Stat 330 (Spring 2010): Homework 1

Due: January 22, 2020

Show all of your work, and *please* staple your assignment if you use more than one sheet. Write your name, the course number and the section on every sheet. Problems marked with * will be graded and one additional randomly chosen problem will be graded.

1. * A coin is tossed three times, and the sequence of heads and tails is recorded.
   (a) Determine the sample space, Ω.
   (b) List the elements that make up the following events: i. A = exactly two tails, ii. B = at least two tails, iii. C = the last two tosses are heads
   (c) List the elements of the following events: i. $\overline{A}$, ii. $A \cup B$, iii. $A \cap B$, iv. $A \cap C$

   Answer:

   (a) Let H and T stand for the events of head and tail, respectively. The sample space is

   $$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

   (b) i. A = exactly two tails = $\{TTH, THT, HTT\}$
   ii. B = at least two tails = $\{TTH, THT, HTT, TTT\}$
   iii. C = the last two tosses are heads = $\{HHH, THH\}$

   (c) i. $\bar{A} = \{HHH, HHT, HTH, THH, TTT\}$ = Elements with 0,1, or 3 tails
   ii. $A \cup B = \{TTH, THT, HTT, TTT\}$ = B since $A \subset B$
   iii. $A \cap B = \{TTH, THT, HTT\} = A$ since $A \subset B$
   iv. $A \cap C = \emptyset$ since they have no elements in common

2. Let a sample space $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let $A = \{1, 3, 5\}$ and $B = \{1, 5, 10\}$ be two events. Verify DeMorgan's Laws on events A and B by showing the events on both sides of the = sign contain the same outcomes.
   (a) $\overline{A \cap B} = \overline{A} \cup \overline{B}$
   Answer: $\overline{A \cap B} = \overline{A} \cup \overline{B} = \{2, 3, 4, 6, 7, 8, 9, 10\}$
   (b) $\overline{A \cup B} = \overline{A} \cap \overline{B}$
   Answer: $\overline{A \cup B} = \overline{A} \cap \overline{B} = \{2, 4, 6, 7, 8, 9\}$

3. Suppose a six sided die is rolled and the probability of each number occurring is proportional to itself, i.e. $\mathbb{P}(1) = 1k, \mathbb{P}(2) = 2k \ldots$. Give the probabilities for each number being rolled so that the axioms of probability are satisfied.

   Answer: Since $P(\Omega) = 1$ we have $P(1 \cup 2 \cup \ldots \cup 6) = 1$. Since the outcome are disjoint, the sum of the individual probabilities must also be 1.

   So, we have $\sum_{i=1}^{6} ik = 1 \Rightarrow k \sum_{i=1}^{6} i = 1 \Rightarrow k * 21 = 1 \Rightarrow k = \frac{1}{21}$.

   Thus $\mathbb{P}(1) = \frac{1}{21}, \mathbb{P}(2) = \frac{2}{21}, \ldots, \mathbb{P}(6) = \frac{6}{21}$

4. Two fair dice are tossed and the number on each die is recorded, e.g. (3,2) indicates the first die had a 3 and the second die had a 2.
   (a) Write down the sample space (Hint: there are 36 outcomes.).
   Assume all outcomes in the sample space are equally likely for the next problems
   (b) What is the probability that the sum of the two numbers is 7?
   (c) What is the probability that the sum of the two numbers is 7 or 11?
   (d) What is the probability of getting an even on the first die or a total of 11?

   Answer:

(a) The sample space is $\{(i,j)|i = 1,\ldots,6, j = 1,\ldots,6\}$. That means Ω is

$$\begin{gathered}\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}.\end{gathered}$$

(b) The sum is 7 if we get the outcome (1,6), (2,5), (3,4), (4,3), (5,2), or (6,1). Thus there are 6 outcomes and the probability is $\frac{6}{36} = \frac{1}{6}$.

(c) The corresponding sums of each outcome is

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

There are 8 outcomes where the sum is either 7 or 11. The probability is 8/36

(d) Let A be the event that the first die rolls an even number, and B be the event that the sum of two rolls is 11. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = \frac{18}{36} + \frac{2}{36} - \frac{1}{36} = \frac{18+2-1}{36} = \frac{19}{36}$

5. Suppose that after 10 years of service, 35% of computers have problems with motherboards (MB), 30% have problems with hard drive (HD), and 20% have problems with both MB and HD.

(a) What is the probability that a 10-year old computer has a problem with MB or HD? **Answer:**

$$\begin{aligned}\mathbb{P}(MB \cup HD) &= \mathbb{P}(MB) + \mathbb{P}(HD) - \mathbb{P}(MB \cap HD)\\&= 0.35 + 0.3 - 0.20\\&= 0.45\end{aligned}$$

(b) What is the probability that a 10-year old computer still has a fully functioning MB and HD? **Answer:**

$$\begin{aligned}\mathbb{P}(\overline{MB} \cap \overline{HD}) &= \mathbb{P}(\overline{MB \cup HD})\\&= 1 - \mathbb{P}(MB \cup HD)\\&= 1 - 0.45\\&= 0.55\end{aligned}$$

6. * The probability that a visit to a physician's office results in neither lab work nor referral to a specialist is 50%. Also, suppose in visits to a physician's office, 30% are referred to specialists and 40% require lab work.

(a) Calculate the probability that a visit to a physician's office results in both lab work and referral to a specialist.

Answer: Let L = requires lab work and S = referred to specialist. We have $\mathbb{P}(L) = 0.4$, $\mathbb{P}(S) = 0.3$, and $\mathbb{P}(\overline{L} \cap \overline{S}) = 0.5$.

$$\begin{aligned}
\mathbb{P}(L \cap S) &= 1 - \mathbb{P}(\overline{L \cap S}) \\
&= 1 - \mathbb{P}(\overline{L} \cup \overline{S}) \\
&= 1 - [\mathbb{P}(\overline{L}) + \mathbb{P}(\overline{S}) - \mathbb{P}(\overline{L} \cap \overline{S})] \\
&= 1 - 0.6 - 0.7 + 0.5 \\
&= 0.2
\end{aligned}$$

Or you could draw a table as shown in class and get the answer much easier.

(b) Calculate the probability that a visit results in lab work or referral to a specialist.
Answer: $\mathbb{P}(L \cup S) = \mathbb{P}(L) + \mathbb{P}(S) - \mathbb{P}(L \cap S) = 0.4 + 0.3 - 0.2 = 0.5$

(c) Calculate the probability that a visit results in only one of the actions (lab work and no referral *or* no lab work and referral).
Answer: We want $\mathbb{P}((L \cap \overline{S}) \cup (\overline{L} \cap S)) = \mathbb{P}(L \cap \overline{S}) + \mathbb{P}(\overline{L} \cap S)$.

$\mathbb{P}(L \cap \overline{S}) = \mathbb{P}(L) - \mathbb{P}(L \cap S)$

$\mathbb{P}(\overline{L} \cap S) = \mathbb{P}(S) - \mathbb{P}(L \cap S)$

So we get $\mathbb{P}(L) + \mathbb{P}(S) - 2\mathbb{P}(L \cap S) = 0.4 + 0.3 - 2(0.2) = 0.3$

Or you could draw a table as shown in class, or use a Venn Diagram and get the answer much easier.