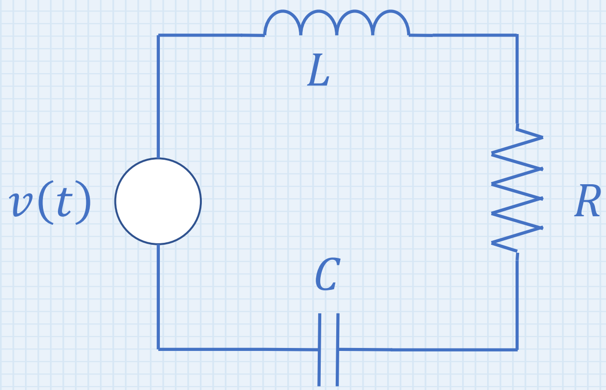
Solving Steady-State Circuits

For a circuit to be in steady state, we assume a sinusoidal excitation as follows:

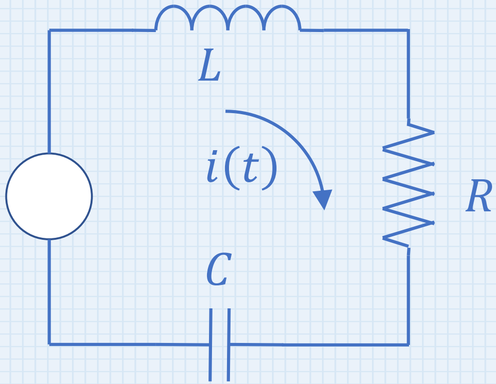


$$v(t) = v_m \cos(\omega t + \theta_v)$$

Note that the applied voltage has both a magnitude, v_m, and a phase, θ_v, both of which are <u>real</u> (non-complex) values. This is the actual voltage applied to the circuit. The current that responds in the circuit, as well as the voltages across any of the elements in the circuit, will all have the exact same frequency dependence, but with different magnitudes and phases. Solving the circuit means finding all of the magnitudes and phases.

$$v_L(t) = v_L \cos(\omega t + \theta_L)$$

$$v(t) = v_m \cos(\omega t + \theta_v)$$



$$v_R(t) = v_R \cos(\omega t + \theta_R)$$

$$i(t) = i_m \cos(\omega t + \theta_i)$$

$$v_C(t) = v_C \cos(\omega t + \theta_C)$$

Here we have explicitly written all of the unknown voltages and currents in terms of the constants we would need to find… the magnitudes and phases of the voltages and the current.

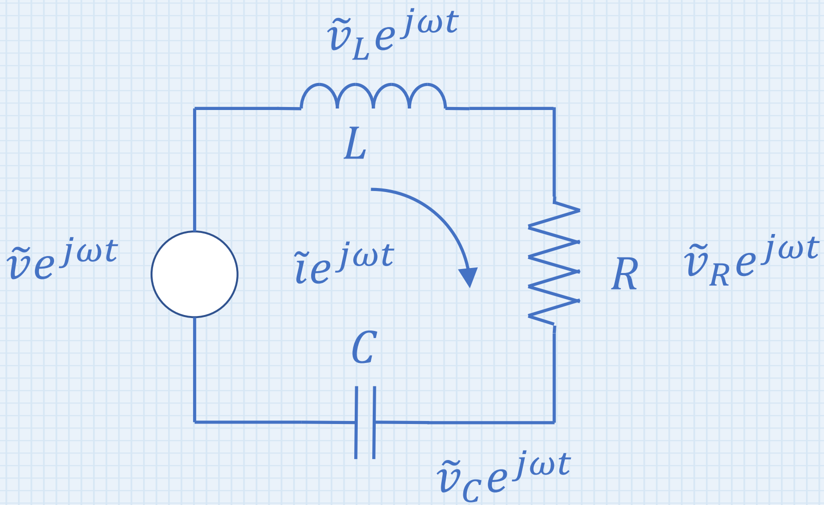
If we tried to solve this problem in the time domain, we would write down a differential equation (second order) and have to deal with messy trigonometric identities.

Instead of solving in the time domain, we can apply Euler's identity to write the voltages and currents in complex form:

$$v(t) = v_m \cos(\omega t + \theta_v) = Re\{v_m e^{j(\omega t + \theta_v)}\} = Re\{v_m e^{j\theta_v} e^{j\omega t}\} = Re\{\tilde{v} e^{j\omega t}\}$$

Here, we have written the voltage as the real part of a complex quantity, with $\tilde{v}$ being the complex voltage. We can then work with the complex variables easily.

The circuit can be redrawn as follows:



All the complex voltages have the same time dependence, so we don't need to think about it anymore. We only need to work with the complex amplitudes, which contain the magnitude and phase information.

We need to know the current/voltage relationships for the complex amplitudes to solve the circuit. These can be found from the time domain equations we've been working with:

Resistor

time domain: $\boxed{v(t) = i(t)R}$

$$v_m \cos(\omega t + \theta_v) = i_m \cos(\omega t + \theta_i)\, R$$

$$Re\{v_m e^{j(\omega t+\theta_v)}\} = Re\{R i_m e^{j(\omega t+\theta_i)}\}$$

$$Re\{v_m e^{j\theta_v} e^{j\omega t}\} = Re\{R i_m e^{j\theta_i} e^{j\omega t}\}$$

$$Re\{\tilde{v} e^{j\omega t}\} = Re\{R\tilde{\imath} e^{j\omega t}\}$$

complex descritption: $\boxed{\tilde{v} = R\tilde{\imath}}$

Capacitor

time domain: $\boxed{q(t) = Cv(t) \quad i(t) = C\frac{dv(t)}{dt}}$

$$v(t) = v_m \cos(\omega t + \theta_v) = Re\{v_m e^{j(\omega t+\theta_v)}\}$$

$$i(t) = i_m \cos(\omega t + \theta_i) = Re\{i_m e^{j\theta_i} e^{j\omega t}\}$$

$$Re\{i_m e^{j\theta_i} e^{j\omega t}\} = \frac{d}{dt} Re\{C v_m e^{j\theta_v} e^{j\omega t}\}$$

$$Re\{i_m e^{j\theta_i} e^{j\omega t}\} = Re\{j\omega C v_m e^{j\theta_v} e^{j\omega t}\}$$

$$Re\{\tilde{\imath} e^{j\omega t}\} = Re\{j\omega C\tilde{v} e^{j\omega t}\}$$

complex descritption: $\boxed{\tilde{\imath} = j\omega C\tilde{v}}$

Inductor

time domain: $\boxed{v(t) = L\frac{di(t)}{dt}}$

$$v(t) = v_m \cos(\omega t + \theta_v) = Re\{v_m e^{j(\omega t+\theta_v)}\}$$

$$i(t) = i_m \cos(\omega t + \theta_i) = Re\{i_m e^{j\theta_i} e^{j\omega t}\}$$

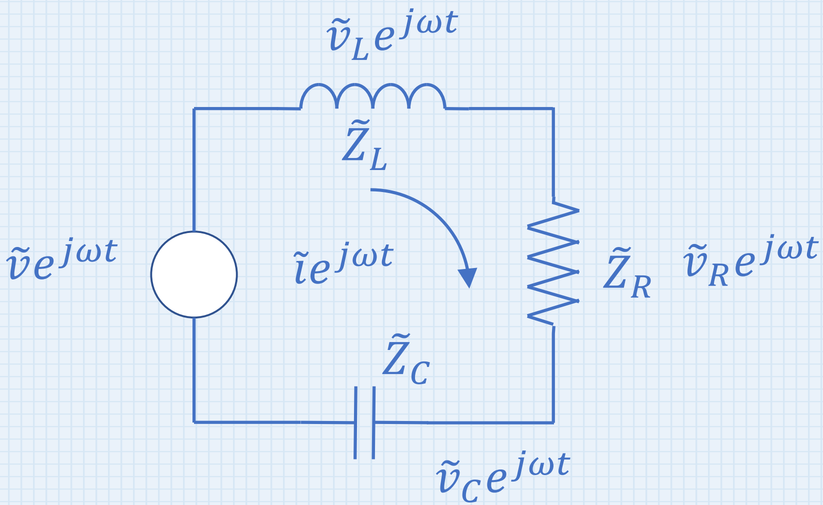
$$Re\{v_m e^{j\theta_v} e^{j\omega t}\} = \frac{d}{dt} Re\{L i_m e^{j\theta_i} e^{j\omega t}\}$$

$$Re\{v_m e^{j\theta_v} e^{j\omega t}\} = Re\{j\omega L i_m e^{j\theta_i} e^{j\omega t}\}$$

$$Re\{\tilde{v} e^{j\omega t}\} = Re\{j\omega L\tilde{\imath} e^{j\omega t}\}$$

complex descritption: $\boxed{\tilde{v} = j\omega L\tilde{\imath}}$

Having the complex equivalents for the circuit elements, we can finally write our circuit as follows:



We have replaced the circuit elements by their complex impedances.

$$\tilde{Z}_R = R$$

$$\tilde{Z}_L = j\omega L$$

$$\tilde{Z}_C = \frac{1}{j\omega C}$$

Once we've arrived at this point, we can apply KVL, KCL and all of the circuit solving methods we've learned. We just have to know how to work with complex variables.

Finding the Magnitude and Phase

The complex notation allows us to manage the magnitude and phase in a simpler way. But, at the end of any calculation, we need to know how to get back the magnitude and phase from our expressions. Here is a review. Let z be a complex variable. The magnitude and phase can be found as follows:

$$z = a + jb \qquad |z| = \sqrt{z^* z} \qquad \tan\theta_z = \frac{Im\{z\}}{Re\{z\}}$$

$$z^* = a - jb$$

$$|z| = \sqrt{z^* z} = \sqrt{(a - jb)(a + jb)} = \sqrt{a^2 + b^2} \qquad \tan\theta_z = \frac{Im\{z\}}{Re\{z\}} = \frac{b}{a}$$

Finding the Magnitude and Phase

If the complex expression has imaginary values in the denominator, then it's best to multiple top and bottom by the complex conjugate of the denominator, so that the denominator is real valued.

$$z = \frac{1}{a + jb}$$

In this example, the denominator is complex. So let's multiply top and bottom by the complex conjugate!

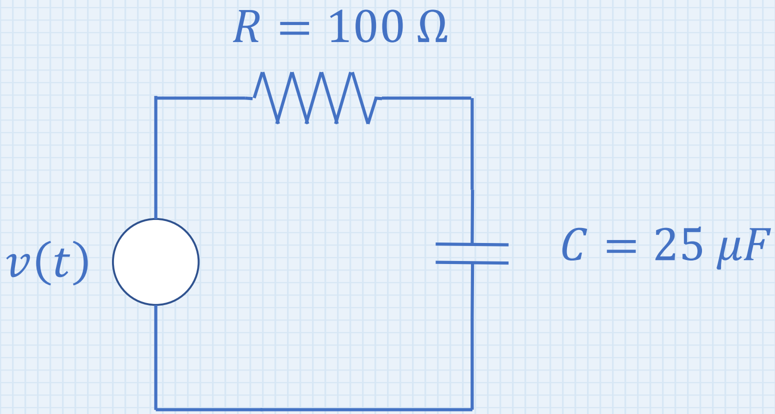
$$z = \frac{1}{a + jb} = \frac{1}{a + jb}\frac{(a - jb)}{(a - jb)} = \frac{a - jb}{(a^2 + b^2)}$$

$$|z| = \sqrt{z^* z} = \sqrt{\left(\frac{1}{a - jb}\right)\left(\frac{1}{a + jb}\right)} = \sqrt{\frac{1}{a^2 + b^2}}$$

$$\tan \theta_z = \frac{Im\{z\}}{Re\{z\}} = -\frac{b}{a}$$

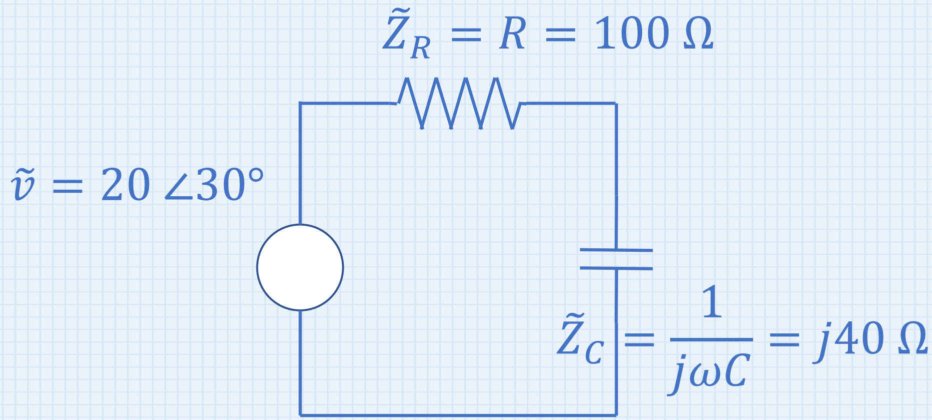
Example: Find the voltage across the capacitor

On the left is the circuit to be solved, as expressed in the time domain. We first rewrite the circuit so that we can solve it using complex analysis.



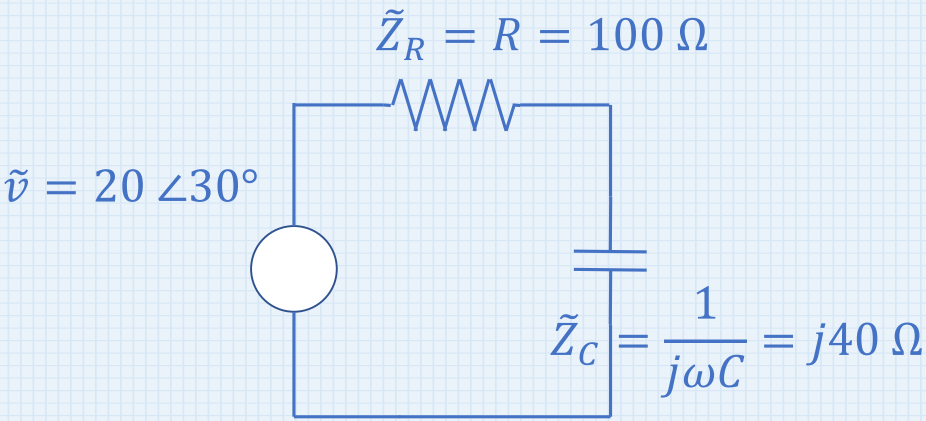
$$v(t) = v_m \cos(\omega t + \theta_v)$$

$$\omega = 1{,}000 \quad \theta_v = 30° \quad v_m = 20\ V$$



Example: Find the voltage across the capacitor

Now we can solve for the voltage across the capacitor. This is now just a voltage divider type of problem, with two complex impedances.



The current that flows through the circuit can be found as:

$$\tilde{\imath} = \frac{\tilde{v}}{\tilde{Z}_R + \tilde{Z}_C}$$

So, the voltage across the capacitor is then:

$$\tilde{v}_C = \tilde{\imath}\tilde{Z}_C = \tilde{v}\frac{\tilde{Z}_C}{\tilde{Z}_R + \tilde{Z}_C}$$

$$= \tilde{v}\frac{1/(j\omega C)}{R + 1/(j\omega C)} = \tilde{v}\frac{1}{1 + j\omega RC}$$

Note that we could have just inserted the numerical values for the impedances above. It would be simpler in this case, but we'll solve symbolically for now.

Example: Find the voltage across the capacitor

We can use this expression to find the magnitude and phase, as we showed on slides 6 and 7 above:

$$\tilde{v}_C = \tilde{v}\frac{1}{1 + j\omega RC}$$

$$|\tilde{v}_C| = \sqrt{\tilde{v}_C{}^* \tilde{v}_C} = \sqrt{\tilde{v}^* \frac{1}{1 - j\omega RC} \cdot \tilde{v}\frac{1}{1 + j\omega RC}} = \sqrt{\tilde{v}^* \tilde{v}} \sqrt{\frac{1}{1 - j\omega RC}\frac{1}{1 + j\omega RC}}$$

$$|\tilde{v}_C| = |\tilde{v}| \sqrt{\frac{1}{1 + \omega^2 R^2 C^2}}$$

To find the phase:

$$\tilde{v}_C = \tilde{v}\frac{1}{1 + j\omega RC} \cdot \frac{1 - j\omega RC}{1 - j\omega RC} = \tilde{v}\overbrace{\left(\frac{1 - j\omega RC}{1 + \omega^2 R^2 C^2}\right)}$$

For the second term, we have

$$\tan(\theta) = -\omega RC$$

Example: Find the voltage across the capacitor

Summarizing, we end up with:

$$\tilde{v}_C = \tilde{v}\frac{1}{1+j\omega RC} = (v_m \angle\theta_v)\left(\sqrt{\frac{1}{1+\omega^2R^2C^2}}\ \angle\tan^{-1}(-\omega RC)\right)$$

That's the expression we get if we do everything symbolically. But we actually have numbers for this problem, so let's plug those in!

$$\sqrt{\frac{1}{1+\omega^2R^2C^2}} = \sqrt{\frac{1}{1+(1000)^2(100)^2(25\times10^{-6})^2}} = 0.371$$

$$\tan^{-1}(-\omega RC) = \tan^{-1}\left(-1000\times100\times\left(25\times10^{-6}\right)\right) = -68°$$

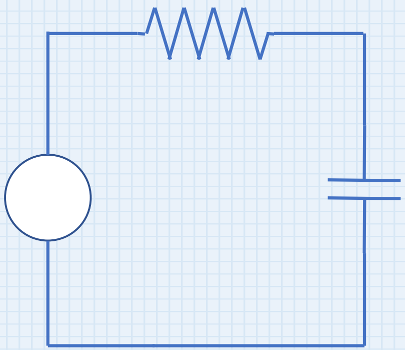
So that, finally, we get

$$\boxed{\tilde{v}_C = (20\angle30°)\,(0.371\angle-68°) = 7.42\ V\angle-38°}$$

Example: Find the voltage across the capacitor

If we want the final expression back in the time domain, it's easy to do since we have solved for the magnitude and phase across the capacitor:

$$\boxed{\tilde{v}_C = (20\angle 30°)\,(0.371\angle -68°) = 7.42\,V\angle -38°}$$



$R = 100\ \Omega$

$v(t)$

$C = 25\ \mu F$

$$\boxed{v_C(t) = 7.42\cos(\omega t - 38°)}$$

$v(t) = 20\cos(\omega t + 30°)$

We have solved for the complete voltage across the capacitor, which requires knowledge of both the phase and the magnitude.

The problem was solved by using the complex description, converting all elements into complex impedances, and the source into a phasor.

Note: We have used degrees inside the cosine just to make it easy to read. Of course, you would actually convert to radians if you were doing an actual calculation.