Linear Algebra

Lecture-14

Singular value decomposition

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Learning Objectives

1. Define SVD and identify its components: V, Σ, and V^T.
2. Compute SVD of a matrix.
3. Explore key properties: orthogonality and diagonalization.
4. Apply SVD in solving systems.

Learning outcomes

1. Define SVD as $A = V\Sigma V^T$ and know its components.
2. Compute SVD using eigenvalues/eigenvectors of A^TA and AA^T.
3. Recognize properties – orthogonality of U,V, and diagonal nature of Σ.
4. Apply SVD in solving systems.

We begin we transpose properties

We start today with a couple of very important properties of matrix transpose. Remember that the transpose operation flips a matrix when given

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \; then \; A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

But what if have matrices A, B, C, D and we want to know

$$(A+B)^T,\ (AB)^T, (A^TC)^T, (B^TCD)^T?$$

Is there any special relation that govern the trace for multiple matrices in parentheses

- with multiplication we flip all the symbols from left to right and all the transposes

$$(AB)^T = B^TA^T, (B^TCD)^T = D^TC^TB,$$

- With addition we flip all the matrices
- $(A+B)^T = A^T + B^T$
- With determinant nothing changes

$$|A|^T = |A|$$

- With inverse we flip the inverse

$$(A^T)^{-1} = (A^{-1})^T$$

- The transpose of a constant is just itself

$$3^T = 3 \ \text{therfore}(3A)^T = 3A^T$$

Inverse properties

1. If BA return I then B must be inverse of A^{-1}

$$BA = I \Rightarrow B = A^{-1}$$

2. When we have the inverse of multiple matrices it has the properties

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

3. Inverse of an inverse is just itself

$$(A^{-1})^{-1} = A$$

4. The transpose of an inverse just the inverse of the transpose

$$(A^{-1})^T = (A^T)^{-1}$$

5. The inverse of a number α time A is

$$(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$$

6. The inverse of a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ C & d \end{bmatrix}^{-1} \Rightarrow \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Practice

Show that for any two matrices A ,B and C of the same size ,verify all Properties of transpose and inverse matrices

$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{3} \\ \mathbf{2} & \mathbf{7} \end{bmatrix}$, B $= \begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{8} \end{bmatrix}$ and C= $\begin{bmatrix} \mathbf{1} & \mathbf{3} \\ \mathbf{5} & \mathbf{20} \end{bmatrix}$

Properties of Diagonal Matrices

- We previously learned about diagonal matrices. These are the matrices of all 0’s except diagonal. For example

$$\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix} \text{ and } \begin{bmatrix}1 & 0 & 0 & 0\\0 & 3 & 0 & 0\\0 & 0 & 7 & 0\\0 & 0 & 0 & 1\end{bmatrix} \text{ and } \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

- A special case of diagonal matrix is identity matrix where the diagonal elements are all ones
- For diagonal matrices it is easy to take power of the matrix. We can just take the power or of each directly for example

$$\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}^2 = \begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1^2 & 0\\0 & 3^2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 9\end{bmatrix}$$

This even extends to the inverse

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1^{-1} & 0 \\ 0 & 3^{-1} \end{bmatrix} \Rightarrow \begin{bmatrix} 1^{-1} & 0 \\ 0 & 3^{-1} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

These properties are only for diagonal matrices and will not work for any regular matrix

- The transpose of a diagonal matrix is just itself

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^T$$

Practice

1. If A= $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then find A^{10}
2. If $B = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$ then find B^2
3. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ then find A^4
4. If $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ then find A^3

5. If C= $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ then find C^3

Practice

Try to find alternative expression for the following

Note: U and V are orthonormal basis. Implying that

- $U^T = U^{-1}$
- $V^T V = I$

Note: Σ is a diagonal matrix

- $(A + B + C)^T$
- $(U^T V)^T$
- $(U \Sigma V^T)^T$
- $((U^{-1} \Sigma V^T)^T)^{-1}$

Practice solution

- $(A + B + C)^T = A^T + B^T + C^T$
- $(U^T V)^T = V^T U$
- $(U\Sigma V^T)^T = V\Sigma U^T$
- $((U^{-1}\Sigma V^T)^T)^{-1} \quad = (V\Sigma(U^{-1})^T)^{-1}$
- $= (V\Sigma(U^T)^{-1})^{-1} = U^T\Sigma^{-1}V^{-1}$
- $= U^T\Sigma^{-1}V^{\mathrm{T}}$

We previously learn to break down matrices via the eigen decomposition

A matrix A can be broken down into the multiplication of multiple matrices. This is called **Matrix Decomposition.** A very important decomposition is called the eigen decomposition it state that

$$\boldsymbol{A = V\Sigma V^{-1}}$$

Where $\boldsymbol{V}$ consist of column eigen vectors

$$V = [v_1, v_2, v_3 \dots]$$

And Σ consist of a diagonal matrix of eigen values

$$\Sigma = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & \dots \\ 0 & 0 & \lambda_3 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

Knowing the eigen values makes the life easier in so many ways

- We can calculate the inverse of a matrix easily

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \qquad \textit{we can get this easily}$$

- We can calculate the power of a matrix easily

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{1000} \quad \textit{we can get this easily}$$

- It makes proving theoretical properties easier, we will see once we break down a matrix A into

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{V}^{-1}$$

Having $\boldsymbol{V}$ as orthonormal matrix make life easier

Example

Solve the Singular Value Decomposition **(SVD)** of the matrix:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

SOLUTION

Firstly we have to find the eigen value and eigen vector for the given matrix

For finding eigen value

$$A - \lambda I = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$$

The

$$\det(A - \lambda I) = (3-\lambda)(3-\lambda) - 1$$

$\lambda^2 - 6\,\lambda + 8 = 0$ so $\lambda = 4, \lambda = 2$

Find the eigen vectors corresponding $\lambda = 4, \lambda = 2$

For $\lambda = 4$

$$A - 4I = \begin{bmatrix} 3-4 & 1 \\ 1 & 3-4 \end{bmatrix}$$

$$A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Now computing null space for

$$(A-4I)x = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As we can clearly see the e_1 is the combination of e_2 so the system has infinite solution as our second column is free column so we can assign any value

$$So\ y = 1$$

$$Then - x + y = 0$$

$$-X\ + 1 = 0$$

$$Then\ x\ = 1$$

So the eigen vector for For $\lambda = 4$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Similarly we can find corresponding eigen vector to $\lambda = 2$

$A - 2I = \begin{bmatrix} 3-2 & 1 \\ 1 & 3-2 \end{bmatrix}$=$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

So Now computing null space for

$$(A-2I) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As we can clearly see the e_1 is the combination of e_2 so the system has infinite solution as our second column is free column so we can assign any value

$$So\ take\ y = 1$$

$$The\ x + y = 0$$

$$X = -y = -1$$

So the eigen vector for For $\lambda = 2$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now find

$$\boldsymbol{A = V\Sigma V^{-1}}$$

A= $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, V=$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\boldsymbol{\Sigma}$=$\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ and $\boldsymbol{V^{-1}}$=$\boldsymbol{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}$=$\boldsymbol{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{-2} & \frac{1}{2} \end{bmatrix}}$

NOW

$$\boldsymbol{A = V\Sigma V^{-1}}$$

WHICH IS $\quad A=\begin{bmatrix}3 & 1\\1 & 3\end{bmatrix}=\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}4 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\-\frac{1}{2} & \frac{1}{2}\end{bmatrix}$

$$=\begin{bmatrix}\mathbf{4} & \mathbf{-2}\\\mathbf{4} & \mathbf{2}\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\-\frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

$$\begin{bmatrix}3 & 1\\1 & 3\end{bmatrix}=\begin{bmatrix}3 & 1\\1 & 3\end{bmatrix}$$

Practice

1. Find the SVD of the matrix

$$A=\begin{bmatrix}3 & 1\\0 & 2\end{bmatrix}$$

2. Compute the SVD of the matrix

$$B=\begin{bmatrix}4 & 0\\3 & -5\end{bmatrix}$$

3. Given $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, compute A^TA, then find its eigenvalues and use them to find the singular values of A.

4. Find the SVD of the diagonal matrix:

$$C = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$$

Eigen decomposition simplifies inverse

If we have the eigen decomposition matrix,

$$A = V\Sigma V^T$$

The inverse of A is just

$$A^{-1} = V\Sigma^{-1}V^T$$

You might be asking ….Really how is this true?

we can easily show this by multiplying them out. If this is true then their product must be identity matrix, right?

$$A\,A^{-1} = I$$

$$(V\Sigma V^T)(V\Sigma^{-1}V^T) = I$$

The key is to remember that since V is an orthonormal matrix $VV^T = V^TV = I$

$$VV^T\Sigma\Sigma^{-1}VV^T = I$$

Powers are easy to find

If we have the eigen decomposition Again,

$$A = V\Sigma V^T$$

Then we can show how to get easily get the power of a matrix given that

$AAA = A^3$ then $AAA\ldots = A^{1000}$

Would be very difficult to calculate. Eigne values five us a theoretical way to easily calculate them

$$(V\Sigma V^T)(V\Sigma V^T) = AA$$

$$V\Sigma\Sigma V^T =$$

$$V\Sigma^2 V^T =$$

From this, we see that multiplying a matrix by itself twice, or A^2can be obtained by taking Σ^2. Again, since Σ is a diagonal matrix, it is easily to find the power

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^2 = \begin{bmatrix} 1^2 & 0 \\ 0 & 2^2 \end{bmatrix}$$

Diagonal matrices are easy to find the power in general we have

$$A^n = V\Sigma^n V^T$$

Solving a linear system

$$Ax = y$$

$$V\Sigma V^T x = y$$

$$(V\Sigma V^T)^{-1}(V\Sigma V^T)x = (V\Sigma V^T)^{-1}y$$

$$x = V\Sigma^{-1}V^T y$$