2(Ax + B) = 3x$$

We need $2A = 3 \Rightarrow A = 3/2$
$3A + 2B = 0 \qquad 2B = -\frac{3A}{2} = -\frac{9}{2\cdot 2}$

$$\therefore y_p = \frac{3}{2}x - \frac{9}{4}$$

Due to the superposition principle, when $g(x)$ is a polynomial, y_p has the format of a polynomial of the same degree (including all terms, a term for each power).

Example. If $g(x) = x^3 - x$ then we have $y_p = Ax^3 + Bx^2 + Cx + D$

- Case $g(x) = ce^{ax}$. Now we would have $y_p = Ae^{ax}$

Example. Find a particular solution to $y'' + 2y' + 2y = 10e^{3x}$

Let $y_p = Ae^{3x}$; $y_p' = 3Ae^{3x}$; $y_p'' = 9Ae^{3x}$ & plug in:

$$9Ae^{3x} + 2(3Ae^{3x}) + 2(Ae^{3x}) = 10e^{3x}$$

$$17Ae^{3x} = 10e^{3x} \quad (e^{3x} \neq 0 \text{ for all } x).$$

$\Rightarrow A = \frac{10}{17}$ $\quad \therefore \; y_p = \frac{10}{17}e^{3x}$

However, there is a glitch in the method!!

Consider the following example: Find a particular solution y_p to the DE

$$y'' - 3y' + 2y = 3e^{2x}.$$

Let $y_p = Ae^{2x}$; $y_p' = 2Ae^{2x}$; $y_p'' = 4Ae^{2x}$ & plug in:

$$4Ae^{2x} - 6Ae^{2x} + 2Ae^{2x} = 3e^{2x}$$

$$0 = 3e^{2x} \; ! \text{(impossible.)}$$

Note that Ae^{2x} is a solution to $y'' - 3y' + 2y = 0$, indeed the auxiliary eqn: $m^2 - 3m + 2 = (m-2)(m-1) = 0 \Rightarrow m_1 = 2, m_2 = 1$

So our y_p should have the form: $y_p = Axe^{2x}$

extra x !

Then $y_p' = Ae^{2x} + 2Axe^{2x}$; $y_p'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}$ & plug in:

$$(4Axe^{2x} + 4Ae^{2x}) - 3(Ae^{2x} + 2Axe^{2x}) + 2(Axe^{2x}) = 3e^{2x}$$

$$Ae^{2x} = 3e^{2x} \quad \Rightarrow A = 3$$

$$\therefore y_p = 3xe^{2x}$$

and the general solution is :

$$y = \underbrace{c_1 e^{2x} + c_2 e^{x}}_{y_c} + \underbrace{3xe^{2x}}_{y_p}$$

Example. Find a particular solution to $y'' - 4y' + 4y = 3e^{2x}$.

Aux. Eqn : $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0 \quad \Rightarrow m = 2$ a repeated root

$\Rightarrow \quad y_1 = e^{2x}$ and $y_2 = xe^{2x}$ so we need $y_p = Ax^2e^{2x}$

$y_p' = 2Axe^{2x} + 2Ax^2e^{2x}$; $y_p'' = 2Ae^{2x} + 4Axe^{2x} + 4Axe^{2x} + 4Ax^2e^{2x}$

plug in: $(4Ax^2e^{2x} + 8Axe^{2x} + 2Ae^{2x}) - 8Axe^{2x} - 8Ax^2e^{2x} + 4Ax^2e^{2x} = 3e^{2x}$

$\Rightarrow 2Ae^{2x} = 3e^{2x} \quad \Rightarrow A = 3/2 \qquad \therefore \underline{y_p = \frac{3}{2}x^2e^{2x}}$

General Sol : $y = \underbrace{c_1e^{2x} + c_2xe^{2x}}_{y_c} + \underbrace{\frac{3}{2}x^2e^{2x}}_{y_p}$

- Case $g(x) = C\sin(\beta x)$ or $g(x) = C\cos(\beta x)$. (Or a linear combination of sines and cosines).

Example. Find y_p for the DE $2y'' - y' = 3\sin(3x)$.

1st guess $y_p = A\sin(3x)$; $y_p' = 3A\cos(3x)$; $y_p'' = -9A\sin(3x)$

plug in: $-18A\sin(3x) - 3A\cos(3x) = 3\sin(3x) + 0\cos(3x)$

we'd need $-3A = 0$ AND $-18A = 3$ can't be . . .

We need $0 \cdot y_p = (A\sin(3x) + B\cos(3x)) \cdot 0$

$-y_p' = -(-3B\sin(3x) + 3A\cos(3x))$

$2y_p'' = 2(-9A\sin(3x)) - 2\cdot 9B\cos(3x)$

$2y_p'' - y_p' = (-18A + 3B)\sin(3x) + (-18B - 3A)\cos(3x) = 3\sin(3x)$

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We need $\left.\begin{array}{l} -18A + 3B = 3 \\ -18B - 3A = 0 \end{array}\right\}$ Solve 2x2 System $A = -\frac{6}{37}$, $B = \frac{1}{37}$

$\therefore y_p = -\frac{6}{37}\sin(3x) + \frac{1}{37}\cos(3x)$

* Note if we want a y_p for $y'' + 9y = 3\sin(3x)$

the l.i. solutions of $y'' + 9y = 0$ are $y_1 = \cos(3x)$ and $y_2 = \sin(3x)$

then the form of $y_p = x\,(A\sin(3x) + B\cos(3x))$

extra x!

We could also have the case $g(x) = e^{2x}\cos x$

$\Rightarrow y_p = e^{2x}(A\sin x + B\cos x)$ as long as $2 \pm i$ is not a root of the auxiliary eqn.

We could also have combinations such as.

$$g(x) = 2x\, e^{3x} \Rightarrow y_p = (Ax+B)\, e^{3x}$$

$$g(x) = 12x^2 (\sin 7x) \Rightarrow y_p = (Ax^2+Bx+C)(D \sin 7x + E \cos 7x)$$

and don't forget the superposition principle.

$$g(x) = x^2 - 2e^{-x} + 3\cos(4x)$$

$$\Downarrow$$

$$y_p = y_{p_1} + y_{p_2} + y_{p_3}$$