In [1]:
```python
import random as rnd

#ber is a function that

def ber(p):
  x=rnd.random()
  if x <= p :
    return 1
  else:
    return 0

[ ber(1/3) for i in range(20)]
```

Out[1]:
```
[1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1]
```

In [2]:
```python
from PIL import Image
from IPython.display import display
```

Exercise 3.1

In [3]:
```python
img = Image.open('ex3-1.png')
display(img)
```

Exercise 3.1: Simulating Discrete Distribution

The `ber(p)` function takes a single probability p and returns 1 and random number drawn uniformly from the interval $[0,1]$ is less than p. Now implement a function `genBer(p)` that take vector $p=[p_0,\dots,p_n]$ or probabilities so $p_0+\cdots+p_n=1$ and returns a number in $\{0,\dots,n\}$ so that $\mathbf{P}(k)=p_k$. show the out put from several runs of you function in each of the following cases: `genBer([.1,.5,.4])` and `genBer([.4,.2,.1,.3])` and `genBer([.3,.3,.1,.1,.1,.1])`.

Hint: Create a function f so that $f(x)=k$ if $x\in[q_k,q_{k+1}]$ where $q_0=0$ and $q_k=p_0+\cdots+p_{k-1}$ for $k>0$. Now ask yourself what is the $\mathbf{P}(\mathrm{rnd.random}()\in(q_k,q_{k+1}])$?

In [4]:

```python
'''
Calculate the cumulative sum of the probabilities:

q0 = 0
q1 = q0 + 0.1 = 0.1
q2 = q1 + 0.2 = 0.3
q3 = q2 + 0.3 = 0.6
q4 = q3 + 0.4 = 1.0

Generate a random number x between 0 and 1.

Determine which interval x falls into:

If x is between 0 and 0.1, return 0.
If x is between 0.1 and 0.3, return 1.
If x is between 0.3 and 0.6, return 2.
If x is between 0.6 and 1.0, return 3.

For example, if the random number x generated is 0.35, it falls between 0.3

genBer([.1,.5,.4]) and genBer([.4,.2,.1,.3]) and genBer([.3,.3,.1,.1,.1,.1])

'''

def genBer(p, printDetails = False):
    x = rnd.random()

    if (printDetails):
        print("x = " + str(x))
        print("p = " + str(p))

    for i in range(0,len(p)):
        x = x - p[i]
        #print(i, x)
        if (x <= 0):
            return i
    return -1
```

In [5]:

```python
ret = genBer([.1,.5,.4], printDetails=True)

print("index = " + str(ret) + ", which means item " + str(ret+1) )
```

```
x = 0.553452985875975
p = [0.1, 0.5, 0.4]
index = 1, which means item 2
```

In [6]:

```python
ret = genBer([.4,.2,.1,.3], printDetails=True)

print("index = " + str(ret) + ", which means item " + str(ret+1) )
```

```
x = 0.9260522513742205
p = [0.4, 0.2, 0.1, 0.3]
index = 3, which means item 4
```

In [7]:
```python
ret = genBer([.3,.3,.1,.1,.1,.1], printDetails=True)

print("index = " + str(ret) + ", which means item " + str(ret+1) )
```

```
x = 0.6491072302125694
p = [0.3, 0.3, 0.1, 0.1, 0.1, 0.1]
index = 2, which means item 3
```

Exercise 3.2

In [8]:
```python
import matplotlib.pyplot as plt

img = Image.open('ex3-2.png')
display(img)
```

Exercise 3.2: Checking Exercise 3.1 : Simulating Discrete Distribution

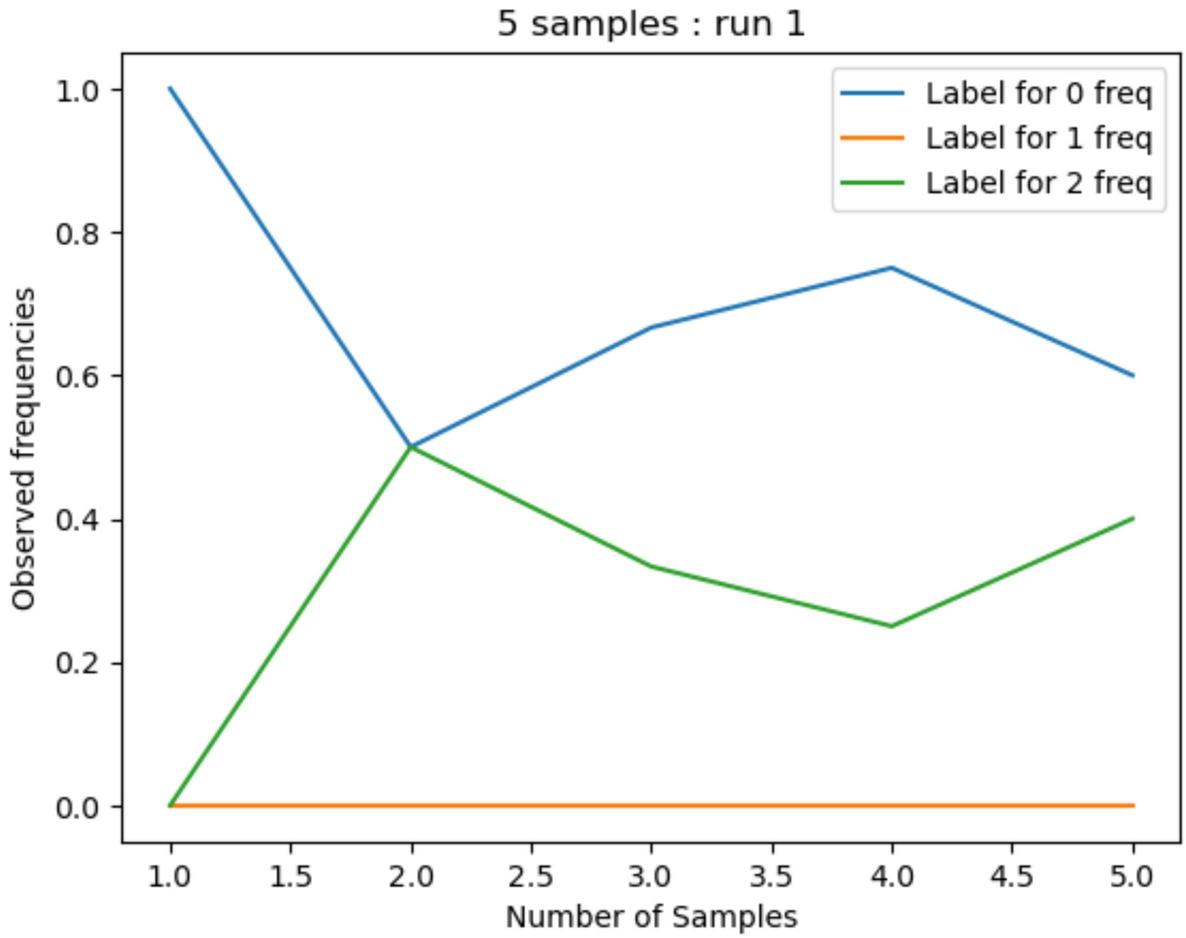
Check that the function `genBer(p)` you made in Exercise 3.1 . Taking $p = [.6, .1, .3]$ use `genBer(p)` to generate 5, 10, 50 100, 500 and 1000 random draws according to the probabilities in p.
1. Check the average number of times you seen the number 0 drawn in each case. Repeat this for 1 and 2. 2. Plot the average times you drew 0, 1 and 3 as a function of the number of draws n for $n = 1, 2, \ldots, 1000$. In other words create plots like we made in Section 1.2. Does it seem like your `genBer(p)` is doing the right thing ? explain.

In [9]:
```python
p = [0.6, 0.1, 0.3]
```

In [10]:
```
# this is a function that can draw numDraws times and take a list of proabil
# such as p above and then print some info + a graph
def drawNPlot(numDraws, probabilityList):
    drawList = []

    _0freq = []
    num0 = 0

    _1freq = []
    num1 = 0

    _2freq = []
    num2 = 0

    for i in range(numDraws):
      draw = genBer(probabilityList)
      drawList.append(draw)

      if draw == 0:
        num0 += 1
      elif draw == 1:
        num1 += 1
      elif draw == 2:
        num2 += 1

      _0freq.append(num0/(i+1)) # this is percentage of all draws that are 0
      _1freq.append(num1/(i+1))
      _2freq.append(num2/(i+1))

    #print out frequency
    print(f"{num0} zeros, {num1} ones, {num2} twos in {numDraws} samples.")
    print("The observed fraction of zeros is {:.4g}".format(num0/numDraws))
    print("The observed fraction of ones is {:.4g}".format(num1/numDraws))
    print("The observed fraction of twos is {:.4g}".format(num2/numDraws))

    # Make Plots
    plt.plot(range(1, numDraws+1), _0freq, label='Label for 0 freq')
    plt.plot(range(1, numDraws+1), _1freq, label='Label for 1 freq')
    plt.plot(range(1, numDraws+1), _2freq, label='Label for 2 freq')

    #labels and stuff
    plt.xlabel("Number of Samples")
    plt.ylabel("Observed frequencies")
    plt.title("{:,} samples : run 1".format(numDraws))

    plt.legend()
    plt.show()
```

In [11]:
```
#running for 5 draws
drawNPlot(5,p)
```

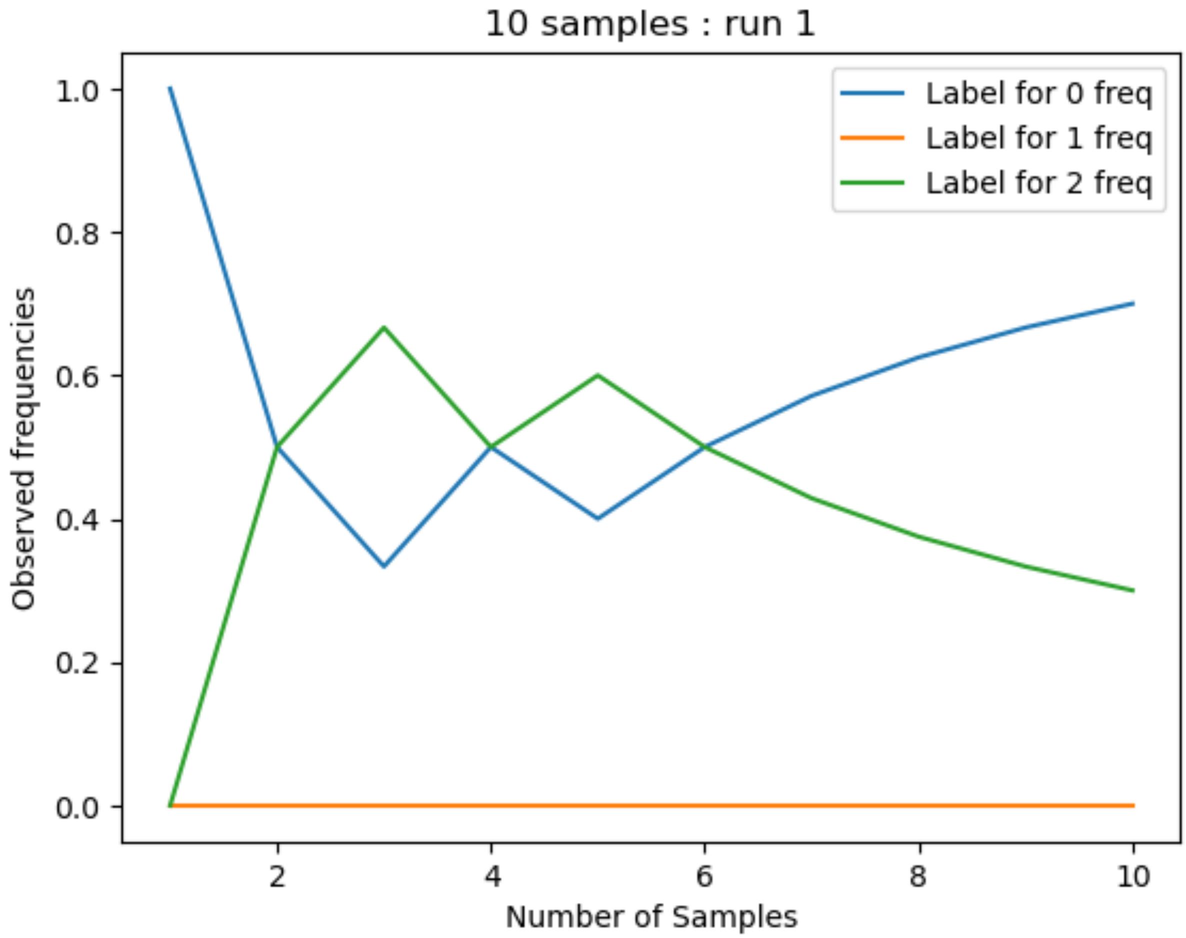
```
3 zeros, 0 ones, 2 twos in 5 samples.
The observed fraction of zeros is 0.6
The observed fraction of ones is 0
The observed fraction of twos is 0.4
```



In [12]:

```
#running for 10 draws
drawNPlot(10,p)
```

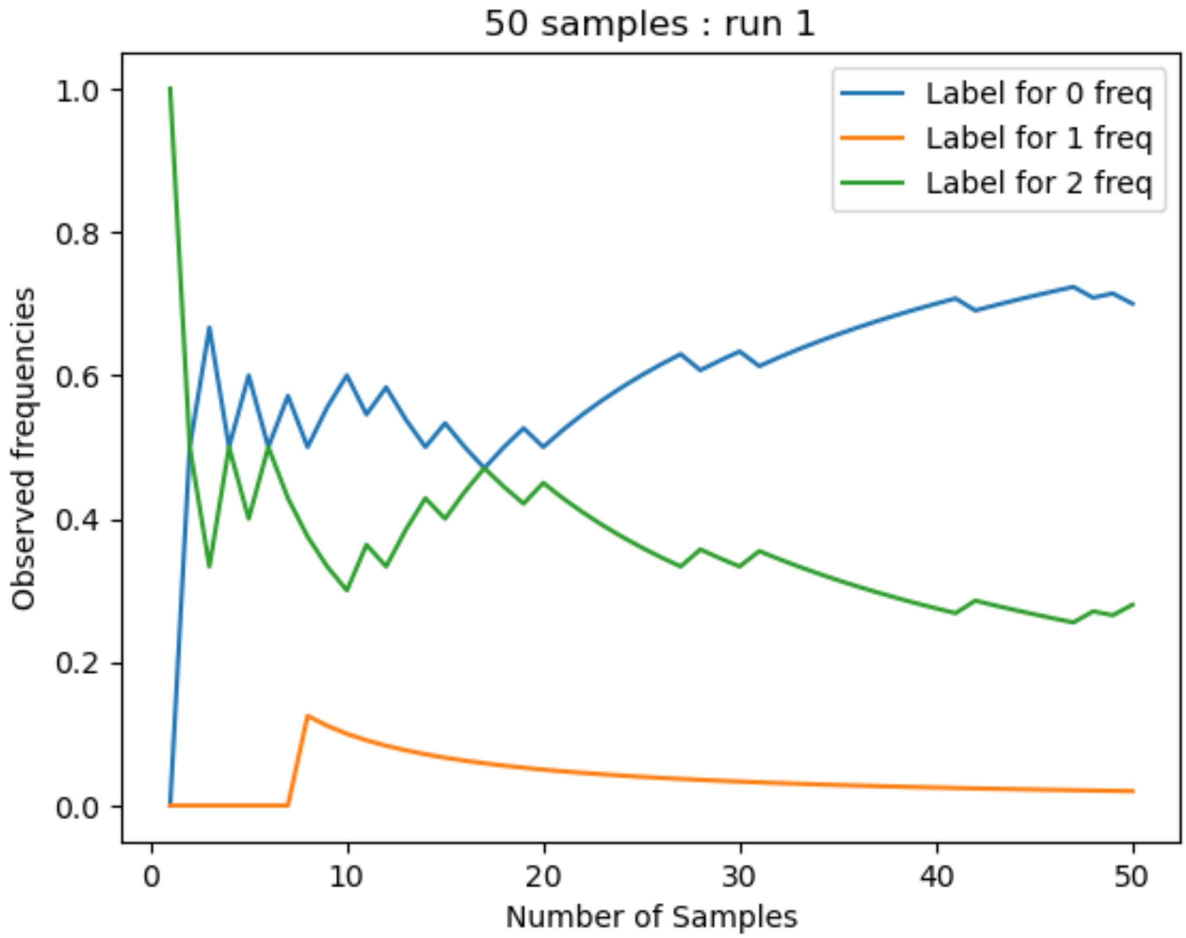
```
7 zeros, 0 ones, 3 twos in 10 samples.
The observed fraction of zeros is 0.7
The observed fraction of ones is 0
The observed fraction of twos is 0.3
```



In [13]:

```
#running for 50 draws
drawNPlot(50,p)
```

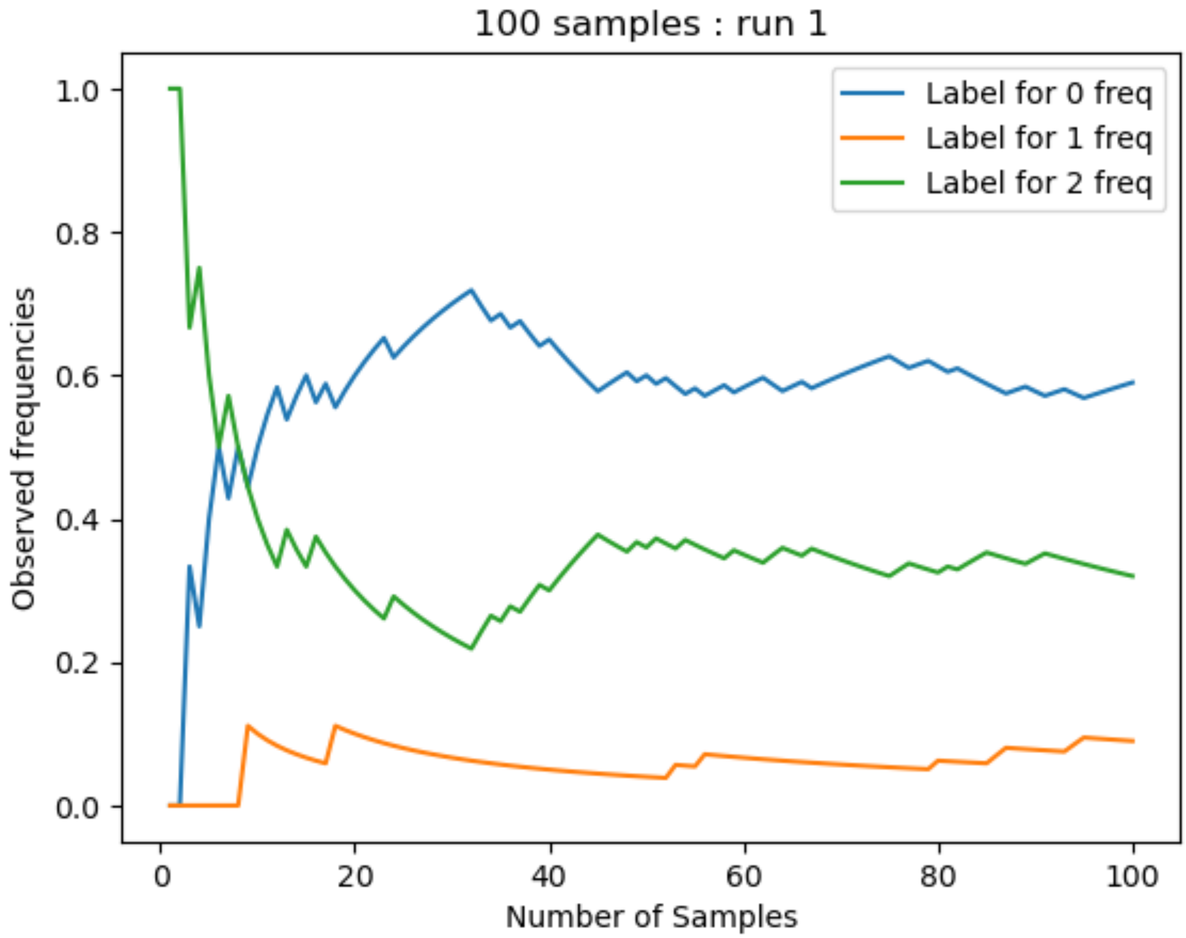
```
35 zeros, 1 ones, 14 twos in 50 samples.
The observed fraction of zeros is 0.7
The observed fraction of ones is 0.02
The observed fraction of twos is 0.28
```



In [14]:

```
#running for 100 draws
drawNPlot(100,p)
```

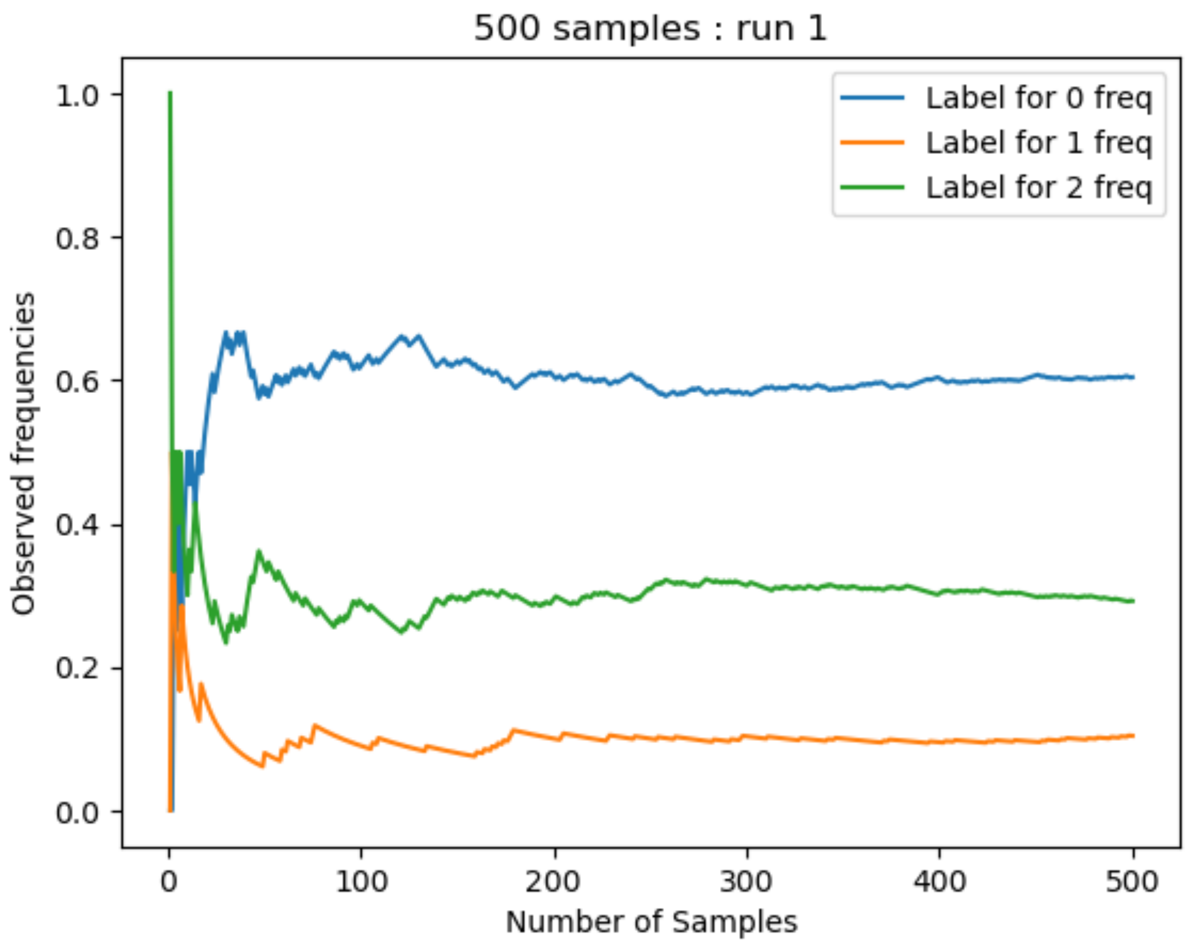
```
59 zeros, 9 ones, 32 twos in 100 samples.
The observed fraction of zeros is 0.59
The observed fraction of ones is 0.09
The observed fraction of twos is 0.32
```



In [15]:

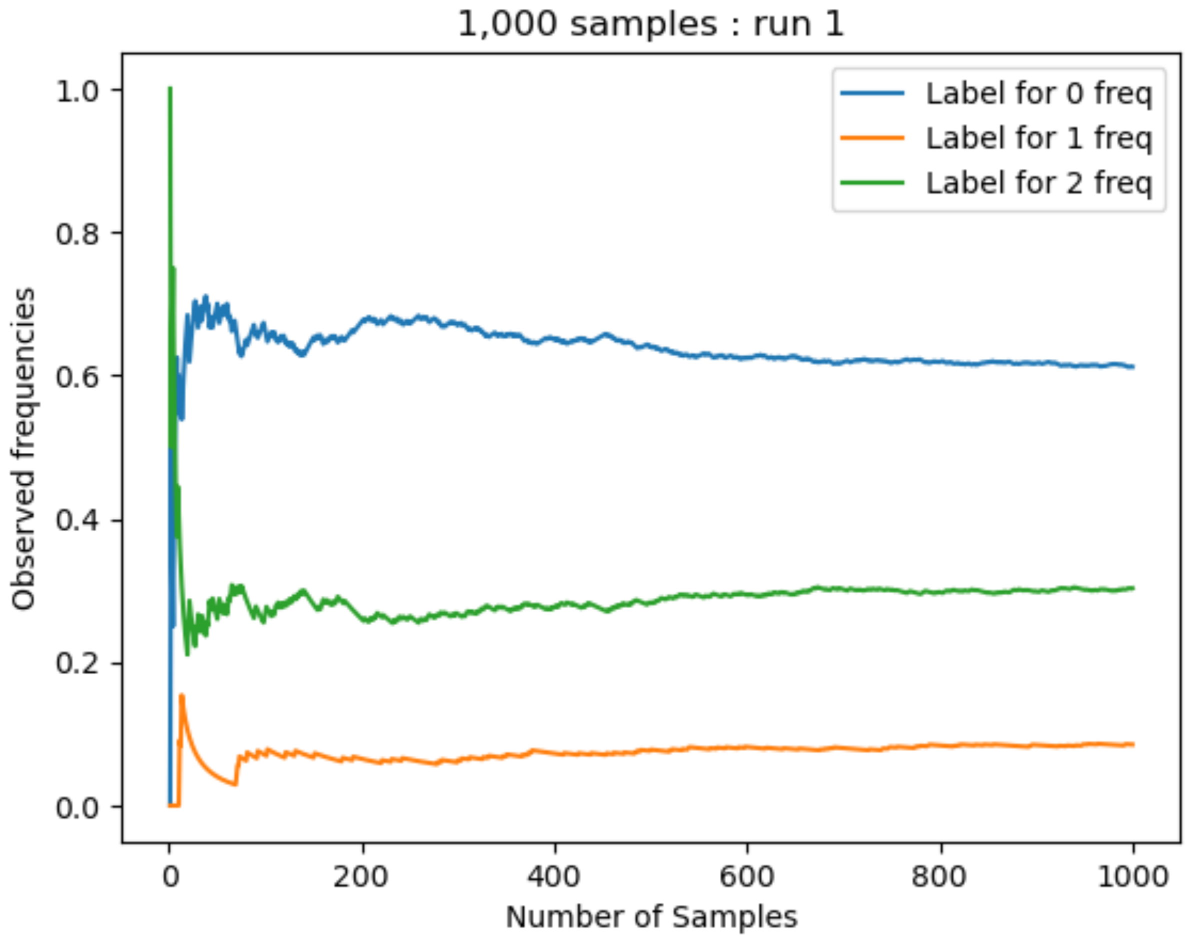
```
#running for 500 draws
drawNPlot(500,p)
```

```
302 zeros, 52 ones, 146 twos in 500 samples.
The observed fraction of zeros is 0.604
The observed fraction of ones is 0.104
The observed fraction of twos is 0.292
```



In [16]:
```
#running for 1000 draws
drawNPlot(1000,p)
```

```
612 zeros, 85 ones, 303 twos in 1000 samples.
The observed fraction of zeros is 0.612
The observed fraction of ones is 0.085
The observed fraction of twos is 0.303
```



This is doing the correct thing, as the number of samples get higher, the probability of each number tends closer to the probability array we defined earlier

Exercise 3.5

In [17]:
```
img = Image.open('ex3-5.png')
display(img)
```

Exercise 3.5: Monte Carlo Integration

Use the Monte Carlo integration method described above to calculate the integral

$$\int_0^{\frac{1}{2}} \sin(cx)^2 dx$$

where $c = 6.14159$. Compare this to the real answer with $c = 2\pi$.

In [18]:

```
def monteCarlo(f, boxX, boxY, area, numberSamples=10000, showPlot=True):
    samplesIn=[]
    samplesOut=[]

    for k in range(numberSamples):
        x = boxX()
        y = boxY()

        if f(x) > y:  # Check to see if point is below curve y=f(x)
            samplesIn.append([x,y]) # point below
        else:
            samplesOut.append([x,y]) # point above

    numIn=len(samplesIn)     # number of points below
    numOut=len(samplesOut)  # number of points above

    ratioIn = numIn/numberSamples # = P(R)
    totalArea = area()

    # P(R) = Area(R) / Area(Box)

    # Area(R) = Area(Box) * P(R)

    # print(totalArea)

    areaRegion = ratioIn * totalArea

    if (showPlot):
        print("Out of {:,} samples, {:,} are in the blue region under \n \t
        print("Hence fraction of the samples below the curve is {:,} \n \t a
        print()
        print("Hence our esitmate of the area under the curve is {:}.".forma

        x_samplesIn=[ p[0] for  p in samplesIn]
        y_samplesIn=[ p[1] for  p in samplesIn]

        x_samplesOut=[ p[0] for  p in samplesOut]
        y_samplesOut=[ p[1] for  p in samplesOut]

        plt.scatter(x_samplesOut,y_samplesOut,color='black',s=3)
        plt.scatter(x_samplesIn,y_samplesIn,color='lightblue',s=3)
        plt.xlabel("x")
        plt.ylabel("y")
        plt.show()

    #monteCarlo estimate of region for num samples

    return areaRegion
```

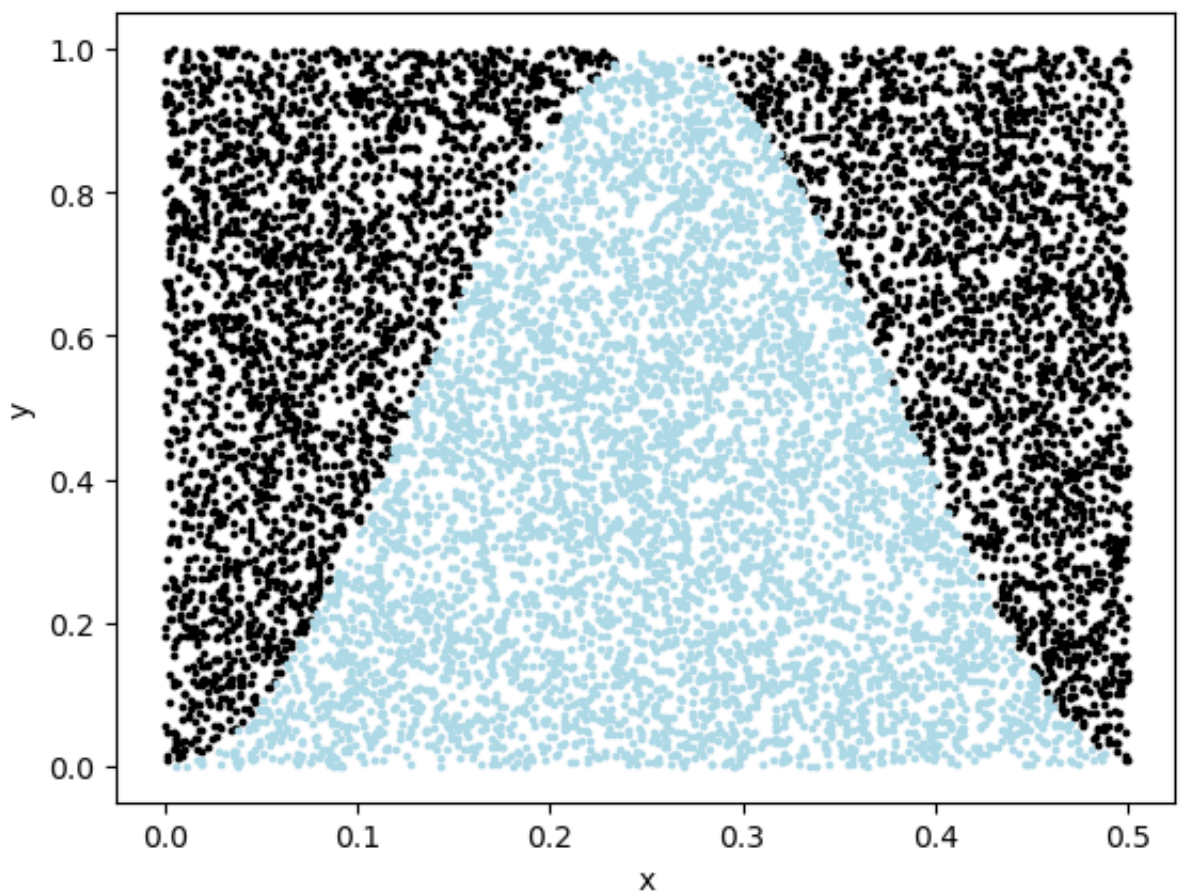
In [19]:

```python
import math

def function(x):
    c = 6.14159
    return math.sin( c * x ) ** 2

#integrate from 0 to half
def boxX():
    return (1/2)*rnd.random()

# goes from 0 to 1 in height
def boxY():
    return rnd.random()

# box area = 1* (1/2)
def area():
    return 1/2
```

In [20]:

```python
monteCarlo(function, boxX, boxY, area)
```

```
Out of 10,000 samples, 5,134 are in the blue region under
         the curve  and 4,866 are above.
Hence fraction of the samples below the curve is 0.5134
         and the fraction above is 0.4866.

Hence our esitmate of the area under the curve is 0.2567.
```



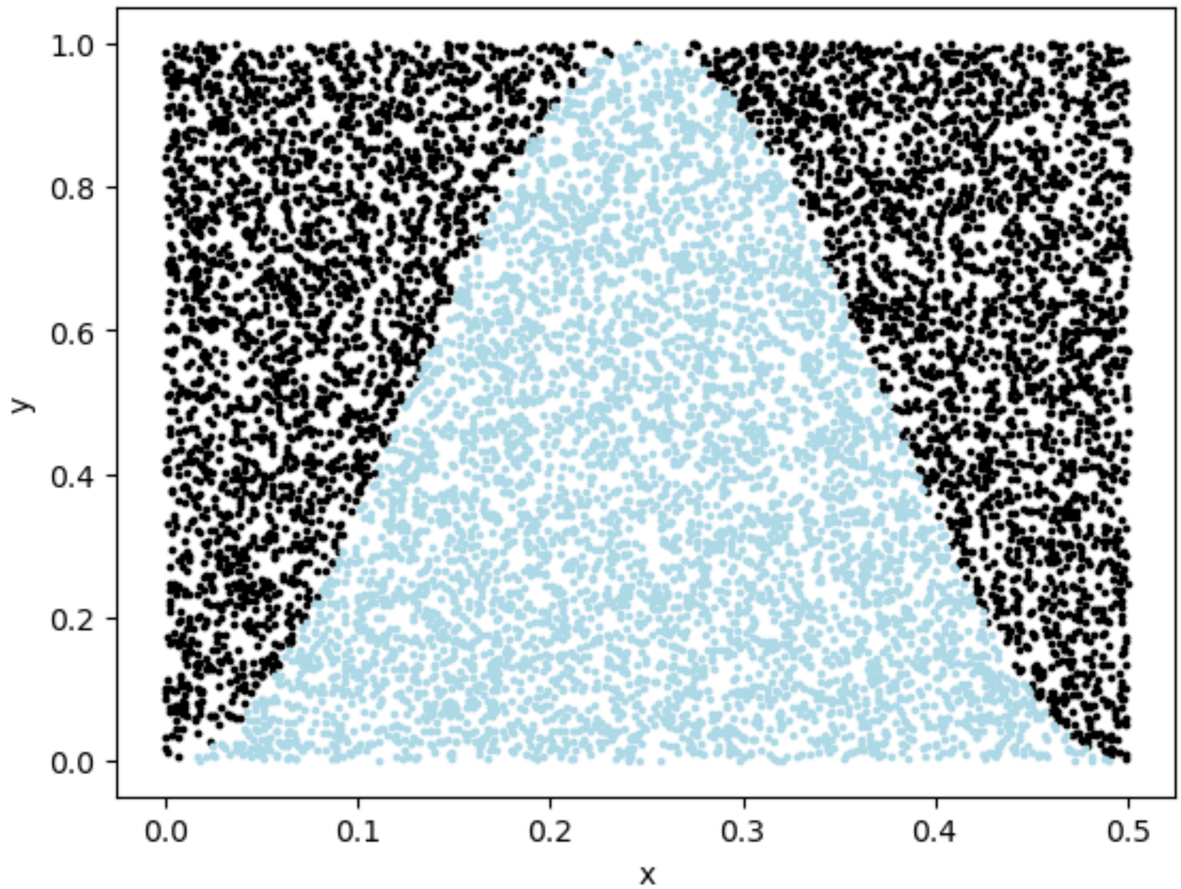
Out[20]:

```
0.2567
```

Actual answer is meant to be 0.25574454803615 with c = 6.14159, so this is pretty good

In [21]:
```
# with c = 2pi

def function(x):
    c = 2* math.pi
    return math.sin( c * x ) ** 2

monteCarlo(function, boxX, boxY, area)
```

```
Out of 10,000 samples, 4,962 are in the blue region under
         the curve  and 5,038 are above.
Hence fraction of the samples below the curve is 0.4962
         and the fraction above is 0.5038.

Hence our esitmate of the area under the curve is 0.2481.
```



Out[21]:
```
0.2481
```

pretty similar. (was c meant to be 6.28... like approx 2*pi?)

Exercise 3.5

In [22]:
```
img = Image.open('ex3-6.png')
display(img)
```

ⓘ **Exercise 3.6: Monte Carlo Convergence Rate**

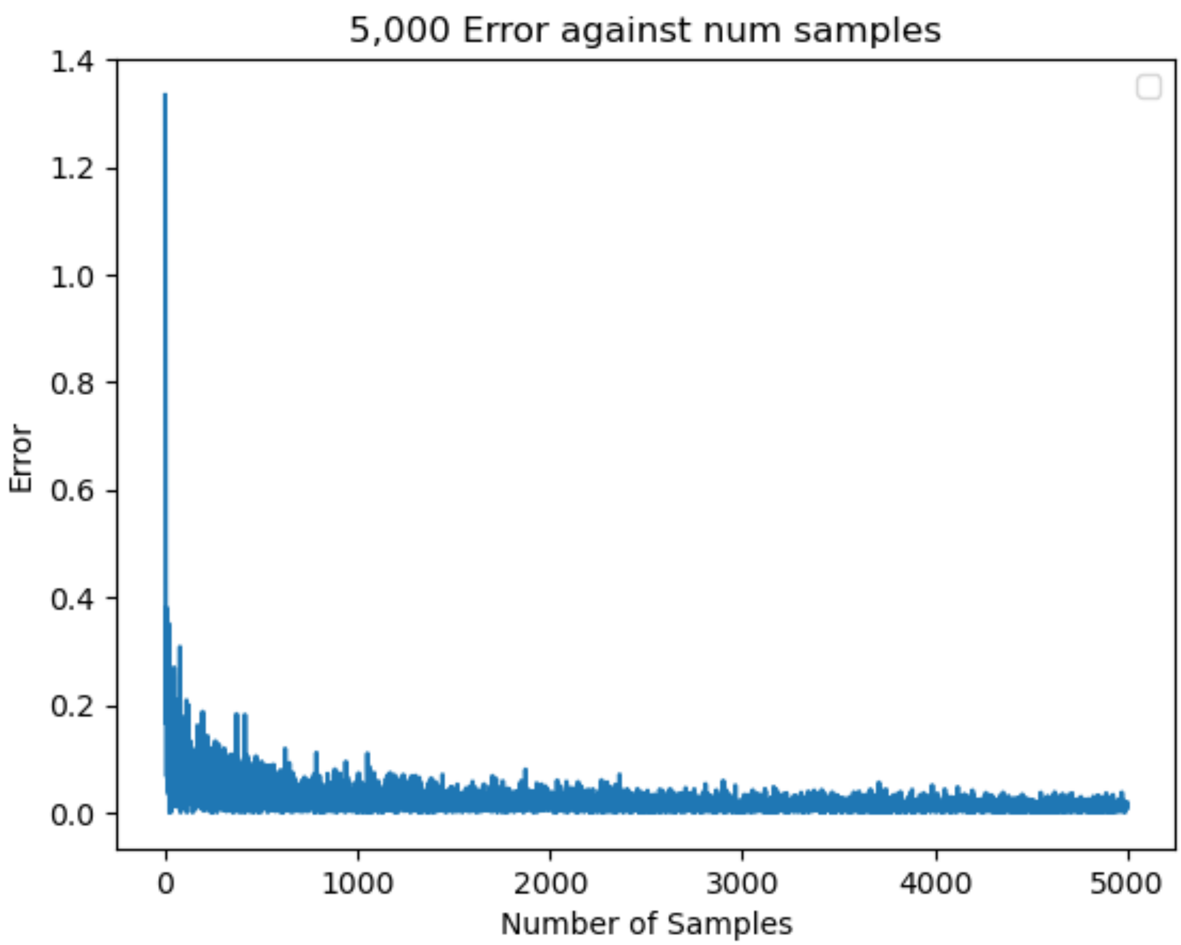
Study the convergence of the Monte Carlo Approximation of the integral of $f(x) = \frac{1}{2}x^3 - x^2 + 1$ over the interval $[0, 2]$. Plot the error verses the size of the sample size used in the Monte Carlo sampling. How does the error scale? Plotting the $\log(\text{error})$ vs $\log(\text{sample size})$ can help identify the relationship.

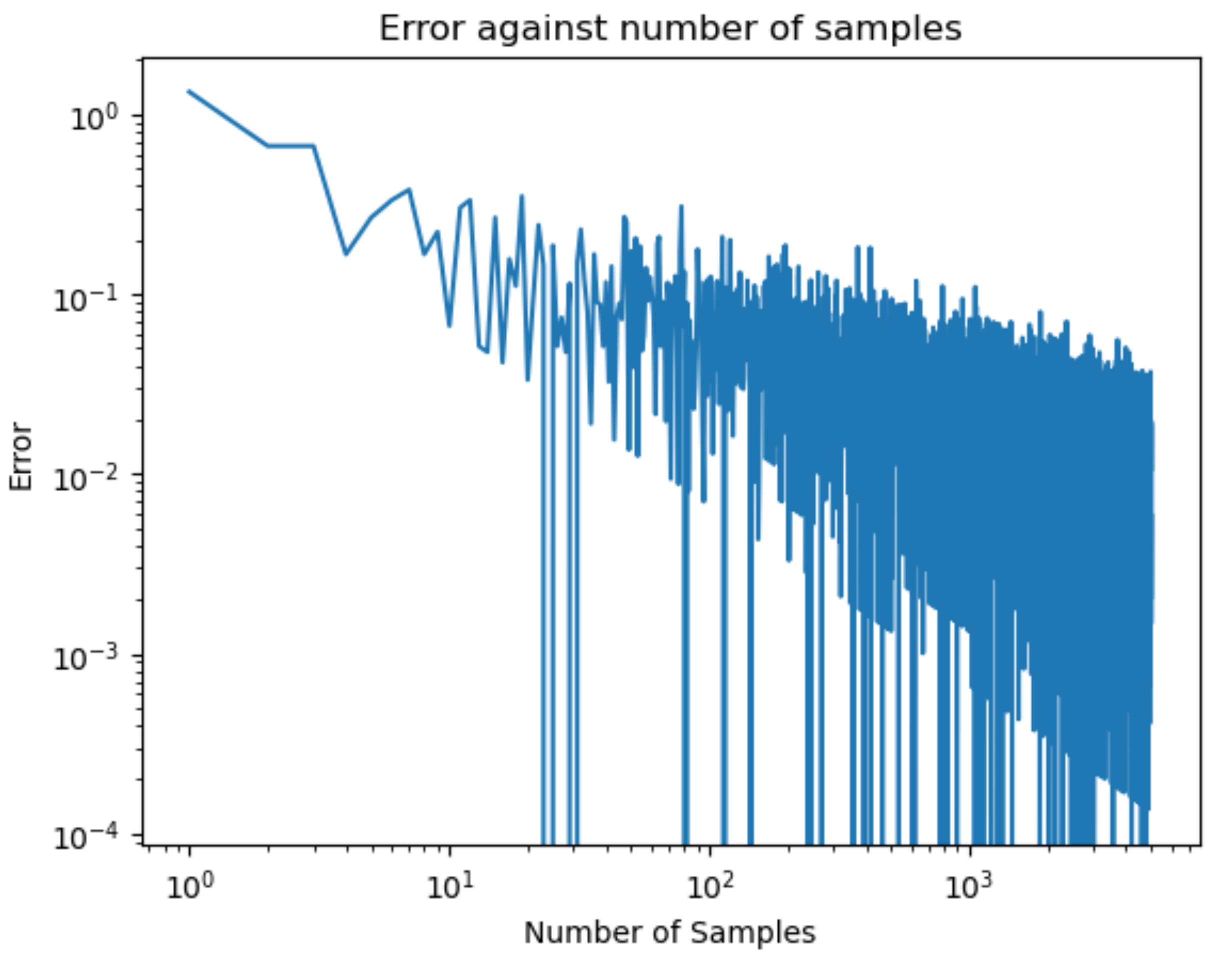
In [23]:
```
import numpy as np

def monteCarloConvergence(f, boxX, boxY, area, realVal, maxSampleSize):
    errorOverSamples = []
    runningAvgError = []

    for i in range(maxSampleSize):
      draw = monteCarlo(f,boxX,boxY,area,numberSamples=i+1, showPlot=False)

      tempError = abs(realVal() - draw)

      errorOverSamples.append(tempError)

      if (len(runningAvgError) != 0):
        runningAvgError.append( ( i * runningAvgError[i-1]+tempError)/i+1 )
      else:
        runningAvgError.append(tempError)

    # #print out frequency
    # print(f"{num0} zeros, {num1} ones, {num2} twos in {numDraws} samples.'
    # print("The observed fraction of zeros is {:.4g}".format(num0/numDraws)
    # print("The observed fraction of ones is {:.4g}".format(num1/numDraws))
    # print("The observed fraction of twos is {:.4g}".format(num2/numDraws))

    # Make Plots
    plt.plot(range(1, maxSampleSize+1), errorOverSamples)

    #labels and stuff
    plt.xlabel("Number of Samples")
    plt.ylabel("Error")
    plt.title("{:,} Error against num samples".format(maxSampleSize))

    plt.legend()
    plt.show()

    plt.loglog(range(1, maxSampleSize+1), errorOverSamples)

    #labels and stuff
    plt.xlabel("Number of Samples")
    plt.ylabel("Error")
    plt.title("Error against number of samples")

    plt.show()
```

In [24]:
```
def function(x):
    return (1/2) * (x**3) - (x**2) + 1

#integrate from 0 to half
def boxX():
    return (2)*rnd.random()

# goes from 0 to 1 in height
def boxY():
    return rnd.random()

# box area = 1* (1/2)
def area():
    return 2

def realVal():
    return 4/3
```

In [25]:
```
monteCarloConvergence(function, boxX, boxY, area, realVal, maxSampleSize=500
```

```
No artists with labels found to put in legend.  Note that artists whose labe
l start with an underscore are ignored when legend() is called with no argum
ent.
```





error very clearly decreases as number of samples increases

Homework question 4

In [26]:
```
img = Image.open('HWq4.png')
display(img)
```

Practice with inclusion, exclusion.

Posted on August 13, 2012 by Jonathan Mattingly | Leave a comment

Events A, B, and C are defined on a probability space. Find expressions for the following probabilities in terms of $\mathbf{P}(A)$, $\mathbf{P}(B)$, $\mathbf{P}(C)$, $\mathbf{P}(AB)$, $\mathbf{P}(AC)$, $\mathbf{P}(BC)$, and $\mathbf{P}(ABC)$.

1. The probability that exactly two of the A, B, C occur.
2. The probability that exactly one of these events occurs.
3. The probability that none of these events occur.

Here the notation AB is short for $A \cap B$ which is the event "both A and B"

([Pitman, p. 31, # 10])

In [27]:
```python
img = Image.open('HWa4.png')
display(img)
```

Practice with inclusion, exclusion.

Posted on August 13, 2012 by Jonathan Mattingly | Leave a comment

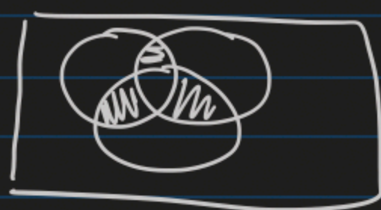
Events A, B, and C are defined on a probability space. Find expressions for the following probabilities in terms of $\mathbf{P}(A)$, $\mathbf{P}(B)$, $\mathbf{P}(C)$, $\mathbf{P}(AB)$, $\mathbf{P}(AC)$, $\mathbf{P}(BC)$, and $\mathbf{P}(ABC)$.

1. The probability that exactly two of the A, B, C occur.
2. The probability that exactly one of these events occurs.
3. The probability that none of these events occur.

Here the notation AB is short for $A \cap B$ which is the event "both A and B"

([Pitman, p. 31, # 10])

1)

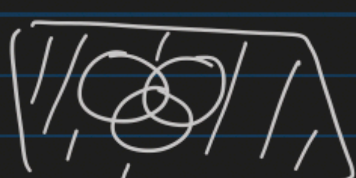


$$\mathbb{P}(\text{event}) = \mathbb{P}(A\cap B) + \mathbb{P}(A\cap C) + \mathbb{P}(B\cap C) - 3\,\mathbb{P}(A\cap B\cap C)$$

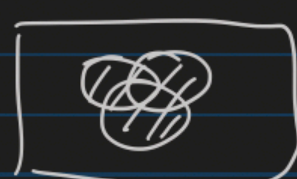
2)

$$\mathbb{P}(\text{event}) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \big(2\mathbb{P}(A\cap B) + 2\mathbb{P}(A\cap C) + 2\mathbb{P}(B\cap C)\big) + 3\mathbb{P}(A\cap B\cap C)$$

3)



$= 1 -$



$$\therefore\ \mathbb{P}(\text{event}) = 1 - \mathbb{P}(A\cup B\cup C)$$

$$\mathbb{P}(A\cup B\cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A\cap B) - \mathbb{P}(A\cap C) - \mathbb{P}(B\cap C)$$

$$+ \mathbb{P}(A \cap B \cap C)$$

$$\therefore \mathbb{P}(\text{event}) = 1 - \left[\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C)\right] + \mathbb{P}(A \cap B \cap C)$$

In [ ]: