. . . . (1) . . . .

GENERAL

Definition 1.1. *$\mathbb{R}[x]$ is the group of all polynomials with real coefficients.*

Theorem 1.1. $\forall p(x) \in \mathbb{R}[x]. \forall z \in \mathbb{C}. p(z) = 0 \iff p(\bar{z})$

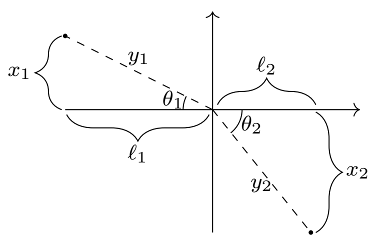
Definition 1.2. *$\mathbb{C}[x]$ is the group of all polynomials with complex coefficient.*

Theorem 1.2. $p(x_0) = 0 \iff (x - x_0) \mid p(x_0)$

Theorem 1.3 (Fundamental theorem of algebra)**.** *For all $p(x) \in \mathbb{C}$ non-constant, exists $z \in \mathbb{C}$ such as $p(z) = 0$.*

Theorem 1.4 (The Politics Assumption)**.** *A problem is dismissed iff it would be became bigger problem in the future.*

Theorem 1.5 (Snell's Law)**.** *Given v_1, v_2 speeds to pass y_1, y_2 repectively;*



We denote $d = x_1 + x_2$. We get that:

$$\theta_2 = \arctan\left(\frac{d - \ell_1 \tan\theta_1}{\ell_2}\right)$$

Hence, when t is the time required to get from one point to another with the given speeds:

$$t(\theta_1) = \frac{\ell_1}{v_1 \cos\theta_1} + \frac{\ell_1}{v_2}\sqrt{1 + \left(\frac{\ell_1 \tan\theta_1 - d}{\ell_2}\right)^2}$$

And t is minimal when:

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

. . . . (2) . . . .

LIMITS ETC.

Definition 2.1 (Dirichlet function)**.**

$$D(x) \equiv \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \end{cases} = \mathbf{1}_{\mathbb{Q}}$$

Theorem 2.1. *Assuming $\lim_{x \to x_0} f(x) = L_f$ and $\lim_{x \to x_0} g(x) = L_g$ where $L_f, L_g \in \mathbb{R} \vee L_f = L_g = \pm\infty$, then $\lim_{x \to x_0} f + g = L_f + L_g$, and $\lim_{x \to x_0} f \cdot g = L_f L_g$.*

Definition 2.2. *A function $f(x) \in \mathbb{R} \to \mathbb{R}$ is continues in x_0 iff $\lim_{x \to x_0} f(x_0) = f(x_0)$.*

Definition 2.3. *$f \in I \to \mathbb{R}$ has the intermediate value property iff $\forall a, b \in I. \exists c \in [a, b]. f(c) \in [f(a), f(b)]$.*

Theorem 2.2 (Intermediate value theorem)**.** *If $f \in I \to \mathbb{R}$ is continues, then f has the intermediate value property.*

Theorem 2.3. *If $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L$ and $\forall x \in [a, b] \neq \emptyset. f(x) \leq g(x) \leq h(x) \wedge x_0 \in [a, b]$, then $\lim_{x \to x_0} g(x) = L$.*

Theorem 2.4 (Lhopital rule)**.** *Assuming $\lim_{x \to x_0} f'(x) = L_f, \lim_{x \to x_0} g'(x) = L_g$ both exists, then:*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L_f}{L_g}$$

Theorem 2.5. *Where f^{-1} is the inverse function of f for a given interval;*

$$f^{-1\prime} = \frac{1}{f' f(^{-1})}$$

. . . . (3) . . . .

TRIGO AND HYPR-TRIGO

Definition 3.1 (Hyperbolic functions)**.**

$$\cosh x = \frac{e^x + e^{-x}}{2}, \ \sinh x = \frac{e^x - e^{-x}}{2}$$

Theorem 3.1.

$$\cosh^2 x - \sinh^2 x = 1,$$
$$\sinh(x + y) = \sinh y \cosh x + \sinh x \cosh y,$$
$$\operatorname{arcsinh} x = \ln(x + \sqrt{x^2 + 1})$$

Theorem 3.2.

$$\operatorname{arccosh} = \ln(x + \sqrt{x^2 - 1}), \ \operatorname{arcsinh} = \ln(x + \sqrt{x^2 + 1})$$

Theorem 3.3. *We denote $\pm = +$ for trigonometric functions, and $\pm = -$ for hyperbolic functions. $\mp$ is the inverse of $\pm$.*

$$\begin{aligned} \arcsin[h] = \sec[h](\arcsin[h]) &= \frac{1}{\sqrt{1 \mp x^2}} \\ \arccos[h] = \mp\csc[h](\arcsin) &= \mp\frac{1}{\sqrt{x^2 - 1}} \\ \arctan[h] = \cos[h]^2(\arctan[h]) &= \frac{1}{1 \pm x^2} \end{aligned}$$

. . . . (4) . . . .

DERIVATIVES

Definition 4.1 (implicit diffrentiation)**.** *diffrentiating both sides of the equation. E.g.:*

$$y = f(x), \ xf(x) = 1 \to f(x) + f'(x)x = 1' = 0 \to y' = -\frac{y}{x}$$

Definition 4.2.

$$e_n \equiv \left(1 + \frac{1}{n}\right)^n$$

Theorem 4.1. *e_n is monotonically increasing*

Definition 4.3.

$$e \equiv \lim_{x \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Theorem 4.2.

$$f(x) = \log_a x \implies f'(x) = \frac{1}{x \ln a}$$
$$f(x) = a^x \implies f'(x) = \ln a \cdot a^x$$

Theorem 4.3.

$$\exists I \text{ interval}.\forall x \in I.f'(x) = f(x) \implies \exists c \in \mathbb{R}.f(x) = ce^x$$

Theorem 4.4 (Darbuax Theorm)**.** *If f diffrentiatable in $I \subseteq \mathbb{R}$, then f' has the intermediate value property.*

. . . . (5) . . . .

INTEGRALS

Definition 5.1. *$F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.*

Theorem 5.1. *If F_1, F_2 antiderivatives of f in $I \subseteq \mathbb{R}$, then exists $C \in \mathbb{R}$ such as $F_1 - F_2 = C$.*

Theorem 5.2.

$$\int (f+g)\,\mathrm{d}x = \int f\,\mathrm{d}x + \int g\,\mathrm{d}x = F + G + C$$
$$\int af(x)\,\mathrm{d}x = a\int f(x)\,\mathrm{d}x = aF(x) + C$$
$$\int f(ax+b)\,\mathrm{d}x = \frac{1}{a}F(ax+b) + C$$

Theorem 5.3.

$$\int f(t(x))t'(x)\,\mathrm{d}x = \int f(t)\,\mathrm{d}t$$

Theorem 5.4 (Lagrange's mean value theorem)**.** *Let F be an antiderivative of f in $[c, d]$. Then exists $c \leq x \leq d$ such as:*

$$f(x) = \frac{F(d) - F(c)}{c - d}$$

Theorem 5.5 (Fundamental theorem of calc)**.**

$$S = \int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

Where S is the area under $f(x)$ in range $[a, b]$.

Theorem 5.6.

$$\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$$

Theorem 5.7 (Integration by parts (IBP))**.**

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

. . . . (6) . . . .

TAYLOW SERIES

Definition 6.1 (Taylow Series)**.** *Taylow series around $x_0 \in \mathbb{R}$ for $f \in \mathbb{R}^{\mathbb{R}}$ is:*

$$\sum_{k=0}^{\infty} \frac{f^{(n)}(x_0)}{k!}(x - x_0)^k$$

Theorem 6.1. *The taylor series for $f(x)$ is equal to $f(x)$ for "many functions".*

Definition 6.2 (Maclaurin Series)**.** *is taylor series around 0:*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Theorem 6.2 (Maclaurin Series for common functions)**.**

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$$
$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}$$
$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$
$$\frac{1}{x} = \sum_{m=0}^{\infty} -(x+1)^m \qquad [-2 \leq x \leq 0]$$
$$\ln(1-x) = \sum_{k=0}^{\infty} -\frac{x^k}{k}$$

Note that Maclaurin Series for $e^{-\frac{1}{x^2}}$ equals 0.

Theorem 6.3 (Weierstrass Substitution)**.** *let $t = \tan\frac{x}{2}$;*

$$\int f(\sin x, \cos x\,\mathrm{d}x)\,\mathrm{d}x = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2\,\mathrm{d}t}{1+t^2}\,\mathrm{d}t$$

. . . . (7) . . . .

COMPLEX NUMBERS

Definition 7.1. *For $z \in \mathbb{C}$, $\sin z, \cos z$ and e^z is defined to be the result of the Maclaurin Series' output for z.*

Theorem 7.1.

$$\forall x, y \in \mathbb{C}, e^{x+y} = e^x + e^y$$

Theorem 7.2 (Euler Formula)**.**

$$\forall x \in \mathbb{C}.\ e^{ix} = \cos x + i \sin x$$

Hence, for all $\mathbb{C} \ni x = a + bi = \Re(x) + \Im(x)i$ where $a, b \in \mathbb{R}$, we know that $r \equiv |e^x| = e^{\Re(x)}$ and the angle of e^x is b. To summrize:

Theorem 7.3.

$$z = r(\cos\theta + i\sin\theta) = e^{\ln r + i\theta} = re^{i\theta}$$

Theorem 7.4.

$$z^n = r^n(\sin\theta + i\cos\theta)$$

Theorem 7.5.

$$u_n := \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2n}{n}\right)$$
$$\forall n \in \mathbb{N}.z \in \{u^m \mid m \in \mathbb{N}\} \iff z^n = 1$$

Course B $\sim$ Shit Cheat Sheet

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