## ~~1~~

Let

$$B = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right\}, \quad C = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right\}$$

Let $T : M_{2\times 2}(R) \to \mathrm{Sym}_2(R)$ be a linear map such that

$$[T]_C^B = \begin{pmatrix} 1 & -2 & 2 & 1 \\ 2 & -4 & 3 & 1 \\ -1 & 2 & 4 & 5 \end{pmatrix}$$

Recall: $\mathrm{Sym}_2(R)$ is the space of $2 \times 2$ real symmetric matrices.

~~(a)~~ Find a basis for $\mathrm{Im}(T)$.

~~(b)~~ Find a basis for $\ker(T)$.

~~(c)~~ Let

$$B' = \left\{ \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}, \quad C' = \left\{ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \right\}$$

Compute $[T]_{C'}^{B'}$.

## ~~2.~~

~~(a)~~ Prove: Two matrices $A_1, A_2 \in \mathbf{M}_{m\times n}(\mathbf{F})$ are row-equivalent **if and only if** there exists a linear transformation $T : \mathbf{F}^n \to \mathbf{F}^m$, a basis $B$ of $\mathbf{F}^n$, and bases $C_1, C_2$ of $\mathbf{F}^m$ such that:

$$A_1 = [T]_{C_1}^B, \quad A_2 = [T]_{C_2}^B$$

~~(b)~~ Let $S, T : V \to U$ be two linear maps. Show that $\dim(\mathrm{Im}(S)) = \dim(\mathrm{Im}(T))$ if and only if there exist bases $B_1, B_2$ of $V$ and $C_1, C_2$ of $U$ such that

$$[S]_{C_2}^{B_2} = [T]_{C_1}^{B_1}$$

.

~~(c)~~ (Exam 2016) Let $T : V \to U$ be a linear transformation. Prove or disprove: $T$ is an isomorphism if and only if there exist bases $B, C$ such that $[T]_C^B = I$

## ~~3~~

Find a spanning set for $U \cap W < \mathbf{R}^4$ where

$$U = \mathrm{span}\left\{ \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \\ 7 \end{pmatrix} \right\}, \quad W = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \\ 7 \end{pmatrix} \right\}$$

## ~~4~~

Find bases for $U \cap W, U + W < \mathbf{Z}_7^4$, where

$$U = \mathrm{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix} \right\}, \quad W = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 5 \\ 3 \\ 2 \end{pmatrix} \right\}$$

### ~~5~~

Find vector subspaces $U_1, U_2, U_3 \subseteq V$ such that:

$$\dim(U_1+U_2+U_3) \neq \dim U_1+\dim U_2+\dim U_3-\dim(U_1\cap U_2)-\dim(U_1\cap U_3)-\dim(U_2\cap U_3)+\dim(U_1\cap U_2\cap U_3)$$

E.g the inclusion-exclusion formula does not always hold in general.

### ~~6~~

Compute the determinant of each of the following real matrices:

~~(a)~~ $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

~~(b)~~ $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ -1 & -2 & -4 \end{pmatrix}$

~~(c)~~ $\begin{pmatrix} 0 & 0 & 4 & 3 \\ 4 & 0 & 7 & 5 \\ -1 & 4 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}$

~~(d)~~ $\begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 2 \end{pmatrix}$

~~(e)~~ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 \\ 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix}$

~~(f)~~ $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -78 & 13 & -43 & 111 & 11 \\ 1 & 1 & 0 & 0 & 0 \\ 235 & 14 & 0 & 0 & 0 \\ 1 & 89 & 0 & 0 & 0 \end{pmatrix}$

### ~~7~~

Express the determinant of the following matrices as a function of the given parameters:

~~(a)~~ The $n \times n$ tridiagonal matrix:

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 & b_1 \\ b_2 & a_2 & \cdots & 0 & 0 \\ 0 & b_3 & a_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n & a_n \end{pmatrix}$$

~~(b)~~ The $4 \times 4$ matrix:

$$\begin{pmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{pmatrix}$$