Week 6

Last Time:

Hadamard Gate

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$H|0\rangle = |+\rangle \qquad H|1\rangle = |-\rangle$

$H|+\rangle = |0\rangle \qquad H|-\rangle = |1\rangle$

ex Show $HZH = X$

$$HZH = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= X \quad \checkmark$$

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3.5 R_ϕ gate

Given some angle ϕ.

R_ϕ gate rotates ϕ about z-axis

$$R_\phi = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$$

eg/

$R_\phi |+\rangle$ for $\phi = \pi$?

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ e^{i\phi} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \underline{|-\rangle}$$

$R_\phi |-\rangle$ for $\phi = \pi$?

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -e^{i\phi} \end{bmatrix}$$

$$\phi = \pi ? \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -(-1) \end{bmatrix} \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underline{|+\rangle}$$

Also

$$\left.\begin{aligned} R_{\phi=\pi}|0\rangle &= |0\rangle \\ R_{\phi=\pi}|1\rangle &= -|1\rangle \end{aligned}\right\} \underline{R_{\phi=\pi} \sim Z}$$

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3.6 Special Cases of R_ϕ

1. Identity

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

"Do nothing"

↳ R_ϕ w/ $\phi = 0, 2\pi, \ldots$

2. S-gate ; $\frac{\pi}{2}$ rotation around z-axis

$$S = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

$$S^{+} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\pi/2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$$

ex

$$S|+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

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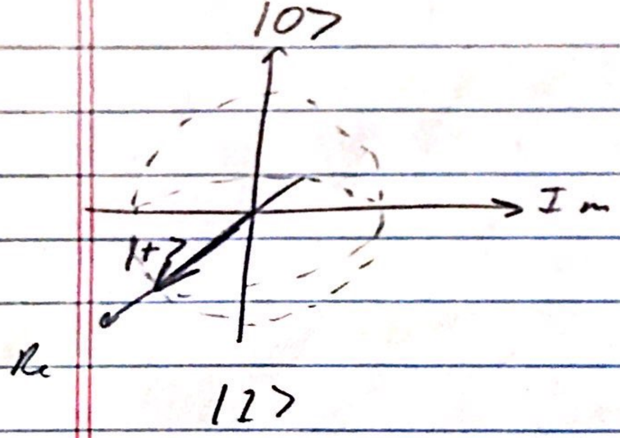
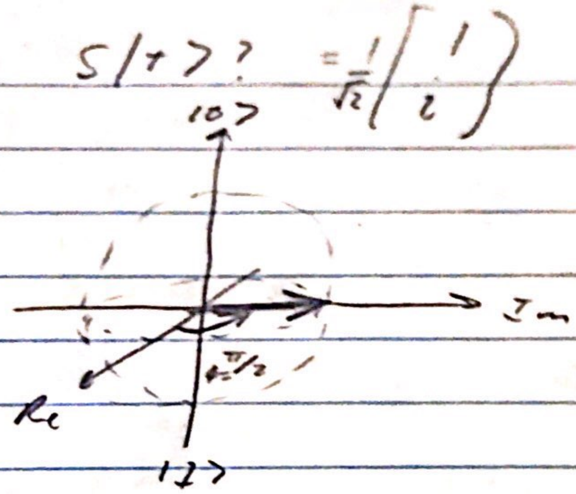
3.6 Gates

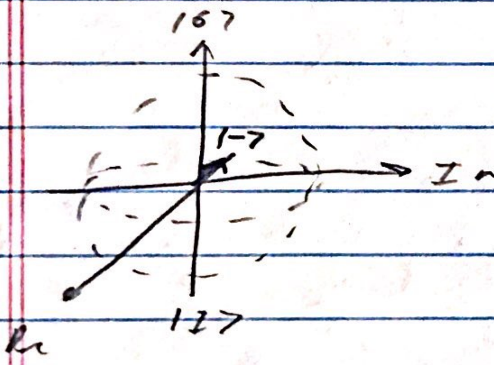
Z Gate) $|+\rangle$?

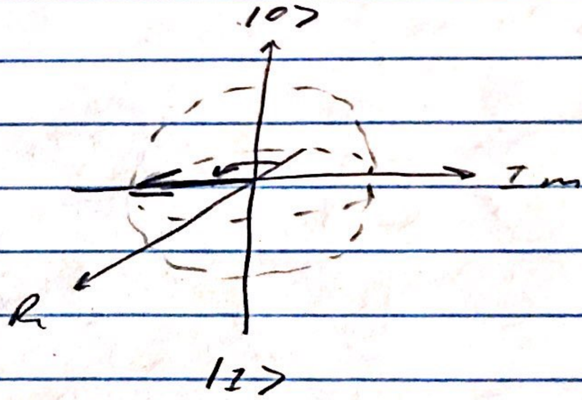


$S|+\rangle$? $= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$



cf $S|-\rangle = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

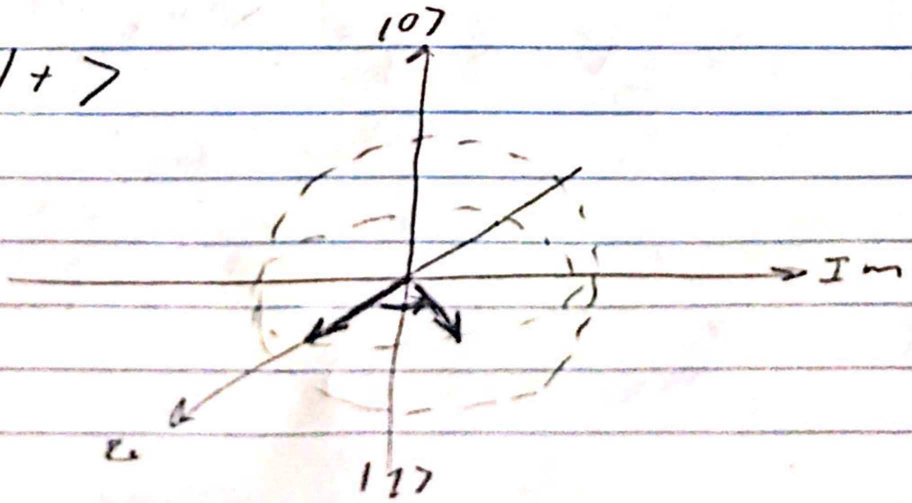




3. T - gate ; $R_\phi = \frac{\pi}{4}$

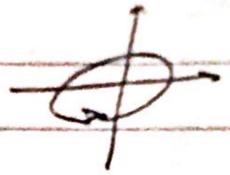
$\frac{\pi}{4}$ rotation about the z-axis.

$S|+\rangle$



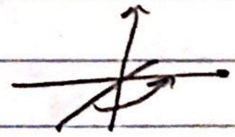
3.6 cont'd

Note → $R_{\phi = 2\pi} = I$

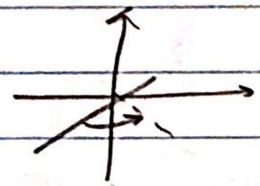


$R_{\phi = \pi} = Z$

$R_{\phi = \frac{\pi}{2}} = S$



$R_{\phi = \frac{\pi}{4}} = T$



$Z^2 = I$; $S^2 = Z$ $T^4 = Z$

3.7 General Unitary Gates.

Unitary $\Longrightarrow$ $U^\dagger = U^{-1}$

or $\det(U) = I$

↳ Rotations and

Reflections

3.7 Contd

Most general Unitary gate is

$$U_3(\theta, \phi, \lambda) = \begin{bmatrix} \cos\frac{\theta}{2} & -e^{i\lambda}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & e^{i(\lambda+\phi)}\cos\frac{\theta}{2} \end{bmatrix}$$

All previously discussed single qubit gates are specific cases of U_3.

Also U_2 and U_1

$$U_3\left(\frac{\pi}{2}, \phi, \lambda\right) = U_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -e^{i\lambda} \\ e^{i\phi} & e^{i\lambda+i\phi} \end{bmatrix}$$

$$U_3(0, 0, \lambda) = U_1 = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\lambda} \end{bmatrix}$$

$$= R_\lambda$$

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4. Multiple Qubits

4.1 Representing Multi-Qubit States

Classical bits: 00 → bit 1 : 0
01 → bit 2 : 0
10 → bit 1 : 0
11 → bit 2 : 1
...

Qubits : $|a\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$

$$= \begin{bmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{bmatrix}$$

$\Longrightarrow$ Basis becomes $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

$|01\rangle$ means qubit 1 is in state $|0\rangle$, and qubit 2 is in state $|1\rangle$

$\Rightarrow$ Same calculation for probabilities

$p(|00\rangle) = |\langle 00|a\rangle|^2$

4.2 Cont'd

Same normalization condition

$$|a_{00}|^2 + |a_{01}|^2 + |a_{10}|^2 + |a_{11}|^2 = 1$$

Given 2 qubits

$$|a\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \qquad |b\rangle = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

the combined 2 qubit state is

$$|a\rangle \otimes |b\rangle = \begin{bmatrix} a_0 \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \\ a_1 \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} a_0 \, b_0 \\ a_0 \, b_1 \\ a_1 \, b_0 \\ a_1 \, b_1 \end{bmatrix}$$

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4.1 Cont'd Tensor Product

Examples

e.g.

$|a\rangle = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$|b\rangle = |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$|a\rangle \otimes |b\rangle = \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \underline{|01\rangle} \checkmark$$

e.g.

$$|+\rangle \otimes |-\rangle = \begin{bmatrix} 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = |00\rangle - |01\rangle + |10\rangle - |11\rangle$$

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4.2 Single Qubit Gates on Multi-Qubit Statevectors

If you want to perform X-gate on qubit 1 and an H on qubit 0,

$$X \otimes H = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$X \otimes H \, |q_0 q_1\rangle$

$= X|q_1\rangle \otimes H|q_0\rangle$

e/

$$X \otimes H|00\rangle = X|0\rangle \otimes H|0\rangle$$

$$= |1\rangle \otimes |+\rangle$$

$$= \begin{bmatrix} 0 \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \\ 1 \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}(|10\rangle + |11\rangle)$$

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4.2 Cont'd

Also $|1\rangle \otimes |+\rangle$

$$= |1\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + |1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|10\rangle + |11\rangle\right)$$

eg $X \otimes I$?

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

Apply to $|01\rangle$

$$= \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |11\rangle$$

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4.2 Cont'd

Also ...

$$X \otimes I \; |01\rangle$$

$$= X|0\rangle \otimes I|1\rangle$$

$$= |1\rangle \otimes |1\rangle$$

$$= \underline{|11\rangle}$$

Also

$$X \otimes I \, |01\rangle$$

$$= X|0\rangle \otimes I|1\rangle$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \underline{|11\rangle}$$

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