Lecture 8

Joint PMF

STAT 330 - Iowa State University

Joint PMF

Joint Probability Mass Function

Motivation:

- Often, real problems deal with more than 1 variable
- Not sufficient to model the variables separately
- Need to consider their *joint* behavior

Definition

For two discrete variables X and Y, the *joint probability mass function (pmf)* is defined as:

"and"

$$p_{X,Y}(x, y) \equiv P(X = x, Y = y) = P(\{X = x\} \cap \{Y = y\})$$

Joint PMF Example

Example 1:

A box contains 5 unmarked processors of different speeds:

speed (mHz)	400	450	500
count	2	1	2

X = speed of the first selected processor

Y = speed of the second selected processor

The *(joint) probability table* below gives the probabilities for each processor combination:

		2nd processor (Y)		
	mHz	400	450	500
	400	0.1	0.1	0.2
1st proc. (X)	450	0.1	0.0	0.1
	500	0.2	0.1	0.1

Joint PMF Example Cont.

1. What is the probability that $X = Y$?

	mHz	2nd processor (Y) 400	450	500
	400	0.1	0.1	0.2
1st proc. (X)	450	0.1	0.0	0.1
	500	0.2	0.1	0.1

$P(X = Y)$

$= p_{X,Y}(400, 400) + p_{X,Y}(450, 450) + p_{X,Y}(500, 500)$

$= 0.1 + 0.0 + 0.1$

$= 0.2$

Joint PMF Example Cont.

2. What is the probability that $X > Y$?

	mHz	2nd processor (Y) 400	450	500
	400	0.1	0.1	0.2
1st proc. (X)	450	0.1	0.0	0.1
	500	0.2	0.1	0.1

In other words, what is the probability that 1^{st} processor has higher speed than 2^{nd} processor?

$P(X > Y)$

$= p_{X,Y}(450, 400) + p_{X,Y}(500, 400) + p_{X,Y}(500, 450)$

$= 0.1 + 0.2 + 0.1$

$= 0.4$

Marginal PMF

Marginal Probability Mass Function

Obtain the *marginal pmf* from the *margins* of the probability table.

- sum up the cells row-wise or column-wise.

Definition

The *marginal probability mass functions* $p_X(x)$ and $p_Y(y)$ can be obtained from the joint pmf $p_{X,Y}(x, y)$ by

$$\left.\begin{aligned} p_X(x) &= \sum_y p_{X,Y}(x, y) \\ p_Y(y) &= \sum_x p_{X,Y}(x, y) \end{aligned}\right\}$$

Due to Law of Total Probability.

Marginal PMF Cont.

	mHz	2nd processor (Y) 400	450	500	$p_X(x)$
1st proc. (X)	400	0.1	0.1	0.2	0.4
	450	0.1	0.0	0.1	0.2
	500	0.2	0.1	0.1	0.4
	$p_Y(y)$	0.4	0.2	0.4	1

marginal PMF of X

marginal PMF of Y

Thus, the marginal pmf are ...

x	400	450	500
$p_X(x)$	0.4	0.2	0.4

y	400	450	500
$p_Y(y)$	0.4	0.2	0.4

Expectation

Expected Value

Definition

The *expected value* of a function of several variables is

a function

$$E[h(X, Y)] \equiv \sum_{x,y} h(x, y) p_{X,Y}(x, y)$$

- The **MOST IMPORTANT** application of this will be for calculating covariance (next slide).
- To calculate the covariance, we will need $E(XY)$.

Take $h(X, Y) = X \cdot Y$, and plug in into expected value formula

$$E(XY) = \sum_{x,y} xy p_{X,Y}(x, y)$$

Covariance

Covariance

For two variables, we can measure how "similar" their values are using *covariance* and *correlation*.

Definition measure of the joint variability of 2 R.Vs

The *covariance* of 2 random variables X, Y is given by

$$Cov(X, Y) = E[(X - E(X))(Y - E(Y))]$$

- This definition is similar to Var(X).
- In fact, $Cov(X, X) = Var(X)$
- In practice, use **SHORT CUT** formula to obtain covariance:

Recall
$Var(X) = E(X^2) - (EX)^2$

$$Cov(X, Y) = \underbrace{E(XY)}_{\text{calculated using Joint PMF of } X \,\&\, Y} - \underbrace{E(X)}_{\text{using marginal PMF of } X}E(Y)$$

Correlation

Correlation

normalized version of the covariance that gives you an idea of the strength & direction of linear relationship b/w X & Y.

Definition

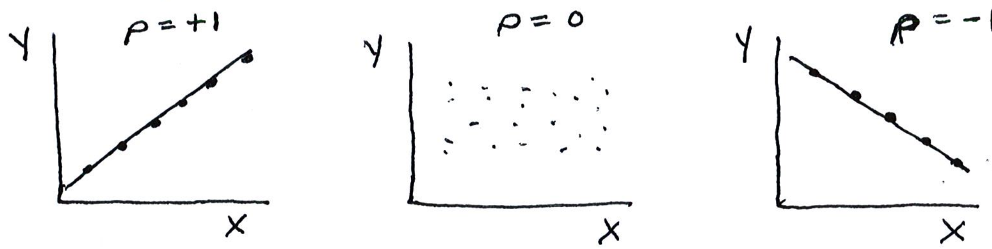
The *correlation* between 2 random variables X, Y is given by

ρ "rho" →
$$\rho = Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X) \cdot Var(Y)}}$$

Properties of Correlation (ρ)

- ρ is a measure of linear association between X and Y.
- $-1 \leq \rho \leq 1$
- ρ near ± 1 indicates a strong linear relationship

  ρ near 0 indicates a lack of linear association.



Correlation Example

Back to Example 1:

Recall $E(X) = E(Y) = 450$; $Var(X) = Var(Y) = 2000$

3. What is the correlation between X and Y? ?

* Calculate Covariance $\quad Cov(X,Y) = E(XY) - \underbrace{E(X)}_{450}\underbrace{E(Y)}_{450}$

$$E(XY) = \sum_{x,y} x \cdot y \, P_{X,Y}(x,y) = (400)(400)(0.1) + (400)(450)(0.1)$$
$$+ (400)(500)(0.2) + (450)(400)(0.1)$$
$$+ (450)(450)(0) + (450)(500)(0.1)$$
$$+ \ldots + (500)(500)(0.1)$$
$$= 202{,}000$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 202{,}000 - (450)(450) = -500$$

* Calculate Correlation

$$\rho = Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{-500}{\sqrt{(2000)(2000)}} = -0.25$$

indicates "weak" negative linear relationship b/w X & Y

Independence

Independence

Recall that random variables X, Y are *independent* if all events of the form $\{X = x\}$ and $\{Y = y\}$ are independent.

For independence, we need

$$p_{X,Y}(x, y) = p_X(x)p_Y(y) \text{ for all } x, y$$

joint probability = product of the marginals

- Check if the above holds for all possible combos of x and y
- If we can find at least one contradiction, then we do not have independence
- If two random variables are independent, then X and Y will always have $Cov(X, Y) = 0$.
- Converse is not always true.
  If $Cov(X, Y) = 0$, then X and Y could be independent or dependent. (need more information)

This implies if we know $Cov(X,Y) \neq 0$, then X & Y could not have been independent.

Checking Independence

all independent R.Vs have $Cov = 0$ So if we know $Cov \neq 0$, then the R.V could not have been independent. (contrapositive argument)

Checking Independence

all independent & some dependent R.Vs have $Cov = 0$. So just knowing $Cov = 0$ is not enough information. to tell independence/dependence.

1. Calculate $Cov(X, Y)$
   - If $Cov(X, Y) \neq 0$, then X and Y are not independent!
   - If $Cov(X, Y) = 0$, then X and Y may or may not be independent. (Need more info)
2. Check if $p_{X,Y}(x, y) = p_X(x)p_Y(y)$ for all x, y pairs.
   - If $p_{X,Y}(x, y) = p_X(x)p_Y(y)$ for all x, y pairs, then X and Y are independent.
   - If $p_{X,Y}(x, y) \neq p_X(x)p_Y(y)$ for at least one x, y pair, then X and Y are not independent.

Independence Example

Back to Example 1: Are X and Y independent?

Previously, we found that $Cov(X, Y) = -500$

- Since $Cov(X, Y) = -500 \neq 0$, X and Y are NOT independent.

Alternatively, we can check whether $p_{X,Y}(x, y) = p_X(x)p_Y(y)$ for x, y pairs.

- $p_{X,Y}(450, 450) = 0 \neq (0.2)(0.2) = p_X(450)p_X(450)$
- X and Y are NOT independent.

Independence Example

Example 2: Consider random variable X and Y where $Y = X^2$

Y \ X	-1	0	1	
0	0.0	0.6	0.0	0.6
1	0.2	0.0	0.2	0.4
	0.2	0.6	0.2	

Are X and Y independent?

Get the Marginal Probalities

x	-1	0	1
$P_X(x)$	0.2	0.6	0.2

y	0	1
$P_Y(y)$	0.6	0.4

Calculate Covariance $\quad Cov(X,Y) = E(XY) - E(X)E(Y)$

$$E(X) = \sum_x x P_X(x) = (-1)(0.2) + (0)\cancel{(0.6)} + (1)(0.2) = 0$$

$$E(Y) = \sum_y y P_Y(y) = (0)\cancel{(0.6)} + (1)(0.4) = 0.4$$

Independence Example

$$E(XY) = \sum_{x,y} x \cdot y \cdot P_{X,Y}(x,y) = \cancel{(-1)(0)(0)} + \cancel{(0)(0)(0.6)} + \cancel{(1)(0)(0)} + (-1)(1)(0.2) + \cancel{(0)(1)(0)} + (1)(1)(0.2)$$
$$= -0.2 + 0.2 = 0$$

$$\rightarrow Cov(X,Y) = E(XY) - E(X)E(Y) = 0 - (0)(0.4) = 0$$

Check whether $P_{X,Y}(x,y) = P_X(x)P_Y(y)$ for every (x,y) pair.

$$P_{X,Y}(-1,0) = 0 \neq (0.2)(0.6) = P_X(-1)P_Y(0)$$

Hence X & Y are dependent (not independent) even though $Cov(X,Y) = 0$.

[Note: If the joint probability equalled the product of the marginal probabilities for every (x,y) pair, the we would have said X & Y are independent.]

More on Expectation and Variance

More on Variance

Definition

constants.

Let X and Y be random variables, and a,b,c be real numbers.

$$Var(aX + bY + c) = a^2 Var(X) + b^2 Var(Y) + 2abCov(X, Y)$$

- Recall that for independent random variables, $Cov(X, Y) = 0$
- Thus if X and Y are independent, this simplifies to

$$Var(aX + bY + c) = a^2 Var(X) + b^2 Var(Y)$$

More on Expected Value

Definition

Let X and Y be random variables.

$$E(XY) = \sum_{x,y} xyp_{X,Y}(x,y)$$

- If X and Y are independent, this simplifies to

$$\begin{aligned} E(XY) &= \sum_{x,y} xyp_X(x)p_Y(y) \\ &= \sum_x xp_X(x) \sum_y yp_Y(y) \\ &= E(X)E(Y) \end{aligned}$$

- If X and Y are independent, E(XY) = E(X)E(Y)

This is exactly why $Cov(X,Y) = 0$ for independent R.Vs

$$Cov(X,Y) = E(XY) - E(X)E(Y) \underset{\text{b/c indep}}{=} E(X)E(Y) - E(X)E(Y) = 0$$

Additional Example

		X = 0	X = 7
Y	9	0.03	0.07
	3	0.27	0.63

x	0	7
$p_X(x)$	0.3	0.7

y	9	3
$p_Y(y)$	0.1	0.9

Since Cov(X,Y) = 0, we can't tell whether X and Y are independent/dependent based on that alone

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \dots = 0$$

Check if $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ for all x,y pairs.

$$p_{X,Y}(0,9) = 0.03 = (0.3)(0.1) = p_X(0)p_Y(9) \checkmark$$

$$\vdots$$

$$p_{X,Y}(7,3) = 0.63 = (0.7)(0.9) = p_X(7)p_Y(3) \checkmark$$

Since this equality holds for all (x,y) pairs, X and Y are independent.