CprE 310 / Spring 2019

Homework 7 - Solutions

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1. **(10 points)** A computer programming team has 15 members.
   (a) How many ways can a group of seven be chosen to work on a project?
   (b) Suppose nine team members are SE students and six are CPRE students.
      i. How many groups of seven can be chosen that contain four SE and three CPRE students?
      ii. How many groups of seven can be chosen that contain at least one SE student?
      iii. How many groups of seven can be chosen that contain at most four CPRE students?

   Solution

   (a) Here, we need to choose 7 people from the group of 15 people. This is simply $\binom{15}{7} = 6435$.

   (b)

   i. We need to choose 4 SE students from the group of 9 SE students - which is $\binom{9}{4}$, **and** 3 CPRE students from the group of 6 CPRE students - which is $\binom{6}{3}$. Please observe the *and* carefully. It suggests that these two numbers are to be multiplied for the total number of possibilities. $\binom{9}{4} \times \binom{6}{3} = 2520$.

   ii. Here, groups of seven containing at least one SE student = total groups - groups without any SE students(all 7 members are chosen from group of 6 CPRE students). Which is, $\binom{15}{7} - \binom{6}{7} = 6435 - 0 = 6435$.

   iii. 'At most 4' suggests that we need groups with 0 CPRE students *or* 1 CPRE students *or* 2 CPRE students *or* 3 CPRE students *or* 4 CPRE students.

   - 0 CPRE students - choose 7 SE students and 0 CPRE students. $\binom{(9}{7)=36}$
   - 1 CPRE student - choose 1 CPRE student from group of 6 CPRE students and choose 6 SE students from group of 9 SE students = $\binom{6}{1} \times \binom{9}{6} = 504$
   - 2 CPRE students - choose 2 CPRE students from group of 6 CPRE students and choose 5 SE students from group of 9 SE students = $\binom{6}{2} \times \binom{9}{5} = 1890$
   - 3 CPRE students - choose 3 CPRE students from group of 6 CPRE students and choose 4 SE students from group of 9 SE students = $\binom{6}{3} \times \binom{9}{4} = 2520$
   - 4 CPRE students - choose 4 CPRE students from group of 6 CPRE students and choose 3 SE students from group of 9 SE students = $\binom{6}{4} \times \binom{9}{3} = 1260$ Observe the OR here. We need to add these possibilities. So, total possibilities = $36 + 504 + 1890 + 2520 + 1260 = 6210$.

2. **(10 points)** If there are 4 colors of jellybeans and you are trying to fill up a jar that holds 100 beans, how many different color combinations exist (assuming no restrictions on the distributions of the colors)?

   Solution

   We can solve this problem with "stars and bars" approach. If you use stars as denoting jellybeans and use bars to distinguish the type of jellybeans, you will need three bars since there are four types. Therefore, you can solve this problem selecting 3 slots for bars out of total 103 (50 for jellybeans + 3 for bars).

$$\binom{103}{3} = \binom{103}{100} = \frac{103!}{3!(103-3)!} = 176851$$

3. **(10 points)** In how many ways can you place 2 identical rooks on an 8×8 chessboard such that they will not be able to capture each other (i.e., they do not share the same row or column).

Solution

$8 \times 8 = 64$ choices for first rook (as it can be placed anywhere on the board). Now, second rook can't be placed in the same row and a column as of the first rook. That makes only 7 columns and 7 rows available for second rook. $7 \times 7 = 49$ choices for second rook. and then divide it by 2! as both rooks are identical. that is, $(64 \times 49)/2! = 1568$.

4. **(20 points)** Here, we prove a deep result in number theory known as *Fermat's Little Theorem*. However, our proof will require very little knowledge of number theory! Instead, we construct a combinatorial proof.

(a) Suppose there are beads available in a different colors for some integer $a > 1$, and let p be a prime number. How many different length p sequences of beads can be strung together?

Solution

We need to strung p beads together. For each bead, we have a different colors to choose from. So, total number of different sequences that can be formed are,

$$a \times a \times \cdots \times a \rightarrow \quad p \; times = a^p$$

(b) How many of them contain beads of at least two different colors? (Hint: Calculate how many beads contain exactly 1 color, and subtract from the first answer.)

Solution

As stated in the hint, let us first calculate how many beads contain exactly 1 color. Here, we have a colors to chose from for the first bead. But all the remaining beads must be of same color as of the first bead. So, we have only 1 choice for second bead onward.

$$a \times 1 \times \cdots \times 1 \rightarrow \quad p \; times = a$$

So, total number of strings containing at least two different colors of beads $= a^p - a$.

(c) Each string of p beads with at least two colors can be made into a bracelet by winding it around a circle in a clockwise manner and tying the two ends of the string together. Two bracelets are the same if one can be rotated to form the other. "Flipping" bracelets or reflecting them is not allowed. Argue that for every bracelet, there are exactly p **distinct** strings of beads that yield it. (Here, you have to use the fact that p is a prime number.)

Solution

Let the given bracelet be made with the string of the beads given as,

$$S_1 = (b_1, b_2, b_3, ..., b_p)$$

Now, similar bracelet can also be made with the string starting with b_2 and ending with b_1. This is similar to rotating the original bracelet by 1 bead.

$$S_2 = (b_2, b_3, b_4, ..., b_p, b_1)$$

We can also rotate the original bracelet by 2 beads, and the string corresponding to that starts at b_3,

$$S_3 = (b_3, b_4, ..., b_p, b_1, b_2)$$

And so on, we can start from $b_4, b_5, b_6, ... $ upto b_p, it means total p strings can form the identical bracelet.

Now, we need to check that each of this $\{S_1, S_2, S_3, ..., S_p\}$ are distinct.

These strings are not distinct if a part of a string repeats itself several times inside a string. Such repetition is possible only if the length of the string can be divided into equal subparts of length, say, $m, m < p$. For that, total length p must be a multiple of m. But as p is a prime number (given), it is not possible to have the value of m other than 1 and p. $m = 1$ would lead to the string having all beads of same color, which is already discarded in part b. $m = p$ is also not possible as $m < p$.

To understand this better, consider a sequence of 6 members in circular manner (clockwise), which is,

$$RGBRGB$$

Now, you rotate the sequence, first 2 rotations will give you different sequences , which are,

$$GBRGBR$$

$$BRGBRG$$

but when you rotate it one more time, it will give you the same sequence as original one.

$$RGBRGB$$

This happened because the part of the sequence (RGB) repeats itself completely. This is essentially possible because 6 is multiple of 3 (number of members of repeating sequence). That's why, here you have to use the fact that the P is prime number, and as P is not a multiple of anything other than 1, no part of the sequence can repeat itself, thus total number of possible strings that can be rotated to make the given string is P.

(d) Use the above result, combined with the Division Rule, to argue Fermat's Little Theorem, which states $a^p - a$ is a multiple of p for any integer $a > 1$ and prime number p.

I recommend actually trying this out with (say) $p = 5$ and $a = 2$. Enumerate all the possible sequences and bracelets, and conclude that this is true.

Solution Since there are $a^p - a$ possible strings of beads and each bracelet can be made with p different sequences, there are $(a^p - a)/p$ bracelets (by the division rule). There cannot be fractional number of bracelets, so $a^p - a$ is divisible by p.