Lecture 2

Combinatorics

STAT 330 - Iowa State University

Equally likely outcomes

Equally Likely Outcomes

Example 1: There are 4 chips in a box; 1 chip is defective. Randomly draw a chip from the box. What is the probability of selecting the defective chip?

$g_1 \quad g_2$
$g_3 \quad d$

- Common sense: $P(\text{draw defective chip}) = \frac{1}{4}$ or 25%
- Using probability theory...

  Sample space:

  $\Omega = \{g_1, g_2, g_3, d\}$
  $|\Omega| = 4$

  Event:

  $A =$ "draw defective chip" $= \{d\}$
  $|A| = 1$

  Probability of event: $P(A) = \frac{|A|}{|\Omega|} = \frac{1}{4}$

Equally Likely Outcomes Cont.

Classical definition of probability.

Theorem

outcome

If events in sample space are equally likely (i.e. $P(\{\omega\})$ is same for all $\omega \in \Omega$), then the probability of an event A is given by:

$$P(A) = \frac{|A|}{|\Omega|},$$

where $|A|$ is the number of elements in set A (cardinality of A).

when number of outcomes is large,
we need techniques to help us count.

Equally Likely Outcomes Cont.

$g_1 \; g_2$
$g_3 \; d$

Example 2: There are 4 chips in a box; 1 chip is defective. Randomly draw 2 chips from the box. What is the probability that defective chip is among the 2 chosen?

Sample space: (All possibilities for drawing 2 chips)

$$\Omega = \{(g_1, g_2), (g_1, g_3), (g_1, d), (g_2, g_3), (g_2, d), (g_3, d)\}$$

$$|\Omega| = 6$$

Event:

$A =$ "defective chip is among the 2 chips drawn"

$= \{(g_1, d), (g_2, d), (g_3, d)\}$

$|A| = 3$

Probability of event: $P(A) = \frac{|A|}{|\Omega|} = \frac{3}{6} = \frac{1}{2}$

Multiplication Principle

Multiplication Principle

Multiplication Principle

If a complex action can be broken down into a series of k component actions, performed one after the other, where ...
(sequentially)

- first action can be performed in n_1 ways
- second action can be performed in n_2 ways

$\vdots$

- last action can be performed in n_k ways

Then, the complex action can be performed in $\underbrace{n_1 n_2 \cdots n_k}_{\text{multiplied together}}$ ways.

Multiplication Principle Cont.

Example 3: Your friend (Kevin, Hamilton) owns 4 shirts (red, blue, green, white), and 2 pants (blue, black). What are all the ways he can create an outfit by choosing a shirt and pants to wear?

Shirt	Pants	
Red	Blue	1
	Black	2
Blue	Blue	3
	Black	4
Green	Blue	5
	Black	6
White	Blue	7
	Black	8

Multiplication Principle

$$\underset{\text{Shirt}}{\underline{4}} \times \underset{\text{Pants}}{\underline{2}} = 8$$

Example 4: Suppose licence plates are created as a sequence of 3 letters followed by 3 numbers. What is $|\Omega|$? (ie. how many license plates are in the sample space?)

$$\underset{L_1}{\underline{26}} \times \underset{L_2}{\underline{26}} \times \underset{L_3}{\underline{26}} \times \underset{N_1}{\underline{10}} \times \underset{N_2}{\underline{10}} \times \underset{N_3}{\underline{10}} = 17{,}576{,}000$$

Sample selection

Sample Selection

Imagine picking k objects from a box containing n objects.

Definitions

with replacement: After each selection, the object is put back in the box. It is possible to select the same object multiple times in the k selections.

without replacement: After each selection, the object is removed from the box. Cannot select the same object again.

ordered sample: Order of selected objects matters.

Example: Passwords ... abc1 $\neq$ c1ba

unordered sample: Order of selected objects doesn't matter.

Example: Selecting people for a study.

(Mary, John, Susan) = (John, Mary, Susan)

3 Main Scenarios

There are *3 main scenarios* we will deal with ...
Consider selecting 2 letters from a box containing "a", "b", "c".

1. Ordered with replacement
   - "with replacement" means repeat letters are allowed.
     $\Omega = \{(a,a),(b,b),(c,c),(a,b),(b,a),$
     $(a,c),(c,a),(b,c),(c,b)\}$
2. Ordered without replacement
   - "without replacement" means repeat letters not allowed.
     $\Omega = \{(a,b),(b,a),(a,c),(c,a),(b,c),(c,b)\}$

ordered
(a, b) and (b, a) are distinct b/c order matters

3. Unordered without replacement
   - "unordered" means (a,b) same as (b,a) - only written once.
   - "without replacement" means repeat letters not allowed.
     $\Omega = \{(a,b),(a,c),(b,c)\}$

unordered
$(a,b) = (b,a)$

Ultimately, we want to count up $|\Omega|$ for these scenarios.

Ordered With Replacement

A box has n items numbered $1, \ldots, n$. Draw k items with replacement. (A number can be drawn ~~twice~~ over again).

Sample Space: $\Omega = \{(x_1, \ldots, x_k) : x_i \in \{1, \ldots, n\}\}$

What is $|\Omega|$?

Break complex action into a series of k single draws.

1. n possibilities for x_1
2. n possibilities for x_2

   $\vdots$

k. n possibilities for x_k

Multiplication principle: $|\Omega| = n \cdot n \cdot n \cdots n = n^k$

$\underset{\text{draw } 1}{\underline{n}} \quad \underset{\text{draw } 2}{\underline{n}} \quad \underset{\text{draw } 3}{\underline{n}} \quad \cdots \quad \underset{\text{draw } k}{\underline{n}} = n^k$

\# of total possibilities for complex action

Permutation

same as Permutation

Ordered Without Replacement

A box has n items numbered $1, \dots, n$. Select k items without replacement. This means once a number is chosen, it can't be selected again.

Sample Space: $\Omega = \{(x_1, \dots, x_k) : x_i \in \{1, \dots, n\}, x_i \neq x_j\}$

What is $|\Omega|$?

Break complex action into a series of k single draws.

1. n possibilities for x_1
2. $n-1$ possibilities for x_2
3. $n-2$ possibilities for x_2

$\vdots$

k. $n-(k-1)$ possibilities for x_k

$$\underset{\text{draw } 1}{\underline{n}} \times \underset{\text{draw } 2}{\underline{(n-1)}} \times \underset{\text{draw } 3}{\underline{(n-2)}} \times \dots \times \underset{\text{draw } k}{\underline{(n-(k-1))}}$$

$$= \frac{n!}{(n-k)!}$$

"permutation number"

Multiplication principle: $|\Omega| = n \cdot (n-1) \cdot (n-2) \cdots (n-(k-1))$

This is equivalent to $\frac{n!}{(n-k)!}$

Permutation

Definition

A *permutation* is an ordering of k distinct objects chosen from n objects. This is another name for the *ordered without replacement* scenario.

Theorem

$P(n, k)$, called the *permutation number*, is the number of permutations of k distinct objects out of n objects.

$$P(n, k) = \frac{n!}{(n-k)!}$$

P^n_k

Note (factorials): $n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$

$0! = 1$

Ex. $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$5! = 5 \cdot 4!$

Permutation Example

Example 5:

Out of a group of 10 students, I choose 3 distinct students to give prizes to. How many ways can I select 3 students?

$n = 10 \qquad k = 3$

$\underset{\text{prize 1}}{\underline{10}} \times \underset{\text{prize 2}}{\underline{9}} \times \underset{\text{prize 3}}{\underline{8}} = 720$

$$P(n, k) = \frac{n!}{(n-k)!}$$

$$P(10, 3) = \frac{10!}{(10-3)!}$$

$$= \frac{10!}{7!}$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}}$$

$$= 10 \cdot 9 \cdot 8 = 720$$

Permutation Example

Example 6:

University phone exchange starts with 641 – _ _ _ _

What is the probability that a randomly selected phone number contains 7 distinct digits?

Sample space: (All possibilities for 4 chosen numbers)

$|\Omega| = \underline{10} \times \underline{10} \times \underline{10} \times \underline{10} = 10^4$

not distinct numbers
all possible numbers

Event: (4 chosen numbers are distinct - no repeats!)

remember that 6, 4, 1 are already taken

$\underline{7} \times \underline{6} \times \underline{5} \times \underline{4} = 840$

$|A| = P(7,4) = \frac{7!}{(7-4)!} = \frac{7!}{3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!}} = 7 \cdot 6 \cdot 5 \cdot 4 = 840$

$P(A) = \frac{|A|}{|\Omega|} = \frac{840}{10^4} = 0.084$

Combination

same as Combination

Unordered Without Replacement

Select k objects out of n objects with *no replacement* where *order does not matter*.

$$\Omega = \{(x_1, \ldots, x_k) : x_i \in \{1, \ldots, n\}, x_i \neq x_j\}$$

To derive $|\Omega|$ for this scenario, we can go back to how it was derived for permutations (where order mattered).

- Step 1: Select k objects from n (order doesn't matter)
- Step 2: Order the objects (there is $k!$ ways to order objects)

$P(n, k) =$ (number of ways to select k objects unordered) $\cdot\ k!$

$\underbrace{\text{Number of ways to select } k \text{ objects unordered}}_{\text{combination}} = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!k!}$

Combination

Definition

A *combination* is a subset of k objects from n objects. This is another name for *unordered without replacement* scenario.

Theorem

$C(n, k)$, called the *combination number*, is the number of combinations of k objects chosen from n.

$$C(n, k) = \binom{n}{k} = \frac{n!}{(n-k)!k!} \qquad C^n_K$$

not a fraction

- $C(n, k)$ or $\binom{n}{k}$ is read "n choose k"

Combination Example

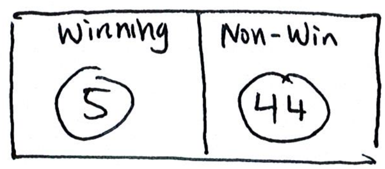
Example 7: Lottery (pick-five)

The lottery picks 5 numbers from $\{1, \ldots, 49\}$ without replacement as the "winning numbers". You choose 5 numbers and win if you pick at least 3 of the winning numbers.

1. What is the probability you match all 5 winning numbers?
2. What is the probability you win?

Easiest way to do combination problems is to draw a picture of the problem by visualizing a box of items you are selecting from. Break the box into sections according to the problems.

Here, we break the box into "winning" and "non-winning" numbers.



Combination Example

1. What is the probability you match all 5 winning numbers?

Event: To match all 5 winning numbers – we need to choose 5 numbers from "winning" and group, and 0 numbers from the "non-winning" group. This is done in ...

$$|A| = \binom{5}{5} \cdot \binom{44}{0} = \frac{5!}{(5-5)!5!} \frac{44!}{(44-0)!0!} = \frac{5!}{0!5!} \frac{44!}{44!0!} = \frac{5!}{1 \cdot 5!} \frac{44!}{44! \cdot 1} = 1$$

Sample Space: How many total ways are there to choose 5 numbers from 49 numbers (all possibilities). This is done in ...

$$|\Omega| = \binom{49}{5} = \frac{49!}{(49-5)!5!} = \frac{49!}{44!5!} = 1,906,884$$

$$P(\text{match all}) = \frac{\binom{5}{5} \cdot \binom{44}{0}}{\binom{49}{5}} = \frac{1}{1,906,884} = 0.000005$$

Combination Example

2. What is the probability you win? (Recall that you win if you match at least 3 "winning" numbers.)

Win	Non Win
5	44

$P(\text{win}) = P(\text{match at least } 3) =$
$P(\text{match } 3) + P(\text{match } 4) + P(\text{match } 5)$

From 5 winning numbers choose 3

From 44 non-winning numbers, choose 2

$$P(\text{match } 3) = \frac{\binom{5}{3}\binom{44}{2}}{\binom{49}{5}}$$

From 49 total ~~things~~ numbers choose 5

From 5 winning numbers, choose 4

From 44 non-win numbers choose 1

$$P(\text{match } 4) = \frac{\binom{5}{4}\binom{44}{1}}{\binom{49}{5}}$$

Combination Example

$$P(\text{match } 5) = \frac{\binom{5}{5}\binom{44}{0}}{\binom{49}{5}}$$

$$\begin{aligned} P(\text{win}) &= P(\text{match at least } 3) \\ &= P(\text{match } 3) + P(\text{match } 4) + P(\text{match } 5) \\ &= \frac{\binom{5}{3}\binom{44}{2}}{\binom{49}{5}} + \frac{\binom{5}{4}\binom{44}{1}}{\binom{49}{5}} + \frac{\binom{5}{5}\binom{44}{0}}{\binom{49}{5}} \end{aligned}$$

Counting Summary

Method	# of Possible Outcomes
Ordered with replacement	n^k
Ordered without replacement	$P(n,k) = \frac{n!}{(n-k)!}$
Unordered without replacement	$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$