L'Hospital Rule

Let $f, g\ :(a,b)\to R$ be diffrentiable at $x_o \epsilon\,(a,b)$. Suppose $f(x_o)=g(x_o)=0$ and $g^{'}(x_o)\ \neq 0$.

Then $lim_{x\to x_o}\frac{f(x)}{g(x)}=\frac{f^{'}(x_o)}{g^{'}(x_o)}$

PROOF : Note that $\frac{lim_{x\to x_o}\frac{f(x)-f(x_o)}{x-x_o}}{lim_{x\to x_o}\frac{g(x)-g(x_o)}{x-x_o}}=lim_{x\to x_o}\frac{f(x)-f(x_o)}{g(x)-g(x_o)}=lim_{x\to xo}\frac{f(x)}{g(x)}$

Strong Version of L'Hospital Rule

Let $f, g:[x_o,b)\to R$. Suppose $f(x_o)=g(x_o)=0$ and f,g are diifrentiable on (x_o,b). Let $g^{'}(x)\neq 0$ for all $x\,\epsilon\,(x_o,b)$. Then

$$lim_{x\to x_o^+}\frac{f(x)}{g(x)}=lim_{x\to x_o}\frac{f'(x)}{g'(x)}$$

provided $\lim_{x\to x_o^+}\frac{f^{'}(x)}{g^{'}(x)} exists.$

THEOREM:

Cauchy Mean Value Theorem

Let f and g be continuous on [a,b] and diffrentiable on (a,b). Suppose that $g^{'}(x)\neq 0$

for all $x\,\epsilon\,(a,b)$. Then there exists $c\,\epsilon\,(a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f^{'}(c)}{g^{'}(c)}.$$

Proof: $F(x)=f(x)-f(a)-\frac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a))$

Then F is continuous on [a,b], diffrentiable on (a,b) and $F(a)=F(b)=0$.

By Rolle's Theorem there exists $c\,\epsilon\,(a,b)$ such that $F^{'}(c)=0$.

This proves the Theorem.