4.2 Reduction of Order

Recall

$$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = 0 \quad (*)$$

has general solution : $y = c_1 y_1 + c_2 y_2$

where y_1, y_2 is a fundamental set, that is, y_1 and y_2 are l.i. solutions of the homogeneous equation $(*)$.

Sometimes we might have one solution but need to find a second one. Say we know y_1, we'd like to find y_2. Note, since we need y_1 and y_2 to be l.i. then

$$\frac{y_2(x)}{y_1(x)} \neq C, \quad \text{that is,} \quad \frac{y_2(x)}{y_1(x)} = u(x)$$

So $y_2 = u(x)\, y_1(x)$, which we can plug into our original equation $(*)$ and try to find a function $u(x)$ which makes equation $(*)$ true.

Before deducing a formula to find y_2, let's work out the following example.

Example: (#1 pg. 132) Note that $y_1 = e^x$ is a solution to the equation

$$y'' - y = 0.$$

Find another solution y_2 linearly independent to y_1.

Assume $y_2 = u(x) y_1(x) = u e^x$; $y_2' = u e^x + u' e^x$; $y_2'' = u e^x + u' e^x + u' e^x + u'' e^x$

plug in: $(u e^x + 2u' e^x + u'' e^x) - u e^x = 0 \iff (u'' + 2u') e^x = 0 \quad (e^x \neq 0 \text{ for all } x)$

$\iff u'' + 2u' = 0$. Let $w = u'$ so $w' = u''$ & substitute

to get $w' + 2w = 0 \iff \dfrac{dw}{dx} = -2w$ (Separable!)

Separate : $\displaystyle\int \frac{1}{w}\, dw = \int -2\, dx$

$\Rightarrow \ln|w| = -2x + C_1 \Rightarrow w = c e^{-2x}$

Back sub $w = u'$:

$$u' = ce^{-2x} \Rightarrow u = \int ce^{-2x}\,dx = -\frac{c}{2}e^{-2x} + k$$

Since we only need one such function $u(x)$ that works, we can pick the constants $c = -2$ and $k = 0$ so that $u(x) = e^{-2x}$

Then $y_2(x) = u(x)\,y_1(x)$ is $y_2(x) = e^{-2x}e^{x} = e^{-x}$.

Indeed we can verify $w(e^{x}, e^{-x}) = \det\begin{bmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{bmatrix} = -e^{x}e^{-x} - e^{x}e^{-x} = -2 \neq 0$ for all x.

And thus the general solution is

$$y = c_1 e^{x} + c_2 e^{-x}.$$

Now we'll deduce a formula for $y_2(x)$, we will work with the standard form of $(*)$:

$$y'' + P\,y' + Q\,y = 0$$

If we know y_1 is a solution, find an expression for the other l.i. solution $y_2 = y_1\,u$.

Need y_2' & y_2'' : $y_2' = y_1'u + y_1u'$; $y_2'' = y_1''u + y_1'u' + y_1'u' + y_1u''$, plug in:

$$(y_1''u + 2y_1'u' + y_1u'') + P(y_1'u + y_1u') + Q(y_1u) = 0$$

$$u\underbrace{[y_1'' + Py_1' + Qy_1]}_{=0 \text{ because } y_1 \text{ is a solution to } *} + \underbrace{u''y_1 + u'(2y_1' + Py_1) = 0}_{**}$$

Let $w = u'$ (so $w' = u''$) and substitute into $**$ to obtain a 1^{st} order DE.

(Solve for ω in:) $\omega' y_1 + \omega(2y_1' + Py_1) = 0 \Leftrightarrow \frac{d\omega}{dx} = -\left(\frac{2y_1'}{y_1} + P\right)\omega$

Separate & Integrate $\int \frac{1}{\omega}\, d\omega = \int -\left(\frac{2y_1'}{y_1} + P\right) dx$

$$\ln|\omega| = -2\ln|y_1| - \int P\,dx + C_1 \Rightarrow \omega = C e^{-2\ln|y_1|} e^{-\int P dx}$$

$$\omega = \frac{C e^{-\int P dx}}{y_1^2}$$ (Back sub of $\omega = u'$)

Pick $C = 1$ & $K = 0$

$$u' = \frac{ce^{-\int P dx}}{y_1^2} \Rightarrow u(x) = \int \frac{ce^{-\int P dx}}{y_1^2}\, dx + K$$

$$\Rightarrow u(x) = \int \frac{e^{-\int P dx}}{y_1^2}\, dx$$

$$\therefore \boxed{y_2 = y_1 \int \frac{e^{-\int P dx}}{y_1^2}\, dx}$$

Example

Find a second linearly independent solution to

$$(1 - x^2)\, y'' + 2xy' = 0\,,$$

if we know $y_1 = 1$ is a solution.

In standard form the equation is: $y'' + \frac{2x}{1-x^2} y' = 0$.

$$-\int P\,dx = -\int \frac{2x}{1-x^2}\, dx = \int \frac{1}{u}\, du = \ln|u| = \ln|1-x^2|$$

$u = 1-x^2$
$du = -2x\,dx$

$$\Rightarrow e^{-\int P dx} = e^{\ln|1-x^2|} = |1-x^2| = 1-x^2 \Rightarrow u(x) = \int \frac{1-x^2}{1^2}\, dx = x - \frac{x^3}{3}$$

$$\therefore y_2 = x - \frac{x^3}{3}$$

(The general sol is: $y = C_1 + C_2\left(x - \frac{x^3}{3}\right)$).

Ex. 2 Find a second l.i. solution to

$$x^2y'' - 3xy' + 4y = 0,$$ if we know that

$y_1 = x^2$ is a sol. in $(0, \infty)$.

In Standard form: $$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

$$e^{-\int P dx} = e^{+\int 3/x\, dx} = e^{3\ln|x|} = |x|^3 = x^3$$

$$\Rightarrow \quad y_2 = y_1 \int \frac{e^{-\int P dx}}{y_1^2} dx = x^2 \int \frac{x^3}{x^4} dx = x^2 \int \frac{1}{x} dx = x^2 \ln|x|$$

We can verify:

$$W(y_1, y_2) = \det \begin{bmatrix} x^2 & x^2 \ln|x| \\ 2x & 2x\ln x + x \end{bmatrix} = 2x^3 \ln x + x^3 - 2x^3 \ln x = x^3 \neq 0$$

for all x in $(0, \infty)$ ✓

And the general sol. is

$$y = c_1 x^2 + c_2 x^2 \ln x$$