Review

Calculus

Integration

Differentiation

for polynomials x, x^2, etc

e

Lecture 9

Continuous Random Variables

STAT 330 - Iowa State University

Continuous Random Variables

Discrete vs. Continuous R.Vs

Discrete Random Variable

Sample space (Ω) maps to finite or countably infinite set in $\Re$

Ex: $\{1,2,3\}$, $\{1,2,3,4,\ldots\}$
(integers)

Continuous Random Variable

Sample space (Ω) maps to an uncountable set in $\Re$.

Ex: $(0,\infty)$, $(10,20)$
(real numbers)

- We have already learned about discrete R.Vs (Lectures 5-8)
- All properties of discrete R.Vs have direct counterparts for continuous R.Vs
- Summations (Σ) used for discrete R.V's are replaced by integrals ($\int$) for continuous R.V's.

Ex: Discrete R.V

$$EX = \sum_x x\, P_X(x)$$

Continuous R.V.

$$EX = \int x\, f(x)\, dx$$

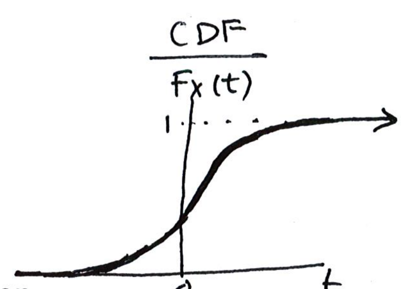
CDF of Continuous Random Variables

Definition

Let X be a continuous random variable. The *cumulative distribution function (cdf)* of X is

$$F_X(t) = P(X \le t)$$

- All cdf properties discussed earlier still hold
  1. $0 \le F_X(t) \le 1$
  2. F_X is non-decreasing (if $a \le b$, then $F_X(a) \le F_X(b)$.
  3. $\lim_{t\to-\infty} F_X(t) = 0$ and $\lim_{t\to\infty} F_X(t) = 1$
  4. F_X is right-continuous with respect to t



- The cdf for continuous R.V is also continuous (not a step function like in discrete case)

PDF $\longleftrightarrow$ CDF

Definition

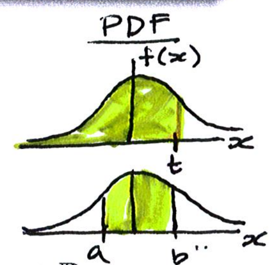
For a continuous variable X with cdf F_X, the *probability density function (pdf)* of Y is defined as:

$$f(x) = F_X'(x) = \frac{d}{dx} F_X(x)$$

Properties of pdf:

1. $f(x) \geq 0$ for all x,
2. $\int_{-\infty}^{\infty} f(x)dx = 1$.

Additionally, for continuous R.V X,



- $F_X(t) = P(X \leq t) = \int_{-\infty}^{t} f(x)dx$ for any $t \in \mathbb{R}$
- $P(a \leq X \leq b) = \int_a^b f(x)dx$ for any $a, b \in \mathbb{R}$
- $P(X = a) = P(a \leq X \leq a) = \int_a^a f(x)dx = 0$ for any $a \in \mathbb{R}$

the probability of X being a single value is 0

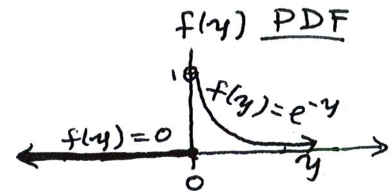
$\Rightarrow \quad P(X \leq a) = P(X < a)$

$P(X \leq t) = P(X < t)$

Examples

Continuous R.V Example

Example 1: Let Y be the time (in yrs) until the first major failure of a new disk drive. Suppose the probability density function (pdf) of ~~X~~ Y is given by



$$f(y) = \begin{cases} 0 & y \leq 0 \\ e^{-y} & y > 0 \end{cases}$$

1. Check whether $f(y)$ is a *valid* density function.

   We need to check the 2 properties of pdfs. ✓

   (1) $f(y)$ is non-negative function on $\Re$ ✓

   (2) $\int_{-\infty}^{\infty} f(y)dy = 1$ ✓

$$\int_{-\infty}^{\infty} f(y)dy = \cancel{\int_{-\infty}^{0} 0\,dy} + \int_{0}^{\infty} e^{-y}\,dy = (-1)e^{-y}\Big|_{0}^{\infty} = -e^{-y}\Big|_{0}^{\infty}$$

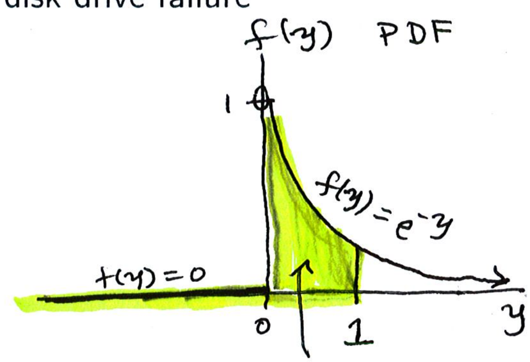
$$= (0) - (-e^{0})$$

$$= 0 + 1$$

$$= 1$$

Continuous R.V Example Cont.

2. What is the probability that the 1^{st} major disk drive failure occurs within the first year?



P(Y ≤ 1) = 0.63

$$P(Y \leq 1) = \int_{-\infty}^{1} f(y)\,dy$$

$$= \cancel{\int_{-\infty}^{0} 0\,dy} + \int_{0}^{1} e^{-y}\,dy$$

$$= -e^{-y}\Big|_{0}^{1}$$

$$= (-e^{-1}) - (-e^{-0})$$

$$= -e^{-1} + 1$$

$$= 1 - e^{-1} = 0.63$$

Continuous R.V Example Cont.

3. What is the probability that the 1^{st} major disk drive failure occurs before the first year?

$$P(Y < 1) = \int_{-\infty}^{1} f(y)\,dy$$

$$= \cancel{\int_{-\infty}^{0} 0\,dy}^{\,0} + \int_{0}^{1} e^{-y}\,dy$$

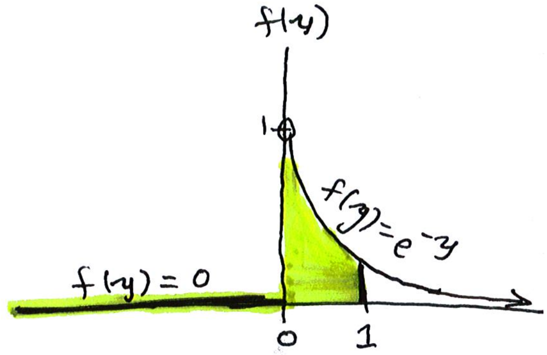
$$= -e^{-y}\Big|_0^1$$

$$= -e^{-1} - -e^{-0}$$

$$= 1 - e^{-1}$$

$$= 0.63$$

(same as $P(Y \leq 1)$ from previous slide)



For continuous dists,
$P(Y \leq a) = P(Y < a)$
b/c $P(Y = a) = 0$

Continuous R.V. Example Cont.

4. What is the probability that the 1^{st} major disk drive failure occurs after the first year?

$$P(Y > 1) = \int_{1}^{\infty} f(y)\,dy$$

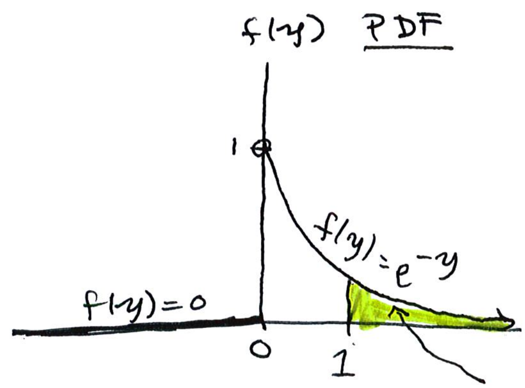
$$= \int_{1}^{\infty} e^{-y}\,dy$$

$$= -e^{-y}\Big|_1^{\infty}$$

$$= 0 - -e^{-1}$$

$$= e^{-1}$$

$$= 0.37$$



We can also get this $P(Y > 1)$ by $P(Y > 1) = 1 - P(Y \leq 1)$

$= 1 - 0.63$

$= 0.37$

Continuous R.V. Example Cont.

5. What is the probability that the 1^{st} major disk drive failure occurs after first year but before second year?

$$P(1 < Y < 2) = \int_1^2 f(y)\,dy$$

$$= \int_1^2 e^{-y}\,dy$$

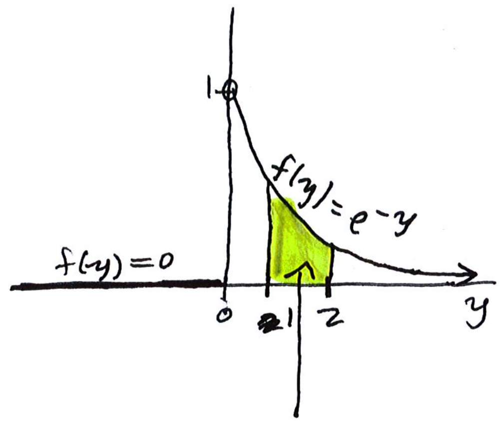
$$= -e^{-y}\Big|_1^2$$

$$= (-e^{-2}) - (-e^{-1})$$

$$= -e^{-2} + e^{-1}$$

$$= 0.23$$

$f(y)$ PDF



$P(1<Y<2)$
$= 0.23$

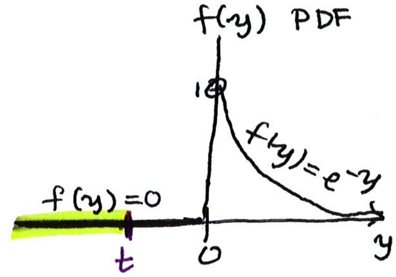
Continuous R.V. Example Cont.

6. What is the cumulative distribution function (cdf) of Y? $F_Y(t) = P(Y \le t)$

$f(y)$ PDF

① For $t \le 0$

$$F_Y(t) = P(Y \le t) = \int_{-\infty}^{t} f(y)\,dy = \int_{-\infty}^{t} 0\,dy = \boxed{0}$$

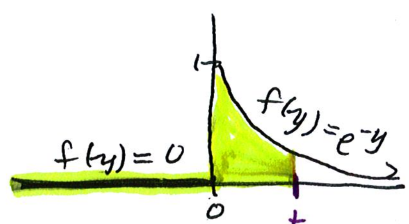


② For $t > 0$

$$F_Y(t) = P(Y \le t) = \int_{-\infty}^{t} f(y)\,dy$$

$$= \cancel{\int_{-\infty}^{0} 0\,dy} + \int_0^t e^{-y}\,dy$$

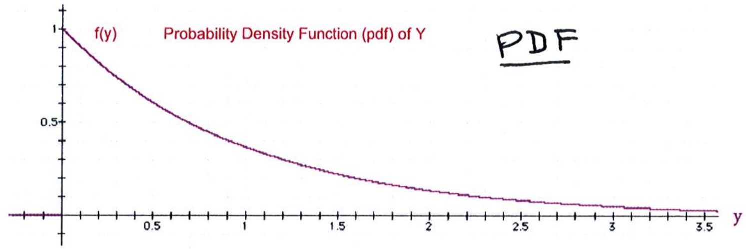
$$= -e^{-y}\Big|_0^t = -e^{-t} - -e^{0} = \boxed{1-e^{-t}}$$

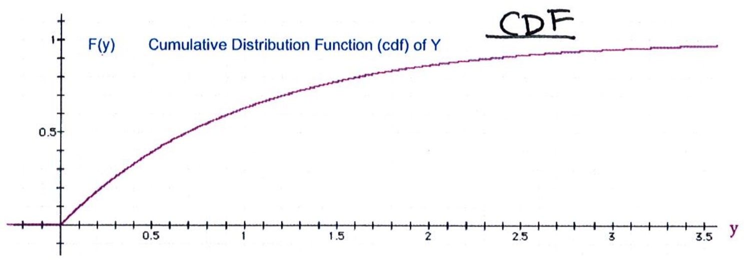


$$\boxed{\boxed{\text{CDF}} \quad F_Y(t) = \begin{cases} 0 & \text{for } t \le 0 \\ 1-e^{-t} & \text{for } t > 0 \end{cases}}$$

Continuous R.V Example Cont.

For Example 1, the pdf and cdf of Y are shown below.





Continuous R.V. Example Cont.

only need to integrate once to get CDF

SHORT CUT: Use the cdf to calculate desired probabilities instead of integrating the pdf for each problem.

- Only need to integrate the pdf once to obtain the cdf
- Write any probability in terms of the cdf and plug in to solve

Back to Example 1: CDF : $F_Y(t) = P(Y \le t) = \begin{cases} 1 - e^{-t} & \text{for } t > 0 \\ 0 & \text{for } t \le 0 \end{cases}$

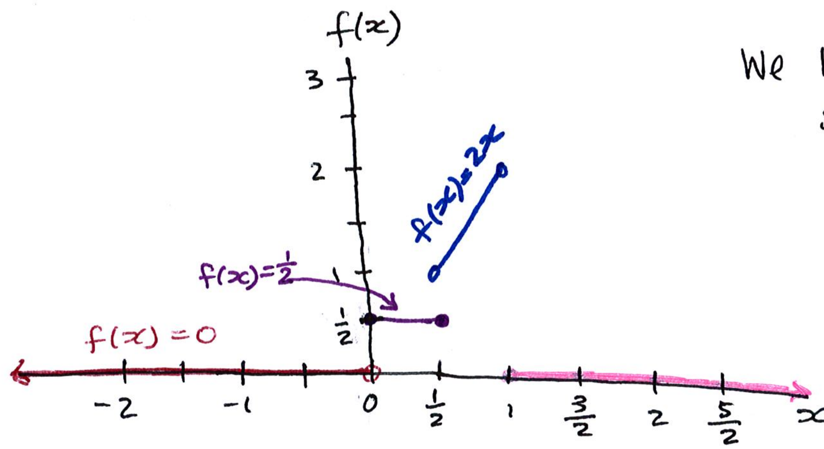
- $P(Y \le 1) = F_Y(1) = 1 - e^{-1} \approx 0.63$
- $P(Y > 1) = 1 - P(Y \le 1) = 1 - F_Y(1) = 1 - 0.63 = 0.37$
- $P(1 < Y < 2) = P(Y < 2) - P(Y \le 1)$
  $= F_Y(2) - F_Y(1)$
  $= (1 - e^{-2}) - (1 - e^{-1})$
  $= -e^{-2} + e^{-1} = 0.23$

Additional Example

Example 2: Let X be a random variable with the following probability density function (pmf):

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2} & \text{for } 0 \le x \le \frac{1}{2} \\ 2x & \text{for } \frac{1}{2} < x < 1 \\ 0 & \text{for } x \ge 1 \end{cases} \iff f(x) = \begin{cases} \frac{1}{2} & \text{for } 0 \le x \le \frac{1}{2} \\ 2x & \frac{1}{2} < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

can be written as



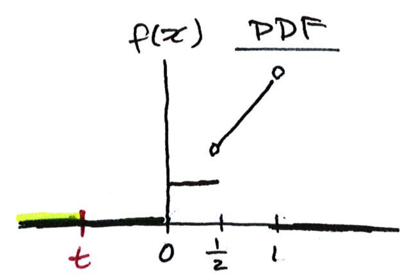
We have 4 sections to consider

Additional Example Cont.

1. Give the cumulative distribution function (cdf) of X (Find the area under the curve)

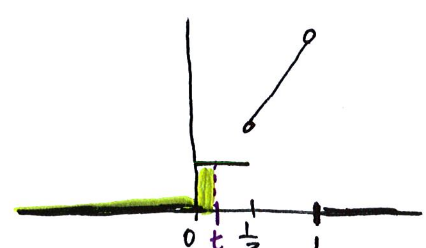
$$F_X(t) = P(X \le t) = \int_{-\infty}^{t} f(x)\,dx$$

Consider 4 sections seperately.



① If $t < 0$

$$F_X(t) = \int_{-\infty}^{t} f(x)\,dx = \int_{-\infty}^{t} 0\,dx = \boxed{0}$$



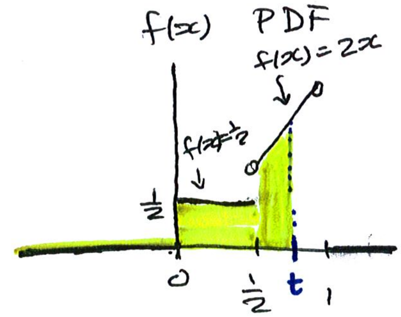
② If $0 \le t \le \frac{1}{2}$

$$F_X(t) = \int_{-\infty}^{t} f(x)\,dx = \cancel{\int_{-\infty}^{0} 0\,dx}^{\,0} + \int_{0}^{t} \frac{1}{2}\,dx$$

$$= \frac{x}{2}\Big|_{0}^{t}$$

$$= \frac{t}{2} - \frac{0}{2} = \boxed{\frac{t}{2}}$$

③ **If $\frac{1}{2} < t < 1$**



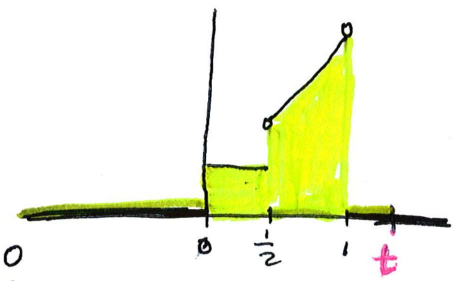
$$F_X(t) = \int_{-\infty}^{t} f(x)\,dx$$

$$= \cancel{\int_{-\infty}^{0} 0\,dx}^{\,0} + \int_0^{\frac{1}{2}} \frac{1}{2}\,dx + \int_{1/2}^{t} 2x\,dx$$

$$= \frac{x}{2}\Big|_0^{\frac{1}{2}} + x^2\Big|_{1/2}^{t}$$

$$= \left(\frac{1/2}{2} - \frac{0}{2}\right) + \left(t^2 - (1/2)^2\right) = \frac{1}{4} + t^2 - \frac{1}{4} \boxed{= t^2}$$

④ If $t \geq 1$



$$F_X(t) = \int_{-\infty}^{t} f(x)\,dx$$

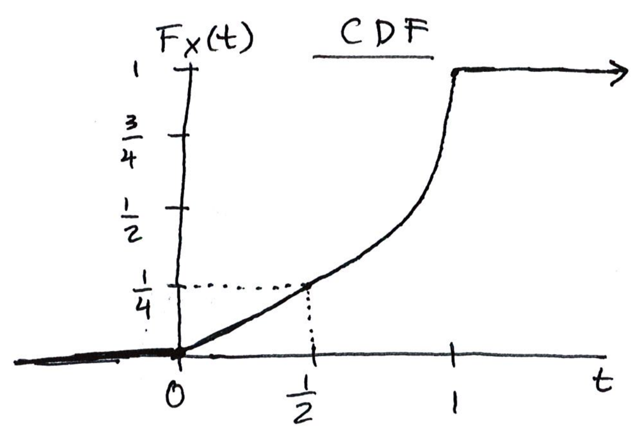
$$= \cancel{\int_{-\infty}^{0} 0\,dx}^{\,0} + \int_0^{1/2} \frac{1}{2}\,dx + \int_{1/2}^{1} 2x\,dx + \cancel{\int_1^{t} 0\,dx}^{\,0}$$

$$= \frac{x}{2}\Big|_0^{1/2} + x^2\Big|_{1/2}^{1}$$

$$= \left(\frac{(1/2)}{2} - \frac{0}{2}\right) + \left(1^2 - (1/2)^2\right) = \frac{1}{4} + 1 - \frac{1}{4} \boxed{= 1}$$

So my CDF of X is . . .

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ t/2 & \text{if } 0 \leq t \leq \frac{1}{2} \\ t^2 & \text{if } \frac{1}{2} < t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$



$$F_X\left(\tfrac{1}{2}\right) = \frac{1}{4}$$

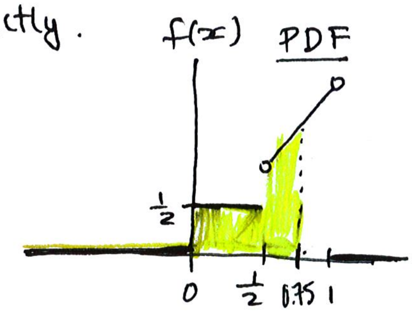
Additional Example Cont.

2. What is the probability that X is less that 0.75? $P(X < 0.75)$

We can plug in directy into CDF

$$P(X < 0.75) = F_X(0.75) = (0.75)^2 = 0.5625$$

OR We can integrate the PDF directly.



$$P(X < 0.75) = \int_{-\infty}^{0.75} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1/2} \frac{1}{2}\,dx + \int_{1/2}^{0.75} 2x$$

$$= \ldots$$

$$= 0.5625$$

Summary of Discrete & Continuous R.V.

Discrete R.V.

- Im(X) finite or countable infinite
- CDF: $F_X(t) = P(X \leq t)$ $= \sum_{x \leq t} p_X(x)$
- PMF: $p_X(x) = P(X = x)$
- $E(h(X)) = \sum_x h(x)p_X(x)$
- $E(X) = \sum_x x p_X(x)$
- $Var(X) = E(X^2) - [E(X)]^2$

Continuous R.V.

- Im(X) uncountable
- CDF: $F_X(t) = P(X \leq t)$ $= \int_{-\infty}^{t} f(x)dx$
- PDF: $f_X(x) = \frac{d}{dx} F_X(x)$
- $E(h(X)) = \int_x h(x)f(x)dx$
- $E(X) = \int_{-\infty}^{\infty} x f(x)dx$
- $Var(X) = E(X^2) - [E(X)]^2$