(1.0) ADTs

List. List(), Retrieve(L, i), Insert(L, b, i), Delete(L, i), Length(L) *optional:* Search(L, b), Concat(L_1, L_2), Plant(L_1, i, L_2), Split(L, i) *special cases:* Retrieve/Insert/Delete-First/Last.

Dictionary. Dictionary(), Insert(D, x), Delete(D, x), Search(D, k), Min(D), Max(D), Successor(D, x), Predecessor(D, x) *(for rank trees):* Select(D, k) [the k^{th} smallest element], Rank(D, x) [the position in sorted order].

Stack. (LIFO) Push(L, b) [=ins.-last], Top(L) [=ret.-last], Pop(L) [=del.-last]. (all $\mathcal{O}(1)$ using arrays)

Queue. (FIFO) Enqueue(L, b) [=ins-last], Head(L) [=ret.-first], Dequeue(L) [=del-first]. (all $\mathcal{O}(1)$ using circular arrays)

Deque. Queue + Stack

Priority Queue. Insert(x, Q), Min(Q), Delete-Min(Q), (optional:) Decrease-Key(x, Q, Δ), Delete(x, Q)

Vector. Vector(m), Get(V, i), Set(V, i, val). (All $\mathcal{O}(1)$ using *legals* and *positions* arrays that reference each other)

```
function isGarbage(i) is
    if 0 ≤ positions[i] < legals.size and legals[positions[i]] = i then
        return false
    end
    return true
end
```

Graph. Edge(i, j), Add-Edge(i, j), Remove-Edge(i, j), InDeg(i), OutDeg(i) etc.

(2.0) Graphs

Definition 1 (Topological sorting algo.). Input: directed graph / Output: numbering $(n_i)_{i=1}^N$ of the graph nodes where $\forall(i,j) \in E: n_i < n_j$.

Theorem 1. Topological Sorting exists iff the graph doesn't contain cycles.

```
k ← 0;
while there are sources do
    find source v;
    n_i ← k;
    k ← k + 1;
    remove v from the graph
end
```

if $k = n$ numbering completed, otherwise isn't possible. building "source queue" takes $\mathcal{O}(n)$, dequeuing source $\mathcal{O}(1)$, and enqueuing new sources to sources-queue $\mathcal{O}(d_{out}(i))$. Total $\mathcal{O}(n+m)$ for topological ordering.

Definition 2 (source). is a node that has no incoming edges.

Remark 1. any DAG has at least one source

(3.0) Complexity

Definition 3. Suppose there's a data structure with k types of operations $(T_i)_{i=1}^k$, then for sequence of operations $(op)_{i=1}^n$, then:

$$\text{time}(op_1 \dots op_n) \le \sum_{i=0}^n \text{bound}(\text{type}(op_i))$$

Where (W.C. bound) worst(T_i) is the maximal time for a single operation type T_i, and (amortized bound) amort(T_i) is a series of bounds for cost of every valid sequence $(op_i)_{i=1}^n$.

Amortization methods. aggregation (regular average), accounting (bank method), and potential function (defined to be the balance of the bank) that satisfies amort(op_i) = time$(op_i) + \Phi_i - \Phi_{i-1}$.

$$\sum_{i=0}^n x^i = \frac{x^{n+1}-1}{x-1} = \Theta(x^n) \quad (x \neq 1)$$

$$\sum_{i=1}^n \frac{1}{i} = H_n = \Theta(\log n)$$

$$\log n! = \Theta(n \log n)$$

$$\alpha + \beta = 1 \wedge T(n) \le cn + T(\alpha) + T(\beta n) \implies T(n) = \mathcal{O}(n)$$

$$\forall \alpha < 1: \quad T(n) = T(\lfloor \alpha n \rfloor) + T(\lfloor (1-\alpha)n \rfloor) + 1 = \mathcal{O}(n)$$

(3.1) Asymptotic Notations

$$f = O(g) \iff \exists n_0, c > 0\, \forall n \ge n_0 : f(n) \le cg(n)$$
$$f = \Omega(g) \iff \exists n_0, c > 0\, \forall n \ge n_0 : f(n) \ge cg(n)$$
$$f = \Theta(g) \iff f = \Omega(g) \wedge f = O(g)$$
$$f = o(g) \iff \forall c\, \exists n_0\, \forall n \ge n_0 : f(n) \le cg(n)$$
$$f = \omega(g) \iff \forall c\, \exists n_0\, \forall n \ge n_0 : f(n) \ge cg(n)$$
$$f = \Omega(g) \iff g = O(f)$$
$$f_1 = O(g_1) \wedge f_2 = O(g_2)$$
$$\implies f_1(n) + f_2(n) = O(\max(g_1(n), g_2(n)))$$
$$f = O(g) \iff \limsup_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$
$$f = \Omega(g) \iff \limsup_{n\to\infty} \frac{f(n)}{g(n)} > 0$$
$$f = o(g) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$
$$f = \omega(g) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

(3.2) Master Theorem

let $f : \mathbb{R} \to \mathbb{R}$ be an function, and let $a \ge 1, b > 1$ be constants, assuming $T : \mathbb{R}_{\ge 0} \to \mathbb{R}$, $T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$, then:

1. $\exists \varepsilon > 0. f(n) = O(n^{\log_b a - \varepsilon})$
   $\implies T(n) = \Theta(n^{\log_b a})$
2. $f(n) = \Theta(n^{\log_b a})$
   $\implies T(n) = \Theta(n^{\log_b a} \cdot \log n)$
3. $\exists \varepsilon > 0. f(n) = \Omega(n^{\log_b a + \varepsilon}) \wedge$
   $\exists c < 1, n_0 \ge 0. \forall n \ge n_0 . a \cdot f(\frac{n}{b}) \le c \cdot f(n)$
   $\implies T(n) = \Theta(f(n))$

Note that $\frac{n}{b}$ *could be* $\left\lfloor \frac{n}{b} \right\rfloor$ *nor* $\left\lceil \frac{n}{b} \right\rceil$

(4.0) Dictionaries

(4.1) General Trees

Definition 4 (full tree). all internal nodes have exactly i children.

Definition 5 (BST). satisfies: $\forall x \forall y$ if y is in the left subtree of x, then $y.key < x.key$, and vise-versa.

Definition 6 (Node's Height). is the maximal length of downward path between that node and a leaf.

Definition 7 (Node's Depth). is the length of the path up the tree to the root.

Theorem 2. minimal height of a tree is $\lfloor \log n \rfloor$

Definition 8 (Balanced BST). if $h = \mathcal{O}(\log n)$.

Theorem 3. for a given set of n distinct keys, there are $\frac{1}{n+1}\binom{2n}{n}$ (catalan number) BSTs.

Theorem 4. the expected search complexity in a random BST is $\le (1 + 4 \log n)$.

Lemma 1. the heights of a binary tree containing ℓ leaves $\ge \log \ell$.

Tree walks. pre: head → SLR, in: LSR, post: LRS

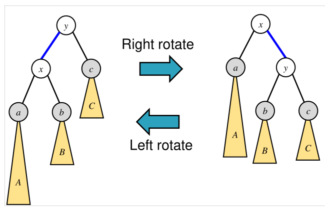
(4.2) AVL trees

Definition 9. BF(v) = h(v.left) − h(v.right)

Definition 10 (AVL Tree). a BST where $\forall v \in V : |\text{BF}(v)| \le 1$

Theorem 5. an AVL tree is balanced. Further more: $h \le \log_\Phi n \approx 1.44 \log n$.

Rotations. see image



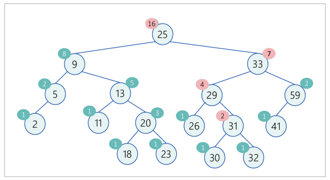
Definition 11 (Fibonacci Tree). F_i is:

$$F_i \to F_{i-1}, \; F_{i-2}$$

Theorem 6. an AVL tree with minimum edges is a fibonacci tree, has a size of $f_{h+3} - 1$.

Definition 12 (Rank Tree). a tree that maintains the size of each subtree, hence supports the rank & select operations in $\mathcal{O}(\log n)$.

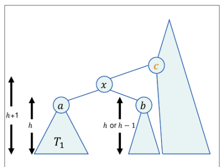
Example. see image (nvm the colors)



Tree-Select(T, k): start with $x \leftarrow T$.root, then let $r \leftarrow x$.left.size + 1, if $k = r$ halt, otherwise if $k < r$ return Select(x.left, k) and if $k > r$ return Select(x.right, $k - r$).

Theorem 7. if the information about a given attribute f defined for each node, can be computed merely from its direct children (local attribute), then we can maintain f in an AVL tree.

Remark 2. The theorem above is sufficient condition but not necessary.



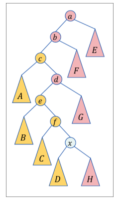
Join(T_1, T_2): where $T_1 < x < T_2$ is done $\mathcal{O}(h_{T_1} + h_{T_2} + 1)$ (see image). Split(T, x): splits T into $T_1 < x < T_2$ in $\mathcal{O}(\log n)$ using joins (image next column).

Lemma 2. the sum of the keys lesser than v, and the sum of the keys in the subtrees, can be implemented both without harming time complexity.

Lemma 3. Between(s, t) = Tree-Rank(t) − Tree-Rank(s) + 1

Definition 13 (Finger Tree). a tree that has a pointer to a specific node.

Split (yellow < x < red):



Theorem 8. in a finger tree Select(T, k) can be implemented in $\mathcal{O}(\log k)$.

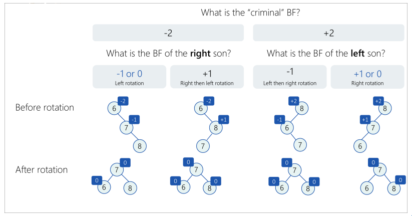
Theorem 9. Given a sorted array, we can create an AVL tree in $\mathcal{O}(n)$ on which $h = \lfloor \log n \rfloor$

Insertion Fix: if |BF| = 2 rotate and terminate, if |BF| < 2 and height hasn't change, terminate, otherwise recursively preform this fix for the parent. (the zero case in blue at the above image doesn't matter for insertions)

Deletion Fix: same as insertions, but with the case for son's BF = 0, and without terminating after rotation (since rotation may not restore the height of the subtree prior the insertion).

Amort. Bounds: in any sequence of insertions only/deletions only, the amortized cost of rebalncing if $\mathcal{O}(1)$ for Φ = #balanced nodes or 1$ on each balanced node.

Insertion Sort to AVL with Max. pointer: has a complexity of $\mathcal{O}\left(n \log\left(\frac{I}{n} + 2\right)\right)$ (see more info under 6.1)



(4.3) B-trees

Definition 14 (B-tree). B-tree (d, $2d$) satisfies:

1. each non-leaf except for the root has $d \le r \le 2d$ children (hence $d-1$ to $2d-1$ keys);
2. all leaves are at the same depth;
3. the root has between 2 and $2d$ children (hence 1 to $2d-1$ keys).

Definition 15 (B^+-tree). a B-tree with keys only on leafs.

Definition 16 (B^*-tree). B-tree with nodes $\frac{2}{3}$ full (instead of $\frac{d}{2d} = \frac{1}{2}$ full).

Theorem 10. at depth h there are at least $2d^{h-1}$ nodes.

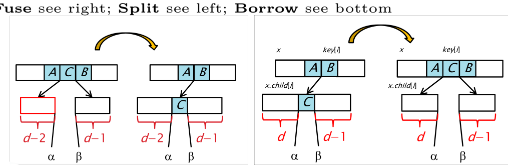
Theorem 11. a B-tree (d, $2d$) with n edges and h height fulfills $n \ge d^h$, $h \le \log_d n$

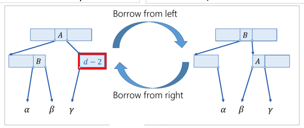
Theorem 12. search in a b-tree requires $\mathcal{O}(\log_d n)$ I/Os, and $\mathcal{O}(\log_2 d \cdot \log_d n) = \mathcal{O}(\log n)$ operations in total.

Lemma 4. In a B-tree #leaves = #internal nodes + 1

Theorem 13. Ins./Del. rebalancing cost is W.C. $\mathcal{O}(\log n)$, and using button-up amort. (ins.+del.) $\mathcal{O}(1)$, using top-down $\Omega(\log_d n)$

Fuse see right; **Split** see left; **Borrow** see bottom





Insertions

Button-Up. Find and insert in the appropriate leaf. If the current node is overflowing: split. If the parent is overflowing: split (etc., recursively). Requires a total of $\mathcal{O}(d \log_d n)$ operations.

Top-Down. if a node is full, we will split it on the way down while searching.

Button-Up non-leaf deletion. replace the item by its predecessor and delete the predecessor (must be a leaf).

Deletions

Button-Up leaf deletion. if the current node is underflowing, borrow and terminate and if not possible fuse and recursively check the if parent if underflowing.

Top-Down leaf deletion. while searching, checking if the items along the way contains d keys, otherwise borrow or fuse.

Top-Down non-leaf deletion. replace the node with its predecessor, while making sure that nodes along the way contains at least d keys.

(5.0) Priority Queues

(5.1) Binary Heap

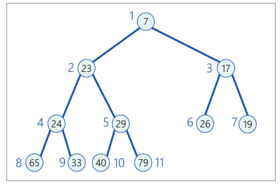
Definition 17 (binary minimum binary heap). an almost perfect BST (only possibly misses nodes at the last level), and satisfies the heap order: the keys at the children of v are greater than they key in v.

Lemma 5. the height of binary heap is $\lfloor \log n \rfloor$

Heap to array. in a d-ary heap representation as an array (in brackets for binary, see image):

$$\text{Left}(i) = dk - (d-2) \quad (2i) \quad \text{Right}(i) = dk + 1 \quad (2i+1)$$

$$\text{Parent}(i) = \left\lfloor \frac{k + (d-2)}{d} \right\rfloor \quad \left(\left\lfloor \frac{i}{2} \right\rfloor\right)$$



Heapify-Down(i): if Parent(i) is bigger, then replace i with Parent(i), and recursively continue on Parent(i).

Heapify-Up(i): exchange with the smallest child until fixed.

Insert: insert on the last place in the array, then heapify up.

Delete: delete the required place in the array (the root) the replace it with the last one, then heapify down until fixed (in d-ary $\mathcal{O}(d \log_d n)$).

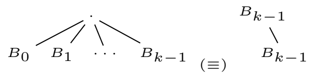
Dec-Key: decrease the key (assumes $\Delta \ge 0$) then heapify up.

Init: iterate over internal nodes bottom-up, and heapify-down each one.

HeapSort: create a min-heap from input, the do delete-min and put the deleted element at the last position of the array. Repeat n times. At the we get a reversely-sorted array (using min-heaps).

(5.2) Binomial Trees

Definition 18. B_k is a binomial tree of degree k if



Theorem 14. (1) The root of B_k has k children (2) B_k contain 2^k nodes (3) its depth is k (4) $\binom{k}{i}$ of the nodes of B_k are at level i.

Definition 19 (Binomial Min-Heap). a list of heap-ordered binomial trees, at most one of each degree, and a pointer to the root with the minimal key.

Remark 3. usually the trees are saved using a linked list.

Lemma 6. There are at most $\lfloor \log n \rfloor + 1$ trees.

Link: if two binomial trees x, y has the same degree, linking could be preformed in $\mathcal{O}(1)$ by attaching y as a child of x and replacing the roots if needed.

Insert: insertion could be done the same way as binary incrementing, where linking≡carrying.

Dec-Key: just heapify up as before.

Meld: link trees with the same degree, like binary addition.

Del-Min: the children of the deleted root are a binomial heap, Meld them into the main tree.

Lazy Binomial Trees adds just B_0-s (allows melding in $\mathcal{O}(1)$), and consolidates when runs delete-min.

Consolidating (on del-min) is the process of taking the nodes and adding them into respected bins (numbered $0 \dots \lfloor \log n \rfloor$), and when two trees are in the same bin – linking them together and moving them into the next bin.

Definition 20. T_0 is #trees before Del-Min, T_1 after Del-Min, and L is the total #Links through consolidating.

Lemma 7. $L \le T_0 + \log n$ (we have at most $\lfloor \log n \rfloor$ trees exposed on Del-Min)

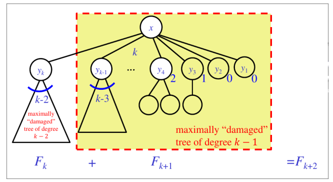
Theorem 15. The cost of consolidating is $T_0 - 1 + \log n + L = \Theta(T_0 + \log n)$.

Theorem 16. Using $\Phi = \#trees$ we get $\Delta\Phi = T_1 - T_0$ hence amort. cost of consolidating is $\mathcal{O}(\log n)$.

Lemma 8. incrementing a binary number has an amortized bound of $O(1)$.

(5.3) Fibonazi Heaps

Dec-Key: using cascading cuts: cut the node. Then if the parent marked as "LOSER" remove him too, otherwise mark it as a LOSER. Outcome: a parent with more than 2 children taken out, is taken out too.



Remark 4. a node in a fibonacci-heap has degree k if the root has k children (see image for a maximally damaged one).

Lemma 9. let x be a node with degree k, and let children $y_1 \dots y_k$ be its children (in the linking order), then y_i's degree is at least $i-2$.

Lemma 10. A node with deg. k has at least $f_{k+2} \geq \phi^k$ descendants (including) .

Lemma 11. in a fib. heap all degrees are at most $\log_\phi n \leq 1.44 \log n$

Theorm 17. For a potential of $\Phi = \#\text{trees} + 2\#\text{markednodes}$, we get `Dec-Key` in amort. $\mathcal{O}(1)$.

***Remark* 5.** actual cost of del-min is $T_0 + \log n$ and of dec-key is $+c$ (c no. newly created trees). With the potential above, for dec-key we get $\Delta = 2(2-c)$.

(6.0) Sorting

(6.1) Comparison-based sorting

Definition 21 (Insertion Sort). at iteration $i \in [n]$, by induction we assume $A[1] \cdots A[i-1]$ is sorted, $A[i]$ "bubble-up" until $A[1] \cdots A[i]$ is sorted ($\mathcal{O}(i)$ per iteration).

***Remark* 6.** can be optimized (in terms of exchanges, but not comparison) if $A[i]$ is saved separately.

Definition 22 (Online sort). a sorting algorithm that doesn't have the whole input at the beginning (e.g. insertion sort)

Theorm 18. insertion sort using AVL tree with insertion from the maximum, and $I > n$ inversions ($I \leq \binom{2n}{n}$) takes $\mathcal{O}\left(n \log \frac{I}{n}\right)$.

Definition 23 (stable sort). a sorting algo. the preserves order of items with the same key.

Definition 24. a comparison-based algo. uses only two-key comparisons to decide on key position.

assumption. two keys can be compared in $\mathcal{O}(1)$, and an item can be moved in $\mathcal{O}(1)$.

Theorm 19. the W.C. and average case of any comparison-base sorting algo. runs in $\Omega(n \log n)$

Lemma 12. comparison trees are a full binary tree, and has $\geq n!$ leafs.

Theorm 20. the worst/best/average case in the comparison-based model is the max/min/average depths of the leafs.

(6.2) Other sorting algos.

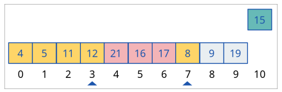
HeapSort: see 5.1.
Count Sort. For dataset A, assumes $\exists R \forall a \in A \leq R$ constant. Counts each element $a \in A$, takes a cumulative sum (a_i), then for all $a \in A$ puts a in a_i and decreases $a_i \leftarrow a_i - 1$. Takes $\mathcal{O}(n+R)$. Stable sort.

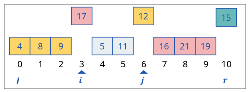
Bin Sort. similar to count sort, takes R bins and throws A into them, then collects them.

Radix sort. For a dataset A sized n, assumes $a \in A$ contains exactly d digit and each digit is bounded by b. Preforms count sort on the LSD $\rightarrow$ MSD. [note: relies on count sort being stable]. Takes $\mathcal{O}(d(n+b))$.

Theorm 21. Radix sort is enough to make IBM.

(6.3) QuickSort





Lomuto's Partition. see left image
Hoare's Partition. see right image

***Remark* 7.** both in place $\mathcal{O}(n)$, lomuto's pivot in the right place while in hoare the pivot is on the extreme right.

Theorm 22. W.C. of quicksort if $\binom{n}{2} = \mathcal{O}(n^2)$.

Theorm 23. Average case of quicksort is $2(n+1)H_n - 4n \approx 1.39 n \log n$.

Lemma 13. Two keys are compared at most once by quicksort.

Lemma 14. The probability that quicksort compares the i^{th} smallest key with the j^{th} where $i < j$ is $\frac{2}{j-i+1}$.

(prove by indicator if i, j compared)

(7.0) Probability

Definition 25 (Experiment). a case where we the result is uncertain.

Definition 26 (Sample Space). the set of all the expected outcomes of a given experiment.

Definition 27. an Event is a subset of the sample space. A singleton subset called a simple event.

Definition 28. *Disjoint Events* are events A, B that fulfills $A \cap B = \varnothing$.

Definition 29 (Probability Function). a function $P\colon S \to [0,1]$ for S sample space, so that $\forall E, F$ disjoint: $P(E \cup F) = P(E) + P(F)$ and $P(S) = 1$.

Definition 30. the Conditional Probability of event E given the event F is $P(E \mid F) := \frac{P(E \cap F)}{P(F)}$.

Theorm 24. for disjoint events $(F_i)_{i=1}^n$, if $\bigcup F_i = E$ then $\forall E\colon P(E) = \sum_{i=0}^n P(E \mid F_i) \cdot P(F_i)$.

Definition 31. events E, F are independent if $P(E \cap F) = P(E) \cdot P(F)$ (iff $P(E \mid F) = P(E)$).

Definition 32 (Random Variable). a function $X\colon S \to \mathbb{R}$.

Definition 33. $X = x$ is the event on which $X(E) = x$, and its probability noted as $P(X = x)$.

Definition 34. the Expectation of a random variable X is $\mathbb{E}[x] = \sum_x x \cdot P(X = x)$.

Theorm 25. the expectation is linear for all constants, and additive for all random variables.

Definition 35. a random variable I is called an Indicator of an event A if $I = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A^c \text{ occurs} \end{cases}$

Lemma 15. $\mathbb{E}[I] = P(A)$.

Definition 36 (Uniform Distribution). of a random variable X occurs when $\exists c \colon \forall x \in \mathbb{R}\colon P(X = x) = c$.

Definition 37 (Geometric Distribution). satisfies $P(x = k) = (1-p)^{k-1} p$, hence $\mathbb{E}[X] = \sum_{k=1}^\infty k(1-p)^{k-1} p = \frac{1}{p}$.

***Remark* 8.** geometric dist. is equal to having the probability of success p and for failure $p-1$, and P is a rand. var. that is equal to the number of required experiments to get to an solution.

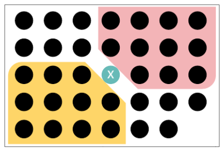
Theorm 26 (The Tail Formula). $\sum_{i=0}^m i \cdot P[X = i] = \sum_{i=1}^m P(x \geq i)$

Theorm 27 (Markov's Inequality). $P[X \leq 2\mathbb{E}[X]] \geq 0.5$

(8.0) Selection

Definition 38. given n numbers, `Select`(n) is defined to return the k^{th} smallest key.

E.g. (width$= \left\lfloor \frac{1}{2} \left\lfloor \frac{n}{5} \right\rfloor \right\rfloor$, height$=3$, total$\geq 3 \cdot \frac{n}{10} - 1 - 3$) This equals for the item in position k, assuming the array was sorted.



Definition 39 (Dynamic settings). assumes one-time building cost (e.g. `Tree-Select`).

Definition 40 (Static settings). not a dynamic setting

Theorm 28. Using min-heap + supporting heap the selection problem is solvable is $\mathcal{O}(n + k \log k)$.

Theorm 29. The expected number of items removed during each quickselect run is $\mathbb{E}[\#\text{removed}] = \frac{k}{2} \cdot \frac{k}{n} + \frac{n-k}{2} \cdot \frac{n-k}{n} \underset{\forall k}{\geq} \frac{n}{4}$.

Theorm 30. The expected runtime of quickselect is $\mathcal{O}(n)$.

Theorm 31. MedofMed cost is W.C. $\mathcal{O}(n)$.

(9.0) Hashing

Direct Addressing. Create a bit vector with the universe size. e.g. `Insert`(D, x) iff `D`$[x.key] \leftarrow x$ etc.

(9.1) Chaining

Lemma 16. There are $|m|^{|U|}$ hashes in $h \in U \to [m]$, hence takes $|U| \log m$ to store.

Chaining. each cell points to a linked list of items.

Definition 41 (load factor). $\alpha := \frac{n}{m}$ where n is the universe, and m is the table size.

Lemma 17. the probability of a random two specific insertions colliding is geometric.

Theorm 32. the expected number of values in each cell is α.

Theorm 33. when $n = \Theta(m)$, with probability $\geq 1 - \frac{1}{n}$, each cell contains at most $\mathcal{O}\left(\frac{\log n}{\log n \log n}\right)$ elements.

Theorm 34. Assuming the keys are distributed ideally (uniformly and independently), and assuming n keys were previously inserted, the expected complexity during search is $\alpha + 1$ for unsuccessful and $\frac{\alpha}{2} + 1$ for a successful search.

(9.2) Open Addressing

Open Addressing. ℓet $h\colon U \times [m] \to [m]$ be a hash function, we'll insert the key k in the first free position in the probing sequence.

***Remark* 9.** make sure to use special marking (not null) for deleted items.

Theorm 35. Under ideal conditions (means $\forall k \in [n]\colon P\left((h(k,i)_{i=0}^{m-1}) = \frac{1}{m}\right)$), the expected time for unsuccessful search is $\frac{1}{1-\alpha}$ and for successful search $\frac{1}{\alpha} \ln \frac{1}{1-\alpha}$.

Theorm 36. under linear probing, unsuccessful search takes $\frac{1}{2}\left(1 + \left(\frac{1}{1-\alpha}\right)^2\right)$ and successful search $\frac{1}{2}\left(1 + \frac{1}{1-\alpha}\right)$.

***Remark* 10.** under linear probing, we can delete by recursively checking if item j can me moved to deleted cell i for all $h'(T[j]) \in [j+1, i]$.

Probing Sequence: $h(k, 0) \dots h(k, m-1)$

Definition 42 (Linear Probing). a hash func $h(k,i) := (h'(k) + i) \bmod m$ (less cache misses + easy to calculate).

Definition 43 (Quadratic Probing). a hash func $h(k,i) := (h'(k) + ic_1 + c_2 i^2) \bmod m$.

Definition 44 (Double Probing). a hash func $h(k,i) := (h'(k) + ih''(k)) \bmod m$.

(9.3) Hash Families

Definition 45. $|Col|$ is the number of collision for a given h hash.

Definition 46. hash family is Universal if $\forall k_1 \neq k_2 \in U\colon P_{h \in H}(h(k_1) = h(k_2)) \leq \frac{1}{m}$.

Theorm 37. For all p prime, $h_{a,b}\colon [p] \to [m]$ defined as $h_{a,b}(x) = ((ax+b) \bmod p) \bmod m$, and $H_{p,m} := \{h_{a,b} \mid a \in [1,p), b \in [0,p)\}$ is a universal hash family.

Theorm 38. for each p prime, ℓet $x_1 \neq x_2 \in [p]$. Then $\forall y_1 \neq y_2 \in [p]\ \exists! a, b \in [p], a \neq 0\colon y_1 \equiv_p a x_1 + b \wedge y_2 \equiv_p a x_2 + b$.

Theorm 39. for a table $m = 2^k$ so $h_a\colon U = [2^w] \to [2^k]$ where w is computer word size, h_a defined as $\left\lfloor \frac{ax \mod 2^w}{2^{w-k}} \right\rfloor$, and H is almost universal.

Definition 47. if $\forall k_1 \neq k_2 \in U\colon P_{h \in H}(h(k_1) = h(k_2)) \leq \frac{2}{m}$ then U is called almost universal.

Theorm 40. using universal hash family, $\mathbb{E}[\text{collisions}] \leq \frac{\binom{n}{2}}{m}$.

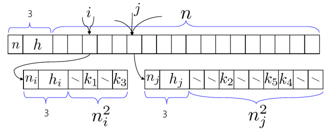
Theorm 41. If $m = n$, then $P(|Col| < n) \geq \frac{1}{2}$

Theorm 42. If $m = n^2$, then $P(|Col| < 1) \geq \frac{1}{2}$

***Remark* 11.** Theorems 40, 41 are derived from theorem 39 using Markov's inequality (see 7.0)

(9.4) Perfect Hashing

Init: choose random $h \in H_{p,n}$ (modular), compute the number of collision, until there are $< n$ collisions (expected 2 attempts). Then for each cell $i \in [n]$ let $n_i := \left|h^{-1}[\{i\}]\right|$, if $n_i > 1$ choose a random $h_i \in H_{p,n_i^2}$ until there are no collisions.



Total size $= 3 + n + 3n + \sum_i n_i^2 = 4n + 3 + \sum(2\binom{n_i}{2} + n_i)$ and since $|col| = \sum_i \binom{n_i}{2}$ we get a total of $\leq 7n + 3$.

(10.0) Other

Reduction. reduction (in our case) is the process of showing the a problem is at least as hard as another problem.
Information Bound. a bound derive by an argument that the algo. has to read a specified amount of the input, in order to get a decision. Notice that in comparison, reading isn't counted.

$$\sum_{i=1}^k \binom{k}{i} = 2^k \qquad \binom{k}{i} = \binom{k-1}{i} + \binom{k-1}{i-1}$$

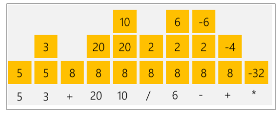
$$a_i = a_1 + (n-1)d \implies \sum_{i=0}^n a_i = \frac{n(a_1 + a_n)}{2}$$

$$a_i = a_1 q^{n-1} \implies \sum_{i=0}^n a_i = \frac{a_1(q^n - 1)}{q-1}$$

Theorm 43. merging k sorted arrays with the total of n items can be done in $\mathcal{O}(n \lfloor \log k \rfloor)$

Theorm 44. Fibonacci closed form. $F_n = \frac{\phi^n - (\bar{\phi})^n}{\sqrt{5}}$ where $\Phi = \frac{1+\sqrt{5}}{2}$

Postfix syntax algo. parse mathematical expressions. For each element e from left to right: (1) if e is operand then push e (2) if e binary operator, pop 2 elements x, y then push $e(x,y)$, and if e unary operator pop 1 element x and push $e(x)$ (see image).



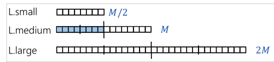
Jensen's inequality. $f(\frac{x_1 + x_2}{2}) \leq \frac{f(x_1) + f(x_2)}{2}$ if f convex.

(10.1) Array Doubling

Potential for doubling by $(1+\alpha)$. $\Phi := \begin{cases} \frac{1+\alpha}{\alpha} n - \frac{M}{\alpha} & n > \frac{M}{\alpha+1} \\ 0 & \text{else} \end{cases}$

yields to amort. bound $\mathcal{O}\left(\frac{1+\alpha}{\alpha} + 1\right)$

Potential for array doubling (including amort deletion: changing the size back when $n = \frac{M}{4}$).



$$\Phi = \begin{cases} 2n - M & \text{if } n \geq \frac{M}{2} \\ \frac{M}{2} - n & \text{if } n < \frac{M}{2} \end{cases}$$

De-Amortized array doubling. see image

(11.0) Complexity Tables

Lists

ℓet mid $:= \min\{i, n-i\} + 1$

	`Ins/Del-Last`	`Ins/Del-First`	`Insert`(i)	`Retrieve`(i)	`Concat`(n_1, n_2)	`Split`(i)
Arrays	$\mathcal{O}(1)$	$\mathcal{O}(n+i)$	$\mathcal{O}(n-i+1)$	$\mathcal{O}(1)$	$\mathcal{O}(n_2+1)$	$\mathcal{O}(n-i+1)$
Circular Arr.	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(\text{mid})$	$\mathcal{O}(1)$	$\mathcal{O}(\min\{n_1, n_2\})$	$\mathcal{O}(\text{mid})$
D-Linked	$\mathcal{O}(1)$	$\mathcal{O}(1)$	$\mathcal{O}(\text{mid})$	$\mathcal{O}(\text{mid})$	$\mathcal{O}(1)$	$\mathcal{O}(\text{mid})$
AVL List	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log i + 1)$	$\mathcal{O}(\log(n_1 + n_2))$	$\mathcal{O}(\log n)$

(in a lazy doubly-linked list, amortized del./ins. $\mathcal{O}(1)$ and ret. $\mathcal{O}(i+1)$)

Priority Queues

	`Insert`	`Minimum`	`Delete-Min`	`Dec.-Key`	`Delete`(pointer)	`Meld`	`Init`
AVL tree	$\mathcal{O}(\log n)$	$\mathcal{O}(1)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(n)$	$\mathcal{O}(n \log n)$
Binary Heap	$\mathcal{O}(\log n)$	$\mathcal{O}(1)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(n)$	$\mathcal{O}(n)$
W.C Binomial Heap	$\mathcal{O}(\log n)^{(*)}$	$\mathcal{O}(1)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(\log n)$	$\mathcal{O}(n)$
Lazy Amort./WC			$\mathcal{O}(\log n)$		$\mathcal{O}(\log n)$		
Binomial Stack	$\mathcal{O}(1)_{W.C.}$	$\mathcal{O}(1)_{W.C.}$	$\mathcal{O}(n)_{W.C.}$	$\mathcal{O}(\log n)_{W.C.}$	$\mathcal{O}(n)_{W.C.}$	$\mathcal{O}(1)_{W.C.}$	$\mathcal{O}(n)_{W.C.}$
Amort./WC			$\mathcal{O}(\log n)$		$\mathcal{O}(\log n)$	$\mathcal{O}(1)$	
Fib. Heap:	$\mathcal{O}(1)_{W.C.}$	$\mathcal{O}(1)_{W.C.}$	$\mathcal{O}(n)_{W.C.}$	$\mathcal{O}(1)_{W.C.}$	$\mathcal{O}(n)_{W.C.}$	$\mathcal{O}(n)_{W.C.}$	$\mathcal{O}(n)_{W.C.}$

$^{(*)}$amortized $\mathcal{O}(1)$ for a sequence of operations from the same type.