

Lecture 13

Normal Distribution

STAT 330 - Iowa State University

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Normal Distribution

Normal Distribution

Setup: The normal distribution is commonly used to model a wide variety of variables (weight, height, temperature, voltage, etc) due to its “bell-shaped” and *symmetric* shape.

If a random variable X follows a *normal distribution*,

$$X \sim N(\mu, \sigma^2)$$

σ^2 “sigma-squared”
 μ “mu”

where μ is the mean, and $\sigma^2 > 0$ is the variance

- Probability Density Function (pdf)

We won't
use this
directly



$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for } -\infty < x < \infty$$

- Expected Value: $E(X) = \mu$
- Variance: $Var(X) = \sigma^2$

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Normal PDF

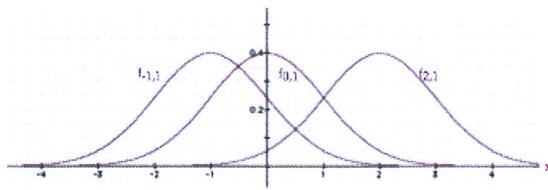
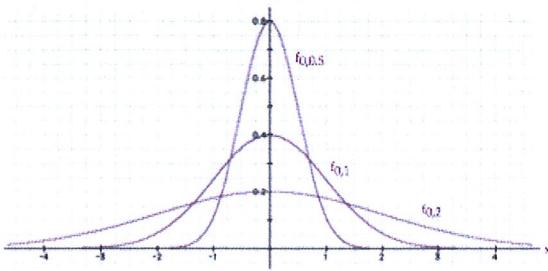


Figure 1: PDFs for normal distribution with various μ and σ^2

μ determines the location of the peak in the x -axis,

σ^2 determines the “width” of the bell shape.

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Normal CDF

- Cumulative Distribution Function (cdf)

$$F_X(t) = \int_{-\infty}^t f(x)dx \quad (\text{no closed form})$$

- Use cdf table (z -table) of *standard normal distribution* (z -table) $N(\mu = 0, \sigma^2 = 1)$ to obtain probabilities.
- Need to *standardize* any normal random variable, X , into standard normal random variable, Z .

Standardization of Normal Distribution

Let $X \sim N(\mu, \sigma^2)$. Then,

Standard deviation

$$\sigma = \sqrt{\sigma^2}$$

1. $Z = \frac{X-\mu}{\sigma}$ is a standard normal random variable

2. $Z \sim N(0, 1)$ (normal distribution with $\mu = 0, \sigma^2 = 1$)

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Standardization

Example 1: Suppose $X \sim N(20, 100)$. What is the probability that X is less than 23.5? $P(X < 23.5) = ?$

To find this probability, we usually ...

- Integrate the PDF (too difficult)
- Plug into CDF (impossible - no closed form for CDF of X)

Instead we standardize X , and obtain probabilities using the standard normal cdf table (z -table)

Normal

- The standardized R.V is $Z = \frac{X-\mu}{\sigma} = \frac{X-20}{\sqrt{100}} \sim N(0, 1)$
- The standardized observation is $z = \frac{x-\mu}{\sigma} = \frac{23.5-20}{\sqrt{100}} = 0.35$
- $P(X < 23.5) = P(Z < 0.35)$ (obtain this from z -table)

$$P(X < 23.5) = P\left(\frac{X-\mu}{\sigma} < \frac{23.5-\mu}{\sigma}\right) = P\left(\frac{X-20}{10} < \frac{23.5-20}{10}\right) = P(Z < 0.35)$$

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Standard Normal Distribution

Standard Normal Distribution

Suppose a random variable, X , follows a $N(\mu, \sigma^2)$ distribution.

Then, $Z = \frac{X - \mu}{\sigma}$ follows a **standard normal distribution**

$$Z = \frac{X - \mu}{\sigma} \quad Z \sim N(0, 1)$$

- Probability Density Function (pdf)

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for } -\infty < z < \infty$$

- Expected Value:

$$E(Z) = E\left(\frac{X - \mu}{\sigma}\right) = \frac{E(X) - \mu}{\sigma} = \frac{\mu - \mu}{\sigma} = 0$$

- Variance:

$$\text{Var}(Z) = \text{Var}\left(\frac{X - \mu}{\sigma}\right) = \text{Var}\left(\frac{X}{\sigma}\right) = \frac{1}{\sigma^2} \text{Var}(X) = \frac{\sigma^2}{\sigma^2} = 1$$

Standard Normal CDF

- Cumulative Distribution Function (cdf)

$$F_Z(t) = \int_{-\infty}^t f(z) dz = \Phi(t) \quad (\text{no closed form})$$

→ Just like the normal cdf, the standard normal cdf does not have a closed form expression.

→ The cdf of $N(0,1)$ random variable is denoted by $\Phi(t)$ (or more commonly $\Phi(z)$)

→ The values of the cdf, $\Phi(z)$, are found in the standard normal table (*z-table*)

$$\begin{aligned} F_Z(t) \\ = P(Z \leq t) \end{aligned}$$

Since the
standard
normal
distribution

is so
common,

its CDF
has its own

symbol $\Phi(z)$

instead of $F_Z(z)$

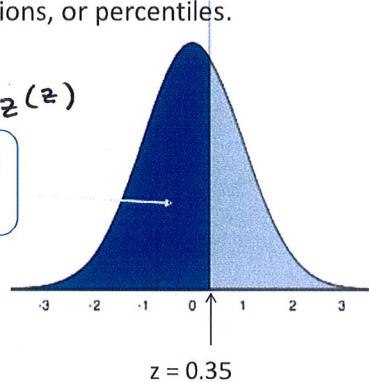
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Z-table (Standard Normal Table)

Z-table

- Z-Table gives proportion of normal curve less than a particular z score
 - Gives **left-hand area** (dark blue shaded region)
 - This is same as the **percentile** value for z
 - Can be referred to as areas, proportions, or percentiles.
 - Denoted $P(Z < z) = P(Z \leq z)$
or $\Phi(z)$ or $F_z(z)$

Proportion of area less than $z=0.35$
Denoted as “ $P(Z < 0.35)$ ”



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How to read the Z-table

- z values range from –3.99 to 3.99 on the z-table
- Row – ones and tenths place for z
- Column – hundredths place for z
- $P(Z < z)$ found inside z-table

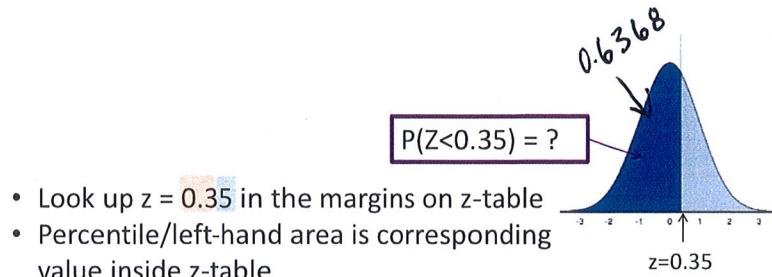
z	Second decimal place in z						
	0.00	0.01	0.02	0.03	0.04	0.05	0.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239
0.1	0.5398	0.5438	0.5478	0.5518	0.5557	0.5596	0.5636
0.2	0.5793	0.5832	0.5871	0.5910	0.5949	0.5987	0.6026
0.3	0.6179	0.6218	0.6257	0.6296	0.6331	0.6368	0.6406
0.4	0.6554	0.6592	0.6630	0.6664	0.6700	0.6736	0.6772

Left-hand area/probability
 $P(Z < z)$

or $P(Z \leq z)$

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How to read the Z-table



- Look up $z = 0.35$ in the margins on z-table
- Percentile/left-hand area is corresponding value inside z-table

z	Second decimal place in z						
	0.00	0.01	0.02	0.03	0.04	0.05	0.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772

$$P(Z < 0.35)$$

$$= P(Z \leq 0.35)$$

$$= 0.6368$$

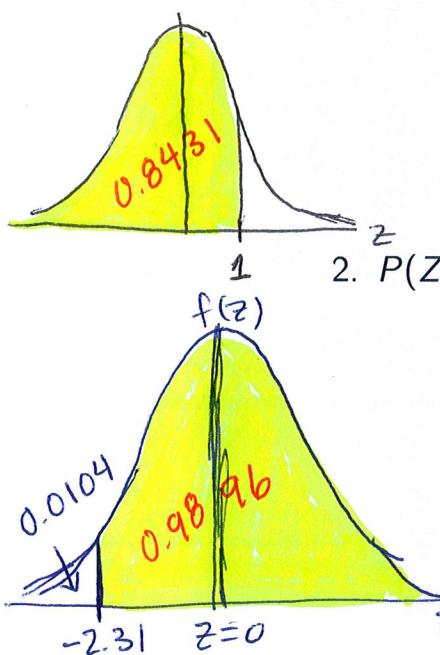
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Examples

Z-table Practice

Suppose $Z \sim N(0, 1)$

$$1. P(Z < 1) = P(Z < 1.00) = \underline{\Phi}(1.00) = 0.8431$$



$$\begin{aligned} 2. P(Z > -2.31) &= 1 - P(Z \leq -2.31) \\ &= 1 - \underline{\Phi}(-2.31) \\ &= 1 - 0.0104 \\ &= 0.9896 \end{aligned}$$

Right-hand area = $1 - \text{left-hand area}$

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Z-table Practice

Suppose $Z \sim N(0, 1)$

$$\begin{aligned} 3. P(0 < Z < 1) &= P(Z < 1) - P(Z \leq 0) \\ &= \underline{\Phi}(1) - \underline{\Phi}(0) \\ &= 0.8413 - 0.5 \\ &= 0.3413 \end{aligned}$$

left-hand area
Bigger area - smaller area

$$4. P(|Z| > 2) = P(Z < -2) + P(Z > 2)$$

$$\begin{aligned} &= 2P(Z < -2) \\ &= 2\underline{\Phi}(-2) \\ &= 2(0.0228) \\ &= 0.0456 \end{aligned}$$

Find the left-hand area and multiply it by 2.

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Normal Distribution Example

$$\mu = 1 \quad \sigma^2 = 2$$

Suppose $X \sim N(1, 2)$, and we want to find $P(1 < X < 2)$.

First, standardize X into std. normal R.V Z

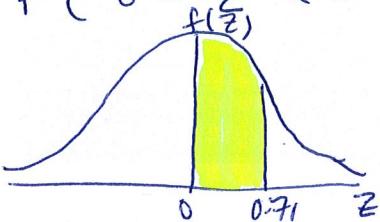
$$Z = \frac{X - \mu}{\sigma} = \frac{X - 1}{\sqrt{2}} \sim N(0, 1)$$

Standardize our observations

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{1 - 1}{\sqrt{2}} = 0$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{2 - 1}{\sqrt{2}} = 0.71 \quad \leftarrow \begin{array}{l} \text{round to 2 decimal} \\ \text{places} \\ \text{b/c that's} \\ \text{as my z-table} \\ \text{goes} \end{array}$$

$$\begin{aligned} P(0 < Z < 0.71) &= P(Z < 0.71) - P(Z \leq 0) \\ &= \Phi(0.71) - \Phi(0) \\ &= 0.7611 - 0.5 \\ &= 0.2611 \end{aligned}$$



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Normal Distribution Example

or do it all in one sweep

$$\begin{aligned} P(1 < X < 2) &= P\left(\frac{1-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{2-\mu}{\sigma}\right) \\ &= P\left(\frac{1-1}{\sqrt{2}} < Z < \frac{2-1}{\sqrt{2}}\right) \\ &= P(0 < Z < 0.71) \\ &= P(Z < 0.71) - P(Z \leq 0) \\ &= \Phi(0.71) - \Phi(0) \\ &= 0.7611 - 0.5 \\ &= 0.2611 \end{aligned}$$

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Example: Start w/ probability and work backwards to get observation.

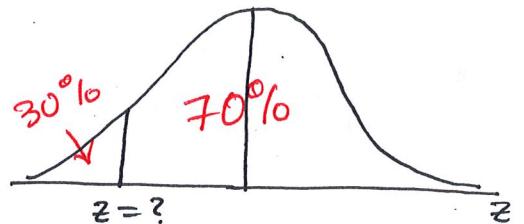
Suppose weight of apples follows a normal distribution w/ mean of 60g and variance of 25g.

70% of apples weigh more than what weight?

$x = \text{apple weight}$

$$x \sim N(60, 25)$$

If 70% of apples weigh more than some weight, then 30% weigh less than ~~some~~ weight.
this



The z-table gives CDF values (left-hand areas). Look up 0.30 (or closest value) inside z-table, and obtain corresponding z-value in margins.

Closest area to 0.30 in size z-table is 0.3015, which has corresponding z-value of $z = -0.52$.

$$\Rightarrow P(z \leq \underbrace{-0.52}_{z}) = 0.3015$$

We need to "reverse standardize" to go from z back to x (apple weight).

$$z = \frac{x - \mu}{\sigma} \quad \leftarrow \begin{matrix} \text{standardization} \\ \text{formula} \end{matrix}$$

$$\Rightarrow x = \mu + z\sigma \quad \leftarrow \begin{matrix} \text{"reverse" standardization} \\ \text{formula.} \end{matrix}$$

$$\Rightarrow x = 60 + (-0.52)(5) = \boxed{57.4 \text{ g}}$$

\Rightarrow 30% of apples weigh less than 57.4g.

\Rightarrow 70% of apples weigh more than 57.4g.