Summary of differential equations techniques – Ahmed Nouralla

- **Separation of variables:**
  - Simplest case: $y' = a(x)b(y) \Longrightarrow \frac{dy}{b(y)} = a(x)dx$, and we integrate both sides.
- **Bernoulli's equation** $y' + p(x)y = q(x)y^k$
  - Solution has the form $y = uy_c$
    - y_c is any non-trivial solution to the homogeneous equation $y' + p(x)y = 0$ "separable"
    - $\frac{du}{u^k} = q(x)y_c^{k-1}\,dx$ "separable"
  - Another way is to introduce $u = y^{1-k}$ then we get $u' + (1-k)p(x)u = (1-k)q(x)$ which can be solved using "Integrating factor" or "Variation of parameters".
- **Riccati's equation** $y' = q_0(x) + q_1(x)y + q_2(x)y^2$
  - **Solution 1:**
    - Find a non-trivial particular solution y_p
      - Try $y_p = ax + b, y_p = \frac{a}{x}, y_p = e^{ax}$
    - Using the substitution $y = y_p + u$
    - We get the Bernoulli equation $u' - \left(q_1 + 2q_2y_p\right)u = q_2u^2$
  - **Solution 2:**
    - $u'' - \left(q_1 + \frac{q_2'}{q_2}\right)u' + (q_2q_0)u = 0 \Longrightarrow y = \frac{-u'}{q_2u}$
- **General tricks:**
  - **Inverting x and y**
    - Sometimes it's easier to solve for $\frac{dx}{dy} = f(x,y)$
  - **Substitutions for some cases**
    - $y' = f(ax + by + c) \Longrightarrow z = ax + by + c$
    - $y' = f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$
      - We check the intersection of lines "numerator and denominator"
      - If they are parallel, use substitution $z = ax + by$
        - a, b are common factors of a_1, a_2 and b_1, b_2 respectively.
      - If they intersect at point (x_0, y_0) we use substitutions
        - $u = x - x_0, v = y - y_0$
  - **Chini equation:** $y' = y^nf(x) + g(x)y + h(x)$
    - Try the substitution $y = \left(\frac{h(x)}{f(x)}\right)^{\frac{1}{n}}v(x)$, you may get a separable equation.
- **Variation of parameters (constants):**
  - A general method to solve the non-homogeneous linear ordinary differential equations.
  - **First order case** on the form $y' + p(x)y = q(x)$
    - Solution has the form $y = u(x)y_c(x)$
      - $y_c(x)$ is any non-trivial solution to $y' + p(x)y = 0$
      - Substitute $y = uy_c, y' = uy_c' + y_cu'$ in the original equation to find u.
  - **Second order case** on the form $p(x)y'' + q(x)y' + r(x)y = g(x)$
    - Detailed below.

- **Integrating factor:**
    - Integrating factor is any expression that a differential equation is multiplied by to facilitate integration. Or a function that yields an exact differential when multiplied by an inexact one.
    - **First order case** on the form $y' + p(x)y = q(x)$
        - $I(x) = e^{\int p dx}, y = \frac{1}{I(x)}\left(\int I(x)q(x) + C\right)$
    - **Exact differential equation:**
        - Exact differential is a differential on the form $d\phi$ for a differentiable function ϕ.
            - For two variable case $\phi = f(x,y) \Longrightarrow d\phi = f_x dx + f_y dy$ is exact
        - **Solve** $F(x,y)dx + G(x,y)dy = 0$
            - **Case 1:** $F_y = G_x \Longrightarrow \phi = C, C \in R$ is an implicit solution.
                - $\phi = \int F dx = \int G dy$
                - To find the constant (function) of integration we either solve
                    - $\left(\int F dx\right)'_y = G$ for $C(y)$
                    - $\left(\int G dy\right)'_x = F$ for $C(x)$
            - **Case 2:** $F_y \neq G_x$ then we need to find a function I such that $(IF)_y = (IG)_x$
                - Typical cases to “guess” the value of I
                    - $\frac{F_y - G_x}{G} = a(x) \Longrightarrow I(x) = e^{\int a(x)dx}$
                    - $\frac{G_x - F_y}{F} = b(y) \Longrightarrow I(x) = e^{\int b(y)dy}$
                    - Let $I = x^m y^n \Longrightarrow (IF)_y = (IG)_x$ and we solve for m, n
                        - If we reached a contradiction, I has another form.
                - Then we apply the same method in case 1.
- **Undetermined coefficients:**
    - Used to find y_p a particular solution to certain types of nonhomogeneous linear ODEs.
    - Solution to $y^{(n+1)} + \sum_{i=0}^{n} c_i y^{(i)} = f(x)$ has the form $y = y_c + y_p$
        - y_c can be calculated using the substitution $y = e^{rx}$.
        - Method works if c_i are constants and $f(x)$ has one of the forms below (or a finite sum of them).
    - For second order case: $ay'' + by' + cy = f(x)$ we use this table to find y_p then its derivatives, then substitute into the original equation and solve for the “undetermined coefficients”.
        - $P_n(x), Q_n(x), R_n(x)$ are polynomials of degree n.
        - If a term in y_p appears in y_c, then y_p should be multiplied by x
            - Repeat until y_p is linearly independent from y_c
        - Replacing sin with cos in $f(x)$ won’t change the guess y_p

$f(x)$	y_p
Ke^{ax}	Ce^{ax}
$P_n(x)$	$Q_n(x)$
$K \sin ax$	$C_1 \cos ax + C_2 \sin ax$
$P_n(x)e^{ax}$	$Q_n(x)e^{ax}$
$Ke^{ax} \sin ax$	$e^{ax}(C_1 \cos ax + C_2 \sin ax)$
$P_n(x) \sin ax$	$Q_n(x) \cos ax + R_n(x) \sin ax$
$P_n(x)e^{ax} \sin bx$	$e^{ax}(Q_n(x) \cos bx + R_n(x) \sin bx)$

- **Second order ODE on the form** $p(x)y'' + q(x)y' + r(x)y = g(x)$
    - First of all, divide equation by $p(x)$ and check the case when $p(x) = 0$
- The solution has the form: $y = y_p + y_c$ “particular solution + complementary solution”.

Form	Solution
“Homogeneous with constant coefficients” $ay'' + by' + cy = 0, \quad a, b, c = const$	**Substitute** $y = e^{ax}$ **TLDR:** Solve the characteristic equation $ar^2 + br + c = 0$ 1. $b^2 - 4ac > 0 \Longrightarrow y_c = C_1 e^{r_1 x} + C_2 e^{r_2 x}$ 2. $b^2 - 4ac = 0 \Longrightarrow y_c = e^{rx}(C_1 + C_2 x)$ 3. $b^2 - 4ac < 0 \Longrightarrow y_c = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x)), \alpha = \frac{-b}{2a}, \beta = \frac{\sqrt{4ac - b^2}}{2a}$ - Notice that $y_p = 0 \Longrightarrow y = y_c$
“Nonhomogeneous with constant coefficients” $ay'' + by' + cy = f(x), \quad a, b, c = const$	**Undetermined coefficients** to find y_p - Then use the method above to find y_c - Solution $y = y_p + y_c$
1st special form: y'' doesn’t depend on y $y'' = f(y', x)$	**Introduce** $t(x) = y' \Longrightarrow y'' = t'$ - Solve for t, then $y = \int t(x)dx$
2nd special form: y'' doesn’t depend on x $y'' = f(y, y')$	**Introduce** $t(y) = y' \Longrightarrow y'' = tt'$ - Solve for t, then $x = \int \frac{dy}{t(y)}$
“Non homogeneous with nonconstant coefficients” $y'' + p(x)y' + q(x)y = g(x)$ **Given** y_1 is a non-trivial solution to the complementary equation (with $g(x) = 0$)	**Reduction of order:** - Substitute $y = v(x)y_1$ and solve for v. - All terms involving v should cancel out - Introduce $t = v'$ and solve for t then integrate to find v. - Solution: $y = vy_1$ **Finding the second solution:** - Calculate $y_2 = uy_1, u = \int \frac{e^{-\int p dx}}{(y_1(x))^2} dx$ - Now you have the case below.
“Non homogeneous with nonconstant coefficients” $y'' + p(x)y' + q(x)y = g(x)$ **Given** y_1, y_2 for the complementary equation (given y_c)	**Variation of parameters:** - Substitute $y_p = C_1(x)y_1 + C_2(x)y_2$ and solve for C_1, C_2 **TLDR:** $y_p = y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)} dx - y_1 \int \frac{y_2 g(x)}{W(y_1, y_2)} dx$ $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$ - For linearly independent y_1, y_2 the Wronskian $W(y_1, y_2) \neq 0$

“Non homogeneous with nonconstant coefficients”

$$y'' + p(x)y' + q(x)y = g(x)$$

Given nothing :)

Notes on power series

Ratio test for convergence of power series:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Converges if

$$x \in (x_0 - R, x_0 + R), R = 1/L$$

$$L = \lim_{N\to\infty} \sum_{n=0}^{N} \left|\frac{a_{n+1}}{a_n}\right|$$

* R is called the radius of convergence

Shifting series index

$$\sum_{n=n_0}^{\infty} a_n (x - x_0)^n = \sum_{n=n_0-k}^{\infty} a_{n+k} (x - x_0)^{n+k}$$

Linear combination of power series:

$$c_1 \sum_{n=0}^{\infty} a_n (x - x_0)^n + c_2 \sum_{n=0}^{\infty} b_n (x - x_0)^n$$
$$= \sum_{n=0}^{\infty} (c_1 a_n + c_2 b_n)(x - x_0)^n$$

*** Valid only for common interval of convergence.**

*** R for the summation = $\boldsymbol{min}(\boldsymbol{R_1}, \boldsymbol{R_2})$**

*** You may need to following formulas**

$$k!! = (1)(3)(5) \dots (k) = 2^k k!$$

$$k!! = (2)(4)(6) \dots (k) = \frac{(2k)!}{2^k k!}$$

Guessing y_1

- Try common forms ($y_1 = ax + b, y_1 = \frac{a}{x}, y_1 = e^{ax}$)
- Then y_2 and y_p can be found using the methods above.

Laplace transformation: In case of IVP: $\begin{cases} y(x_0) = y_0 \\ y'(x_0) = y_1 \end{cases}$

- Take Laplace transformation for both sides (check below).
- Isolate $L(y) = F(s) \Rightarrow y = L^{-1}\big(F(s)\big)$
- For the IVP with $x_0 \neq 0$ substitute $z = x - x_0$
- **Laplace for derivatives:**
  - $L(y') = sL(y) - f(0)$
  - $L(y'') = s^2 L(y) - sy(0) - y'(0)$
  - $L\big(y^{(n)}\big) = s^n L(y) - \sum_{k=1}^{n} s^{n-k} y^{(k-1)}(0)$

Power series approach: substitute

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

- x_0 is any point such that derivatives of p, q at $x = x_0$ exists.
  - In case of IVP, use $\begin{cases} y(x_0) = a_0 \\ y'(x_0) = a_1 \end{cases}$
- The goal is to reach the form

$$g(a_{i<k}, x - x_0) + \sum_{n=k}^{\infty} f(a_{i \geq k})(x - x_0)^n = 0$$

  - Use index shifting to get the single power of $(x - x_0)$.
  - Extract k terms out (if necessary) to get a single sum.
  - Then find the recurrence relation for a_n by solving $\begin{cases} f(a_i) = 0, n < k \\ g(a_i) = 0, n \geq k \end{cases}$
  - If required, solve the recurrence to get an explicit form (check)
  - **Note:** You may get a finite number of terms as an answer (i.e. $y = a_0 + a_1(x - x_0)^1 + \cdots + a_k(x - x_0)^k$

- **Laplace transformation (Table)**
  - We say that $F(s)$ is the Laplace transformation of $f(t)$ and we write $f(t) \leftrightarrow F(s)$ if f is a piecewise continuous function and:
    - $L\big(f(t)\big) = F(s) = \int_0^{+\infty} f(t)e^{-st}dt$
    - $L^{-1}\big(F(s)\big) = f(t)$
  - **Properties:**
    - **Linearity**
      - $L(c_1y_1 + \cdots + c_ny_n) = c_1L(y_1) + \cdots + c_nL(y_n)$, same with $L^{-1}(y)$
    - **First shifting theorem**
      - $L\big(e^{at}f(t)\big) = F(s-a) \Longrightarrow L^{-1}\big(F(s-a)\big) = e^{at}L^{-1}\big(F(s)\big)$
      - The substitution $S = s - a$ in the transform corresponds to the multiplication of the original function by e^{at}
    - **Second shifting theorem**
      - $L(f(t-a)I_{t>a}) = e^{-as}\,L\big(f(t)\big) = e^{-as}F(s)$
      - Laplace transformation of a function with shifted parameter $T = t - a$ is equal to the transformation of the unshifted function multiplied by e^{-as}
    - **Change of scale property**
      - $L\big(f(at)\big) = \frac{1}{a}F\left(\frac{s}{a}\right) \Longrightarrow L^{-1}\left(F\left(\frac{s}{a}\right)\right) = af(at)$

- **Numerical Methods: for approximating a solution to the IVP** $y' = f(x,y), y(x_0) = y_0$
  - **Euler method (RK1)**
    - $x_{i+1} = x_i + h$
    - $y_{i+1} = y_i + hf(x_i, y_i)$
    - $LTE \sim O(h^2), GTE \sim O(h)$
  - **Heun/Improved Euler method (RK2)**
    - $y_{i+1} = y_i + \frac{h}{2}\Big(f(x_i,y_i) + f\big(x_i + h, y_i + hf(x_i,y_i)\big)\Big)$
    - $LTE \sim O(h^3), GTE \sim O(h^2)$
  - **Runge-Kutta (RK) methods:**
    - A family of implicit and explicit iterative methods
    - $LTE \sim O(h^{n+1}), GTE \sim O(h^n)$, for RK(n)
    - The original/classic/most used one is RK4, uses 4 tangent lines and calculate their average slope to be used in the approximation.
    - **RK4:**
      - $k_1 = f(x_i, y_i)$
      - $k_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right)$
      - $k_3 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_2\right)$
      - $k_4 = f(x_i + h, y_i + hk_3)$
      - $y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

- **Numerical methods Errors:** Suppose we are using some numerical method defined by the following rule: $y_{i+1} = N(y_i)$ with step size $= h$
  - **Global truncation error (GTE): the error caused by one iteration**
    - The difference between the approximated value calculated using our numerical method (at some point) **and** the exact value of the function (at the same point).
    - $GTE\ (on\ step\ i) = |y(i) - y_i|$
  - **Local truncation error (LTE): the cumulative error caused by many iterations.**
    - Same as GTE, but when calculating the approximated value at some point, we don't rely on the previous approximated value we got at the previous step, but rather use the exact value.
    - $LTE\ (on\ step\ i) = \left|y(i) - N\big(y(i-1)\big)\right|$
  - **Note that these definitions can vary between textbooks**

- **Applications:**
  - Newton's law of cooling
  - Predator-Prey model
  - Exponential growth and decay