Lecture 6

Bernoulli and Binomial Distributions

STAT 330 - Iowa State University

Discrete Distributions

Common distributions for discrete random variables

- Bernoulli distribution

R.V "is distributed"

$$X \sim Bern(p)$$

parameter

- Binomial distribution

$$X \sim Bin(n, p)$$

parameters

- Geometric distribution

$$X \sim Geo(p)$$

- Poisson distribution

$$X \sim Pois(\lambda)$$

We will also discuss *joint distributions* for 2 or more discrete random variables

Bernoulli Distribution

Bernoulli Distribution

Bernoulli Experiment: Random experiment with only 2 outcomes:

- Success (S)
- Failure (F)

where P(Success) = P(S) = p for $p \in [0, 1]$

Example 1: (Bernoulli experiments):

1. Flip a coin: S = heads, F = tails
2. Watch stock prices: S = increas~~e~~, F = decrease
   stays same
3. Cancer screening: S = cancer, F = no cancer

Working with Bernoulli Random Variable

Suppose we have a Bernoulli experiment (only 2 outcomes: "success", "failure").

We obtain the outcome "success" with probability p

When random variable X follows a *Bernoulli Distribution*, we write

R.V

$$X \sim Bern(p)$$

parameter that determines my distribution

"is distributed"

- Define a random variable X

$$X = \begin{cases} 1 & \text{Success (S)} \\ 0 & \text{Failure (F)} \end{cases}$$

Bernoulli Random Variable Cont.

- Probability Mass Function (pmf)
  1. $Im(X) = \{\overset{F}{0}, \overset{S}{1}\}$
  2. $P(X = 1) = P(S) = p$
     $P(X = 0) = P(F) = 1 - p$

  The pmf can be written in tabular form:

	(F)	(S)
x	0	1
$p_X(x)$	$1 - p$	p

  The pmf can be written as a function:

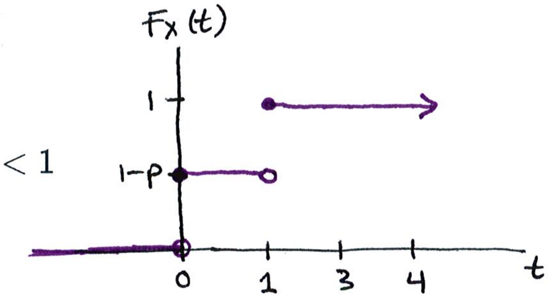
  $$p_X(x) = \begin{cases} p^x(1-p)^{1-x} & x \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases}$$

  Typically, we use the above functional form to describe the *probability mass function (pmf)* of Bernoulli random variable.

Bernoulli Random Variable Cont.

- Cumulative distribution function (cdf)

$$F_X(t) = P(X \le t) = \begin{cases} 0 & t < 0 \\ 1-p & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases}$$



- Expected Value: $E(X) = p$

$$E(X) = \sum_{x \in \{0,1\}} xP(X = x) = 0(1-p) + 1(p) = p$$

- Variance: $Var(X) = p(1-p)$

$$E(X^2) = \sum_{x=0,1} x^2 P(X=x) = (0)^2(1-p) + (1)^2(p) = p$$

$$E(X) = p$$

$$\underbrace{Var(X) = E(X^2) - (E(X))^2}_{\text{short cut formula for variance}} = p - p^2 = p(1-p)$$

Binomial Distribution

Binomial Distribution

Set up: Conduct multiple trials of *identical* and *independent* Bernoulli experiments

- Each trial is independent of the other trials
- $P(Success) = p$ for each trial

We are interested in the number of success after n trials. The random variable X is

$$X = \text{" \# of successes in } n \text{ trials"}$$

This random variable X follows a *Binomial Distribution*

$$X \sim Bin(n, p)$$ parameters that determine your distribution

where n is the number of trials, and p is the probability of success for each trial.

Binomial Distribution Cont.

Example 2: Flip a coin 10 times, and record the number of heads.

Success = "heads"; $P(\text{Success}) = p = 0.5$ (fair coin)

- Define the random variable X

$$X = \text{" \# of heads in } n = 10 \text{ trials"}$$

- The distribution of X is ...

$p = 0.5$

$$X \sim Bin(10, 0.5)$$

$n = 10$

Derivation of Binomial pmf

- Probability Mass Function (pmf)
  1. $Im(X) = \{0, 1, 2, 3, 4, \ldots, n\}$ = $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
  2. $P(X = x) = ?$

Recall $P(\text{Success}) = P(S) = p$, $P(\text{Failure}) = P(F) = 1 - p$

$\boxed{\text{Case: } X = 0}$ $\underline{F}\ \underline{F}\ \underline{F} \cdots \underline{F}$

$P(X = 0) = (1-p)^n$

$\boxed{\text{Case: } X = 1}$ $\longrightarrow$

$$\left\{\begin{array}{l} \underline{S}\ \underline{F}\ \underline{F}\ \underline{F} \ldots \underline{F} \\ \underline{F}\ \underline{S}\ \underline{F}\ \underline{F} \ldots \underline{F} \\ \underline{F}\ \underline{F}\ \underline{S}\ \underline{F} \ldots \underline{F} \\ \vdots \\ \underline{F}\ \underline{F}\ \underline{F}\ \underline{F} \ldots \underline{S} \end{array}\right\} \binom{n}{1} \text{ ways to have 1 success in } n \text{ trials}$$

$P(X = 1) = \binom{n}{1} p^1 (1-p)^{n-1}$

$\boxed{\text{Case: } X = 2}$

$P(X = 2) = \binom{n}{2} p^2 (1-p)^{n-2}$

Binomial Random Variables

In general, the *probability mass function (pmf)* of a Binomial R.V can be written as:

$$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1, 2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

- Cumulative distribution function (cdf)

$\lfloor t \rfloor$ = "floor" of t

$$F_X(t) = P(X \le t) = \sum_{x=0}^{\lfloor t \rfloor} \binom{n}{x} p^x (1-p)^{n-x}$$

$\lfloor 5 \rfloor = 5$
$\lfloor 4.99\bar{9} \rfloor = 4$
$\lfloor 4.43 \rfloor = 4$
$\lfloor 5.1 \rfloor = 5$

(Add up the pmfs to obtain the cdf)

- Expected Value: $E(X) = np$
- Variance: $Var(X) = np(1-p)$

} derivation using indep. & identical R.V.

(independent & identically distributed)

IID Random Variables

Properties of IID Random Variables

Independent and identically distributed (iid) random variables have properties that simplify calculations

Suppose $Y_1, \ldots, Y_n$ are iid random variables

- Since they are *identically* distributed,
  - $E(Y_1) = E(Y_2) = \ldots = E(Y_n)$
    $\rightarrow \underbrace{E(\sum Y_i) = \sum E(Y_i)}_{\text{always true}} \underbrace{= nE(Y_1)}_{\text{since } Y_i\text{'s identical}}$
  - $Var(Y_1) = Var(Y_2) = \ldots = Var(Y_n)$
- Since they are also *independent*,
  - $\rightarrow \underbrace{Var(\sum Y_i) = \sum Var(Y_i)}_{\text{since } Y_i\text{'s are independent, we don't have any covariance}} \underbrace{= nVar(Y_1)}_{\text{since } Y_i\text{'s are identical.}}$

$Var(Y_1 + Y_2) = Var(Y_1) + Var(Y_2)$

Working with IID Random Variables

A Binomial random variable, X, is the sum of n *independent and identically distributed (iid)* Bernoulli random variables, Y_i.

Let Y_i be a sequence of iid Bernoulli R.V. For $i = 1, \ldots, n$,

$$Y_i \overset{iid}{\sim} Bern(p)$$

with $E(Y_i) = p$ and $Var(Y_i) = p(1-p)$. Then,

" is distributed "

$$X = \sum_{i=1}^{n} Y_i \sim Bin(n, p)$$

Then, we obtain E(X) and Var(X) using properties of iid R.V.s

$$E(X) = nE(Y_1) = np$$
$$Var(X) = nVar(Y_1) = np(1-p)$$

Examples

Binomial Distribution Examples

Example 3: A box contains 15 components that each have a defective rate of 5%. What is the probability that ...

1. exactly 2 out of 15 components are defective?
2. at most 2 components are defective?
3. more than 3 components are defective?
4. more than 1 but less than 4 components are defective?

How to approach solving these types of problems?

1. Define the random variable
2. Determine the R.V's distribution (and values for the parameters)
3. Use appropriate pmf/cdf/E(X)/Var(X) formulas to solve

Binomial Distribution Examples Cont.

Define the R.V: $X = \#$ defective out of $n = 15$ components
State the Distribution of X: $X \sim Bin(\underset{n=15}{15}, \underset{p=0.05}{0.05})$
$n = 15,\ p = 0.05$

1. What is the probability that exactly 2 out of 15 components are defective?

$$P(X=2) = p_X(2) = \binom{15}{2}(0.05)^2(0.95)^{15-2}$$
$$= \binom{15}{2}(0.05)^2(0.95)^{13}$$
$$= \frac{15!}{13!\,2!}(0.05)^2(0.95)^{13}$$
$$= (105)(0.05)^2(0.95)^{13}$$
$$= 0.1348$$

Binomial Distribution Examples Cont.

2. What is the probability that at most 2 components are defective?

(using PMF) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= P_X(0) + P_X(1) + P_X(2)$

$= \binom{15}{0}(0.05)^0(0.95)^{15}$

$+ \binom{15}{1}(0.05)^1(0.95)^{15-1}$

$+ \binom{15}{2}(0.05)^2(0.95)^{15-2}$

$= 0.9638$

(using CDF) $P(X \leq 2) = F_X(2) = 0.9638$ (using Appendix A Binomial CDF table)

How to Use Binomial CDF Table (Appendix A)

Suppose we have random variable $X \sim Bin(n = 15, p = 0.05)$.

$P(X \leq 2) = ?$

- Find the $n = 15$ sub-table
- $P(X \leq 2)$ is found inside the table corresponding to $p = 0.05$ (column) and $x = 2$ (row).

$\boxed{P(X \leq 2) = 0.9637998}$

n=15	p=0.01	0.05	0.1	0.15	1/6
x=0	0.8600584	0.4632912	0.2058911	0.08735422	0.06490547
1	0.9903702	0.8290475	0.5490430	0.31858598	0.25962189
2	0.9995842	$\boxed{0.9637998}$	0.8159389	0.60422520	0.53222487
3	0.9999875	0.9945327	0.9444444	0.82265520	0.76848078
4	0.9999997	0.9993853	0.9872795	0.93829461	0.91023433
5	1.0000000	0.9999472	0.9977503	0.98318991	0.97260589
6	1.0000000	0.9999965	0.9996894	0.99639441	0.99339642
7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Binomial Distribution Examples Cont.

3. What is the probability that more than 3 components are defective? $P(X>3) = ? \quad = 1 - P(X \le 3)$

using PMF

$$P(X \le 3) = P_X(0) + P_X(1) + P_X(2) + P_X(3)$$
$$= \binom{15}{0}(0.05)^0(0.95)^{15} + \binom{15}{1}(0.05)^1(0.95)^{14} + \binom{15}{2}(0.05)^2(0.95)^{13} + \binom{15}{3}(0.05)^3(0.95)^{12}$$
$$= 0.9945$$
$$\rightarrow P(X>3) = 1 - P(X \le 3) = 1 - 0.9945 = 0.0055$$

using CDF

$$P(X \le 3) = F_X(3) = 0.9945$$ (Appendix A Binom. Table)

$$\rightarrow P(X>3) = 1 - P(X \le 3) = 1 - 0.9945 = 0.0055$$

Binomial Distribution Examples Cont.

4. What is the probability that more than 1 but less than 4 components are defective?

using PMF

$$P(1 < X < 4) = P(X=2) + P(X=3)$$
$$= P_X(2) + P_X(3)$$
$$= \binom{15}{2} 0.05^2\, 0.95^{13} + \binom{15}{3} 0.05^3\, 0.95^{12}$$
$$= 0.1655$$

using CDF

$$P(1 < X < 4) = P(X<4) - P(X \le 1)$$
$$= P(X \le 3) - P(X \le 1)$$
$$= F_X(3) - F_X(1)$$
$$= 0.9945 - 0.8290$$
$$= 0.1655$$

To find CDF, we need probabilities with "$\le$" sign

CDF table only given $P(X \le t)$