Be aware of model capacity when talking about generalization in modern machine learning

Fanghui Liu

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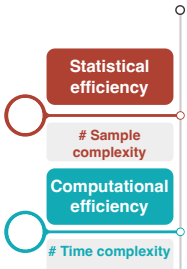
Centre for Discrete Mathematics and its Applications (DIMAP), Warwick

at Department of Statistics, University of Oxford

My research

❒ Research interests

- Foundations of machine learning (ML)
- Theory-grounded efficient algorithm design
- Trustworthy ML



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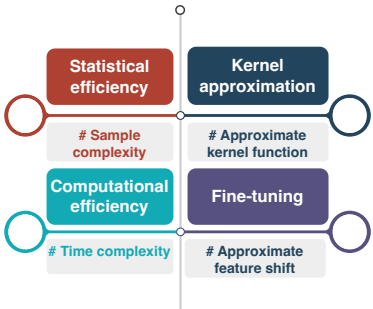

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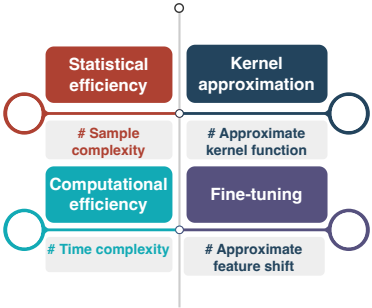
- characterize **learning efficiency** in theory
- contribute to practice



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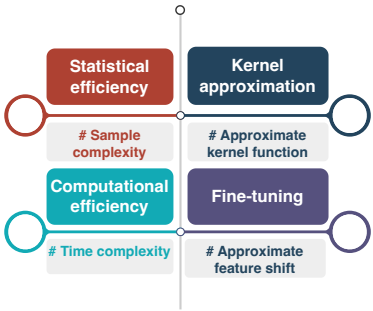

Learning efficiency (Curse of Dimensionality, CoD)

Machine learning works in **high dimensions** that can be a **curse**!

— David Donoho, 2000. (Richard E. Bellman, 1957)

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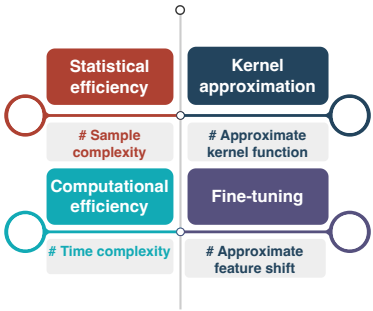

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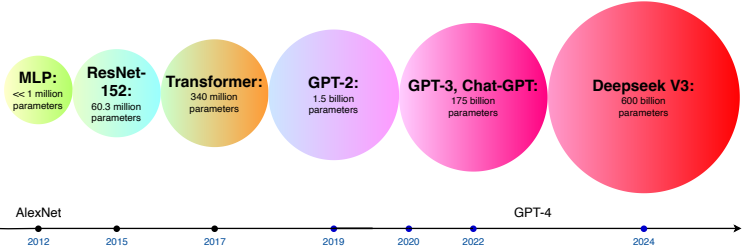
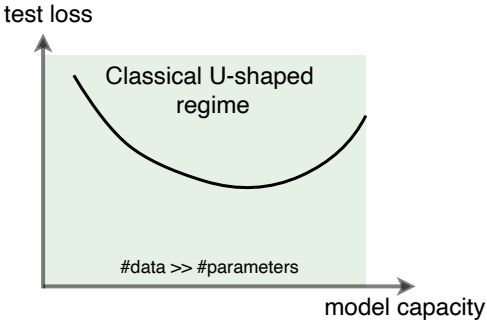
In the era of machine learning

Prefer more data and larger model to obtain better performance...

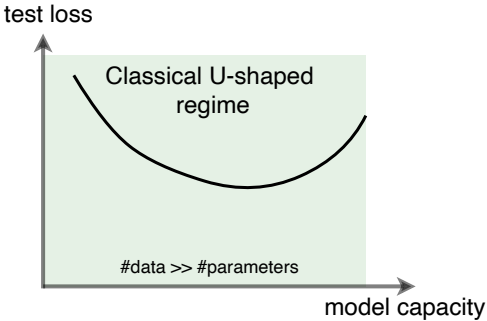


ML textbooks: Larger models tend to overfit!



ML textbooks: Larger models tend to overfit!

Practice of deep learning: bigger models perform better!



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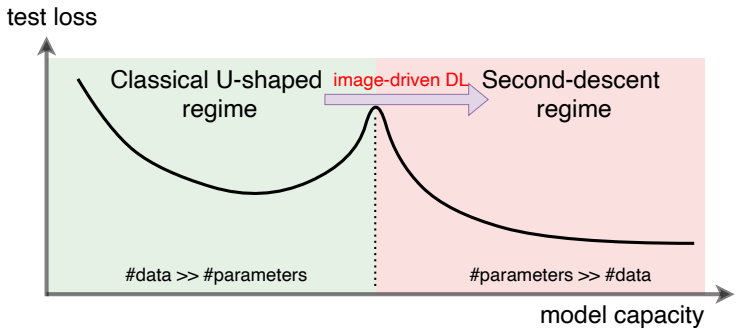

Proposed explanation: double descent (Belkin et al., 2019)

Learning paradigm in the past twenty years

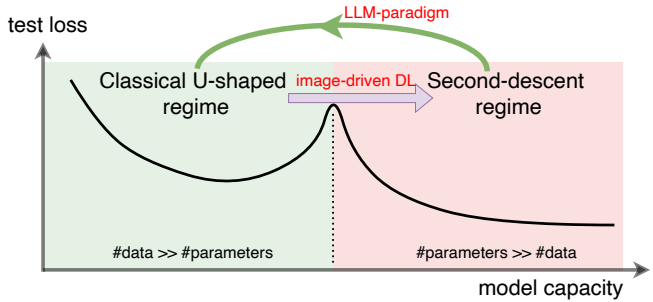

Figure 1: Paradigm among test loss, data, and model capacity.

Scaling law (Kaplan et al., 2020) in the era of LLMs

$$\text{test loss} = \text{A} \times \text{Model Size}^{-a} + \text{B} \times \text{Data Size}^{-b} + \text{C}$$

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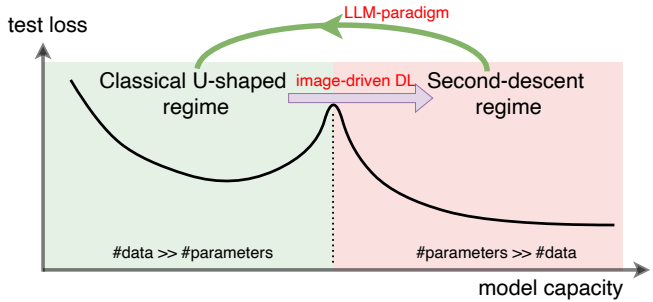

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A fundamental concept in machine learning: model capacity

Too many learning curves...

- U-shaped curve (bias-variance trade-offs) (Vapnik, 1995; Hastie et al., 2009)
- double (multiple) descent (Belkin et al., 2019; Liang et al., 2020)
- scaling law (Kaplan et al., 2020; Paquette et al., 2024)

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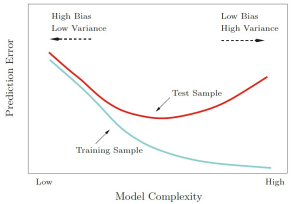
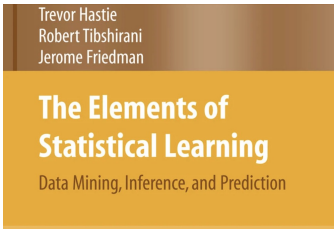
Bias-variance decomposition

$$\text{Test error} = \text{Bias}^2 + \text{Variance}$$



(Hastie et al., 2009, Figure 2.11)



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"Remove bias-variance trade-offs from ML textbooks"

Trade-off is a **misnomer**, by Geman et al. (1992); Neal (2019); Wilson (2025).

I can define **model capacity** at random and see whatever curve I want to see.

— Ben Recht, 2025

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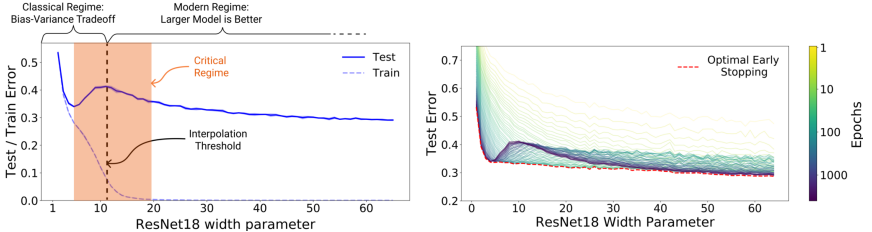
Double descent can disappear for the same architecture!



(a) Results on ResNet18 (Nakkiran et al., 2019) **(b)** Optimal early stopping (Nakkiran et al., 2019).

Today's talk: Learning with norm-based capacity

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- Theoretical studies (Neyshabur et al., 2015; Savarese et al., 2019)
- Min-norm solution (Hastie et al., 2022)
- Applications: neural networks pruning (Molchanov et al., 2017), lottery ticket hypothesis (Frankle and Carbin, 2019)

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How these learning curves behave under a more suitable model capacity?

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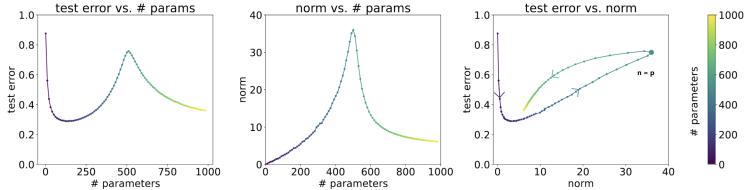

Figure 3: Stanford CS229 lecture notes (Ng and Ma, 2023, Figure 8.12).

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❒ How to **precisely** characterize the relationship under norm-based model capacity?

- Reshape bias-variance trade-offs, double descent, scaling law under ℓ_2 norm-based capacity!
- Yichen Wang, Yudong Chen, Lorenzo Rosasco, **Fanghui Liu**. *The shape of generalization through the lens of norm-based capacity control.* 2025. arXiv

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❒ What is the induced function space and statistical/computational efficiency under norm-based capacity?

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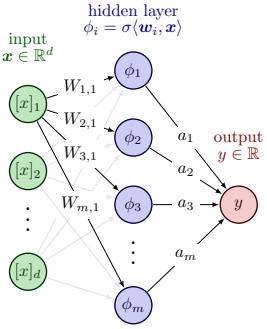
❒ What is the induced function space and statistical/computational efficiency under norm-based capacity?

- Which function class can be **efficiently** learned by neural networks?
- **Fanghui Liu**, Leello Dadi, and Volkan Cevher. *Learning with norm constrained, over-parameterised, two-layer neural networks.* JMLR 2024.

Background: Random features ridge regression



$$f_m(\boldsymbol{x}; \boldsymbol{\theta}) = \sum_{i=1}^{m} a_i \phi(\boldsymbol{x}, \boldsymbol{w}_i), \quad \boldsymbol{\theta} := \{(a_i, \boldsymbol{w}_i)\}_{i=1}^m$$

- $\phi : \mathcal{X} \times \mathcal{W} \to \mathbb{R}$, e.g., ReLU: $\phi(\boldsymbol{x}, \boldsymbol{w}) = \max(\langle \boldsymbol{x}, \boldsymbol{w}\rangle, 0)$
- Random features models (RFMs) Rahimi and Recht (2007):
  - $\{\boldsymbol{w}_i\}_{i=1}^m \overset{iid}{\sim} \mu$ for a given $\mu \in \mathcal{P}(\mathcal{W})$
  - only train the second layer

$$\hat{\boldsymbol{a}} := \operatorname*{argmin}_{\boldsymbol{a} \in \mathbb{R}^p} \left\{ \sum_{i=1}^{n} (y_i - f(\boldsymbol{x}_i; \boldsymbol{a}))^2 + \lambda \|\boldsymbol{a}\|_2^2 \right\} = \left(\boldsymbol{Z}^\top \boldsymbol{Z} + \lambda \boldsymbol{I}_p \right)^{-1} \boldsymbol{Z}^\top \boldsymbol{y} \,.$$

  - $\boldsymbol{Z} \in \mathbb{R}^{n \times p}$ with $[\boldsymbol{Z}]_{ij} = \frac{1}{\sqrt{p}} \phi(\boldsymbol{x}_i, \boldsymbol{w}_j)$.
- Norm over the first-layer (untrained) $\|\boldsymbol{W}\|_{\mathrm{F}}$
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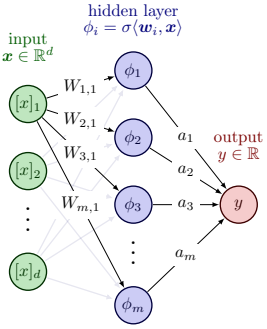

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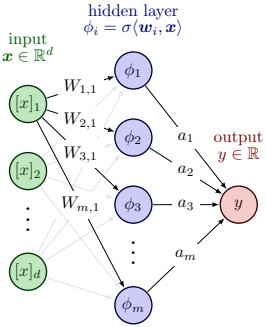

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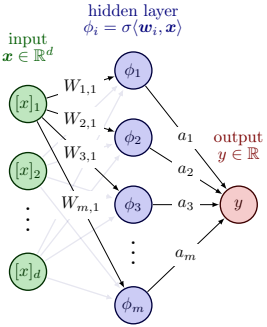

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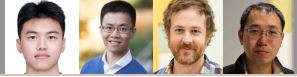
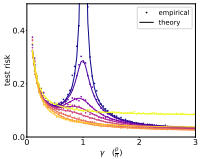
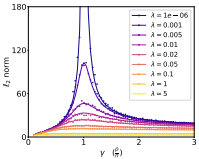
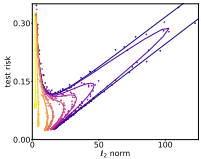




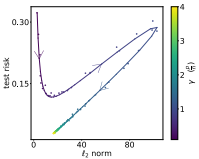
(a) Test Risk vs. γ



(b) ℓ_2 norm vs. γ



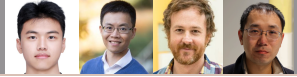
(c) Test Risk vs. norm

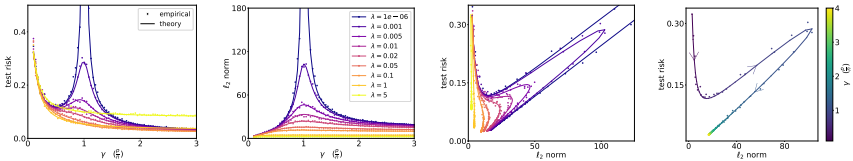


(d) $\lambda = 0.001$

- $\gamma := p/n$, p : model size (width), n : data size

(Wang, Chen, Rosasco, Liu, 2025)

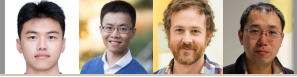


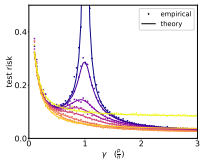


(a) Test Risk vs. γ **(b)** ℓ_2 norm vs. γ **(c)** Test Risk vs. norm **(d)** $\lambda = 0.001$

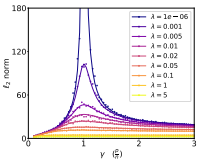
- $\gamma := p/n$, p : model size (width), n : data size

- Phase transition exists but double descent does not exist
- More close to **U-shaped** instead of double descent

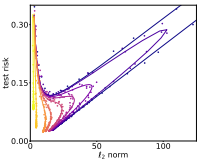




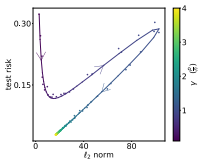
(a) Test Risk vs. γ



(b) ℓ_2 norm vs. γ



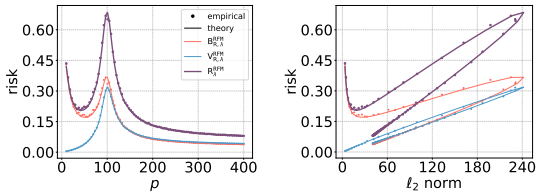
(c) Test Risk vs. norm



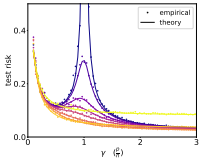
(d) $\lambda = 0.001$

- $\gamma := p/n$, p : model size (width), n : data size

$$\text{Test error} = \text{Bias}^2 + \text{Variance}$$



(Wang, Chen, Rosasco, Liu, 2025)



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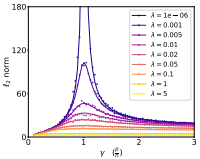

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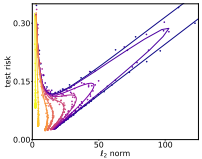

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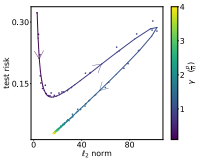

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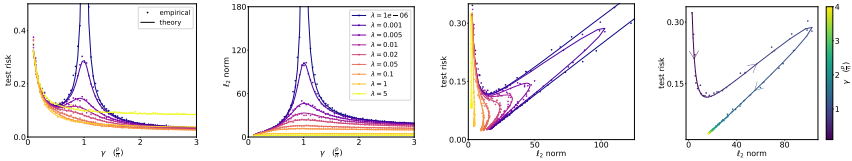
(Wang, Chen, Rosasco, Liu, 2025)



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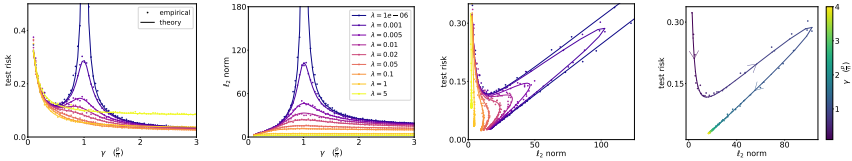

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Control norm by tuning λ: L-curve (Hansen, 1992)

Explicit (model size) *vs.* Implicit (norm)

One-to-one mapping between norm and λ

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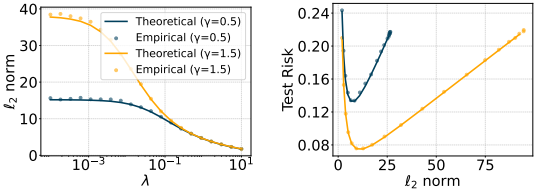

(a) Norm vs. λ (varying λ) **(b)** Risk vs. Norm (varying λ)

An example of linear regression: Textbook level and beyond

- n i.i.d. samples $\{(\boldsymbol{x}_i, y_i)\}_{i=1}^n$ with $\boldsymbol{x}_i \in \mathbb{R}^d$, $y_i \in \mathbb{R}$
- $y = \langle \boldsymbol{\beta}_*, \boldsymbol{x} \rangle + \varepsilon$, $\mathbb{E}(\varepsilon) = 0$ and $\mathbb{V}(\varepsilon) = \sigma^2$, covariance matrix $\boldsymbol{\Sigma} = \mathbb{E}[\boldsymbol{x}\boldsymbol{x}^\top]$
- ridge regression: $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^\top \boldsymbol{X} + \lambda \boldsymbol{I})^{-1} \boldsymbol{X}^\top \boldsymbol{y}$

Target: precise analysis

The expected test risk $\mathbb{E}_\varepsilon \|\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_*\|_{\boldsymbol{\Sigma}}^2$ vs. the norm $\mathbb{E}_\varepsilon \|\hat{\boldsymbol{\beta}}\|_2^2$

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❒ Deterministic equivalence (Cheng and Montanari, 2024; Misiakiewicz and Saeed, 2024): law of large samples/dimensions in random matrix theory

The empirical spectral measure converges to a deterministic limit.

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$$\mathrm{Tr}\Big(\boldsymbol{X}^\top \boldsymbol{X} (\boldsymbol{X}^\top \boldsymbol{X} + \lambda)^{-1}\Big) \sim \mathrm{Tr}\big(\boldsymbol{\Sigma}(\boldsymbol{\Sigma} + \lambda_*)^{-1}\big)\,, w.h.p.$$

- $\sim$ can be **asymptotic** or **non-asymptotic** at the rate of $\mathcal{O}(1/\sqrt{n})$.
- λ_* is the non-negative solution to the self-consistent equation $n - \frac{\lambda}{\lambda_*} = \mathrm{Tr}(\boldsymbol{\Sigma}(\boldsymbol{\Sigma} + \lambda_*)^{-1})$.

Our results

Theorem (Deterministic equivalence of estimator's norm)

We have a bias-variance decomposition $\mathbb{E}_\varepsilon \|\hat{\beta}\|_2^2 = \mathcal{B}_{\mathcal{N},\lambda} + \mathcal{V}_{\mathcal{N},\lambda}$.

For well-behaved data, we have

$$\mathrm{B}_{\mathrm{N},\lambda} := \langle \beta_*, \Sigma^2(\Sigma+\lambda_*)^{-2}\beta_* \rangle + \frac{\mathrm{Tr}(\Sigma(\Sigma+\lambda_*)^{-2})}{n} \frac{\lambda_*^2 \langle \beta_*, \Sigma(\Sigma+\lambda_*)^{-2}\beta_* \rangle}{1-\frac{1}{n}\mathrm{Tr}(\Sigma^2(\Sigma+\lambda_*)^{-2})},$$

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Remark: Which model capacity suffices to characterize the test risk?

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- effective dimension-style $\mathrm{Tr}(\Sigma(\Sigma+\lambda I)^{-1})$: ✗ ☹

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Example: Relationship under isotropic features ($\Sigma = I_d$)

❒ Test risk R_λ and norm N_λ formulates a cubic curve (complex but precise).

- min-norm interpolator ($\lambda = 0$):

$$\mathrm{R}_0 = \begin{cases} \mathrm{N}_0 - \|\beta_*\|_2^2; \text{in under-parameterized regimes} \\ \sqrt{\left[\mathrm{N}_0 - (\|\beta_*\|_2^2 - \sigma^2)\right]^2 + 4\|\beta_*\|_2^2\sigma^2} - \sigma^2. \end{cases}$$

- optimal regularization $\lambda = \frac{d\sigma^2}{\|\beta^*\|_2^2}$ (Wu and Xu, 2020): $\mathrm{R}_\lambda = \|\beta_*\|_2^2 - \mathrm{N}_\lambda$

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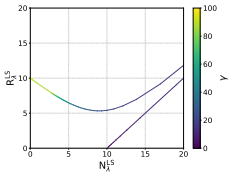

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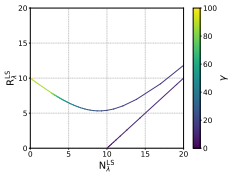

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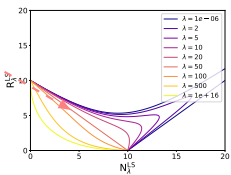

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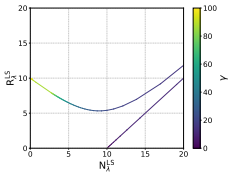

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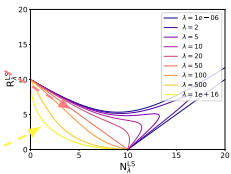

Precise analysis via deterministic equivalence

- Precisely describe the learning curve.
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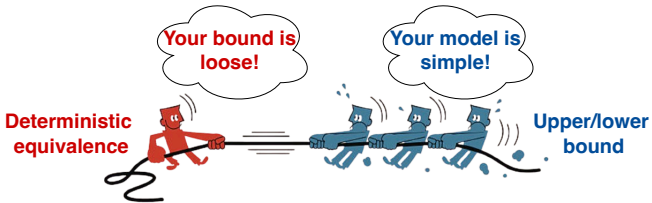

Which model capacity is suitable (for neural networks)?

Table 1: Complexity measures compared in the empirical study (Jiang et al., 2020), and their correlation with generalization.

name	definition	rank correlation
Parameter Frobenius norm	$\sum_{i=1}^{L} \|\boldsymbol{W}_i\|_F^2$	0.073
Frobenius distance to initialization	$\sum_{i=1}^{L} \|\boldsymbol{W}_i - \boldsymbol{W}_i^0\|_F^2$	−0.263
Spectral complexity	$\prod_{i=1}^{L} \|\boldsymbol{W}_i\| \left(\sum_{i=1}^{L} \frac{\|\boldsymbol{W}_i\|_{2,1}^{3/2}}{\|\boldsymbol{W}_i\|^{3/2}} \right)^{2/3}$	−0.537
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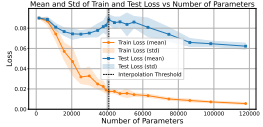
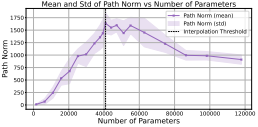
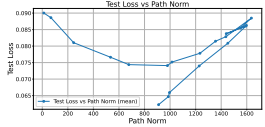
Path-norm	$\sum_{(i_{\mathbf{0}},\ldots,i_L)} \prod_{j=\mathbf{1}}^{L} \left(\boldsymbol{W}_{i_j, i_{j-\mathbf{1}}} \right)^2$	0.373



(a) Test (training) Loss vs. p



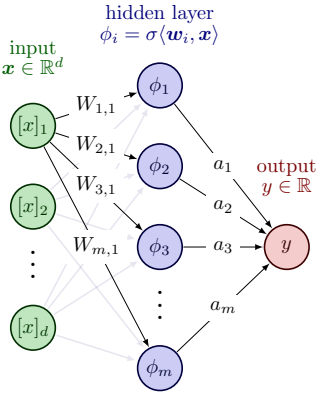
(b) Path-norm vs. p



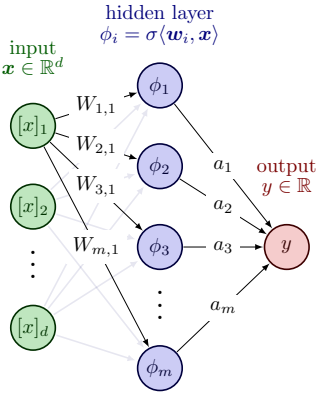
(c) Test Loss vs. Path-norm

Figure 5: Experiments on two-layer neural networks.

Two-layer neural networks, path norm



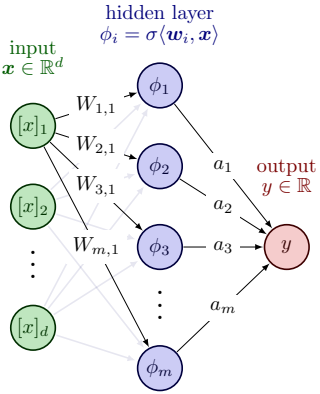
Two-layer neural networks, path norm



ℓ_1-path norm (Neyshabur et al., 2015)

$\|\boldsymbol{\theta}\|_{\mathcal{P}} := \frac{1}{m} \sum_{k=1}^{m} |a_k| \|\boldsymbol{w}_k\|_1$

Two-layer neural networks, path norm



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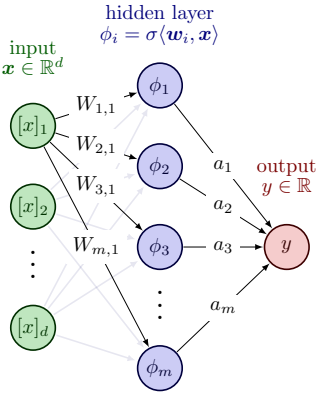
Two-layer neural networks, path norm



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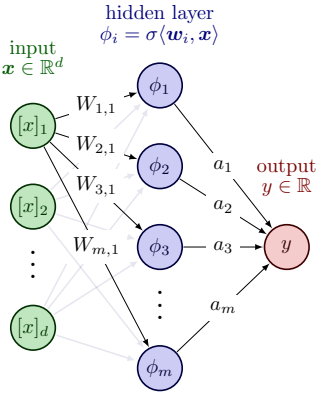
Two-layer neural networks, path norm



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- Variation in only a few directions (Parhi and Nowak, 2022)

Can neural networks identify this structure?

(Liu, Dadi, Cevher, JMLR'24)

Theorem (Informal, sample complexity of learning $f^\star \in \mathcal{B}$)

To achieve ϵ-excess risk,

- *Kernel methods* *require* $\Omega(\epsilon^{-d})$ *samples.*
- *Two-layer neural networks* *require* $\Omega(\epsilon^{-\frac{2d+2}{d+2}})$ *samples.* *smaller than* ϵ^{-2}

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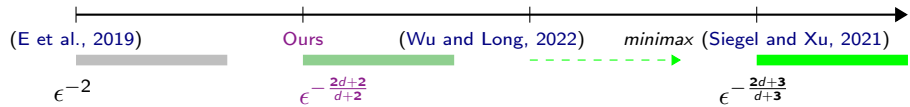
❒ Track sample complexity (via metric entropy) and dimension dependence



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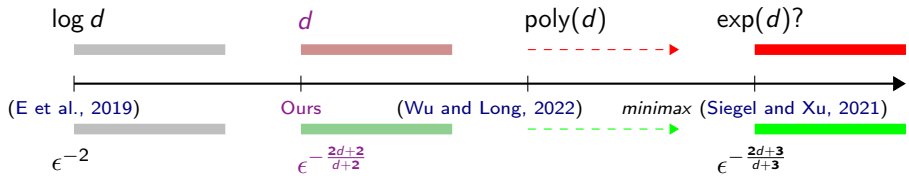

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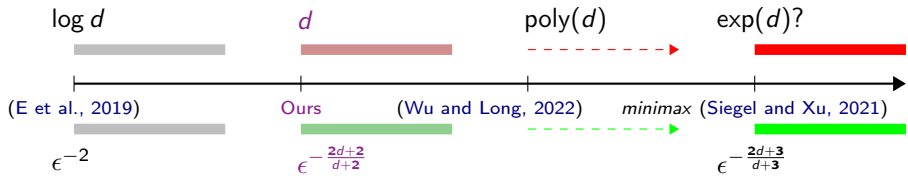
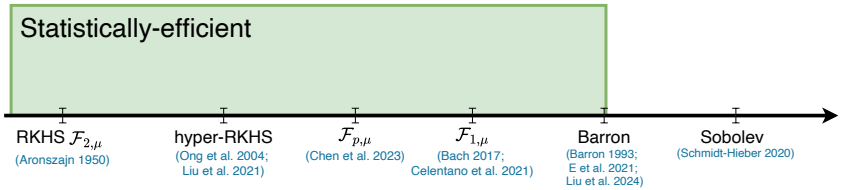


The "best" trade-off between ϵ and d.

Which function class can be efficiently learned by neural networks



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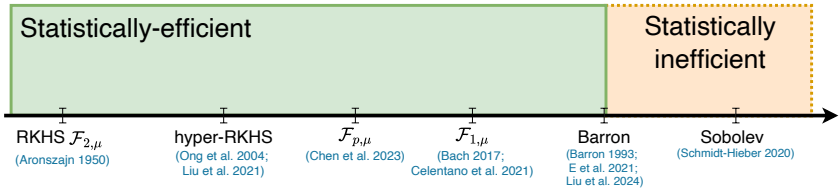

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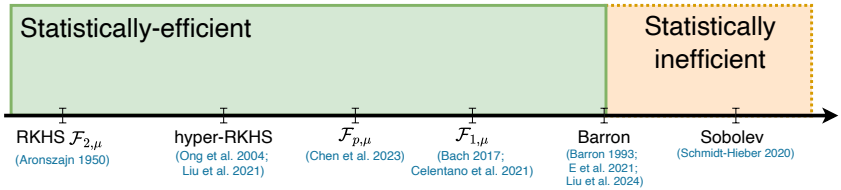

Optimization in Barron spaces is NP hard: curse of dimensionality!
(Bach, 2017)

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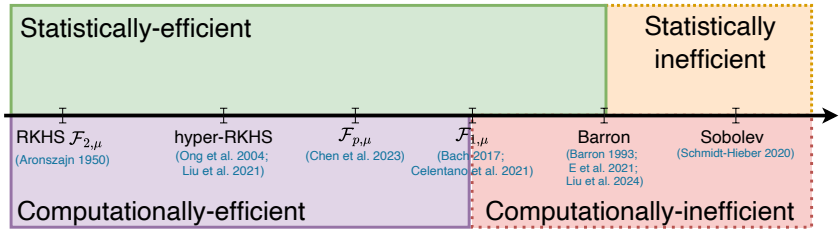

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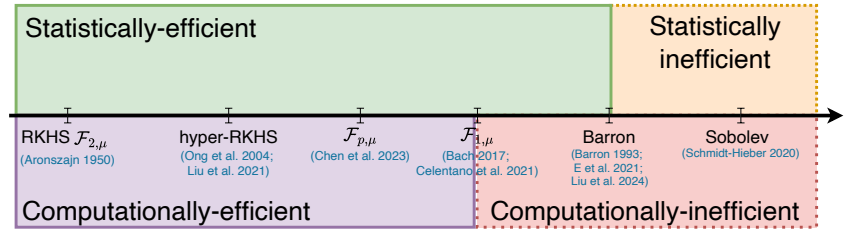
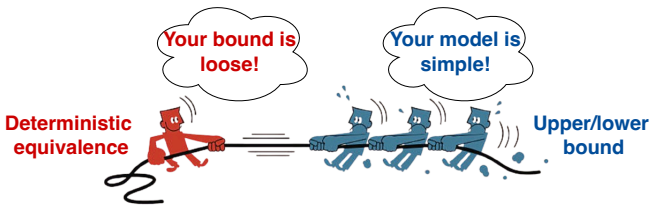


- ReLU neurons (Chen and Narayanan, 2023)
- Low-dimensional polynomials (Arous et al., 2021; Lee et al., 2024)

Takeaway messages

Deep learning phenomena $\Rightarrow$ interesting mathematical problems

- Be aware of model capacity!
  - Reshape bias-variance trade-offs, double descent, scaling law under proper ℓ_2 norm-based capacity via deterministic equivalence.



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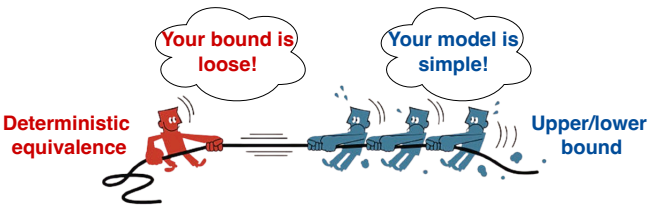
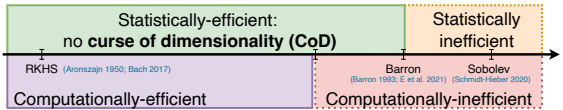


- Which function class can be **efficiently** learned by neural networks?
  - Neural networks can adapt to low-dimensional structure and avoid CoD!



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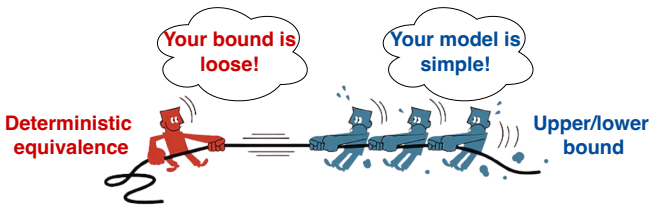

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Theoretical advances ⇒ principled guidance in practical problems

- ❒ How does our theory contribute to practical fine-tuning problems?
  - One-step full gradient can be sufficient! [GitHub]

References

Gerard Ben Arous, Reza Gheissari, and Aukosh Jagannath. Online stochastic gradient descent on non-convex losses from high-dimensional inference. *Journal of Machine Learning Research*, 22(106):1–51, 2021.

Francis Bach. Breaking the curse of dimensionality with convex neural networks. *Journal of Machine Learning Research*, 18(1):629–681, 2017.

Andrew R Barron. Universal approximation bounds for superpositions of a sigmoidal function. *IEEE Transactions on Information theory*, 39(3): 930–945, 1993.

Peter Bartlett. The sample complexity of pattern classification with neural networks: the size of the weights is more important than the size of the network. *IEEE Transactions on Information Theory*, 44(2):525–536, 1998.

Mikhail Belkin, Daniel Hsu, Siyuan Ma, and Soumik Mandal. Reconciling modern machine-learning practice and the classical bias–variance trade-off. *the National Academy of Sciences*, 116(32):15849–15854, 2019.

Sitan Chen and Shyam Narayanan. A faster and simpler algorithm for learning shallow networks. *arXiv preprint arXiv:2307.12496*, 2023.

Chen Cheng and Andrea Montanari. Dimension free ridge regression. *The Annals of Statistics*, 52(6):2879–2912, 2024.

Weinan E, Chao Ma, and Lei Wu. A priori estimates of the population risk for two-layer neural networks. *Communications in Mathematical Sciences*, 17 (5):1407–1425, 2019.

Weinan E, Chao Ma, and Lei Wu. The barron space and the flow-induced function spaces for neural network models. *Constructive Approximation*, pages 1–38, 2021.

Jonathan Frankle and Michael Carbin. The lottery ticket hypothesis: Finding sparse, trainable neural networks. In *International Conference on Learning Representations*, 2019.

Stuart Geman, Elie Bienenstock, and René Doursat. Neural networks and the bias/variance dilemma. *Neural computation*, 4(1):1–58, 1992.

Per Christian Hansen. Analysis of discrete ill-posed problems by means of the l-curve. *SIAM Review*, 34(4):561–580, 1992.

Trevor Hastie, Robert Tibshirani, Jerome H Friedman, and Jerome H Friedman. *The elements of statistical learning: data mining, inference, and prediction*, volume 2. Springer, 2009.

Trevor Hastie, Andrea Montanari, Saharon Rosset, and Ryan J Tibshirani. Surprises in high-dimensional ridgeless least squares interpolation. *Annals of Statistics*, 50(2):949–986, 2022.

Yiding Jiang, Behnam Neyshabur, Hossein Mobahi, Dilip Krishnan, and Samy Bengio. Fantastic generalization measures and where to find them. In *International Conference on Learning Representations*, 2020.

Jared Kaplan, Sam McCandlish, Tom Henighan, Tom B Brown, Benjamin Chess, Rewon Child, Scott Gray, Alec Radford, Jeffrey Wu, and Dario Amodei. Scaling laws for neural language models. *arXiv preprint arXiv:2001.08361*, 2020.

Jason D Lee, Kazusato Oko, Taiji Suzuki, and Denny Wu. Neural network learns low-dimensional polynomials with sgd near the information-theoretic limit. *arXiv preprint arXiv:2406.01581*, 2024.

Tengyuan Liang, Alexander Rakhlin, and Xiyu Zhai. On the multiple descent of minimum-norm interpolants and restricted lower isometry of kernels. In *Conference on Learning Theory*, pages 2683–2711, 2020.

Theodor Misiakiewicz and Basil Saeed. A non-asymptotic theory of kernel ridge regression: deterministic equivalents, test error, and gcv estimator. *arXiv preprint arXiv:2403.08938*, 2024.

Pavlo Molchanov, Stephen Tyree, Tero Karras, Timo Aila, and Jan Kautz. Pruning convolutional neural networks for resource efficient inference. In *International Conference on Learning Representations*, 2017.

Preetum Nakkiran, Gal Kaplun, Yamini Bansal, Tristan Yang, Boaz Barak, and Ilya Sutskever. Deep double descent: Where bigger models and more data hurt. In *International Conference on Learning Representations*, 2019.

Brady Neal. On the bias-variance tradeoff: Textbooks need an update. *arXiv preprint arXiv:1912.08286*, 2019.

Behnam Neyshabur, Ryota Tomioka, and Nathan Srebro. Norm-based capacity control in neural networks. In *Conference on Learning Theory*, pages 1376–1401. PMLR, 2015.

Andrew Ng and Tengyu Ma. CS229 lecture notes. 2023. URL https://cs229.stanford.edu/main_notes.pdf.

Elliot Paquette, Courtney Paquette, Lechao Xiao, and Jeffrey Pennington. 4+3 phases of compute-optimal neural scaling laws. *arXiv preprint arXiv:2405.15074*, 2024.

Rahul Parhi and Robert D Nowak. Near-minimax optimal estimation with shallow ReLU neural networks. *IEEE Transactions on Information Theory*, 2022.

Ali Rahimi and Benjamin Recht. Random features for large-scale kernel machines. In *Advances in Neural Information Processing Systems*, pages 1177–1184, 2007.

Pedro Savarese, Itay Evron, Daniel Soudry, and Nathan Srebro. How do infinite width bounded norm networks look in function space? In *Conference on Learning Theory*, pages 2667–2690. PMLR, 2019.

Jonathan W Siegel and Jinchao Xu. Sharp bounds on the approximation rates, metric entropy, and n-widths of shallow neural networks. *arXiv preprint arXiv:2101.12365*, 2021.

Vladimir N. Vapnik. *The Nature of Statistical Learning Theory*. Springer, 1995.

Andrew Gordon Wilson. Deep learning is not so mysterious or different. *arXiv preprint arXiv:2503.02113*, 2025.

Denny Wu and Ji Xu. On the optimal weighted ℓ_2 regularization in overparameterized linear regression. In *Advances in Neural Information Processing Systems*, pages 10112–10123, 2020.

Lei Wu and Jihao Long. A spectral-based analysis of the separation between two-layer neural networks and linear methods. *Journal of Machine Learning Research*, 119:1–34, 2022.