- **Maxima and minima of multivariable functions**
  - Put first order derivatives to 0
  - Solve all equations simultaneously to find stationary points
  - Use either second differential test or Sylvester criterion on Hessian matrix.
- **Method 1: Second differential test**
  - Calculate d^2u
    - $d^2u = U_{xx}.dx^2 + 2U_{xy}dxdy + U_{yy}.dy^2$ (2D case)
    - $d^2u = U_{xx}.dx^2 + U_{yy}.dy^2 + U_{zz}.dz^2 + 2U_{xy}dxdy + 2U_{xz}dxdz + 2U_{yz}dydz$ (3D case)
  - If d^2u at point P is:
    - Positive definite (> 0 for all (x, y) besides (0, 0)) ⇒ P is a local min.
    - Negative definite (< 0 for all (x, y) besides (0, 0)) ⇒ P is a local max.
    - Indefinite ($\exists$ (x_1, y_1), (x_2, y_2) | $u(x_1, y_1) < 0$ and $u(x_1, y_1) > 0$) ⇒ P is a saddle point.
    - ± Semi-definite ($\exists$ (x, y) ≠ (0, 0) | u(x, y) = 0).
      - Test is inconclusive. (ex. $\pm(x+2y)^2$)
- **Method 2: Find the matrix Hessian Matrix H.**
  - H > 0 ⇒ P is local min
  - H < 0 ⇒ P is local min

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Theorem 1.1 (Sylvester's criterion). *Let A be an $n \times n$ symmetric matrix.*

- *$A \succ 0$ if and only if $\Delta_1 > 0, \Delta_2 > 0, \ldots, \Delta_n > 0$.*
- *$A \prec 0$ if and only if $(-1)^1\Delta_1 > 0, (-1)^2\Delta_2 > 0, \ldots, (-1)^n\Delta_n > 0$.*
- *A is indefinite if the first Δ_k that breaks both patterns is the wrong sign.*
- *Sylvester's criterion is inconclusive (A can be positive or negative semidefinite, or indefinite) if the first Δ_k that breaks both patterns is 0.*

- **Extrema for a multivariable function u subject to constraints.**
  - If constraints have the form (f(x, y) = 0)
    - Use Lagrange multiplier test
  - If constraints have the form (a ≤ x ≤ b), (c ≤ y ≤ d)
    - Calculate all critical points as usual and exclude those who are not in the boundaries.
    - Calculate function at the boundaries themselves
    - Calculate extrema for f(a, y), f(b, y), f(x, c), f(x, d) (Use calculus I methods by finding first derivative and put it to zero and comparing all values)
    - Compare all values you got from step (2) and (3) to determine max and min.
  - If constraints have the form (a ≤ f(x, y) ≤ b)
    - Suppose f(x, y) – c = 0
    - Use Lagrange multiplier test.
    - Solve for x = g(c), y = h(c)
    - Substitute in the original function to get u = u(c)
    - Get extrema for u using Calculus I method
    - Test all function values to get max and min
      - Exclude values of 'c', which are not in the range [a, b].
- **Lagrange multiplier test: if we have n constraint functions on the form $f_{1..n} = 0$**
  - Calculate Lagrange function $L = u - \lambda_1 f_1 - \lambda_2 f_2 - \ldots - \lambda_n f_n$
  - Solve for these equations simultaneously to get stationary points
    - grad(L) = 0
    - $f_1 = f_2 = \ldots = f_n = 0$
  - Determine definiteness of d^2L

- **Gradient vector** of a function: a vector of partial derivatives: grad(f(x, y)) = < f_x, f_y >
- **Directional derivative** of some function with respect to a vector v: $D_v f$ = grad(f).u
    - u is a unit vector in the direction of v (u = $\frac{v}{|v|}$) and '.' is the dot product.
- **Limits of multivariable functions:** $\lim_{(x,y)\to(a,b)} f(x,y)$
    - Try direct substitution.
    - Try to use any line that passes through the point where you are taking the limit.
        - Try these lines in order x=a, y=b, x=y, y=x2, y2=x
        - If any two of them gave different results → limit DNE
        - If all of them gave the same result. It's extremely possible that the limit is this point, but you can't conclude that unless you prove it, by either transforming to polar coordinates, or using the definition of the limit.
- **Solving equations introducing new variables or transforming to polar coordinates:**
    - Describe the relation between the old variables and the new ones.
    - Find the partial derivative with respect to old variables in terms of the new variables.
    - **Example:**
        - z = z(x, y), Introduce new variables u, v where u = u(x, y) and v = v(x, y)
            - $z_x = z_u u_x + z_v v_x$ and we can calculate u_x and v_x
            - $z_{xx} = (z_u u_x + z_v v_x)_x = z_u u_{xx} + u_x z_{ux} + z_v v_{xx} + v_x z_{vx}$
            - Note: $z_{ux} = z_{uu} u_x + z_{uv} v_x$
    - Substituting into the given equation yields the answer for transformation.
    - If during substitution you find that $z_u = 0$ then z depends only on v ⇒ z = f(v)
        - where f is an arbitrary continuously differentiable function.
    - **Polar coordinates:**
        - x = r.cos(t), y = r.sin(t) ⇒ r = $\sqrt{}(x^2+y^2)$, t = arctan(y/x)
- **Taylor expansion for multivariable functions:**
    - $f(x_0 + dx, y_0 + dy) = \sum_{k=0}^{n} \frac{f^{(n)}(x_0,y_0)}{k!}$