Week 2: 0.3 cont'd

How to describe a linear transformation?

$$\begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix} \rightarrow ??? \rightarrow \begin{bmatrix} x_{out} \\ y_{out} \end{bmatrix}$$

↳ Only need to record where basis vectors, $\hat{i}$ and $\hat{j}$ each land.

$$\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -1(\hat{i}) + 2(\hat{j})$$

↓

Transformed $\vec{v}$ = -1 (transformed $\hat{i}$)

+ 2 (transformed $\hat{j}$)

if

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Longrightarrow \text{New } \hat{i} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{New } \hat{j} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

0.3 Cont'd

$$\begin{bmatrix} x \\ y \end{bmatrix} = x\hat{i} + y\hat{j}$$

$$\hookrightarrow x\begin{bmatrix} 1 \\ -2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1x \\ -2x \end{bmatrix} + \begin{bmatrix} 3y \\ 0y \end{bmatrix} = \begin{bmatrix} 1x + 3y \\ -2x + 0y \end{bmatrix}$$

Similarly

$$\underbrace{\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}}_{\text{Matrix.}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1x + 3y \\ -2x + 0y \end{bmatrix}$$

SAME THING!

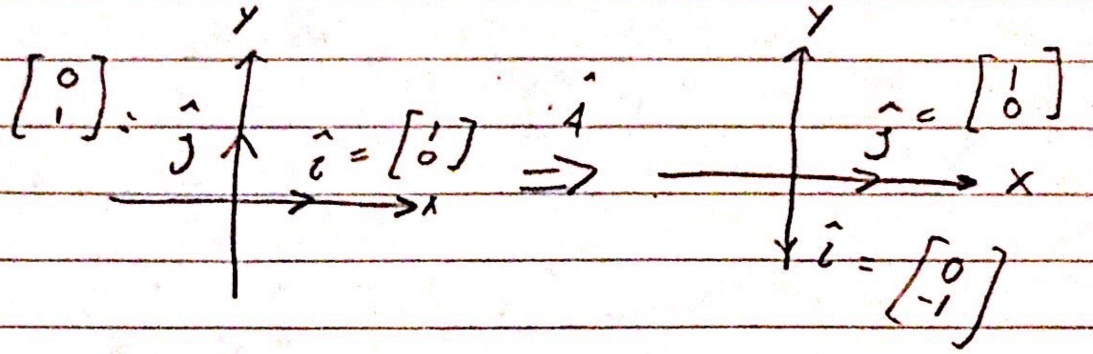
Matrix has columns

$$\text{Matrix} = \begin{bmatrix} | & | \\ \text{Transformed} & \text{transformed} \\ \hat{i} & \hat{j} \\ | & | \end{bmatrix}$$

6.3 Cont'd

ex 1

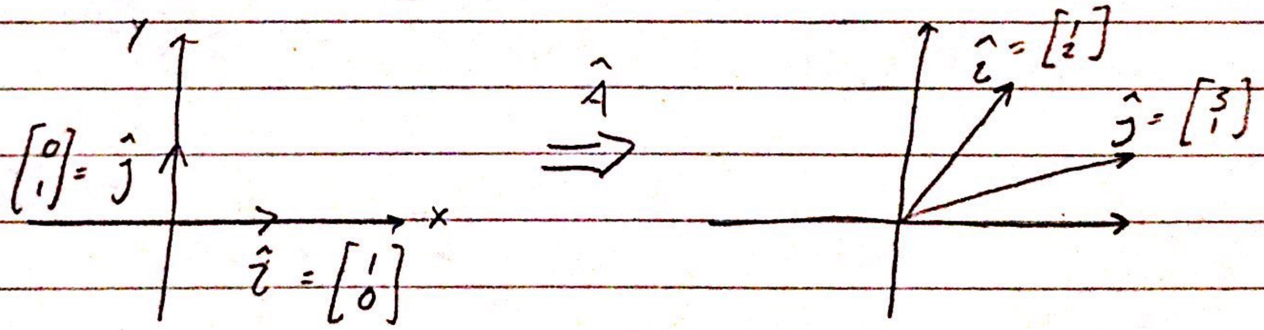
Rotate 90° clockwise



$$\hat{A} = \text{Matrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

ex 2

Given $\hat{A} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$



6.4 Matrix Multiplication

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

$\hat{i}$ goes to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\hat{j}$ goes to $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$

then

$$\hat{i} : \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}^{*}$$

$$\hat{j} : \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}^{**}$$

*

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

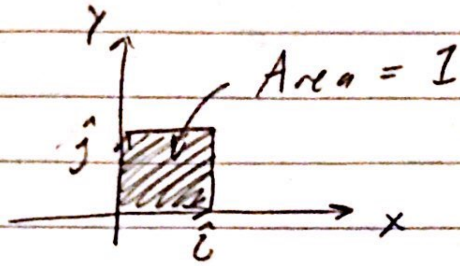
**

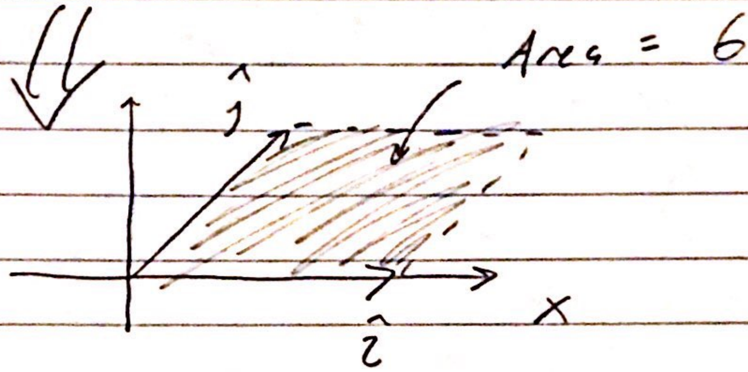
$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} = -2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

0.5 Determinant

How to characterize a matrix?

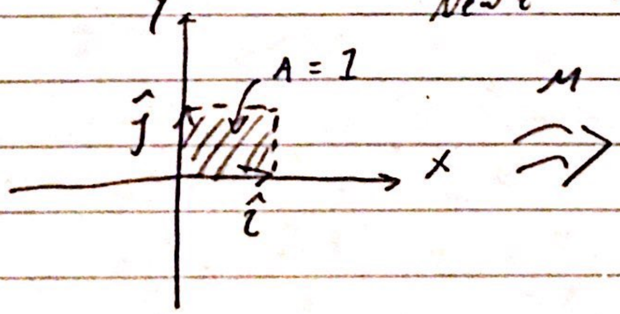


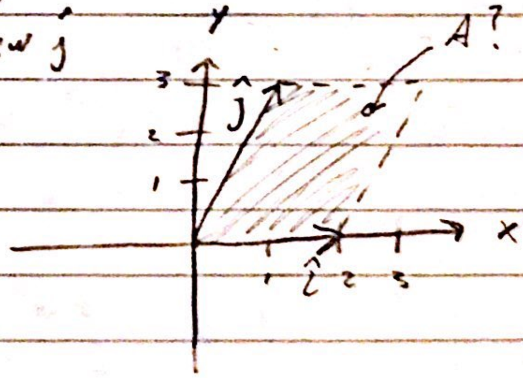


For

$$M = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

New $\hat{i}$ New $\hat{j}$





New Area $=$ base $\times$ height

$= 2 \times 3 = \boxed{6}$

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0.5 Determinant Cont'd

Also

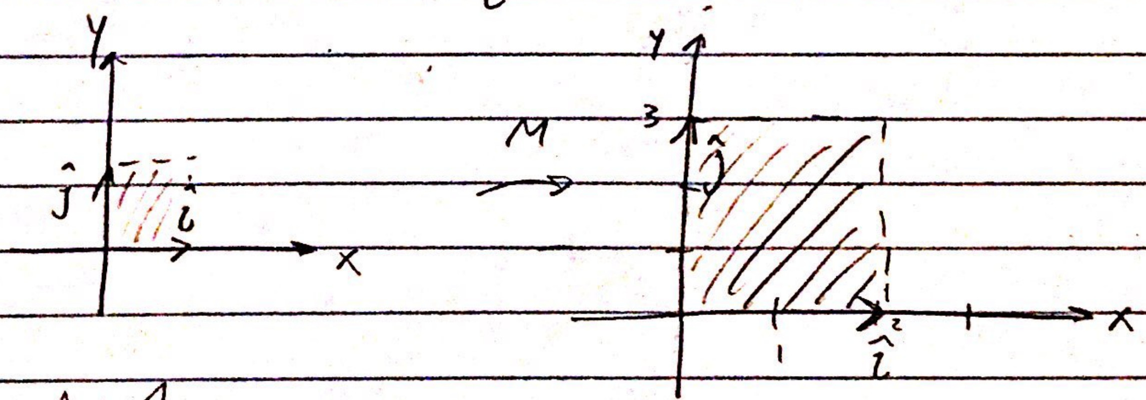
$$M = \begin{bmatrix} 2 & \boxed{1} \\ \boxed{0} & 3 \end{bmatrix}$$

$$\det(M) = 2 \times 3 - \boxed{0 \times 1}$$

$$= 6$$

What about

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$



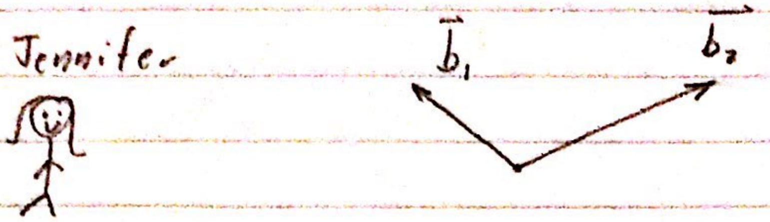
A = 1

A = 6

$$\det(M) = 2 \times 3 - \boxed{0 \times 0}$$

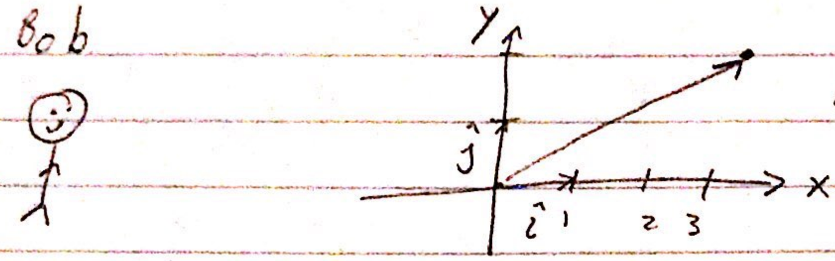
0.6 Change of Basis

Jennifer



Bob sees this as $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Bob



Jennifer sees this as $\begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix}$

$$\begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix} = \underline{\frac{5}{3}\vec{b}_1 + \frac{1}{3}\vec{b}_2}$$

But to us

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \underline{3\hat{i} + 2\hat{j}}$$

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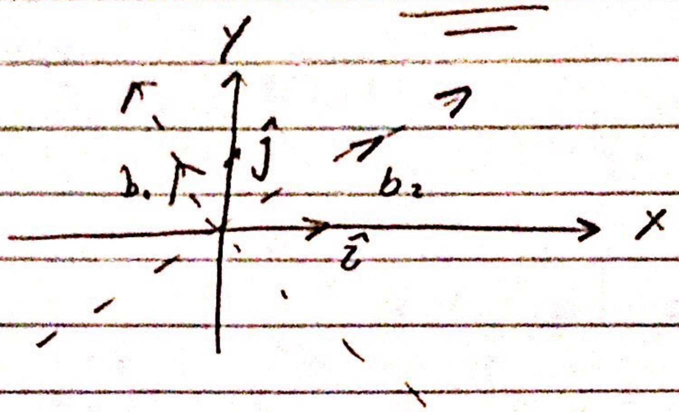
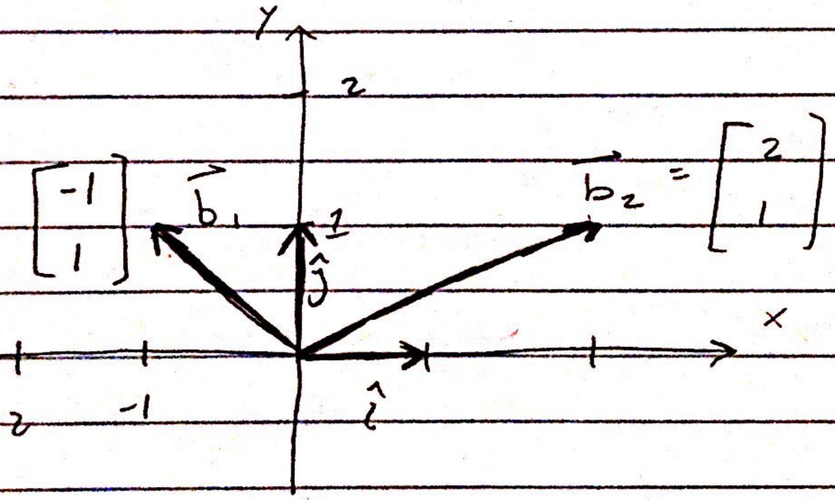
0.6 Cont'd

Origin is the same



Translate from Jennifer's coordinates to our coordinates?



We say Jennifer's basis vectors in our frame are $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

0.6 Cont'd

Jennifer sees $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$= -1\vec{b}_1 + 2\vec{b}_2$$

$$= -1\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$\boxed{\text{We see } \begin{bmatrix} 5 \\ 1 \end{bmatrix}}$

There is then a matrix which takes us from our basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

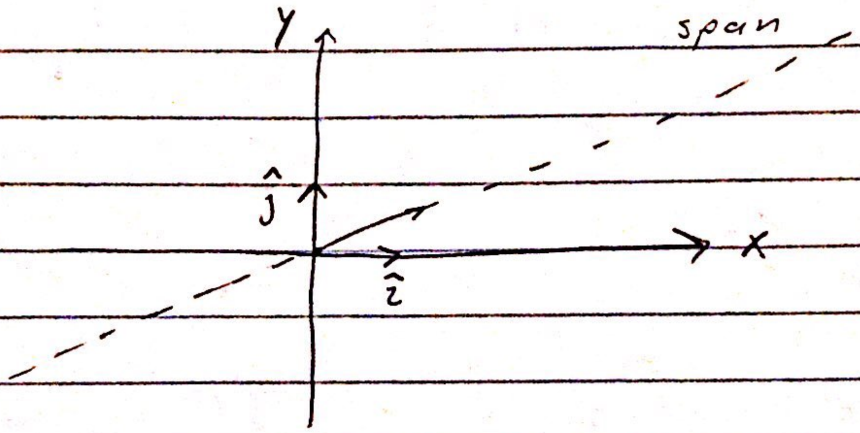
To Jennifer's basis $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

$$\boxed{M = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}}$$

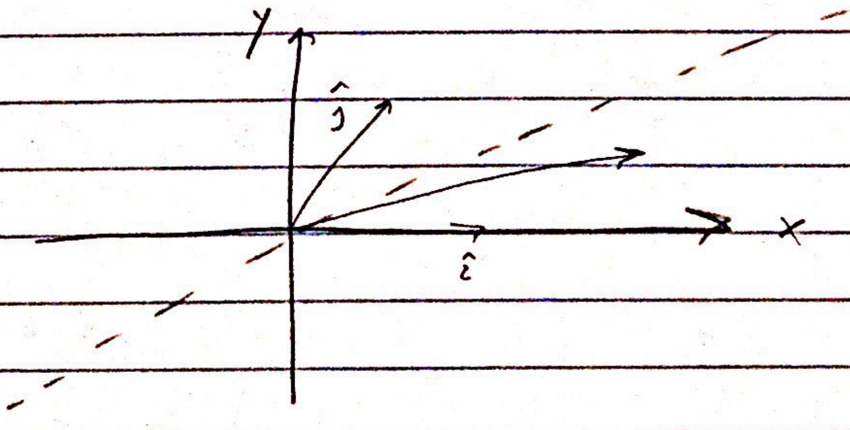
0.7 Eigenvectors and Eigenvalues

Think of a Transformation

$$M = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$



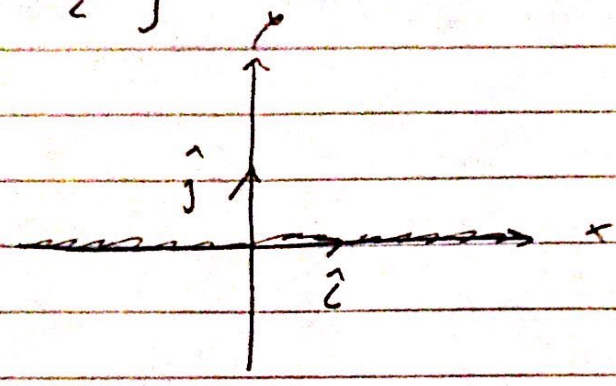
$\Downarrow M$



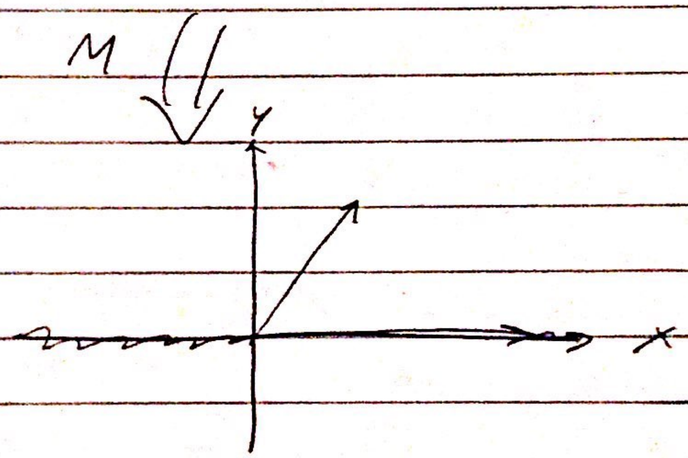
"It would be coincidental if that vector remained on its own span"

0.7 Cont'd

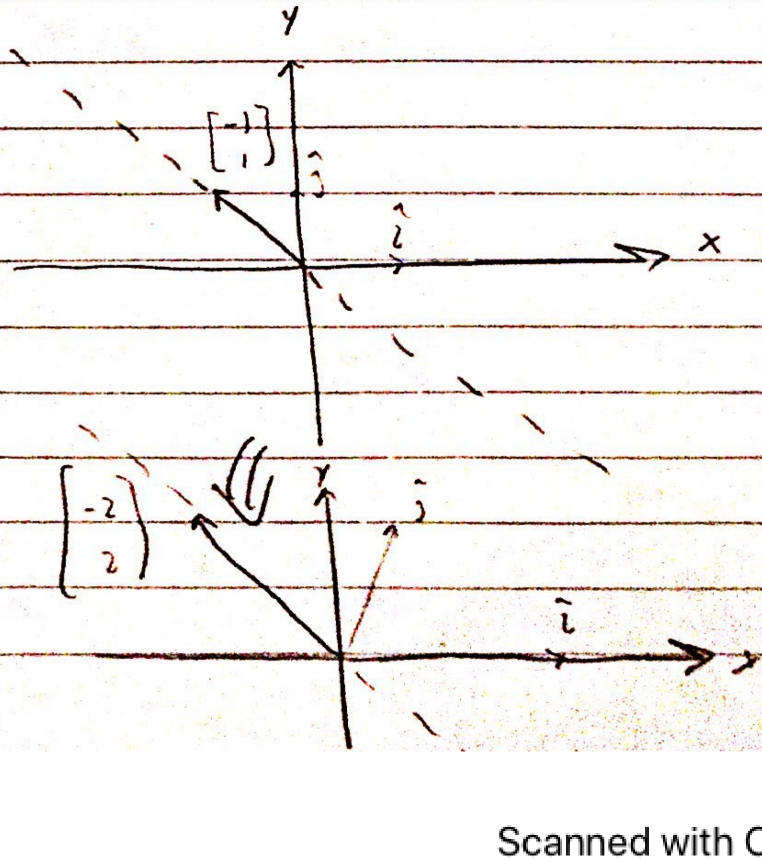
$$M = \begin{array}{c} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \\ \begin{array}{cc} \hat{i} & \end{array} \end{array} \text{ has eigen vector } \begin{array}{c} \hat{i} \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array}$$



* video 1.14
2:25



Sneakier eigen vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$



0.7 Cont'd

Those two vectors $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

are the <u>only</u> eigenvectors of

$$M \equiv \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

Eigenvectors:
↳ Vectors which, when transformed, stay on their own span.

Eigen values:
↳ The amount by which an ~~vector~~ eigen vector gets stretched by

Symbolically:

$$\hat{A}\vec{v} = \lambda\vec{v}$$ ← (eigen vector)

(matrix)

(eigen vector)

(eigen value)

0.7 Cont'd

$$\underline{A \vec{v} = \lambda \vec{v}}$$

$$\hookrightarrow \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= \underline{\lambda \hat{\mathbb{I}} \vec{v}}$$

$$\hookrightarrow \hat{A} \vec{v} - \lambda \hat{\mathbb{I}} \vec{v} = \vec{0}$$

or $$(\hat{A} - \lambda \hat{\mathbb{I}}) \vec{v} = \vec{0}$$

0.7 Gn/id

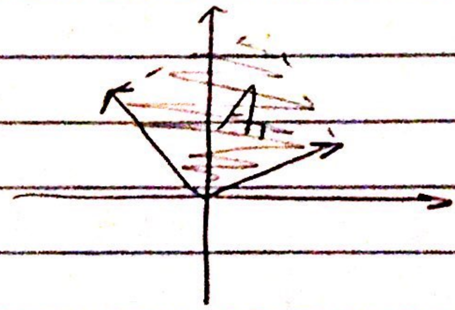
True if ...

$$(A - \lambda \hat{\mathbb{I}}) \times \vec{v} = \underline{0}$$

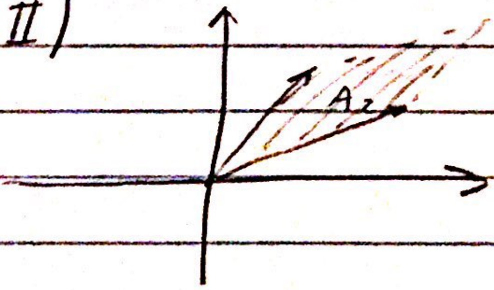
↳ Happens when $\underline{\det(A - \lambda \hat{\mathbb{I}}) = 0}$

* Video 7:35 *

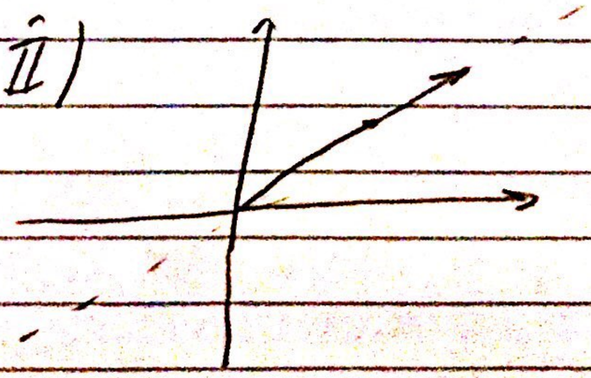
$\det(A)$
$= A_1$



$\det(A - \lambda \hat{\mathbb{I}})$
$= A_2$



$\det(A - \lambda \hat{\mathbb{I}})$
$= 0$



0.7 Cont'd

if you find $\vec{v}$ such that

$$(A - \lambda \hat{I})\,\vec{v} = \vec{0},$$

you found the eigen vector $\vec{v}$ of A, with eigen value λ.

ex/

$$M = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

λ?

$$M - \lambda \hat{I} = \begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}$$

determinant?

$$(3-\lambda)(2-\lambda) - 0 \times 1$$

Set equal to 0 to get eigen value

Q. 7 Con't'd

solve $(3-\lambda)(2-\lambda) = 0$

$\hookrightarrow \boxed{\lambda = 2}$ ⭑

What is the eigenvector?

$$\begin{bmatrix} 3-2 & 1 \\ 0 & 2-2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

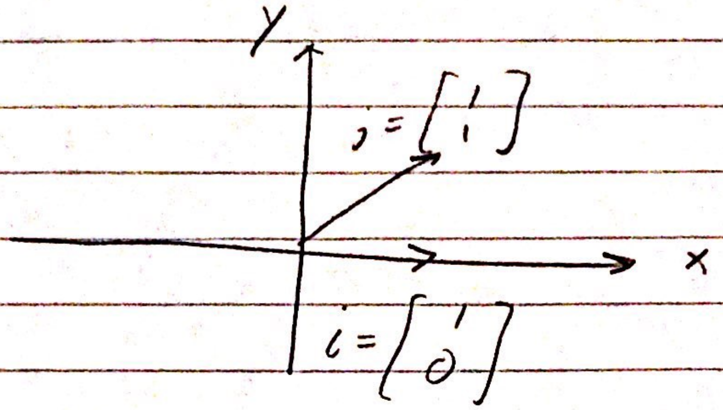
$\longrightarrow V_1 + V_2 = 0$ or

$\underline{V_1 = -V_2}$

$$\boxed{\text{so } \quad \vec{V} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$ ⭑

0.7 Cont'd

ex 2



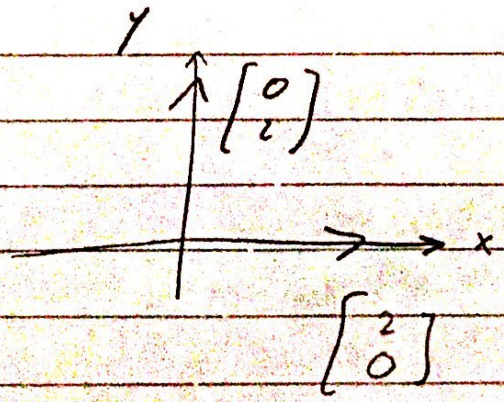
$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

eigen vector is $\underline{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$ with

eigen value $\underline{\lambda = 1}$

ex 3

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$



Every vector is an eigen-vector! w/ eigenvalue 2!

Ex 3 Con't'd   0.7 Con't'd

$$\begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(2-\lambda)^2 = 0$

$\underline{\lambda = 2}$

v_1 and v_2 can be

<u>anything</u>.

Look again at $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

w/ eigen vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

and eigen values ~~is~~ 3 and 2.