Exam 1 — Wed, Feb 12, in regular lecture room/time
— bring a calculator

Coverage — Lecture notes 1 – 4 (Basic Probabity — Bayes Rule)

Study — Practice Exam (to be posted)
— Hwk 1 – 3
— Bring 1 page (front & back) note sheet.

1) General Prob Question (Ω, $P(A) = \frac{|A|}{|\Omega|}$)
2) General Prob. question Venn Diagram/Table
3) Counting Quest
 · multiplication rule
 · Permutation/Combination
4) Bayes
5) Reliability Question

Lecture 5

Discrete Random Variables

STAT 330 - Iowa State University

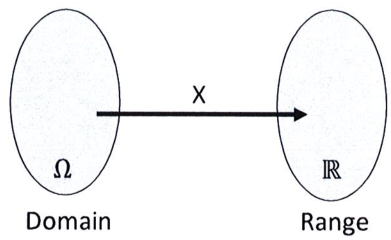
Random Variable

Random Variable

Definition

A *random variable (R.V.)* is a function that maps the sample space (Ω) to real numbers ($\Re$)

$$X : \Omega \to \Re$$



- Random variables (R.V.) connect random experiment to data
- Denote random variables with capital letters (X, Y, Z, etc)
- The values of a R.V. are determined by the outcome of a random experiment.

Random Variable Cont.

Example 1: Suppose you toss 3 coins, and observe the face up for each flip. $\Omega = \{HHH, HHT, \ldots, TTT\}; |\Omega| = 8$
We are interested in the number of heads we obtain in 3 coin tosses.

What is the random variable X?
$X = \#$ of heads in 3 coin tosses

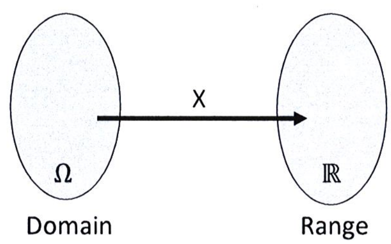
Notation:
$X \equiv$ Random variable
$x \equiv$ Realized value
$X = x \to$ "random variable X takes on the value x".
$\{X = x\}$ is just an event

union "or"

Consider the event 1 or 2 heads. This is $\{X = 1\} \cup \{X = 2\}$

Types of Random Variables

Types of Random Variables



Two types of random variables:

Discrete Random Variable

Sample space (Ω) maps to finite or countably infinite set in $\Re$

Ex: $\{1, 2, 3\}$, $\{1, 2, 3, 4, \ldots\}$

First we'll learn about discrete R.Vs

Continuous Random Variable

Sample space (Ω) maps to an uncountable set in $\Re$.

Ex: $(0, \infty)$, $(10, 20)$

Image of a Random Variable

Definition

The *image* of a random variable is defined as the values the random variable can take on.

$$Im(X) = \{x : x = X(\omega) \text{ for some } \omega \in \Omega\}$$

↑ all possible values that my RV can possibly take on

Example 2:

1. Put a disk drive into service. Let Y = time till the first major failure. $Im(Y) = (0, \infty)$.
   Image of Y is an interval (uncountable)
   $\rightarrow$ Y is a continuous random variable.

   intervals are continuous

2. Flip a coin 3 times. Let X = # of heads obtained.
   $Im(X) = \{0, 1, 2, 3\}$. Image of X is a finite set
   $\rightarrow$ X is a discrete random variable.

Probability Mass Function (PMF)

Probability Mass Function

Two things to know about a random variable X:

(1) What are the values X can take on? (what is its image?)

(2) What is the probability that X takes on each value?

(1) and (2) together gives the *probability distribution* of X.

Breakdown of X and their corresponding probabilities

Definition

Let X be a discrete random variable.

The *probability mass function (pmf)* of X is $p_X(x) = P(X = x)$.

little x represents realized values

Capital X represents my R.V.

Properties of pmf:

1. $0 \leq p_X(x) \leq 1$
2. $\sum_x p_X(x) = 1$

Probability Mass Function Cont.

Example 3: Which of the following are *valid* probability mass functions (pmfs)?

1.

x	-3	-1	0	5	7
$p_X(x)$	0.1	0.45	0.15	0.25	0.05

can also be written as a function

$$P_X(x) = \begin{cases} 0.1 & \text{for } x = -3 \\ 0.45 & x = -1 \\ 0.15 & x = 0 \\ 0.25 & x = 5 \\ 0.05 & x = 7 \end{cases}$$

not valid

2.

y	-1	0	1.5	$\cancel{3}$	4.5
$p_Y(y)$	0.1	0.45	0.25	$\cancel{-0.05}$	0.25

$P_Y(3) = -0.05$ but probabilities must be b/w 0 & 1.

not valid

3.

z	0	1	3	5	7
$p_Z(z)$	0.22	0.18	0.24	0.17	0.18

should sum to 1

$\sum_z P_Z(z) = 0.99 \neq 1$

Probability Mass Function Cont.

Example 4: Suppose you toss 3 coins, and observe the face up for each flip. We are interested in the number of heads we obtain in 3 coin tosses.

1. Define the random variable X.

   X = # of heads in 3 coin tosses

2. What is the image of X?

   $Im(X) = \{0, 1, 2, 3\}$

3. What is the pmf of X? (find $p_X(x)$ for all x)

   $P(X=0) = P(TTT) = (1/2)(1/2)(1/2) = (1/2)^3 = 1/8$

   $P(X=1) = P(HTT) + P(THT) + P(TTH) = 3(1/2)^3 = 3/8$

   $P(X=2) = P(HHT) + P(HTH) + P(THH) = 3(1/2)^3 = 3/8$

   $P(X=3) = P(HHH) = (1/2)^3 = 1/8$

Probability Mass Function Cont.

We can write the PMF as a table

x	0	1	2	3
$P_X(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

or write PMF as a function

$$P_X(x) = \begin{cases} 1/8 & \text{for } x = 0, 3 \\ 3/8 & \text{for } x = 1, 2 \end{cases}$$

- What is the probability that I obtain 2 heads?

  $P(X=2) = P_X(2) = 3/8$

- What is the prob. of 2 or 3 heads?

  $P(\{X=2\} \cup \{X=3\}) = P(X=2) + P(X=3)$

  $= \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$

Cumulative Distribution Function (CDF)

Cumulative Distribution Function

Definition

The *cumulative distribution function (cdf)* of X is

$$F_X(t) = P(X \le t)$$

- The pmf is $P_X(x) = P(X = x)$, the probability that R.V. X is equal to value x.
- The cdf is $F_X(t) = P(X \le t)$, the probability that R.V. X is less than or equal to t.

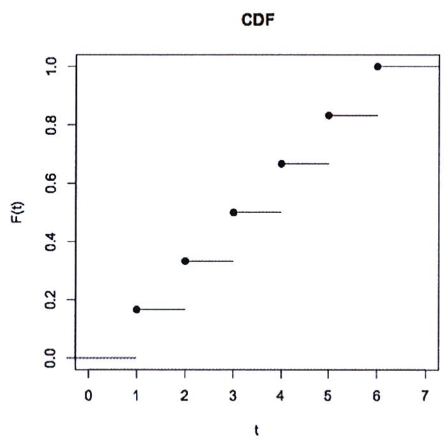
Relationship between pmf and cdf

- $F_X(t) = P(X \le t) = \sum_{x \le t} p_X(x) = \sum_{x \le t} P(X = x)$

Properties of CDFs

Properties of CDFs

1. $0 \leq F_X(t) \leq 1$
2. F_X is non-decreasing (if $a \leq b$, then $F(a) \leq F(b)$.
3. $\lim_{t \to -\infty} F_X(t) = 0$ and $\lim_{t \to \infty} F_X(t) = 1$
4. F_X is right-continuous with respect to t

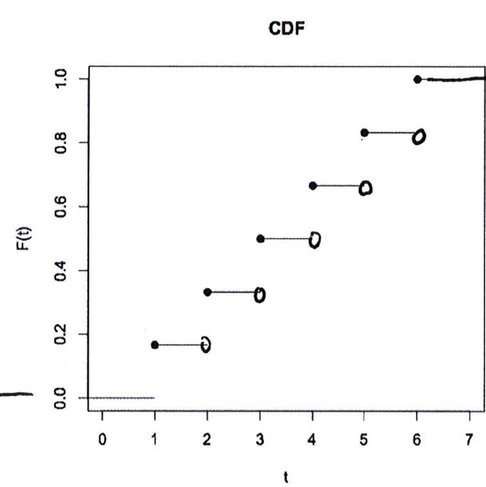


CDFs for discrete R.V are step functions

Cumulative Distribution Function Cont.

Example 5: Roll a fair die. Let X = the number of dots on face up

x	1	2	3	4	5	6	7	8
(pmf) $p_X(x)$	1/6	1/6	1/6	1/6	1/6	1/6	0	0
(cdf) $F_X(x)$	1/6	2/6	3/6	4/6	5/6	1	1	1

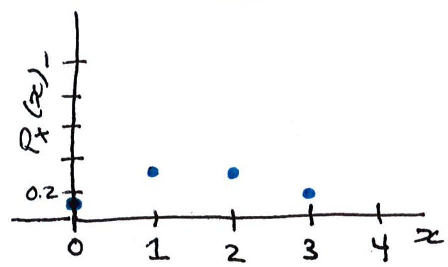


Example 6: Suppose you toss 3 coins, and observe the face up for each flip. We are interested in the number of heads we obtain in 3 coin tosses.

From Example 4, the pmf is

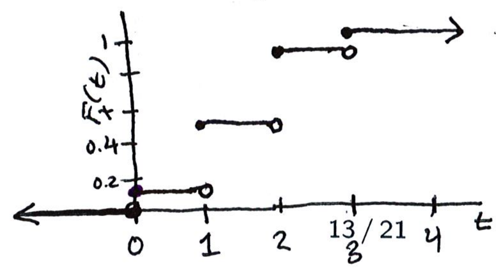
x	0	1	2	3
(pmf) $p_X(x)$	1/8	3/8	3/8	1/8
(cdf) $F_X(x)$	1/8	4/8	7/8	1

PMF



What is the cdf of X?

CDF



Expected Value

Expected Value

Example 7: Flip a coin 3 times. Let $X = \#$ of heads obtained in 3 flips. The probability mass function (pmf) of X is

x	0	1	2	3
$p_X(x)$	1/8	3/8	3/8	1/8

PMF

What number of heads do we "expect" to get?

0 obtained $\frac{1}{8}$ of the time

1 obtained $\frac{3}{8}$ of the time

2 obtained $\frac{3}{8}$ of the time

3 obtained $\frac{1}{8}$ of the time

Intuitively, we can think about taking $0(\frac{1}{8}) + 1(\frac{3}{8}) + 2(\frac{3}{8}) + 3(\frac{1}{8})$ as the "expected" number of heads

Expected Value

Definition

function of X

Let X be a discrete random variable. The *expected value* or *expectation* of $h(X)$ is

$$E\left[h(X)\right] = \sum_x h(x)p_X(x) = \sum_x h(x)P(X = x)$$

- The **MOST IMPORTANT** version of this is when $h(x) = x$ identity function

$$E(X) = \sum_x xp_X(x) = \sum_x xP(X = x)$$

- $E(X)$ is usually denoted by μ "mu"
- $E(X)$ is the weighted average of the x's, where the weights are the probabilities of the x's.

Expected Value Cont.

Example 8: Flip a coin 3 times. Let $X = \#$ of heads obtained in 3 flips. The probability mass function (pmf) of X is

x	0	1	2	3
$p_X(x)$	1/8	3/8	3/8	1/8

Calculate the expected value of X.

$$
\begin{aligned}
E(X) &= \sum_x x p_X(x) \\
&= 0P(X=0) + 1P(X=1) + 2P(X=2) + 3P(X=3) \\
&= (0)(1/8) + (1)(3/8) + (2)(3/8) + (3)(1/8) \\
&= 1.5
\end{aligned}
$$

Variance (how spread apart are my x's)

Variance & Standard Deviation

Definition "sigma squared"

The *variance* (σ^2) of a random variable X is

$$Var(X) = E\left[(X - E(X))^2\right] = \sum_x (x - E(X))^2 \cdot p_X(x)$$

The *standard deviation* (σ) of a random variable X is

"sigma"

$$\sigma = \sqrt{Var(X)}$$

- Units for variance is squared units of X.
- Units for standard deviation is same as units of X.

SHORT CUT (usually more convenient)

$$Var(X) = E(X^2) - [E(X)]^2$$ ← don't forget to square

(usually easier to calculate)

$$= \sum_x x^2 P(X = x) - \left[\sum_x xP(X = x)\right]^2$$

Variance Cont.

Example 9: Flip a coin 3 times. Let $X = \#$ of heads obtained in 3 flips. The probability mass function (pmf) of X is

x	0	1	2	3
$p_X(x)$	1/8	3/8	3/8	1/8

Calculate the variance and standard deviation of X.

use shortcut formula

$Var(X) = E(X^2) - (E(X))^2$

- $E(X) = \sum_x x p_X(x) = 1.5$ (example 8)
- $E(X^2) = \sum_x x^2 p_X(x) = (0)^2(1/8) + (1)^2(3/8) + (2)^2(3/8) + (3)^2(1/8) = 3$
- $Var(X) = E(X^2) - [E(X)]^2 = 3 - (1.5)^2 = 0.75$ dont forget to square
- $\sigma = \sqrt{Var(X)} = \sqrt{0.75} = 0.866$

↑ standard deviation

Operations involving E(X) & Var(X)

Operations

X, Y are random variables; a, b are constants.

Operations with $E(\cdot)$

- $E(aX) = aE(X)$
- $E(aX + b) = aE(X) + b$
- $E(aX + bY) = aE(X) + bE(Y)$

Example if $EX = 6$ and $E(Y) = 2$

- $E(4X) = 4E(X) = 4(6) = 24$
- $E(4X - 3Y + 1) = 4E(X) - 3E(Y) + 1$
$= 4(6) - 3(2) + 1$
$= 24 - 6 + 1$
$= 19$

Operations Cont.

X, Y are random variables; a, b, are constants.

<u>**Operations with $Var(\cdot)$**</u>

· square constants when you pull it out of variance

· ignore any constants that are not attached to a R.V

- $Var(aX) = a^2 Var(X)$
- $Var(aX + b) = a^2 Var(X)$
- $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + \boxed{2abCov(X, Y)}$
  (when X,Y are independent, Cov(X,Y) = 0. We'll discuss more about independence and define covariance later)

Example : $Var(X) = 10$

- $Var(4X) = (4)^2 Var(X) = (16)(10) = 160$
- $Var(-4X) = (-4)^2 Var(X) = (16)(10) = 160$
- $Var(-4X - 3) = (-4)^2 Var(X) = (16)(10) = 160$

If $Var(X) = 10$ and $Var(Y) = 5$ and X, Y are indep.

- $Var(-4X + 3Y) = (-4)^2 Var(X) + (3)^2 Var(Y)$
  $= (16)(10) + (9)(5) = 205$

Chebyshev's Inequality

Chebyshev's Inequality:

For any positive real number k (constant), and a random variable X with variance σ^2:

$$P(|X - E(X)| \leq k\sigma) \geq 1 - \frac{1}{k^2}$$

- bounds the probability that X lies within a certain number of standard deviations from $E(X)$