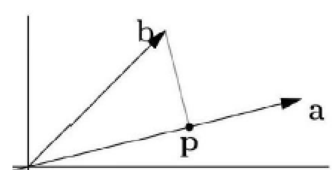
- The SLE $Ax = b$ has a solution $\Longleftrightarrow \text{rank}A = \text{rank}(A, b)$.
- The SLE $Ax = b$ has a unique solution $\Longleftrightarrow \text{rank}A = \text{rank}(A, b) = n$.
- The SLE $Ax = b$ has infinitely many solutions $\Longleftrightarrow \text{rank}A = \text{rank}(A, b) = k < n$.

- **Row echelon form:** The matrix after applying Gaussian elimination on it.
  - All zero-rows should be at the bottom
  - The leading 1 for each non-zero row should be strictly at the right of the leading 1 in the preceding row.
- **Reduced row echelon form:** Row echelon form with two extra restrictions:
  - First non-zero number from the left (pivot/leading-coefficient) for each row is 1.
  - Each pivot is the only non-zero element in his column.
- **Algorithms for solving a SLE in the form Ax = b**
  - **Gaussian elimination:** Reduce the **augmented matrix** [A | b] using **elementary row operations** until it becomes in **row echelon form**. Then solve equations from bottom to top.
    - **Gauss-Jordan elimination:** When you represent augmented matrix in reduced row echelon form. Then solution is the last column.
    - **Elementary row operations:** Multiply a row by a non-zero scalar (Scalar M) - Swap two rows (Permutation M) - Add one row to a multiple of another (Elimination M).
  - **Cramer rule: $x = \{x_i\}$, i: 1…n**
    - A_i is a modification of A where you replace the i-th column with b. $x_i = \dfrac{\det(A_i)}{\det(A)}$
  - **Matrix solution: $Ax = b \rightarrow x = A^{-1}.b$**

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A). \quad \operatorname{adj}(\mathbf{A}) = \mathbf{C}^\mathsf{T}. \quad \mathbf{C} = \left((-1)^{i+j}\mathbf{M}_{ij}\right)_{1\le i,j\le n}$$

  - $M_{i,j}$ is the (i,j)-th minor = det of the matrix. results from deleting the i-th row and j-th column.
- **Gaussian elimination applications:**
  - **Computing determinant: let d=1, apply elementary row operations until you reach an upper or lower triangular matrix then use the formula.** $\det(A) = \dfrac{\prod \operatorname{diag}(B)}{d}$.
    - Swapping rows → d* = -1, multiply row by s → d *= s.
    - Lower triangular → all elements above the main diagonal are zeros.
  - **Computing inverse:** Apply elementary row operations to get from [A | I] → [I | A^{-1}]
  - **Rank of a matrix:** the number of non-zero rows when the matrix is in row echelon form.
- **A = LU decomposition:**
  - Apply Gaussian elimination on A to get U (upper triangular matrix)
  - Calculate L which will be a lower triangular matrix with all elements on the main diagonal = 1
  - Elements under the main diagonal are coefficients that were used in the elimination * -1.
- **A = LDU decomposition:**
  - Apply Gaussian elimination you get a matrix R in REF
  - Factorize R into two matrices D and U
    - D has the same diagonal as R with all rest as zeros
    - U is R with each row divided by the corresponding diagonal element. (Suppose 0/0=0)
    - Same L as LU decomposition.



$$p = \hat{x}a = \frac{a^\mathrm{T}b}{a^\mathrm{T}a}a$$

$$p = A\hat{x} = A(A^\mathrm{T}A)^{-1}A^\mathrm{T}b = Pb$$

- **Projection problem, given a, b, find p. Example in 2D:**
  - Vector e = b-p is the approximation error.
- **General case: Solving Ax=b when b is not in C(A).**
  - Ax=b is not solvable, But Ax' = p is solvable.
  - E^2 is minimal when x = x', P is the projection of b onto C(A).

$$E^2 = \|Ax - b\|^2$$

- **Least square problem:** find the closest line b = C+Dt to a set of points {(t, b)}. **Solution:** instead of solving for Ax=b, solve for:

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{b}.$$

$$l_{11} = \sqrt{a_{11}}, \quad l_{j1} = \frac{a_{j1}}{l_{11}}, \quad j \in [2, n], \qquad d_{jj} = a_{jj} - \sum_{k=1}^{j-1} l_{jk}^2 d_{kk}$$

$$l_{ii} = \sqrt{a_{ii} - \sum_{p=1}^{i-1} l_{ip}^2}, \quad i \in [2, n], \quad l_{ji} = \left(a_{ji} - \sum_{p=1}^{i-1} l_{ip} l_{jp}\right) / l_{ii}, \qquad l_{ij} = \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} d_{kk} l_{jk}\right) \times \left(\frac{1}{d_{jj}}\right)$$

- **A = LL^T = U^{T}U (Cholesky) decomposition**
  - Then solve **L.y = b** for y by forward substitution, solve **L^T.x = y** for x by back substitution.
  - **Works for symmetric, positive definite matrices only**
    - Matrix is positive definite if all the upper-left submatrices have positive determinants.
- **A = LDL^T decomposition: Same conditions as Cholesky, except that it doesn't require the matrix to be positive definite.**
  - Then solve **L.z = b** for z by forward substitution, then solve **D.y = z** for y (trivially)
  - Finally solve **L^T .x = y** for x by back substitution.
- A **system** of **linear equations** is **homogeneous** if all of the constant terms are zero (b is zero matrix).
- **Change of coordinates formula: d = A.d' + b, b = oo'**
  - d is the point in old coordinate system, d' is the point in new coordinate system.
  - Unless stated otherwise, old coordinate system is: < o(0, 0), **i**(1, 0), **j**(0, 1) >
  - A is the transformation matrix obtained from coefficients when representing new vector bases in terms of old ones.
- Vectors form a **subspace** when they're closed under addition and multiplication by any real number.
- **The four fundamental subspaces:**
  - **The Column Space** of some matrix A is the subspace that contain all linear combinations of the matrix column vectors. (The span of independent column vectors of A).
    - Rank(A) = number of factor(pivot) columns of A = dimension of C(A).
    - Vectors from the original matrix that correspond to pivot pos. in RREF forms C(A)
  - **The NullSpace** of some matrix A is the subspace which contain all solution to Ax = 0
    - Columns with no pivots → free variables. Other columns → pivot variables.
    - Dimension of N(A) = number of free variables = n – r.
  - **The Row Space** of A = C(A^T) is perpendicular to the NullSpace of A.
  - **The Left NullSpace** of A = N(A^T) is perpendicular to the Column Space of A.
- **Given a set of vectors.** Vectors are linearly dependent when:
    - Any of them can be expressed as a linear combination of the others.
    - There is a non-trivial solution to the equation C_1**V_1** + C_2**V_2 + .. = 0**
    - **The null space of the matrix made out of these vectors is not the zero vector.**
  - Vectors form a **basis** when they're linearly independent.
  - **Spanning** these vectors means providing all possible linear combinations of these vectors.
- **Finding complete solutions to Ax = b when there are infinitely many.**
  - Find pivots and free variables from RREF of the matrix (N(A) = Xn = Xs)
    - Xs = {x | x = x1.v1 + x2.v2 + .. } x1,x2,.. free variables, v1, v2, .. constant vectors.
  - Set all free variables to 0 in the original equation and solve to find Xp.
  - X = Xp + Xs, Xp is a matrix, Xs is a set (Subspace).
- **Orthonormal Matrix Q:** Matrix with orthonormal column vectors (pairwise perpendicular). $Q^TQ = I$
- **Orthogonal Matrix Q**: A square orthonormal matrix, $Q^T = Q^{-1}$

- **A = QR decomposition, Q: Orthogonal Matrix, R: Upper triangular Matrix.**
- **General case** $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$,

$$\mathrm{proj}_{\mathbf{u}}\,\mathbf{a} = \frac{\langle \mathbf{u}, \mathbf{a}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}\mathbf{u}$$

$$Q = [\mathbf{e}_1, \ldots, \mathbf{e}_n]$$

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{a}_1, & \mathbf{e}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \\ \mathbf{u}_2 &= \mathbf{a}_2 - \mathrm{proj}_{\mathbf{u}_1}\,\mathbf{a}_2, & \mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \\ \mathbf{u}_3 &= \mathbf{a}_3 - \mathrm{proj}_{\mathbf{u}_1}\,\mathbf{a}_3 - \mathrm{proj}_{\mathbf{u}_2}\,\mathbf{a}_3, & \mathbf{e}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \end{aligned}$$

$$R = \begin{pmatrix} \langle \mathbf{e}_1, \mathbf{a}_1\rangle & \langle \mathbf{e}_1, \mathbf{a}_2\rangle & \langle \mathbf{e}_1, \mathbf{a}_3\rangle & \ldots \\ 0 & \langle \mathbf{e}_2, \mathbf{a}_2\rangle & \langle \mathbf{e}_2, \mathbf{a}_3\rangle & \ldots \\ 0 & 0 & \langle \mathbf{e}_3, \mathbf{a}_3\rangle & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$$\begin{aligned} \mathbf{a}_1 &= \langle \mathbf{e}_1, \mathbf{a}_1\rangle \mathbf{e}_1 \\ \mathbf{a}_2 &= \langle \mathbf{e}_1, \mathbf{a}_2\rangle \mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{a}_2\rangle \mathbf{e}_2 \\ \mathbf{a}_3 &= \langle \mathbf{e}_1, \mathbf{a}_3\rangle \mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{a}_3\rangle \mathbf{e}_2 + \langle \mathbf{e}_3, \mathbf{a}_3\rangle \mathbf{e}_3 \end{aligned}$$

- **Special case when A is [a b c]**
  - **A = a** **B = b – Proj$_A$(b)** **C = c – Proj$_A$(c) – Proj$_B$(c)**
  - **Proj$_b$(a) =** $\frac{b.a}{b.b}b$ **Q = [** $\frac{A}{||A||}$ $\frac{B}{||B||}$ $\frac{C}{||C||}$ **] = [q1 q2 q3]**
  - **R =** $\begin{bmatrix} \mathbf{q1}.a & \mathbf{q1}.b & \mathbf{q1}.c \\ \mathbf{0} & q2.b & q2.c \\ \mathbf{0} & \mathbf{0} & q3.c \end{bmatrix}$

Methods for solving Ax = b iteratively:

- **Jacobi Method**

$$A = D + L + U \qquad \text{where} \qquad D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \text{and } L + U = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix}$$

The solution is then obtained iteratively via

$$\mathbf{x}^{(k+1)} = D^{-1}(\mathbf{b} - (L + U)\mathbf{x}^{(k)}),$$

- **Seidel Method**

$$A = L_* + U \qquad \text{where} \qquad L_* = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad U = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

$$\mathbf{x}^{(k+1)} = L_*^{-1}(\mathbf{b} - U\mathbf{x}^{(k)}).$$