**Fourier Series for f(x),** $-L \le x \le L$

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \qquad B_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$A_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx$$

**Note:**

- If the function f(x) is **odd** then $A_n$ is zero for all n and the **sin** integral from -L to L can be replaced by 2 * the same integral from 0 to L.
- If the function f(x) is **even** then $B_n$ is zero for all n and the **cos** integral from -L to L can be replaced by 2 * the same integral from 0 to L.

**Useful identities:**

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 2L & \text{if } n = m = 0 \\ L & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} L & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0 \qquad \cos(n\pi) = (-1)^n \qquad \sin(n\pi) = 0$$

**Graphing Fourier function on the whole real line.**

- The graph of Fourier approximation for f(x) at (-L, L) is the same as the graph for f(x).
- If f(L) = f(-L) then Fourier function at L and -L has the same value as f(L).
- If f(L) ≠ f(-L) then Fourier function at L and -L has the value of (f(L) + f(-L)) / 2.
- Outside the interval [-L, L] the graph is repeated infinitely because Fourier series is a periodic function.

**Complex form of Fourier series:**

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}, \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\,e^{-\frac{in\pi x}{L}}\,dx,$$

- **Dirichlet's integral:**
    - $\int_0^{+\infty} \frac{\sin(\alpha x)}{x} dx = \frac{\pi}{2} sgn(\alpha)$
- $\int_a^b dx \int_c^d f(x,y)dy \ \ =^? \ \int_c^d dy \int_a^b f(x,y)dx$
    - **If both integrals are proper**
        - It's sufficient that f(x, y) is continuous for any x, y ∈ [a, b], [c, d]
    - **If both integrals are improper**
        - f(x, y) should be continuous for any x, y ∈ [a, b), [c, d)
        - $\int_a^b |f(x,y)|dx$ and $\int_c^d |f(x,y)|dy$ both should converge uniformly with respect to their parameters on any subinterval ⊂ the domain of the variable of integration.
        - At least one integral (the LHS or the RHS) converge.
    - **If one of them is proper**
        - Continuity condition should hold.
        - The improper one should converge uniformly.
- $\lim\limits_{\omega\to+\infty} \int_0^a \frac{\sin\omega x}{x} dx = \frac{\pi}{2}$ if a > 0
- If h(x) is integrable on [a, b] then $\lim\limits_{\omega\to\infty} \int_a^b h(x)\cos(\omega x)\, dx \ = 0$, same for sin(ωx)
- **Fourier integral:**
    - $\frac{f(x^+)+f(x^-)}{2} = \frac{1}{\pi}\int_0^{+\infty} a(y)\cos(xy) + b(y)\sin(xy)\, dy =$ f(x) if f is continuous at x
        - $a(y) = \int_{-\infty}^{+\infty} f(t)\cos(yt)\, dt,\ b(y) = \int_{-\infty}^{+\infty} f(t)\sin(yt)\, dt$
- **Complex form:**
    - f(x) = $\frac{1}{2\pi}\int_{-\infty}^{+\infty} [\int_{-\infty}^{+\infty} f(t)e^{iyt}dt].e^{iyx}dy$, only the expression in brackets should be calculated.
- **Fourier transform:**
    - **Direct transform:** $\hat{f}(y) = \frac{1}{\sqrt{(2\pi)}} \mathrm{P.V}\int_{-\infty}^{+\infty} f(x)e^{-ixy}dx$
    - **Inverse transform:** $\tilde{f}(y) = \frac{1}{\sqrt{(2\pi)}} \mathrm{P.V}\int_{-\infty}^{+\infty} f(y)e^{ixy}dy$
- P.V is the principal value of integral (Check snippets folder).
    - Usually it's equal to the integral itself, if there are no points of singularity other that -∞ and +∞.
- **Fourier transform of derivatives:** $\mathrm{F}[\mathrm{f}^{\mathrm{n}}(\mathrm{x})](\mathrm{y}) = (\mathrm{iy})\mathrm{n}.\mathrm{F}[\mathrm{f}(\mathrm{x})](\mathrm{y})$
- **Derivative of Fourier transform:** $\frac{d^nF[f(x)](y)}{dy^n} = F[(ix)^n f(x)](y)$
    - F[f(x)] denotes the Fourier transform applied on f(x).
- **You may need this:**

$$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$$

$$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$$

$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$$

**Sum and Difference Identities**

$$\sin(a+b) = \sin a\cos b + \cos a\sin b$$

$$\sin(a-b) = \sin a\cos b - \cos a\sin b$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$\cos(a-b) = \cos a\cos b + \sin a\sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1-\tan a\tan b}$$

$$\tan(a-b) = \frac{\tan a - \tan b}{1+\tan a\tan b}$$

$$sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$$

$$cos(x) = \frac{e^{ix}+e^{-ix}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$