Review for Final Exam Part 1

1st Order Differential Equations

- Separable Equations
  - Definition. $\dfrac{dy}{dx} = f(x,y)$ is separable when we can write: $f(x,y) = g(x)\,h(y)$
  - Method. 1st: Separate Variables
    2nd: Integrate both Sides
    3rd: Solve for y

    $$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

  - Example. Solve $y^{-1}dy + (ye^{\cos x}\sin x)dx = 0$ $\quad y^{-1}\,dy = -y\,e^{\cos x}\sin x\,dx$

Separate: Integrate

$$\int y^{-2}\,dy = -\int e^{\cos x}\sin x\,dx$$

$$-y^{-1} = e^{\cos x} + C$$

$$\boxed{y = -\frac{1}{e^{\cos x} + C}}$$

← General Solution.

- Linear Equations
  - Definition/Form: A 1st order DE is linear if it is of the form:
    $a_1(x)\dfrac{dy}{dx} + a_0(x)\,y = g(x)$ and the standard form is: $\dfrac{dy}{dx} + P(x)\,y = Q(x)$.
  - Method. 1st: Find the integrating factor $\mu(x) = e^{\int P\,dx}$
    2nd: Multiply both sides of the equation by $\mu(x)$ so the equation becomes: $(\mu(x)\,y(x))' = \mu(x)\,Q(x)$
    3rd: Integrate both sides and solve for y $\quad y(x) = \dfrac{1}{\mu}\left(\int \mu Q\,dx + C\right)$
  - Example. Solve the IVP: $xy' + 3y = \dfrac{e^{3x}}{x}$; $y(1) = e^3$

In standard form is: $y' + \frac{3}{x}y = \frac{e^{3x}}{x^2}$; $\mu(x) = e^{\int \frac{3}{x}dx} = e^{3\ln|x|} = e^{\ln|x|^3} = |x|^3 = x^3$

New Equation: $(x^3 y)' = xe^{3x} \Rightarrow x^3 y = \int xe^{3x}\,dx = \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + C$

Solve for y: $y(x) = \dfrac{e^{3x}}{3x^2} - \dfrac{e^{3x}}{9x^3} + \dfrac{C}{x^3}$ $\quad$ (solve for c) $C = \frac{7}{9}e^3$

Solve IVP (plug initial cond.): $y(1) = \dfrac{e^3}{3} - \dfrac{e^3}{9} + C = e^3$

$$\text{Sol: } \boxed{y = \frac{e^{3x}}{3x^2} - \frac{e^{3x}}{9x^3} + \frac{7e^3}{9x^3}}$$

- Exact Equations
  - Definition/Form. The DE: $M(x,y)dx + N(x,y)dy = 0$ is exact iff there exists a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = M \text{ and } \frac{\partial f}{\partial y} = N \qquad \underline{\text{Test:}} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{Exact.}$$

  - Method. 1st: find f(x,y) 2nd: Solution is $f(x,y) = c$
  - Example. $\dfrac{dy}{dx} = \dfrac{\cos(xy) - xy\sin(xy) - y^2e^{-x}}{x^2\sin(xy) - 2ye^{-x}}$

$$[x^2\sin(xy) - 2ye^{-x}]dy = [\cos(xy) - xy\sin(xy) - y^2e^{-x}]dx$$

$$\underbrace{[-\cos(xy) + xy\sin(xy) + y^2e^{-x}]}_{M}dx + \underbrace{[x^2\sin(xy) - 2ye^{-x}]}_{N}dy = 0$$

$$\frac{\partial M}{\partial y} = x\sin(xy) + x\sin(xy) + x^2y\cos(xy) + 2ye^{-x}$$

$$\frac{\partial N}{\partial x} = 2x\sin(xy) + x^2y\cos(xy) + 2ye^{-x}$$

> Equal, thus exact.

Next find $f(x,y)$ such $\frac{\partial f}{\partial x} = M$ (1) and $\frac{\partial f}{\partial y} = N$ (2)

From (2): $f(x,y) = \int N dy = \int x^2\sin(xy) - 2ye^{-x} dy = -x\cos(xy) - y^2e^{-x} + g(x)$

And (1) $\Rightarrow \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(-x\cos(xy) - y^2e^{-x} + g(x)\right) = -\cos(xy) + xy\sin(xy) + y^2e^{-x} + g'(x) = M$ ↖ must

$\Rightarrow g'(x) = 0 \Rightarrow g(x) = C_1 = 0$

(↖ we pick $C_1 = 0$)

Then update $f(x,y) = -x\cos(xy) - y^2e^{-x}$

Finally set $f = c$

General Solution: $\boxed{-x\cos xy - y^2e^{-x} = c}$

- Solutions by Substitutions (3 kinds)

Kind	Form	Substitutions	New Equation
Homogeneous	$\frac{dy}{dx} = G(\frac{y}{x}) = f(x,y)$ $f(tx,ty) = t^{\alpha} f(x,y)$	$y = ux$ $\frac{dy}{dx} = u + x\frac{du}{dx}$	separable
Bernoulli	$\frac{dy}{dx} + P(x)y = f(x)y^n$ $\boxed{y^{-n}\frac{dy}{dx}} + P\boxed{y^{1-n}} = f(x)$	$u = y^{1-n}$ $\frac{du}{dx} = (1-n)\boxed{y^{-n}\frac{dy}{dx}}$	linear
Of the form $\frac{dy}{dx} = f(Ax + By + C)$ $B \neq 0$		$u = Ax + By + C$ $\frac{du}{dx} = A + B\frac{dy}{dx}$	separable

- Example. $\sqrt{x}\, y' = \frac{y}{\sqrt{x}} + \sqrt{y-2x} \iff y' = \frac{y}{x} + \sqrt{\frac{y}{x} - 2}$

$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

Homogeneous, Then $u = \frac{y}{x}$ or $y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx} (= y')$

Substitute: $\cancel{u} + x\frac{du}{dx} = \cancel{u} + \sqrt{u-2} \iff x\frac{du}{dx} = \sqrt{u-2}$ (separable)

Separate: $\int \frac{1}{\sqrt{u-2}}\, du = \int \frac{1}{x}\, dx \Rightarrow 2\sqrt{u-2} = \ln|x| + C$

Solve for u: $u = \left(\frac{\ln|x| + C}{2}\right)^2 + 2$ & plug back in $(u = \frac{y}{x})$

$\frac{y}{x} = \left(\frac{\ln|x| + C}{2}\right)^2 + 2 \Rightarrow \boxed{y(x) = x\left(\frac{\ln|x| + C}{2}\right)^2 + 2x}$

General Solution.

- Applications
  - Population Dynamics $\dfrac{dP}{dt} = K P(t) \Rightarrow P(t) = P_0 e^{Kt}$
    $K > 0$
  - Radioactive Decay $\dfrac{dA}{dt} = -KA(t) \Rightarrow A(t) = A_0 e^{-Kt}$
    $K > 0$
  - Mixing Problems $\dfrac{dA}{dt} = (\text{Input rate}) - (\text{output rate})$
  - Newton's Law of Cooling $\dfrac{dT}{dt} = -K(T - T_m)$