Ahmed Nouralla – B19-02 – Control Theory – Assignment 1

Problem 1: Algorithm:

$$a_n x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_2\ddot{x} + a_1\dot{x} + a_0 x = b, \qquad a_n \neq 0$$

$$\text{substitute: } \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(n-1)} \end{bmatrix} \Longrightarrow \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} z_1 \\ \dot{z}_1 \\ \vdots \\ \dot{z}_{n-1} \end{bmatrix}$$

$$\dot{z}_n = x^{(n)} = \frac{b - a_0 x - a_1\dot{x} - \cdots - a_{n-1}x^{(n-1)}}{a_n} = \frac{b - a_0 z_1 - a_1 z_2 - \cdots - a_{n-1}z_n}{a_n}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} 0 & & \mathrm{I}_{(n-1)\,\mathrm{x}\,(n-1)} \\ 0 & & \\ \vdots & & \\ -a_0/a_n & \cdots & -a_{n-1}/a_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b/a_n \end{bmatrix}$$

1.1)

$$\ddot{z} + 4z = 0 \Longrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

1.2)

$$2z^{(4)} - 7z^{(3)} + 2\ddot{z} + 3\dot{z} + z = 0 \Longrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.5 & -1.5 & -1 & 3.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

1.3)

$$z^{(3)} + 2\ddot{z} + 6\dot{z} + 3z = 2 \Longrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

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2)

$$\begin{cases} J\ddot{\theta} = K_m i - b\dot{\theta} \\ Li' = -Ri + V - K_e\dot{\theta} \end{cases} \Longrightarrow \begin{cases} \ddot{\theta} = -\dfrac{b}{J}\dot{\theta} + \dfrac{K_m}{J} i \\ i' = -\dfrac{K_e}{L}\dot{\theta} - \dfrac{R}{L} i + \dfrac{V}{L} \end{cases}$$

$$\dot{x} = Ax + Bu \Longrightarrow \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ i' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\dfrac{b}{J} & \dfrac{K_m}{J} \\ 0 & -\dfrac{K_e}{L} & -\dfrac{R}{L} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{L} \end{bmatrix} V$$

$$y = Cx + Du \Longrightarrow \theta = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ i \end{bmatrix} + [0]\, V$$

Problem 2: Algorithm:

- $\dot{\boldsymbol{x}} = A\boldsymbol{x}$
  - Autonomous LTI is asymptotically stable ⇔ real parts of eigenvalues of A are **negative**.
  - Autonomous LTI is stable ⇔ real parts of eigenvalues of A are **non-positive.**
  - Unstable otherwise
- Eigenvalues (λ) can be computed by solving $det(A - \lambda I) = 0$

1.1)

$$\begin{vmatrix} -\lambda & 1 \\ -5 & -2-\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(-2-\lambda) + 5 = 0 \Rightarrow \lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = -1 \pm 2i \Longrightarrow \text{LTI system is asymptotically stable}$$

1.2)

$$\begin{vmatrix} -\lambda & 8 \\ 6 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 = 48 \Rightarrow \lambda = \pm 4\sqrt{3} \Longrightarrow \text{LTI system is unstable}$$

1.3)

$$\begin{vmatrix} -\lambda & 1 \\ -3 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 = -3 \Rightarrow \lambda = \pm \sqrt{3}i \Longrightarrow \text{LTI system is stable}$$

2.1)

$$3z^{(2)} - 7z = 0 \Longrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 7/3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ 7/3 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 = 7/3 \Rightarrow \lambda = \pm\sqrt{7/3} \Longrightarrow \text{LTI system is unstable}$$

2.2)

$$10z^{(4)} - 7z^{(3)} + 2\ddot{z} + 0.5\dot{z} + 4z = 0 \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.4 & -0.05 & -0.2 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ -0.4 & -0.05 & -0.2 & 0.7-\lambda \end{vmatrix} = 0 \Longrightarrow \lambda \cong \begin{cases} 0.74 + 0.6i \\ 0.74 - 0.6i \\ -0.39 + 0.54i \\ -0.39 - 0.54i \end{cases} \Longrightarrow \text{LTI system is unstable}$$

2.3)

$$z^{(3)} - 3\ddot{z} + 2z = 0 \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & 0 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = \begin{cases} 1 + \sqrt{3} \\ 1 \\ 1 - \sqrt{3} \end{cases} \Longrightarrow \text{LTI system is unstable}$$

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3)

$$m\ddot{y} + b\dot{y} + ky = 0 \Rightarrow \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\mathrm{k/m} & -\mathrm{b/m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad m \neq 0$$

$$\begin{vmatrix} -\lambda & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} - \lambda \end{vmatrix} = 0 \Longrightarrow \lambda\left(\frac{b}{m} + \lambda\right) + \frac{k}{m} = 0 \Longrightarrow \lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} = 0 \Longrightarrow$$

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4km}}{2m}, \qquad \lambda_2 = \frac{-b - \sqrt{b^2 - 4km}}{2m}$$

- For this system to be **asymptotically stable**, we have to have $Re(\lambda_1) < 0, Re(\lambda_2) < 0$, which means that one of the following must be true:
  - $b > 0,\ k > 0,\ m > 0$
  - $b < 0,\ k < 0,\ m < 0$
- For this system to be **Lyapunov stable**, we have to have both eigenvalues purely imaginary.
  - $Re(\lambda_1) = Re(\lambda_2) = 0 \Longrightarrow b = 0, k < 0, m < 0$ **OR** $b = 0, k > 0, m > 0$
    - **Excluding** the case when $\lambda_1 = 0, \lambda_2 = 0 \Longrightarrow b = k = 0$
- If $b < 0, m > 0, k > 0$ we will have $-\frac{b}{2m} > 0$, at least one positive real part of one eigenvalue (λ_1) is a sufficient condition for instability.
  - The root term $\sqrt{b^2 - 4km}$ is either positive or imaginary, so it won't affect the result.

Problem 3:

1)

- Dynamics of the mass-damper system is given by:

$$m\ddot{y} + b\dot{y} = F(t) = F_c\delta(t), \qquad F_c = \text{const}, \qquad b > 0, \qquad m > 0, \qquad y(0) = 0, \qquad \mathcal{L}(\delta(t)) = 1$$

- Taking Laplace transform for both sides:

$$ms^2Y(s) + bsY(s) = F_c$$
$$Y(s) = \frac{F_c}{ms^2 + bs}$$

- Taking the Inverse Laplace transform for both sides:

$$LHS = \mathcal{L}^{-1}(Y(s)) = y(t)$$

$$RHS = \mathcal{L}^{-1}\left(\frac{F_c}{ms^2 + bs}\right) = \frac{F_c}{m}\mathcal{L}^{-1}\left(\frac{1}{s^2 + \frac{b}{m}s + \frac{b^2}{4m^2} - \frac{b^2}{4m^2}}\right)$$
$$= \frac{F_c}{m}\mathcal{L}^{-1}\left(\frac{1}{\left(s + \frac{b}{2m}\right)^2 - \frac{b^2}{4m^2}}\right) =$$
$$= \frac{F_c}{m} * \frac{2m}{b}\mathcal{L}^{-1}\left(\frac{\left(\frac{b}{2m}\right)}{\left(s + \frac{b}{2m}\right)^2 - \left(\frac{b}{2m}\right)^2}\right) = \frac{2F_c}{b}e^{\frac{-bt}{2m}}\sinh\left(\frac{bt}{2m}\right) \xRightarrow{\text{simplification}} \frac{F_c}{b}\left(1 - e^{\frac{-bt}{m}}\right)$$

From Laplace transformation table:

$$e^{At}\sinh(Bt) \leftrightarrow \frac{B}{(s - A)^2 - B^2}$$

In our case:

$$A = -\frac{b}{2m}, B = \frac{b}{2m}$$

- Since $LHS = RHS$

$$y(t) = \frac{F_c}{b}\left(1 - e^{\frac{-bt}{m}}\right) \Longrightarrow y(\infty) = \lim_{t\to\infty}\frac{F_c}{b}\left(1 - e^{\frac{-bt}{m}}\right) = \frac{F_c}{b}$$

- **Answer:** $y(\infty) = \frac{F_c}{b}$

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2)

$$\begin{cases} a_3x^{(3)} + a_2\ddot{x} + a_1\dot{x} + a_0x = b_1\dot{u} - b_0u \\ y = c_1\dot{x} + c_0x + u \end{cases}$$

- Taking the Laplace transformation for both sides of the 1st equation:

$$a_3s^3X(s) + a_2s^2X(s) + a_1sX(s) + a_0X(s) = b_1sU(s) - b_0U(s)$$

$$X(s) = \frac{b_1s - b_0}{a_3s^3 + a_2s^2 + a_1s + a_0}U(s) \qquad \textbf{(1)}$$

- Taking the Laplace transformation for both sides of the 2nd equation:

$$Y(s) = c_1sX(s) + c_0X(s) + U(s) \qquad \textbf{(2)}$$

- Substitute the value of $X(s)$ from **(1)** into **(2)** and simplify:

$$Y(s) = c_1s\left(\frac{b_1s - b_0}{a_3s^3 + a_2s^2 + a_1s + a_0}U(s)\right) + c_0\left(\frac{b_1s - b_0}{a_3s^3 + a_2s^2 + a_1s + a_0}U(s)\right) + U(s)$$

$$Y(s) = \left(\frac{c_1s(b_1s - b_0)}{a_3s^3 + a_2s^2 + a_1s + a_0} + \frac{c_0(b_1s - b_0)}{a_3s^3 + a_2s^2 + a_1s + a_0} + 1\right)U(s)$$

- **Therefore:**

$$W(s) = \left(\frac{c_1s(b_1s - b_0)}{a_3s^3 + a_2s^2 + a_1s + a_0} + \frac{c_0(b_1s - b_0)}{a_3s^3 + a_2s^2 + a_1s + a_0} + 1\right)$$
