TAYLOR'S THEOREM :

Let $f : [a, b] \to R, f, f^{'} f^{''}, ..., f^{(n-1)}$ *be continuous on* $[a, b]$ *and suppose* $f^{(n)}$ *exists on* (a, b). *Then there exists* $c \, \epsilon \, (a, b)$ *such that*

$f(b) = f(a) + f^{'}(a)(b-a) + \frac{f^{''}(a)}{2!}(b-a)^2 + ... + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + \frac{f^{(n)}(c)}{n!}(b-a)^n$.

PROOF: $F(x) = f(b) - f(x)(b-x) - \frac{f^{''}(x)}{2!}(b-x)^2 - ... - \frac{f^{(n-1)}}{(n-1)!}(b-x)^{n-1}$.

We will show that $F(a) = \frac{(b-a)^n}{n!} f^{(n)}(c)$ for some $c \, \epsilon \, (a, b)$, which will prove the theorem.

Note that

$$F^{'}(x) = -\frac{f^{(n)}(x)}{(n-1)!}(b-x)^{n-1}$$

Define $g(x) = F(x) - (\frac{b-x}{b-a})^n F(a)$. now we know that g(a) = g(b) = 0 and hence by Rolle's theorem

there exists some $c \, \epsilon \, (a, b)$ such that

$$g^{'}(c) = F^{'}(c) + \frac{n(b-c)^{n-1}}{(b-a)^n} F(a) = 0$$

from (1) and (2) we obtain that $\frac{f^{(n)}(c)}{(n-1)}(b-c)^{n-1} = \frac{n(b-c)^{n-1}}{(b-a)n} F(a)$. This implies

$F(a) = \frac{(b-a)^n}{n!} f^{(n)}(c)$. This proves the theorem.