Linear Algebra
Lecture Note 3

Multiplication (Hadamard Product, Dot Product), of Matrices, The diagonal and Trace of a Matrix, and inner product of matrices of Matrices

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Matrix Hadamard Product

$$X = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, then\ Z = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} X \odot Z = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 3 \end{bmatrix}$$

$$Y = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, then\ Z = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} Y \odot Z = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \odot \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1a & 3b \\ 2c & 3d \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{bmatrix} B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} A \odot B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{bmatrix} \odot \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Hadamard product can be performed on any two vectors of the same dimension

$$2A \odot B = 2 \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{bmatrix} \odot \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 2 \\ 0 & 6 \end{bmatrix} \odot \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

Practice

Given two matrices A and B:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

Find A∘B

Given two matrices A and B:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find A∘B

Matrix Dot Product

We pair up every combinations of vectors with each other.

Matrix dot product is the default multiplication. So dot is not necessary. It is always often omitted.

$$X = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, Z = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} \; then \; XZ = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1*1+0*2 & 1*3+0*3 \\ 2*1+1*2 & 2*3+1*3 \end{bmatrix}$$

Unlike the Hadamard product the size does not have to be the same.

Instead, the number of columns on the left must be the same as the number of rows on the right

$$X = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}, Z = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} \; then \; XZ = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 9 \\ 2 & 3 \end{bmatrix}$$

The transposed of two matrices multiplied together is the transpose of each individual matrix switched in location

$$(XY)^T = Y^T X^T$$

Find the dot Product of metrics

1.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = ?$$

2.

$$\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} = ?$$

3.

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ 5 & 4 \end{bmatrix} = ?$$

4.

$$\begin{bmatrix} 7 & 2 \\ -3 & 4 \\ 5 & 1 \end{bmatrix} \times \begin{bmatrix} 6 & -2 & 3 \\ 0 & 5 & 1 \end{bmatrix} = ?$$

The order of matrix dot product cannot be flipped.

$$X = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$XY = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

$$YX = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$$

$$XY \neq YX$$

The diagonal and Trace of a Matrix

Defination (Matrix Diagonal): given a square Matrix X the diagonal of a matrix is a vector form from the diagonal elements of a matrix

Example. Given a matrix X defined as

$$X = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 7 & 3 & 9 \end{bmatrix}, \quad Diag = (X) = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} \quad Z = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}, \quad Diag(Z) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Definition (Matrix Trace): given a square matrix X the Trace of a matrix is the sum of all elements in the diagonal of a matrix

Example.

$\mathrm{Tr}(X) = 1 + 4 + 9$ $\qquad\qquad$ $\mathrm{Tr}(Z) = 1 + 4 = 5$

Practice

1.Given matrix A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find the diagonal elements of A

$\mathbf{2.}$ Given matrix A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find The trace of A

Matrix inner product

The matrix inner product is different from the vector inner product with two matrices X and Y their inner product is defined as

$$\langle X, Y \rangle = \mathrm{Tr}(XY^T) \quad (1)$$

For example, given

$$X = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \; Y = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \quad (2)$$

Then

$$\langle X, Y \rangle = Tr\left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^T\right) = Tr\left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\right) \quad (3)$$

$$= Tr\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = 1 + 4 = 5 \quad (4)$$

Take a minute and see if you can find $\langle Y, X \rangle$

Matrix Inner Product Example

$$\langle X, Y \rangle = \mathrm{Tr}(XY^T) \tag{1}$$

For example, given

$$X = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \tag{2}$$

Then

$$\langle Y, X \rangle = Tr\left(\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^T \right) = Tr\left(\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right) \tag{3}$$

$$= Tr\left(\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \right) = 1 + 4 = 5 \tag{4}$$

A faster Trick. We previously knew that $XY^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, we now need to know YX^T , we can find it easily where since

$$YX^T = (XY^T)^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Matrix/Vector Multiplication Practice

Given

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, y = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, Y = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 3 & 1 \end{bmatrix}, z = \begin{bmatrix} 1 & 3 & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

1. $Y^T x$
2. $\mathrm{Z} \odot \mathrm{Z}$
3. $2Z\mathrm{y}$
4. $\langle X, X \rangle$
5. $x \otimes y$
6. $Diag(X)$
7. $\mathrm{Tr}(Z)$
8. $x^T\, Y z^T$

In addition, given

$$u = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}, v = \begin{bmatrix} x \\ y \end{bmatrix}$$

Solve for

$1 \cdot v \otimes v$

2. $\mathrm{Tr}(v \otimes v)$

3. $Z^T u$

4. Zu

Matrix/Vector Multiplication Practice

Given

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, y = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, Y = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 3 & 1 \end{bmatrix}, z = \begin{bmatrix} 1 & 3 & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

1. $Y^T x$
2. $Z \odot Z$
3. $2Zy$
4. $\langle X, X \rangle$
5. $x \otimes y$
6. $Diag(X)$
7. $\mathrm{Tr}(Z)$
8. $x^T \, Y z^T$

In addition, given

$$u = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}, v = \begin{bmatrix} x \\ y \end{bmatrix}$$

Solve for

$1 \cdot v \otimes v$

2. $\mathrm{Tr}(v \otimes v)$

3. $Z^T u$

4. Zu

Solution to exercise

1. $Y^T \ x = \begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1*1 & + & 2*2 \\ 3*1 & + & 3*2 \\ 0*1 & + & 1*2 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}$

2. $\mathrm{Z} \odot \mathrm{Z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

3. $2Zy + 1 = 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} + 1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 14 \end{bmatrix} + 1 = \begin{bmatrix} 7 \\ 11 \\ 15 \end{bmatrix}$

4. $\langle X, X \rangle = \mathrm{Tr}(XX^T) = Tr\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \mathrm{Tr} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 + 1 = 2$

5. $x \otimes y = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \otimes [3 \ 1 \ 3] = \begin{bmatrix} 3 & 1 & 3 \\ 6 & 2 & 6 \end{bmatrix}$

6. $Diag(X) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Solution to exercise

$$7.\ \mathrm{Tr}(Z) = \mathrm{Tr}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 1 + 2 + 1 = 4$$

$$8.\ x^T\ Y z^T = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & 9 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = 32$$

Solution to symbolic Representation

$$3.\ Z^T u = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^T \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \alpha + \gamma \\ 2\beta + \gamma \\ \beta + \gamma \end{bmatrix}$$

$$4.\ Yu = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \alpha + 3\beta \\ 2\alpha + 3\beta + \gamma \end{bmatrix}$$