Recitation 10 Solutions

- Here is a set of additional problems. They range from being very easy to very tough. The best way to learn the material in 310 is to solve problems on your own.
- Feel free to ask (and answer) questions about this problem set on Piazza.
- This is an **optional** problem set; do not turn this in for grading.
- While you don't have to turn this in, be warned that this material **can** appear in a quiz or exam.

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1. Recall that the complete graph over n nodes is defined as the graph in which every pair of nodes are connected via an undirected edge.

   a. Find a recurrence relation for the number of edges, $e(n)$, in this graph.

   Solution

   $e(n) = e(n-1) + (n-1)$ and $e(1) = 0$.

   b. Previously, we used the first degree theorem to find a closed form expression for $e(n)$. This time, prove it via induction using your answer from part a.

   Solution

   Let $P(n)$ is true if the complete graph with n nodes have $n(n-1)/2$ nodes.

   Base case: For $n = 1$, $1(1-1)/2 = 0$.

   Induction Step: Suppose $P(k)$ is true for arbitrary k. Then, for some $k \in \mathbb{N}$, we have the following:

   $e(k) = k(k-1)/2$.

   From induction hypothesis, we want to show that $P(k+1)$ is true. Using the recurrence relation found in part a) as a fact,

$$\begin{aligned} e(k+1) &= e(k) + k \\ &= \frac{k(k-1)}{2} + k \ (\because \mathit{Induction\ Hypothesis}) \\ &= \frac{k(k+1)}{2} \end{aligned}$$

   Via induction, the $P(n)$ is true for all positive integers n.

2. A *Koch snowflake* is created by starting with an equilateral triangle with sides one unit in length. Then, on each side of the triangle, a new equilateral triangle is created on the middle third of that side. This process is repeated continuously, as shown in Figure 1 below.

Prove that the number of sides (colored in black) for the n^{th} Koch snowflake is given by $3 \cdot 4^n$.

Solution

Let $P(n)$ is true if the number of sides of n^{th} Koch snowflake is $e(n) = 3 \cdot 4^n$.

Base case: For $n = 0$, $3 \cdot 4^0 = 3$.

Induction Step: Suppose $P(k)$ is true for arbitrary k. Then, for some nonnegative k, $e(k) = 3 \cdot 4^k$.

From induction hypothesis, we want to show that $P(k+1)$ is true. Using the fact that each side is multiplied by 4 on the next step of Koch snowflake,

$$\begin{aligned} e(k+1) &= 4 \cdot e(k) \\ &= 4 \cdot 3 \cdot 4^k \ (\because Induction\ Hypothesis) \\ &= 3 \cdot 4^{k+1} \end{aligned}$$

Via induction, the $P(n)$ is true for all nonnegative integers n.

3. Let $f : \mathbb{N} \to \mathbb{N}$ such that $f(0) = 1$, $f(1) = 2$, and $f(a+b) = f(a)f(b)$ for all $a, b \in \mathbb{N}$. Prove via induction that
$$f(n) = 2^n.$$

Solution

Base cases: For this one, we are given two base cases, so we need to check both of them. For $n = 0$, $1 = f(0) = 2^0$. Similarly, for $n = 1$, $2 = f(1) = 2^1$.

Induction Hypothesis: For some arbitrary $k \in \mathbb{N}$, $f(k) = 2^k$.

Induction Step: Suppose the induction hypothesis is true. Then, for some $k \in \mathbb{N}$, we have the following:

$f(k) = 2^k$.

The key here is to use the given information about the function f. Note that we can apply the property that $f(a + b) = f(a)f(b)$ with $a = k$ and $b = 1$. This gives us the following:

$f(k + 1) = f(k)f(1)$.

This is useful because we know the values of both $f(k)$ and $f(1)$ based on the induction hypothesis and the base cases, respectively. Thus, we obtain the final result:

$f(k + 1) = f(k)f(1) = (2^k)(2) = 2^{k+1}$.

As this equation simply replaces k by $k + 1$ in the induction hypothesis, the proof is finished via induction.