Recitation 13 - Solutions

1. Let's do some card counting practice. Suppose that two identical 52-card decks are mixed together to form a deck of 104 cards. Find the number of distinct permutations, formed by shuffling the 104 cards and laying them out in order. (Hint: Different shuffles may lead to the same permutation, so you will need to use the Division Rule.)

   Solution

   There are total 104 cards and there exists two pairs of 52 cards. Therefore, you need to apply the division rule for for 52 cards.

   $\frac{104!}{2^{52}}$

2. More card counting. A standard card deck consists of 52 cards, with 13 ranks (Ace through King) and 4 suits (clubs, hearts, spades, diamonds). Five-card draw is a variant of poker where each player is dealt a hand of 5 cards from the deck (the order of the cards in the hand doesn't matter).

   (a) What is the number of possible hands in Five-card draw?

   (b) A *four-of-a-kind* is a 5-card hand where 4 cards have the same rank. (e.g., 8Spades-AceHearts-8Clubs-8Diamonds-8Hearts is a four-of-a-kind since we have 4 eights). How many hands contain a four-of-a-kind? Divide this number by the number you got in part (a) to get a sense of how rare four-of-a-kind hands are.

   (c) A *full house* is a 5-card hand where 3 cards share one rank and the remaining two share the other hand. (e.g., 3Spades-3Diamonds-KingDiamonds-KingClubs-KingHearts) is a full house. How many hands are a full house?

   (d) A *two-pairs* is a 5-card hand where there are two cards of one rank, two cards of a second rank, and 1 card of a third-rank. (4Hearts-4Clubs-QSpades-QClubs-9Diamonds) is a two-pairs. How many hands have two-pairs?

   Solution

   (a) $\binom{52}{5}$

   (b) $\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{1}$

   (c) $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$

   (d) $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1}$

3. One urn contains two black balls (labeled B1 and B2) and one white ball A second urn contains one black ball and two white balls (labeled W1 and W2) suppose the following experiment is performed: One of the two urns is chosen at random Next a ball is randomly chosen from the urn Then a second ball is chosen at random from the same urn without replacing the first ball

(a) What is the total number of outcomes of this experiment?
(b) What is the number of ways that two black balls are chosen?
(c) What is the number of ways that two balls of opposite colors are chosen?

Solution (a) There are total 6 outcomes: $\{B1, B2\}, \{B1, W\}, \{B2, W\}, \{B, W1\}, \{B, W2\}$ and $\{W1, W2\}$

(b) There are 1 ways that two black balls are chosen: $\{B1, B2\}$
(c) There are 4 ways that two balls of opposite colours are chosen: $\{B1, W\}, \{B2, W\}, \{B, W1\}$ and $\{B, W2\}$

4. Six people attend the theater together:

(a) How many ways can they be seated in a row?
(b) Suppose one of the six is a doctor who must sit on the aisle in case he is paged How many ways can the people be seated in a row of seats if exactly one of the s eats is on the aisle and the doctor is in the aisle seat?

Solution

(a) Apply the product rule:

- Choose a person for the 1^{st} seat: 6 ways
- Choose a person for the 2^{nd} seat: 5 ways

. . . .

- Choose a person for the 5^{th} seat: 2 ways
- Choose a person for the 6^{th} seat: 1 ways

Totally, there are $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720$ ways

(b) Apply the product rule:

- Place the doctor to aisle seat: 1 ways
- Choose a person for the 1^{st} seat: 5 ways
- Choose a person for the 2^{nd} seat: 4 ways

. . . .

- Choose a person for the 5^{th} seat: 1 ways

Totally, there are $1 \times 5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$ ways