- **Functional sequence is a sequence of different functions defined on some interval Δ:**
  - Example: $f_n(x) = x^n$, $x \in \Delta$
    - $f_1(x) = x$, $f_2(x) = x^2$, and so on…
  - Calculating the limit of the functional sequence means getting the limit function f(x)
    - Limit function is a single function that the sequence converges to, letting $n \to \infty$
      - For the above example if $\Delta = [0, 1]$, then
      - $f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$
- **Pointwise convergence:**
  - The functional sequence $f_n(x)$ converges pointwise if $\lim_{n\to\infty} f_n(x) = f(x)\ \forall x \in \Delta$
  - $\forall x \in \Delta, \forall \epsilon > 0, \exists n \ge N \text{ s.t } |f_n(x) - f(x)| < \epsilon$
  - For any point we choose in the interval, we can find its own epsilon (If we can find some value for epsilon for all x's in Δ then convergence is uniform).
- **Uniform convergence:**
  - $\forall \epsilon > 0, \exists n \ge N, \forall x \in \Delta \text{ s.t } |f_n(x) - f(x)| < \epsilon$
  - For any epsilon we choose, there is a starting point after which the functional sequence is bounded (will not exceed some fixed value ϵ)
  - There is one neighborhood "tube" that bound the functional sequence for all points in Δ
  - To use the above definition, we need to find the supremum of $|f_n(x) – f(x)|$ and prove that it's a fixed value.
- **Useful properties of uniform convergence:**
  - If $f_n(x)$ converges uniformly to f(x), $x \in \Delta = [a, b]$, $f_n(x)$ is continuous $\forall n$, then
    - f(x) is also continuous $\forall x \in \Delta$
    - $\int_a^b \lim_{n\to\infty} f_n(x)\, dx = \lim_{n\to\infty} \int_a^b f_n(x) dx$
    - In the same way we can swap taking limit and first derivative.
- **Functional series: An infinite series of functions.**
  - $\sum_{n=1}^{\infty} a_n(x)$ converges to S(x) if the functional sequence of partial sums $S_n(x) = \sum_{k=1}^{n} a_k(x)$ converges to S(x).
  - All properties of functional sequence apply to series.
- **Cauchy criterion of convergence of $\boldsymbol{\sum_{k=1}^{\infty} a_k(x)}$**
  - $\forall \epsilon > 0\ \exists N: \forall n \ge N,\ \forall p \ge 0 \Rightarrow |\sum_{k=n}^{n+p} a_k(x)| < \epsilon$
  - For any small epsilon we choose we can always find a starting point N that, starting from it, the partial sums of any length p are all bounded from above and below by epsilon.
  - **Useful for proving that convergence is not uniform.**
  - $\sum_{k=1}^{\infty} a_k(x)$ does not converge uniformly $\Leftrightarrow \exists \epsilon > 0, \forall N: \exists n \ge N, \exists p \ge 0\ \exists x \in \Delta: |\sum_{k=n}^{n+p} a_k(x)| \ge \epsilon$
  - **Practically**, If $\lim_{n\to\infty} \sup_{x\in\Delta} |f_n(x) - f(x)| \ne 0$ then series doesn't converge uniformly, otherwise, we don't know and we have to use another test.
- **Weierstrass M-test:**
  - The series $\sum_{n=1}^{\infty} a_n(x)$ converges uniformly and absolutely if we know that $|a_n(x)| \le b_n\ \forall n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n$ converges.
  - We can find b_n by taking the derivative of a_n with respect to x, and find which x yields the absolute max, we set x to that value in a_n and that will be b_n.

- **Uniform convergence of functional series:**
  - We fix x to some value in Δ and check that the series converges pointwise using any test from last semester (divergence/limit/root/ratio/Cauchy/…).
  - If series diverges, then it cannot converge uniformly.
  - If it converges -pointwise- we need to check convergence of its max series (Weierstrass) which we can get using first derivative test (finding critical points of x by taking the derivative of $f_n(x)$ with respect to x and substitute x with the point that yields absolute max of a_n the result will be the series b_n), if b_n converges then a_n converges uniformly.
- **Dirichlet's test for uniform convergence:**
  - If we have the series $\Sigma\, a_n(x)b_n(x)$, $x \in \Delta$ and we know that:
    - Partial sums of a_n are uniformly bounded, that is $| \Sigma a_n(x) | \leq M$
    - $b_n(x)$ is monotonic, $b_n(x)$ converges uniformly to 0
  - Then the original series converges uniformly on Δ