Linear Regression and Classification

Outline

I. Line fitting and gradient descent

II. Multivariable linear regression

III. Linear classifiers

IV. Logistic regression

* Figures are from the textbook site.

I. Linear Regression

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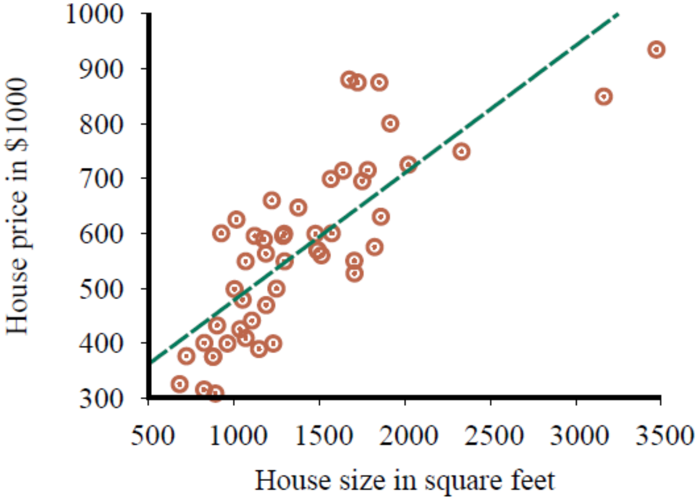

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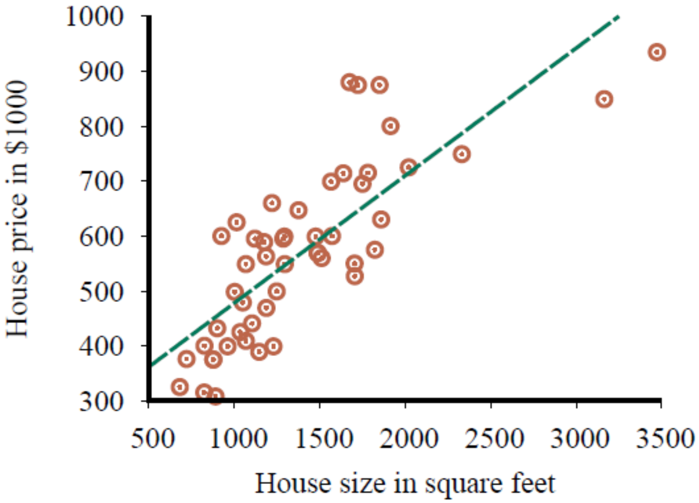

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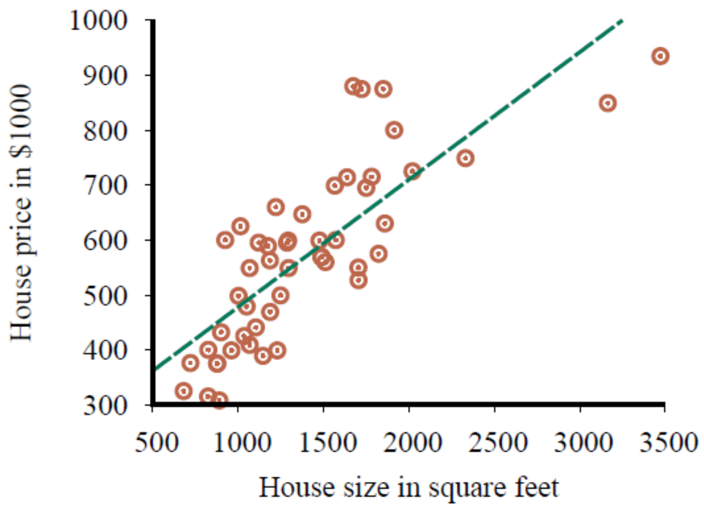

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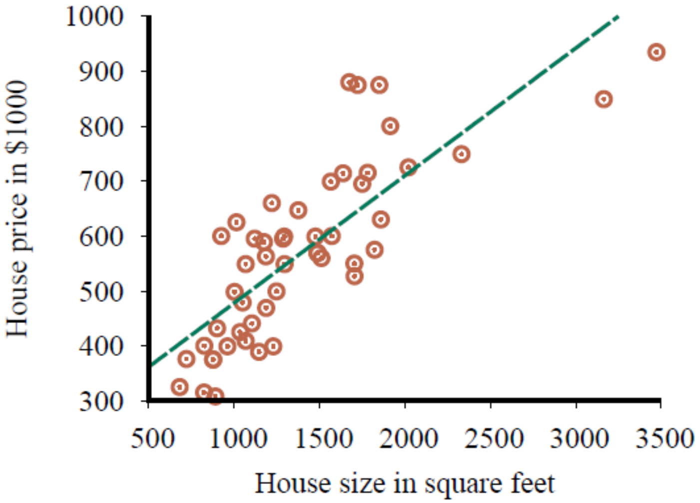

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Linear regression: Find the $h_{\boldsymbol{w}}$ that best fits the data.

Line Fitting

We find the weights (w_0, w_1) that minimizes the empirical loss.

Use the squared-error loss $L_2(y, h_{\boldsymbol{w}}) = (y - h_{\boldsymbol{w}})^2$, summed over all the points.

$$Loss(h_{\boldsymbol{w}}) = \sum_{j=1}^{N} L_2(y_j, h_{\boldsymbol{w}}(x_j))$$

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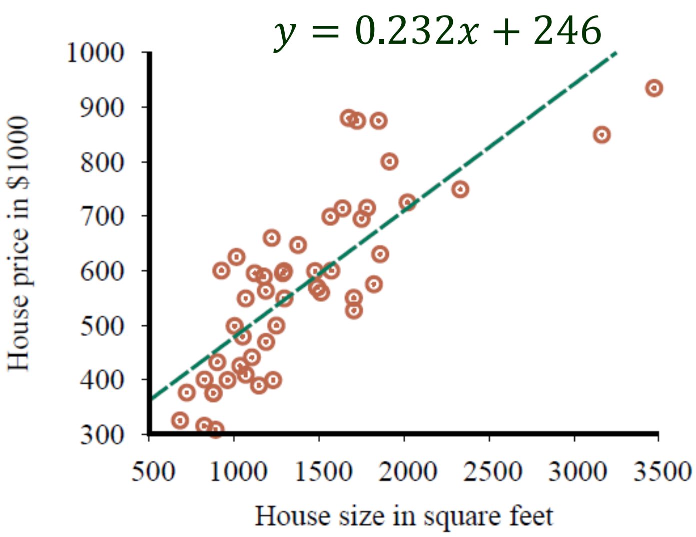
$$\boldsymbol{w}^* = \operatorname*{argmin}_{\boldsymbol{w}^*} \mathit{Loss}(h_{\boldsymbol{w}})$$

Vanishing of Partial Derivatives

At the minimizing $\boldsymbol{w}$, the gradient of $Loss(h_{\boldsymbol{w}})$ must vanish:

$$\nabla Loss(h_{\boldsymbol{w}}) = \left(\frac{\partial Loss}{\partial w_0}, \frac{\partial Loss}{\partial w_1}\right) = 0$$



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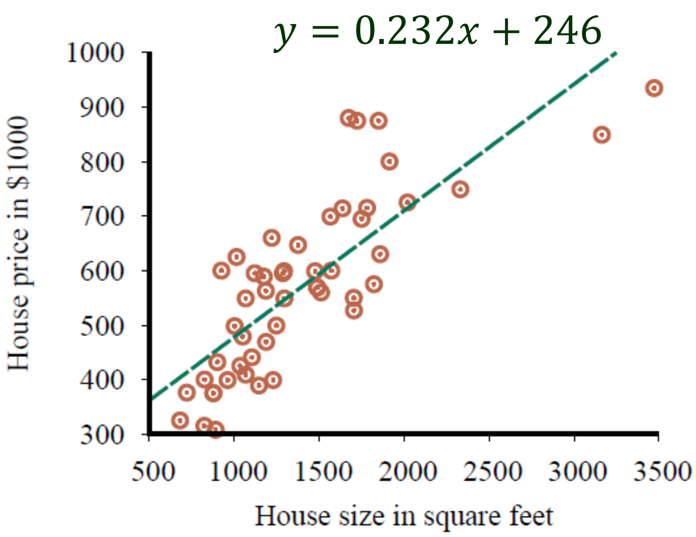

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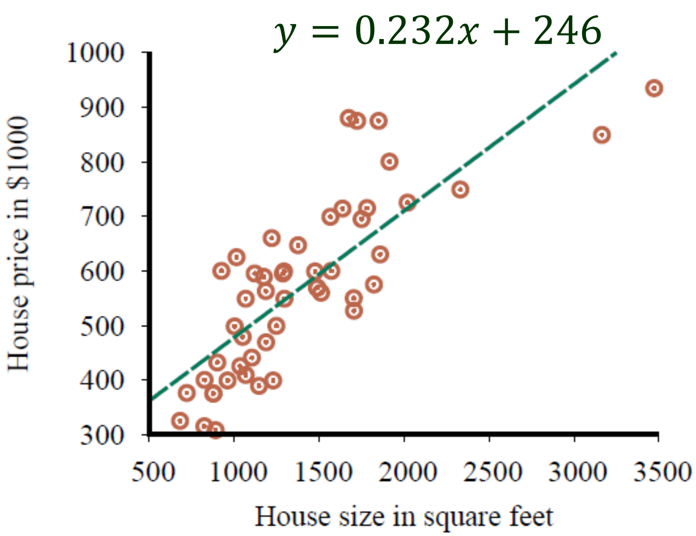

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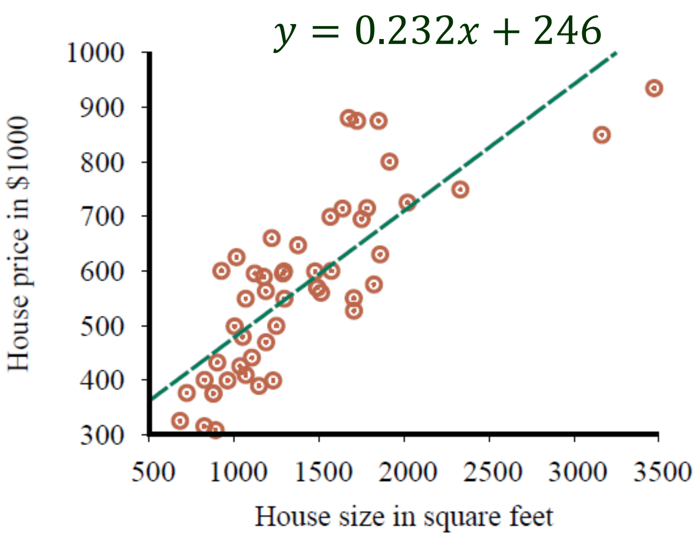

Note: the best-fit line does not minimize the sum of squares of distances of the data points to the line. It is inferior to a method used in computer vision for the purpose of extracting edges from an image. One reason is that the model cannot represent a vertical line.

Plot of the Loss Function

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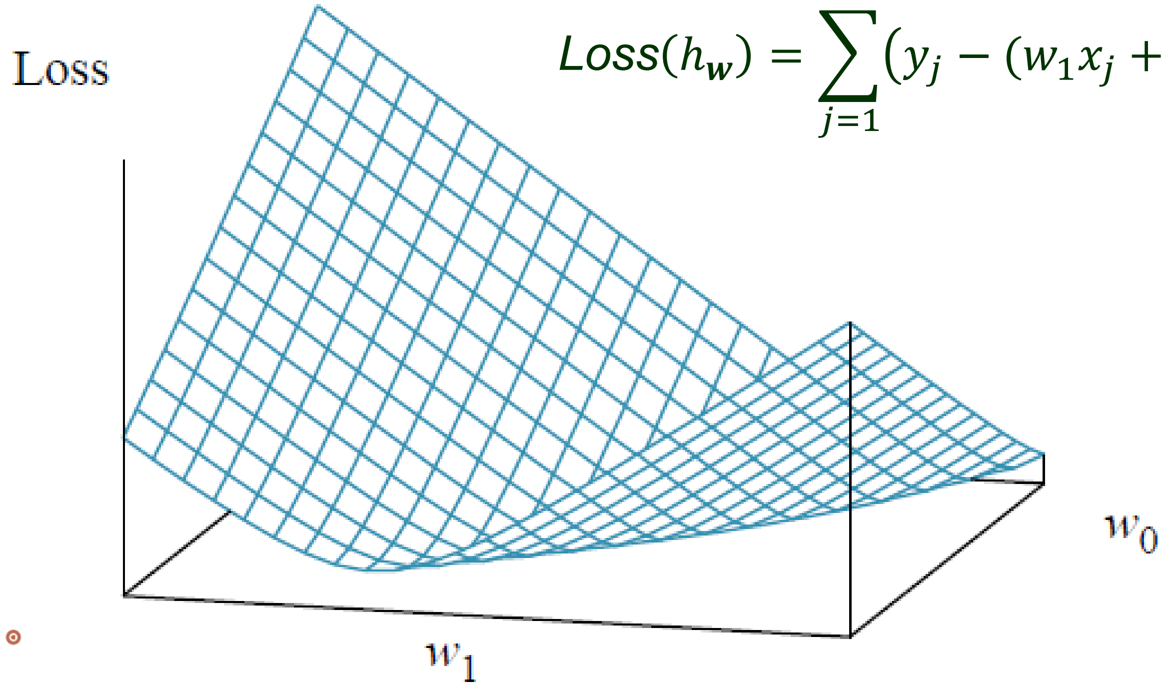

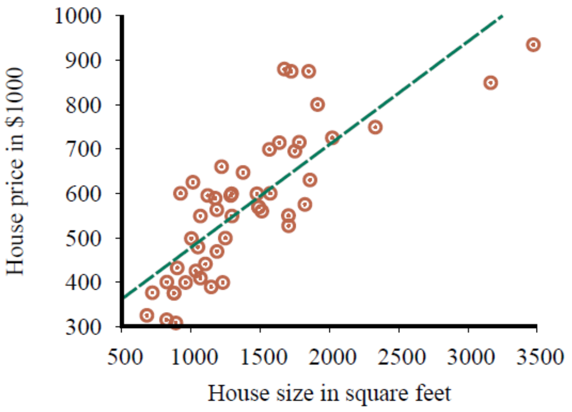

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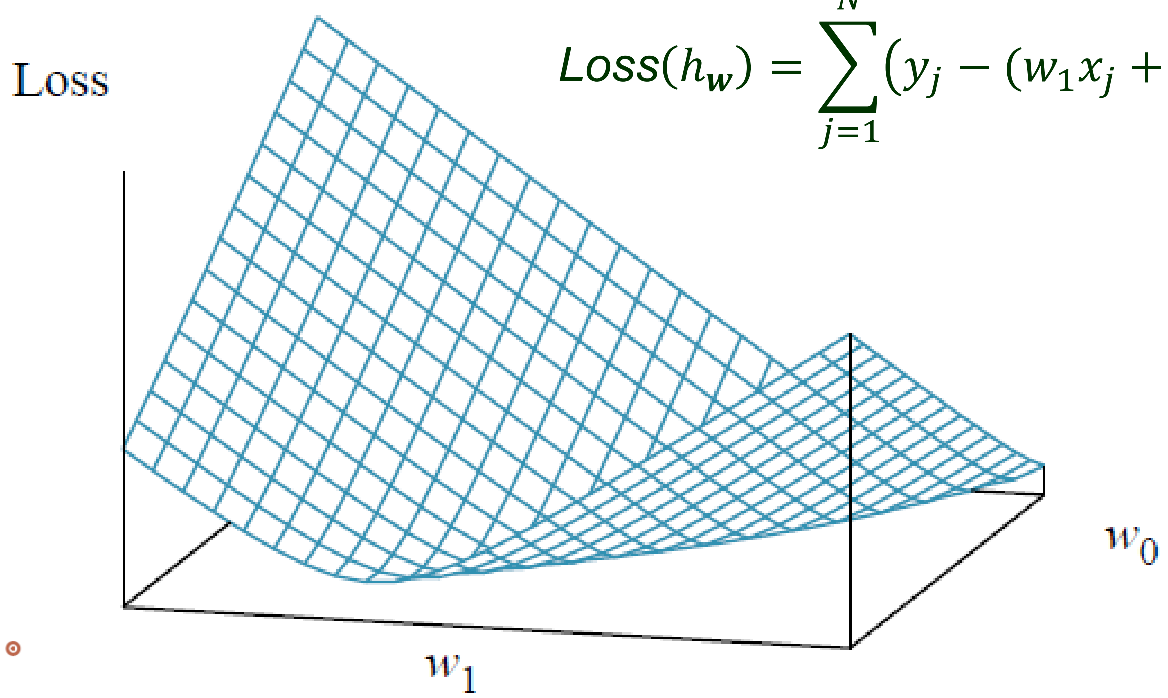

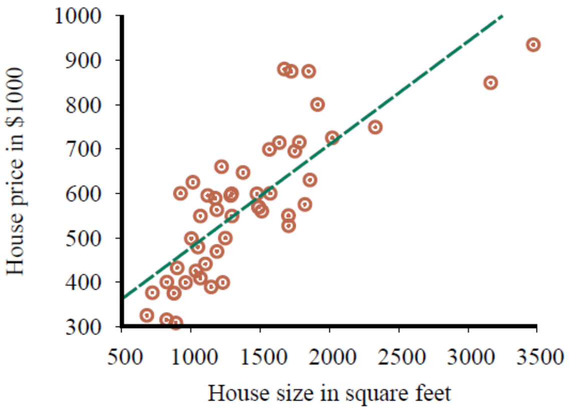

- Convex function with no local minima.

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** To see how gradient descent works, see Section 4 of http://web.cs.iastate.edu/~cs577/handouts/nonlinear-program.pdf.

Multivariable Linear Regression

- An example is represented by an n-vector $\boldsymbol{x}_j = (x_{j,1}, \ldots, x_{j,n})$.
- Hypothesis space:

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- Best weight vector:

$$\boldsymbol{w}^* = \underset{\boldsymbol{w}}{\operatorname{argmin}} \sum_j L_2(y_j, \boldsymbol{w} \cdot \boldsymbol{x}_j)$$

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- Write $\boldsymbol{w}$ as a column vector, i.e., $\boldsymbol{w} = (w_0, w_1, \dots, w_n)^{\boldsymbol{T}}$.
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- L_1 $(q = 1)$ regularization tends to produce a sparse model (in which many weights are set to zero) because it takes the $w_0, w_1, \ldots, w_n$ axes seriously.

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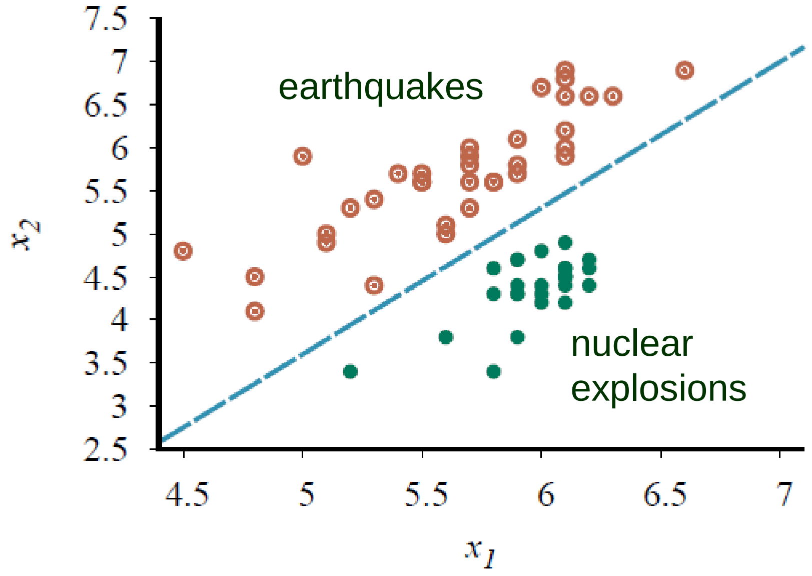
- ♦ L_1 $(q = 1)$ regularization tends to produce a sparse model (in which many weights are set to zero) because it takes the $w_0, w_1, \ldots, w_n$ axes seriously.

- ♠ L_2 $(q = 2)$ regularization takes the dimension axes arbitrarily.

III. Linear Classifiers



Seismic data for earthquakes and nuclear explosions: x_1 and x_2 respectively refer to body and surface wave magnitudes computed from the seismic signal.

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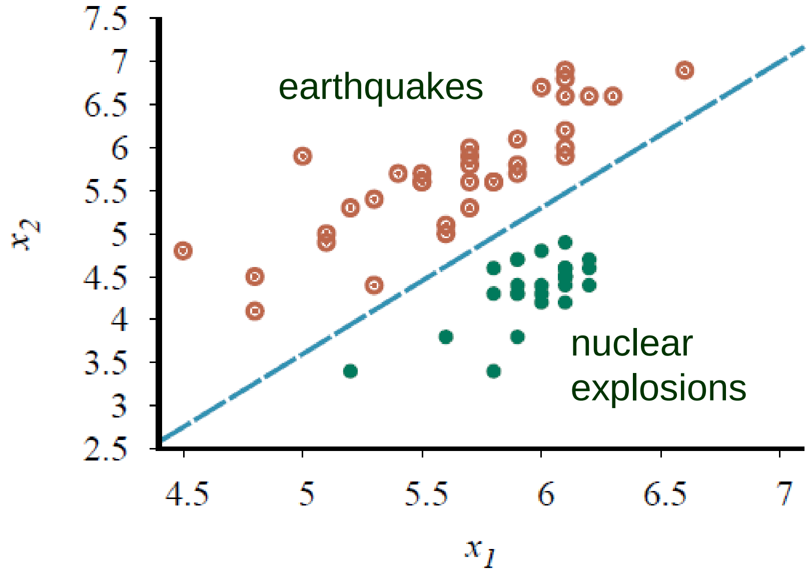

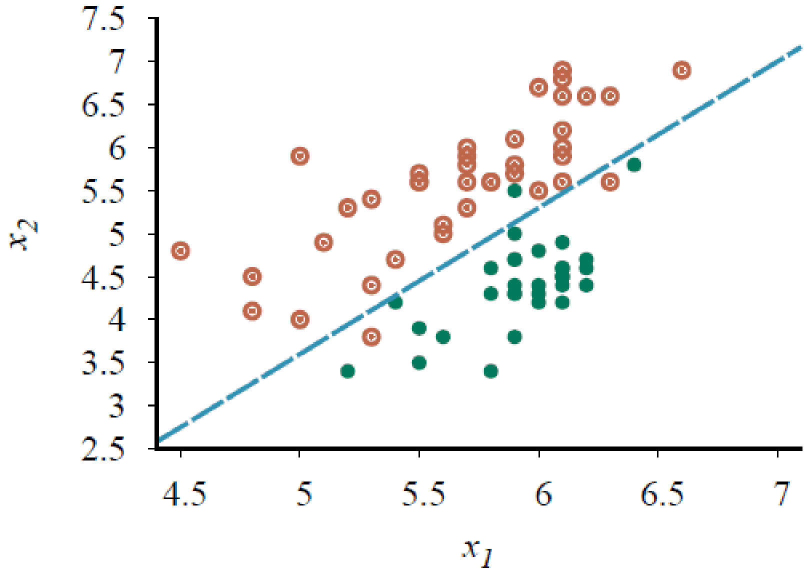

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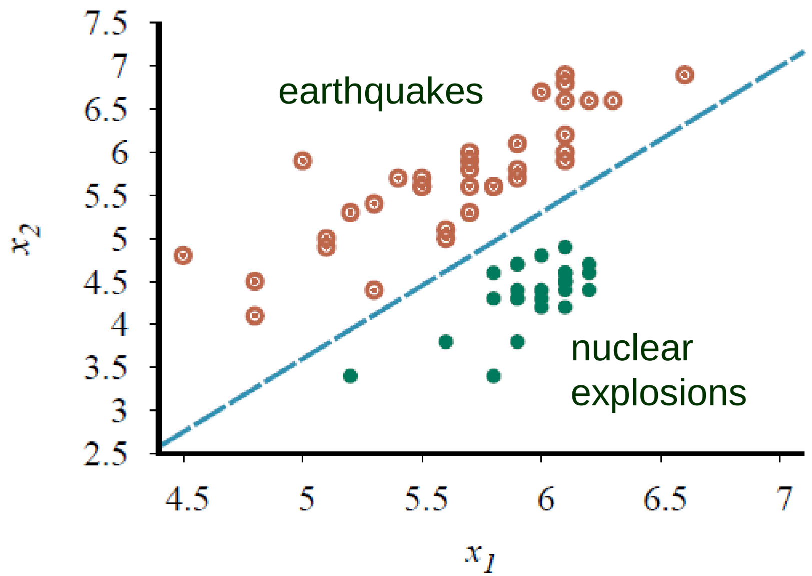

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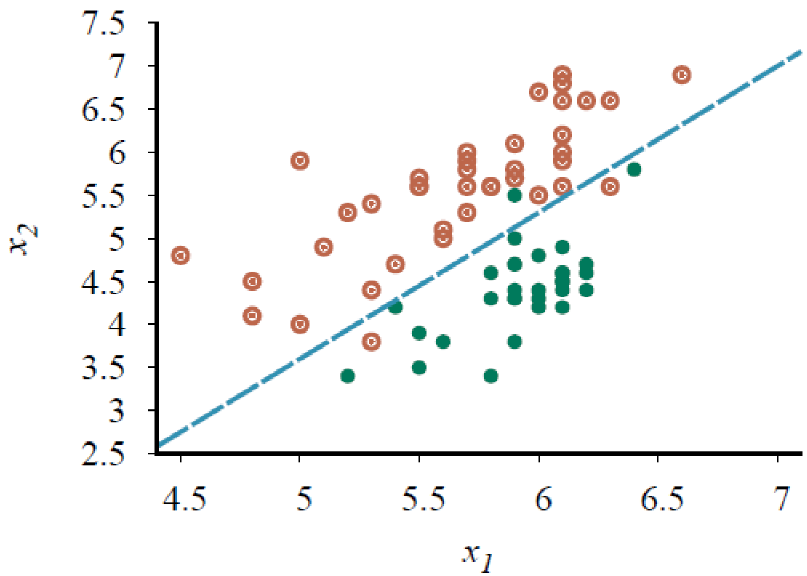


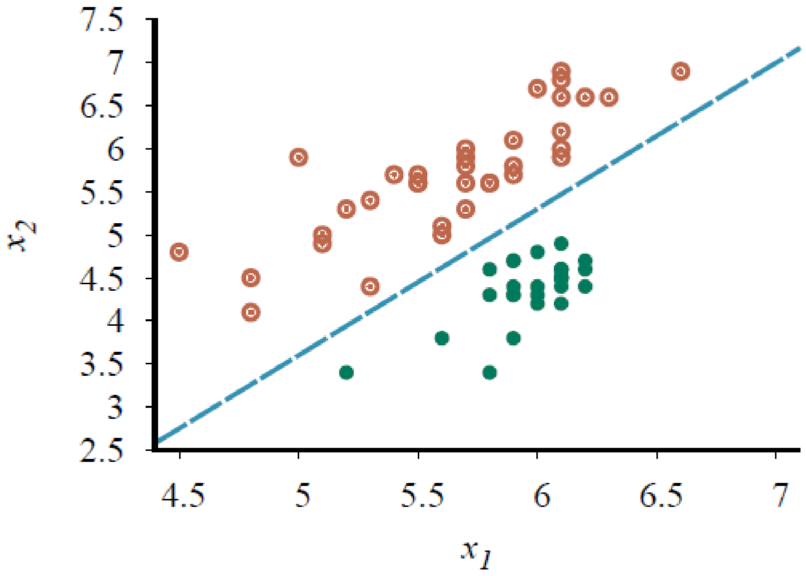
Same domain with more data points.

Task Learn a hypothesis that will take new (x_1, x_2) points and return 0 for earthquakes and 1 for explosions.

Linear Separator

A *decision boundary* is a line that separates two classes.

A *linear separator* is a linear decision boundary.



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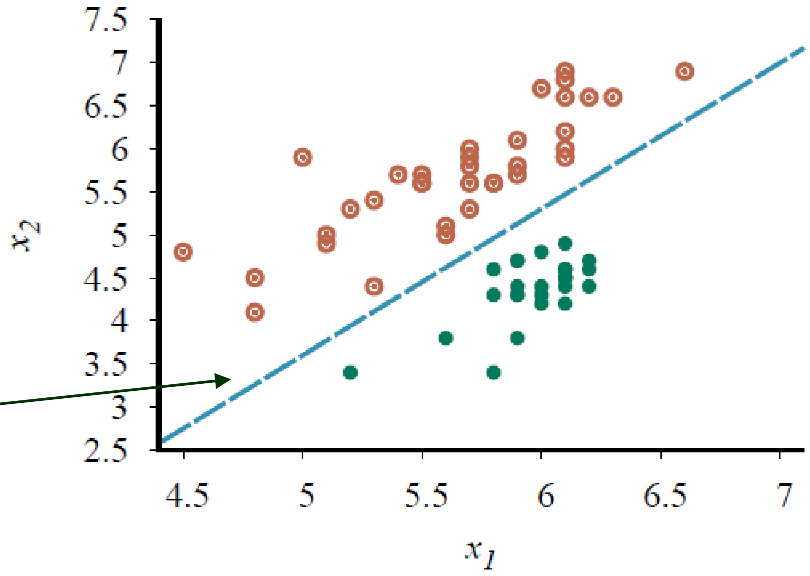
e.g., $-4.9 + 1.7x_1 - x_2 = 0$



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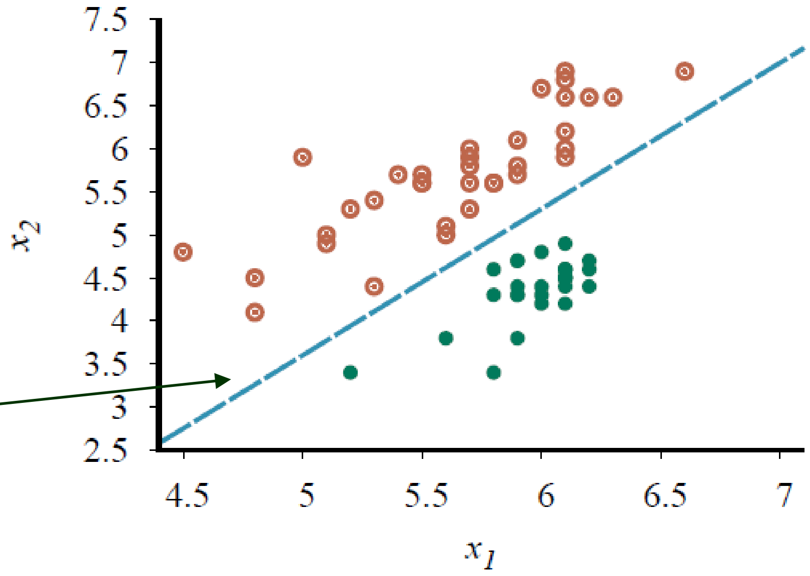

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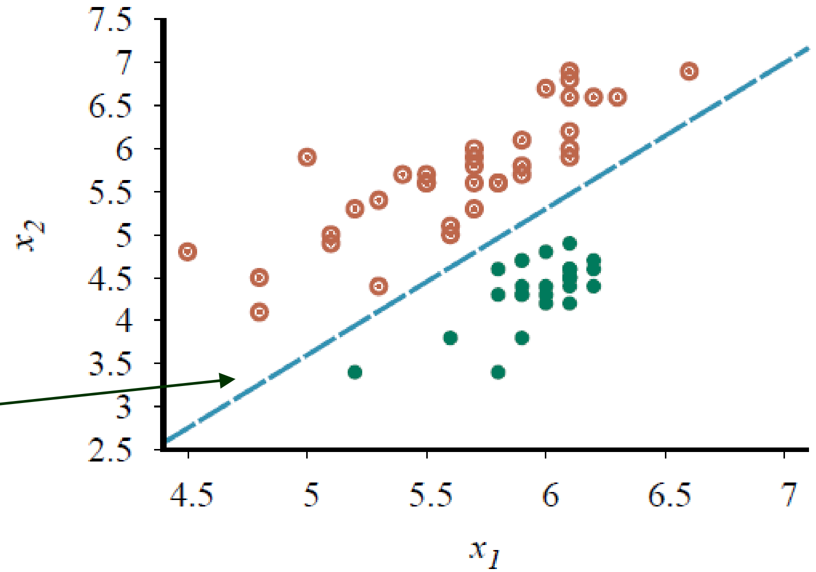

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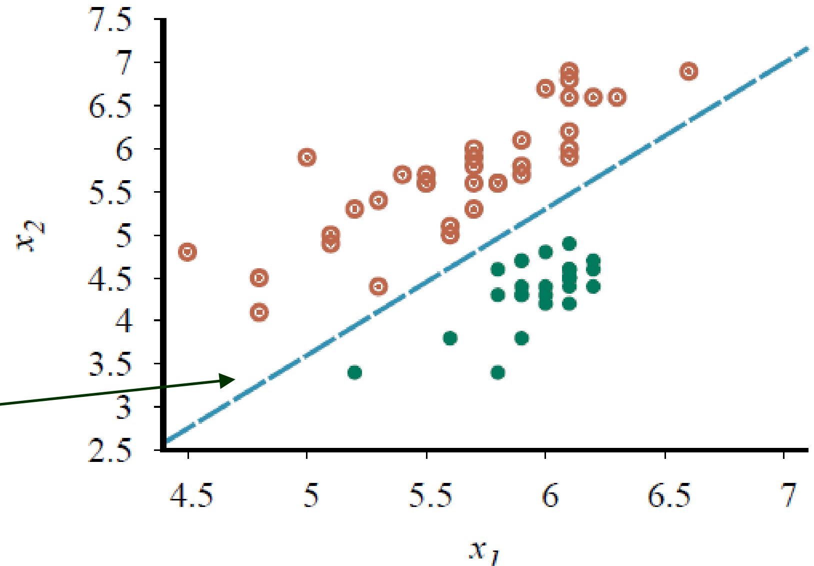

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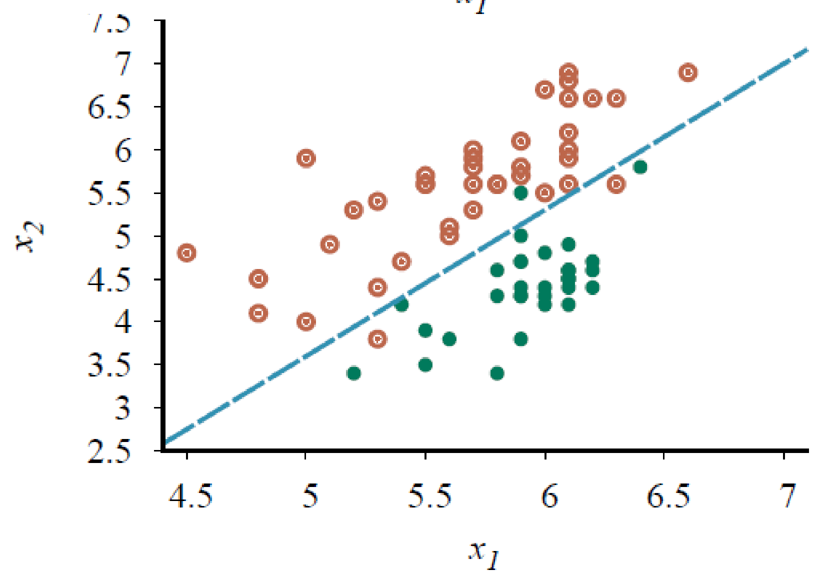

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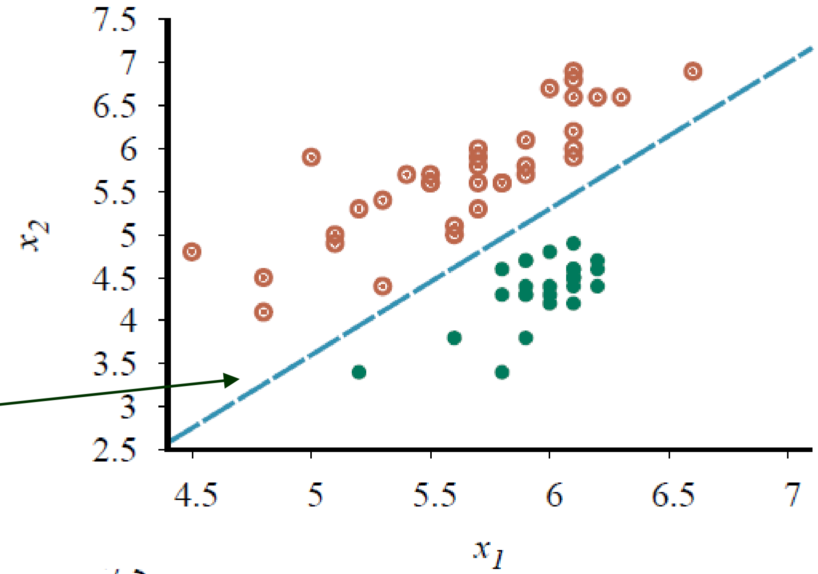

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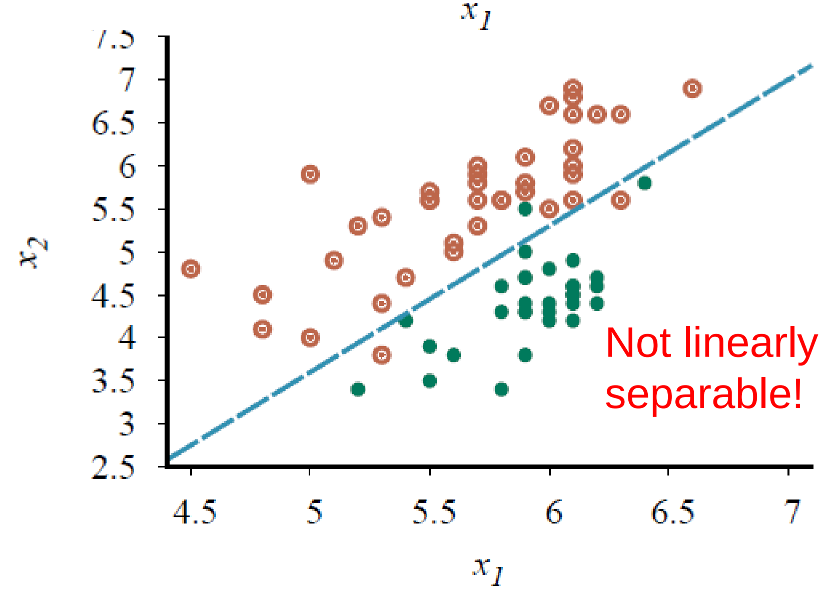

Learning Rule

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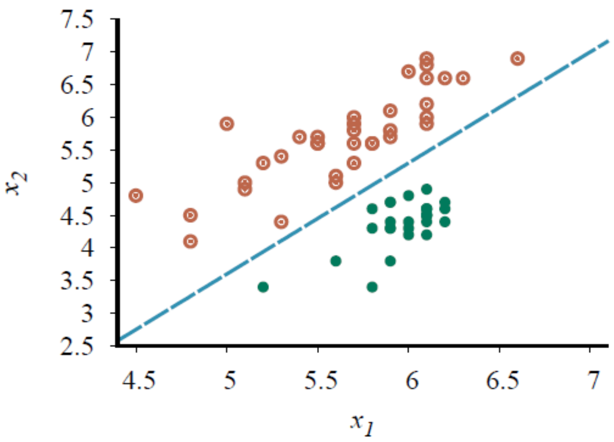
- $y_j = 0$ but $h_{\boldsymbol{w}}(\boldsymbol{x}_j) = 1$. w_i is decreased if $x_{j,i} > 0$ and increased if $x_{j,i} < 0$. In both situations, $\boldsymbol{xw}$ decreases with the intention to output 0.

Training Curves for Perceptron Learning

- The learning rule is applied one example at a time.
- A *training curve* measures the classifier performance on a fixed training set as learning proceeds one example at a time on the same set.



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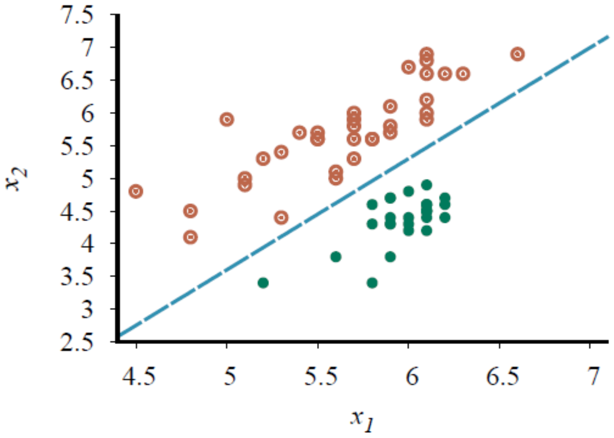

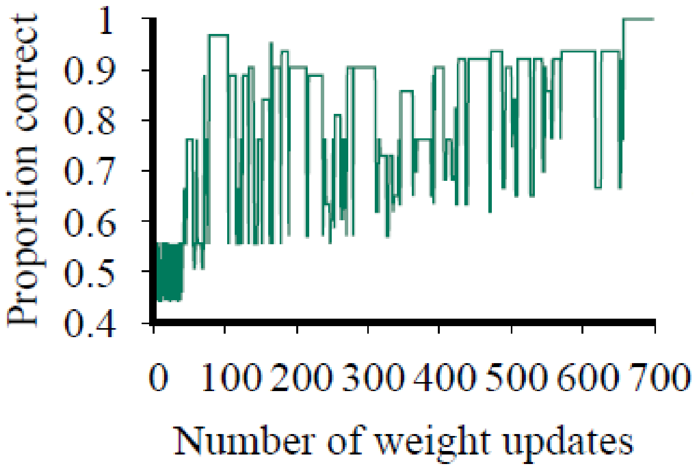

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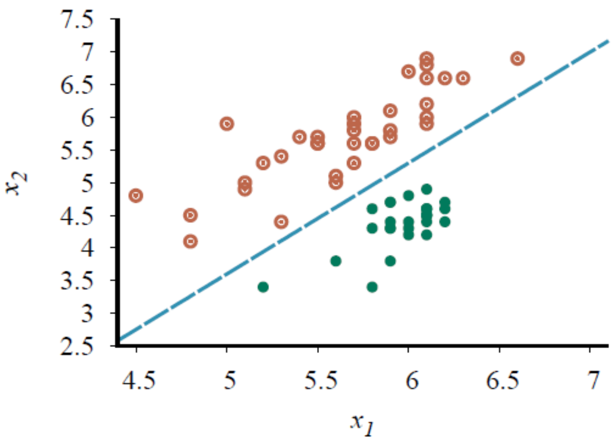

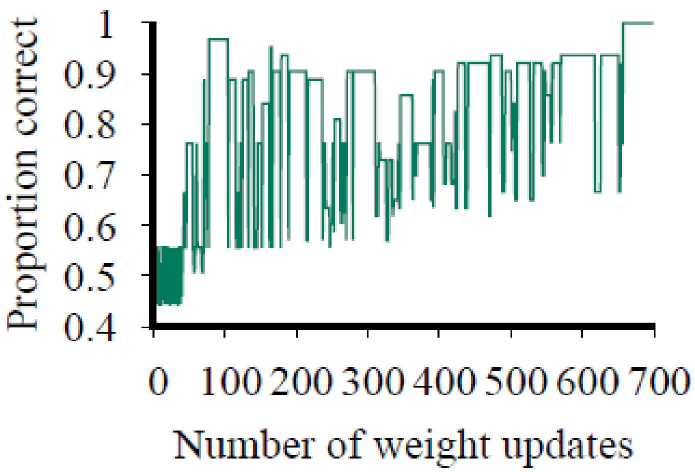

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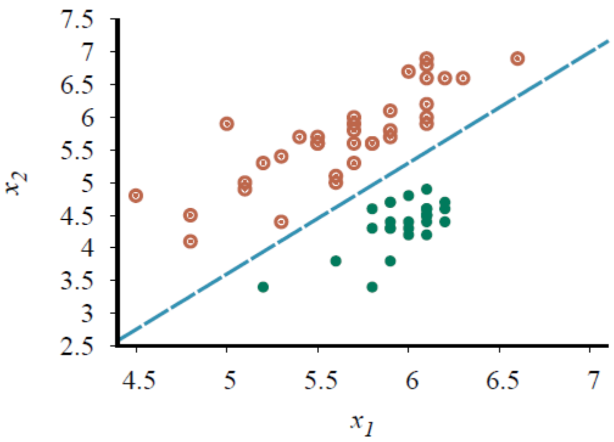
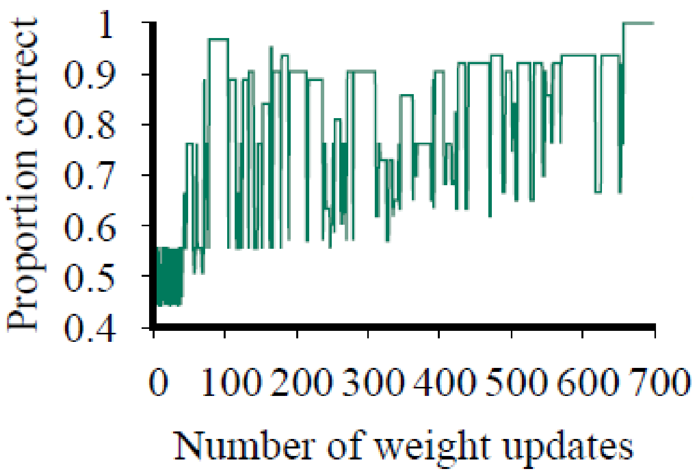
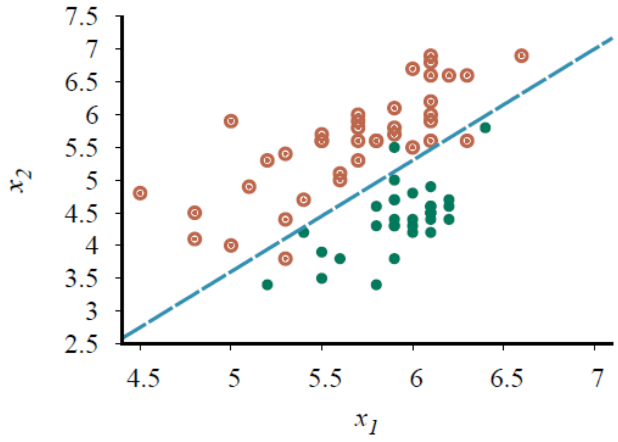


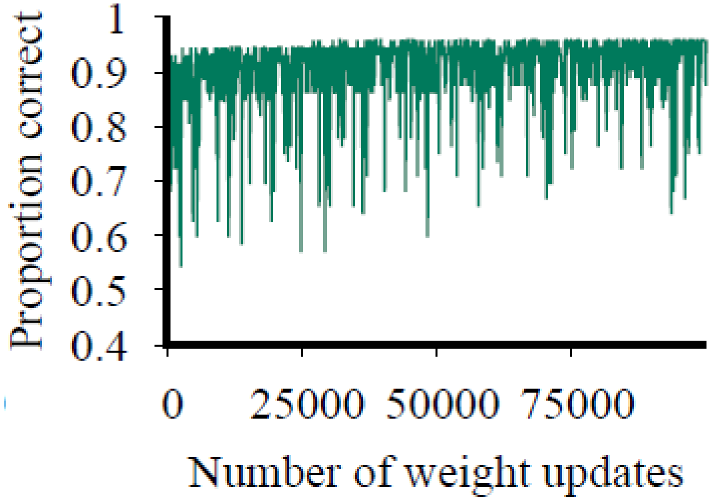


- 657 steps before convergence
- 63 examples, each used 10 times on average

Training Curves (cont'd)



Data not linearly separable.



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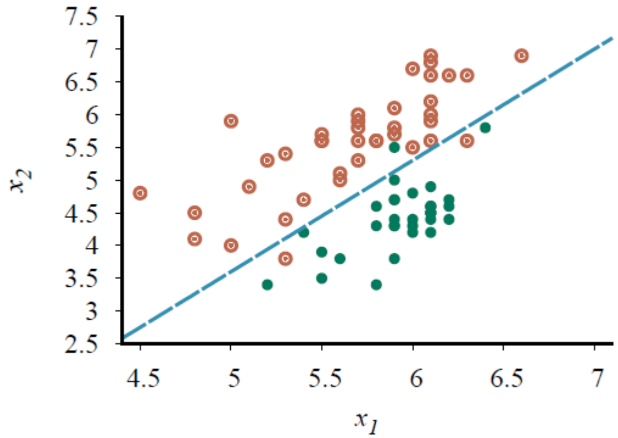

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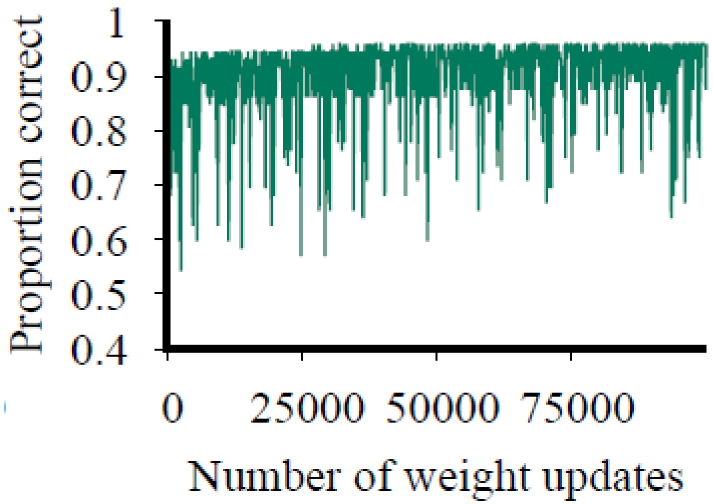
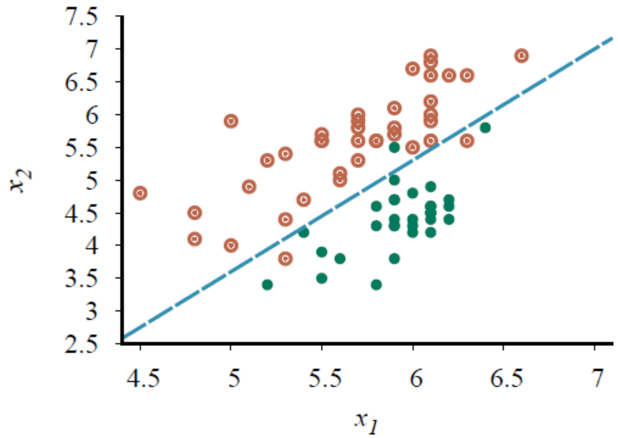


- Fails to converge after 10,000 steps.

Training Curves (cont'd)



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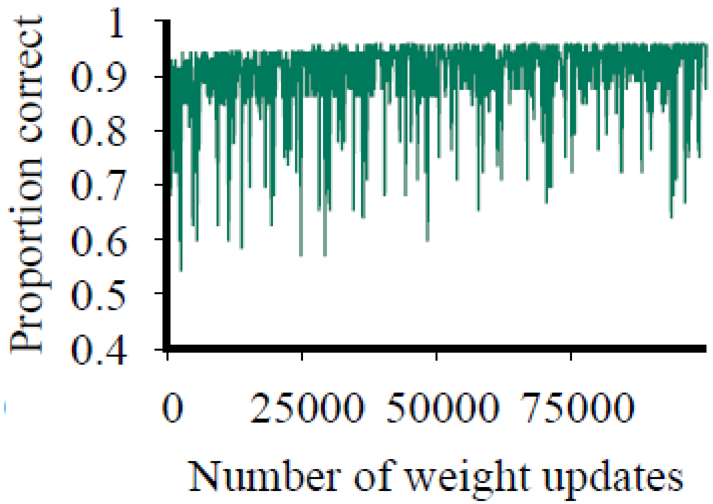

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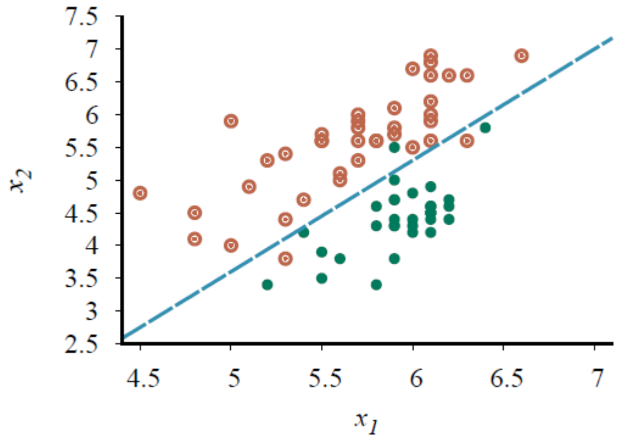

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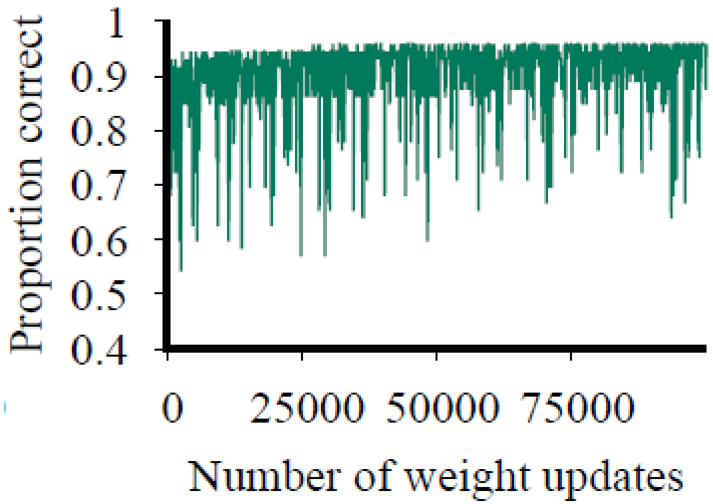

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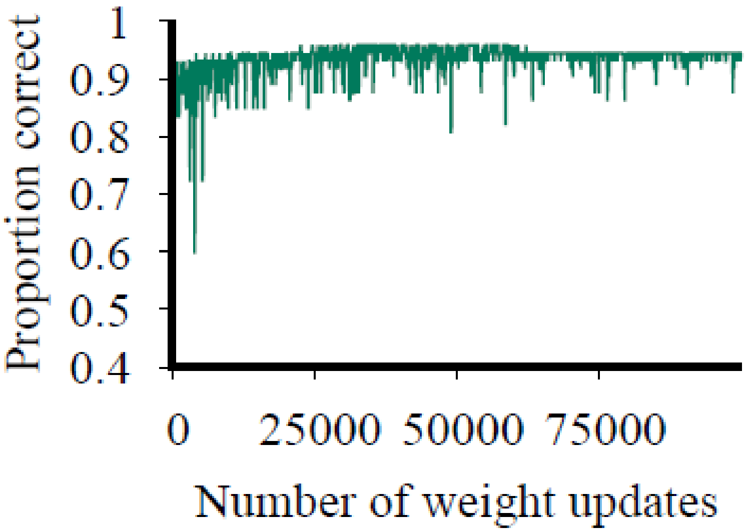
e.g., $\alpha(t) = 1000/(1000 + t)$



IV. Logistic Function

- Current hypothesis function is not continuous, let alone differentiable.
- This makes learning with the perceptron rule very unpredictable.
- It would be better if some examples could be classified as unclear borderline cases.

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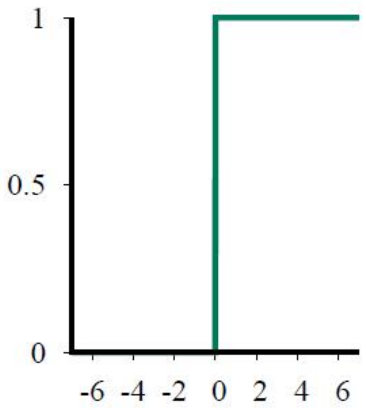

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♦ Use a continuous, differential function to soften the threshold

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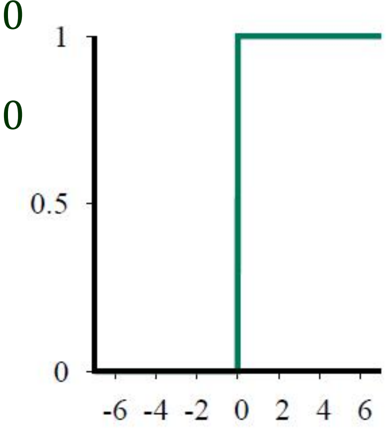

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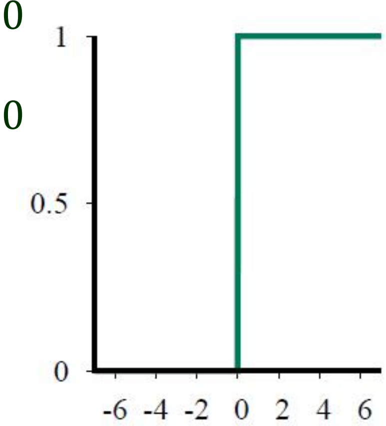

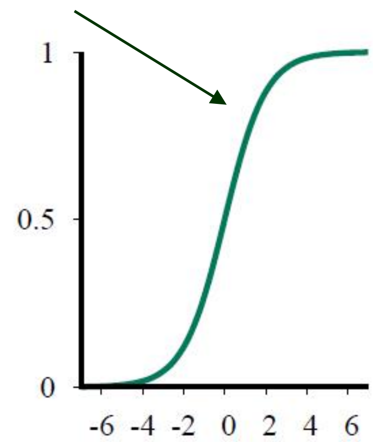

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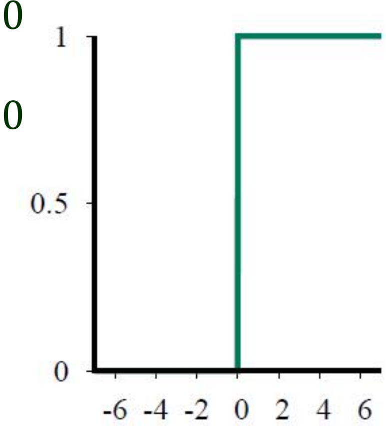

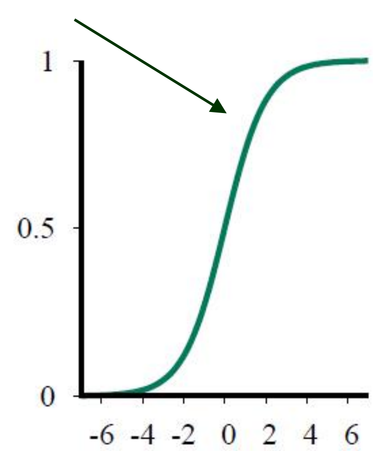

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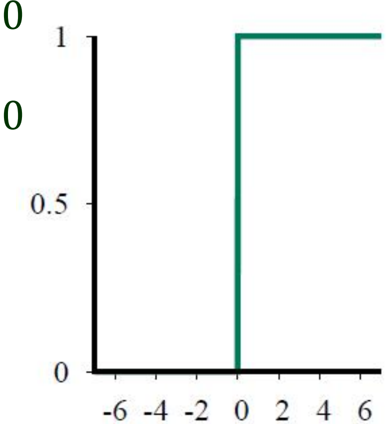

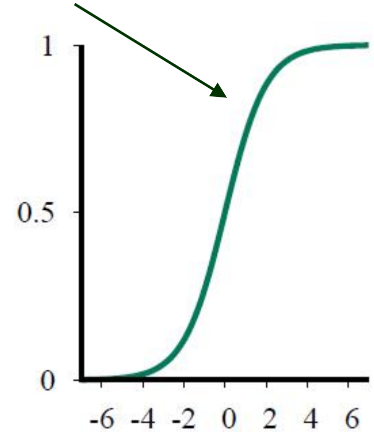

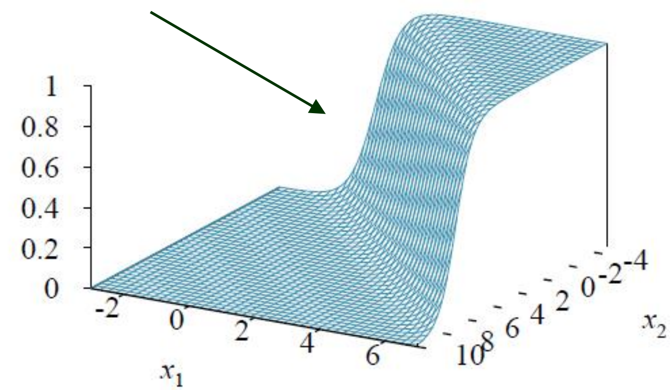

Logistic Regression

Fit the model $h_{\boldsymbol{w}}(\boldsymbol{x}) = \textit{Logistics}(\boldsymbol{xw})$ to minimize loss on a data set.

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Still apply gradient descent.

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$$\frac{\partial}{\partial w_i} \mathit{Loss}(\boldsymbol{w}) = \frac{\partial}{\partial w_i} \left(y - h_{\boldsymbol{w}}(\boldsymbol{x})\right)^2$$

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$$\begin{aligned} g'(\boldsymbol{xw}) &= g(\boldsymbol{xw})(1 - g(\boldsymbol{xw})) \\ &= h_{\boldsymbol{w}}(\boldsymbol{x})(1 - h_{\boldsymbol{w}}(\boldsymbol{x})) \end{aligned}$$

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Fit the model $h_{\boldsymbol{w}}(\boldsymbol{x}) = \mathit{Logistics}(\boldsymbol{xw})$ to minimize loss on a data set.

Still apply gradient descent.

$$\mathit{Logistics}(z) = g(z) = \frac{1}{1 + e^{-z}}$$

$$\frac{\partial}{\partial w_i} \mathit{Loss}(\boldsymbol{w}) = \frac{\partial}{\partial w_i} \left(y - h_{\boldsymbol{w}}(\boldsymbol{x})\right)^2$$

$$\vdots$$

$$= -2(y - h_{\boldsymbol{w}}(\boldsymbol{x})) \cdot g'(\boldsymbol{xw}) \cdot x_i$$

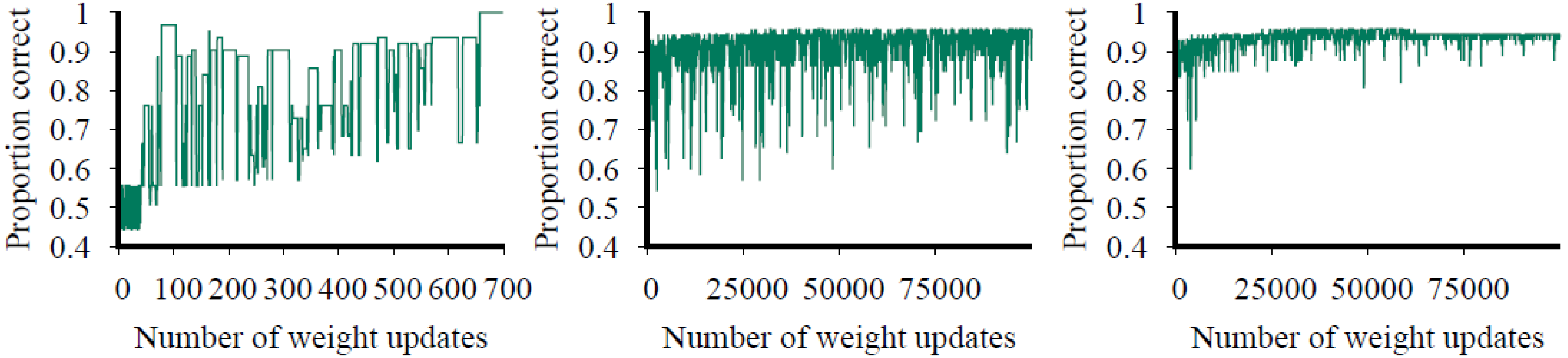
$$g'(\boldsymbol{xw}) = g(\boldsymbol{xw})(1 - g(\boldsymbol{xw}))$$
$$= h_{\boldsymbol{w}}(\boldsymbol{x})(1 - h_{\boldsymbol{w}}(\boldsymbol{x}))$$

$$= -2(y - h_{\boldsymbol{w}}(\boldsymbol{x})) \cdot h_{\boldsymbol{w}}(\boldsymbol{x})(1 - h_{\boldsymbol{w}}(\boldsymbol{x})) \cdot x_i$$

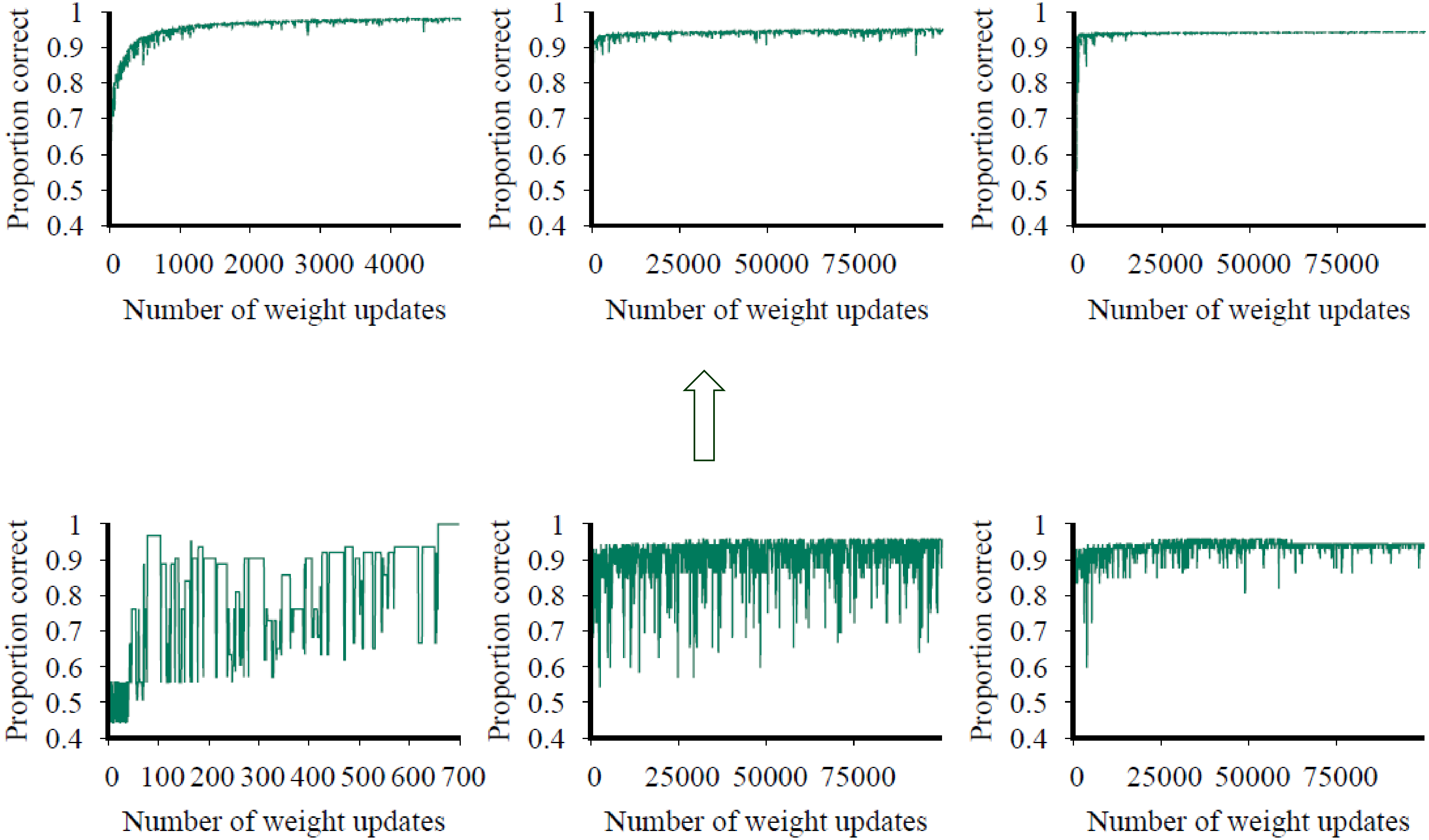
Weight update:

$$w_i \leftarrow w_i + \alpha(y - h_{\boldsymbol{w}}(\boldsymbol{x})) \cdot h_{\boldsymbol{w}}(\boldsymbol{x})(1 - h_{\boldsymbol{w}}(\boldsymbol{x})) \cdot x_i$$

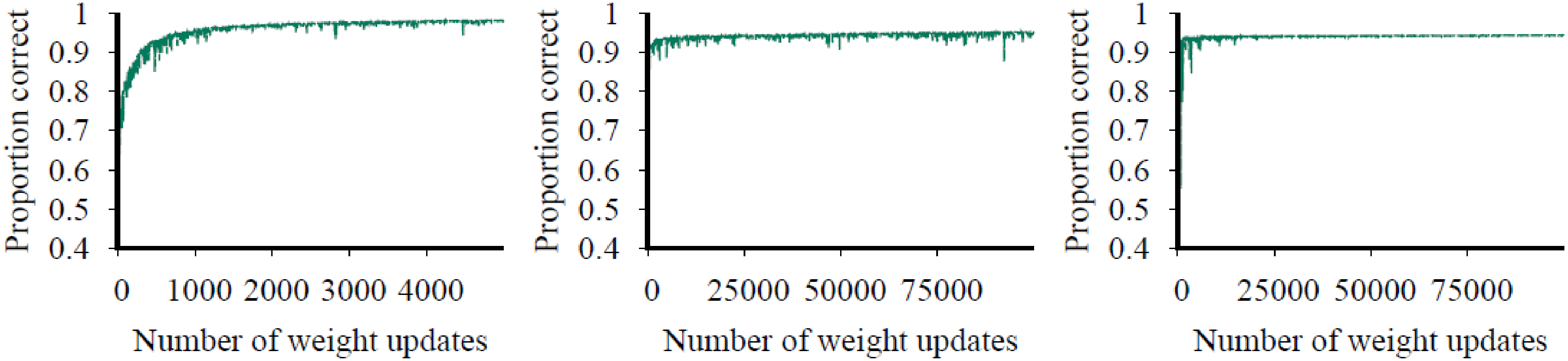
Improvements on Training Results



Improvements on Training Results



Improvements on Training Results



Logistic regression converges
far more quickly and reliably.