Bases

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February 23, 2022

Duke UNIVERSITY MATH

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Fundamental Subspaces
- Bases
- Examples

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- Bases of Null(A)
- Bases of Null(A^T)

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We think of a basis as a *minimal spanning set*.

Example

Consider the vector space $V \subset \mathbb{R}^3$ given by

$$V = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} \right\}$$

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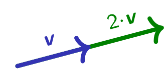

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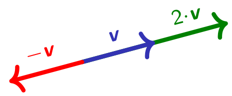

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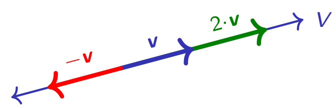

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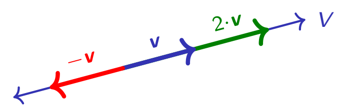

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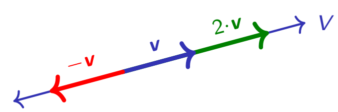

These vectors are all *dependent*. Choosing any one gives a basis of V.

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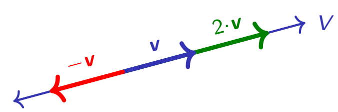

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Vector spaces have infinitely many bases!

Example

To find vectors $\boldsymbol{v} \in \mathrm{Null}(A)$, we must solve $A\boldsymbol{v} = \boldsymbol{O}$ for $\boldsymbol{v}$.

$$\mathrm{rref}\begin{bmatrix} 2 & 1 & 5 & 25 \\ -1 & -12 & 9 & 68 \\ 3 & 5 & 4 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 16 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Here, we have shown that

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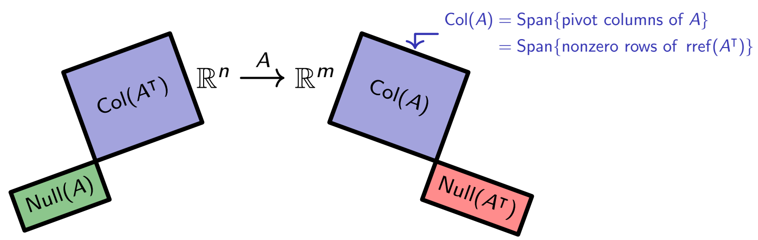
This is called the *pivot basis* of Null(A).

Fundamental Subspaces

Bases

Theorem

We may systematically construct bases of the four fundamental subspaces.



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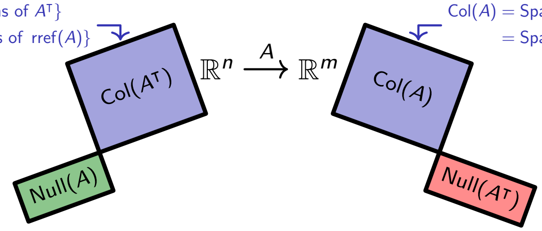

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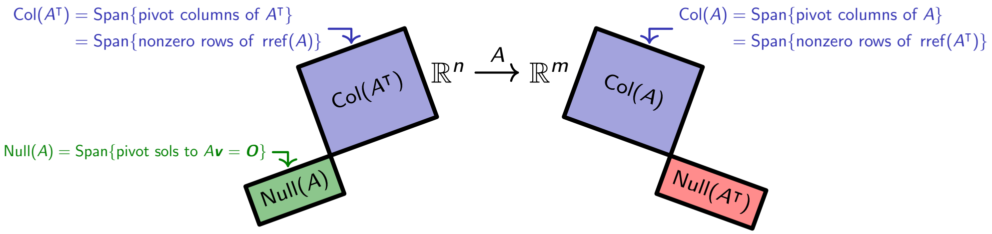

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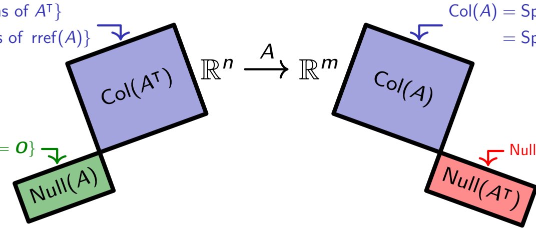

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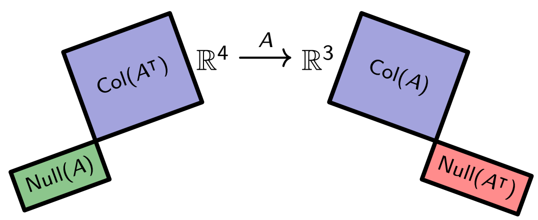

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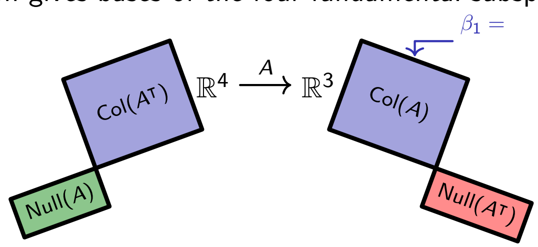

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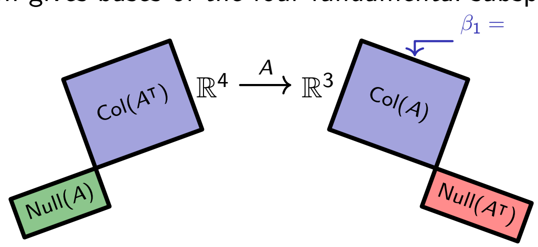

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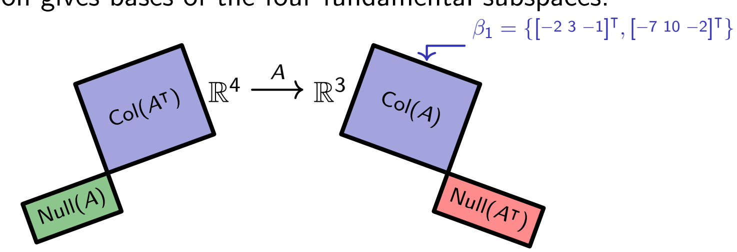

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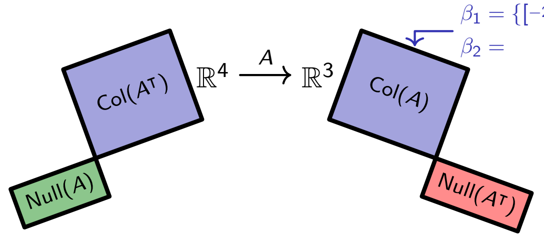

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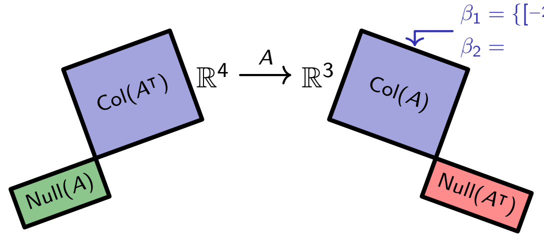

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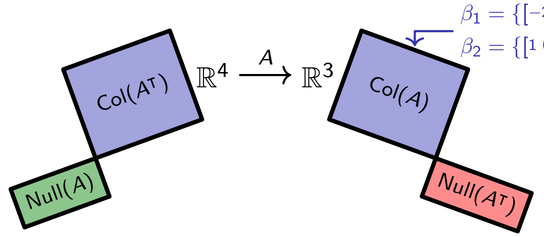

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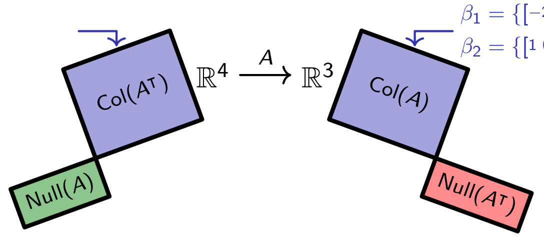

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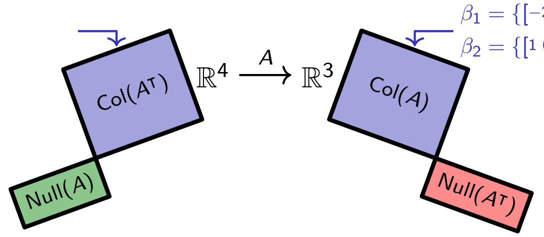
$\beta_2 = \{[1\ 0\ -4]^\intercal, [0\ 1\ -3]^\intercal\}$



Fundamental Subspaces

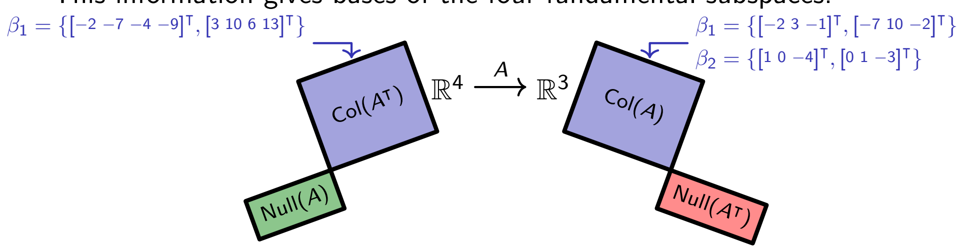
Examples

Example

Consider the calculations

$$\operatorname{rref}\overset{A}{\begin{bmatrix} -2 & -7 & -4 & -9 \\ 3 & 10 & 6 & 13 \\ -1 & -2 & -2 & -3 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \operatorname{rref}\overset{A^\intercal}{\begin{bmatrix} \boxed{-2} & \boxed{3} & -1 \\ \boxed{-7} & \boxed{10} & -2 \\ \boxed{-4} & \boxed{6} & -2 \\ \boxed{-9} & \boxed{13} & -3 \end{bmatrix}} = \begin{bmatrix} \boxed{1} & 0 & -4 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This information gives bases of the four fundamental subspaces.



Fundamental Subspaces

Examples

Example

Consider the calculations

$$\operatorname{rref}\overbrace{\begin{bmatrix} -2 & -7 & -4 & -9 \\ 3 & 10 & 6 & 13 \\ -1 & -2 & -2 & -3 \end{bmatrix}}^{A} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \operatorname{rref}\overbrace{\begin{bmatrix} -2 & 3 & -1 \\ -7 & 10 & -2 \\ -4 & 6 & -2 \\ -9 & 13 & -3 \end{bmatrix}}^{A^\intercal} = \begin{bmatrix} \boxed{1} & 0 & -4 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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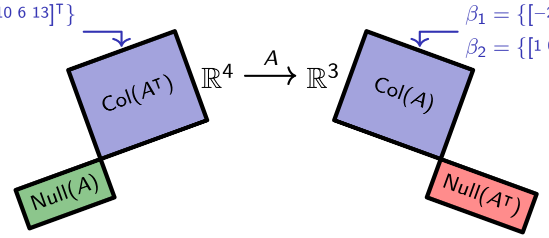

Fundamental Subspaces

Examples

Example

Consider the calculations

$$\operatorname{rref}\begin{array}{c} A \\ \begin{bmatrix} -2 & -7 & -4 & -9 \\ 3 & 10 & 6 & 13 \\ -1 & -2 & -2 & -3 \end{bmatrix}\end{array} = \begin{bmatrix} \boxed{1\ 0\ 2\ 1} \\ \boxed{0\ 1\ 0\ 1} \\ 0\ 0\ 0\ 0 \end{bmatrix} \qquad \operatorname{rref}\begin{array}{c} A^\intercal \\ \begin{bmatrix} -2 & 3 & -1 \\ -7 & 10 & -2 \\ -4 & 6 & -2 \\ -9 & 13 & -3 \end{bmatrix}\end{array} = \begin{bmatrix} \boxed{1} & 0 & -4 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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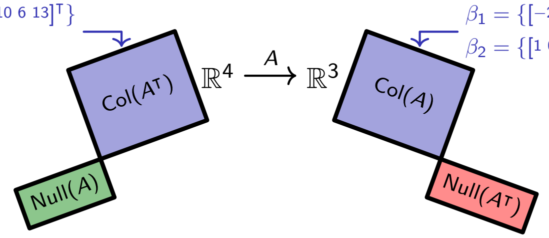

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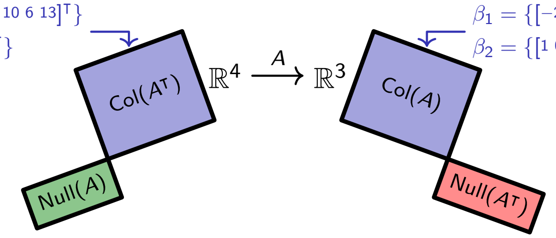

Fundamental Subspaces

Examples

Example

Consider the calculations

$$\operatorname{rref} \overset{A}{\begin{bmatrix} -2 & -7 & -4 & -9 \\ 3 & 10 & 6 & 13 \\ -1 & -2 & -2 & -3 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \operatorname{rref} \overset{A^\intercal}{\begin{bmatrix} -2 & 3 & -1 \\ -7 & 10 & -2 \\ -4 & 6 & -2 \\ -9 & 13 & -3 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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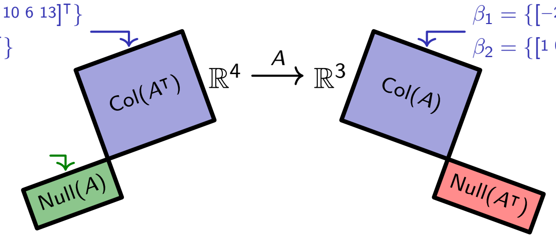

Fundamental Subspaces

Examples

Example

Consider the calculations

$$\operatorname{rref}\overbrace{\begin{bmatrix}-2 & -7 & -4 & -9\\ 3 & 10 & 6 & 13\\ -1 & -2 & -2 & -3\end{bmatrix}}^{A} = \boxed{\begin{bmatrix}1 & 0 & 2 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}} \qquad \operatorname{rref}\overbrace{\begin{bmatrix}-2 & 3 & -1\\ -7 & 10 & -2\\ -4 & 6 & -2\\ -9 & 13 & -3\end{bmatrix}}^{A^{\intercal}} = \begin{bmatrix}1 & 0 & -4\\ 0 & 1 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

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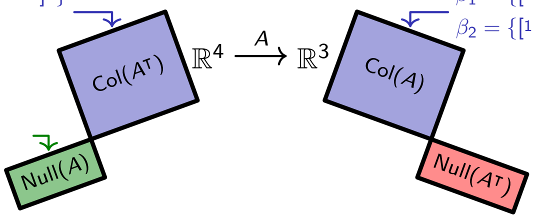

$\beta =$

Fundamental Subspaces

Examples

Example

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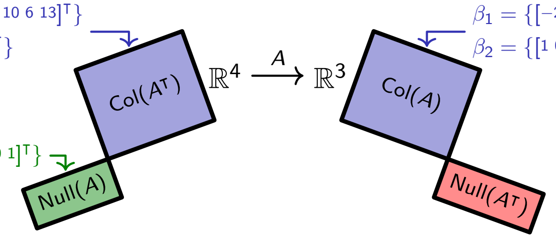

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$$\operatorname{rref}\overset{A}{\begin{bmatrix} -2 & -7 & -4 & -9 \\ 3 & 10 & 6 & 13 \\ -1 & -2 & -2 & -3 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \operatorname{rref}\overset{A^\intercal}{\begin{bmatrix} -2 & 3 & -1 \\ -7 & 10 & -2 \\ -4 & 6 & -2 \\ -9 & 13 & -3 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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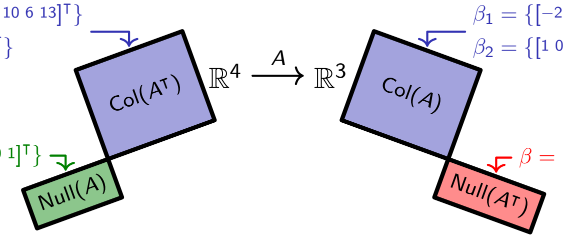

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Fundamental Subspaces

Examples

Example

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$$\operatorname{rref}\overbrace{\begin{bmatrix}-2 & -7 & -4 & -9\\ 3 & 10 & 6 & 13\\ -1 & -2 & -2 & -3\end{bmatrix}}^{A} = \begin{bmatrix}1 & 0 & 2 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix} \qquad \operatorname{rref}\overbrace{\begin{bmatrix}-2 & 3 & -1\\ -7 & 10 & -2\\ -4 & 6 & -2\\ -9 & 13 & -3\end{bmatrix}}^{A^\intercal} = \boxed{\begin{bmatrix}1 & 0 & -4\\ 0 & 1 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}}$$

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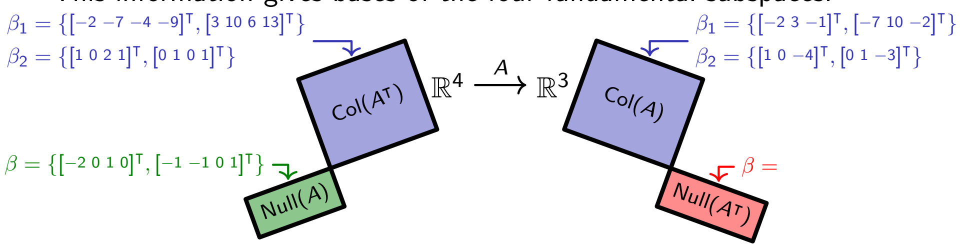

Fundamental Subspaces

Examples

Example

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$$\operatorname{rref}\overset{A}{\begin{bmatrix} -2 & -7 & -4 & -9 \\ 3 & 10 & 6 & 13 \\ -1 & -2 & -2 & -3 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \operatorname{rref}\overset{A^\intercal}{\begin{bmatrix} -2 & 3 & -1 \\ -7 & 10 & -2 \\ -4 & 6 & -2 \\ -9 & 13 & -3 \end{bmatrix}} = \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 & -4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}} \end{bmatrix}$$

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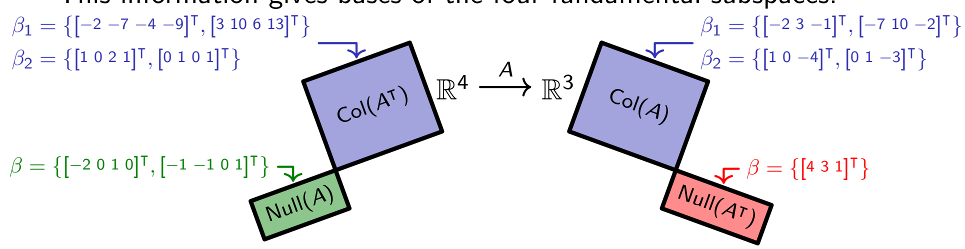
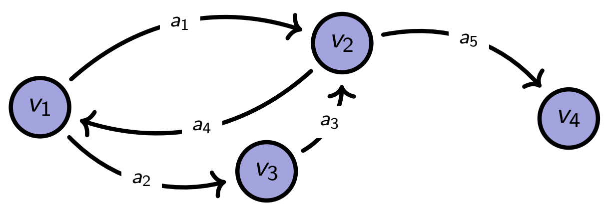


Digraphs

Bases of Null(A)

Theorem

Purge the minimum number of arrows necessary to break all cycles.



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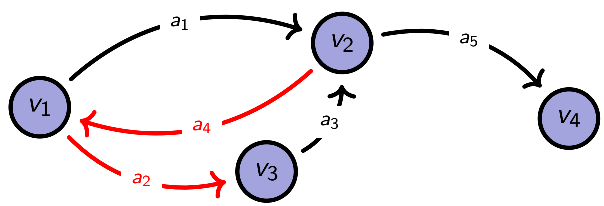

Digraphs

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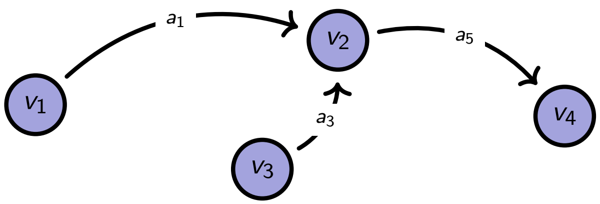

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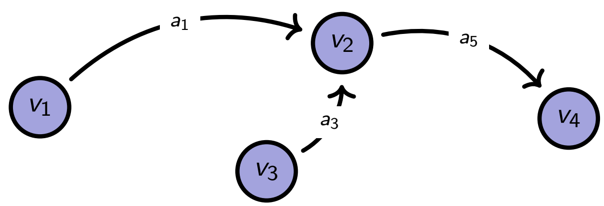

Let $\{c_1, \ldots, c_k\}$ be cycles, each passing through exactly one purged vector.

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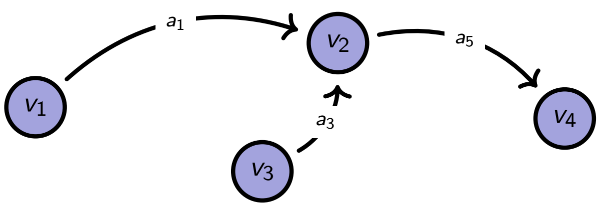

Let $\{c_1, \ldots, c_k\}$ *be cycles, each passing through exactly one purged vector.*

$$\boldsymbol{v}_{c_1} = \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Digraphs

Bases of Null(A)

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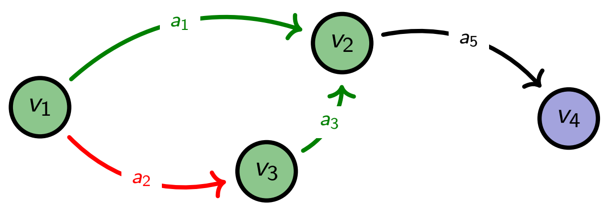

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Digraphs

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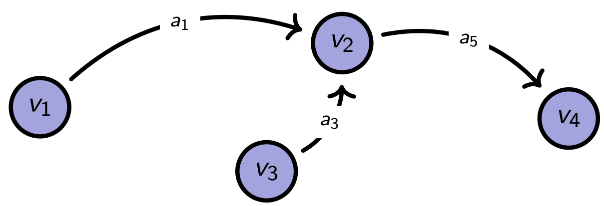

Let $\{c_1, \ldots, c_k\}$ *be cycles, each passing through exactly one purged vector.*

$$\boldsymbol{v}_{c_1} = \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \end{matrix} \\ \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Digraphs

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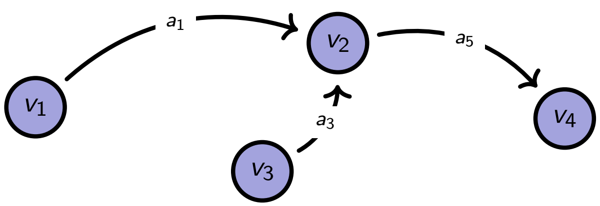

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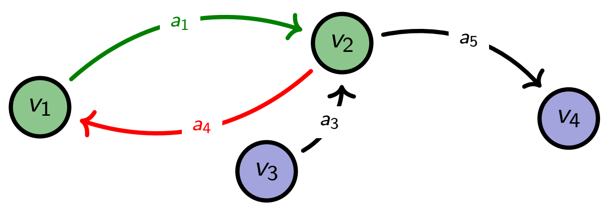

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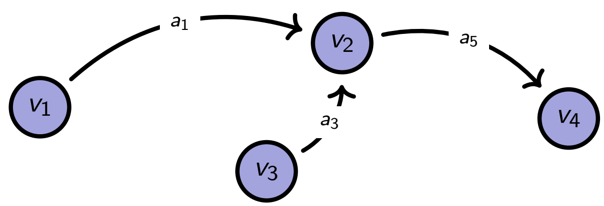

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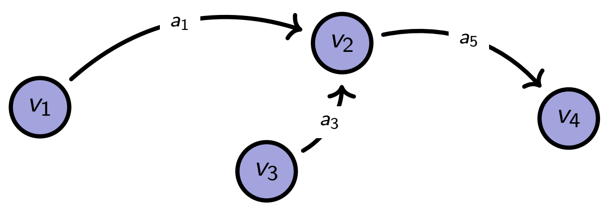
Then $\{\boldsymbol{v}_{c_1}, \ldots, \boldsymbol{v}_{c_k}\}$ *is a basis of* $\mathrm{Null}(A)$.

Digraphs

Bases of Null(A)

Theorem

Purge the minimum number of arrows necessary to break all cycles.



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$$\begin{array}{c} \\ \boldsymbol{v}_{c_1} = \end{array} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \\ \left[-1\right. & 1 & 1 & 0 & \left.0\right] \end{array} \qquad \begin{array}{c} \\ \boldsymbol{v}_{c_2} = \end{array} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \\ \left[1\right. & 0 & 0 & 1 & \left.0\right] \end{array}$$

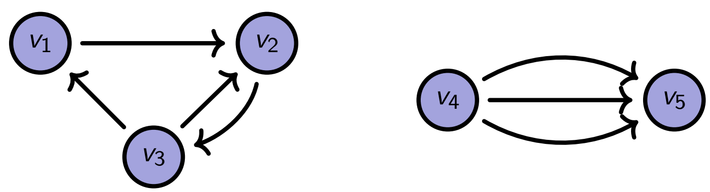
Then $\{\boldsymbol{v}_{c_1}, \ldots, \boldsymbol{v}_{c_k}\}$ *is a basis of* $\mathrm{Null}(A)$*. Here,* $\mathrm{Null}(A) = \mathrm{Span}\{\boldsymbol{v}_{c_1}, \boldsymbol{v}_{c_2}\}$.

Digraphs

Bases of Null($A^\intercal$)

Theorem

Let $\{G_1, \ldots, G_k\}$ be the connected components of a digraph G.



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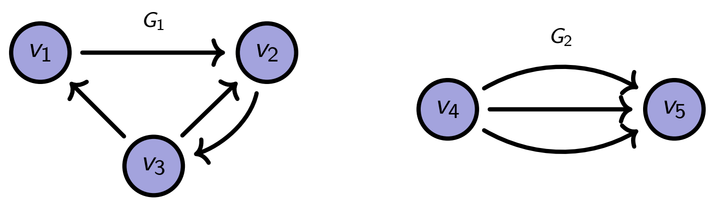

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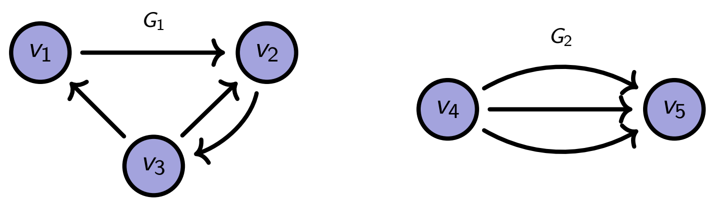

Consider the associated classification vectors $\{\boldsymbol{v}_{G_1}, \ldots, \boldsymbol{v}_{G_k}\}$.

$$\boldsymbol{v}_{G_1} = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \end{bmatrix}^\intercal \end{array}$$

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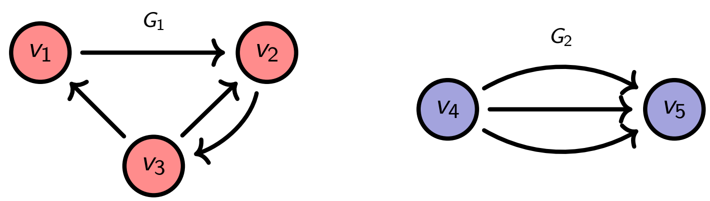

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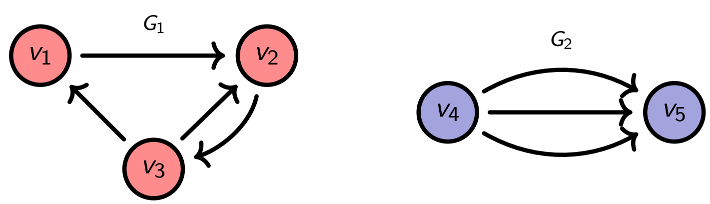

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$$\boldsymbol{v}_{G_1} = \begin{array}{c} \begin{matrix} \color{red}{v_1} & \color{red}{v_2} & \color{red}{v_3} & v_4 & v_5 \end{matrix} \\ \begin{bmatrix} \color{red}{1} & \color{red}{1} & \color{red}{1} & 0 & 0 \end{bmatrix} \end{array}^\intercal$$

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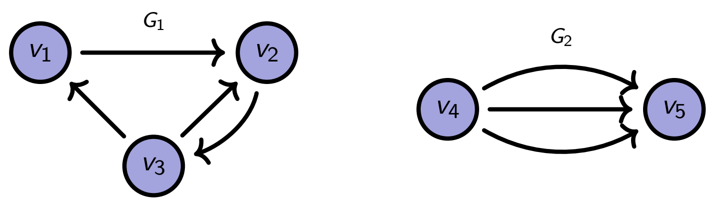
Let $\{G_1, \dots, G_k\}$ be the connected components of a digraph G.



Consider the associated classification vectors $\{\boldsymbol{v}_{G_1}, \dots, \boldsymbol{v}_{G_k}\}$.

$$\boldsymbol{v}_{G_1} = \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}^{\intercal} \qquad \boldsymbol{v}_{G_2} = \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}^{\intercal}$$

Theorem

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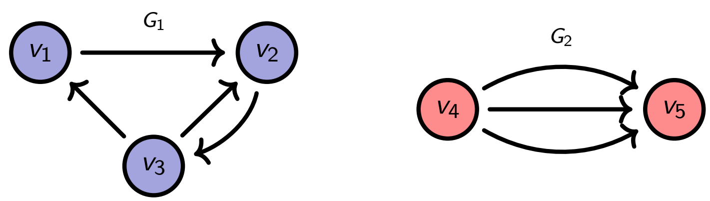

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Digraphs

Bases of $\operatorname{Null}(A^\intercal)$

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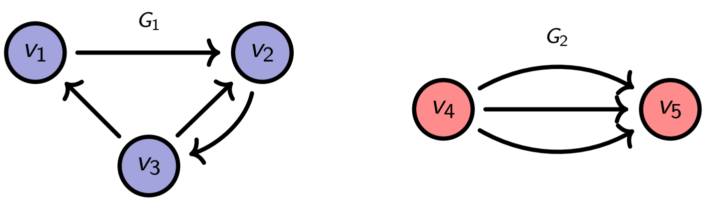

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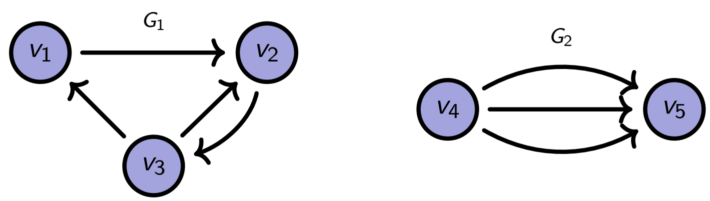

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These vectors form a basis of $\mathrm{Null}(A^\intercal)$.

Digraphs

Bases of $\mathrm{Null}(A^\intercal)$

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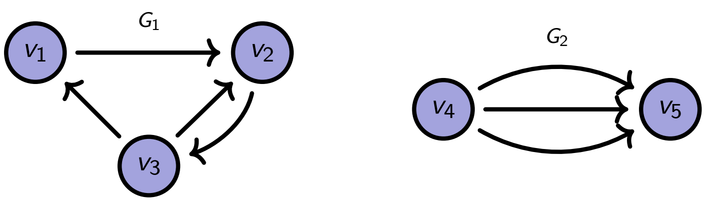

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These vectors form a basis of $\mathrm{Null}(A^\intercal)$. *Here,* $\mathrm{Null}(A^\intercal) = \mathrm{Span}\{\boldsymbol{v}_{G_1}, \boldsymbol{v}_{G_2}\}$.