Unit - IV

Second and Higher Order Partial Differential Equation and Application of differential Equation

Partial Differential Equation of Second Order

A partial differential equation of second order can be expressed as,

$$F(x, y, z, p, q, r, s, t) = 0$$

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}$$

* Equation reduced to Linear equation

Que $t - xq = x^2$

Sol^n:- $t = \frac{\partial^2 z}{\partial y^2}$, $t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \Rightarrow t = \frac{\partial q}{\partial y}$

Given equation can be written as,

$$\frac{\partial q}{\partial y} - xq = x^2$$

which is linear in q and y

$$I.f = e^{\int -x\,dy} = e^{-xy}$$

Solution,

$$q \cdot (I.f) = \int x^2 \cdot e^{-xy}\,dy + f(x)$$

$f(x)$ is arbitrary function of x

$$q \cdot e^{-xy} = x^2 \cdot \frac{e^{-xy}}{-x} + f(x)$$

$$q e^{-xy} = -x e^{-xy} + f(x)$$

$$q = -x + e^{xy} f(x)$$

Integrating w.r.t y gives,

$$\frac{\partial z}{\partial y} = q = -xy + e^{xy} \cdot \frac{1}{x} f(x) + F(x)$$

$$z = -xy + e^{xy} \cdot \phi(x) + F(x)$$

where $\phi(x) = \frac{1}{x} f(x)$ and $F(x)$ are two arbitrary function of x.

* Equation integrable by Lagrange's Method

<u>Ques</u> Solve :- $t + s + q = 0$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial y}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z}{\partial x} \cdot \frac{\partial}{\partial y} = \frac{\partial p}{\partial y}$$

$$q = \frac{\partial z}{\partial y}$$

from given equation,

$$\frac{\partial q}{\partial y} + \frac{\partial p}{\partial y} + \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial}{\partial y}(q + p + z) = 0$$

On integration, $q + p + z = f(x)$

$$p + q = f(x) - z$$

$$p \cdot 1 + q \cdot 1 = f(x) - z$$

$\therefore$ Lagrange's auxiliary equation

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{f(x) - z}$$

from first two ratio,

$$dx = dy$$

Integrating, $x = y + a \Rightarrow x - y = a$

from first and third ratio,

$$dx = \frac{dz}{f(x) - z}$$

$$\frac{dz}{dx} + z = f(x)$$

$$I.f = e^{\int dx} = e^x$$

Solution,

$$z \cdot e^x = \int f(x) \cdot e^x \, dx + b$$

$$z e^x = \phi(x) + b \qquad \text{where } \phi(x) = \int f(x) \cdot e^x dx$$

Required Solution, $b = F(a) \Rightarrow z e^x - \phi(x) = F(x - y)$

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* Equation Integrable by inspection

Ques Solve :- $ys + p = \cos(x+y) - y\sin(x+y)$

Solⁿ:- Rewriting the given equation,

$$y\frac{\partial q}{\partial x} + \frac{\partial z}{\partial x} = \cos(x+y) - y\sin(x+y)$$

Integrating w.r.t 'x', we get

$$yq + z = \sin(x+y) + y\cos(x+y) + f(y)$$

$\left\{q = \frac{\partial z}{\partial y}\right\}$

$$y\frac{\partial z}{\partial y} + z = \sin(x+y) + y\cos(x+y) + f(y)$$

$$\frac{\partial}{\partial y}(zy) = \frac{\partial}{\partial y}[y\sin(x+y)] + f(y)$$

Integrating w.r.t 'y' we get

$$zy = y\sin(x+y) + \phi(y) + F(x)$$

where, $\phi(y) = \int f(y)\,dy$

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Ques Solve :- $r = 6x$

Solⁿ:- $r = \frac{\partial^2 z}{\partial x^2}$

Given equation, $\frac{\partial^2 z}{\partial x^2} = 6x$

Integrating w.r.t 'x'

$$\frac{\partial z}{\partial x} = 3x^2 + f(y)$$

where constant of integration is taken as function of y.

Again Integrating w.r to x we get

$$z = x^3 + xf(y) + \phi(y)$$

Required Solution.

Ques Solve :- $s = 2x + 2y$

Solⁿ:- Given equation can be written as,

$$\frac{\partial^2 z}{\partial x \partial y} = 2x + 2y$$

Integrating w.r.t 'x',

$$\frac{\partial z}{\partial y} = x^2 + 2xy + f(y)$$

Again integrating w.r.t 'y'

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$$z = x^2y + xy^2 + \phi(y) + F(x)$$

where $\phi(y) = \int f(y)\,dy$

which is required solution

Ques Solve :- $yt - q = xy$

Sol^n : Given equation can be written as,

$$t - \frac{q}{y} = x$$

$$\frac{\partial q}{\partial y} - \frac{1}{y}q = x$$

which is linear in q

$$I.F = e^{\int -\frac{1}{y}dy} = e^{-\log y} = \frac{1}{y}$$

Solution,

$$q \cdot \frac{1}{y} = \int x \cdot \frac{1}{y}\,dy + f(x)$$

$$\frac{q}{y} = x\log y + f(x)$$

$$q = xy\log y + yf(x)$$

$$\frac{\partial z}{\partial y} = xy\log y + y f(x)$$

Integrating w.r.t 'y' we get

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$$z = x\left[\int y\log y\,dy\right] + f(x)\int y\,dy + F(x)$$

$$z = x\left[\frac{y^2}{2}\log y - \int \frac{1}{y}\cdot\frac{y^2}{2}dy\right] + \frac{y^2}{2}f(x) + F(x)$$

$$z = \frac{1}{2}xy^2\log y - \frac{1}{4}xy^2 + \frac{y^2}{2}f(x) + F(x) \quad \text{Ans}$$

Required Solution

Ques $\log s = x + y$

Sol^n : Given equation can be written as,

$$s = e^{x+y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = e^{x+y}$$

Integrating w.r.t 'x' we get

$$\frac{\partial z}{\partial y} = e^{x+y} + f(y)$$

Again integrating w.r.t y, we have

$$z = e^{x+y} + \int f(y)\,dy + F(x)$$

$$z = e^{x+y} + \phi(y) + F(x)$$

Required Solution.

Reduction To Canonical Forms

↳ $Ar + 2Bs + Ct + f(x, y, z, p, q) = 0$

↳ Compare above equation with given equation

$A = a \ , \ B = b \ , \ C = c$

↳ Quadratic Equation ; $A\lambda^2 + 2B\lambda + C = 0$

↳ Roots are real and distinct,

$\frac{dy}{dx} + \lambda_1 = 0 \ , \ \frac{dy}{dx} + \lambda_2 = 0$

↳ By Solving above equation, we get c_1 and c_2

↳ Assume, $c_1 = u$ and $c_2 = v$

↳ Find the value of p, q, r, s, t and put in the given equation.

Ques $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = 0$

Soln:-

Canonical Equation.

$$A\frac{\partial^2 z}{\partial x^2} + 2B\frac{\partial^2 z}{\partial x \partial y} + C\frac{\partial^2 z}{\partial y^2} + f(x, y, z, p, q) = 0$$

Comparing with given equation,

$A = 1 \ , \ B = \frac{1}{2} \ , \ C = -2$

$B^2 - AC = \frac{1}{4} + 2$

$\frac{9}{4} > 0$

Given equation is hyperbolic

Ques Classify and Solve :- $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$

or $r - t = 0$

Solve:- Given Equation,

$r - t = 0$ —— (1)

Comparing (1) with

$Ar + 2Bs + Ct + f(x, y, z, p, q) = 0$, we get

$A = 1 \ , \ B = 0 \ , \ C = -1$

Since, $B^2 - AC = 0 - 1.(-1)$

$1 > 0$

Hence the given equation is hyperbolic

from λ - quadratic equation

$$A\lambda^2 + 2B\lambda + C = 0$$
$$\lambda^2 - 1 = 0$$
$$\lambda = \pm 1$$
$$\lambda_1 = 1 \text{ and } \lambda_2 = -1$$

Hence, $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$

$$\frac{dy}{dx} + 1 = 0 \quad \text{and} \quad \frac{dy}{dx} - 1 = 0$$

$$dx + dy = 0 \quad \text{and} \quad dx - dy = 0$$

On integration, $x + y = c_1$ and $x - y = c_2$

Let $u = x + y$ and $v = x - y$

$\therefore\ p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $t = \frac{\partial^2 z}{\partial y^2}$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 1$$

$$\frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}\cdot 1 + \frac{\partial z}{\partial v}\cdot 1$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}\cdot 1 + \frac{\partial z}{\partial v}\cdot(-1)$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$= \frac{\partial^2 z}{\partial u^2} + \frac{2\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\cdot 1 + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)(-1)$$

$$= \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} = \frac{\partial^2 z}{\partial u^2} - \frac{2\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

Substituting the value of r and t in eqn (1)

$$\cancel{\frac{\partial^2 z}{\partial u^2}} + 2\frac{\partial^2 z}{\partial u \partial v} + \cancel{\frac{\partial^2 z}{\partial v^2}} - \cancel{\frac{\partial^2 z}{\partial u^2}} + 2\frac{\partial^2 z}{\partial u \partial v} - \cancel{\frac{\partial^2 z}{\partial v^2}} = 0$$

$$4\frac{\partial^2 z}{\partial u \partial v} = 0$$

or $\boxed{\frac{\partial^2 z}{\partial u \partial v} = 0}$ Ans

which is canonical form of equation (1).

Que Classify and Solve $\frac{\partial^2 z}{\partial x^2} + x^2\frac{\partial^2 z}{\partial y^2} = 0$

Solⁿ:- Given equation,

$$r + x^2 t = 0 \quad -(1)$$

Comparing eqn (1) with

$$Ar + 2Bs + Ct + f(x, y, z, p, q) = 0$$ we get

$A = 1$, $B = 0$, $C = x^2$

Since, $B^2 - AC = 0 - x^2$

$= -x^2 < 0$

Hence given equation is elliptic

from λ - quadratic equation,

$$A\lambda^2 + 2B\lambda + C = 0$$

$$\lambda^2 + x^2 = 0$$

$$\lambda^2 - i^2x^2 = 0$$

$$\lambda = \pm ix$$

$\lambda_1 = ix$ and $\lambda_2 = -ix$

Hence, $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$

$\frac{dy}{dx} + ix = 0$ and $\frac{dy}{dx} - ix = 0$

On integration, $dy + ix\,dx = 0$ and $dy - ix\,dx = 0$

$y + \frac{ix^2}{2} = c_1$ and $y - \frac{ix^2}{2} = c_2$

Let, $u = y + \frac{ix^2}{2} = \alpha + i\beta$

$v = y - \frac{ix^2}{2} = \alpha - i\beta$

Solving for α and β we get

$\alpha = y$ and $\beta = \frac{1}{2}x^2$

$\frac{\partial \alpha}{\partial x} = 0$, $\frac{\partial \alpha}{\partial y} = 1$

$\frac{\partial \beta}{\partial x} = x$, $\frac{\partial \beta}{\partial y} = 0$

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$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x}$$

$$= \frac{\partial z}{\partial \alpha}.0 + \frac{\partial z}{\partial \beta}.x$$

$$\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial \beta}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y}$$

$$= \frac{\partial z}{\partial \alpha}.1 + \frac{\partial z}{\partial \beta}.0$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial \beta}\right)$$

$$= 1.\frac{\partial z}{\partial \beta} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$

$$= \frac{\partial z}{\partial \beta} + x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right]$$

$$= \frac{\partial z}{\partial \beta} + x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right).0 + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)x\right]$$

$$= \frac{\partial z}{\partial \beta} + x\left[0 + x\frac{\partial^2 z}{\partial \beta^2}\right]$$

$$= \frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)$$

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$$= \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial y}$$

$$= \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right).1 + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right).0$$

$$= \frac{\partial^2 z}{\partial \alpha^2}$$

Substituting the value of r and t in eqn (1), we get

$$\frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2} + x^2\frac{\partial^2 z}{\partial \alpha^2} = 0$$

Or $$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{x^2}\frac{\partial z}{\partial \beta}$$

or $$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{\beta}\frac{\partial z}{\partial \beta}$$ Ans $\left\{\because \beta = x^2\right\}$

which is canonical form of equation (1)

Homogeneous And Non - Homogeneous Equation with Constant Coefficients

$$F(D, D')z = 0$$

$D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$, $DD' = \frac{\partial^2}{\partial x \partial y}$

$D = m$ and $D' = 1$

$$A_0 m^n + A_1 m^{n-1} + \dots A_n = 0$$

$m_1, m_2, m_3 \dots m_n$ are roots

Case I :- If roots are distinct,

$m_1, m_2, m_3 \dots m_n$

$$C.F = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \dots + \phi_n(y + m_n x)$$

Case II :- Repeated Roots

- If roots are equal m_1, m_1

$$C.F = \phi_1(y + m_1 x) + x\phi_2(y + m_1 x)$$

- If r roots are equal $m_1 = m_2 = m_3 = \dots = m_n$

$$C.F = \phi_1(y + m_1 x) + x\phi_2(y + m_1 x) + x^2\phi_3(y + m_1 x) + \dots + x^{r-1}\phi_r(y + m_1 x)$$

Ques Solve :- $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$

Sol^n:- Let, $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$

from given equation,

$$(D^2 - D'^2)z = 0 \quad - (1)$$

$D = m$ and $D' = 1$

from eqn (1), $m^2 - 1 = 0$ { Auxiliary Equation }

$m = \pm 1$

$m = 1, -1$

Required General Solution of given equation

$$z = \phi_1(y + x) + \phi_2(y - x) \quad \text{Ans}$$

where ϕ_1 and ϕ_2 are arbitrary function

Ques Solve :- $25r - 40s + 16t = 0$

Sol^n:- Given equation can be written as,

$$(25D^2 - 40DD' + 16D'^2)z = 0$$

Auxiliary Equation, $D = m$ and $D' = 1$

$$25m^2 - 40m + 16 = 0$$

$$(5m)^2 - 2.5.4m + (4)^2 = 0$$

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$$(5m-4)^2 = 0$$

$$m = \frac{4}{5}, \frac{4}{5}$$

Required General Solution

$$z = \phi_1\left(y + \frac{4}{5}x\right) + x\phi_2\left(y + \frac{4}{5}x\right)$$

$$z = \phi(5y+4x) + x\,\phi_2(5y+4x)$$ Ans

Que Solve :- $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0$

Solⁿ :- Given equation can be written as,

$$(D^4 - D'^4)z = 0$$

Auxiliary equation, $D = m$, $D' = 1$

$$m^4 - 1 = 0$$

$$(m^2-1)(m^2+1) = 0$$

$$(m^2-1)(m^2-i^2) = 0$$

$$m = \pm i, \; m = \pm 1$$

Required Solution,

$$z = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y+ix) + \phi_4(y-ix)$$ Ans

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Que Solve :- $\frac{\partial^4 z}{\partial x^4} - \frac{2\,\partial^4 z}{\partial x^3 \partial y} + \frac{2\,\partial^4 z}{\partial x \partial y^3} - \frac{\partial^4 z}{\partial y^4} = 0$

Solⁿ :- Given equation can be written as,

$$(D^4 - 2D^3D' + 2DD'^3 - D'^4)z = 0$$

Auxiliary equation, $D = m$ and $D' = 1$

$$m^4 - 2m^3 + 2m - 1 = 0$$

$$m^4 - 1 - 2m^3 + 2m = 0$$

$$(m^4-1) - 2m(m^2-1) = 0$$

$$(m^2-1)(m^2+1) - 2m(m^2-1) = 0$$

$$(m^2-1)(m^2-2m+1) = 0$$

$$m = \pm 1, \; m = 1, 1$$

Required Solution,

$$z = \phi_1(y \mp x) + \phi_2(y+x) + x\phi_3(y+x) + x^2\phi_4(y+x)$$

Que Solve :- $\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = x$

Solⁿ :- Given equation can be written as,

$$(D^2 - a^2D'^2) = x$$

Auxiliary equation

$D = m$ and $D' = 1$

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$$m^2 - a^2 = 0$$
$$m = \pm a$$
$$m = a, -a$$

$$C.F = \phi_1(y + ax) + \phi_2(y - ax)$$

$$P.I = \frac{1}{D^2 - a^2D'^2}\,x$$
$$= \frac{1}{D^2\left(1 - \frac{a^2D'^2}{D^2}\right)}\,x = \frac{1}{D^2}\left[1 - \left(\frac{aD'}{D^2}\right)^2\right]^{-1}x$$
$$= \frac{1}{D^2}\left[1 + \frac{a^2D'^2}{D^2} + \dots\right]x$$
$$= \frac{1}{D^2}\,x + \frac{a^2D'^2}{D^4}\,x$$
$$= \frac{1}{D}\cdot\frac{x^2}{2} + 0$$
$$= \frac{x^3}{6}$$

Required General Solution

$$y = C.F + P.I$$
$$y = \phi_1(y + ax) + \phi_2(y - ax) + \frac{x^3}{6}$$ Ans

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Ques Solve :- $(D^2 + 3DD' + 2D'^2)z = x + y$

Sol^n:-

Auxiliary equation,

$$D = m \text{ and } D' = 1$$
$$m^2 + 3m + 2 = 0$$
$$m^2 + 2m + m + 2 = 0$$
$$m(m+2) + 1(m+2) = 0$$
$$(m+1)(m+2) = 0$$
$$m = -1, -2$$

$$C.F = \phi_1(y - x) + \phi_2(y - 2x)$$

$$P.I = \frac{1}{D^2 + 3DD' + 2D'^2}(x + y)$$
$$= \frac{1}{D^2\left(1 + \frac{3D'}{D} + \frac{2D'^2}{D^2}\right)}(x+y)$$
$$= \frac{1}{D^2}\left(1 + \frac{3D'}{D} + \frac{2D'^2}{D^2}\right)^{-1}(x+y)$$
$$= \frac{1}{D^2}\left(1 - \frac{3D'}{D} - \dots\right)(x+y)$$
$$= \frac{1}{D^2}(x+y) - \frac{3}{D^3}D'(x+y)$$
$$= \frac{1}{D}\left(\frac{x^2}{2} + xy\right) - \frac{3}{D^3}\,1$$
$$= \frac{x^3}{6} + \frac{x^2y}{2} - \frac{3}{D^2}\,x$$
$$= \frac{x^3}{6} + \frac{1}{2}x^2y - \frac{3x^3}{6} = \cancel{\frac{x^3}{6}} + \frac{1}{2}x^2y \cancel{\frac{x^3}{2}}$$

$$= -\frac{1}{3}x^3 + \frac{1}{2}x^2y$$

Required General Solution,

$$z = \phi_1(y-x) + \phi_2(y-2x) - \frac{1}{3}x^3 + \frac{1}{2}x^2y \quad \underline{\text{Ans}}$$

Finding Particular Integral

$F(D, D')$ is homogeneous function

$$\frac{1}{F(D, D')} \phi(ax + by) = \frac{1}{F(a, b)} \iint \cdots\cdots \int \phi(v) \cdot (dv)^n$$

$$V = ax + by \quad , \quad F(a, b) \neq 0$$

* If $F(a, b) = 0$

$$\frac{1}{(bD - aD')^n} \phi(ax + by) = \frac{x^n}{b^n \, n!} \phi(ax + by)$$

* $$\frac{1}{F(D, D')} e^{ax + by} = \frac{1}{F(a, b)} e^{ax + by} \quad , \quad F(a, b) \neq 0$$

* $$\frac{1}{F(D^2, DD', D'^2)} \cos(ax + by) = \frac{1}{(-a^2, -ab, -b^2)} \cos(ax + by)$$

* $$\frac{1}{F(D^2, DD', D'^2)} \sin(ax + by) = \frac{1}{(-a^2, ab, -b^2)} \sin(ax + by)$$

* $$\frac{1}{(D - mD')} f(x, y) = \int f(x, a - mx)\, dx$$

* $$\frac{1}{D + mD'} f(x, y) = \int f(x, a + mx)\, dx$$

Que Solve :- $(D^2 - 5DD' + 6D'^2)z = e^{x+y}$

Solⁿ:- Auxiliary Equation,

$$D = m \quad , \quad D' = 1$$

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$m = 2, 3$$

$$C.F = \phi_1(y+2x) + \phi_2(y+3x)$$

$$P.I = \frac{1}{D^2 - 5DD' + 6D'^2} \cdot e^{x+y}$$

$$= \frac{1}{1^2 - 5.1.1 + 6.1^2} e^{x+y}$$

$$= \frac{1}{2} e^{x+y}$$

Required General Solution

$$z = \phi_1(y+2x) + \phi_2(y+3x) + \frac{1}{2} e^{x+y}$$

Ans

Ques find Particular Integral of

$$(D^3 - 10D^2D' + D'^3)z = \cos(2x+3y)$$

Solⁿ:-

$$P.I = \frac{1}{(D^3 - 10D^2D' + D'^3)} \cdot \cos(2x+3y)$$

$$V = 2x + 3y \quad , \quad n = 3$$

$$\iiint \cos v (dv)^3 = \iint \sin v (dv)^2 = -\int \cos v \, dv$$

$$= -\sin v$$

$$= \frac{1}{2^3 - 10.2^2.3 + 3^2} \cdot \iiint \cos v (dv)^3$$

$$= \frac{1}{8 - 120 + 9} \cdot -\sin v$$

$$= + \frac{1}{85} \cdot + \sin(2x+3y) = \frac{1}{85} \sin(2x+3y)$$

Ans

Ques Find General Solution

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \cos mx \sin ny$$

Solⁿ:- Given equation,

$$(D^2 + D'^2)z = \cos mx \sin ny$$

Auxiliary equation,

$$D = m \quad \text{and} \quad D' = 1$$

$$m^2 + 1 = 0$$

$$m = \pm i$$

$$C.F = \phi_1(y + ix) + \phi_2(y - ix)$$

Here, $f(x,y) = \cos mx \sin ny$

$$= \frac{1}{2}[2\cos mx \sin ny]$$

$$= \frac{1}{2}[\sin(mx + ny) - \sin(mx - ny)]$$

$$P.I = \frac{1}{D^2 + D'^2} \cdot \frac{1}{2}[\sin(mx + ny) - \sin(mx - ny)]$$

$$= \frac{1}{2} \cdot \frac{1}{D^2 + D'^2} \cdot \sin(mx + ny) - \frac{1}{2} \cdot \frac{1}{D^2 + D'^2} \cdot \sin(mx - ny)$$

$$u = mx + ny, \quad v = mx - ny$$

$$= \frac{1}{2} \cdot \frac{1}{m^2 + n^2} \cdot \iint \sin u \,(du)^2 - \frac{1}{2} \frac{1}{m^2 + (-n)^2} \cdot \iint \sin v \,(dv)^2$$

$$= \frac{1}{2} \frac{1}{m^2 + n^2} \cdot \int (-\cos u)\, du - \frac{1}{2} \cdot \frac{1}{m^2 + n^2} \cdot \int (-\cos v)\, du$$

$$= \frac{1}{2} \frac{1}{m^2 + n^2} - \sin u + \frac{1}{2} \frac{1}{m^2 + n^2} \cdot \sin v$$

$$= \frac{1}{2} \frac{1}{m^2 + n^2} \sin(mx + ny) - \frac{1}{2} \frac{1}{m^2 + n^2} \sin(mx + ny)$$

General Solution,

$$z = C.f + P.I$$

$$z = \phi_1(y + ix) + \phi_2(y - ix) + \frac{1}{2} \cdot \frac{1}{m^2 + n^2} \sin(mx - ny) - \frac{1}{2} \frac{1}{m^2 + n^2} \cdot \sin(mx + ny)$$

Ans

Ques Solve :- $r - t = x - y$

Sol^n:- Given equation

$$(D^2 - D'^2)z = x - y$$

Auxiliary equation,

$$D = m \text{ and } D' = 1$$

$$m^2 - 1 = 0$$

$$m = \pm 1$$

$$C.F = \phi_1(y - x) + \phi_2(y + x)$$

$$P.I = \frac{1}{(D^2 - D'^2)} \cdot x - y$$

$$= \frac{1}{(D + D')(D - D')} \cdot (x - y)$$

$$= \frac{1}{D + D'} \left\{ \frac{1}{1 - (-1)} \cdot \int v \, dv \right\} \quad \text{where } v = x - y$$

$$= \frac{1}{D + D'} \cdot \frac{1}{2} \cdot \frac{v^2}{2}$$

$$= \frac{1}{D + D'} \cdot \frac{1}{4} \cdot (x - y)^2 = -\frac{1}{4} \cdot \frac{1}{(-1)D - 1D'} \cdot (x - y)^2$$

$$\because \frac{1}{(bD - aD')^n} \phi(ax + by) = \frac{x^n}{b^n \, n!} \phi(ax + by)$$

$$= -\frac{1}{4} \frac{x}{(-1)^1 \cdot 1!} (x - y)^2$$

$$= \frac{x}{4} (x - y)^2$$

Required General Solution.

$$Z = C.F + P.I$$

$$Z = \phi_1(y - x) + \phi_2(y + x) + \frac{x}{4}(x - y)^2 \quad \underline{\text{Ans}}$$

Ques. Solve :- $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} - \frac{\partial^3 z}{\partial x \partial y^2} + 2\frac{\partial^3 z}{\partial y^3} = e^{x+y}$

Solⁿ :- Given equation,

$$(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = e^{x+y}$$

Auxiliary equation,

$$D = m \;,\; D' = 1$$

$$m^3 - 2m^2 - m + 2 = 0$$

$$(m-1)(m-2)(m+1) = 0$$

$$m = 1, 2, -1$$

$$C.F = \phi_1(y+x) + \phi_2(y+2x) + \phi_3(y-x)$$

$$P.I = \frac{1}{(D-D')(D-2D')(D+D')} \cdot e^{x+y}$$

$$= \frac{1}{D-D'}\left[\frac{1}{(D-2D')(D+D')} \cdot e^{x+y}\right]$$

$$= \frac{1}{D-D'} \cdot \left[\frac{1}{(1-2)(1+1)} \cdot e^{x+y}\right]$$

$$= \frac{1}{D-D'} \cdot \left(-\frac{1}{2} e^{x+y}\right)$$

$$= -\frac{1}{2} \frac{1}{D-D'} \cdot e^{x+y}$$

$$= -\frac{1}{2} \cdot \frac{x}{1 \cdot 1!} e^{x+y}$$

$$= -\frac{1}{2} x e^{x+y}$$

General Solution,

$$Z = C.f + P.I$$

$$Z = \phi_1(y+x) + \phi_2(y+2x) + \phi_3(y-x) - \frac{1}{2} x e^{x+y}$$

Ans

Que Solve :- $(D^2 - 2DD' + D'^2)z = \tan(y+x)$

Solⁿ :- Given equation,

$$(D^2 - 2DD' + D'^2)z = \tan(y+x)$$

Auxiliary Equation,

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$$C.F = \phi_1(y+x) + x\phi_2(y+x)$$

$$P.I = \frac{1}{(D-D')^2} \cdot \tan(y+x)$$

$$= \frac{x^2}{1^2 \cdot 2!} \tan(y+x)$$

$$= \frac{x^2}{2} \tan(x+y)$$

General Solution

$$z = \phi_1(y+x) + x\phi_2(y+x) + \frac{x^2}{2}\tan(x+y)$$

Ans

Que Solve :- $(D^2 + 3DD' + 2D'^2)z = 2x+3y$

Solⁿ :- Auxiliary Equation,

$$m^2 + 3m + 2 = 0$$

$$m^2 + 2m + m + 2 = 0$$

$$(m+1)(m+2) = 0$$

$$m = -1, -2$$

$$C.F = \phi_1(y-x) + \phi_2(y-2x)$$

$$P.I = \frac{1}{(D^2 + 3DD' + 2D'^2)} \cdot 2x+3y$$

$$V = 2x + 3y, \quad n = 2$$

$$= \frac{1}{2^2 + 3.2.3 + 2.3^2} \cdot \iint v \, (dv)^2$$

$$= \frac{1}{40} \cdot \int \frac{v^2}{2} = \frac{1}{240} v^3 = \frac{1}{240}(2x+3y)^3$$

Required Solution,

$$z = \phi_1(y-x) + \phi_2(y-2x) + \frac{1}{240}(2x+3y)^3$$

Ans

Solve :- $(D^2 + 3DD' + 2D'^2)z = x + y$

Sol:- Given equation,

$$(D^2 + 3DD' + 2D'^2)z = x + y$$

Auxiliary equation,

$$D = m, \quad D' = 1$$

$$m^2 + 3m + 2 = 0$$

$$m^2 + 2m + m + 2 = 0$$

$$m(m+2) + 1(m+2) = 0$$

$$(m+1)(m+2) = 0$$

$$m = -1, -2$$

C.F $= \phi_1(y - x) + \phi_2(y - 2x)$

P.I $= \dfrac{1}{(D + 2D')(D + D')} \cdot x + y$

$$= \frac{1}{(D + 2D')}\left[\frac{1}{(D + D')} x + y\right]$$

$$= \frac{1}{(D + 2D')} \int x + a + x \, dx \qquad \{y = a + x\}$$

$$= \frac{1}{(D + 2D')} \int 2x + a \, dx$$

$$= \frac{1}{(D + 2D')} (x^2 + ax)$$

$$= \frac{1}{(D + 2D')} \cdot (x^2 + (y - x)x) \qquad \{a = y - x\}$$

$$= \frac{1}{(D + 2D')} \cdot (\cancel{x^2} + xy - \cancel{x^2})$$

$$= \frac{1}{(D + 2D')} \cdot (xy)$$

Que Solve :- $(D^2 + DD' - 2D'^2)z = \sqrt{2x+y}$

Solⁿ :- Auxiliary Equation

$$m^2 + m - 2 = 0$$

$$(m-1)(m+2) = 0$$

$$m = 1, -2$$

$$C.F = \phi_1(y+x) + \phi_2(y-2x)$$

$$P.I = \frac{1}{(D+2D')(D-D')}\sqrt{2x+y}$$

$$= \frac{1}{D+2D'}\left[\frac{1}{D-D'}\sqrt{2x+y}\right]$$

$$= \frac{1}{D+2D'}\int \sqrt{x+a}\,dx \qquad \{y = a - x\}$$

$$= \frac{1}{D+2D'}\cdot\frac{2}{3}(x+a)^{3/2}$$

$$= \frac{1}{D+2D'}\cdot\frac{2}{3}(x+y+x)^{3/2} \qquad \{a = x+y\}$$

$$= \frac{1}{D+2D'}\cdot\frac{2}{3}(2x+y)^{3/2}$$

$$= \int \frac{2}{3}(2x+b+2x)^{3/2}dx \qquad \{y = b+2x\}$$

$$= \frac{2}{3}\int (4x+b)^{3/2}dx$$

$$= \frac{\cancel{2}}{3}\cdot\frac{\cancel{2}}{5}\times\frac{1}{\cancel{4}}(4x+b)^{5/2}$$

$$= \frac{1}{15}(4x+b)^{5/2}$$

Complete Solution,

$$Z = \phi_1(y+x) + \phi_2(y-2x) + \frac{1}{15}(4x+b)^{5/2}$$

Ans

1) Solve :- $(D^2 - 2DD' - 15D'^2)z = 12xy$

Sol:- Auxiliary Equation

$$m^2 - 2m - 15 = 0$$

$$m^2 - 5m + 3m - 15 = 0$$

$$m(m-5) + 3(m-5) = 0$$

$$(m-5)(m+3) = 0$$

$$m = -3, 5$$

$$C.F = \phi_1(y - 3x) + \phi_2(y + 5x)$$

$$P.I = \frac{1}{(D^2 - 2DD' - 15D'^2)} \cdot 12xy$$

$$= \frac{1}{(D+3D')(D-5D')} \cdot 12xy$$

$$= \frac{12}{(D+3D')}\left[\frac{1}{(D-5D')} \cdot xy\right]$$

$$= \frac{12}{(D+3D')} \int x(a-5x)\,dx \qquad \{y = a - 5x\}$$

$$= \frac{12}{(D+3D')} \cdot \int (ax - 5x^2)\,dx$$

$$= \frac{12}{D+3D'} \cdot \frac{ax^2}{2} - \frac{5}{3}x^3$$

$$= \frac{12}{D+3D'} \cdot \frac{(y+5x)x^2}{2} - \frac{5x^3}{3} \qquad \{a = y + 5x\}$$

$$= \frac{12}{D+3D'} \cdot \frac{1}{6}\left[3x^2y + 15x^3 - 10x^3\right]$$

$$= \frac{2}{(D+3D')} \cdot (3x^2y + 5x^3)$$

$$y = 3x + b$$

$$= 2\int (3x^2(3x+b) + 5x^3)\,dx$$

$$= 2\int (9x^3 + 3x^2b + 5x^3)\,dx$$

$$= 2\int (14x^3 + 3x^2b)\,dx$$

$$= 2 \cdot \frac{14}{4} \cdot x^4 + x^3 b$$

$$= 7x^4 + x^3 b$$

$$= 7x^4 + 2x^3(y - 3x) \qquad \{ b = y - 3x \}$$

$$= 7x^4 + 2x^3 y - 6x^4$$

$$= x^4 + 2x^3 y$$

Complete Solution,

$$z = \phi_1(y + 5x) + \phi_2(y - 3x) + x^4 + 2x^3 y \quad \underline{\text{Ans}}$$

3) Solution of Linear Non-Homogeneous D.E

$$F(D, D') = 0$$

$$(D - mD' - \alpha)z = 0$$

$$(D - m_1 D' - \alpha_1)(D - m_2 D' - \alpha_2) + \dots (D - m_n D' - \alpha_n)$$

$$z = e^{\alpha_1 x} \phi_1(y + m_1 x) + e^{\alpha_2 x} \phi_2(y + m_2 x) + \dots$$

If Roots are repeated,

$$z = e^{\alpha_1 x} [\phi_1(y + m_1 x) + x\phi_2(y + m_1 x) + x^2 \phi_3(y + m_1 x)]$$

Ques. Solve :- $(D - D' - 1)(D - D' - 2)z = e^{2x - y}$

Sol^n:- Given equation,

$$(D - D' - 1)(D - D' - 2)z = e^{2x - y}$$

$$\alpha_1 = 1, \quad \alpha_2 = 2$$

$$m_1 = 1, \quad m_2 = 1$$

$$\text{C.f}, \; z = e^{x} \phi_1(y + x) + e^{2x} \phi_2(y + x)$$

$$\text{P.I}, = \frac{1}{(D - D' - 1)(D - D' - 2)} \cdot e^{2x - y}$$

$$= \frac{1}{(2 - (-1) - 1)(2 + 1 - 2)} \cdot e^{2x - y} = \frac{1}{2} e^{2x - y}$$

General Solution

$$Z = e^{x}\phi_1(y+x) + e^{2x}\phi_2(y+x) + \frac{1}{2}e^{x+y}$$ Ans

Ques Solve :- $(D^2 + 2DD' + D' - 1) = \sin(x+2y)$

Sol^n:- Given equation,

$$(D^2 + 2DD' + D' - 1)z = \sin(x+2y)$$

~~Auxiliary~~ equation

$$[(D^2-1) + (D+1)D']z = \sin(x+2y)$$

$$[(D+1)(D-1) + (D+1)D']z = \sin(x+2y)$$

$$(D+1)(D+D'-1)z = \sin(x+2y)$$

$$C.F = e^{-x}\phi_1(y) + e^{x}\phi_2(y-x)$$

$$P.I = \frac{1}{(D^2 + 2DD' + D' - 1)} \cdot \sin(x+2y)$$

$$= \frac{1}{-1^2 + (-1).2 + D' - 1}\sin(x+2y)$$

$$= \frac{1}{D' - 4}\sin(x+2y)$$

$$= \frac{D'+4}{(D'-4)(D'+4)}\sin(x+2y)$$

$$= \frac{D'+4}{D'^2 - 16}\sin(x+2y)$$

$$= \frac{D'+4}{-2^2 - 16}\sin(x+2y)$$

$$= \frac{1}{-20}[D'\sin(x+2y) + 4\sin(x+2y)]$$

$$= -\frac{1}{20}[\cos(x+2y) + 4\sin(x+2y)]$$

General Solution,

$$Z = e^{-x}\phi_1(y) + e^{x}\phi_2(y-x) - \frac{1}{20}[\cos(x+2y) + 4\sin(x+2y)]$$

Ans

Ques Solve :- $x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = xy$

Soln :-

$x\frac{\partial z}{\partial x} = Dz \quad , \quad x^2\frac{\partial^2 z}{\partial x^2} = D(D-1)$

$y\frac{\partial z}{\partial y} = D'z \quad , \quad y^2\frac{\partial^2 z}{\partial y^2} = D'(D'-1)$

from given equation,

$[D(D-1) - D'(D'-1)] = xy$

$x = e^X \quad , \quad y = e^Y$

$X = \log x \quad , \quad Y = \log y$

$[D^2 - D - D'^2 + D'] = e^X . e^Y$

$[D^2 - D'^2 - D + D'] = e^{X+Y}$

$[(D-D')(D+D') - (D-D')] = e^{X+Y}$

$(D-D')(D+D'-1)z = e^{X+Y}$

$C.F = \phi_1(Y+X) + e^X\phi_2(Y-X)$

$= \phi_1(\log y + \log x) + e^x\phi_2(\log y - \log x)$

$= \phi_1\log(xy) + e^x\phi_2\log(y/x)$

$P.I = \frac{1}{(D-D')(D+D'-1)} . e^{X+Y}$

$= \frac{1}{(D-D')(1+1-1)} . e^{X+Y} = \frac{1}{(D-D')} . e^{X+Y}$

$= e^{X+Y} . \frac{1}{D}\left[1 - \frac{D'}{D}\right]^{-1} . 1$

$= e^{X+Y}\left[\frac{1}{D}\left(1 + \frac{D'}{D}\right)\right]1$

$= e^{X+Y}\frac{1}{D}(1) = Xe^{X+Y}$

$= X e^X . e^Y$

$= X.(xy) = \log x(xy)$

Monge's Methods

$$Rr + Ss + Tt = V \qquad - (1)$$

where R, S, T and V are function of x, y, z, p and q.

we know that, $dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy$

$$dp = r\,dx + s\,dy \qquad - (2)$$

And, $dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy$

$$dq = s\,dx + t\,dy \qquad - (3)$$

Substituting these value in eqn (1),

from eqn (1) & (2)

$$r = \frac{dp - s\,dy}{dx} \quad \text{and} \quad t = \frac{dq - s\,dx}{dy}$$

$$R\left(\frac{dp - s\,dy}{dx}\right) + Ss + T\left(\frac{dq - s\,dx}{dy}\right) = V$$

$$R\,dy\,(dp - s\,dy) + Ss\,dx\,dy + T(dq - s\,dx) = V\,dx\,dy$$

$$(R\,dp\,dy + T\,dq\,dx - V\,dx\,dy) - s(R\,dy^2 - S\,dx\,dy + T\,dx^2) = 0$$

$$R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0 \qquad - (4)$$

$$R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0 \qquad - (5)$$

Equation (4) and (5) are Monge's Subsidiary Equation

Ques Solve :- $pt - qs = q^3$

Sol^n:- Given equation,

$$pt - qs = q^3 \quad - (1)$$

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy$$

$$dp = r\,dx + s\,dy \quad - (2)$$

$$dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy$$

$$dq = s\,dx + t\,dy \quad - (3)$$

from eqn (2) and (3)

$$r = \frac{dp - s\,dy}{dx}, \quad t = \frac{dq - s\,dx}{dy}$$

Substituting the value of t in eqn (1)

$$p \cdot \frac{dq - s\,dx}{dy} - qs = q^3$$

$$p\,dq - ps\,dx - qs\,dy = q^3 dy$$

$$(p\,dq - q^3 dy) - s(p\,dx + q\,dy) = 0$$

$\therefore$ Monge's subsidiary equation,

$$p\,dq - q^3 dy = 0 \quad - (4)$$

$$p\,dx + q\,dy = 0 \quad - (5)$$

from eqn (5) $\quad dz = 0$

On integration, $z = A$

from eqn (4) and (5)

$$q\,dy = -p\,dx$$

$$p\,dq - q^2(q\,dy) = 0$$

$$p\,dq - q^2(-p\,dx) = 0$$

$$p(dq + q^2 dx) = 0$$

$$dq + q^2 dx = 0$$

$$\frac{dq}{q^2} + dx = 0$$

On integration,

$$-\frac{1}{q} + x = B \quad \Rightarrow \quad \frac{1}{q} - x = -B$$

$$q = \frac{\partial z}{\partial y}$$

$$\frac{\partial y}{\partial z} - x = f(z)$$

$$dy - x\,dz = f(z)\,dz$$

On integration, $y - xz = \int f(z)\,dz$

$$y - xz = \phi(z) + c$$

where $\int f(z)\,dz = \phi(z)$

$$\cancel{y = x} \quad y = xz + \phi(z) + c \qquad \underline{\text{Ans}}$$

Ques Solve :- $x^2 r + 2xys + y^2 t = 0$

Sol^n :-

$$dp = r\,dx + s\,dy$$

$$dq = s\,dx + t\,dy$$

$$\therefore \quad r = \frac{dp - s\,dy}{dx}, \quad t = \frac{dq - s\,dx}{dy}$$

Substituting these value in given equation,

$$x^2\left(\frac{dp - s\,dy}{dx}\right) + 2xys + y^2\left(\frac{dq - s\,dx}{dy}\right) = 0$$

$$x^2 dp\,dy - x^2 s\,dy^2 + 2xys\,dx\,dy + y^2 dq - y^2 s\,dx = 0$$

$$(x^2 dp\,dy + y^2 dq\,dx) - s(x^2 dy^2 - 2xy\,dx\,dy + y^2 dx^2) = 0$$

Monge's Subsidiary Equation.

$$x^2 dp\,dy + y^2 dq\,dx = 0 \quad - (1)$$

$$x^2 dy^2 - 2xy\,dx\,dy + y^2 dx^2 = 0 \quad - (II)$$

Equation (II) can be written as,

$$(x\,dy - y\,dx)^2 = 0$$

$$x\,dy - y\,dx = 0 \quad \Rightarrow \quad x\,dy = y\,dx \quad - (III)$$

$$\frac{dy}{y} - \frac{dx}{x} = 0$$

On integration,

$$\log y - \log x = \log A$$

$$\frac{y}{x} = A$$

from eq^n (I) and (III)

$$x\,dp\,(x\,dy) + y\,dq\,(y\,dx) = 0$$

$$x\,dp + y\,dq = 0$$

Adding $p\,dx + q\,dy$ both side,

$$x\,dp + y\,dq + p\,dx + q\,dy = p\,dx + q\,dy$$

$$d(px) + d(qy) = dz$$

On integration, $px + qy = z + B$

$$xp + yq = z + f\left(\frac{y}{x}\right)$$

$$xp + yq = z + f(A)$$

from lagrange's subsidiary equation,

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z + f(A)}$$

from first two ratio,

$$\frac{y}{x} = A$$

from first and third ratio,

$$\log x + \log c = \log\,[z + f(A)]$$

$$cx = z + f(A)$$

$$z = cx - f(A)$$

or $$z = x f_1\left(\frac{y}{x}\right) - f_2\left(\frac{y}{x}\right)$$

Ques Solve :- $2r + te^x - (rt - s^2) = 2e^x$

Sol^n :-

$$r = \frac{dp - s\,dy}{dx} \quad , \quad t = \frac{dq - s\,dx}{dy}$$

Putting these value in given equation,

$$2 \cdot \frac{dp - s\,dy}{dx} + \frac{dq - s\,dx}{dy} \cdot e^x - \left(\frac{dp - s\,dy}{dx} \cdot \frac{dq - s\,dx}{dy} - s^2 \right) = 2e^x$$

$$\frac{2\,dp\,dy - 2s\,dy^2 + e^x dq\,dx - e^x s\,dx^2}{dx\,dy} - \left(\frac{dp\,dq - s\,dx\,dp - s\,dy\,dq + s^2 dx\,dy}{dx\,dy} - s^2 \right) = 2e^x$$

$$2dp\,dy - 2s\,dy^2 + e^x dq\,dx - e^x s\,dx^2 - (dp\,dq - s\,dx\,dp - s\,dy\,dq + \cancel{s^2 dx\,dy} - \cancel{s^2 dx\,dy}) = 2e^x dx\,dy$$

$$2dp\,dy - 2s\,dy^2 + e^x dq\,dx - e^x s\,dx^2 - dp\,dq + s\,dx\,dp + s\,dy\,dq - 2e^x dx\,dy = 0$$

$$2\,dp\,dy + e^x dq\,dx - dp\,dq - 2e^x dx\,dy - s(2dy^2 + e^x dx^2 - dp\,dx - dq\,dy) = 0$$

$\therefore$ Monge's Subsidiary Equation,

$$2dp\,dy + e^x dq\,dx - dp\,dq - 2e^x dx\,dy = 0 \quad -(1)$$

$$2\,dy^2 + e^x dx^2 - dp\,dx - dq\,dy = 0$$

from equation (1),

$$dp(2dy - dq) - e^x dx(-dq + 2dy) = 0$$

$$(dp - e^x dx)(2dy - dq) = 0$$

$$dp - e^x dx = 0 \quad \text{and} \quad 2dy - dq = 0$$

On integration,

$$p - e^x = A \quad \text{and} \quad 2y - q = B$$

$$p = e^x + A \quad \text{and} \quad q = 2y - B$$

Substituting these value of p ad q is

$$dz = p\,dx + q\,dy$$

$$dz = (e^x + A)\,dx + (2y - B)\,dy$$ Ans

Integrating, we get

$$\boxed{z = e^x + Ax + y^2 - By + C}$$ Ans