Lecture 3

Conditional Probability & Independence

STAT 330 - Iowa State University

Contingency Table (Two Way Table)

Contingency Table

Definition

A *contingency table* gives the distribution of 2 variables.

Example 1: Suppose in a small college of 1000 students, 650 students own Iphones, 400 students own MacBooks, and 300 students own both.

Define events: I = "owns Iphone", and M = "owns MacBook".

$|M \cap I| = |I \cap M|$ = # that own iphone and Mac

Phone \ Computer	M	$\overline{M}$	Total
I	300	?	650
$\overline{I}$	?	?	?
Total	400	?	1000

$|I|$ = # that own iphone

$|M|$ = # that own Mac

$|\Omega|$ = total # in sample

Contingency Table

- Fill in rest of table
- Inner cell add to margin
- Margin side adds to Total

Phone \ Computer	M	$\overline{M}$	Total
I	300	350	650
$\overline{I}$	100	250	350
Total	400	600	1000

Marginal Probability

Marginal Probability

Definition

The *marginal probability* is the probability of a variable. It can be obtained from the *margins* of contingency table.

Phone \ Computer	M	$\overline{M}$	Total
I	300	350	650
$\overline{I}$	100	250	350
Total	400	600	1000

What is the probability of owning a Mac? (ie marginal probability of owning a Mac)

$P(M) = \frac{400}{1000} = 0.40$

$P(M) = \frac{|M|}{|\Omega|} = \frac{400}{1000} = 0.40$

Conditional Probability

Conditional Probability

Does knowing someone owns an Iphone change the probability they own a Mac?

Informally, conditional probability is updating the probability of an event given information about another event.

If we *know* that someone owns an Iphone, then we can narrow our sample space to just the "owns Iphone" case (highlighted blue row) and ignore the rest!

Phone \ Computer	M	$\overline{M}$	Total
I	300	350	650
$\overline{I}$	100	250	350
Total	400	600	1000

Conditional Probability Cont.

narrow our data to just the iphone cases.

What is the probability of owning a Mac *given* they own an Iphone?

Phone \ Computer	M	$\overline{M}$	Total
I	300	350	650
~~$\overline{I}$~~	~~100~~	~~250~~	~~350~~
~~Total~~	~~400~~	~~600~~	~~1000~~

$P(M|I) = \frac{300}{650} = 0.46$

$P(M|I)$ = probability of M given I

"given"

Conditional Probability Cont.

Definition

The *conditional probability* of event A given event B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

"given"

provided $P(B) \neq 0$.

It can be obtained from the *rows/columns* of contingency table.

Back to Example 1 ...

What is the probability of owning a Mac *given* they own an Iphone?

$$P(M|I) = \frac{P(I \cap M)}{P(I)} = \frac{0.3}{0.65} = 0.46$$

Consequences of Conditional Probability

The definition of conditional probability gives useful results:

1.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \rightarrow P(A \cap B) = P(B)P(A|B)$$

2.

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \rightarrow P(A \cap B) = P(A)P(B|A)$$

This gives us two additional ways to calculate probability of intersections. Putting it together . . .

$$\boxed{P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)}$$

Probability Calculations

Probability Calculations

A contingency table can also be written with probabilities instead of counts. This is called a *probability table*. / joint prob. table

Inner cells give "joint probabilities" → probability of intersections

- $P(A \cap B), P(\overline{A} \cap B)$, etc

Margins give "marginal probabilities" → probability of variables

- $P(A), P(B), P(\overline{A})$, etc

$P(I \cap M) = \frac{|I \cap M|}{|\Omega|} = \frac{300}{1000} = 0.3$

Phone \ Computer	M	$\overline{M}$	Total
I	0.30	0.35	0.65
$\overline{I}$	0.10	0.25	0.35
Total	0.40	0.60	1

← $P(I) = \frac{|I|}{|\Omega|} = \frac{650}{1000} = 0.65$

← $P(\Omega) = \frac{|\Omega|}{|\Omega|} = 1$

Probability Calculations Cont.

Phone \ Computer	M	$\overline{M}$	Total
I	0.30	0.35	0.65
$\overline{I}$	0.10	0.25	0.35
Total	0.40	0.60	1

$P(\overline{I}) = 0.35$

$P(M) = 0.40$

$P(\overline{I} \cap M) = 0.1$

probability of owning a Mac given they don't own an iphone.

$P(M|\overline{I}) = \frac{P(M \cap \overline{I})}{P(\overline{I})} = \frac{0.10}{0.35} = 0.29$

$P(\overline{I}|M) = \frac{P(M \cap \overline{I})}{P(M)} = \frac{P(\overline{I} \cap M)}{P(M)} = \frac{0.10}{0.40} = 0.25$

probability of not owning an iphone given they own a Mac.

Independence

Independence of Events

In Example 1, knowing an event occurred changed the probability of another event occurring.

However, sometimes knowing an event occurs *doesn't change* the probability of the other event.
In this case, we say the events are *independent*.

Definition
Events A and B are *independent* if ...

1. $P(A \cap B) = P(A)P(B)$

   or equivalently

2. $P(A|B) = P(A)$ if $P(B) \neq 0$

Knowing B occurs has no impact on the probability of A.

Independence of Events Cont.

Example 2: Check if events are independent

Is owning an Iphone and owning MacBook independent?
Recall that $P(I) = 0.65$, $P(M) = 0.4$, $P(I \cap M) = 0.35$

check if $P(I \cap M) \stackrel{?}{=} P(I)P(M)$

or check if
$P(M|I) \stackrel{?}{=} P(M)$
$P(I|M) \stackrel{?}{=} P(I)$

$P(I \cap M) = 0.35$

$P(I) \cdot P(M) = (0.65)(0.4) = 0.26$

Since $P(I \cap M) \neq P(I)P(M)$,

I, M are not independent.

Independence of Events Cont.

Example 3: Using independence to simplify calculations

If A, B independent $\rightarrow P(A \cap B) = P(B)P(A|B) = P(B)P(A)$

Roll a die 4 times. Assuming that rolls are independent, what is the probability of obtaining at least one '6'?

$P(\text{at least 1 '6'}) = 1 - P(\text{No '6's})$

$= 1 - P(\text{no '6' on roll 1} \cap \text{no '6' on roll 2} \cap \cdots \cap \text{no '6' on roll 4})$

$= 1 - [P(\text{no 6 on roll 1})P(\text{no 6 on roll 2}) \cdots P(\text{no 6 on roll 4})]$

$= 1 - [(5/6)(5/6)(5/6)(5/6)]$

$= 1 - (5/6)^4$

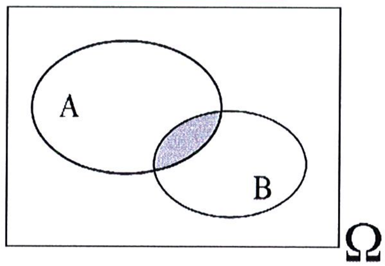
$= 0.518$

Independent vs. Disjoint

Independent ≠ Disjoint!!!

Completely different concepts!

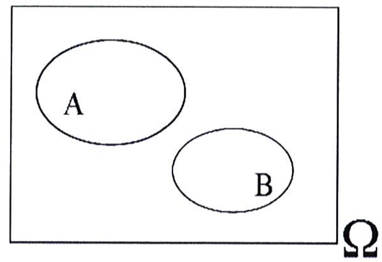
Independent:



$P(A \cap B) = P(A)P(B)$

↙ overlap region

Disjoint:



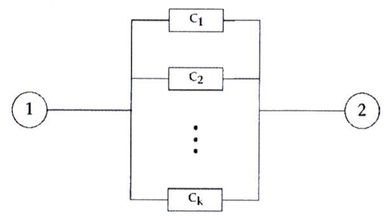
$P(A \cap B) = P(\emptyset) = 0$

↙ no overlap

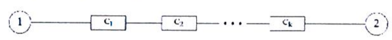
System Reliability

Application: System Reliability

Parallel: A parallel system consists of k components $(c_1, \ldots, c_k)$ arranged such that the system works if and only if at least one of the k components functions properly.



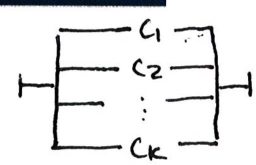
Series: A series system consists of k components $(c_1, \ldots, c_k)$ arranged such that the system works if and only if ALL components function properly.



Reliability: Reliability of a system is the probability that the system works.

Reliability of Parallel System

Example 4:



Let $c_1, \ldots, c_k$ denote the k components in a *parallel* system. Assume the k components operate independently, and $P(c_j \text{ works }) = p_j$. What is the reliability of the system?

$P(c_j \text{ fails}) = P(\overline{c_j \text{ works}}) = 1 - P(c_j \text{ work}) = 1 - p_j$

$$
\begin{aligned}
P(\text{system works}) &= P(\text{at least one component works}) \\
&= 1 - P(\text{all components fail}) \\
&= 1 - P(c_1 \text{ fails} \cap c_2 \text{ fails} \cap \cdots \cap c_k \text{ fails}) \\
&= 1 - [P(C_1 \text{ fails})P(C_2 \text{ fails}) \cdots P(C_k \text{ fails})] \quad \text{Indep} \\
&= 1 - \prod_{j=1}^{k} P(c_j \text{ fails}) \\
&= 1 - \prod_{j=1}^{k} (1 - p_j)
\end{aligned}
$$

Reliability of Series System

Example 5:

$\vdash c_1 - c_2 - \cdots - c_k \dashv$

Let $c_1, \ldots, c_k$ denote the k components in a *series* system. Assume the k components operate independently, and $P(c_j \text{ works }) = p_j$. What is the reliability of the system?

$$
\begin{aligned}
P(\text{system works}) &= P(\text{all components work}) \\
&= P(c_1 \text{ works} \cap c_2 \text{ works} \cap \cdots \cap c_k \text{ works}) \\
&= P(c_1 \text{ works}) P(c_2 \text{ works}) \cdots P(c_k \text{ work}) \quad \text{Indep} \\
&= \prod_{j=1}^{k} P(c_j \text{ works}) \\
&= \prod_{j=1}^{k} p_j
\end{aligned}
$$

Reliability Example

Example 6: Suppose a base is guarded by 3 radars (R_1, R_2, R_3), and the radars are independent of each other. The detection probability are ...

$P(R_1) = P(R_1 \text{ detects}) = 0.95$

$P(R_2) = P(R_2 \text{ detects}) = 0.98$

$P(R_3) = P(R_3 \text{ detects}) = 0.99$

Does a system in *parallel* or *series* have higher reliability for this scenario?

Known

$P(R_1) = 0.95$ $\qquad P(\bar{R}_1) = P(R_1 \text{ fails}) = 1 - 0.95 = 0.05$

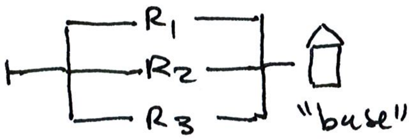
$P(R_2) = 0.98$ $\qquad P(\bar{R}_2) = 1 - 0.98 = 0.02$

$P(R_3) = 0.99$ $\qquad P(\bar{R}_3) = 1 - 0.99 = 0.01$

R_1, R_2, R_3 are all indep.

Reliability Example

Parallel

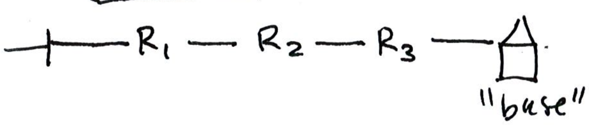


Reliability = probability that the system works.

$$
\begin{aligned}
P(\text{sys works}) &= P(\text{at least 1 radar works}) \\
&= 1 - P(\text{none works}) \\
&= 1 - P(\bar{R}_1 \cap \bar{R}_2 \cap \bar{R}_3) \quad \text{"and"} \\
&= 1 - [P(\bar{R}_1)P(\bar{R}_2)P(\bar{R}_3)] \quad \text{indep} \\
&= 1 - [(0.05)(0.02)(0.01)] \\
&= 0.99999
\end{aligned}
$$

Reliability Example

Series



$$
\begin{aligned}
P(\text{sys. works}) &= P(\text{all radars work}) \\
&= P(R_1 \cap R_2 \cap R_3) \quad \text{Indep} \\
&= P(R_1)P(R_2)P(R_3) \\
&= (0.95)(0.98)(0.99) \\
&= 0.922
\end{aligned}
$$

(Parallel system has higher reliability than the series system)