- **If** we have a function $I(\alpha) = \int_a^b f(x,\alpha)dx, \;\; \alpha_1 \le \alpha \le \alpha_2$, and we know that f(x, α) is continuous on x ∈ [a, b] and α in [α_1, α_2]
  - **Then** I(α) is also continuous on the same domain and we have $I'(\alpha) = \int_a^b f_\alpha(x,\alpha)dx$
- **If** the function is not continuous on some x ∈ [a, b] but has a finite limit at this point
  - **Then** the theorem still works, because we can redefine the function on this specific point.
- **If** we have $I(\alpha) = \int_{g(\alpha)}^{h(\alpha)} f(x,\alpha)dx, \;\; \alpha_1 \le \alpha \le \alpha_2$
  - **Then** $I'(\alpha) = f(h(\alpha),\alpha).h'(\alpha) - f(g(\alpha),\alpha).g'(\alpha) + \int_{g(\alpha)}^{h(\alpha)} f_\alpha(x,\alpha)dx$
- The improper integral $\int_a^\infty f(x)dx$ converges if $\lim_{t\to\infty} \int_a^t f(x)dx$ is finite.
- The improper integral $\int_a^\infty f(x,\alpha)dx$ converges **uniformly** if $\lim_{t\to\infty} \int_a^t f(x,\alpha)dx = h(\alpha)$ and the convergence is uniform.
- Sometimes, when calculating $I(\alpha) = \int_a^b f(x,\alpha)dx$, it's easier to calculate I'(α) using the theorem above, then we integrate the result indefinitely with respect to α to find the original integral.
  - To find the value of C, Find any special point that is easy to find from the original problem and substitute to get C, for example, I(0).
- **Γ function:** $\Gamma(x) = \int_0^{+\infty} t^{x-1}.e^{-t}dt, x > 0$
  - Γ(x) is continuous and converges for all x > 0
  - For any x ∈ [c, d] ⊂ (0, +∞), the Γ-function is uniformly convergent
    - It is not uniformly convergent on (0, +∞).
  - Γ function is infinitely differentiable.
    - $\Gamma^{(n)}(x) = \int_0^{+\infty} t^{x-1}.e^{-t}.\ln^n(t)\,dt$
  - $\Gamma(x+1) = x\,\Gamma(x)$, thus, $\Gamma(n) = (n-1)!$
    - This equality can be used to define Γ for non-integer negative values of x.
  - $\Gamma(x) \sim \frac{1}{x}$ as $x \to 0^+, \Gamma(0.5) = \sqrt{\pi}$
- **B function:** $B(x,y) = \int_0^1 t^{x-1}.(1-t)^{y-1}dt = \int_0^{+\infty} \frac{t^{x-1}}{(1+t)^{x+y}}dt = \int_0^1 \frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}}dt \;, x > 0,\; y > 0$
  - The integral is proper and can be calculated if x, y > 1. It's improper otherwise.
  - $B(x,y) = B(y,x) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$
  - $B(x, 1-x) = \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, 0 < x < 1$
- **More formulas** (not sure whether we can use all of them or not):

$$\int_0^\infty \frac{x^{m-1}}{(x^n+1)^\alpha}dx = \frac{\Gamma(m/n)\Gamma(\alpha - m/n)}{n\Gamma(\alpha)} \qquad \int_0^\infty \frac{x^{m-1}\ln x}{x^n+1}dx = -\frac{\pi^2}{n^2}\csc\frac{m\pi}{n}\cot\frac{m\pi}{n}$$

$$\int_0^\infty \frac{x^{m-1}}{x^n+1}dx = \frac{\pi}{n}\csc\frac{m\pi}{n} \qquad \int_0^\infty \frac{x^{m-1}\ln^2 x}{x^n+1}dx = \frac{\pi^3}{n^3}\csc\frac{m\pi}{n}\left(2\csc^2\frac{m\pi}{n} - 1\right)$$

$$\frac{\Gamma(1+\epsilon)}{\Gamma(1/2+\epsilon)} = \frac{2^{2\epsilon}}{\sqrt{\pi}}\frac{\Gamma^2(1+\epsilon)}{\Gamma(1+2\epsilon)} \qquad B(x+1,y) = B(x,y)\frac{x}{x+y}. \qquad B(x,y) = \int_0^{\pi/2} 2\sin^{2x-1}(t)\cos^{2y-1}(t)\,dt$$

$$B(x+1,y) + B(x,y+1) = B(x,y).$$

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