Linear Algebra

Lecture 9

The null space , Linearly dependent and independence, The formal definition of linear independence, Using different basis to represent the same point , Unit vectors are the preferred basis , Equation for the unit vector .

By Noorullah

Learning objectives

1. **Null Space**
   Find all solutions to $Ax=0$; understand the structure of solution sets.
2. **Linear Dependence/Independence**
   Determine if vectors are dependent or independent using equations or row reduction.
3. **Formal Definition of Linear Independence**
   Know that a set is independent if only the trivial solution exists for
   $c_1v_1+c_2v_2+c_3v_3+\ldots\ldots\ldots\ldots..+c_nv_n=0$
4. **Different Bases, Same Point**
   Represent the same vector using different bases; convert between them.
5. **Unit Vectors as Preferred Basis**
   Use unit vectors for simplicity and clarity in coordinate systems.
6. **Equation for a Unit Vector**

Learning outcomes

1. **Null Space**
   Find and interpret the set of solutions to Ax=0.
2. **Linear Dependence/Independence**
   Determine if vectors are dependent or independent using equations or row operations.
3. **Formal Definition of Linear Independence**
   Apply the definition to test if only the trivial solution exists.
4. **Different Bases, Same Point**
   Represent a vector in different bases and convert between them.
5. **Unit Vectors as Preferred Basis**
   Use unit vectors for simpler, standard representation in coordinate systems.
6. **Equation for Unit Vector**.

The null space

- Given a set of vectors v_1, v_2 ... and we let these vectors be the columns of a matrix $A = [v_1 \ \ v_2 \ \ v_3 \ldots]$.
- **Definition:** the null space is the set of all possible solutions for x such that

$$x_1 v_1 + x_2 v_2 + \cdots = 0 \quad \text{or } Ax = 0$$

- To find the null space we simply create the augmented matrix such that

$$[A|0]$$

The point of this augmented matrix is that we are trying to find the solution x that will give you the output of 0 (this is the definition of null space)

For example, the matrix A would be converted to augmented matrix as

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

We turn it into the RREF and it gives us the solution for x

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$$

This is an example of trivial solution when $x = 0$
Because obviously

$$A0 = 0$$

However, once we have reduced the augmented matrix down to RREF, you might get a non-trivial solution

$$\underbrace{\left[\begin{array}{cc|c}1&2&0\\0&0&0\end{array}\right]}_{\text{RREF}} \xrightarrow{\textit{to equations}} \underbrace{\begin{bmatrix}1&2\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}}_{\text{Infinite solutions}} \xrightarrow{\textit{implying}} \underbrace{\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix}}_{\text{non-trivial solutions}} \qquad x_2 \text{ is free variable}$$

Solution here is a space defined by the vector, hence , **non-trivial solution**.

Example Find the null space of the matrix:

$$A = \begin{bmatrix}1&2\\3&6\end{bmatrix}$$

To find the null space, solve:

$$\begin{bmatrix}1&2\\3&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

This gives:

$$\begin{cases} x_1 + 2x_2 = 0 \\ 3x_1 + 6x_2 = 0 \end{cases} \Rightarrow$$

second equation is a multiple of the first

So the system have infinite solution so we are free to give any value to x_2

Suppose $x_2 = 1$

$$x_1 + 2x_2 = 0$$

$$x_1 + 2(1) = 0$$

$$x_1 = -2$$

So, the null space is:

$$Null\ space\ of\ (A) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Example Find the null space of the matrix:

$$B = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{bmatrix}$$

Solve $Bx = 0$. Notice each row is a multiple of the first, so the system has infinite solutions.

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So 2^{nd} and 3 rd columns are free

Therefore we can assign any value to x_2 and x_3

Suppose x_2=1 and $x_3 = 2$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The $x_1 + 2(1) - 1(2) = 0$

$$x_1 = 0$$

So the null space is:

$$\text{Null (B)}=\begin{bmatrix}0\\1\\2\end{bmatrix}$$

Practice

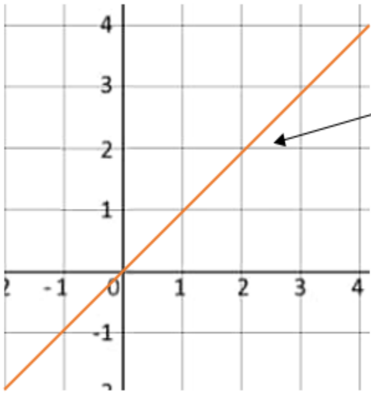
1) Find the null space of the matrix:

$$B = \begin{bmatrix}1 & 2 & -1\\0 & 0 & 0\\3 & 6 & -3\end{bmatrix}$$

2) Find the null space of the matrix:

$$D = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 3 & 9 & 6 \end{bmatrix}$$

We also learned about the span in the last lecture



This is the span of $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

This is the set of all possible points reachable by one or multiple scaled vectors

This notation is called the set builder notation

It defines a set such that α can be any real number

You may sometime see a line instead

With a vector v its span is

$$\boldsymbol{S} = \text{span}(v) = \{\alpha v : \alpha \in \mathbb{R}\}$$

But you can also have multiple vectors $v_1, v_2, v_3, \ldots,$ its span would be

$$S = \text{span}(v, v_2, v_3 \ldots,) = \{(av_1, bv_2 \ldots) : a, b \in \mathbb{R}\}$$

If we have 2 vectors it can at most span 2d

Given 2 vectors

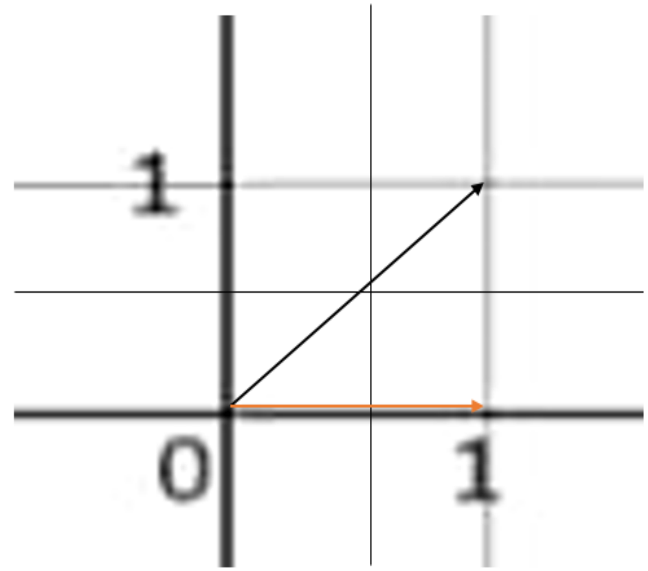
$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Their span would cover the entire 2D space
For example, the point d is in the span because

$$d = v_1 + {}^{1}/_{2}\, v_2$$

$$\begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

The vectors used to create the span of a space are called base



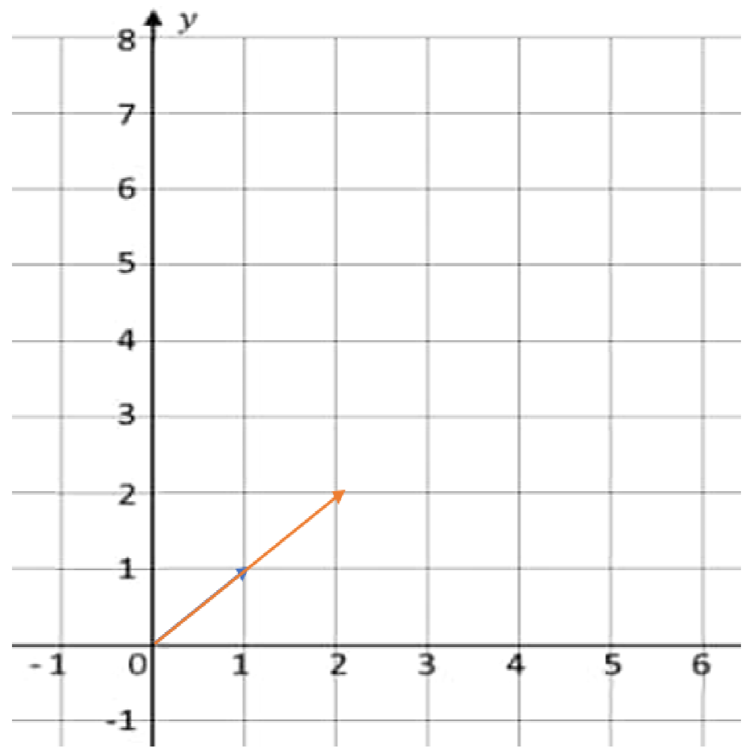
Every single point on this 2D plane can be achieved by some combination of the 2 vectors

2 vector does not imply 2d span

Given 2 vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Their span would cover only 1 dimension



The dimension span by a set of vectors is called the **Rank**

This lead to have two possible cases

The rank is same as the number of bases

This happen when each vector is contributing to the new dimension

The rank is fever than the number of bases

This happen when we have redundant vectors covering the same space

These two cases have names

The rank is same as the number of basis

[every vector is expanding the span larger]

These vectors are called linearly independent

The rank is fever than the number of basis

[having redundant vectors]

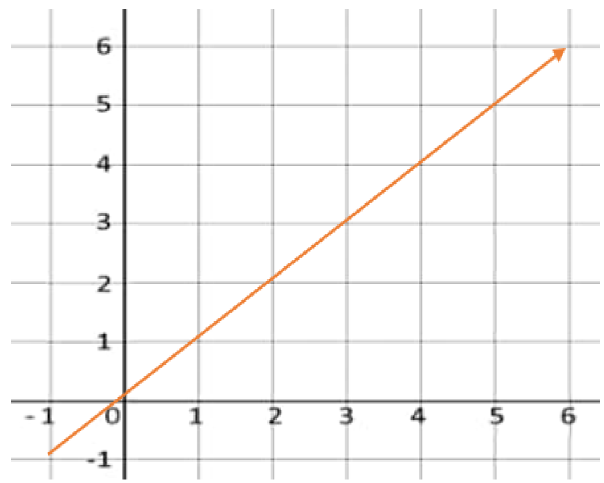
These vectors are called linearly dependent

Linearly dependent and independence

Given 2 vectors

$$v_1 = \begin{bmatrix}1\\1\end{bmatrix}, \qquad v_2 = \begin{bmatrix}2\\2\end{bmatrix}$$

Linearly dependent

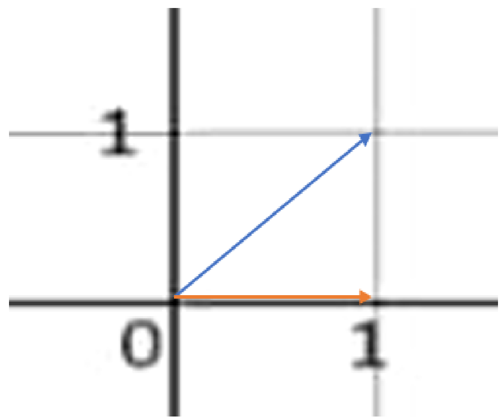


Their span would cover only 1 dimension

Given 2 vectors

$$v_1 = \begin{bmatrix}1\\0\end{bmatrix} v_2 = \begin{bmatrix}1\\1\end{bmatrix}$$

Linearly independent



Their span would cover the entire 2D space

The formal definition of linear independence

A set of vectors is linearly independent if and only if the equation:

$$c_1v_1 + c_2v_2 + \cdots c_kv_k = 0$$

Has only the trivial solution. What that mean is that these vectors are linearly independent when $c_1 = c_2 = \cdots = 0$ is the only possible solution to the vectors equation

If a set of vectors is not linearly independent we say that they are linearly dependent. Then we can write a linear dependence relation showing how one vector is combination of the other vectors

Given this definition it implies that if we have a set of vectors v_1v_2, v_3 ... they are

- **Linearly independent** : if their null space solution is the trivial solution
- **Linearly dependent** : if their null space solution is non-trivial solution

Why linear independence?

Linear independence is a very important concept in data representation

$$v_1 = \begin{bmatrix}1\\2\\1\\0\end{bmatrix}, \quad v_2 = \begin{bmatrix}2\\4\\2\\0\end{bmatrix}, \quad v_3 = \begin{bmatrix}4\\8\\4\\0\end{bmatrix},$$

$$v_4 = \begin{bmatrix}6\\12\\6\\0\end{bmatrix}, \quad v_5 = \begin{bmatrix}1\\2\\1\\0\end{bmatrix}$$

We can represent the data as a matrix

$$A = \begin{bmatrix}1 & 2 & 4 & 6 & 1\\2 & 4 & 8 & 12 & 2\\1 & 2 & 4 & 6 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

We can assume the first column as basis and represent the data

$\begin{bmatrix}1 & 2 & 4 & 6 & 1\end{bmatrix}$ become $\begin{bmatrix}1v_1 & 2v_1 & 4v_1 & 6v_1 & 1v_1\end{bmatrix}$

Or we can use the first row as basis and represent the data as

$\begin{bmatrix}1 & 2 & 1 & 0\end{bmatrix}$ become X = $\begin{bmatrix}1r_1\\2r_1\\1r_1\\0r_1\end{bmatrix}$

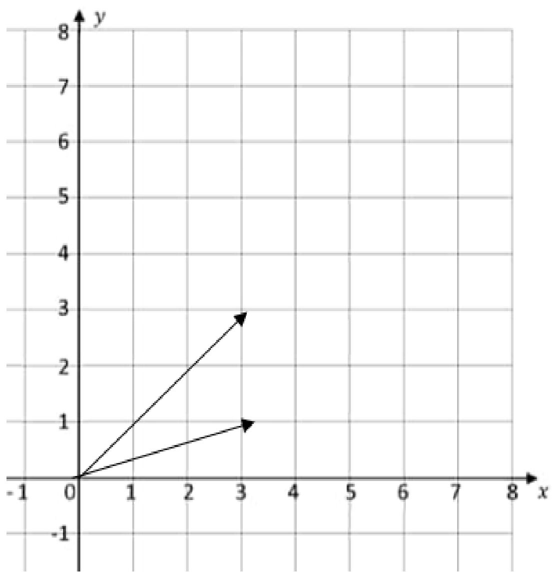
Much compact way to represent data compare to matrix

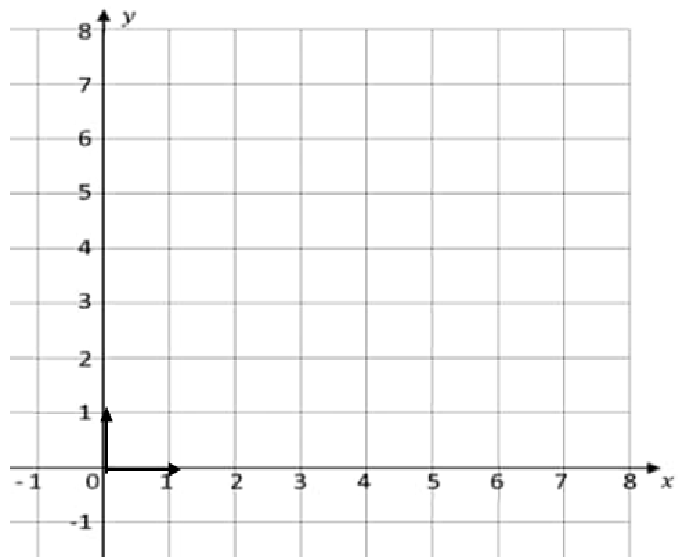
Using different basis to represent the same point

Below we show 2 possible basis to reach target D

Both basis span the same 2D space

but which basis are easier to identify the combination to reach point D?





In general, the basis the are perpendicular with a size are the nicest and easiest one to deal with

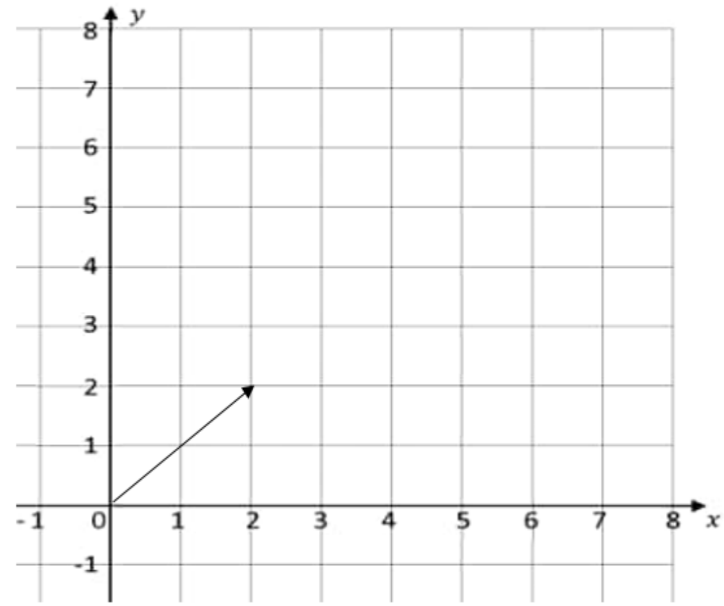
Unit vectors are the preferred basis

We often represent data using independent bases
As a convention, we prefer to use **unite vectors.**
[A unite vector is a vector with length 1]

Examples of unite vectors

$$c = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \\ 0 \end{bmatrix}$$

$$\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{\frac{2}{2}} = \sqrt{1} = 1$$



Example of not a unit vector

$$c = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\sqrt{(2)^2 + (1)^2} = \sqrt{4+1} = \sqrt{5} \neq 1$$

Equation for the unit vector

Example of NOT Unite Vector

$$C = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

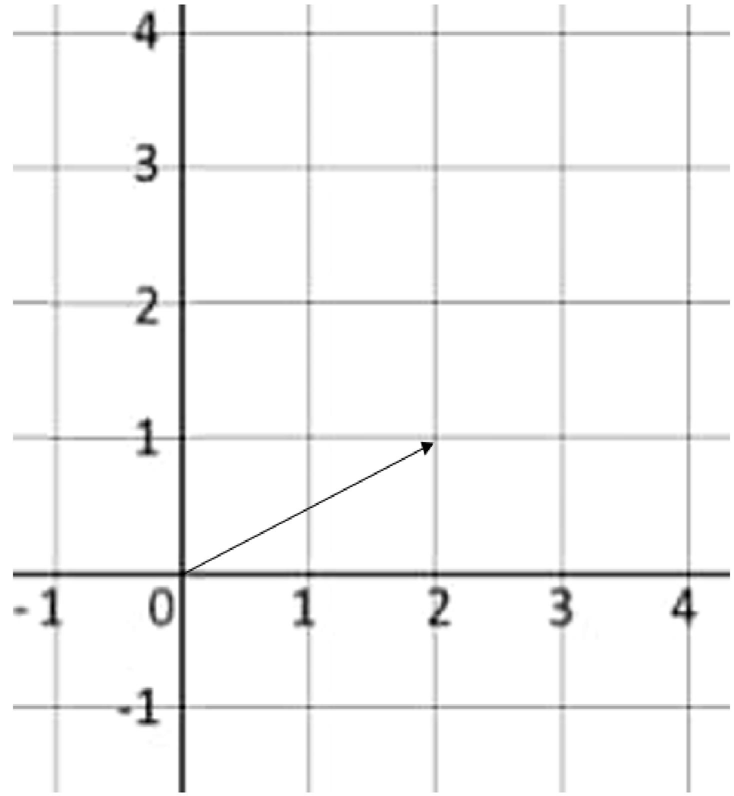
$$\sqrt{(2)^2 + (1)^2} = \sqrt{5} \neq 1$$

How do we turn this vector into a normal vector

Divide the vector by the length !!!

unite vector $\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \frac{1}{length}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2^2+1^2}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

You can double check your result the new length should be 1



$$\sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = 1$$

Equation for the unit vector

The act of turning a vector into a unite vector is called **normalization**

Normalized vectors are normally denoted by hat

$$u \to \hat{u}$$
normalized

$$\text{length}(u) \neq 1$$

$$\text{Length } (\hat{u}) = 1$$

We refers to unite vector as **normalized vector**

How do we turn this vector into a normal vector

Divide the vector by the length !!!

$$\text{unite vector}\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \frac{1}{length}\begin{bmatrix}2\\1\end{bmatrix} = \frac{1}{\sqrt{2^2+1^2}}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{bmatrix}$$

You can double check your result the new length should be 1