DEFINITION 1.5

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
2. Σ is a finite set called the ***alphabet***,
3. $\delta \colon Q \times \Sigma \longrightarrow Q$ is the ***transition function***,[1]
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.[2]

DEFINITION 1.16

A language is called a ***regular language*** if some finite automaton recognizes it.

THEOREM 1.26

The class of regular languages is closed under the concatenation operation.

In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

To prove this theorem, let's try something along the lines of the proof of the union case. As before, we can start with finite automata M_1 and M_2 recognizing the regular languages A_1 and A_2. But now, instead of constructing automaton M to accept its input if either M_1 or M_2 accept, it must accept if its input can be broken into two pieces, where M_1 accepts the first piece and M_2 accepts the second piece. The problem is that M doesn't know where to break its input (i.e., where the first part ends and the second begins). To solve this problem, we introduce a new technique called nondeterminism.

THEOREM 1.39

Every nondeterministic finite automaton has an equivalent deterministic finite automaton.

PROOF IDEA If a language is recognized by an NFA, then we must show the existence of a DFA that also recognizes it. The idea is to convert the NFA into an equivalent DFA that simulates the NFA.

Recall the "reader as automaton" strategy for designing finite automata. How would you simulate the NFA if you were pretending to be a DFA? What do you need to keep track of as the input string is processed? In the examples of NFAs, you kept track of the various branches of the computation by placing a finger on each state that could be active at given points in the input. You updated the simulation by moving, adding, and removing fingers according to the way the NFA operates. All you needed to keep track of was the set of states having fingers on them.

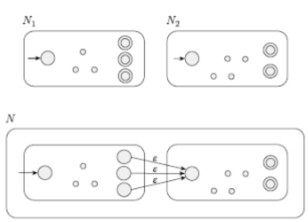
If k is the number of states of the NFA, it has 2^k subsets of states. Each subset corresponds to one of the possibilities that the DFA must remember, so the DFA simulating the NFA will have 2^k states. Now we need to figure out which will be the start state and accept states of the DFA, and what will be its transition function. We can discuss this more easily after setting up some formal notation.

THEOREM 1.47

The class of regular languages is closed under the concatenation operation.

PROOF IDEA We have regular languages A_1 and A_2 and want to prove that $A_1 \circ A_2$ is regular. The idea is to take two NFAs, N_1 and N_2 for A_1 and A_2, and combine them into a new NFA N as we did for the case of union, but this time in a different way, as shown in Figure 1.48.

Assign N's start state to be the start state of N_1. The accept states of N_1 have additional ε arrows that nondeterministically allow branching to N_2 whenever N_1 is in an accept state, signifying that it has found an initial piece of the input that constitutes a string in A_1. The accept states of N are the accept states of N_2 only. Therefore, it accepts when the input can be split into two parts, the first accepted by N_1 and the second by N_2. We can think of N as nondeterministically guessing where to make the split.



DEFINITION 1.23

Let A and B be languages. We define the regular operations ***union***, ***concatenation***, and ***star*** as follows:

- **Union:** $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
- **Concatenation:** $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$.
- **Star:** $A^* = \{x_1x_2 \ldots x_k \mid k \geq 0 \text{ and each } x_i \in A\}$.

EXAMPLE 1.24

Let the alphabet Σ be the standard 26 letters $\{\mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}\}$. If $A = \{\mathtt{good}, \mathtt{bad}\}$ and $B = \{\mathtt{boy}, \mathtt{girl}\}$, then

$A \cup B = \{\mathtt{good}, \mathtt{bad}, \mathtt{boy}, \mathtt{girl}\}$,

$A \circ B = \{\mathtt{goodboy}, \mathtt{goodgirl}, \mathtt{badboy}, \mathtt{badgirl}\}$, and

$A^* = \{\varepsilon, \mathtt{good}, \mathtt{bad}, \mathtt{goodgood}, \mathtt{goodbad}, \mathtt{badgood}, \mathtt{badbad}, \mathtt{goodgoodgood}, \mathtt{goodgoodbad}, \mathtt{goodbadgood}, \mathtt{goodbadbad}, \ldots\}$.

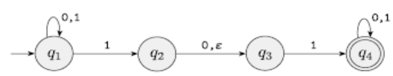
DEFINITION 1.37

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta \colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

EXAMPLE 1.38

Recall the NFA N_1:



The formal description of N_1 is $(Q, \Sigma, \delta, q_1, F)$, where

1. $Q = \{q_1, q_2, q_3, q_4\}$,
2. $\Sigma = \{\mathtt{0}, \mathtt{1}\}$,
3. δ is given as

	0	1	ε
q_1	$\{q_1\}$	$\{q_1, q_2\}$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\{q_3\}$
q_3	$\emptyset$	$\{q_4\}$	$\emptyset$
q_4	$\{q_4\}$	$\{q_4\}$	$\emptyset$,

4. q_1 is the start state, and
5. $F = \{q_4\}$.

The formal definition of computation for an NFA is similar to that for a DFA. Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA and w a string over the alphabet Σ. Then we say that N ***accepts*** w if we can write w as $w = y_1y_2 \cdots y_m$, where each y_i is a member of Σ_ε and a sequence of states $r_0, r_1, \ldots, r_m$ exists in Q with three conditions:

1. $r_0 = q_0$,
2. $r_{i+1} \in \delta(r_i, y_{i+1})$, for $i = 0, \ldots, m-1$, and
3. $r_m \in F$.

Condition 1 says that the machine starts out in the start state. Condition 2 says that state r_{i+1} is one of the allowable next states when N is in state r_i and reading y_{i+1}. Observe that $\delta(r_i, y_{i+1})$ is the *set* of allowable next states and so we say that r_{i+1} is a member of that set. Finally, condition 3 says that the machine accepts its input if the last state is an accept state.

DEFINITION 1.52

Say that R is a ***regular expression*** if R is

1. a for some a in the alphabet Σ,
2. ε,
3. $\emptyset$,
4. $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,
5. $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or
6. (R_1^*), where R_1 is a regular expression.

In items 1 and 2, the regular expressions a and ε represent the languages $\{a\}$ and $\{\varepsilon\}$, respectively. In item 3, the regular expression $\emptyset$ represents the empty language. In items 4, 5, and 6, the expressions represent the languages obtained by taking the union or concatenation of the languages R_1 and R_2, or the star of the language R_1, respectively.

THEOREM 1.25

The class of regular languages is closed under the union operation.

In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

PROOF IDEA We have regular languages A_1 and A_2 and want to show that $A_1 \cup A_2$ also is regular. Because A_1 and A_2 are regular, we know that some finite automaton M_1 recognizes A_1 and some finite automaton M_2 recognizes A_2. To prove that $A_1 \cup A_2$ is regular, we demonstrate a finite automaton, call it M, that recognizes $A_1 \cup A_2$.

This is a proof by construction. We construct M from M_1 and M_2. Machine M must accept its input exactly when either M_1 or M_2 would accept it in order to recognize the union language. It works by *simulating* both M_1 and M_2 and accepting if either of the simulations accept.

PROOF

Let M_1 recognize A_1, where $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, and
M_2 recognize A_2, where $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$.

Construct M to recognize $A_1 \cup A_2$, where $M = (Q, \Sigma, \delta, q_0, F)$.

1. $Q = \{(r_1, r_2) \mid r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$.
   This set is the ***Cartesian product*** of sets Q_1 and Q_2 and is written $Q_1 \times Q_2$. It is the set of all pairs of states, the first from Q_1 and the second from Q_2.
2. Σ, the alphabet, is the same as in M_1 and M_2. In this theorem and in all subsequent similar theorems, we assume for simplicity that both M_1 and M_2 have the same input alphabet Σ. The theorem remains true if they have different alphabets, Σ_1 and Σ_2. We would then modify the proof to let $\Sigma = \Sigma_1 \cup \Sigma_2$.
3. δ, the transition function, is defined as follows. For each $(r_1, r_2) \in Q$ and each $a \in \Sigma$, let
   $$\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a)).$$
   Hence δ gets a state of M (which actually is a pair of states from M_1 and M_2), together with an input symbol, and returns M's next state.
4. q_0 is the pair (q_1, q_2).
5. F is the set of pairs in which either member is an accept state of M_1 or M_2. We can write it as
   $$F = \{(r_1, r_2) \mid r_1 \in F_1 \text{ or } r_2 \in F_2\}.$$
   This expression is the same as $F = (F_1 \times Q_2) \cup (Q_1 \times F_2)$. (Note that it is *not* the same as $F = F_1 \times F_2$. What would that give us instead?[3])

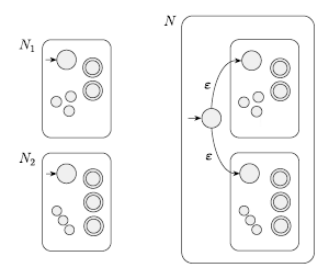
THEOREM 1.45

The class of regular languages is closed under the union operation.

PROOF IDEA We have regular languages A_1 and A_2 and want to prove that $A_1 \cup A_2$ is regular. The idea is to take two NFAs, N_1 and N_2 for A_1 and A_2, and combine them into one new NFA, N.

Machine N must accept its input if either N_1 or N_2 accepts this input. The new machine has a new start state that branches to the start states of the old machines with ε arrows. In this way, the new machine nondeterministically guesses which of the two machines accepts the input. If one of them accepts the input, N will accept it, too.

We represent this construction in the following figure. On the left, we indicate the start and accept states of machines N_1 and N_2 with large circles and some additional states with small circles. On the right, we show how to combine N_1 and N_2 into N by adding additional transition arrows.

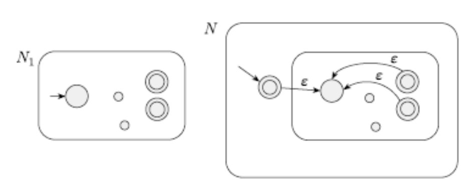


THEOREM 1.49

The class of regular languages is closed under the star operation.

PROOF IDEA We have a regular language A_1 and want to prove that A_1^* also is regular. We take an NFA N_1 for A_1 and modify it to recognize A_1^*, as shown in the following figure. The resulting NFA N will accept its input whenever it can be broken into several pieces and N_1 accepts each piece.

We can construct N like N_1 with additional ε arrows returning to the start state from the accept states. This way, when processing gets to the end of a piece that N_1 accepts, the machine N has the option of jumping back to the start state to try to read another piece that N_1 accepts. In addition, we must modify N so that it accepts ε, which always is a member of A_1^*. One (slightly bad) idea is simply to add the start state to the set of accept states. This approach certainly adds ε to the recognized language, but it may also add other, undesired strings. Exercise 1.15 asks for an example of the failure of this idea. The way to fix it is to add a new start state, which also is an accept state, and which has an ε arrow to the old start state. This solution has the desired effect of adding ε to the language without adding anything else.



THEOREM 1.54

A language is regular if and only if some regular expression describes it.

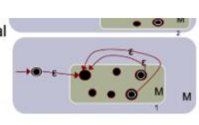
This theorem has two directions. We state and prove each direction as a separate lemma.

LEMMA 1.55

If a language is described by a regular expression, then it is regular.

PROOF IDEA Say that we have a regular expression R describing some language A. We show how to convert R into an NFA recognizing A. By Corollary 1.40, if an NFA recognizes A then A is regular.

- $L(M) = A_1^*$ Make a new start/final state for M and connect it to the start state of M_1 by ε transitions and then connect every final state of M_1 to it's *old* start state by ε transitions. Final states for M_1 are still final states.



LEMMA 1.60

If a language is regular, then it is described by a regular expression.

PROOF IDEA We need to show that if a language A is regular, a regular expression describes it. Because A is regular, it is accepted by a DFA. We describe a procedure for converting DFAs into equivalent regular expressions.

DEFINITION 1.64

A ***generalized nondeterministic finite automaton*** is a 5-tuple, $(Q, \Sigma, \delta, q_{\text{start}}, q_{\text{accept}})$, where

1. Q is the finite set of states,
2. Σ is the input alphabet,
3. $\delta \colon (Q - \{q_{\text{accept}}\}) \times (Q - \{q_{\text{start}}\}) \longrightarrow \mathcal{R}$ is the transition function,
4. q_{start} is the start state, and
5. q_{accept} is the accept state.