Review for Final Exam Part 2

Higher Order DEs (Constant Coefficients)

Most General Problem: A non-homogeneous IVP.

$$ay'' + by' + cy = f(x)\ ; \quad y(x_0) = y_0,\ y'(x_0) = y_1$$

STEPS

1 Find y_c, the general solution of the associated homogeneous problem
(that is, if $f(x) = 0$. $y_c = c_1 y_1 + c_2 y_2$)

2 Find y_p, a particular solution.

- Undetermined Coefficients
- Variation of Parameters

⋆ Superposition Principle

3 Write the general solution: $y = y_c + y_p = c_1 y_1 + c_2 y_2 + y_p$

4 Plug in initial conditions into the general solution
to find the undetermined constants.

To find y_c: Use $ay'' + by' + cy = 0$

Auxiliary Equation → $am^2 + bm + c = 0$

Find its roots → m_1 and m_2.

Cases:

1. $m_1 \neq m_2$ (both real) $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$
2. $m_1 = m_2$ (real) $y_1 = e^{m_1 x}$ and $y_2 = x e^{m_1 x}$
3. $m_{1,2} = \alpha \pm i\beta$ (complex conjugates)

$y_1 = e^{\alpha x} \cos(\beta x) \quad y_2 = e^{\alpha x} \sin(\beta x)$

Examples

Find the general solution of the following:

a) $36y'' + 4y' = 0$

Auxiliary Equation: $36m^2 + 4m = 0$

$4m(9m+1) = 0 \Rightarrow m_1 = 0, \quad m_2 = -1/9$

$\Rightarrow$ General Sol: $y(x) = C_1 + C_2 e^{-x/9}$

b) $2y'' - 18y = 0$

Auxiliary Equation: $2m^2 - 18 = 0$

$2(m^2 - 9) = 0$

$2(m-3)(m+3) = 0 \Rightarrow m_1 = 3$ and $m_2 = -3$

Gen. Sol: $y(x) = C_1 e^{3x} + C_2 e^{-3x}$

c) $y'' - 4y' + 4y = 0$

Auxiliary Equ: $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0 \Rightarrow m_1 = m_2 = 2$

Gen. Sol: $y(x) = C_1 e^{2x} + C_2 x e^{2x}$

d) $y'' + 16y' + 68y = 0$

Aux. Eq: $m^2 + 16m + 68 = 0$

$m^2 + 16m + 64 = -4$

$(m+8)^2 = -4 \Rightarrow m = -8 \pm 2i$

$\therefore \alpha = -8$

$\beta = 2$

Gen. Sol: $y(x) = e^{-8x}(C_1 \cos 2x + C_2 \sin 2x)$

To find y_p: Suppose we have $ay'' + by' + cy = f_1(x) + f_2(x)$

I. Undetermined Coefficients (and superposition principle)

- "Guess/propose" the form of y_{p_i} according to the form of f_i.
- Examples. $f(x) = 2 \Rightarrow y_p = A$
  $f(x) = x^3 + x \Rightarrow y_p = A + Bx + Cx^2 + Dx^3$
  $f(x) = 3e^{7x} \Rightarrow y_p = Ae^{7x}$
  $f(x) = \sin(3x)$ or $f(x) = \cos(3x)$
  OR $f(x) = 7\sin 3x - \cos 3x$
  $\Rightarrow y_p = A\cos 3x + B\sin 3x$
- Remember sometimes is necessary to multiply by an extra x (or x^2).
- Determine the coefficients of each y_{p_i} separately by plugging it into

$$ay'' + by' + cy = f_i(x).$$

- By superposition principle $y_p = y_{p_1} + y_{p_2}$

Example

Find a particular solution for $y'' + y' - 6y = \overset{f_1}{3x} + \overset{f_2}{\cos(2x)} + \overset{f_3}{e^{2x}}$

Aux. Eqn: $m^2 + m - 6 = (m+3)(m-2) = 0 \Rightarrow m_1 = -3,\ m_2 = 2.$

y_{p_1}: Let $y_{p_1} = Ax + B$; $y_{p_1}' = A$; $y_{p_1}'' = 0$

$0 + A - 6(Ax + B) = 3x$ Need: $-6A = 3$ and $A - 6B = 0$

$-6Ax + (A - 6B) = 3x$ $\Rightarrow A = -\frac{1}{2}$ $B = \frac{A}{6} = -\frac{1}{12}$

$y_{p_1} = -\frac{1}{2}x - \frac{1}{12}$

y_{p_2}: Let $-6y_{p_2} = -6(A\cos 2x + B\sin 2x)$

$+\ y_{p_2}' = 2B\cos 2x - 2A\sin 2x$

$+\ y_{p_2}'' = -4A\cos 2x - 4B\sin 2x$

$(2B - 10A)\cos 2x + (-10B - 2A)\sin 2x = \cos 2x$

Need $\left.\begin{array}{l} 2B - 10A = 1 \\ -10B - 2A = 0 \end{array}\right\}$ Solve 2x2 System $A = -\frac{5}{52}$, $B = \frac{1}{52}$

$\therefore \quad y_{p_2} = -\frac{5}{52}\cos 2x + \frac{1}{52}\sin 2x$

y_{p_3} : Let $-6\, y_{p_3} = -6Axe^{2x}$

$+ \; y_{p_3}' = Ae^{2x} + 2Axe^{2x}$

$+ \; y_{p_3}'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}$

$5Ae^{2x} - \cancel{6Axe^{2x}} + \cancel{6Axe^{2x}} = e^{2x} \quad \Rightarrow 5A = 1 \Rightarrow A = 1/5$

$\therefore \quad y_{p_3} = \frac{1}{5}xe^{2x}$

Finally $y_p = \underbrace{-\frac{1}{2}x - \frac{1}{12}}_{y_{p_1}} + \underbrace{-\frac{5}{52}\cos 2x + \frac{1}{52}\sin 2x}_{y_{p_2}} + \underbrace{\frac{1}{5}xe^{2x}}_{y_{p_3}}$

II. Variation of Parameters

Recall we look for a particular solution of the form $y_p = u_1y_1 + u_2y_2$. Where y_1 and y_2 are the l.i. solutions from y_c and u_1, u_2 are:

$$u_1 = -\int \frac{f y_2}{W}\,dx \qquad u_2 = \int \frac{f y_1}{W}\,dx$$

and $W = \text{wronskian}(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$

This method is most useful when $f(x)$ is not a polynomial, a sine or a cosine or an exponential.

Example

Use variation of parameters to find a particular solution to

$$y'' - 2y' + y = \frac{e^x}{x^2}, \quad \text{and solve the IVP:} \quad y(1) = 5e, \; y'(1) = 6e.$$

Aux. Eqn: $m^2 - 2m + 1 = (m-1)^2 = 0 \Rightarrow m_1 = m_2 = 1$ (repeated)

$y_1 = e^x \quad y_2 = xe^x \qquad y_c = c_1 e^x + c_2 x e^x$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} + xe^{2x} - xe^{2x} = e^{2x}$$

$$u_1 = -\int \frac{\cancel{e^x}}{x^2} \cdot \frac{x\cancel{e^x}}{\cancel{e^{2x}}}\, dx = -\int \frac{1}{x}\, dx = -\ln|x|$$

$$u_2 = \int \frac{\cancel{e^x}}{x^2} \cdot \frac{\cancel{e^x}}{\cancel{e^{2x}}}\, dx = \int \frac{1}{x^2}\, dx = -\frac{1}{x}$$

$$y_p = u_1 y_1 + u_2 y_2 = -\ln|x|\, e^x - \frac{1}{x} x e^x = (-\ln|x| - 1)e^x$$

General Solution: $y(x) = c_1 e^x + c_2 x e^x + (-\ln|x| - 1)e^x$

$$y'(x) = c_1 e^x + c_2 e^x + c_2 x e^x + \left(-\frac{1}{x}\right)e^x + (-\ln|x| - 1)e^x$$

Plug initial Conditions:

$y(1) = (c_1 + c_2 - 1)e = 5e \iff c_1 + c_2 - 1 = 5 \iff c_1 + c_2 = 6$

$y'(1) = (c_1 + 2c_2 - 2)e = 6e \qquad c_1 + 2c_2 - 2 = 6 \qquad c_1 + 2c_2 = 8$

$\Rightarrow \quad c_1 = 4$ and $c_2 = 2$.

$$y(x) = [4 + 2x + (-\ln|x| - 1)]e^x$$

Application

Mass-Spring oscillator system: We can model the position of a mass m attached to a spring with constant k.

Without Damping: $y'' + \omega^2 y = 0$ $\omega = \sqrt{\frac{k}{m}}$

Sol $y(x) = A\sin(\omega t) + B\cos(\omega t)$; Amplitude $= \sqrt{A^2 + B^2}$

With Damping: $y'' + \lambda y' + \omega^2 y = 0$

According to the nature of the roots in this second type we have three cases:

underdamped, critically damped, overdamped.

$\downarrow$ complex conj. roots.

$\downarrow$ repeated real roots

$\downarrow$ distinct real roots