Last Time

Single Qubit Gates on Multi Qubit state vectors.

eg/ X - gate on qubit 0
H on qubit 1

1) $X \otimes H \; |01\rangle$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \left[\begin{array}{cc|cc} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ \hline 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} |10\rangle - \frac{1}{\sqrt{2}} |11\rangle$$

$$= |1\rangle \otimes \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$$

2) $X \otimes H$

$|01\rangle$

$= X|0\rangle \otimes H|1\rangle \ldots$

Last Time:

$$X|0\rangle \otimes H|1\rangle$$

$$= |1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$= \frac{1}{\sqrt{2}}\{|10\rangle - |11\rangle\}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \quad \#$$

4.3 Multi-Qubit Gates

CNOT Gate: Performs an X-gate on qubit 1 (second qubit) if qubit 0 (first qubit) is in the $|1\rangle$ state.

ex

CNOT

$|00\rangle \longrightarrow |00\rangle$

$|01\rangle \longrightarrow |01\rangle$ #

$|10\rangle \longrightarrow |11\rangle$ depends on which qubit is "target" #

$|11\rangle \longrightarrow |10\rangle$

4.3 Gates

qubit 0 control?

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

ex

$|01\rangle$?

$$CNOT\,|01\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= |01\rangle$$

$|10\rangle$?

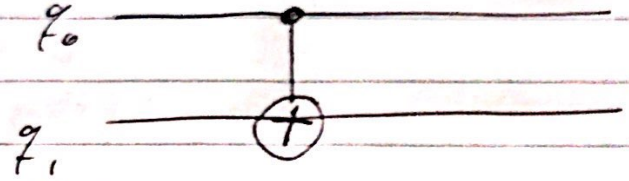
$$CNOT\,|10\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \underline{|11\rangle}$$ ✓

"check if first qubit is 1, then flip the second qubit"

4.3 cont'd

Use qubit 1 as control?



$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

ex/ $|01\rangle$?

$$CNOT\,|01\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |11\rangle \quad \checkmark$$

Using general prefactors q_{00} q_{01} q_{10} q_{11}

CNOT switches q_{01} and q_{10}

4.3 Gate

$$|q\rangle = \begin{pmatrix} q_{00} \\ q_{01} \\ q_{10} \\ q_{11} \end{pmatrix} \qquad CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$CNOT\,|q\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_{00} \\ q_{01} \\ q_{10} \\ q_{11} \end{pmatrix}$$

$$= \begin{pmatrix} q_{00} \\ q_{11} \\ q_{10} \\ q_{01} \end{pmatrix}$$

* q_{11} and q_{01} get flipped!

ex)

$|0+\rangle$?

$$= |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Apply CNOT

$$|0+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle$$

$$= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

4.3 Gates

$$\text{CNOT}\,|0+\rangle = \begin{bmatrix} 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Before?

$|01\rangle + |00\rangle$

After?

$|00\rangle + |11\rangle$ ...

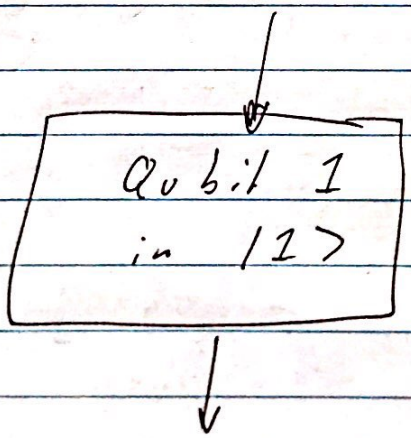
Given state

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

50% probability of being in $|00\rangle$, 50% prob. of being in $|11\rangle$.

↳ Measure qubit 1?

$$|00\rangle + |11\rangle$$



$|11\rangle$

conclusion on state of qubit 0 by measurement of qubit 1!

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Alice has qubit 1

↓ Measure state of qubit 1.

$|1\rangle$

Bob has qubit 0

↓

$|1\rangle$

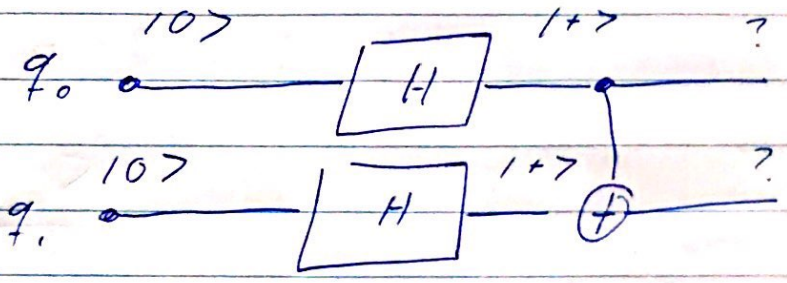
Bob must also have qubit 0 in state $|1\rangle$.

"Measuring the state of 1 qubit will collapse the state of the other."

4.5 Phase Kickback

CNOT on a control qubit in superposition?

ex/ start w/ $|++\rangle$



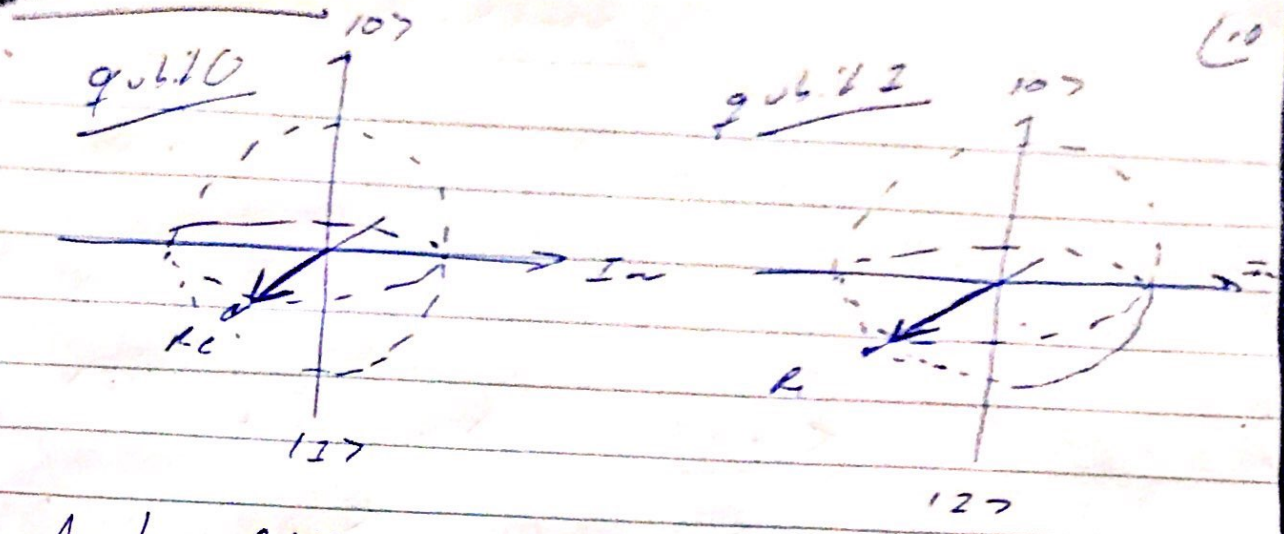
start with

$$|++\rangle = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\frac{1}{2}\left[|00\rangle + |01\rangle + |10\rangle + |11\rangle\right]$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Block Sphere ...



Apply CNOT

$$CNOT\,|++\rangle = CNOT\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

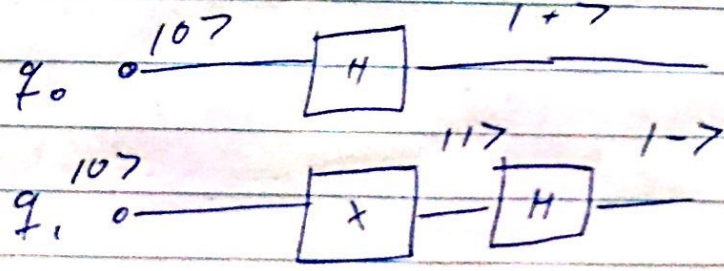
Flip q_{01} and q_{11}

$$= \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

stays the same!

Now Try ~~not~~ $|-+\rangle$?

Initialize



4.5 Cont'd

$|q_1 q_0\rangle = |- -\rangle$

$= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$= \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)$

$= \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$

Apply CNOT?

$CNOT\,|- +\rangle = CNOT \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$ Flip q_{01} and q_{11}

$= \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$

$= \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$

$|- = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

4.5 Con...

$$\therefore \text{CNOT}\,|-+\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$= |--\rangle \quad *$$

Before?

$$\text{CNOT}\,|10\rangle = |10\rangle$$

(control qubit = $|1\rangle$; stays the same.)

$$\text{CNOT}\,|11\rangle = |01\rangle$$

(control qubit = $|1\rangle$; stays the same)

Now?

$$\text{CNOT}\,|-+\rangle = |--\rangle$$

(control qubit; unchanged; gets flipped!)