Linear Algebra
Lecture Note 6

Matrix as a function of vectors, Multiplication by 0, Matrix version of multiplying 1, multiplying by a value, Rotation of vector, Permutation Matrix, Transform multiple points simultaneously, dealing with multiple samples, averaging a vector.

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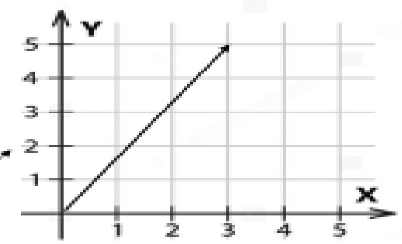
Previously: a vector is a set of numbers enclosed in a bracket

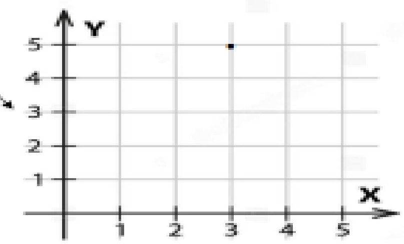
$x = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$

One way to interpret vector is to view theme as an arrow with a direction and length(magnitude)

$x = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$



Yet, another interpretation is just a point in space



Practice 1

1) Plot the vector **x**= $\begin{bmatrix} \mathbf{3} \\ \mathbf{4} \end{bmatrix}$ from the origin

2) Plot the vector **y**= $\begin{bmatrix} \mathbf{-4} \\ \mathbf{4} \end{bmatrix}$ from the origin

3) Plot the vector **w**= $\begin{bmatrix} \mathbf{2} \\ \mathbf{4} \\ \mathbf{0} \end{bmatrix}$ from the origin

4) Plot the vector z $\begin{bmatrix} \mathbf{2} \\ \mathbf{-2} \\ \mathbf{5} \end{bmatrix}$ from the origin

5) Plot the vector z $\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{5} \end{bmatrix}$ from the origin

6) Plot the vector z $\begin{bmatrix} \mathbf{-1} \\ \mathbf{-2} \\ \mathbf{5} \end{bmatrix}$ from the origin

7) Plot the vector z $\begin{bmatrix} \mathbf{-1} \\ \mathbf{5} \\ \mathbf{5} \end{bmatrix}$ from the origin

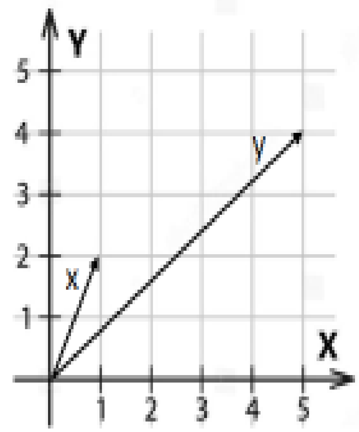
8) Plot the vector z $\begin{bmatrix} \mathbf{5} \\ \mathbf{5} \\ \mathbf{5} \end{bmatrix}$ from the origin

If you interpret a vector as an arrow what is happening if we multiply a vector bv a matrix?

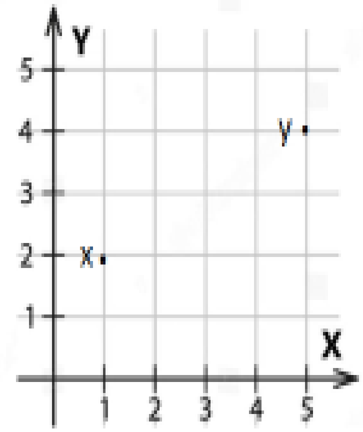
$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$Ax = y$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$



It transform to another vector



It move to another location

In both cases, the matrix A acts as a function it takes vector x as input and output vector y as output

Today's lecture is all about the different ways we can transform a vector into another vector!!!

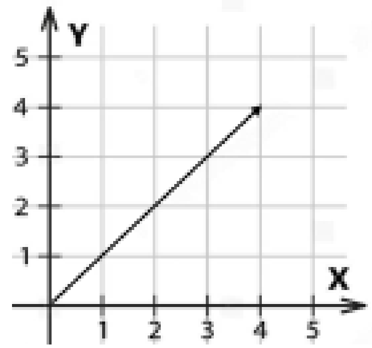
Multiplication by 0

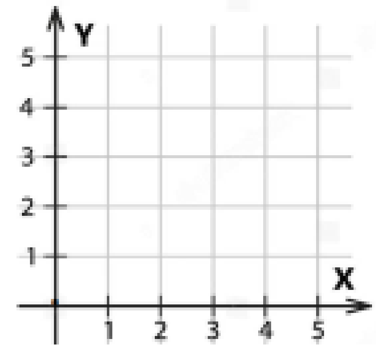
$$Ax = y$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

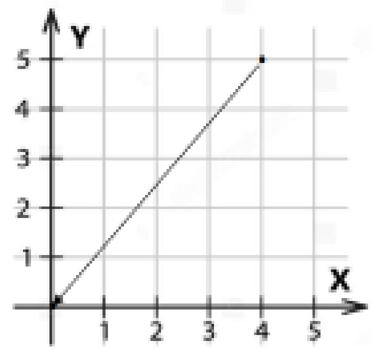
The symbol for a 0 matrix is **0** where **0**x = **0**

It is the analogy of a number multiply be 0
It always results to 0





A vector shrink to size 0



Point (4,5) become(0,0)

Just like multiplying any number by 0 gives 0, multiplying any vector by a ***zero matrix*** gives the ***zero vector***

When a vector is multiplied by a ***zero matrix***:

- It ***loses all direction and length***
- It becomes the ***zero vector***: a point at the origin (0, 0)
- In visual terms, it's like the vector ***shrinks or collapses*** into nothingness

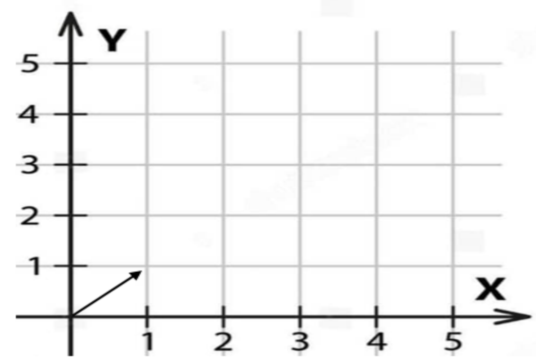
Matrix version of multiplying 1

$$Ax = y \tag{1}$$

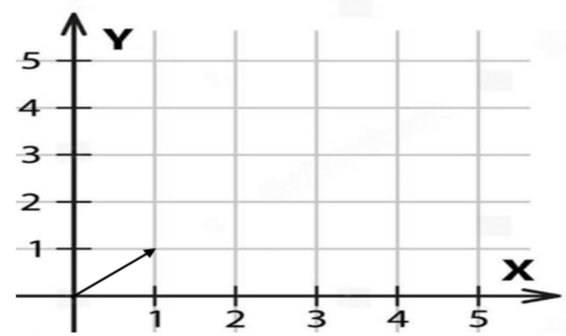
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2}$$

- A matrix that output the same vector as the input is called **The identity Matrix**
- Identity matrix is always 0's everywhere and 1's on the diagonal entries
- An identity matrix is always a square matrix
- The symbol for identity matrix is

$$\mathbf{I} \; where \; \mathbf{I}x = x \tag{3}$$



Nothing change with the vector



Multiplying by a value

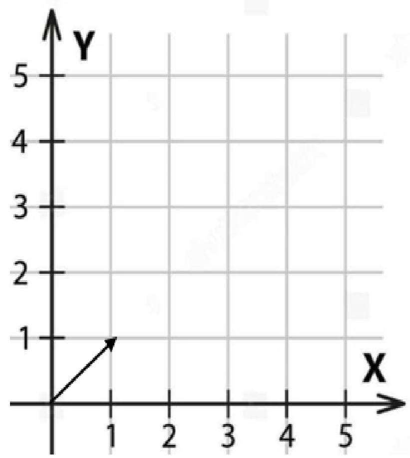
$$Ax = y$$

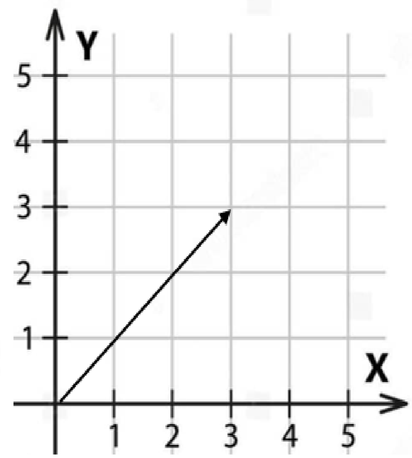
$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

A matrix that stretches a vector without any rotation is called scalar matrix, but instead of 1's in the diagonal it has a different values in the diagonal

The symbol for scalar matrix is just the number itself

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$





It is the analogy of a number multiplying by a constant value

It always result in itself but different

Practice 2

1)let

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

then find I X

2) let A vector $\vec{v}=\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ undergoes a transformation using a matrix A and comes out unchanged.
What kind of matrix is A? Why?

3) let $A=\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and vector $x = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ the find AX

4) Let

$$A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

What is the result of $A\vec{x}$?

What happens to the vector length?

5) let

$$A = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Find $A\vec{x}$.
Describe the change in direction.

6) let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Find $A\vec{x}$ and explain the geometric effect.

7) let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Find $A\vec{x}$

Rotation of vector

$$Ax = y$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = y$$

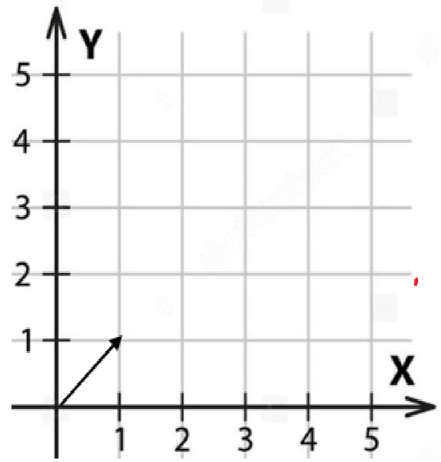
$$\begin{bmatrix} \cos 45 & \sin 45 \\ -\sin 45 & \cos 45 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

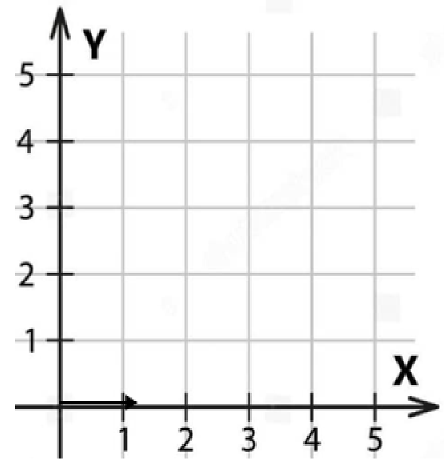
We set that as the angle of the rotation

the symbol for rotation matrix is

R_θ *where* $R_{\pi/4} = y$

y is x rotated by 45 degree





The vector is rotated by 45 degree

Practice 2

1) **Rotate vector v =** $\begin{bmatrix} \mathbf{2} \\ \mathbf{4} \end{bmatrix}$ **by 45° anti-clockwise**

2) **Rotate vector v =** $\begin{bmatrix} \mathbf{2} \\ \mathbf{4} \end{bmatrix}$ **by 45° clockwise**

3) **Rotate vector v =** $\begin{bmatrix} \mathbf{3} \\ \mathbf{7} \end{bmatrix}$ **by 90° anti-clockwise**

4) **Rotate vector v =** $\begin{bmatrix} \mathbf{3} \\ \mathbf{7} \end{bmatrix}$ **by 90° clockwise**

5) Rotate vector v = $\begin{bmatrix} \mathbf{1} \\ \mathbf{8} \end{bmatrix}$ **by 180° anti-clockwise**

6) Rotate vector v = $\begin{bmatrix} \mathbf{1} \\ \mathbf{8} \end{bmatrix}$ **by 180° anti-clockwise**

7) Rotate vector v = $\begin{bmatrix} \mathbf{3} \\ \mathbf{4} \\ \mathbf{0} \end{bmatrix}$ **by 90° anti-clockwise**

8) Rotate vector v = $\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$ by 90° clockwise

9) Rotate vector v = $\begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$ by 90° anti-clockwise

10) Rotate vector v = $\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$ by 90° clockwise

11) Rotate vector v = $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ by 180° clockwise

12) Rotate vector v = $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ by 60° anti-clockwise

13) Rotate vector v = $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$ by 120° anti-clockwise

14) Rotate vector v = $\begin{bmatrix} 8 \\ 0 \end{bmatrix}$ by 135° anti-clockwise

Permutation Matrix

The permutation matrix rearrange the order of elements of input vector.

All 6 Possible 3×3 Permutation Matrices:

1. **Identity permutation (no change)**

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. **Swap row 1 and row 2**

$$P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Swap row 1 and row 3

$$P_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

4. Swap row 2 and row 3

$$P_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

5. Cycle down: (1→2, 2→3, 3→1)

$$P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

6. Cycle up: (1→3, 3→2, 2→1)

$$P_6 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$Ax = y$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

We basically put a one in the location where we want the item to move the item to

Dealing with multiple samples

Given $X \in R^{7\times1000}$

The matrix represent the running record if 1000 peoples in this week(by miles)

$$X = \begin{bmatrix} 2 & 1 & 0 & \dots & 3 \\ 3 & 4 & 3 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 4 & \dots & 6 \end{bmatrix}$$

If Y represent the total miles ran in this week and $AX = Y$, then what is A?

Notice if you write $R^{7\times1000}$ matrix it would be annoying

The matrix notation allows us to conceptualize and manipulate large data easily

You can accomplish this with A as a matrix of all 1s.

$$A \in R^{1\times7} = [1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1]$$

A matrix of 1s is used very often for

- Summation
- Average
- Subtraction
- Centering

It has its own special symbols

$$\mathbf{1} \in R^{1\times7}$$

This example show us that how matrix dot product can be used to acquire necessary information from a large data

Moving a point vs object

If you have a point, and you would like to move that somewhere you can add that vector to another vector. Example given

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Then we can move it to point $y = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ by

$$y = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

If you want to move multiple points simultaneously by $z = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$ it has to be done slightly differently. Given 7 points

$$x_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, x_4 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$x_5 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, x_6 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, x_7 = \begin{matrix} 3 \\ 3 \end{matrix}\Bigg]$$

We would first stack theme together

$$X^T = \begin{bmatrix} 2 & 4 & 3 & 3 & 2 & 4 & 3 \\ 0 & 0 & 1 & 2 & 2 & 2 & 3 \end{bmatrix}$$

Then we perform the addition to $z\mathbf{1}_7^T$

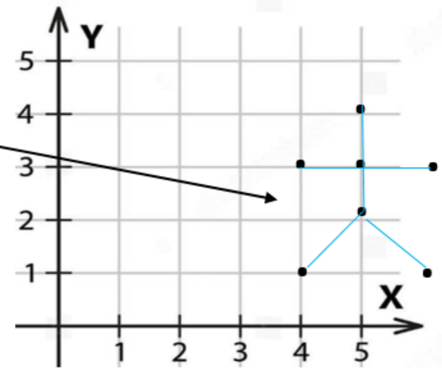
$$Y^T = z\mathbf{1}_7^T + X^T$$

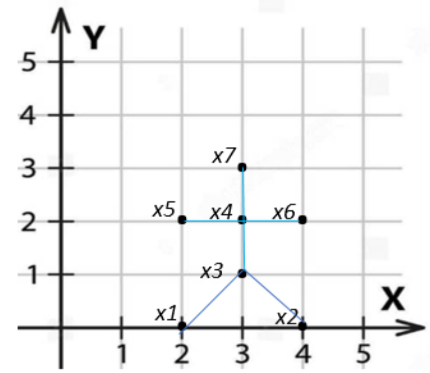
$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 3 & 3 & 2 & 4 & 3 \\ 0 & 0 & 1 & 2 & 2 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 3 & 3 & 2 & 4 & 3 \\ 0 & 0 & 1 & 2 & 2 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 & 5 & 5 & 4 & 0 & 5 \\ 1 & 1 & 2 & 3 & 3 & 3 & 4 \end{bmatrix}$$

The result of the above transformation is shown in this graph





Practice 3

1. If $x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and the movement vector is $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$, what is the new position y?
2. A point starts at $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$ and moves to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. What vector was added?
3. Find the final position of a point originally at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ after moving by $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$.
4. What is the movement vector that takes a point from $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ to $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$?

1) Given

$$X^T = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 2 & 2 & 3 \end{bmatrix}, \quad Z = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Then Compute the translated matrix Y^T

2) Given

$$X^T = \begin{bmatrix} 5 & 4 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Find $Y^T = Z \cdot \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} + X^T$

3) let

$$X^T = \begin{bmatrix} 7 & 8 & 9 & 10 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

And translation vector

$$Z = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Compute Y^T.

Combination of transformation

Remember when we rotated a vector

$$\begin{bmatrix} \cos 45 & \sin 45 \\ -\sin 45 & \cos 45 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

We can combine operations together, here we first rotate a vector then and then scale it by 2

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos 45 & \sin 45 \\ -\sin 45 & \cos 45 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

This is equivalent to multiplying the 2 matrices together first

$$\begin{bmatrix} 2\cos 45 & 2\sin 45 \\ -2\sin 45 & 2\cos 45 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}$$

Here we multiply the 2 vectors by 2 then we permuted the order

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

This is equivalent to multiplying the 2 matrices first

$$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Centering Matrix

Often you need to move an object to the center at(0,0). This can be done via the centering matrix. Again, given the stick figure data we can find the center of each dimension by finding the average of each column .

$$X - 1_7\, c^T = \begin{bmatrix} 2 & 0 \\ 4 & 0 \\ 3 & 1 \\ 3 & 2 \\ 2 & 2 \\ 4 & 2 \\ 3 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 1.4 \\ 3 & 1.4 \\ 3 & 1.4 \\ 3 & 1.4 \\ 3 & 1.4 \\ 3 & 1.4 \\ 3 & 1.4 \end{bmatrix} = \begin{bmatrix} -1.00 & -1.43 \\ 1.00 & -1.43 \\ 0.00 & -0.43 \\ 0.00 & 0.57 \\ -1.00 & 0.57 \\ 1.00 & 0.57 \\ 0.00 & 1.57 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 & 0 \\ 4 & 0 \\ 3 & 1 \\ 3 & 2 \\ 2 & 2 \\ 4 & 2 \\ 3 & 3 \end{bmatrix} \longrightarrow c^T = \frac{1}{\mathbf{7}} \mathbf{1}_7^T X = [3 \quad 1.4]$$

We next shift the entire data by subtracting the center.

The stick figure should end up at the center (0,0)