Linear Algebra

Lecture 9-10-11

The null space , Linearly dependent and independence, The formal definition of linear independence, Using different basis to represent the same point , Unit vectors are the preferred basis , Equation for the unit vector, Linear independence, Rank, Orthogonality,Vector Normalization

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Learning objectives

1. **Null Space**
   Find all solutions to Ax=0; understand the structure of solution sets.
2. **Linear Dependence/Independence**
   Determine if vectors are dependent or independent using equations or row reduction.
3. **Formal Definition of Linear Independence**
   Know that a set is independent if only the trivial solution exists for
   $c_1v_1+c_2v_2+c_3v_3+\ldots\ldots\ldots\ldots+c_nv_n=0$
4. **Different Bases, Same Point**
   Represent the same vector using different bases; convert between them.
5. **Unit Vectors as Preferred Basis**
   Use unit vectors for simplicity and clarity in coordinate systems.
6. **Equation for a Unit Vector**

7. Linear Independence
Identify if vectors are independent.
Understand its role in forming a basis.

8. Rank
Find the rank of a matrix.
Use rank to understand solutions of linear systems.

9. Orthogonality
Recognize orthogonal vectors (dot product = 0).
Use orthogonality in simplifying vector problems.

10. Vector Normalization
Convert a vector to unit length.

Learning outcomes

1. **Null Space**
   Find and interpret the set of solutions to Ax=0.
2. **Linear Dependence/Independence**
   Determine if vectors are dependent or independent using equations or row operations.
3. **Formal Definition of Linear Independence**
   Apply the definition to test if only the trivial solution exists.
4. **Different Bases, Same Point**
   Represent a vector in different bases and convert between them.
5. **Unit Vectors as Preferred Basis**
   Use unit vectors for simpler, standard representation in coordinate systems.
6. **Equation for Unit Vector**.
7. **Linear Independence**

   Determine whether a set of vectors is linearly independent.
   Explain the importance of linear independence in forming a basis for a vector space.

8. **Rank**
   Calculate the rank of a matrix.
   Interpret how rank affects the solution set of a system of linear equations.

9. **Orthogonality**
   Identify orthogonal vectors using the dot product.

10. **Vector Normalization**

   Normalize vectors to unit length.

The null space

- Given a set of vectors v_1, v_2 ... and we let these vectors be the columns of a matrix $A = [v_1 \;\; v_2 \;\; v_3 \dots]$.
- **Definition:** the null space is the set of all possible solutions for x such that

$$x_1 v_1 + x_2 v_2 + \cdots = 0 \quad \text{or } Ax = 0$$

- To find the null space we simply create the augmented matrix such that

$$[A|0]$$

The point of this augmented matrix is that we are trying to find the solution x that will give you the output of 0 (this is the definition of null space)

For example, the matrix A would be converted to augmented matrix as

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

We turn it into the RREF and it gives us the solution for x

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

This is an example of trivial solution when $x = 0$
Because obviously

$$A0 = 0$$

However, once we have reduced the augmented matrix down to RREF, you might get a non-trivial solution

$$\underbrace{\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]}_{\text{RREF}} \xrightarrow{\text{to equations}} \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\text{Infinite solutions}} \xrightarrow{\text{implying}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\text{non-trivial solutions}} \qquad x_2 \text{ is free variable}$$

Solution here is a space defined by the vector, hence , **non-trivial solution**.

Example Find the null space of the matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

To find the null space, solve:

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives:

$$\begin{cases} x_1 + 2x_2 = 0 \\ 3x_1 + 6x_2 = 0 \end{cases} \Rightarrow$$

Second equation is a multiple of the first

So the system have infinite solution so we are free to give any value to x_2

Suppose $x_2 = 1$

$$x_1 + 2x_2 = 0$$

$$x_1 + 2(1) = 0$$

$$x_1 = -2$$

So, the null space is:

$$Null\ space\ of\ (A) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Example Find the null space of the matrix:

$$B = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{bmatrix}$$

Solve $Bx = 0$. Notice each row is a multiple of the first, so the system has infinite solutions.

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So 2nd and 3 rd columns are free

Therefore we can assign any value to x_2 and x_3

Suppose x_2=1 and $x_3 = 2$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The $x_1 + 2(1) - 1(2) = 0$

$$x_1 = 0$$

So the null space is:

$$\text{Null (B)}=\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Practice

a. Find the null space of the matrix:

$$B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 3 & 6 & -3 \end{bmatrix}$$

b. Find the null space of the matrix:

$$D = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 3 & 9 & 6 \end{bmatrix}$$

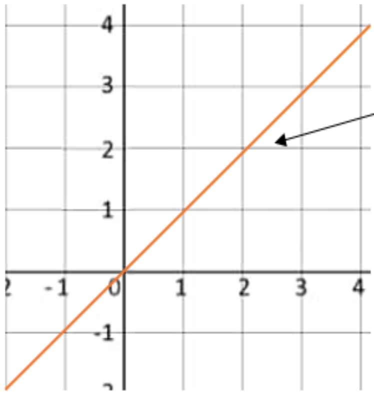
c. Find the null space of the matrix:

$$E = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

d. Find the null space of the matrix:

$$A_1 = \begin{bmatrix} 1 & 2 & 4 & 6 & 1 \\ 2 & 4 & 8 & 12 & 2 \\ 1 & 2 & 4 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We also learned about the span in the last lecture



This is the span of $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

This is the set of all possible points reachable by one or multiple scaled vectors

This notation is called the set builder notation

It defines a set such that α can be any real number

You may sometime see a line instead

With a vector v its span is

$$\boldsymbol{S} = \text{span}(v) = \{\alpha v : \alpha \in \mathbb{R}\}$$

But you can also have multiple vectors $v_1, v_2, v_3, \ldots,$ its span would be

$$S = \text{span}(v, v_2, v_3 \ldots,) = \{(av_1, bv_2 \ldots): a, b \in \mathbb{R}\}$$

If we have 2 vectors it can at most span 2d

Given 2 vectors

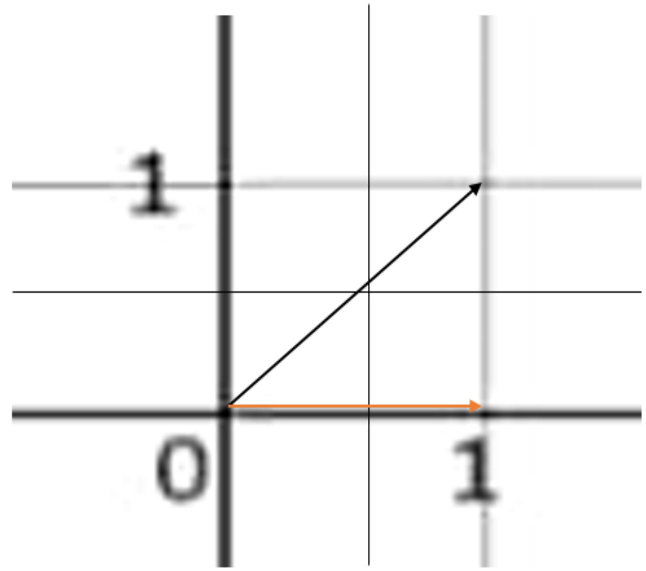
$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Their span would cover the entire 2D space
For example, the point d is in the span because

$$d = v_1 + {}^1\!/_2\, v_2$$

$$\begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

The vectors used to create the span of a space are called base



Every single point on this 2D plane can be achieved by some combination of the 2 vectors

Example 1: Vectors that span R^2

Let: $v1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- These are the standard basis vectors in R^2
- They are **linearly independent**.

They span the entire plane R^2

Example 2: Vectors that also span $\mathbf{R^2}$

Let:

$$v1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad v2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

These are not scalar multiples**.**

So, they are **linearly independent.**

They span R^2

Example 3: Vectors that DO NOT span R^2

Let:

$$v1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad v2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- $v1 = 2 \cdot v2$
- These vectors are **linearly dependent**.
- They only span a **line** in R^2, not the full plane.

Example 4: Vectors that do NOT span R^2

Let:

$$v1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad v2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- $V1 = 2 \cdot v2$
- **Linearly dependent**

Only span a **line** through $\begin{bmatrix}1\\1\end{bmatrix}$

Key concepts :

- **2 independent vectors → span R^2**
- **Less than 2 independent vectors → a line**

Example 5: Vectors that span R^3

Let: $v1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $v1 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $v1 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

- These are the standard basis vectors.
- They are **linearly independent**.

They span the entire 3D space R^3

Example 6: Vectors that span R^3

Let $v1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $v1 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$ $v1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

- Try forming a linear combination like $av1 + bv2 + cv3 = 0.$
- If only the **trivial solution** exists (a = b = c = 0), they are independent.

These vectors are linearly independent → they span R^3

Example 7: Vectors that DO NOT span R^3

let $v1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $v1 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ $v1 = \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$

- Each vector is just a **multiple** of the others.
- They're **linearly dependent**.

They only span a line — not the whole 3D space.

Example 4: Vectors in the same plane

Let $$\boldsymbol{v1} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} \ \boldsymbol{v1} = \begin{bmatrix} \mathbf{2} \\ \mathbf{0} \\ \mathbf{2} \end{bmatrix} \ \boldsymbol{v1} = \begin{bmatrix} \mathbf{3} \\ \mathbf{0} \\ \mathbf{3} \end{bmatrix}$$

All vectors lie in the **xz-plane** (y = 0).

Also, they are scalar multiples → **linearly dependent**.

They span a **plane**, not R^3.

Key concepts :

3 independent vectors → span R^3

Less than 3 independent vectors → span line or plane

How Many Vectors Are Needed to Span R^n

- To span $\mathbf{R^n}$, you need **at least n linearly independent vectors**.
- So

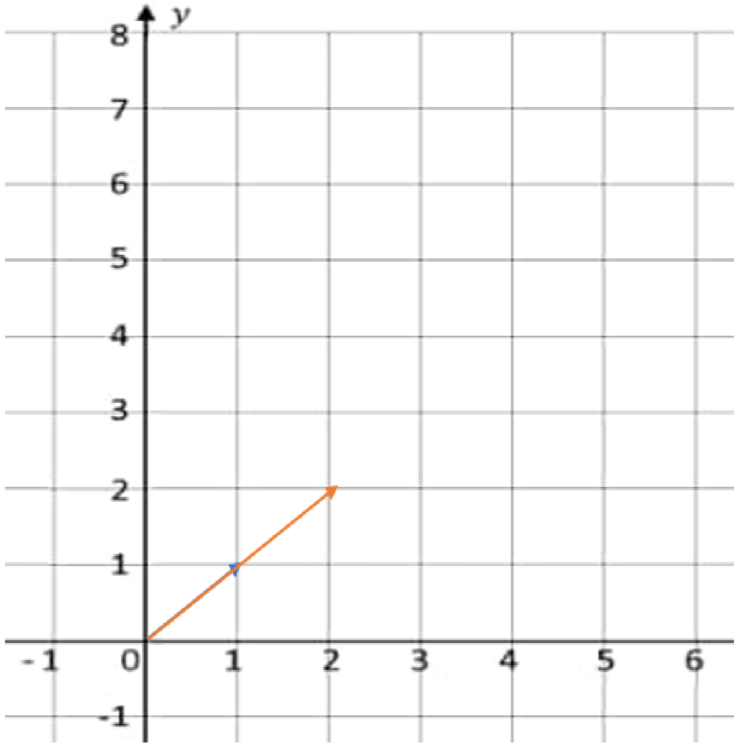
n **linearly independent vectors** → span the entire space R^n.
Fewer than n → span a subspace (like a line or plane).
More than n vectors → must be **linearly dependent** in R^n (they don't add new directions).

2 vector does not imply 2d span

Given 2 vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Their span would cover only 1 dimension



The dimension span by a set of vectors is called the

Rank

Rank OF a matrix

The ***rank*** of a matrix A, denoted as rank (A), is:

The maximum number of linearly independent **rows** or **columns** in the matrix.

Or The rank gives the number of **non-zero rows** in the matrix after it's reduced to **row echelon form** (REF) or reduced row echelon form (RREF).

Full Rank:

Full rank matrices have linearly independent rows and columns.

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow \text{rank}(A) = 2$$

Rank-Deficient:

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \Rightarrow \text{rank}(B) = 1$$

Explanation: Second row is a multiple of the first → not independent.

Example:

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

The columns of A are

$$\text{Col}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \text{Col}_2 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} \quad \text{Col}_3 = \begin{bmatrix} 4 \\ 8 \\ 0 \end{bmatrix}$$

As we clearly Notice:

That $Col_2=3\cdot Col_1$ $Col_3=4\cdot Col_1$

All columns are linearly dependent on Column 1.
so **Rank (A)= 1**

Example :

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Columns of B are :

$$\mathrm{Col}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathrm{Col}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathrm{Col}_3 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

We notice that

$$\mathrm{Col}_3 = 2 \cdot \mathrm{Col}_1 + 3 \cdot \mathrm{Col}_2$$

So Column 3 is **dependent** on Columns 1 and 2.
Columns 1 and 2 are **independent**.

so **Rank(B) = 2**

Example

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

We want to:

- Use row operations to bring this matrix to row echelon form (REF).
- Then count the number of leading (non-zero) rows to get the rank.

- Then identify which columns are linearly dependent.

  Start with the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

  Eliminate entries below the pivot in column 1

$-2e_1 + e_2 \rightarrow e_2$

$e_3 - e_1 \rightarrow e_3$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & -4 \end{bmatrix}$$

  Eliminate below the pivot in column 2

Pivot = -1 in e_3, col_2.
Now the matrix is in row echelon form:

$$\text{REF:}\quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & -4 \end{bmatrix}$$

Now make row 3's second element into 1:

$-1 \times e_3 \rightarrow e_3$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 4 \end{bmatrix}$$

Flip e_2 to e_3

So the REF $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Now Count Non-Zero Rows → Rank

Non-zero rows = 2 So **rank (A) = 2**

This lead to have two possible cases

The rank is same as the number of bases

This happen when each vector is contributing to the new dimension

The rank is fever than the number of bases

This happen when we have redundant vectors covering the same space

These two cases have names

The rank is same as the number of basis

[every vector is expanding the span larger]

These vectors are called linearly independent

The rank is fever than the number of basis

[having redundant vectors]

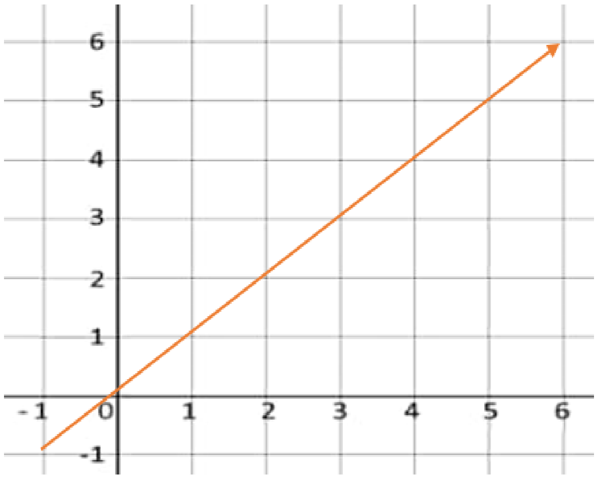
These vectors are called linearly dependent

Linearly dependent and independence

Given 2 vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Linearly dependent

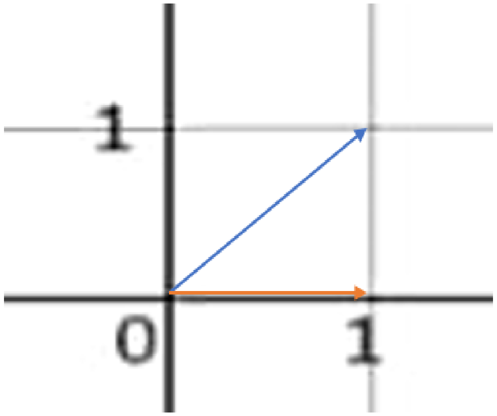


Their span would cover only 1 dimension

Given 2 vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Linearly independent



Their span would cover the entire 2D space

The formal definition of linear independence

A set of vectors is linearly independent if and only if the equation:

$$c_1 v_1 + c_2 v_2 + \cdots c_k v_k = 0$$

Has only the trivial solution. What that mean is that these vectors are linearly independent when $c_1 = c_2 = \cdots = 0$ is the only possible solution to the vectors equation

If a set of vectors is not linearly independent we say that they are linearly dependent. Then we can write a linear dependence relation showing how one vector is combination of the other vectors

Given this definition it implies that if we have a set of vectors $v_1 v_2, v_3 \ldots$ they are

- **Linearly independent** : if their null space solution is the trivial solution
- **Linearly dependent** : if their null space solution is non-trivial solution

Why linear independence?

Linear independence is a very important concept in data representation

$$v_1 = \begin{bmatrix}1\\2\\1\\0\end{bmatrix}, \quad v_2 = \begin{bmatrix}2\\4\\2\\0\end{bmatrix}, \quad v_3 = \begin{bmatrix}4\\8\\4\\0\end{bmatrix},$$

$$v_4 = \begin{bmatrix}6\\12\\6\\0\end{bmatrix}, \quad v_5 = \begin{bmatrix}1\\2\\1\\0\end{bmatrix}$$

We can represent the data as a matrix

$$A = \begin{bmatrix}1 & 2 & 4 & 6 & 1\\2 & 4 & 8 & 12 & 2\\1 & 2 & 4 & 6 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

We can assume the first column as basis and represent the data

$[1\ \ 2\ \ 4\ \ 6\ \ 1]$ become $[1v_1\ \ 2v_1\ \ 4v_1\ \ 6v_1\ \ 1v_1]$

Or we can use the first row as basis and represent the data as

$[1\ \ 2\ \ 1\ \ 0]$ become X = $\begin{bmatrix}1r_1\\2r_1\\1r_1\\0r_1\end{bmatrix}$

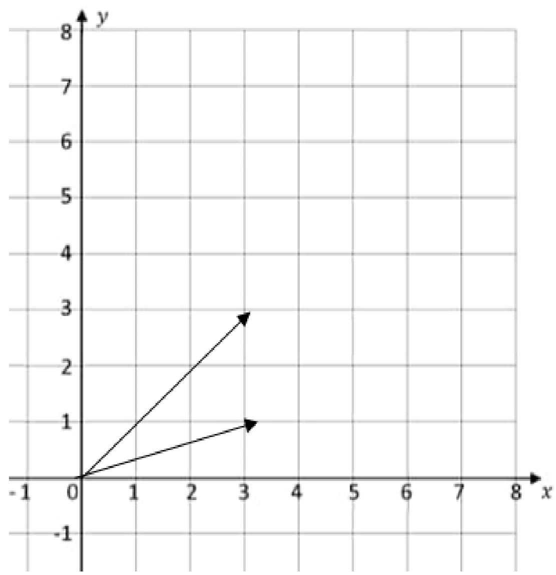
Much compact way to represent data compare to matrix

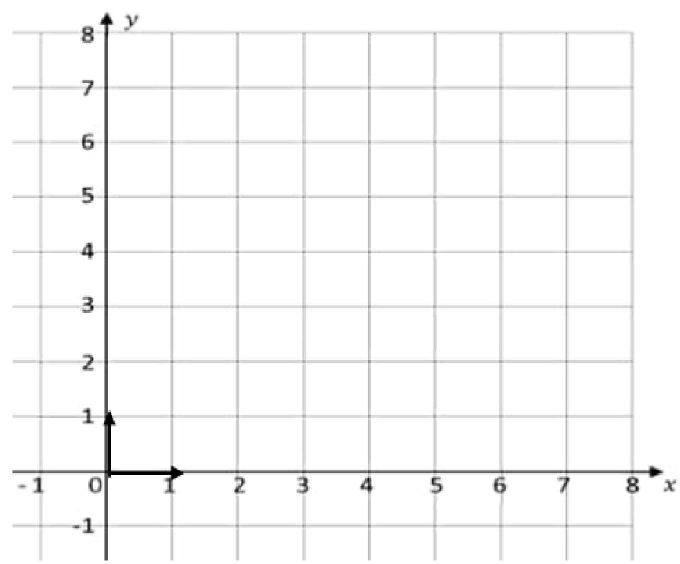
Using different basis to represent the same point

Below we show 2 possible basis to reach target D

Both basis span the same 2D space

but which basis are easier to identify the combination to reach point D?





In general, the basis the are perpendicular with a size are the nicest and easiest one to deal with

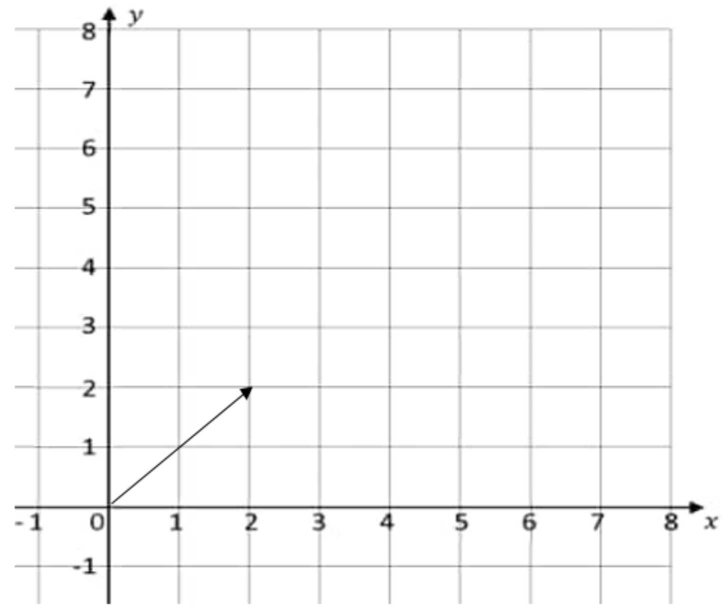
Unit vectors are the preferred basis

We often represent data using independent bases
As a convention, we prefer to use **unite vectors.**
[A unite vector is a vector with length 1]

Examples of unite vectors

$$c = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \\ 0 \end{bmatrix}$$

$$\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{\frac{2}{2}} = \sqrt{1} = 1$$



Example of not a unit vector

$$c = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\sqrt{(2)^2 + (1)^2} = \sqrt{4+1} = \sqrt{5} \neq 1$$

Equation for the unit vector

Example of NOT Unite Vector

$$C = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\sqrt{(2)^2 + (1)^2} = \sqrt{5} \neq 1$$

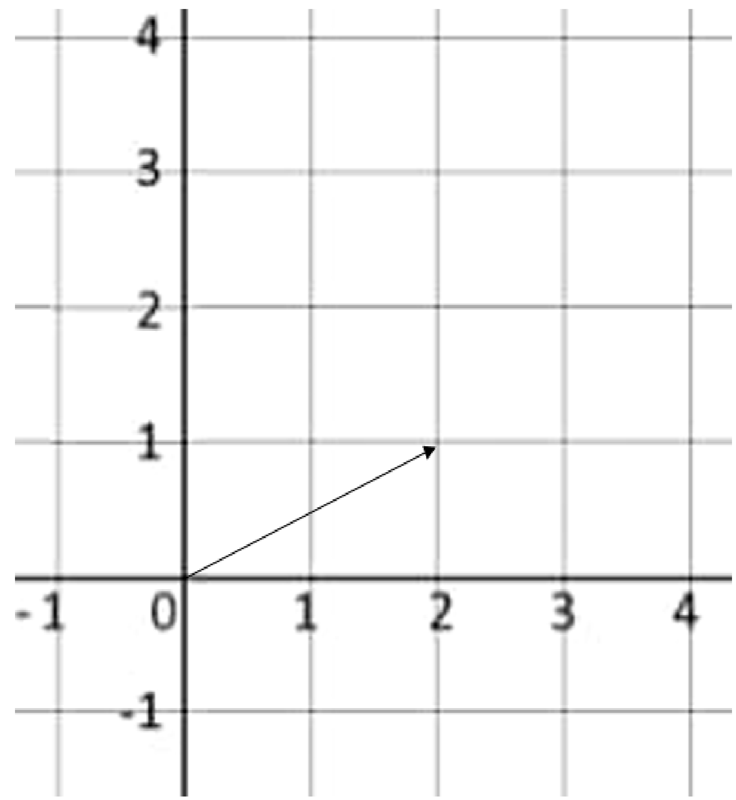
How do we turn this vector into a normal vector

Divide the vector by the length !!!

unite vector $\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \frac{1}{length}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2^2+1^2}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

You can double check your result the new length should be 1

$$\sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = 1$$



Equation for the unit vector

The act of turning a vector into a unite vector is called **normalization**

Normalized vectors are normally denoted by hat

$$u \to \hat{u}$$

normalized

length$(u) \neq 1$

Length $(\hat{u}) = 1$

We refers to unite vector as **normalized vector**

How do we turn this vector into a normal vector

Divide the vector by the length !!!

unite vector $\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \frac{1}{length}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2^2+1^2}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$

You can double check your result the new length should be 1

PRACTICE 1

Convert these 2dim vectors to UNITE vector

1) $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

2) $c = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

3) $d = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

4) $e = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

5) $f = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

6) $x = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$

7) $y = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

8) $z = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Formula to find the length of $x = \begin{bmatrix} c1 \\ c2 \end{bmatrix}$

IS $\sqrt{{C_1}^2 + {C_2}^2}$

Practice 2

Convert these 3 dim vectors to UNITE vector

Formula to find the length of $x = \begin{bmatrix} C1 \\ C2 \\ C3 \end{bmatrix}$

IS $\sqrt{{C_1}^2 + {C_2}^2 + {C_3}^2}$

1) $\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$

2) $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

3) $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$

4) $\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$

Practice 3

Convert these 4 dim vectors to UNITE vector

$$1)\begin{bmatrix} 9 \\ 3 \\ -2 \\ -0 \end{bmatrix}$$

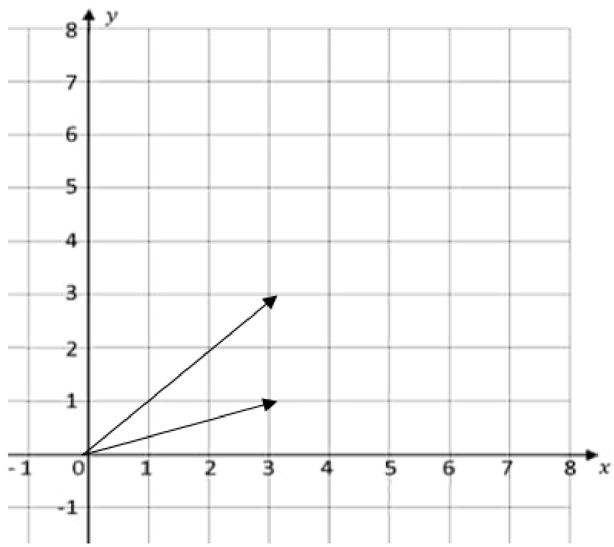
$$2)\begin{bmatrix} -2 \\ 0 \\ 9 \\ -1 \end{bmatrix}$$

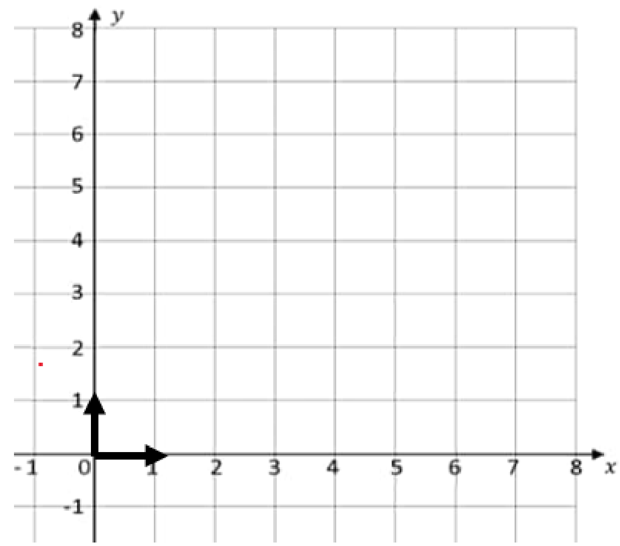
$$3)\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$4)\begin{bmatrix} 1 \\ -2 \\ 2 \\ 2 \end{bmatrix}$$

Besides having nice unite vector, we also want theme to be perpendicular

In linear algebra the basis that are perpendicular are called **orthogonal basis**



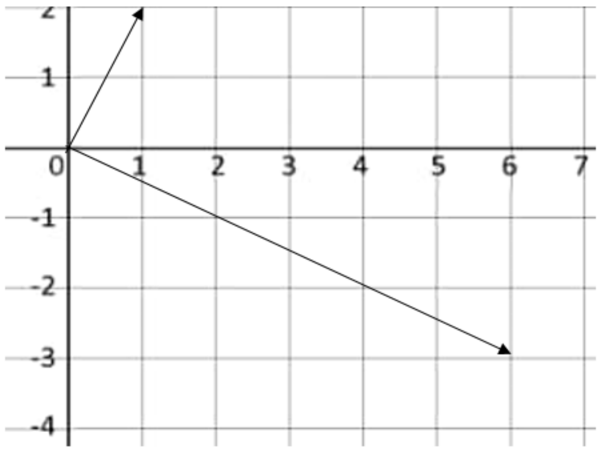


Let’s go over the definition of orthogonality

The final concept of the day: orthogonality

Two vector $\vec{v}$ and $\vec{w}$ are called orthogonal if their do product is zero $\vec{v} \cdot \vec{w} = 0$

$\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}6\\-3\end{bmatrix}$ are orthogonal in $\mathbb{R}^2$



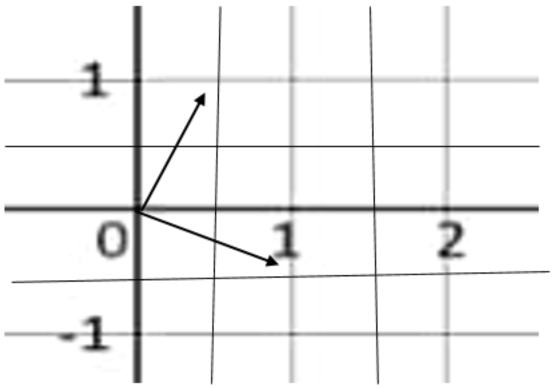
In $\mathbb{R}^2$ vector are orthogonal if they are perpendicular to eachother

Taking the dot product and getting 0 tell you if the vectors are orthogonal

Two vectors are called orthonormal if they are orthogonal and both vectors are normalized

Orthonormal Pair

$$\hat{v}_1 = \begin{bmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}\frac{6}{\sqrt{45}}\\ \frac{-3}{\sqrt{45}}\end{bmatrix}, v_1^T v_2 = 0$$



Definition : Two vectors are orthonormal if they are both **orthogonal** (dot product is 0) and **unit vectors** (magnitude = 1).

Practice

Determine whether the following vectors are orthogonal:

1)$v1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, v2 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

2)$v1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, v2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

3)$v1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, v2 = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$

4)$v1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v2 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

5)$v1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, v2 \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$

6)$v1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, v2 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

Practice

Determine if the following vectors are orthonormal:

1)

$$\vec{u} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

2)

$$\vec{u} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix}$$

3) $$v1 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, v2 = \begin{bmatrix} \frac{-2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

In general, we prefer orthonormal pairs as basis

Orthonormal vectors as basis make life easier

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} \frac{6}{\sqrt{45}} \\ -\frac{3}{\sqrt{45}} \end{bmatrix} \xrightarrow[\textit{As column vectors}]{} X = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{6}{\sqrt{45}} \\ \frac{2}{\sqrt{5}} & -\frac{3}{\sqrt{45}} \end{bmatrix}$$

We can identify linear independence immediately by multiplying X by itself transposed

$$X^T X = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{6}{\sqrt{45}} & -\frac{3}{\sqrt{45}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{6}{\sqrt{45}} \\ \frac{2}{\sqrt{5}} & -\frac{3}{\sqrt{45}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix formed by orthonormal basis are called **orthonormal matrix**

If we get identity matrix the column vectors are automatically **independent**

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Very important observation

Orthonormal vectors as basis make life easier

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This is very important we have previously seen

$$XX^{-1} = I$$

Therefore if $XX^{-1} = I$, then we can conclude that for orthonormal matrix

$$X^{-1} = X^T$$

This makes orthonormal matrices very special and easy to handel

Practice

Determine if the following matrix is orthonormal matrix

$$x = \begin{bmatrix} \frac{2}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$