Linear Algebra
Lecture Note 7

Matrix as a function of vectors(Moving a point vs object , Combination of transformation , Centering Matrix , Scaling the size of object , Special case where a matrix is 2x2)

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Moving a point vs object

If you have a point, and you would like to move that somewhere you can add that vector to another vector. Example given

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Then we can move it to point $y = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ by

$$y = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

If you want to move multiple points simultaneously by $z = [2 \;\; 1]^T$ it has to be done slightly differently. Given 7 points

$$x_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, x_4 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$x_5 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, x_6 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, x_7 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

We would first stack theme together

$$X^T = \begin{bmatrix} 2 & 4 & 3 & 3 & 2 & 4 & 3 \\ 0 & 0 & 1 & 2 & 2 & 2 & 3 \end{bmatrix}$$

Then we perform the addition to $z\mathbf{1}_7^T$

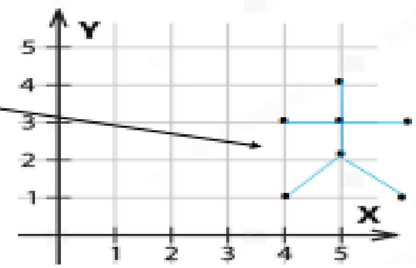
$$Y^T = z\mathbf{1}_7^T + X^T$$

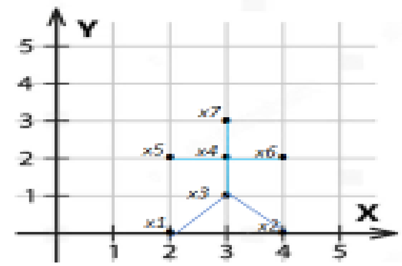
$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} [1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1] + \begin{bmatrix} 2 & 4 & 3 & 3 & 2 & 4 & 3 \\ 0 & 0 & 1 & 2 & 2 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 3 & 3 & 2 & 4 & 3 \\ 0 & 0 & 1 & 2 & 2 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 & 5 & 5 & 4 & 0 & 5 \\ 1 & 1 & 2 & 3 & 3 & 3 & 4 \end{bmatrix}$$

The result of the above transformation is shown in this graph





Practice 1

1) If x= $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ move point x to y= $\begin{bmatrix} 6 \\ 9 \end{bmatrix}$
2) If x= $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ move point x to y= $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$
3) If x=$\begin{bmatrix} 2 \\ -1 \\ -6 \end{bmatrix}$ move point x to y=$\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$

4) If x=$\begin{bmatrix} 3 \\ -6 \\ -6 \end{bmatrix}$ move point x to y=$\begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$

Practice 2

1) if x=$\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ -1 & 6 \end{bmatrix}$ move multiple points simultaneously by $z = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$

2) if x=$\begin{bmatrix} 0 & 3 \\ 2 & 3 \\ -1 & 3 \end{bmatrix}$ move multiple points simultaneously by $z = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$

3) If x=$\begin{bmatrix} 2 & 3 & -1 \\ 2 & 1 & 0 \\ 4 & 5 & 7 \end{bmatrix}$ move multiple points simultaneously by Z=$\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

4) If x=$\begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 7 \\ 2 & 5 & 7 \end{bmatrix}$ move multiple points simultaneously by Z=$\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$

Centering Matrix

Often you need to move an object to the center at(0,0). This can be done via the centering matrix. Again, given the stick figure data we can find the center of each dimension by finding the average of each column .

$$X = \begin{bmatrix} 2 & 0 \\ 4 & 0 \\ 3 & 1 \\ 3 & 2 \\ 2 & 2 \\ 4 & 2 \\ 3 & 3 \end{bmatrix} \longrightarrow c^T = \frac{1}{\mathbf{7}} \mathbf{1}_7^T X = [3 \quad 1.4]$$

We next shift the entire data by subtracting the center.
The stick figure should end up at the center (0,0)

$$X - 1_7\, c^T = \begin{bmatrix} 2 & 0 \\ 4 & 0 \\ 3 & 1 \\ 3 & 2 \\ 2 & 2 \\ 4 & 2 \\ 3 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 1.4 \\ 3 & 1.4 \\ 3 & 1.4 \\ 3 & 1.4 \\ 3 & 1.4 \\ 3 & 1.4 \\ 3 & 1.4 \end{bmatrix} = \begin{bmatrix} -1.00 & -1.43 \\ 1.00 & -1.43 \\ 0.00 & -0.43 \\ 0.00 & 0.57 \\ -1.00 & 0.57 \\ 1.00 & 0.57 \\ 0.00 & 1.57 \end{bmatrix} \tag{2}$$

The interesting thing about this derivation is that we have the equation $c^T = \frac{1}{7}\mathbf{1}_7^T X$, this means that we can plug c^T into equation 1 to get

$$X - 1_7\, c^T = X - \mathbf{1}_7 \frac{1}{7} \mathbf{1}_7^T X \qquad (2)$$

$$= X - \frac{1}{7} \mathbf{1}_7 \mathbf{1}_7^T X \qquad (3)$$

$$= X - \frac{1}{7} \mathbf{1}_{7,7} X \qquad (4)$$

$$= (\mathbf{I} - \frac{1}{7} \mathbf{1}_{7,7} \mathbf{I}) X \qquad (5)$$

$$= (\mathbf{I} - \frac{1}{7} \mathbf{1}_{7,7}) X \qquad (6)$$

$$= HX \qquad (7)$$

Practice 3

Center these matrices

1) A= $\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ -1 & 6 \end{bmatrix}$

2) B= $\begin{bmatrix} 0 & 3 \\ 2 & 3 \\ -1 & 3 \end{bmatrix}$

3) C= $\begin{bmatrix} 2 & 3 & -1 \\ 2 & 1 & 0 \\ 4 & 5 & 7 \end{bmatrix}$

4) W= $\begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 7 \\ 2 & 5 & 7 \end{bmatrix}$

5) X= $\begin{bmatrix} 2 & -5 & -1 \\ 3 & 3 & 0 \\ 4 & 5 & 9 \end{bmatrix}$

6) Y= $\begin{bmatrix} 2 & 3 & -2 \\ 2 & 2 & 0 \\ 2 & 5 & 2 \end{bmatrix}$

Scaling the size of object

If we have multiple points that connect to an object, you can scale the size by

1. Move the object to the center
2. Multiply the object matrix by the scaling value

Let's say we have a triangle with points at

$$X = \begin{bmatrix} 3 & 0 \\ 5 & 0 \\ 4 & 1 \end{bmatrix}$$

To double the size, we first move it to the center

$$\bar{X} = (\mathbf{I} - \frac{1}{3}\mathbf{1}_{3,3})X$$

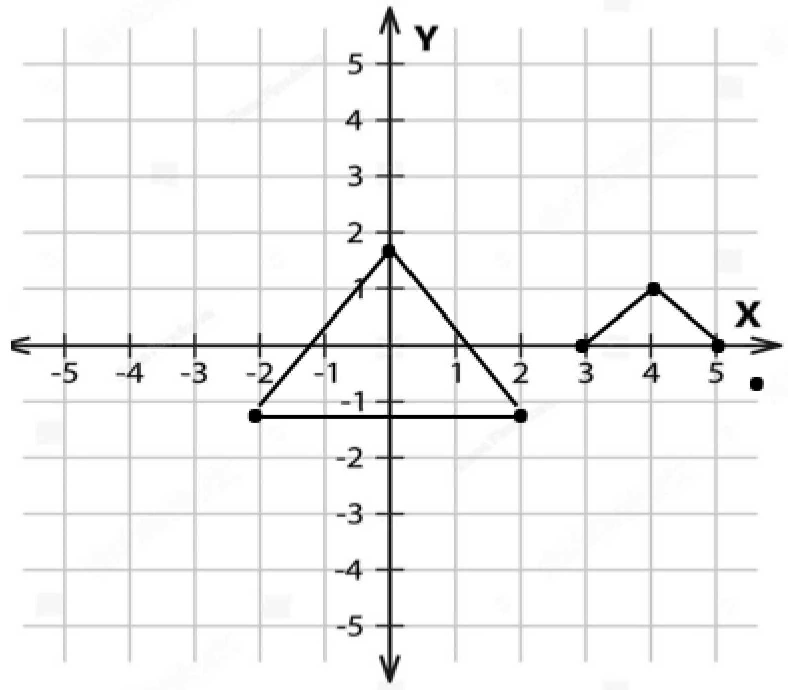
$$= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right) \begin{bmatrix} 3 & 0 \\ 5 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1/3 \\ 1 & -1/3 \\ 0 & 2/3 \end{bmatrix}$$

We next multiply the centered shape 2 to double the size

$$2 \begin{bmatrix} -1 & -1/3 \\ 1 & -1/3 \\ 0 & 2/3 \end{bmatrix} = \begin{bmatrix} -2 & -2/3 \\ 2 & -2/3 \\ 0 & 4/3 \end{bmatrix}$$

We can scale both the scaling and centering as follow

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right) = 2H$$



Y
X
5
4
3
2
1
-1
-2
-3
-4
-5
-5
-4
-3
-2
-1
1
2
3
4
5

Viewing vectors and matrices as scalars

We have so far seen a lot of matrix transformation

$0x = 0$ like multiplying a number by 0

$1x = x$ like multiplying a number 1

$2Ix = 2x$ like multiplying a number 2

$R_\pi x = -x$ if you rotate bt pi

All of these transformation allow us to almost think of

x and matrix A as just number

There is one more transformation to learn that is, the **inverse.**

If you have a matrix A the inverse of

A written as A^{-1} defined as

$$AA^{-1} = I$$

The matrix inverse is the continuation of treating vectors

and matrices as numbers.

If we pretend A as a number = 5 then

$$5^{-1}5 = \frac{1}{5}5 = 1$$

Note that

$A^{-1}A = AA^{-1}$ where $\frac{1}{5}5 = 5\frac{1}{5} = 1$

Finding the inverse by Gaussian Elimination

Given a matrix A, what is A^{-1}

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

We know that

$$A^{-1}A = I, \qquad A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

We can use Gaussian Elimination to solve for the inverse First, we create the augmented matrix

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

We then manipulate the equations so we have the identity matrix on the left side

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-2e_2 + e_1 \to e_1} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

Once you have an identity matrix on the left the matrix on the right is the inverse

$$A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

You can always double check

$$A^{-1}A = I, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Another example

Given the matrix A what is A^{-1}

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

We know that

$$AA^{-1} = I \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can use Gaussian Elimination to solve for the inverse first we create the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

We then manipulate the equations so we have the identity matrix on the left side

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[e_3 + e_1 \to e_1]{-e_1 + e_3 \to e_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[-1/2 e_3 + e_2 \to e_2]{e_2/2 \to e_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & -1/2 \\ 1 & 0 & 1 \end{bmatrix}$$

Singular matrices are not invertible

Singular matrices occur when the system

- Does not have a solution
- Have infinite solutions

A singular matrix is the left-hand side of the augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Left portion}} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$$

- If after reducing down to RREF, you get a row that is all 0's you have a singular matrix
- You can not take the inverse of a singular matrix. Try it yourself

Special case where a matrix is 2x2

For the case of $A \in \mathbb{R}^{2\times 2}$ matrices there is a special equation for it.

So Gaussian Elimination is not necessary

$$\text{Given } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Remember the last example

$$\left[\begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix}\middle|\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right] \xrightarrow{-2e_2 + e_1 \to e_1} \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\middle|\begin{matrix} 1 & -2 \\ 0 & 1 \end{matrix}\right]$$

Once you have the identity matrix on the left the matrix on the right-hand is the inverse

$$A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

Since this is a 2x2 matrix we can use the equation

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{1(1)-2(0)}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

Notice the trick can give us exactly the same result

Practice 4

1) A= $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ find A^{-1} and find $A\,A^{-1}$ = identity matrix

2) B= $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ find B^{-1} and find BB^{-1} = identity matrix

3) C= $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ find C^{-1} and find CC^{-1} = identity matrix

4) D= $\begin{bmatrix} 4 & 6 \\ 2 & 7 \end{bmatrix}$ find D^{-1} and find DD^{-1} = identity matrix

Find the inverse of the given 3×3 matrices

1) Find the inverse matrix of using Gaussian Elimination

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

2) Find the inverse matrix of using Gaussian Elimination

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 4 & 0 & 5 \end{bmatrix}$$

3) Find the inverse matrix of using Gaussian Elimination

$$C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 5 & 2 \end{bmatrix}$$

4) Find the inverse matrix of using Gaussian Elimination

$$D = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 0 \\ 4 & 1 & 1 \end{bmatrix}$$

Solving a linear system with matrix inverse

Everything we are doing is to allow us to conceptually think of matrices and vectors

- Almost as just regular numbers
- With adding 0
- With multiplying 1
- With an inverse

If we know the inverse, we can solve the system of equations easily

This idea is very useful when solving problems

It is very obvious how we can solve this problem

$$5x = 1$$

$$(5^{-1})5x = 5^{-1}(1)$$

$$x = 5^{-1}$$

We can now think of matrices similarly the matrix A is analogues to 5

$$Ax = y$$

$$A^{-1}Ax = A^{-1}y$$

$$Ix = A^{-1}y$$

$$x = A^{-1}y$$