Linear Algebra

Lecture-12,13

Eigen Decomposition

By Noorullah ibrahimi

Learning Objectives of Eigen Decomposition

By the end of studying Eigen Decomposition, a learner should be able to:

1. Define Eigenvalues and Eigenvectors
    - Understand what they represent and how they relate to linear transformations.
2. Compute the Eigen Decomposition of a Matrix
    - Find eigenvalues and corresponding eigenvectors.

Learning outcomes

1. Define Eigenvalues and Eigenvectors

- Understand the meaning of eigenvalues and eigenvectors.
- Recognize how they relate to the equation $Av=\lambda v$.

2. Compute Eigen Decomposition

- Find eigenvalues and their eigenvectors.

Vector Norms

Norm: measures the size or "length" of vectors/matrices

- Generalized form of a vector p-norm:

$$||X||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

1-norm/L 1 norm/Manhattan norm:

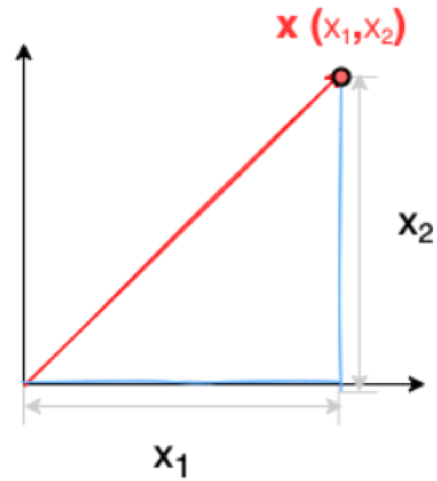
$$||X||_1 = \sum_{i=1}^{n} |x_i|$$

$$||x||_1 = |x_1| + |x_2|$$

2-norm/L 2 norm/Euclidean norm:

$$||X||_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

$$||x||_2 = \sqrt{x_1^2 + x_2^2}$$



- L 2 norm is shortest distance from the origin.

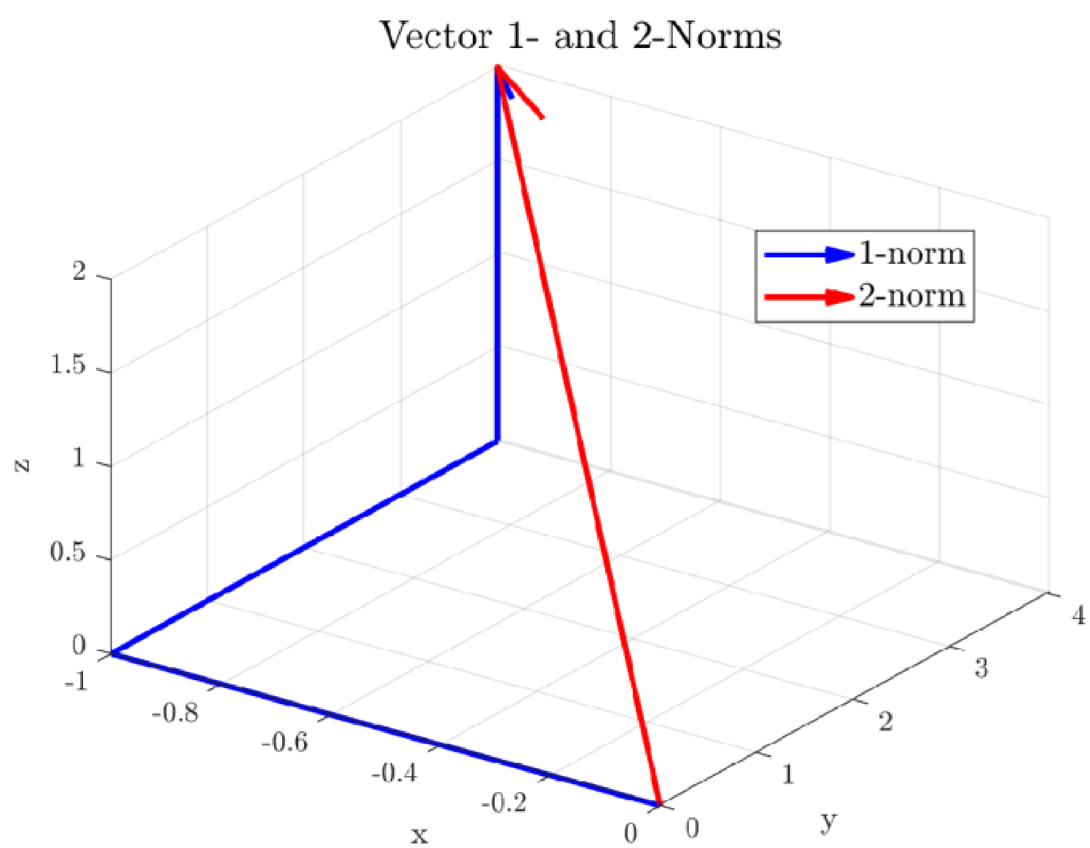
Example: Given $X = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$, calculate the 1- norm and 2-norm.

- **1-norm:**

$$||X||_1 = \sum_{i=1}^{n} |x_i| = 1 + 4 + 2 = \boxed{7}$$

- 2-norm:

$$||X||_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \sqrt{(-1)^2 + 4^2 + 2^2} = \boxed{4.58}$$



Forbenius norm

$$\|\boldsymbol{A}\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}$$

The frobenius norm is the extension of L2 norm

Remember that given a vector $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ then the L2 norm is $\|x\|_2 = \sqrt{2^2 + 1^2}$ The Frobenius Norm is the equivalent matrix version

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\|A\|_F = \sqrt{a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2}$$

It means we square every element in the matrix and square root it sum

Example Given

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Then

$$\|A\|_F = \sqrt{1^2 + 2^2 + 1^2 + 3^2}$$

The symbol for determinant of A is $|A|$, the absolute value symbol

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \mathrm{d} & \mathrm{e} & \mathrm{f} \\ \mathrm{g} & \mathrm{h} & \mathrm{i} \end{vmatrix} = \mathrm{a}ei - \mathrm{afh} + b\,di - bgf + cdh - cge$$

Given a 2x2 and 3x3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{72} \end{bmatrix} \Rightarrow$$

The determinant of 2x2 matrix

$$|A| = a_{11} * a_{22} - a_{12*}a_{21}$$

The determinant of a 3x3 matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow$$

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21}a_{23} \\ a_{31}a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

For matrices bigger than 3x3 we normally use computers to calculate their determinant

But we should at least know the equations for 2x2 and 3x3 matrices.

Knowing the determinant is useful because $|A| = 0$ implies

1. There is no unique solution.
2. The matrix is singular.
3. Inv(A) does not exist.
4. Column vectors are linearly dependent.

For smaller matrices it is easier to find the determinant than Gaussian elimination.

The general formula for finding the det of any square matrix is

$$\det(A) = \sum(-1)^{i+j} a_{ij} \det(M_{ij})$$

Example: Find the determinant for each matrix

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 2 \\ -3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \quad |A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} a_{23} \\ a_{31} a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \Rightarrow$$

$$\det\begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix} = 2\begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} - (-3)\begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} + 1\begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix}$$

$$= 2[0 - (-4)] + 3[10 - (-1)] + 1[8 - 0]$$

$$= 8 + 33 + 8$$

$$\det\begin{bmatrix} 1 & 3 & 2 \\ -3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix} = 1\begin{vmatrix} -1 & -3 \\ 3 & 1 \end{vmatrix} - 3\begin{vmatrix} -3 & -3 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} -3 & -1 \\ 2 & 3 \end{vmatrix}$$

$$= 1[-1 - (-9)] - 3[-3 - (-6)] + 2[-9 - (-2)]$$

$$= 8 - 9 - 14 = -15$$

Practice find the determinant and measurement of the following matrices

1) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

2) $\begin{bmatrix} 3 & 0 \\ -1 & -2 \end{bmatrix}$

3) $\begin{bmatrix} -1 & 0 \\ 4 & -3 \end{bmatrix}$

4) $\begin{bmatrix} 5 & -3 \\ 7 & 4 \end{bmatrix}$

5) $\begin{bmatrix} 8 & 23 \\ 9 & 12 \end{bmatrix}$

6) $\begin{bmatrix} 0 & 2 \\ 5 & 6 \end{bmatrix}$

7) $\begin{bmatrix} 4 & -8 \\ 3 & -2 \end{bmatrix}$

8) $\begin{bmatrix} -2 & 5 \\ 6 & 3 \end{bmatrix}$

9) $\begin{bmatrix} 4 & 8 \\ 7 & -3 \end{bmatrix}$

10) $\begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 1 \\ 4 & -3 & 3 \end{bmatrix}$

11) $\begin{bmatrix} 4 & 1 & 0 \\ 8 & 2 & 2 \\ 12 & 3 & -5 \end{bmatrix}$

12) $\begin{bmatrix} 0 & -11 & 5 \\ 3 & 0 & 1 \\ 5 & 3 & 4 \end{bmatrix}$

13) $\begin{bmatrix} 2 & 1 & 5 \\ 0 & 5 & 0 \\ 7 & 3 & 0 \end{bmatrix}$

14) $\begin{bmatrix} \frac{-1}{2} & \frac{3}{2} & \frac{3}{4} \\ 5 & -1 & 5 \\ 0 & 8 & 3 \end{bmatrix}$

15) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

16) $\begin{bmatrix} \frac{2}{3} & 5 & 0 \\ 2 & \frac{3}{2} & 2 \\ -1 & 6 & 1 \end{bmatrix}$

Previously we saw that matrix transformations vectors

- We have so far seen a lot of matrix transformation.
- $0x = 0$ like multiplying a number by 0
- $Ix = x$ like multiplying a number by 1
- $2Ix = 2x$ like multiplying a number by 2
- $R_\pi x = -x$ like rotating by π, 1→-1

All these transformation allow us to almost think of a x and matrix A just as number.

These transformation are from vectors perspective.

- It tell us how a matrix can change a vector

Now we are going to look from the matrix perspectives.

- Given a matrix what special number and vectors are there

Special vectors: Eigen vectors

Special number: Eigen value

Eigne values and eigne vectors are very special

So However, we have learned the concept of independence

If a set of bases is independent there is no redundancy

If a set of basis is dependent, there is redundancy

The existence of redundancy is what allow data compression

Eigenvalues and vectors allow to identify the minimum set of vectors that could span the same rank

For example, you could start off with 100 basis vectors. Once you have identified the eigenvectors and its associated eigenvalues, we might get

$$(v_1, \lambda_1 = 0.5), (v_2, \lambda_2 = 0.3)(v_3, \lambda_3 = 0.2), (v_4, \lambda_4 = 0.0001), (v_5, \lambda_5 = 0.0), (v_6, \lambda_6 = 0.0), (\dots, \lambda_{100} = 0.0)$$

The eigen values associated with the eigenvectors tells you the size of the vector in contributing to the span

Eigenvalues and vectors

Given a matrix A the eigenvectors of A are vectors that treat A as a number. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Notice that if you multiply it by 2 vectors it is equivalent to a number multiplying the vector

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

These vectors $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $are\ special\ to\ A\ because\ they\ treat\ as\ a\ consntant\ multiplier\ of\ 1\ and\ 3$

- these vectors are called eigne vectors
- The multiplier values are called the eigne values

- The definition of eigenvalues and vectors is therefore

$$Av = \lambda v$$

- Our goal today is to learn how to find these vectors and value

How do we find the eigen values

If we follow the definition of the eigenvalues are eigenvectors we have

$$Av = \lambda v$$

This implies that we have

$$Av - \lambda I v = 0$$

Since we have a 2x2 where matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$$

Then the expression becomes

$$\left| \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) v \right| = 0$$

$$\left| \begin{bmatrix} a_{11} - \lambda & a_{21} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \right| = 0$$

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12^*} a_{21}$$

This expression is called the characteristic equation/ polynomial.

The eigenvalue is the solution of the characteristic polynomial.

Example

Consider the matrix we previously saw

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\left|\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right| = (2-\lambda)(2-\lambda) - 1$$

$$\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$$

From this, we can see that the solution for the characteristic equation is 3 and 1 they eigen values

Once we have the eigen values we can substitute it back in the equation let's first look at the case where the $\lambda = 1$

$$A - \lambda I = 0$$

$$\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} v = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} v = 0$$

This implies that the null space the solution for v, we know that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{e1 - e2} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

This implies that

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$v_1 = -v_2$$

$$V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} v_2$$

The space spanned by [-1 1] is the null space and the eigenvector corresponding to $\lambda = 1$.

Consider the matrix we previously saw

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\left|\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right| = (2-\lambda)(2-\lambda) - 1$$

$$\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$$

From this, we can see that the solution for the characteristic equation is 3 and 1 they eigen values.

Once we have the eigen values we can substitute it back in the equation let's first look at the case where the $\lambda = 3$

$$A - \lambda I = 0$$

$$\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} v = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} v = 0$$

This implies that the null space the solution for v, we know that

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \overrightarrow{e1 + e2} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

This implies that

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$-v_1 = -v_2$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_2$$

The space spanned by [1 1] is the null space and the eigenvector corresponding to $\lambda = 3$.

Final Answers:

Eigenvalues:

$$\lambda 1 = 1, \qquad \lambda 2 = 3$$

Corresponding Eigenvectors: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Obtain different eigenvectors

Our eigenvalues and vectors for $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ are

$\lambda_1 = 1, v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\lambda_2 = 3, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Ad a standard when we are asked for the eigenvalue we present them in a sorted fashion (normally from the largest to the smallest)

So we have

$$\lambda = (3,1)$$

$$v = [v_2, v_1] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Notice that v2 is the first column Lastly notice that v1 and v2 are not the solutions the solutions is the span of the v1 and v2.

Example Given the matrix

$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

a) Find the eigenvalues of matrix A.
b) Find the corresponding eigenvectors for each eigenvalue.

SOLUTION

Step 1: Find the Eigenvalues

$$\mathrm{Av} = \lambda \mathrm{v}$$

IF a matrix is multiplied with identity matrix nothing will change

$$\mathrm{Iv} = \mathrm{v}$$

$$\mathrm{Av} = \lambda \mathrm{Iv}$$

$$(\mathrm{A} - \lambda \mathrm{I})\mathrm{v} = 0$$

We solve the characteristic equation:

$$det(A-\lambda I)=0 \qquad \begin{bmatrix}4 & 1\\ 2 & 3\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}4 & 1\\ 2 & 3\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}=\begin{bmatrix}4-\lambda & 1\\ 2 & 3-\lambda\end{bmatrix}$$

$$det(A-\lambda I)=0 \qquad (4-\lambda)(3-\lambda)-(2)(1)=(4-\lambda)(3-\lambda)-2$$

$So,$

$$\det(A-\lambda I)=\lambda 2-7\lambda+12-2=\lambda 2-7\lambda+10$$

Set the determinant to zero:

$$\lambda 2-7\lambda+10=0$$

Factor the quadratic:

$$(\lambda-5)(\lambda-2)=0$$

Eigenvalues:

$$\lambda 1=5\,, \qquad \lambda 2=2$$

For λ=5

Solve $(A-5I)=0$ $\begin{bmatrix}4-5 & 1\\ 2 & 3-5\end{bmatrix}=\begin{bmatrix}-1 & 1\\ 2 & -2\end{bmatrix}$

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} [v] = 0$$

Solve: $$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the first equation: $-v1 + v2 = 0 \Rightarrow v2 = v1$

So the eigenvectors corresponding to $\lambda = 5$ is

$$v1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad v1\ not\ equal\ to\ 0$$

For λ=2

Solve (A−2I)=0 $$\begin{bmatrix} 4-2 & 1 \\ 2 & 3-2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

Solve: $$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This reduces to:

$$2x + y = 0 \Rightarrow \quad y = -2x$$

So the eigenvectors corresponding to λ=2 $$v1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Final Answers:

Eigenvalues:

$$\lambda 1 = 5, \qquad \lambda 2 = 2$$

Corresponding Eigenvectors: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Example

Let $\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

a) Find the eigenvalues of matrix A.

b) Find the eigenvectors corresponding to each eigenvalue.

Solution:

Step 1: Find Eigenvalues

Solve the characteristic equation:

det$(A - \lambda I)$=0

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

$$det\ (A - \lambda I) = (4-\lambda)(2-\lambda)(3-\lambda) = 0$$

So the eigenvalues are:

$$\lambda 1 = 4 \qquad \lambda 2 = 2 \qquad \lambda 3 = 3$$

Step 2: Find Eigenvectors

For λ=4

Solve: $(A - 4I)x = 0$

$$A - 4I = \begin{bmatrix} 4-4 & 1 & 0 \\ 0 & 2-4 & 0 \\ 0 & 0 & 3-4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

From the system:

- x_2=0
- x_3=0
- x_1 is free

So one eigenvector is: $\boldsymbol{v_1} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$

For λ=2

$$A - 2I = \begin{bmatrix} 4-2 & 1 & 0 \\ 0 & 2-2 & 0 \\ 0 & 0 & 3-2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solve:

- $2x_1+x_2=0 \Rightarrow x_2=-2x_1$
- $x_3=0$

So one eigenvector is:

$$\boldsymbol{v_2} = \begin{bmatrix} \mathbf{1} \\ \mathbf{-2} \\ \mathbf{0} \end{bmatrix}$$

For λ=3

Solve:

- $x_2=0$
- x_1 is free
- x_3 is free

So one eigenvector is:

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Final Answer:

Eigenvalues: λ=4,2,3

Corresponding Eigenvectors:

For $\lambda = 4$ is $\boldsymbol{v_1} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$

For $\lambda = 2$ is $\qquad \boldsymbol{v_2} = \begin{bmatrix} \mathbf{1} \\ \mathbf{-2} \\ \mathbf{0} \end{bmatrix}$

For $\lambda = 3$ is $\qquad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Example

Let $\qquad A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

1) Find the **Eigenvalues**

2) Find the **Eigenvectors** for each eigenvalue

Solution

step 1: Find the Characteristic Polynomial

We solve:

det(A−λI)=0

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix}$$

$$\det(A-\lambda I)=0 \Rightarrow \det\left[\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix}\right]$$

or

$$(1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ 2 & 3-\lambda \end{vmatrix} + (-1)\begin{vmatrix} 1 & 2-\lambda \\ 2 & 2 \end{vmatrix}$$

$$(1-\lambda)[\lambda^2 - 5\lambda + 4] - 0 - 1[2 - 2(2-\lambda)]$$

After few steps we get

$$-\lambda^3 + 6\lambda^2 - 11\,\lambda + 6 \ldots\ldots\ldots\ldots\ldots (1)$$

This gives the characteristic polynomial: $\lambda^3 - 6\lambda^2 + 11\,\lambda - 6$

Step 2: Solve for Eigenvalues

Factoring: $\lambda^3 - 6\lambda^2 + 11\,\lambda - 6 = (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$

How we can factor a cubic equation

Put $\lambda = 1$ in eqution 1

$So\ 1 - 6 + 11 - 6 = 0$

$\lambda = 1$ is root of equation (1)

so the remaining root of 1 are

$\lambda^2 - 5\lambda + 6 = 0$ or $(\lambda - 3)(\lambda - 2) = 0$

$\lambda_2 = 3, \lambda_3 = 2,$ as 1 is also the root of 1 equation then all root of 1 are

$$, \lambda_2 = -1\ \ , \lambda_2 = 3, \lambda_3 = 2,$$

Eigenvalues:

λ_1=1 **λ_2=2** **λ_3=3**

Step 3: Find Eigenvectors

We solve $(A - \lambda I)x$ for each eigenvalue:

For λ=1:

Solve: $(A - 1I) = 0 \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

So now computing $(A - 1I)X$=$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Reduced Row Echelon Form (RREF) gives:

$$e1 \leftrightarrow e3 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2e_1 + e_3 \rightarrow e_3 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}$$

So $x_1 + x_2 + x_3 = 0$

And $-x_3 = 0$ that mean $x_3 = 0$

And $x_1 + x_2 + x_3 = 0$ putting the value of $x_3 = 0$ in equation

We get $x_1 + x_2 + 0 = 0 \Rightarrow x_1 = -x_2$

Take $x_2 = 1 \quad \Rightarrow \quad x_1 = -1$

The corresponding eigen vector is $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Find the eigen vectors corresponding to eigen values $\lambda_2 = 2$ $\lambda_3 = 3$

Eigen decomposition Only Works on Square Matrix

- Eigen decomposition only work on square matrices
- This is major drawback since most of the data are not perfectly square
- But that’s okay you can always turn a non-square matrix into square matrix by multiplying it with itself transposed

$$A \rightarrow AA^T$$

- You might be thinking but then we are not getting the eigenvalues and vectors of A anymore we are getting them for $A^T A$
- This is okay the goal was never to get the eigenvectors and values of A
- The goal was to identify the minimum set of basis that span the same span as the column of space of A
- We discovered that the orthonormal basis generated by $A^T A$ has the same span
- This is why we can generate the appropriate orthonormal basis with $A^T A$ instead. The eigenvectors will span the same space anyway

Practice

1) Find Eigen vectors of the given matrix A

$$A = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

the matrix has the characteristics equation-

$$|\lambda I - A| = \begin{vmatrix} \lambda+4 & -1 & 0 \\ 0 & \lambda+3 & -1 \\ 0 & 0 & \lambda+2 \end{vmatrix}$$

$$= (\lambda+4)(\lambda+3)(\lambda+2) = 0$$

Find the corresponding eigen vectors to , $\lambda_2 = -4$, $\lambda_2 = -3, \lambda_3 = -2$

SOLUTION:

For $\lambda_3 = -2$ the eigen vector is $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

For , $\lambda_2 = -4$, $\lambda_2 = -3$ the corresponding eigen vectors are $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

2) Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

The characteristic polynomial is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6$$

Its factors are

$$(\lambda - 1)(\lambda - 2)(\lambda - 3)$$

So the eigenvalues are 1, 2, and 3.

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

3) Find the eigenvalues and corresponding eigenvectors of

$$A=\begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{-1} & \mathbf{1} & \mathbf{0} \\ \mathbf{5} & \mathbf{3} & \mathbf{3} \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda-2 & 0 & 0 \\ 1 & \lambda-1 & 0 \\ -5 & -3 & \lambda+3 \end{vmatrix} = (\lambda-2)(\lambda-1)(\lambda+3) = 0$$

$$\Rightarrow \lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = -3$$

Find the corresponding eigen vectors to each λ=1,2,-3

4) Determine the characteristic root(Eigen values) and characteristic vectors(eigen vectors) of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

the Eigen values of A are **For λ=0,3,15**

find corresponding eigen vector **characteristic vectors**

5) Determine the Eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

the Eigen values of A are **For λ=2,2,8**

find corresponding eigen vector **characteristic vectors**

6) Find the Eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

the Eigen values of A are **For λ=2,3,5**

find corresponding eigen vector **characteristic vectors**