Linear Algebra
Lecture Note1

Addition, Subtraction and Multiplication of Vectors

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A vector is a set of numbers enclosed in a bracket.

Example 1

$x = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$

Example 2

$\mathbf{x} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$

We denote the dimension of vectors with the notation

$\mathbf{x} \in R^2$

$x \in R^3$

Example 3

$y = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

Example 4

$\alpha = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}$

For this class: we will use cursive or bold font to denote a vector but most of the time we will use cursive notation

The notation $x \in R^3$ says that the vector x will have three real numbers of **dimensions.**

Of course, a vector does not need to be denoted by a letter x it can be any letter like y or even Greek , α

A vector by itself does not tell as anything

$$x = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

🧑‍🏭We human should attach meaning to that numbers. For example these numbers can be

1. The number of point score by three people
2. The number of programming languages you have learned in the last three months

The possibilities are endless…………

If we tell that the vector has the values of [1, 5, 3], it does not really mean anything to you.

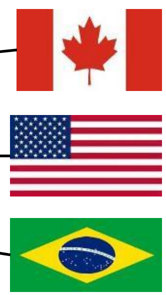
Meaning of an object only arise from the consciousness that is capable of giving meaning . For without it, itself can not exist.

Once meaning is given to a vector, it becomes data

Giving meaning to vector…..

These are the scores of three soccer teams

$$x = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$



The three numbers are no longer just numbers they tell us a lot about the three teams

Once a vector become data we can. We can pose various queries(questions) about the information it holds. For example…..

1. What is the highest score?
2. What is the lowest score?
3. What is the average score?

We are going to call the three questions

q1,q2,q3

So once a vector becomes data we can use a queries to take the vector and return a meaningful value.

q1 using x $\longrightarrow$ meaningful number

{here q1 will give us number 5}

We can translate the queries into mathematical language

q1 using x ---> meaningful number

Think of function like a query that take x and return a meaningful value

q1(x) = some meaningful value

q1 here is equal to the max function

$$q_1\left(\begin{bmatrix}1\\5\\3\end{bmatrix}\right) = \max(x) = 5$$

This analogy extend to other query q2, q3

$$q_2\left(\begin{bmatrix}1\\5\\3\end{bmatrix}\right) = \min(x) = 1$$

Here we, establish that a function/query is just a way to extract information from vector

$$q_3\left(\begin{bmatrix}1\\5\\3\end{bmatrix}\right) = \arg(x)$$

Vector addition

In linear algebra, vectors are often represented as **column matrices** or **row matrices**, and vector addition follows **component-wise addition**

For any **n-dimensional vectors**:

$$\mathbf{A} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_n \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_n \end{bmatrix}$$

The sum of these vectors is:

$$\mathbf{R} = \mathbf{A} + \mathbf{B} = \begin{bmatrix} A_1 + B_1 \\ A_2 + B_2 \\ A_3 + B_3 \\ \vdots \\ A_n + B_n \end{bmatrix}$$

This equation works for **any dimension** n.

Example in 2D (n = 2)

$$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, y = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, then\ \ x + y = \ x + y = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Example in 3D (n = 3)

$$\mathbf{A} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 1+4 \\ 2+(-1) \\ 3+2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix}$$

PRACTICE

FIND A+B

$$\mathbf{A} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 \\ -4 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 \\ 6 \\ -1 \\ 5 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 6 \\ -2 \\ 8 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -3 \\ 4 \\ -5 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -2 \\ 5 \\ 0 \\ -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 4 \\ -1 \\ 7 \\ 2 \end{bmatrix}$$

Vector subtraction

Vector subtraction follows the same rules as vector addition but involves **subtracting** corresponding components

$$\mathbf{A} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_n \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_n \end{bmatrix}$$

$$\mathbf{R} = \mathbf{A} - \mathbf{B} = \begin{bmatrix} A_1 - B_1 \\ A_2 - B_2 \\ A_3 - B_3 \\ \vdots \\ A_n - B_n \end{bmatrix}$$

2D Vector Subtraction

$$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, y = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, then\ x - y = \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Find $\mathbf{A} - \mathbf{B}$, where:

$$\mathbf{A} = \begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 10 \\ -3 \\ 7 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 \\ -3 \\ 6 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 4 \\ -1 \\ 3 \\ 7 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 \\ 5 \\ -2 \\ 4 \end{bmatrix}$$

The transpose operation

The transpose operation is when you flip the vector

$$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, then\ x^T = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, then\ y^T = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$$

$$z = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}, then\ z^T = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Practice

Find the transpose **of the following column vector:**

$$\mathbf{A} = \begin{bmatrix} 5 \\ -2 \\ 8 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} -7 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

Find the transpose of the row vector

$$\mathbf{B} = \begin{bmatrix} 3 & 6 & -4 & 2 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 9 & -5 & 2 & 6 & -1 \end{bmatrix}$$

Vector Multiplication, Hadamard Product, and inner Product

Vector Multiplication Hadamard Product

$$\mathbf{A} \circ \mathbf{B} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \circ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 b_1 \\ a_2 b_2 \\ a_3 b_3 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, y = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \; then \; x \odot y = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \odot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Practice

$$\mathbf{A} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} -2 \\ 0 \\ 5 \\ 3 \end{bmatrix}$$

Vector dot Product

For two vectors **A** and **B** in $\mathbb{R}^n$:

$$\mathbf{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

The **dot product** is calculated as:

$$\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i$$

$$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, y = \begin{bmatrix} 2 \\ 2 \end{bmatrix} x \cdot y = x^T y = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 1 \times 2 + 3 \times 2 = 8$$

Practice

Compute the dot product:

$$\mathbf{A} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 \\ k \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Outer Product

$$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, y = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \ then\ x \otimes y = xy^T = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \otimes \begin{bmatrix} 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 6 & 6 \end{bmatrix}$$

Vector multiplication by a constant value

Multiplication by a constant

$$\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha & x_1 \\ \alpha & x_l \end{bmatrix}$$

For example

$$3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & x & 1 \\ 3 & x & 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$