

Lecture 1

Basic Probability and Set Theory

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STAT 330 - Iowa State University

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Course Info

Course Info

STAT 330: Probability and Statistics for Computer Science

General Info

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Course Setup

1. Intro to probability & random variables
2. Applications of probability
3. Statistics

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Introduction

Definitions

- *Probability* is a mathematical theory for modeling processes where outcomes occur randomly
- *Statistics* is learning about the real world from data under the assumption that the data was generated by a random process

Goals

1. Probability: model and assign probabilities to outcomes
2. Statistics: use probability to draw conclusions

Random Experiment

Random Experiment

Definition

A *random experiment* is an experiment (or process) for which outcome cannot be predicted with complete certainty

Example 1: Various random experiments

- A message takes 2 network routers to reach a recipient computer. We record the status of router 1, the status of router 2, and the status of the recipient computer, where the status is either up (U) or down (D).
- Record the time for a web page to respond.
- Roll a die and record the face up.
- Flip a coin until you get a head. Record all the faces that you obtain.

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Components of a Random Experiment

Outcome

Definition ω *lower case omega*

The **outcome** (ω) is the result of an experiment

Example 2: Outcomes

- **Network Routers:** $\omega = (\text{router 1 down, router 2 down, recipient computer up}) = \text{DDU}$
- **Access web page:** $\omega = 2.576 \text{ sec}$
- **Roll a die:** $\omega = 1$
- **Toss coin until head:** $\omega = TTH$

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Sample space

Definition Ω *capital Omega*

The **sample space** (Ω) is the set of **ALL** possible outcomes

Example 3: Sample spaces

- **Network Routers:** $\Omega = \{\text{DDD, DDU, DUD, UDD, UUD, UDU, DUU, UUU}\}$
- **Access web page:** $\Omega = (0, \infty)$
- **Roll a die:** $\Omega = \{1, 2, 3, 4, 5, 6\}$
- **Toss coin until head:** $\Omega = \{H, TH, TTH, TTTH, \dots\}$

$|\Omega| = \# \text{ of outcomes in sample space.}$

For the network routers example, $|\Omega| = 8$

Note $|A| = \text{"cardinality"} \text{ of } A$
 $= \# \text{ of elements in set } A$

$|\Omega| = \# \text{ of outcomes in sample space}$
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Types of sample space

Sample space can be ...

- finite → discrete
- countable infinite → discrete
- uncountable infinite → not discrete

Example 4: Discrete/not discrete sample spaces

- Network Routers:

$$\Omega = \{\text{DDD}, \text{DDU}, \text{DUD}, \text{UDD}, \text{UUD}, \text{UDU}, \text{DUU}, \text{UUU}\}$$

→ discrete

- Access web page: $\Omega = (0, \infty)$ → not discrete

- Roll a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$ → discrete

- Toss coin until head: $\Omega = \{H, TH, TTH, TTTH, \dots\}$

→ discrete

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Event

Definition subset of sample space

An event (A, B, C , etc) is a collections of outcomes from the sample space that we are interested in. $A \subseteq \Omega$

Example 5: Events

- Network Routers: A message is transmitted successfully if at least one router is up and recipient computer is up.

$$A = \text{successful transmission} = \{\text{DUU}, \text{UDU}, \text{UUU}\}$$

- Access web page: $B = \text{More than 10 seconds} = (10, \infty)$

- Roll a die: $C = \text{Rolling even \#} = \{2, 4, 6\}$

- Toss coin until head: $D = \text{getting 1}^{\text{st}} \text{ head in 2-5 tosses}$
 $= \{TH, TTH, TTTH, TTTTH\}$

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Probability

Probability

- Let A be an event (set of outcomes from the sample space)
- Then, probability of event A is written as $P(A)$

Example 6:

- Consider event C as successful transmission in the Network Router example
- Suppose the chance that a message is successfully transmitted is 90%
- $P(C) = 0.90$

To calculate probability of events, start with understanding set theory

Set Theory

Set Notation

Review symbols $\in, \notin, \subset, \supset, \subseteq, \supseteq$.

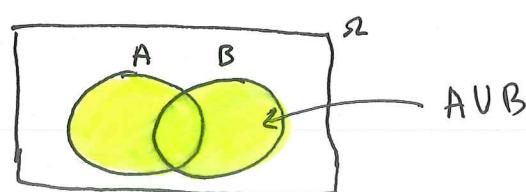
- If x is an element of set B , this is denoted $x \in B$ "x in B"
- If y is **not** an element of set B , this is denoted $y \notin B$ "y not in B"
- If every element of set A is also an element of set B , then A is a subset of B . $A \subseteq B$

Let A and B be two events ...

"or"

- **Union** (\cup): $A \cup B$ is the event consisting of all outcomes in A or in B or in both.

$$A \cup B = \{\omega \mid \omega \in A \text{ or } \omega \in B\}$$

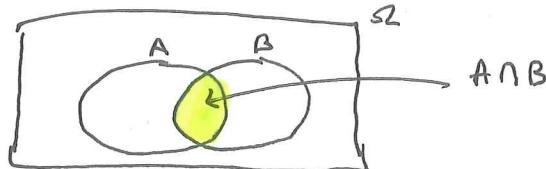


Set Notation

"and"

- **Intersection (\cap)**: $A \cap B$ is the event consisting of all outcomes simultaneously in A and in B.

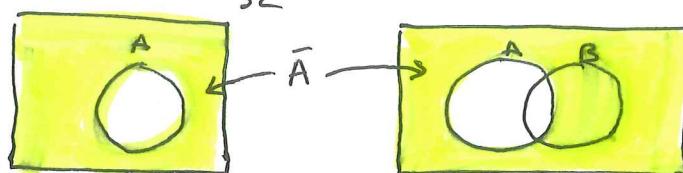
$$A \cap B = \{\omega \mid \omega \in A \text{ and } \omega \in B\}$$



- **Complement (\bar{A})**: The complement of an event A (\bar{A}) is the event consisting of all outcomes not in A.

$$\bar{A} = \{\omega \mid \omega \notin A\}$$

$$\bar{A} = A^c$$



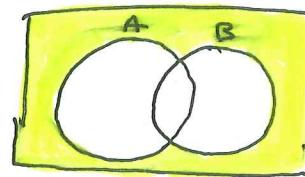
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Set Notation

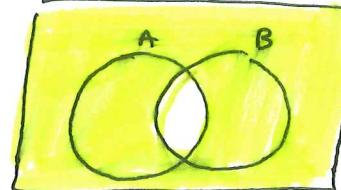
* • **De Morgan's laws: (Important)**

$$(\overline{A \cup B}) = \bar{A} \cap \bar{B} \quad (\overline{A \cap B}) = \bar{A} \cup \bar{B}$$

$$(1) \quad \overline{A \cup B} = \bar{A} \cap \bar{B}$$



$$(2) \quad \overline{A \cap B} = \bar{A} \cup \bar{B}$$

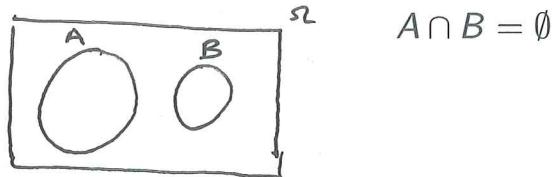


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Set Notation

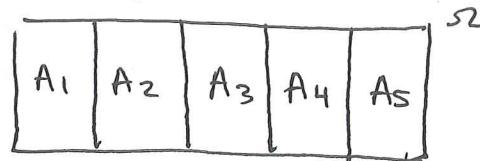
- **Empty set (\emptyset)**: \emptyset is a set containing no elements, usually denoted by $\{\}$. The empty set is a subset of every set:
"null set" $\emptyset \subseteq A$

- **Disjoint/mutually exclusive sets**: Sets A, B are disjoint if their intersection is empty:



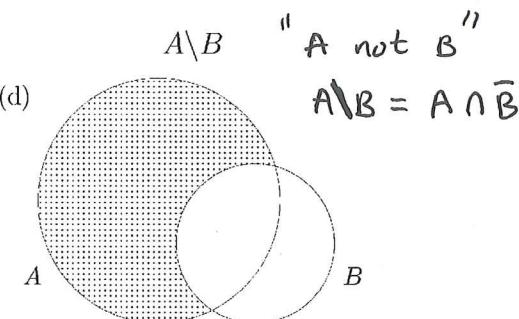
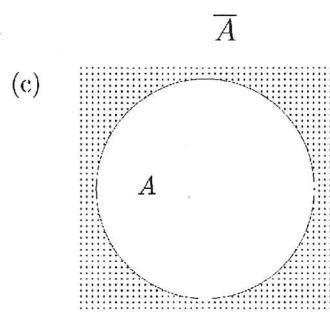
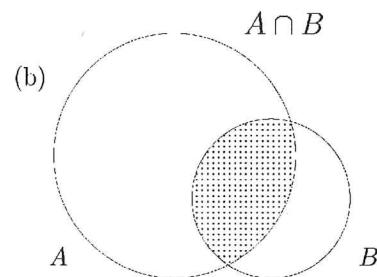
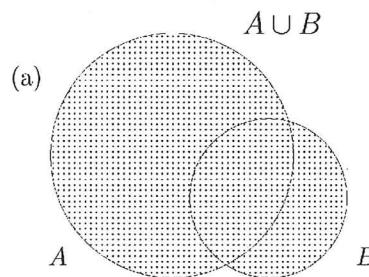
$$A \cap B = \emptyset$$

- **Pairwise disjoint sets**: Sets A_1, A_2, A_3, \dots are pairwise disjoint if $A_i \cap A_j = \emptyset$ for any $i \neq j$



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Venn Diagrams



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Set Notation Cont.

Example 7:

$$\Omega = \{1, 2, 3, 4, 5\}$$

$$A = \{1, 2, 3\}$$

$$B = \{2, 3, 4\}$$

$$C = \{4, 5\}$$

$$1. \bar{A} = \{4, 5\}$$

$$2. A \cup B = \{1, 2, 3, 4\}$$

$$3. A \cap B = \{2, 3\}$$

$$4. A \cap C = \emptyset$$

5. Are A and B disjoint? no , b/c $A \cap B \neq \emptyset$

6. Are A and C disjoint? yes , b/c $A \cap C = \emptyset$

7. Are A,B,C pairwise disjoint? no

at least one pair has elements in common.

Kolmogorov's Axioms

Kolmogorov's Axioms

- Recall: $P(A)$ is the probability that event A occurs
- Want to assign probabilities to events as a measure of their likelihood of occurring
- A *probability model* is an assignment of numbers $P(A)$ to events $A \subseteq \Omega$ such that *Kolmogorov's axioms* are satisfied.

Kolmogorov's Axioms

1. $0 \leq P(A) \leq 1$ for all A
2. $P(\Omega) = 1$
3. If A_1, A_2, A_3, \dots are pairwise disjoint, then

$$P(A_1 \cup A_2 \cup \dots) = P(A_1) + P(A_2) + \dots = \sum_i P(A_i)$$

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Kolmogorov's Axioms Cont.

Kolmogorov's axioms...

- Give the logical framework that probability assignment must follow
- But don't tell us what probabilities to assign to events

Example 8: Draw a single card from a standard deck of playing cards: $\Omega = \{\text{red}, \text{black}\}$

Two different probability models are:

Model 1	Model 2
$P(\Omega) = 1$	$P(\Omega) = 1$
$P(\text{red}) = 0.5$	$P(\text{red}) = 0.3$
$P(\text{black}) = 0.5$	$P(\text{black}) = 0.7$

Both are *valid* probability models. However, real world experience tells us model 1 is more accurate for the scenario.

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Consequences of Kolmogorov's Axioms

Let $A, B \subseteq \Omega$.

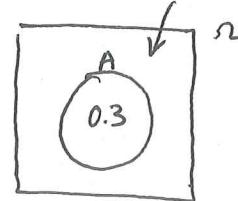
$$P(A^c) = P(\bar{A}) = 0.7$$

A. Probability of the Complementary Event:

$$P(\bar{A}) = 1 - P(A)$$

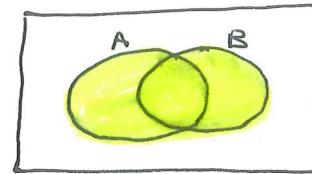
Corollary: $P(\emptyset) = 0$

$$\bar{\Omega} = \emptyset \quad P(\emptyset) = 1 - P(\Omega) = 1 - 1 = 0$$



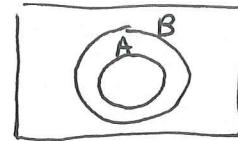
B. Addition Rule of Probability

★ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



C. If $A \subseteq B$, then $P(A) \leq P(B)$.

Corollary: For any A , $P(A) \leq 1$.



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Using Kolmogorov's axioms

There are 2 main approaches to assign probabilities to events at this point.

1. When we know events are disjoint (easy!).

- Let A be a collection of k outcomes $(\omega_1, \dots, \omega_k)$ that are all *pairwise disjoint*.
- Use Kolmogorov's axiom 3: $P(A) = P(\bigcup_{i=1}^k \omega_i) = \sum_{i=1}^k P(\omega_i)$.

Example 9: Roll a die. Suppose event A is rolling an even number.

(Assume all numbers are equally likely $\rightarrow P(\omega) = \frac{1}{6}$ for all ω)

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$A = \{2, 4, 6\}$$

$$\begin{aligned} P(A) &= P(\text{"2" or "4" or "6"}) \\ &= P(\text{"2"} \cup \text{"4"} \cup \text{"6"}) \\ &= P(2) + P(4) + P(6) \\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \\ &= \frac{3}{6} = \frac{1}{2} \end{aligned}$$

Note "2", "4", "6" are pairwise disjoint so probability of union is sum of individual probabilities

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Using Kolmogorov's axioms cont.

2. When events may or may not be disjoint (harder).

- Start with known probability of some of the events.
- Use this information and Kolmogorov's axioms to deduce probabilities of other events.
- Drawing Venn diagrams will simplify the problem.
(or using table)

Example 10: Suppose in a small college of 1000 students, 650 students own Iphones, 400 students own MacBooks, and 300 students own both.

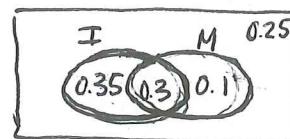
Define events: I = "owns Iphone", and M = "owns MacBook".

Known

$$P(I) = 0.65$$

$$P(M) = 0.40$$

$$P(I \cap M) = 0.30$$



	I	\bar{I}	
M	0.3	0.1	0.4
\bar{M}	0.35	0.25	0.6
0.65	0.35	1	

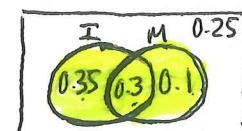
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Using Kolmogorov's axioms cont.

union

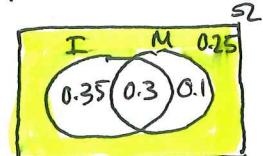
a. What is the probability of owning an Iphone or a MacBook?

$$\begin{aligned} P(I \cup M) &= P(I) + P(M) - P(I \cap M) \\ &= 0.65 + 0.4 - 0.3 \\ &= 0.75 \end{aligned}$$



b. What is the probability of owning neither an Iphone nor a MacBook?

$$\begin{aligned} P(\bar{I} \cap \bar{M}) &\rightarrow \text{de Morgan's Law} \\ &= P(\bar{I} \cup \bar{M}) \\ &= 1 - P(I \cup M) \rightarrow \text{complement} \\ &= 1 - 0.75 \\ &= 0.25 \end{aligned}$$



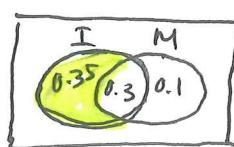
	I	\bar{I}	
M	0.3	0.1	0.4
\bar{M}	0.35	0.25	0.6
0.65	0.35	21 / 22	

$$P(I) = P(I \cap M) + P(I \cap \bar{M})$$

$$\Rightarrow P(I) - P(I \cap M) = P(I \cap \bar{M})$$

- c. What is the probability of owning only an Iphone? (ie. owning an iPhone and no MacBook)

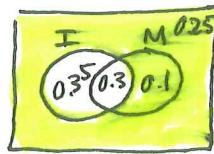
$$\begin{aligned} P(I \setminus M) &= P(I \cap \bar{M}) \\ &= P(I) - P(I \cap M) \\ &= 0.65 - 0.3 \\ &= 0.35 \end{aligned}$$



	I	\bar{I}	
M	0.3	0.1	0.4
\bar{M}	0.35	0.25	0.6
	0.65	0.35	

- d. What is the probability of not owning an iPhone?

$$\begin{aligned} P(\bar{I}) &= 1 - P(I) \\ &= 1 - 0.65 \\ &= 0.35 \end{aligned}$$



	I	\bar{I}	
M	0.3	0.1	0.4
\bar{M}	0.35	0.25	0.6
	0.65	0.35	

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Table Method (useful when dealing w/ 2 sets)

Suppose you're given $P(A) = 0.7$ $P(B) = 0.5$ $P(\bar{A} \cap \bar{B}) = 0.1$

	A	\bar{A}	
B			0.5
\bar{B}			0.1
	0.7		

$\nwarrow P(B)$

$\nearrow P(A)$

	A	\bar{A}	
B	0.3	0.2	0.5
\bar{B}	0.4	0.1	0.5
	0.7	0.3	1

$$P(A \cap B) = 0.3$$

$$P(A \cap \bar{B}) = 0.4$$

$$P(\bar{A} \cap B) = 0.2$$

$$P(\bar{A} \cap \bar{B}) = 0.1$$

$$P(A) = 0.7$$

$$P(\bar{A}) = 0.3$$

$$P(B) = 0.5$$

$$P(\bar{B}) = 0.5$$

- Need to know one value for each margin, and at least one cell value
- Then you can deduce all other cell values (intersections)

- cells add to margin
- margin adds to 1
- all cells add to 1

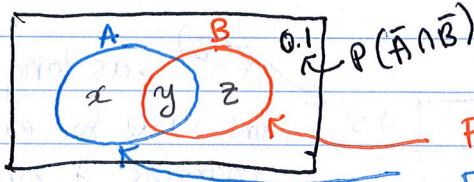
How to obtain $P(A \text{ and } B)$ using Venn Diagram & system of equations
Useful when dealing with more than 2 sets

What if we were given $P(A)$, $P(B)$, $P(\bar{A} \cap \bar{B})$

Given
$P(A) = 0.7$
$P(B) = 0.5$
$P(\bar{A} \cap \bar{B}) = 0.1$

$$P(\bar{A} \cap \bar{B}) = P(\bar{A} \cup \bar{B})$$

↑
probability of neither
A nor B
de morgan's law



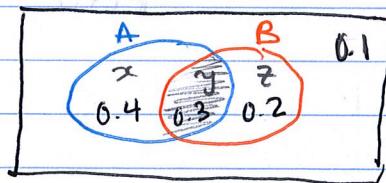
$$P(B) = y + z = 0.5$$

$$P(A) = x + y = 0.7$$

What is $P(A \cap B)$?
Can solve using system of Eq's

$$\begin{aligned} x + y + z &= 0.9 \\ x + y &\stackrel{?}{=} 0.7 \end{aligned}$$

$$\begin{aligned} 0.7 + z &= 0.9 \\ z &= 0.2 \end{aligned}$$



What is $P(A \cap B)$

$$P(B) = y + z = 0.5$$

$$y + 0.2 = 0.5$$

$$y = 0.3$$

$$P(A \cap B) = 0.3$$

$$P(A) = x + y = 0.7$$

$$x + 0.3 = 0.7$$

$$x = 0.4$$