Graph is a pair of sets

G = (V, E)

V = {v1, v2, v3, ...} - set of elements representing vertices

E = { (vx1, vy1), (vx2, vy2), ...} - set of pairs of vertices

* in case of undirected graph we have E as a set of sets, each containing two elements

**** Euler formula f + v - e = 2**

1) Graph order |V| : Number of vertices

2) Loop: Non-empty path where only repeated vertices are the first and the last

When you can go from a node to itself using two different paths

3) Cycle: Self-loop

4) Path/trail:

Sequence (v1, e1, v2, e2, ...)

Length of path: number of edges in it

Distance b/w two paths: length of the shortest path b/w them

Diameter: largest distance b/w two nodes

5) Node Neighborhood: Set of all adjacent nodes to it

6) Node degree: Cardinality of Node Neighborhood

Sum of degrees of all nodes in a graph = 2 * number of edges

7) Graph complement G = (V, E) **->** !G = (V, !E) -> !E = {(u, v) | u, v ∈ V ^ u ≠ v ^ (u, v) ∉ E}

8) Almost all graphs

phi: a property, phi (<= n): set of all graphs of order <= n that enjoy phi

TRUE (<= n) set of all graphs of order <= n

If limit of |phi(<=n)| / |TRUE(<=n)| as n -> infinity

= 1 then almost all graphs enjoy property phi

= 0 then almost all graphs doesn't enjoy property phi

9) Face of a plane graph: A region bounded by edges

** The infinitely large outer region is considered a face

10) Subgraph of a graph: Another graph formed from a subset of the vertices of the graph and all of the edges connecting pairs of vertices in that subset.

11) Subdivision of a graph: The result of inserting vertices into edges

** The graph is planer iff it doesn't contain a subgraph that it a subdivision of K(5) OR K(3, 3)

12) Forest: A graph with no loops and no cycles

13) Tree: A connected forest, any two nodes are connected by a unique path, e = v – 1

14) Terminal node: node with degree 1, A tree has at least 2 terminal nodes.

15) Spanning tree: A tree that contains all nodes of the graph.

1) Digraphs: Directed graphs (can have loops/can be infinite/can't have multiple edges)

Number of digraphs of order n = 2^(n^2) - assuming digraphs have no multiple edges

2) Undirected graph (graph): Digraphs with relation on its E the relation is

1- symmetric (if there is a path from A to B then there is one from B to A)

2- irreflexive (No loops (Self-Cycles))

Number of graphs of order n is 2^(n(n-1)/2)

3) Multi-(bi/di/complete) graphs Can have multiple edges between the same two nodes

The set of edges is multi-set

multiset is a set of pairs: each pair contain the element and it's multiplicity (possibly zero)

4) Connected graph: Every two nodes has a path b/w them

5) Isomorphic graphs:

- Two graphs are said to be isomorphic iff they have an isomorphism (bijective function) F between their sets of vertices such that (v', v'') ∈ E1 iff (F(v'), F(v'')) ∈ E2

- For every vertex in the first graph there is a corresponding one in the other graph that have the same connections as the original one

- You can construct a bijective function that links each vertex with that corresponding vertex in the other graph

Auto morphism: An isomorphism of a graph with itself

6) Self-complementary graph: A graph that is isomorphic to its complement.

**Either a graph or it's complement is connected

To draw a complement for any graph with n vertices

1- Draw K(n), **2-** Remove edges that are present in the original graph

Example: Inverse of P(4) 'N' is another graph which looks like 'Z'

7) Special Graphs:

O(n): Empty graph of order n (has no edges)

K(n): Complete Graph of order n (has all possible edges - not multi, have no self-cycles)

Complete graph of order n has n(n-1)/2 edges = C(n, 2

Complete digraph of order n has n(n-1) edges

Number of cycles in K(n):

{summation from i=3 -> i=n} { n!/((n-i)!.2i) }

C(n) : Elementary loop (has n edges)

E = { (1, 2), (2, 3), .., (n-1, n), (n, 1) }

P(n) : Elementary chain (has n-1 edges)

E = { (1, 2), (2, 3), .., (n-1, n) }

8) Bipartite graphs(Bigraphs):

1- Vertices can be divided into two set s.t no connection between elements in the same set

2- Can be colored with two colors in such a way that no connection between two nodes with the same color

3- Has no cycles of odd length

4- Order of bigraph is a pair (m, n)

n = #nodes in one set, m = #nodes in the other set

5- Bigraph can be complete K(m, n) or multi

K(m, n) has m*n edges

Petri-net:

Bipartite directed (Multi) graph where one set nodes are called places and the other one nodes are called transitions

There are 2^(mn) petri nets (Cartesian Product of a complete bigraph)

9) Euler graphs: Graphs that has an Euler path.

Euler path: A path that uses all edges of the graph

Can use the same node multiple times

Euler loop (Euler circuit): a Euler path that starts and ends in the same node

Euler Theorem: Euler loop exists in a graph iff the graph is connected and degrees of all nodes are even.

** A graph has an Euler path iff it is connected and has no more than 2 nodes with odd degrees.

case 0 it's Euler path because no edges to traverse

case 1 it's P(OO)

10) Planar graphs: Graphs that can be drawn without edges crossing

Examples: K(4) and C(5), Counter-Examples: K(5), K(3, 3)

Proof that K(5) is not planar

** same for K(3, 3) but using 4 instead of 3 because in order to have a face in bigraph you need at least 4 edges

1- Assume K(5) is planar

2- It should obey Euler relationship f + v – e = 2

3- f + 5 – 10 = 2 -> f = 7

4- each face is bounded by 3 edges minimum -> there are at least 3f boundaries

5- each edge separates two faces maximum -> there are at least 3f/2 edges -> there are at least 11 edge but there is only 10 -> Contradiction

11) Plane graph: Graphs that has no edges crossing

* For any planar graph with order >= 3, the inequality e <= 3v – 6 holds

* An infinite tree contains either a vertex of infinite degree or an infinite simple path

Prove that for any connected planar graph $G = (V, E)$ with $e \geq 3$, $v - e + r = 2$, where $v = |V|$, $e = |E|$, and r is the number of regions in the graph.

Inductive Hypothesis:

$S(k) : v - e + r = 2$ for a graph containing $e = k$ edges.

Basis of Induction:

$S(3)$: A graph G with three edges can be represented by one of the following cases:

1. G will have one vertex x and three loops $\{x, x\}$. For this case, $v = 1$, $e = 3$, $r = 4$, and $v - e + r = 1 - 3 + 4 = 2$
2. G will have two vertices x, y, one edge $\{x, y\}$, and two loops $\{x, x\}$ (or $\{y, y\}$). For this case, $v = 2, e = 3, r = 3$, and $v - e + r = 2 - 3 + 3 = 2$
3. G will have two vertices x, y, one edge $\{x, y\}$, one loop $\{x, x\}$ and one loop $\{y, y\}$. For this case, $v = 2, e = 3, r = 3$, and $v - e + r = 2 - 3 + 3 = 2$
4. G will have two vertices x, y, two edges $\{x, y\}$ and one loop $\{x, x\}$ (or $\{y, y\}$). For this case, $v = 2, e = 3, r = 4$, and $v - e + r = 2 - 3 + 4 = 2$
5. G will have three vertices x, y, z and three edges: $\{x, y\}, \{y, z\}, \{z, x\}$. For this case, $v = 3, e = 3, r = 2$ and $v - e + r = 3 - 3 + 2 = 2$

Inductive Step:

$S(k + 1)$: Assume $S(k)$ to be true. Then for a connected planar graph $G = (V, E)$ with $e \geq 3$, $v - e + r = 2$. From $S(k)$, move to $S(k + 1)$ by adding one edge to G. Call this new graph H. Let v', e', and r' represent the number of vertices, edges, and regions in H, respectively. Now, be created in one of the following ways:

1. Add a loop to some $v \in V$. This divides on region bordering v into two regions. Then, $v' = v, e' = e + 1, r' = r + 1$, and $v' - e' + r' = v - (e + 1) + (r + 1) = (v - e + r) - 1 + 1 = 2$
2. Add an edge $\{x, y\}$ to E for some $x, y \in V, x \neq y$. x and y must border a similar region t, or the edge $\{x, y\}$ will violate the planarity of H. This new edge will divide region t into two regions. So, $v' = v, e' = e + 1, r' = r + 1$, and $v' - e' + r' = v - (e + 1) + (r + 1) = (v - e + r) - 1 + 1 = 2$
3. Add an edge $\{x, y\}$ to E and vertex y to V for some $x \in V, y \notin V$. y must be added in a region t that borders x, or the new edge will violate the planarity of H. So, $v' = v + 1, e' = e + 1, r' = r$, and $v' - e' + r' = (v + 1) - (e + 1) + r = (v - e + r) + 1 - 1 = 2$