Exam 2 : Wed, March 11

Lecture 10

Uniform Distribution

STAT 330 - Iowa State University

Continuous Distributions

Continuous Distributions

Common distributions for continuous random variables

- Uniform distribution

$$X \sim Unif(a, b)$$

- Exponential distribution

$$X \sim Exp(\lambda)$$

- Gamma distribution

$$X \sim Gamma(\alpha, \lambda)$$

- Normal distribution

$$X \sim Normal(\mu, \sigma^2)$$

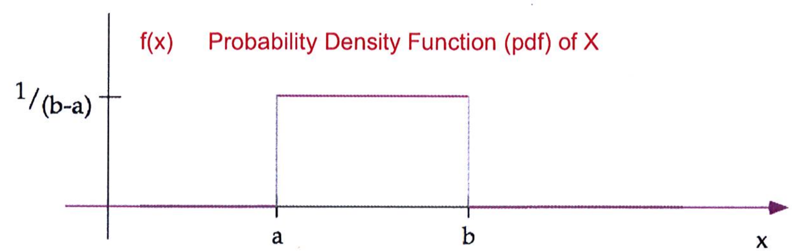
Uniform Distribution

Uniform Distribution

If a random variable follows a *uniform distribution*, then the R.V has constant probability between values a and b.

$$X \sim Unif(a, b)$$

- Probability Density Function (pdf)
  - Im(X) = (a,b)
  - $f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a < x < b \\ 0 & \text{otherwise} \end{cases}$



Uniform Distribution Cont.

- Cumulative Distribution Function (cdf) $F_X(t) = P(X \le t)$

$$F_X(t) = \begin{cases} 0 & \text{for } t \le a \\ \frac{t-a}{b-a} & \text{for } a < t < b \\ 1 & \text{for } t \ge b \end{cases}$$

- Expected Value: $E(X) = \frac{a+b}{2}$

$$E(X) = \int_a^b \underbrace{\frac{x}{b-a}}_{x \cdot f(x)} dx = \frac{1}{b-a}\left(\frac{x^2}{2}\right)\Big|_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$$

- Variance: $Var(X) = \frac{(b-a)^2}{12}$

$$Var(X) = \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a} dx = \ldots = \frac{(b-a)^2}{12}$$

Can also get variance by $Var(X) = E(X^2) - [E(X)]^2$

$E(X^2) = \int_a^b x^2 f(x)\,dx = \ldots$

Example

Uniform Distribution Example

Example 1: A basic (pseudo) random number generator creates realizations of Unif(0, 1) random variables.

X = number obtained from the random number generator.

$X \sim \text{Unif}(0, 1)$ — called "Standard Uniform Distribution" special case of Unif. Dist where $a=0, b=1$

$a=0$ $b=1$

1. What is Im(X)?

   $Im(X) = (0, 1)$

2. Give the pdf and cdf of X

PDF

$$f(x) = \begin{cases} \frac{1}{b-a} = \frac{1}{1-0} = 1 & \text{for } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ $\boxed{\text{PDF}}$

$$f(x) = \begin{cases} 1 & \text{for } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

CDF

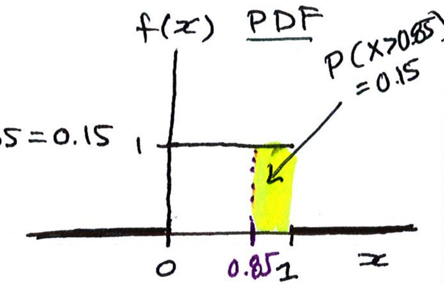
$$F_X(t) = \begin{cases} 0 & t \le a \\ \frac{t-a}{b-a} = \frac{t-0}{1-0} & a<t<b \\ 1 & t \ge b \end{cases}$$

$\Rightarrow$ $\boxed{\text{CDF}}$

$$F_X(t) = \begin{cases} 0 & t \le 0 \\ t & 0<t<1 \\ 1 & t \ge 1 \end{cases}$$

Uniform Distribution Example

3. What is the probability that it generates a number greater than 0.85?



Integrate PDF

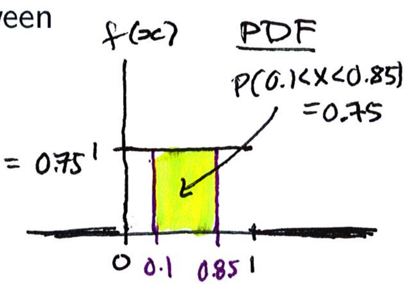
$$P(X>0.85) = \int_{0.85}^{\infty} f(x)\,dx = \int_{0.85}^{1} 1\,dx = x\Big|_{0.85}^{1} = 1-0.85 = 0.15$$

or Plug into CDF

$$P(X>0.85) = 1 - P(X \le 0.85)$$
$$= 1 - F_X(0.85)$$
$$= 1 - 0.85$$
$$= 0.15$$

Uniform Distribution Example

3. What is the probability that it generates a number between 0.1 and 0.85?



Integrate PDF

$$P(0.1 < X < 0.85) = \int_{0.1}^{0.85} f(x)\,dx = \int_{0.1}^{0.85} 1\,dx = x\Big|_{0.1}^{0.85} = 0.85 - 0.1 = 0.75$$

or Plug into CDF

$$P(0.1 < X < 0.85) = P(X < 0.85) - P(X \le 0.1)$$
$$= F_X(0.85) - F_X(0.1)$$
$$= 0.85 - 0.1$$
$$= 0.75$$

4. What is the expected value?

$$EX = \frac{a+b}{2} = \frac{0+1}{2} = 0.5$$

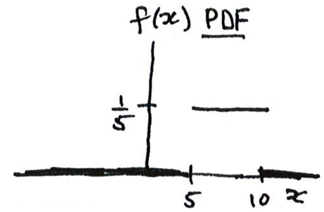
5. What is the variance?

$$Var(X) = \frac{(b-a)^2}{12} = \frac{(1-0)^2}{12} = \frac{1}{12}$$

Uniform Distribution Example

Example 2: Suppose X has a uniform distribution between 5 and 10. Calculate

1. $P(X < 7) =$
2. $P(6 < X < 7) =$



$a=5$, $b=10$

$X \sim Unif(5, 10)$

PDF

$$f(x) = \begin{cases} \frac{1}{b-a} = \frac{1}{10-5} = \frac{1}{5} & \text{for } 5<x<10 \\ 0 & \text{o.w} \end{cases}$$

CDF

$$F_X(t) = \begin{cases} 0 & t \le 5 \\ \frac{t-a}{b-a} = \frac{t-5}{10-5} & 5<t<10 \\ 1 & t \ge 10 \end{cases}$$

$\Rightarrow$

$\boxed{\text{PDF}}$

$$f(x) = \begin{cases} \frac{1}{5} & 5<x<10 \\ 0 & \text{else} \end{cases}$$

$\boxed{\text{CDF}}$

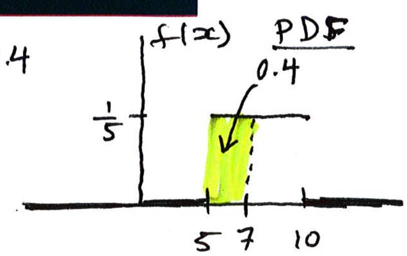
$$F_X(t) = \begin{cases} 0 & t \le 5 \\ \frac{t-5}{10-5} & 5<t<10 \\ 1 & t \ge 10 \end{cases}$$

Uniform Distribution Example

① $P(X<7) = \int_{-\infty}^{7} f(x)\,dx = \int_{5}^{7} \frac{1}{5}\,dx = \frac{x}{5}\Big|_5^7 = \frac{7}{5} - \frac{5}{5} = 0.4$

or

$P(X<7) = F_X(7) = \frac{7-5}{10-5} = 0.4$



② $P(6<X<7) = \int_6^7 f(x)\,dx = \int_6^7 \frac{1}{5}\,dx = \frac{x}{5}\Big|_6^7 = \frac{7}{5} - \frac{6}{5} = \frac{1}{5}$

or

$$\begin{aligned} P(6<X<7) &= P(X<7) - P(X \le 6) \\ &= F_X(7) - F_X(6) \\ &= \left(\frac{7-5}{10-5}\right) - \left(\frac{6-5}{10-5}\right) \\ &= 2/5 - 1/5 \\ &= 1/5 = 0.2 \end{aligned}$$