Lecture 16

Stochastic Process & Markov Chain

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Stochastic Process

Stochastic Processes

Definitions

A *stochastic process* is a random variable that also depends on time. It is written as

$$X_t(\omega) = X(t,\omega) \text{ for } t \in \mathcal{T}, \omega \in \Omega$$

where $\mathcal{T}$ is a set of possible times. e.g. $[0,\infty), \{0,1,2,\ldots\}$ and Ω is the whole sample space.

- $X_t = X_t(\omega)$ is the random variable
- t is time
- ω is the "state"
- The *state space* is the collection of values the R.V X_t can take on: $\cup_{t\in\mathcal{T}} Im(X_t)$

Types of Stochastic Processes

Types of Stochastic Processes: $X_t(\omega)$ can be

- Continuous-time (t) continuous-state (ω)
- Discrete-time (t) continuous-state (ω)
- Continuous-time (t), discrete-state (ω)
- Discrete-time (t), discrete-state (ω)

We will only look at these two types

Types of Stochastic Processes

Examples:

1. Let X_t be the result of tossing a fair coin ($0 =$ tails, $1 =$ heads) in the t^{th} trial.

at each trial (t) X_t can take on values 0 or 1

   - The time (trial) $t \in \mathcal{T}$ where $\mathcal{T} = \{1, 2, 3, \ldots\}$
   - $Im(X_t) = \{0, 1\}$
   - This is an example of discrete time, discrete state stochastic process.

2. Let X_t be the number of customers in a store at time t.

at any time (t) X_t can take on values $\{0,1,2,3,\ldots\}$

   - The time $t \in \mathcal{T}$ where $\mathcal{T} = (0, \infty)$
   - $Im(X_t) = \{0,1,2,3, \ldots\}$
   - This is an example of continuous time, discrete state stochastic process.

Markov Chain

Markov Chain (MC) and Markov Property

Markov Property

A stochastic process X_t satisfies the *Markov property* if for any $t_1 < t_2 < \ldots < t_n < t$ and any sets A; $A_1, \ldots, A_n$:

$$P\{X_t \in A | X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n\} = P\{X_t \in A | X_{t_n} \in A_n\}.$$

- The probability distribution of X_t at time t only depends on its previous state. (what happened right before it)
- If the above is satisfied, then X_t is called a Markov Chain.

Markov Property Examples

1. A (fair) coin in flipped over and over: If coin lands on "heads", you win \$1. If coin lands on "tails", you lose \$1. Let X_t be your profit after t flips.

   H → +\$1
   T → - \$1

   - $P(X_5 = 3 | X_4 = 2) =$ 0.5
   - $P(X_5 = 3 | X_4 = 2, X_3 = 1, X_2 = 2, X_1 = 1) =$ 0.5

   X_t follows markov property → X_t is a markov chain

2. An urn contains 2 red balls, and 1 green ball. A ball is drawn (without replacement) from the urn yesterday and today. Another ball will be drawn tomorrow. Suppose you drew a red ball yesterday, and a red ball today.
   - $P($Red tomorrow$|$Red today$) =$ 0.5
   - $P($Red tomorrow$|$Red today, Red yesterday$) =$ 0

   X_t is not markov chain
   b/c markov property not satisfied.

Discrete-Time Discrete-State MC

Discrete-Time Discrete-State Markov Chain (MC)

Suppose we have a Markov chain with time set $\mathcal{T} = \{0, 1, 2, \ldots\}$ and state space $\{0, 1, 2, \ldots\}$ Two things we need to know about X_t:

1. **Initial distribution (P_0)**: $P_0(x) = P(X_0 = x)$ usually given as a vector of probabilities for the initial states of X_t. "starting probabilities"
   Ex: State space $= \{0, 1, 2\}$; $P_0 = \{0.3, 0.4, 0.3\}$
2. **Transition probabilities**:
   move forward one step (ie 1 day, 1 hour, 1 trial etc) → 1-step transition probability: probability of moving from state i to state j in 1 step.
   $p_{ij} = P(X_{t+1} = j | X_t = i)$
   h-step transition probability: probability of moving from state i to state j in h steps.
   $p_{ij}^{(h)} = P(X_{t+h} = j | X_t = i)$

Ex $P_{ij}^{(2)}$ = probabity of moving from state i to j in 2 steps

Discrete-Time Discrete-State Markov Chain (MC)

- We assume that the Markov Chain (MC) is ***homogeneous.***
  (ie) transition probabilities p_{ij} are independent of t.
  $\rightarrow$ For all times $t_1, t_2 \in \mathcal{T}$, $p_{ij}(t_1) = p_{ij}(t_2)$.
- Then, the distribution of a homogeneous MC is completely determined by the initial distribution (P_0) and one-step transition probability (p_{ij}).

Main Idea: Start with an initial distribution P_0. Then use the one-step transition probability p_{ij} to "jump" forward to the next step. Then, we can keep going forward one step at a time.

$$X_0 \xrightarrow{p_{ij}} X_2 \xrightarrow{p_{ij}} X_3 \xrightarrow{p_{ij}} X_4 \xrightarrow{p_{ij}} X_5 \rightarrow$$

Examples

Example

Example 1: In the summer, each day in Ames is either sunny or rainy. A sunny day is followed by another sunny day with probability 0.7, whereas a rainy day is followed by a sunny day with probability 0.4. It rains on Monday. Make weather forecasts for Tuesday and Wednesday.
Let 1 = "Sunny" and 2 = "Rainy".

To simplify and solve these types of problems, use transition matrices and matrix multiplication.

First we will solve w/o matrices. Then show that it's much easier w/ matrices.

RV: X_t = weather on day t

state space = $\{1, 2\}$
sunny, rainy

Start on Monday: We know it rains on Monday

(Mon) x_0	(sunny) 1	(rainy) 2
$P(x_0)$	0	1

- Initial Distribution P_0 of X_0 (monday)

$$P_0 = [\,0 \quad 1\,]$$

- Transition Probabilities (p_{ij} where i = current, j = future)
  - $p_{11} = P(X_{t+1} = 1 \mid X_t = 1) = 0.7$ P(Sunny | Sunny)
  - $p_{12} = P(X_{t+1} = 2 \mid X_t = 1) = 1 - p_{11} = 0.3$ P(Rainy | Sunny)
  - $p_{21} = P(X_{t+1} = 1 \mid X_t = 2) = 0.4$ P(Sunny | Rainy)
  - $p_{22} = P(X_{t+1} = 2 \mid X_t = 2) = 1 - p_{21} = 0.6$ P(Rainy | Rainy)
- Forecast for Tuesday (1-step ahead)

P(Tues Sunny | Mon Rainy) $= 0.4 = p_{21}$

P(Tues Rainy | Mon Rainy) $= 0.6 = p_{22}$

(Tues) x_1	(sunny) 1	(rainy) 2
$P(x_1)$	0.4	0.6

~~Pred~~ Prediction: 60% chance of rain on Tues
40% chance of sun on Tues.

• Forecast for wednesday

Mon → Tues → Wed

$R \xrightarrow{P_{22}} R \xrightarrow{P_{22}} R$

Mon → Tues → Wed

$R \xrightarrow{P_{21}} S \xrightarrow{P_{12}} R$

given

$P(\text{Wed Rainy} \mid \text{Mon Rainy})$

$= P_{22}P_{22} + P_{21}P_{12}$

$= (0.6)^2 + (0.4)(0.3)$

$= 0.48$

(Wed) x_2	(sun) 1	(rain) 2
$p(x_2)$	0.52	0.48

$P(\text{Wed Sunny} \mid \text{Mon Rainy})$

$= 1 - 0.48$

$= 0.52$

48% chance of rain on wednesday
52% chance of sun on wednesday

We can continue like this (moving forward one step at a time) to make predictions for all future days
But it will get increasingly complicated.

Simplify & avoid mistakes by using <u>transition matrices</u>.

1-Step Transition Probability Matrix

For a homogeneous MC with state space $\{1, 2, ..., n\}$, the ***1-step transition probability matrix*** is:

j

$$P = \begin{array}{c} \\ 1 \\ 2 \\ i \\ n \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\ \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} \end{array}.$$

Inside is P_{ij} where $i=$ current $j=$ future states

The element from the i-th row and j-th column is p_{ij}, which is the transition probability from state i to state j.

h-Step Transition Probability Matrix

Similarly, one can define a h-step transition probability matrix

$X_t =$ RV

state space $= \{1, 2, 3\}$

$P_0 = [0.3 \;\; 0.3 \;\; 0.4] \leftarrow$ Initial Distribution

$$P^{(h)} = \begin{pmatrix} p_{11}^{(h)} & p_{12}^{(h)} & \cdots & p_{1n}^{(h)} \\ p_{21}^{(h)} & p_{22}^{(h)} & \cdots & p_{2n}^{(h)} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}^{(h)} & p_{n2}^{(h)} & \cdots & p_{nn}^{(h)} \end{pmatrix}.$$

1-step transition

$P = \begin{bmatrix} 0.1 & 0.3 & 0.6 \\ 0.2 & 0.2 & 0.6 \\ 0.4 & 0.6 & 0 \end{bmatrix}_{3\times3}$

Using the matrix notation the following results follow:

- 2-step transition matrix $P^{(2)} = P \cdot P = P^2$
- h-step transition matrix $P^{(h)} = P^h = \overbrace{P \cdot P \cdots P}^{n}$
- The initial distribution of X_0 is written as row vector P_0. The distribution of X_h (h-steps in the future) is $\underbrace{P_h = P_0 P^h}_{\text{Ending Distridution}}$

Predict 1-step ahead $= P_0 P$

Predict 2 step ahead $= P_0 P^{(2)} = P_0 \cdot P \cdot P$

Example

Back to Example 1: We can solve the problem much more easily by using transition matrices . . . .

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

$$P^{(2)} = P \cdot P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} \cdot \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix}$$

$$P^{(3)} = P \cdot P \cdot P = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix} \cdot \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = \begin{pmatrix} 0.583 & 0.417 \\ 0.556 & 0.444 \end{pmatrix}$$

Recall : state space $= \{ \underset{\text{sun}}{1} \quad \underset{\text{rain}}{2} \}$

Mon : Initial Distribution P_0 (We know 100% rain on monday)

$P_0 = [0 \quad 1]$

x_0	1 (sun)	2 (rain)
$P(x_0)$	0	1

Tues 1-step ahead Prediction

$$P_0 \cdot P = \underset{1\times 2}{[0 \quad 1]} \underset{2\times 2}{\begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}} = \underset{1\times 2}{\overset{\text{(sun)} \quad \text{(rain)}}{[0.4 \quad 0.6]}}$$

x_1	1 (sun)	2 (rain)
$P(x_2)$	0.4	0.6

Wed 2-step ahead Prediction

$$P_0 P^{(2)} = P_0 \cdot P \cdot P$$

$$= [0 \quad 1] \begin{bmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{bmatrix} = [0.52 \quad 0.48]$$

(wed) x_2	1 (sun)	2 (rain)
$P(x_2)$	0.52	0.48

P(Wed Rain | Mon Rain) = 0.48

P(Wed Sun | Mon Rain) = 0.52

Thurs 3-steps ahead Prediction

$$P_0 P^{(3)} = P_0 \cdot P \cdot P \cdot P \overset{\cdots}{=} [0.556, 0.444]$$