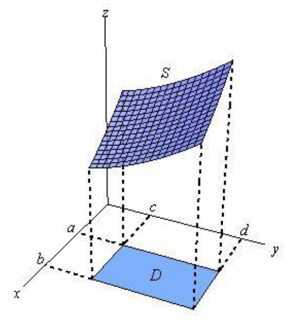
Illustration of surface integral with an example.

Underlined words are prerequisites of understanding what is happening here.

Calculate the surface integral $\iint_S (x+y+z)dS$, where S is the part of the plane $x + 2y + 4z = 4, x \geq 0, y \geq 0, z \geq 0$

- Surface integral means the we are integrating over a surface (which is not necessarily flat).
- In the usual double integration, we integrate over the some "Flat Area" of the xy-plane.
- Consider some function f(x, y) of some 3D shape S.
- This double integral $\iint_D f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx$ Will give the area of this part of S.
    - Here we say that we integrate over the domain D.
    - There are several cases when D is not bounded by a rectangle we use Fubini's Theorem.

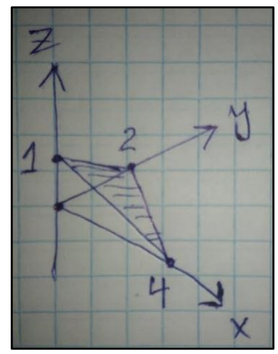


- Now our problem is to integrate some surface given by f(x, y, z) = 0, where the domain is a part of this surface, we call this part S and it can be determined with an equation and a set of constraints.
    - There are some general formulas for 3D surfaces that you should be familiar with to imagine what kind of surface you are working with.
- But we cannot simply calculate that, because the usual integration methods require the domain to be "flat".
- So we use the following formula that relates the surface integral with the double integral.
    - $\iint_S \boldsymbol{f(x,y,z)dS} = \iint_D f(\mathrm{r}(\mathrm{u},\mathrm{v})).\, ||\mathrm{r_u} \times r_v||\, dA$
- **FAQ about the above formula:**
    - **What is D again?**
        - D is the projection "shadow" of S on the 2D plane.
    - **What does this r mean?**
        - It's a 2-parameter function that results from parametrization of the original function.
    - **How can we parametrize f?**
        - We can parametrize by just substituting x = u, y = v and calculate z(u, v)
            - But this doesn't always work, and not very useful in some cases
                - Consider $f(x, y, z) = x^2 + y^2 + z^2$.
        - Some known useful parametrizations are polar, spherical, cylindrical coordinates. That are used in these complex cases.
    - **What about this operator ||**
        - It's the vector magnitude. For 3D case where x(x, y, z) we have
            - $||x|| = \sqrt{(x^2+y^2+z^2)}$
    - **What about the ×**
        - It's the cross product of two vectors, that can be calculated using determinants.
    - **Why this vector $\mathbf{r_u} \times \mathbf{r_v}$**
        - Because it's a general form for the normal vector to any surface parametrized as r(u, v).

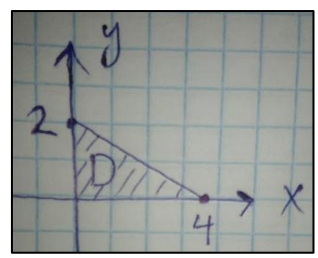
- So, to calculate a surface integral generally, we need to be able to parametrize functions, find normal vector to a surface, calculate the cross product using determinants, convert the double integral to iterated integral, then evaluate it.
    - That's a very long process, that's why we have many more formulas for special cases.
- **One special case** is when we can represent f(x, y, z) = 0 as z = z(x, y), that is already a parametrization, think of z = r, x = u, y = v, we did the first step.
    - Now we can calculate $\| \mathbf{r_u} \times \mathbf{r_v} \|$ using this fact
    - After some calculations we get $\| \mathbf{r_u} \times \mathbf{r_v} \| = \sqrt{(z_x^2 + z_y^2 + 1)}$.
- So the above formula is now reduced to be:
    - $\iint_{\boldsymbol{S}} \boldsymbol{f(x,y,z)dS} \ = \ \iint_{\boldsymbol{D}} f(x,y,z(x,y)).\sqrt{(z_x^2 + \ z_y^2 + 1)}\ dA$

Back to the problem again, after all this discussion

- **This is the problem** $\iint_{S}(x+y+z)dS$
- **This is S, the part of the surface we are integrating over:**
    - $x + 2y + 4z = 4, x \geq 0, y \geq 0, z \geq 0$
- Now go ahead and draw that part of "Plane"
    - Hint: put x, y = 0 find z, put x, z = 0 find y, put y, z = 0 find x.



- **Now you have S, D is the projection of this onto S.**
- **This is the special case we just considered.**
    - **z = z(x, y) = 1 – x/4 – y/2**
    - $\sqrt{(z_x^2 + \ z_y^2 + 1)} = \frac{\sqrt{21}}{4}$
    - $\iint_{\boldsymbol{S}} \boldsymbol{(x+y+z)dS} \ = \ \frac{\sqrt{21}}{4} \iint_{\boldsymbol{D}} x + y + \left(1 - \frac{x}{4} - \frac{y}{2}\right) dA$
- **Solving the double integral "Using Fubini's Theorem"**
    - **Get the line equation from the picture on the right: y = - ½ x + 2**



- $\iint_{\boldsymbol{D}} x + y + \left(1 - \frac{x}{4} - \frac{y}{2}\right) dA = \int_{x=0}^{x=4} dx \int_{y=0}^{y=-\frac{1}{2}x+2} (\frac{3}{4}x + \frac{1}{2}y + 1)\ dy$
- $\int_0^4 \left(-\frac{5}{16}x^2 + \frac{1}{2}x + 3\right) dx \ = \frac{28}{3}$
- **Final answer** = $\frac{\sqrt{21}}{4} * \frac{28}{3} = \frac{7\sqrt{21}}{3}$