Linear Algebra Lecture Note 5

The Row Echelon Form (REF), The Reduced Row Echelon Form (RREF)
A linear system that is not possible, More Equation than Variables, infinite solutions

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The Row Echelon Form (REF).

We have achieved the REF when

1. The bottom triangle are all 0s
2. The diagonal are all 1s
3. Notice that the upper triangle does not have to have all 0s

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & -3 & -5 \\ 0 & 0 & 1 & -5.5 & -11 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

This is the minimum manipulation that allows us to easily obtain the solutions.

If we write the equations back out, we get.

$$1x_1 + 0x_2 + 0x_3 + 1x_4 = 3$$

$$0x_1 + 1x_2 + 1x_3 - 3x_4 = -5$$

$$0x_1 + 0x_2 + 1x_3 - 5.5x_4 = 11$$

$$0x_1 + 0x_2 + 0x_3 + 1x_4 = 2$$

1. Notice that x4 is already known by e4
2. If we know x4 it is very easy to get x3 by e3
3. Similarly, if we know x4, x3, we can easily get e2
4. Finaly since we know x2, x3, x4 we can easily get x1 from e1

Practice

1) Convert the following matrix to Row Echelon Form (R.E.F)

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & 7 \\ -1 & -2 & 5 & -1 \end{bmatrix}$$

2) Convert the following matrix to Row Echelon Form (R.E.F)

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

3)Convert the following matrix to Row Echelon Form (R.E.F)

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 4 \\ -1 & -3 & 1 \end{bmatrix}$$

The Reduced Row Echelon Form (RREF)

Once we get the REF then it is very easy to obtain the final version, we want the Reduced Row Echelon Form (RREF)
Looking at the previous matrix by using e4 we can easily 0 out the column 4

- $\boldsymbol{e_1 - e_4 \to e_1}$ $\quad\quad$ $\boldsymbol{.5.5e_4 + e_3 \to e_3}$
- $\boldsymbol{e_2 + 3e_4 \to e_2}$

- Finaly we can remove the extra 1 in row 2 by $e_2 - e_3 \to e_2$ now we have the RREF

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

We now know that the final solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

Practice

1) Reduce the matrix to **R.R.E.F**

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$$

2) Reduce the matrix to **R.R.E.F**

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 6 & 9 \end{bmatrix}$$

3) Reduce the matrix to **R.R.E.F**

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & 1 \\ 2 & -1 & 1 & 3 \end{bmatrix}$$

Update of Notation

In traditional textbooks, the notation for converting one matrix into another matrix is like $e_1 - e_2$ for example

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 2 & 1 \end{array}\right] \xrightarrow{e_1 - e_2} \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

- This notation was how I was taught
- However, after our last class, I realized this notation is the reason why I was so confused
- This is why going forward, we are going to follow a different notation (instead of only declaring the operation, we are also going to identify the row it replace)
- So the same operation is going to be

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 2 & 1 \end{array}\right] \xrightarrow{e_1 - e_2 \to e_1} \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

We learned Gaussian Elimination in the last Lecture

$$\begin{cases} x_1 + 2x_2 = 1 \\ 2x_1 + 4x_2 = 3 \end{cases}$$

a linear system may not always have a unique solution or even have a solution at all.

Today, we are going to look at the 3 possible cases.

1. A good unique solution (learned in the last class)
2. No solution
3. Infinite solutions

Take out a piece of paper and try to solve this problem and identify which of the 3 cases is associated with this problem.

A linear system that is not possible

$$\begin{cases} x_1 + 2x_2 = 1 \\ 2x_1 + 4x_2 = 3 \end{cases}$$

You could have also seen that by multiplying e1 by 2 you get

$$\begin{cases} 2x_1 + 4x_2 = 2 \\ 2x_1 + 4x_2 = 3 \end{cases}$$

Since $2x_1 + 4x_2 = 2$ can not simultaneously be 2 and 3, it is not possible

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4 & 3 \end{array}\right] \xrightarrow{2e_1 - e_2 \to e_2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & -1 \end{array}\right]$$

The second of the augmented matrix states that

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} x_1 + 2x_2 = 1 \\ 0x_1 + 0x_2 = -1 \end{cases}$$

No solution can satisfy the linear system of equations

and therefor

$$0x_1 + 0x_2 = -1$$

or 0 = -1 which is not possible

Example of More Equation than Variables

$$\begin{cases} x_1 + 3x_2 = 1 \\ 2x_1 + x_2 = -3 \\ 2x_1 + 2x_2 = -2 \end{cases} \xrightarrow[-2e_1 + e_3 \to e_3]{-2e_1 + e_2 \to e_2} \begin{cases} x_1 + 3x_2 = 1 \\ -5x_2 = -5 \\ -4x_2 = -4 \end{cases}$$

$$\xrightarrow{-{}^{4}/_{5}\, e_2 + e_3 \to e_3} \begin{cases} x_1 + 3x_2 = 1 \\ -5x_2 = -5 \\ 0 = 0 \end{cases}$$

- we have an equation where $0 = 0$, this is not a contradiction, so it is okay.
- A contradiction only happen if there is a contradicting logic, for example
  - $5 = 2$
  - two equations that simultaneously equals to different things.

$$\begin{cases} 2x_1 = 2 \\ 2x_1 = 3 \end{cases}$$

Notice the key difference between the 2 cases

look at the 2 cases this is what we just saw, it has a solution

$$\begin{cases} x_1 + 3x_2 = 1 \\ -5x_2 = -5 \\ -4x_2 = -4 \end{cases} \xrightarrow{-\,^4/_5\, e_2 + e_3 \to e_3} \begin{cases} x_1 + 3x_2 = 1 \\ -5x_2 = -5 \\ 0 = 0 \end{cases}$$

This case does not have a solution

$$\begin{cases} x_1 + 2x_2 = 1 \\ 2x_1 + 4x_2 = 3 \end{cases} \xrightarrow{2e_1 - e_2 \to e_2} \begin{cases} x_1 + 2x_2 = 1 \\ 0x_1 + 0x_2 = -1 \end{cases}$$

Obvious contradiction

If there is a contradiction you can not have a solution.

In this example, we have infinite solutions

$$\begin{aligned} x + y + z - w &= 1 \\ y - z + w &= -1 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

Variables in the beginning of the echelon form are called the **Pivot/leading** terms

The variables that are not leading terms are called the **free variables**

After reducing the system of linear equations into the echelon form, if you have more variables than equations, then it means you have **infinite solutions**

Here is another example

$$\begin{bmatrix} 0 & x_2 & x_3 & x_4 & x_5 \\ 0 & 0 & 0 & x_4 & x_5 \end{bmatrix}$$

x2 and x4 are leading variables
And x3 and x 5 are free variables

Example

Given the linear system of equations

1. We first convert it into augmented matrix format
2. We then convert it to RREF

$$\begin{cases} x + y + z - w = 1 \\ y - z + w = -1 \\ 3x + 6z - 6w = 6 \\ -y + z - w = 1 \end{cases} \xrightarrow{\text{Augmented Matrix}} \begin{bmatrix} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 3 & 0 & 6 & -6 & 6 \\ 0 & -1 & 1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 3 & 0 & 6 & -6 & 6 \\ 0 & -1 & 1 & -1 & 1 \end{bmatrix} \xrightarrow[-3e_1 + e_3 \to e_3,]{e_2 + e_4 \to e_4,\ 3e_2 + e_3 \to e_3} \begin{bmatrix} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 + x_2 + x_3 - x_4 = 1 \\ x_2 - x_3 + x_4 = -1 \end{cases} \qquad \begin{cases} x_1 = 1 - x_1 - x_3 + x_4 \\ x_2 = -1 + x_3 - x_4 \end{cases}$$

Example

$$\begin{cases} x_1 + 3x_3 = 7 \\ 2x_1 + x_2 = 2 \\ 2x_2 + 4x_1 + x_3 = 4 \end{cases} \xrightarrow{\text{Augmented matrix}} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 2 & 1 & 0 & 2 \\ 2 & 4 & 1 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 2 & 1 & 0 & 2 \\ 2 & 4 & 1 & 4 \end{array}\right] \xrightarrow[-2e_1 + e_2 \rightarrow e_2]{-2e_1 + e_3 \rightarrow e_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & -6 & -12 \\ 0 & 4 & -5 & -10 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & -6 & -12 \\ 0 & 4 & -5 & -10 \end{array}\right] \xrightarrow{-4e_2 + e_3 \rightarrow e_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & -6 & -12 \\ 0 & 0 & 19 & 38 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & -6 & -12 \\ 0 & 0 & 19 & 38 \end{array}\right] \xrightarrow{\frac{e_4}{19}} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & -6 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & -6 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[-3e_3 + e_1 \to e_1]{6e_3 + e_2 \to e_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Example

$$\begin{cases} x_1 + 3x_3 = 1 \\ 2x_1 + x_2 + x_3 + x_4 = 3 \\ 2x_1 + 4x_2 + x_3 = 2 \\ x_3 + x_4 = 1 \end{cases} \xrightarrow{\text{Augmented matrix}} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 2 & 1 & 1 & 1 & 3 \\ 2 & 4 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 2 & 1 & 1 & 1 & 3 \\ 2 & 4 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[-2e_1 + e_2 \to e_2]{-2e_1 + e_3 \to e_3} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -5 & 1 & 1 \\ 0 & 4 & -5 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -5 & 1 & 1 \\ 0 & 4 & -5 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{-4e_2 + e_3 \to e_3} \begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -5 & 1 & 1 \\ 0 & 0 & 15 & -4 & -4 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix}1&0&3&0&1\\0&1&-5&1&1\\0&0&15&-4&-4\\0&0&1&1&1\end{bmatrix}\xrightarrow{e_{3\leftrightarrow}e_4}\begin{bmatrix}1&0&3&0&1\\0&1&-5&1&1\\0&0&1&1&1\\0&0&15&-4&-4\end{bmatrix}$$

$$\begin{bmatrix}1&0&3&0&1\\0&1&-5&1&1\\0&0&1&1&1\\0&0&15&-4&-4\end{bmatrix}\xrightarrow{-15e_3+e_4\rightarrow e_4}\begin{bmatrix}1&0&3&0&1\\0&1&-5&1&1\\0&0&1&1&1\\0&0&0&-19&-19\end{bmatrix}$$

$$\begin{bmatrix}1&0&3&0&1\\0&1&-5&1&1\\0&0&1&1&1\\0&0&0&-19&-19\end{bmatrix}\xrightarrow{e_4/-19}\begin{bmatrix}1&0&3&0&1\\0&1&-5&1&1\\0&0&1&1&1\\0&0&0&1&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&3&0&1\\0&1&-5&1&1\\0&0&1&1&1\\0&0&0&1&1\end{bmatrix}\xrightarrow[e_3-e_4\rightarrow e_3]{e_2-e_4\rightarrow e_2}\begin{bmatrix}1&0&3&0&1\\0&1&-5&0&0\\0&0&1&0&0\\0&0&0&1&1\end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[e_2 + 5e_3 \to e_2]{e_1 - 3e_3 \to e_1} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Example

$$\begin{cases} 2x - 2y & = 0 \\ z + 3w & = 2 \\ 3x - 3y & = 0 \\ x - y + 2z + 6w & = 4 \end{cases} \xrightarrow{\text{Augmented matrix}} \begin{bmatrix} 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 2 \\ 3 & -3 & 0 & 0 & 0 \\ 1 & -1 & 2 & 6 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 2 \\ 3 & -3 & 0 & 0 & 0 \\ 1 & -1 & 2 & 6 & 4 \end{bmatrix} \xrightarrow{e_{1\leftrightarrow}e_4} \begin{bmatrix} 1 & -1 & 2 & 6 & 4 \\ 0 & 0 & 1 & 3 & 2 \\ 3 & -3 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 6 & 4 \\ 0 & 0 & 1 & 3 & 2 \\ 3 & -3 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-2e_1 + e_4 \rightarrow e_4]{-3e_1 + e_3 \rightarrow e_3} \begin{bmatrix} 1 & -1 & 2 & 6 & 4 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & -6 & -18 & -12 \\ 0 & 0 & -4 & -12 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 6 & 4 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & -6 & -18 & -12 \\ 0 & 0 & -4 & -12 & -8 \end{bmatrix} \xrightarrow[e_{4/-4}]{e_{3/-6}} \begin{bmatrix} 1 & -1 & 2 & 6 & 4 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 6 & 4 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 & 2 \end{bmatrix} \xrightarrow[e_4 - e_3 \to e_4,\, e_1 - 2e_3 \to e_1]{e_2 - e_3 \to e_2} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 - x_2 = 0 \\ x_3 + 3x_4 = 2 \end{cases} \Rightarrow \begin{matrix} x_1 = x_2 \\ x_3 = 2 - 3x_4 \end{matrix}$$

Practice

1) Solve the following system of equations. State whether it has one solution, no solution, or infinitely many solutions

$$\begin{aligned} x + 2y - z &= 4 \\ 2x + 4y - 2z &= 8 \\ 3x + 6y - 3z &= 12 \end{aligned}$$

3) Solve the following system of equations. State whether it has one solution, no solution, or infinitely many solutions

$$\begin{aligned} x + 2y + 3z &= 0 \\ 2x + 4y + 6z &= 0 \\ -x - 2y - 3z &= 0 \end{aligned}$$

3)Solve the following system of equations. State whether it has one solution, no solution, or infinitely many solutions

$$\begin{aligned} x + y + z &= 4 \\ 2x + 2y + 2z &= 8 \\ x + y + z &= 4 \end{aligned}$$