Linear Algebra Lecture Note 4

Gaussian Elimination

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Notice Large equations can be compressed into matrix format

Notice how we can rewrite an equation with vector matrix format

$$f(x) = 1x_1 + 3x_2 = [1 \;\; 3]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ax$$

Imagine if we have 1000 variables, it would be annoying to write it each time

but with matrix notation we can write it compactly

$$f(x) = 1x_1 + 2x_2 + 3x_3 + \cdots + 1000x_{1000} = [1 \quad 2 \quad 3 \cdots 1000]\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_{1000} \end{bmatrix}$$

$$= \alpha x$$

This is a function where $f: \mathbb{R}^{1000} \to \mathbb{R}$ but the same idea apply if we have output of multiple dimensions. $f: \mathbb{R}^{1000} \to \mathbb{R}^{20}$

This is an example of function with multiple inputs and multiple outputs

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$f(x) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1x_1 + 2x_2 \\ 0x_1 + 1x_2 \\ 3x_1 + 2x_2 \end{bmatrix}$$

If you put an input of $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ of 2 dimensions, you will get a three dimensions output

$$(x) \quad \begin{bmatrix} \\ \\ \\ \end{bmatrix} \quad f \quad = \begin{matrix} 1 \\ 0 \\ 3 \end{matrix}$$

In this case we can represent this systems of linear equations simply as

$$f(x) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax$$

Compact way of representing thousands of equations simultaneously

Here is a problem you can see this type of problem in the SAT

A concert charges \$20 for adult tickets and \$10 for student's tickets. If the concert sold 100 tickets and made \$2000. how many of each type of tickets were sold?

The problem gives us two pieces of information

- Let a be the number of adult tickets sold
- Let s be the number of students tickets sold

Then, we know that

1. adult + student is 100 tickets or a + s = 100
2. $20a + 10s = 2000$

With two equations and two unknowns, you should be able to find a and s. this is called **linear system of equations** or just **linear system**

$$a + s = 100$$
$$20a + 10s = 2000$$

The standard way we learn to this is through substitution

Given the linear system of

$a + s = 100$

$20a + 10s = 2000$

we let a be

$a = 100 - s$

And we substitute a into the 2nd equations

$20(100 - s) + 10s = 2000$

this allow us to solve for s

$2000 - 20s + 10s = 2000$

$-10s = 0$

$s = 0$

This approach is useful with only a few variables to solve, but what if we have a bigger problem?

It would take a long time to solve this. What is the alternative?

Gaussian Elimination.

$$
\begin{aligned}
2x_1 - 3x_2 + 4x_3 + 5x_4 - 6x_5 + 7x_6 - 8x_7 + 9x_8 - 10x_9 + 11x10 &= 12 \\
-3x_1 + 4x_2 - 5x_3 + 6 \times 4 - 7x_5 + 8x_6 - 9x_7 + 10x_8 - 11x_9 + 12x_{10} &= 13 \\
4x_1 - 5x_2 + 6x_3 - 7x_4 + 8x_5 - 9x_6 + 10x_7 - 11x_8 + 12x_9 - 13x_{10} &= 14 \\
5x_1 - 6x_2 + 7x_3 - 8x_4 + 9x5 - 10x_6 + 11x_7 - 12x_8 + 13x_9 - 14x_{10} &= 15 \\
-6x_1 + 7x_2 - 8x_3 + 9x_4 - 10x_5 + 11x_6 - 12x_7 + 13x_8 - 14x_9 + 15x_{10} &= 16 \\
7x - 8x_2 + 9x_3 - 10x_4 + 11x_5 - -12x_6 + 13x_7 - 14x_8 + 15x_9 - 16x_{10} &= 17 \\
-8x_1 + 9\chi_2 - 10x_3 + 11x_4 - 12x_5 + 13x_6 - 14\chi_7 + 15x_8 - 16x_9 + 17x_{10}18 & \\
9x_1 - 10x_2 + 11 \times 3 - 12x_4 + 13x_5 - 14x_6 + 15x_7 - 16x_8 + 17x_9 - 18x_{10} &= 19 \\
-10x_1 + 11x_2 - 12x_3 + 13x_4 - 14x_5 + 15x_6 - 16x_7 + 17x_8 - 18x_9 + 19x_{10} &= 20 \\
11x_1 - 12x_2 + 13x_3 - 14x_4 + 15x_5 - 16x_8 + 17x_7 - 18x_8 + 19x_9 + 20x_{10} &= 21
\end{aligned}
$$

Manipulating the truth

Let's say have a vector x and two function defined as

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \; and \; \begin{matrix} 2x_1 + x_2 = 1 \\ -x_2 = 1 \end{matrix}$$

In this case we are telling you that these 2 equations are true given some x

We can also see that these statements both are true if we put $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ into the equations

$$\begin{matrix} (\;)\quad (2\;1) + \;-1 \;= 1 \rightarrow 1 = 1 \\ (-1)\; - \qquad = 1 \rightarrow 1 = 1 \end{matrix}$$

An interesting fact about true statements is that they can be manipulated and remain true. For example we can multiply both sides of the equation by a number……

$$\begin{matrix} 2x_1 + x_2 & & = 1 \\ (2x_1 + x_2) & 2(1) & = 2 \end{matrix}$$

Let us verify the last equation if it is still true for $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$2\big(2(1) + (-1)\big) = 2(1) \Rightarrow 4 - 2 = 2 \Rightarrow 2 = 2$$

Even though we have two statements that look different actually they are actually the same statements presented in 2 different ways

$$2x_1 + x_2 = 1 \quad \text{and} \quad 4x_1 + 2x_2 = 2$$

We are going to symbolically represent what we did to equation 1 as

$$2e_1$$

These are a couple of fact to note

1. A true statement requires an equal sign like 5 = 5 we can very easily see

   that 3(5) = 3(5)

2. These mathematical "True" statements are only true given a specific x. note if we use a

   different x the statements are no longer true. For example given

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; then \; \square \; 2 \;\; 1 \quad 0 \neq 1$$

$$(\;) + (\;) \neq 1 \quad (0)$$

However under correct condition the truth can be manipulated and remain true. This allow us to conclude the firs observation

Conclusion1: the result of multiplication and division of a true statement on both side of the equation is still true

The 2nd conclusion

let’s we have a vector x and two functions defined as follow

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \square \begin{matrix} 2x_1 + x_2 = 1 \\ -x_2 = 1 \end{matrix}$$

Notice the order of the true statements does not matter

$$\square \begin{matrix} 2x_1 + x_2 = 1 \\ -x_2 = 1 \end{matrix} \quad \textit{This is the original true statement}$$

$$\square \begin{matrix} -x_2 = 1 \\ 2x_1 + x_2 = 1 \end{matrix} \quad this\ is\ the\ fillped\ \ statement\ but\ they\ still\ true$$

Here we see that equation 1 e1 become e2. we are going to mathematically represent this as

$$e_1 \leftrightarrow\ e_2$$

The 3rd conclusion

let’s we have a vector x and two functions defined as follow

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \square \begin{matrix} 2x_1 + x_2 = 1 \\ -x_2 = 1 \end{matrix}$$

Notice if we combine two true statements the result is still true

$$\begin{array}{r} 2x_1 + x_2 = 1 \\ -x_2 = 1 \\ \hline 2x_1 + 0x_2 = 2 \end{array}$$

We are going to symbolically represent what we just did as

$$e_1 + e_2$$

The combination of equation 1 (e1) and equation 2 (e2) yields a brand new equation

$$2x_1 = 2$$

and this equation is still true for x1 = 1 let's check

$$2(1) = 2 \Rightarrow 2 = 2$$

Final conclusion: The combination of two true statements yields another true statement

Combining the three conclusions

By combining the three conclusion it allow us to solve really complex problems. Let's say that we first replace equation 1 (e1) with e1+ e2 to get two new statements

$$2x_1 + 0x_2 = 20x_1 - x_2 = 1$$

Now, let's perform $0.5e_1$ and $-1e_2$ to get yet another statement as

$$\begin{cases} 2x_1 + 0x_2 = 2 \\ 0x_1 - x_2 = 1 \end{cases} \xrightarrow[-1e_2]{0.5e_1} \begin{cases} x_1 + 0x_2 = 1 \\ 0x_1 + x_2 = -1 \end{cases} = \begin{cases} x_1 = 1 \\ x_2 = -1 \end{cases}$$

Take a note and study what just happened

In this special example, we already know x1 and x2. However if we don't know this solution, we can manipulate the existing equations to tell us the solutions for x1 and x2
This work for any size of the problem, the goal is to manipulate the equations to get them into a special form that look like this

$$\begin{aligned} x_1 + 0x_2 + 0x_3 + 0x_4 &= 3 \\ 0x_1 + x_2 + 0x_3 + 0x_4 &= 2 \\ 0x_1 + 0x_2 + x_3 + 0x_4 &= 1 \\ 0x_1 + 0x_2 + 0x_3 + x_4 &= 1 \end{aligned}$$

Once we get into this format the solution for x1, x2, x3, x4 are automatically known. This process is called Gaussian Elimination

Let's look at a simple example

Given the following system of linear equations we first multiply e2 by 2

$$\begin{matrix} e_1: 2x_1 + 2x_2 = 2 \\ e_2: x_1 - x_2 = 1 \end{matrix} \xrightarrow[2e2]{} \begin{matrix} \text{e1}: 2x_1 + 2x_2 = 2 \\ \text{e2}: 2x_1 - 2x_2 = 2 \end{matrix}$$

now we can combine the two equations

$$\begin{matrix} \text{e1}: 2x_1 + 2x_2 = 2 \\ \text{e2}: 2x_1 - 2x_2 = 2 \end{matrix} \xrightarrow[e_1 + e_2 \to e_1]{} \begin{matrix} \text{e1}: 4x_1 + 0x_2 = 4 \\ e2: 2x_1 - 2x_2 = 2 \end{matrix}$$

Notice that we eliminated the blue term. We cam also further simplify the e1 by dividing it by 4

$$\begin{matrix} \text{e1}: 4x_1 + 0x_2 = 4 \\ \text{e2}: 2x_1 - 2x_2 = 2 \end{matrix} \xrightarrow[e_1\text{T}4]{} \begin{matrix} \text{e1}: x_1 + 0x_2 = 1 \\ \text{e2}: 2x_1 - 2x_2 = 2 \end{matrix}$$

To remove x1 from e2 we multiply e1 by -2 and add to e2

$$\begin{matrix} e1: x_1 + 0x_2 = 1 \\ e2: 2x_1 - 2x_2 = 2 \end{matrix} \xrightarrow[-2e_1 + e_2 \to e_2]{} \begin{matrix} \text{e1}: x_1 + 0x_2 = 1 \\ \text{e2}: 0x_1 - 2x_2 = 0 \end{matrix}$$

After dividing e2 by 2 we have essentially solved the problem

$$\begin{aligned} \text{e1}&: x_1 + 0x_2 = 1 \\ \text{e2}&: 0x_1 - 2x_2 = 0 \end{aligned} \xrightarrow[e_2\text{T}-2]{} \begin{aligned} e1&: x_1 + 0x_2 = 1 \\ \text{e2}&: 0x_1 + x_2 = 0 \end{aligned}$$

We have x1 = 1 and x2 = 0

To double check our solution, we just plug it back in

$$\text{e1: } 2(1) + 2(0) = 2$$
$$e2: (0)(1) - \quad = 1$$

Notation Simplification

Notice that when we use Gaussian Elimination so far, we have to copy the x 1 and x 2 over and over again. This is really not necessary. Instead, we can simplify the copying by writing the system in matrix format.

Instead of writing

$$\begin{aligned} \text{e1}&: \quad 2x_1 + x_2 = 1 \\ \text{e2}&: \quad\quad\quad -x_2 = 1 \end{aligned}$$

The matrix notation becomes

$$\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We now can ignore x1, x2 and rewrite it as

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & -1 & 1 \end{array}\right]$$

We call this notation **augmented matrix**

All of the row operations and combination stay the same

Now let's solve a system of four equations.

$$\begin{aligned} 2x_1 + 4x_2 + 2x3 + x_4 &= 8 \\ 3x_1 + 1x_2 + 1x_3 + 0x_4 &= 4 \\ 0x_1 + 2x_2 + 2x_3 + 0x_4 &= 2 \\ 1x_1 + 0x_2 + 0x_3 + 1x_4 &= 3 \end{aligned} \xrightarrow{\text{To augmented matrix}} \left[\begin{array}{cccc|c} 2 & 4 & 2 & 1 & 8 \\ 3 & 1 & 1 & 0 & 4 \\ 0 & 2 & 2 & 0 & 2 \\ 1 & 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{\text{Final goal}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & - \\ 0 & 1 & 0 & 0 & - \\ 0 & 0 & 1 & 0 & - \\ 0 & 0 & 0 & 1 & - \end{array}\right]$$

Here are the first steps we take

1. Notice the e4 already has 1 at the start, so the first thing we are going to do is to swap e1 and e4 $e_1 \leftrightarrow e_4$
2. Once we have 1 at the top left it is easy to make the rest of the column 0, let's perform two operations to make the the rest of the column 0

- $-3e_1 + e_2 \to e_2$
- $-2e_1 + e_4 \to e_4$

$$\left[\begin{array}{cccc|c} 2 & 4 & 2 & 1 & 8 \\ 3 & 1 & 1 & 0 & 4 \\ 0 & 2 & 2 & 0 & 2 \\ 1 & 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{e1 \leftrightarrow e4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 3 & 1 & 1 & 0 & 4 \\ 0 & 2 & 2 & 0 & 2 \\ 2 & 4 & 2 & 1 & 8 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 3 & 1 & 1 & 0 & 4 \\ 0 & 2 & 2 & 0 & 2 \\ 2 & 4 & 2 & 1 & 8 \end{array}\right] \xrightarrow[-3e_1 + e_2 \to e_2]{-2e_1+e_4 \to e_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 2 & 2 & 0 & 2 \\ 0 & 4 & 2 & -1 & 2 \end{array}\right]$$

Once we have zero out the entire column one we now have a smaller problem shown in red.

We repeat the process with a smaller matrix.
Given the smaller matrix we try to get a 1 at the top left of the red portion. Luckily it is already a 1.

1. We now proceeds to 0 out the rest of the column. To do so we perform two more operations $-2e_2 + e_3 \to e_3$ and $-4e_2 + e_4 \to e_4$
2. Remember we are trying to make all the lower triangle values to be 0. since e3 is already in good shape we swap that with the e4 in the second step

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 2 & 2 & 0 & 2 \\ 0 & 4 & 2 & -1 & 2 \end{bmatrix} \xrightarrow[-4e_2 + e_4 \to e_4]{-2e_2 + e_3 \to e_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 0 & 0 & 6 & 12 \\ 0 & 0 & -2 & 11 & 22 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 0 & 0 & 6 & 12 \\ 0 & 0 & -2 & 11 & 22 \end{bmatrix} \xrightarrow{e_3 \leftrightarrow e_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 0 & -2 & 11 & 22 \\ 0 & 0 & 0 & 6 & 12 \end{array}\right]$$

1. Since we want the diagonal elements to be all 1s we divide e3 by -2 and e4 by 6

$$\xrightarrow[e_4 \mathrm{T} 6 \rightarrow e_4]{e_3 \mathrm{T}(-2) \rightarrow e_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 0 & 1 & -5.5 & -11 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \longleftarrow$$

We achieved the minimum form to obtain the solution. This form is called the Row Echoeln Form