מתמטיקה B $\sim$ תרגיל בית 4

שחר פרץ

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(1)

.1

$$\int \cos^3 x \sin x \,\mathrm{d}x = \begin{bmatrix} u = \cos x & u' = -\sin x \\ \mathrm{d}u = -\sin x \,\mathrm{d}x & \end{bmatrix} = \int -u^3 = -\frac{1}{4}u^4 = \boldsymbol{-\frac{\cos^4 x}{4} + C}$$

.2

$$\int \sqrt{\frac{\arcsin x}{1-x^2}}\,\mathrm{d}x = \int \sqrt{\arcsin x}\arcsin'\,\mathrm{d}x = \begin{bmatrix} \theta = \arcsin x & \theta' = \arcsin' \\ \mathrm{d}\theta = \arcsin'\,\mathrm{d}x & \end{bmatrix} = \int \sqrt{\theta}\,\mathrm{d}\theta = \frac{2}{3}\theta^{1.5} = \boldsymbol{\frac{\arcsin^{1.5} x}{1.5} + C}$$

.3

$$\int \frac{\ln^2 x}{x}\,\mathrm{d}x = \begin{bmatrix} u = \ln x & u' = \frac{1}{x} \\ \mathrm{d}u = \frac{1}{x}\,\mathrm{d}x & \end{bmatrix} = \int u^2\,\mathrm{d}u = \frac{1}{3}u^3 = \boldsymbol{\frac{\ln^3 x}{3} + C}$$

.4

$$\int \frac{\mathrm{d}x}{\sqrt{x}+\sqrt[3]{x}} = \begin{bmatrix} u = x^{\frac{1}{6}} & u' = \frac{1}{6}x^{-\frac{5}{6}} \\ \mathrm{d}u = \frac{1}{6}x^{-\frac{5}{6}}\,\mathrm{d}x & \mathrm{d}x = 6u^5\,\mathrm{d}u \end{bmatrix} = \int \frac{6u^5\,\mathrm{d}u}{u^3+u^2} = \int \frac{\cancel{u^2}6u^3\,\mathrm{d}u}{\cancel{u^2}(1+u)} = \begin{bmatrix} t = u+1 & t' = u \\ \mathrm{d}t = u\,\mathrm{d}u & \end{bmatrix}$$

$$= \frac{6t^2\,\mathrm{d}t}{t} = 6\int t\,\mathrm{d}t = 3t^2 = 3(u+1)^2 = 3u^2+6u+1 = \boldsymbol{3\sqrt[3]{x} + 6\sqrt[6]{x} + 1 + C}$$

.5

$$\int x^3(3x^2-1)^{15}\,\mathrm{d}x = \begin{bmatrix} x = \frac{1}{\sqrt{3}}\sin\theta & x' = \frac{1}{\sqrt{3}\cos t} \\ \mathrm{d}x = \frac{1}{\sqrt{3}\cos t}\,\mathrm{d}t & \end{bmatrix} = \int \frac{1}{9\sqrt{3}}\sin^3 t\cdot(\sin^2 -1)^{15}\frac{1}{\sqrt{3}}\cos t\,\mathrm{d}t = \int 27^{-1}\sin^3 t\cos^{31} t\,\mathrm{d}t$$

$$= \begin{bmatrix} \theta = \sin t & \theta' = \cos t \\ \mathrm{d}\theta = \cos t\,\mathrm{d}t & \end{bmatrix} = \int 27^{-1}\theta^3\cos^{30}(\arcsin\theta)\,\mathrm{d}\theta = \frac{1}{27}\int \theta^3(1-\sin^2\arcsin\theta)^{15}\,\mathrm{d}\theta = \frac{1}{27}\int \theta^3(1-\theta^2)^{15}\,\mathrm{d}\theta$$

$$= \int \theta^5((1-\theta^2))^5\,\mathrm{d}\theta = \begin{bmatrix} u = 1-\theta^2 & x = \sqrt{1-u} \\ \mathrm{d}u = 2\theta\,\mathrm{d}\theta & \end{bmatrix} = \int u^5(1-u)^2\ 0.5\,\mathrm{d}u = \frac{1}{2}\int u^7 - \int u^6 + \frac{1}{2}\int u^5$$

$$= \frac{u^8}{14} - \frac{u^6}{6} + \frac{u^5}{10} + C = \frac{(1-\theta^2)^8}{14} - \frac{(1-\theta^2)^6}{6} + \frac{(1-\theta^2)^5}{10} + C = \frac{\cos^{16} t}{14} - \frac{\cos^{12} t}{6} + \frac{\cos^{10} t}{10} + C$$

$$= \boldsymbol{\frac{\cos^{16}\left(3^{-0.5}x\right)}{14} - \frac{\cos^{12}\left(3^{-0.5}x\right)}{6} + \frac{\cos^{10}\left(3^{-0.5}x\right)}{10} + C}$$

.6

$$\int \frac{x}{(x+3)^{\frac{1}{5}}}\,\mathrm{d}x = \begin{bmatrix} u = x+3 \\ \mathrm{d}u = \mathrm{d}x \end{bmatrix} = \int \frac{u-3}{\sqrt[5]{u}}\,\mathrm{d}u = \int u^{\frac{4}{5}}\,\mathrm{d}u - 3\int u^{-\frac{1}{5}}\,\mathrm{d}u = \boldsymbol{\frac{5}{9}u^{1.8} - 3.75u^{\frac{4}{5}} + C}$$

. . . . (2) . . . .

. . . . (3) . . . .

.a

$$\int \frac{\sqrt{25x^2-4}}{x}\,\mathrm{d}x = \begin{bmatrix} x = 0.4\sinh x & x' = 0.4\cosh x \\ \mathrm{d}x = 0.4\cosh x\,\mathrm{d}\theta & \end{bmatrix} = \int \frac{\sqrt{4(6.25\cdot 0.4^2\sinh^2 x - 1)}}{0.4\sinh x} 0.4\cosh x\,\mathrm{d}\theta$$

$$= \int \frac{\cancel{0.4}\sqrt{2}\sqrt{\sinh^2 x - 1}}{\cancel{0.4}\sinh x}\cosh x = \sqrt{2}\cosh x \frac{\cosh x}{\sinh x} = \boldsymbol{\sqrt{2}\cosh x\coth x}$$

.b למען הנוחות, נגדיר $a = 1 + \frac{3}{\sqrt{2}}$:

$$\int \frac{x}{\sqrt{2x^2-4x-7}}\,\mathrm{d}x = \frac{x}{\sqrt{\left(x-1-\frac{3}{\sqrt{2}}\right)\left(x-1+\frac{3}{\sqrt{2}}\right)}}\,\mathrm{d}t = \begin{bmatrix} t = x-1 & t' = 1 \\ \mathrm{d}t = 1\,\mathrm{d}x & \end{bmatrix} = \int \frac{t+1}{t^2+a^2}\,\mathrm{d}t$$

$$= \int \frac{1}{t^2+a^2}\,\mathrm{d}t + \int \frac{t}{t^2+a^2}\,\mathrm{d}t = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + \int \frac{t}{t^2+a^2}$$

נפתור את האינטגרל שנותרנו עימו בנפרד:

$$\int \frac{t}{t^2+a^2}\,\mathrm{d}t = \begin{bmatrix} u = t & v = \arctan t \\ du = 1 & dv = \frac{1}{t^2+a^2} \end{bmatrix} = t\arctan t - \int \arctan t\,\mathrm{d}t$$

כאשר האינטגרל של $\arctan$:

$$\int \arctan x\,\mathrm{d}x = \begin{bmatrix} x = \tan\theta\,\mathrm{d}x \\ \mathrm{d}x = \frac{1}{\cos^2\theta}\,\mathrm{d}\theta \end{bmatrix} = \int \arctan\tan\theta \cdot \frac{\mathrm{d}\theta}{\cos^2\theta} = \int \frac{\theta\,\mathrm{d}\theta}{\cos^2\theta} = \begin{bmatrix} u = \theta & v = \tan\theta \\ du = 1 & dv = \sec^2\theta \end{bmatrix}$$

$$= \theta\tan\theta - \int \frac{\sin\theta}{\cos\theta}\,\mathrm{d}\theta = \begin{bmatrix} t = \cos\theta \\ \mathrm{d}t = -\sin\theta\,\mathrm{d}\theta \end{bmatrix} = \theta\tan\theta - \underbrace{\int -\frac{1}{t}\,\mathrm{d}t}_{-\ln|t|} = \theta\tan\theta + \ln|\cos\theta| + C$$

$$= \arctan x \cdot (\tan\arctan x) + \ln(\cos(\arctan x)) + C = x\arctan x + \ln\left(\frac{1}{\sqrt{1+x^2}}\right) + C = x\arctan x - 0.5\ln(1+x^2) + C$$

זאת כי נזכר שהוכח בשיעורי בית 2 כי $\arctan' = \cos^2(\arctan) = \frac{1}{x^2+1}$, כלומר $\cos(\arctan x) = \frac{1}{\sqrt{x^2+1}}$. סה"כ, הראנו כי:

$$\int \frac{t}{t^2+a^2} = \cancel{+\,t\arctan t - t\arctan t} - 0.5\ln(1+t^2)$$

ניזכר למה עשינו את זה מלכתחילה, ונציב באינטגרל המקורי:

$$\cdots = a^{-1}\arctan\left(\frac{x}{a}\right) - 0.5\ln(1+t^2) = \left(1+\frac{3}{\sqrt{2}}\right)^{-1}\arctan\left(\frac{(3+\sqrt{2})(x-1)}{\sqrt{2}}\right) - 0.5\ln(1+(t-1)^2)$$

$$= \boldsymbol{\frac{\sqrt{2}}{\sqrt{2}+3}\arctan\left(1+\frac{3(x-1)}{\sqrt{2}}\right) - 0.5\ln(x^2-2x+2)}$$

.c

$$\int e^{4x}\sqrt{1+e^{2x}} = \begin{bmatrix} t = e^x \\ \mathrm{d}t = e^x\,\mathrm{d}x \end{bmatrix} = \int t^3\sqrt{1+t^2} = \begin{bmatrix} t = \tan\theta \\ \mathrm{d}t = \sec^2\,\mathrm{d}t \end{bmatrix} = \int \tan^3 \cdot \overbrace{\sqrt{1+\tan^2\theta}}^{\sec\theta}\sec^2 x\,\mathrm{d}\theta = \int \tan^3\theta\sec^3\theta\,\mathrm{d}\theta$$

$$= \int (1-\sec^2\theta)\sec^2\theta\tan\theta\sec\theta\,\mathrm{d}\theta \begin{bmatrix} u = \sec\theta \\ \mathrm{d}u = \sec^2\tan\theta\,\mathrm{d}\theta \end{bmatrix} = \int (1-u^2)u^2\,\mathrm{d}u = \int u^2 - \int u^4 = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sec^3\theta}{3} - \frac{\sec^5\theta}{5} + C$$

$$= \frac{\sec^3(\arctan t)}{3} - \frac{\sec^5(\arctan t)}{5} + C = ((1-t^2)^{3/2}3^{-1} - (1-t^2)^{5/2}5^{-1}) + C = \boldsymbol{(1-t^2)^{1.5}3^{-1} + (1-t^2)^{2.5}5^{-1} + C}$$