Unit - III

First Order Partial Differential Equation

Partial differential equation

A differential equation, containing one or more partial derivate is called a partial differential equation

Equation - $x\frac{dz}{dx} + y\frac{dz}{dy} + z = 0$

Notation - $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$, $\frac{\partial^2 z}{\partial x^2} = r$, $\frac{\partial^2 z}{\partial x \partial y} = s$, $\frac{\partial^2 z}{\partial y^2} = t$

Derivation of Partial differentiation Equation By Elimination of Arbitrary Constant

Que find P.D.E by eliminating a & b

$$z = ax + by + ab$$

Solⁿ:- Given equation,

$$z = ax + by + ab \quad - (1)$$

Differentiating (1) partially w.r.t 'x' and 'y'

$$p = \frac{\partial z}{\partial x} = a$$

$$q = \frac{\partial z}{\partial y} = b$$

Eliminating 'a' and 'b' from eqn (1)

$$z = px + qy + pq \quad \text{Ans}$$

Que Eliminate a, b, c from algebric equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Solⁿ:- Given equation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad - (i)$$

Differentiating (i) partially w.r.t x and y

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \quad - (ii)$$

$$\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0 \quad - (iii)$$

Again differentiating (ii) w.r.t x

$$\frac{2}{a^2} + \frac{2}{c^2}\left\{\frac{\partial z}{\partial x}\frac{\partial z}{\partial x} + z\frac{\partial^2 z}{\partial x^2}\right\} = 0$$

$$\frac{1}{a^2} + \frac{1}{c^2}\left\{\left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2}\right\} = 0$$

$$\frac{c^2}{a^2} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0 \quad - (iv)$$

from eqn (ii) $\quad \frac{c^2}{a^2} = -\frac{z}{x}\frac{\partial z}{\partial x}$

Substituting in eqn (iv)

$$\frac{-z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0$$

$$-z\frac{\partial z}{\partial x} + x\left(\frac{\partial z}{\partial x}\right)^2 + xz\frac{\partial^2 z}{\partial x^2} = 0 \quad - (v)$$

Differentiating (iii) partially w.r.t 'y'

$$\frac{2}{b^2} + \frac{2}{c^2}\left\{\frac{\partial z}{\partial y}\cdot\frac{\partial z}{\partial y} + z\frac{\partial^2 z}{\partial y^2}\right\} = 0$$

$$\frac{1}{b^2} + \frac{1}{c^2}\left\{\left(\frac{\partial z}{\partial y}\right)^2 + z\frac{\partial^2 z}{\partial y^2}\right\} = 0$$

$$\frac{c^2}{b^2} + \left(\frac{\partial z}{\partial y}\right)^2 + z\frac{\partial^2 z}{\partial y^2} = 0 \quad - (vi)$$

from eqn (iii) $\quad \frac{c^2}{b^2} = -\frac{z}{y}\frac{\partial z}{\partial y}$

Substituting in eqn (vi)

$$\frac{-z}{y}\frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial y}\right)^2 + z\frac{\partial^2 z}{\partial y^2} = 0$$

$$-z\frac{\partial z}{\partial y} + y\left(\frac{\partial z}{\partial y}\right)^2 + yz\frac{\partial^2 z}{\partial y^2} = 0 \quad - (vii)$$

Equation (v) & (vii) is Required Partial differential equation.

Ques find P.D.E by eliminating a and b

$$z = (x-a)^2 + (y-b)^2$$

Sol^n:- Given equation,

$$z = (x-a)^2 + (y-b)^2 \quad -(1)$$

Differentiating (1) partially w.r.t x and y, we have

$$\frac{\partial z}{\partial x} = 2(x-a) = p \quad \Rightarrow \quad x-a = \frac{p}{2}$$

$$\frac{\partial z}{\partial y} = 2(y-b) = q \quad \Rightarrow \quad y-b = \frac{q}{2}$$

Eliminating $(x-a)$ & $(y-b)$ from eq^n (1)

$$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$$4z = p^2 + q^2 \quad \text{An}$$

Required Partial differential equation.

Ques find P.D.E by eliminating a and b

$$z = (x^2+a)(y^2+b)$$

Sol^n:- Given equation,

$$z = (x^2+a)(y^2+b) \quad -(1)$$

Differentiating (1) partially w.r.t x and y we get

$$\frac{\partial z}{\partial x} = 2x(y^2+b) = p \quad \Rightarrow \quad y^2+b = \frac{p}{2x}$$

$$\frac{\partial z}{\partial y} = 2y(x^2+a) = q \quad \Rightarrow \quad x^2+a = \frac{q}{2y}$$

Eliminating (x^2+a) and (y^2+b),

$$z = \frac{q}{2y} \cdot \frac{p}{2x}$$

$$4xyz = pq \quad \text{An}$$

Required Partial differential equation

(#) Elimination of Arbitrary function

Ques Obtain the P.D.E by eliminating the arbitrary function f

$$z = f\left(\frac{y}{x}\right)$$

Sol^n:- Given equation,

$$z = f\left(\frac{y}{x}\right) \quad -(1)$$

Differentiating (1) partially w.r.t x and y

$$p = \frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right) \quad -(2)$$

$$f'\left(\frac{y}{x}\right) = -\frac{x^2 p}{y}$$

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$$q = \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right) \cdot \frac{1}{x}$$

$$f'\left(\frac{y}{x}\right) = xq \qquad - (3)$$

Eliminating f' from eqn (2) & (3)

$$-\frac{x^2 p}{y} = xq$$

$$xp + yq = 0$$

Req. Partial differential equation.

Que Eliminate the arbitrary function

$$z = f(x + ay)$$

Sol:- Given eqn, $z = f(x + ay) \quad - (1)$

Differentiating (1) partially w.r.t x and y

$$p = \frac{\partial z}{\partial x} = f'(x + ay) \cdot 1 \Rightarrow p = f'(x + ay) \qquad - (ii)$$

$$q = \frac{\partial z}{\partial y} = f'(x + ay) \cdot a \Rightarrow q/a = f'(x + ay) \qquad - (iii)$$

Eliminating arbitrary function from eqn (ii) & (iii)

$$p = \frac{q}{a}$$

$ap = q$ Req. Partial differential equation.

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Que (ii) $y = f(x - at) + \phi(x + at)$

Sol:- Given equation, $y = f(x - at) + \phi(x + at)$

Differentiating Partially w.r.t x and t we get

$$\frac{\partial y}{\partial x} = f'(x - at) + \phi'(x + at) \qquad - (i)$$

$$\frac{\partial y}{\partial t} = -a f'(x - at) + a \phi'(x + at) \qquad - (ii)$$

Again differentiating partially eqn (i) and (ii) we get

$$\frac{\partial^2 y}{\partial x^2} = f''(x - at) + \phi''(x + at) \qquad - (iii)$$

$$\frac{\partial^2 y}{\partial t^2} = a^2 f''(x - at) + a^2 \phi(x + at) \qquad - (iv)$$

from eqn (iii) & (iv) we get

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Required Partial differential different equation

Que :- Solve :- $xzp + yzq = xy$

Solⁿ:- Here, $P = xz$, $Q = yz$, $R = xy$

Lagrange Equation,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$$

Taking first two equation, we get

$$\frac{dx}{x} = \frac{dy}{y}$$

On integration, $\log x = \log y + \log c_1$

$$\frac{x}{y} = c_1$$

Taking last two equation, we get

$$\frac{dy}{z} = \frac{dz}{yc_1} \qquad \left\{ \because \frac{x}{y} = c_1 \right\}$$

$$c_1 y \, dy = z \, dz$$

On integration, $c_1 y^2 - z^2 = c_2$

$$\frac{x}{y} \cdot y^2 - z^2 = c_2$$

$$xy - z^2 = c_2$$

Required General Solution,

$$\phi\left(\frac{x}{y}, xy - z^2\right) = 0 \qquad \text{Ans}$$

Que Solve :- $x^2p + y^2q = nxy$

Solⁿ:- Here, $P = x^2$, $Q = y^2$, $R = nxy$

Lagrange's Equation,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{nxy}$$

from first two equation, we get

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

On integration, $-\frac{1}{x} = -\frac{1}{y} + a$

$$a = \frac{1}{y} - \frac{1}{x} \qquad \Rightarrow \frac{1}{x} = \frac{1}{y} - a$$

Taking last two fraction, we get

$$\frac{dy}{y^2} = \frac{dz}{nxy} \Rightarrow \frac{dy}{y} = \frac{dz}{nx}$$

$$n\frac{dy}{y} = \left(\frac{1}{y} - a\right) dz$$

$$dz = \frac{n}{(1-ay)} dy$$

On integration, $\int dz = \int \frac{n}{(1-ay)} dy$

$$z = -\frac{1}{a} \cdot n \cdot \log(1-ay) + b$$

$$z = -\frac{n}{\frac{1}{y} - \frac{1}{x}} \cdot \log\left[1 - \left(\frac{1}{y} - \frac{1}{x}\right)y\right] + b$$

$$z = -\frac{nxy}{x-y} \log\left[1 - 1 + \frac{y}{x}\right] + b$$

$$z = \frac{nxy}{y-x} \log\left(\frac{y}{x}\right) + b$$

$$b = z - \frac{nxy}{y-x} \log\frac{y}{x}$$

Hence, Required General Solution

$$\phi(a, b) = \phi\left(\frac{1}{y} - \frac{1}{x},\ z - \frac{nxy}{y-x} \log\frac{y}{x}\right) = 0$$

where ϕ is an arbitrary function

Ques Solve : $p + q = x + y + z$

Solⁿ:- Given equation,

$$P + q = x + y + z$$

Here, $P = 1$, $Q = 1$, $R = x + y + z$

Lagrange's Equation,

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{x+y+z}$$

from first two fraction,

$$dx = dy$$

On Integration,

$$x = y + c_1$$

$$x - y = c_1$$

from Second and third fraction,

$$dy = \frac{dz}{x+y+z}$$

$$\frac{dz}{dy} = x + y + z$$

$$\frac{dz}{dy} - z = x + y$$

$$\frac{dz}{dy} - z = c_1 + 2y$$

Integrating factor, $I.F = e^{\int -dy} = e^{-y}$

Solution, $z \cdot e^{-y} = \int (c_1 + 2y) e^{-y} dy + c_1$

$$z \cdot e^{-y} = (c_1 + 2y)\int e^{-y} dy - \int\left[\frac{d(c_1+2y)}{dy}\int e^{-y} dy\right] dy$$

$$z e^{-y} = (c_1 + 2y)(-e^{-y}) - \int 2 \cdot (-e^{-y})\, dy$$

$$z \cdot e^{-y} = (c_1 + 2y)(-e^{-y}) + 2(-e^{-y}) + c_2$$

$$z e^{-y} = -e^{-y}(c_1 + 2y + 2) + c_2$$

$$z e^{-y} = -e^{-y}(x - y + 2y + 2) + c_2$$

$$z e^{-y} + e^{-y}(x + y + 2) = c_2$$

$$e^{-y}(z + x + y + 2) = c_2$$

Hence Required General Solution,

$$\phi(c_1, c_2) = \phi(x - y,\ z + x + y + 2) = 0$$

Que Solve :- $(y+z)p + (z+x)q = x+y$

Solⁿ:- Here, $P = y+z$, $Q = z+x$, $R = x+y$

Lagrange's Equation,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

$$\therefore \frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{-(x-y)} = \frac{dy-dz}{-(y-z)}$$

Taking first two fraction,

$$\frac{dx+dy+dz}{x+y+z} = -2\,\frac{dx-dy}{x-y}$$

On integration,

$$\log(x+y+z) = -2\log(x-y) + \log c_1$$

$$\log(x+y+z) = -\log(x-y)^2 + \log c_1$$

$$\log(x+y+z) + \log(x-y)^2 = \log c_1$$

$$(x+y+z)(x-y)^2 = c_1$$

Taking last two fraction,

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$

On integration,

$$\log(x-y) = \log(y-z) + \log c_2$$

$$\log(x-y) - \log(y-z) = \log c_2$$

$$\frac{x-y}{y-z} = c_2$$

Required General Solution,

$$\phi(c_1, c_2) = \phi\left((x+y+z)(x-y)^2, \frac{x-y}{y-z}\right) = 0$$

Ans

Que Solve :- $(mz-ny)p + (nx-lz)q = ly - mx$

Solⁿ:- Here, $P = mz-ny$, $Q = nx-lz$, $R = ly-mx$

Lagrange Equation,

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

Using x, y, z multiplier,

$$\frac{x\,dx + y\,dy + z\,dz}{0} = 0$$

$$x\,dx + y\,dy + z\,dz = 0$$

On integration,

$$x^2 + y^2 + z^2 = c_1$$

Again using l, m, n multiplier,

$$\frac{l\,dx + m\,dy + n\,dz}{0} = 0$$

$$l\,dx + m\,dy + n\,dz = 0$$

On integration,

$$lx + my + nz = c_2$$

Hence General Solution,

$$\phi(x^2+y^2+z^2,\ lx+my+nz) = 0$$

Ans

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* Standard form I

Equation of the form

$$f(p, q) = 0 \quad -(1)$$

Complete integral of eqn (1)

$$z = ax + by + c \quad -(2)$$

where a, b are connected by relation

$$f(a, b) = 0 \quad -(3)$$

from eqn (2) $p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$

Let $b = \phi(a)$

from eqn (2) $z = ax + \phi(a) + c \quad -(4)$

Taking $c = \psi(a)$, where ψ is arbitrary function

Equation (4) becomes

$$z = ax + \phi(a) + \psi(a)$$

General Integral,

Differentiating w.r.t 'a'

$$0 = x + \phi'(a) + \psi'(a)$$

Singular Integral,

Diff. Partially eq (4) w.r.t 'a' and 'c'

$$z = ax + \phi(a) + c$$

$$0 = x + \phi'(a)\, y$$

Que Solve :- $p^2 + q^2 = m^2$

Solⁿ: Given equation is in standard form I, in the form $f(p,q)=0$

Hence its solution is,

$$z = ax + by + c \quad -(1)$$

$$p = \frac{\partial z}{\partial x} = a, \quad q = \frac{\partial z}{\partial y} = b$$

from given equation

$$a^2 + b^2 = m^2 \Rightarrow b = \sqrt{m^2 - a^2} \quad -(2)$$

Complete integral,

$$z = ax + \sqrt{m^2 - a^2}\cdot y + c \quad -(3)$$

General integral,

$c = \psi(a)$, where ψ is arbitrary function

from eqn (3)

$$z = ax + \sqrt{m^2 - a^2}\cdot y + \psi(a) \quad -(4)$$

Differentiating (4) w.r.t a, we get

$$0 = x - \frac{2a}{2\sqrt{m^2 - a^2}}\cdot y + \psi'(a)$$

$$0 = x - \frac{a}{\sqrt{m^2 - a^2}}\cdot y + \psi(a) \quad -(5)$$

The general integral is obtained by eliminating 'a' from equations (4) and (5)

Que Solve : $p + q = pq$

Solⁿ:-

Given equation, $p + q = pq$

Complete integral is given by,

$$z = ax + by + c \quad - (1)$$

from given equation, $a + b = ab \Rightarrow a = ab - b$

$$b = a/a-1$$

Putting the value of b in eqn (1)

$$z = ax + \frac{ay}{(a-1)} + c \quad - (2)$$

General Integral, Let $c = \varphi(a)$, where φ is arbitrary function

Equation (2) becomes, $z = ax + \frac{ay}{(a-1)} + \varphi(a) \quad - (3)$

Differentiating (3) w.r.t 'a'

$$0 = x - \frac{y}{(a-1)^2} + \varphi'(a) \quad - (4)$$

General Integral can be obtained by eliminating 'a' from (3) & (4).

Que Solve :- $x^2p^2 + y^2q^2 = z^2$

Solⁿ:- Given equation,

$$x^2p^2 + y^2q^2 = z^2$$

Let $x = e^X \Rightarrow X = \log x$

$y = e^Y \Rightarrow Y = \log y$

$z = e^Z \Rightarrow Z = \log z$

from given equation,

$$\frac{x^2p^2}{z^2} + \frac{y^2q^2}{z^2} = 1$$

$$\left(\frac{xp}{z}\right)^2 + \left(\frac{yq}{z}\right)^2 = 1$$

$$\left(\frac{x}{z} \cdot \frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z} \cdot \frac{\partial z}{\partial y}\right)^2 = 1 \quad - (1)$$

Now, $\partial X = \frac{1}{x}\partial x$, $\partial Y = \frac{1}{y} \cdot \partial y$, $\partial Z = \frac{1}{z} \cdot \partial z$

Equation (1) becomes

$$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1$$

$$P^2 + Q^2 = 1$$

where, $P = \frac{\partial Z}{\partial X}$, $Q = \frac{\partial Z}{\partial Y}$

Complete integral is given by,

$$Z = aX + bY + c_1 \quad - (11)$$

$$a^2 + b^2 = 1 \Rightarrow b = \sqrt{1 - a^2}$$

Putting the value of b in eqn (11), we get

$$Z = aX + \sqrt{1-a^2}\,Y + c_1$$

$$\log z = a\log x + \sqrt{1-a^2}\,\log y + c_1$$

putting $a = \cos\alpha$ and $c_1 = \log c$

$$\log z = \cos\alpha \log x + \sin\alpha \log y + \log c$$
$$\log z = \log x^{\cos\alpha} + \log y^{\sin\alpha} + \log c$$
$$z = c\,x^{\cos\alpha} y^{\sin\alpha} \quad - \text{(iii)}$$

General Integral, $c = \psi(a)$ where ψ is arbitrary constant

Equation (iii) becomes,

$$z = \psi(a)\,x^{\cos\alpha} y^{\sin\alpha}$$

Differentiating w.r.t 'a' we get

$$0 = \psi'(a)\,x^{\cos\alpha} y^{\sin\alpha} + \psi(a)\,x^{\cos\alpha} y^{\sin\alpha}(-\sin\alpha)\log x + \psi(a)\,x^{\cos\alpha} y^{\sin\alpha}\cos\alpha \log y \quad - \text{(iv)}$$

General integral is obtained by eliminating α from eqn (iii) & (iv)

Singular Integral - Obtained by eliminating α,

$$z = c\,x^{\cos\alpha} y^{\sin\alpha}$$
$$\frac{\partial z}{\partial \alpha} = 0 = -c\sin\alpha\, x^{\cos\alpha} y^{\sin\alpha}\log x + c\cos\alpha \cdot x^{\cos\alpha} \cdot y^{\sin\alpha}\log y$$
$$\frac{\partial z}{\partial c} = 0 = x^{\cos\alpha} y^{\sin\alpha}$$

Thus Singular integral is $z = 0$

Ques Solve :- $(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1$

Solⁿ:- Let, $x+y = X^2 \Rightarrow X = \sqrt{x+y}$

$x-y = Y^2 \Rightarrow Y = \sqrt{x-y}$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y}$$
$$= \frac{\partial z}{\partial X}\cdot\frac{1}{2\sqrt{x+y}} + \frac{\partial z}{\partial Y}\cdot\frac{1}{2\sqrt{x-y}}$$
$$= \frac{1}{2X}\cdot\frac{\partial z}{\partial X} + \frac{1}{2Y}\cdot\frac{\partial z}{\partial Y}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y}$$
$$= \frac{\partial z}{\partial X}\cdot\frac{1}{2\sqrt{x+y}} - \frac{\partial z}{\partial Y}\cdot\frac{1}{2\sqrt{x-y}}$$
$$= \frac{1}{2X}\cdot\frac{\partial z}{\partial X} - \frac{1}{2Y}\cdot\frac{\partial z}{\partial Y}$$

$p+q = \frac{1}{X}\cdot\frac{\partial z}{\partial X}$ and $p-q = \frac{1}{Y}\frac{\partial z}{\partial Y}$

Substituting the value in given equation,

$$X^2\cdot\frac{1}{X^2}\cdot\left(\frac{\partial z}{\partial X}\right)^2 + Y^2\cdot\frac{1}{Y^2}\cdot\left(\frac{\partial z}{\partial Y}\right)^2 = 1$$
$$\left(\frac{\partial z}{\partial X}\right)^2 + \left(\frac{\partial z}{\partial Y}\right)^2 = 1$$
$$P^2 + Q^2 = 1 \quad - \text{(1)}$$

$P = \frac{\partial z}{\partial x}$, $Q = \frac{\partial z}{\partial y}$

Eqn (1) is standard form I.

Complete integral is given by,

$$z = ax + by + c \quad - (2)$$

$$a^2 + b^2 = 1 \rightarrow b = \sqrt{1-a^2}$$

from eqn (2), $z = ax + \sqrt{1-a^2} \cdot y + c$

$$z = a\sqrt{x+y} + \sqrt{1-a^2} \cdot \sqrt{x-y} + c$$ Ans

* Standard form II

Equation of the form,

$$f(z, p, q) = 0 \quad - (1)$$

Let $z = f(X)$ be a solution

$X = x + ay$, where a is constant

$$p = \frac{\partial z}{\partial x} = \frac{dz}{dX} \cdot \frac{\partial X}{\partial x} = \frac{dz}{dX}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{dX} \cdot \frac{\partial X}{\partial y} = a \frac{dz}{dX}$$

from eqn (1) $f\left(z, \frac{dz}{dX}, a \cdot \frac{dz}{dX}\right) = 0$

Diff. Eqn. of first Order

General Integral, $b = \psi(a)$

$F = 0$ complete Solution.

$\frac{dF}{da} = 0$

Singular Integral, $F = 0$, $\frac{\partial F}{\partial a} = 0$ and $\frac{\partial F}{\partial b} = 0$

Ques Solve : $z = p^2 + q^2$

Solⁿ:

Given equation,

$$z = p^2 + q^2 \quad - (1)$$

Equation is standard form II in the form $f(z, p, q)$.

Let $z = f(X)$ where $X = x + ay$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{dX}, \quad q = \frac{\partial z}{\partial y} = a\frac{dz}{dX}$$

Equation (1) becomes

$$z = \left(\frac{dz}{dX}\right)^2 + a^2\left(\frac{dz}{dX}\right)^2$$

$$z = (1 + a^2)\left(\frac{dz}{dX}\right)^2$$

$$\frac{dz}{dX} = \frac{\sqrt{z}}{\sqrt{1+a^2}}$$

On Seperating the variable, we have

$$\frac{dz}{\sqrt{z}} = \frac{dX}{\sqrt{1+a^2}}$$

On integration, $2\sqrt{z} = \frac{1}{\sqrt{(1+a^2)}} \cdot x + c$

$$2\sqrt{z}\cdot\sqrt{(1+a^2)} = x + c\sqrt{1+a^2}$$

$$2\sqrt{z}\sqrt{1+a^2} = x + b \qquad \{b = c\sqrt{1+a^2}\}$$

$$4z(1+a^2) = (x+b)^2$$

$$4z(1+a^2) = (x+ay+b)^2 \qquad -(2)$$

Singular Solution,

Differentiating eqn (2) partially w.r.t a and b

$$8az = 2(x+ay+b)$$

And $0 = 2(x+ay+b).1$

On subtracting, the Required singular solution is

$$8az = 0 \quad , \quad z = 0$$

Que Find CI of $9(p^2z + q^2) = 4$

Solⁿ:- Given equation,

$$9(p^2z + q^2) = 4 \Rightarrow p^2z + q^2 = \frac{4}{9} \qquad -(1)$$

Equation is standard form II, in the form $f(z, p, q)$.

Let $z = f(X)$ where $X = x + ay$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{dX} \quad , \quad q = \frac{\partial z}{\partial y} = \frac{a\,dz}{dX}$$

from eqn (1) we get

$$z\left(\frac{dz}{dX}\right)^2 + a^2\left(\frac{dz}{dX}\right)^2 = \frac{4}{9}$$

$$(z+a^2)\left(\frac{dz}{dX}\right)^2 = \frac{4}{9}$$

$$\sqrt{z+a^2}\cdot\frac{dz}{dX} = \frac{2}{3}$$

$$dX = \frac{3}{2}\sqrt{z+a^2}\,dz$$

On integration, $X + b = \frac{3}{2}\cdot\frac{(z+a^2)^{1/2+1}}{1/2+1}$

$$X + b = \frac{3}{2}\times\frac{2}{3}(z+a)^{3/2}$$

$$X + b = (z+a)^{3/2}$$

$$(x+ay+b)^2 = (z+a)^3$$

Required complete integral

Que find the complete and Singular solution

$$z^2(p^2z^2 + q^2) = 1$$

Solⁿ:-

Given equation, $z^2(p^2z^2 + q^2) = 1 \qquad -(1)$

Let $z = f(X)$ where $X = x + ay$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{dX} \quad , \quad q = \frac{\partial z}{\partial y} = \frac{a\,dz}{dX}$$

Substituting these value in eqn (1) we get

$$z^2\left[\left(\frac{dz}{dx}\right)^2 z^2 + a^2\left(\frac{dz}{dx}\right)^2\right] = 1$$

$$z^2(z^2+a^2)\left(\frac{dz}{dx}\right)^2 = 1$$

$$z\sqrt{z^2+a^2}\,\frac{dz}{dx} = 1$$

$$\int z\sqrt{z^2+a^2}\,dz = \int dx$$

On integration, Let $z^2+a^2 = t$

$$2z\,dz = dt \Rightarrow z\,dz = dt/2$$

$$\int \sqrt{t}\,\frac{dt}{2} = x+b$$

$$\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{3} t^{3/2} = x+b$$

$$\frac{1}{3}(z^2+a^2)^{3/2} = x+b$$

$$(z^2+a^2)^{3/2} = 3(x+ay+b)$$

$$9(x+ay+b)^2 = (z^2+a^2)^3$$

Required C.I — (2)

Singular Solution - Differentiating (2) partially w.r.t a and b,

$$18(x+ay+b)y = 6(z^2+a^2)^2 \cdot a \quad — (3)$$

$$18(x+ay+b) = 0 \quad — (4)$$

from (3) & (4)

$a = 0$ and ~~and~~ $z = 0$

* Standard form III

Equation of the form, $f(x,p) = f(y,q)$

$$f(x,p) = f(y,q) = a \text{ (say)}$$

Now, $p = f_1(x,a)$ and $q = f_2(y,a)$

$$dz = p\,dx + q\,dy$$

$$dz = f_1(x,a)\,dx + f_2(y,a)\,dy$$

Integrating $z = \int f_1(x,a)\,dx + \int f_2(y,a)\,dy + b$

Complete integral will be obtain

- There is no Singular Integral in this case

Ques find C.I of $p^2+q^2 = x+y$

Solⁿ:-

~~but~~ Given equation can be written as,

$$p^2 - x = y - q^2$$

Let, $p^2 - x = y - q^2 = a$

$p^2 - x = a$ and $y - q^2 = a$

$p = \sqrt{x+a}$ and $q = \sqrt{y-a}$

Now, $dz = p\,dx + q\,dy$

$$dz = \sqrt{x+a}\,dx + \sqrt{y-a}\,dy$$

On integration, we get

$$z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{2/3} + b$$

Required Complete integral

Que Solve :- $pe^{y} = qe^{x}$

Solⁿ:-

Given equation can be written as,

$$pe^{-x} = qe^{-y}$$

Let $pe^{-x} = qe^{-y} = a \Rightarrow p = ae^{x}, \; q = ae^{y}$

We know that, $dz = p\,dx + q\,dy$

$$dz = ae^{x}dx + ae^{y}dy$$

On integration, $z = ae^{x} + ae^{y} + b$

$$z = a(e^{x} + e^{y}) + b$$

Required Complete integral

Que find Complete integral : $z(p^2 - q^2) = x - y$

Solⁿ:-

Given equation can be written as

$$(\sqrt{z}\,p)^2 - (\sqrt{z}\,q)^2 = x - y$$

$$\left(\sqrt{z}\frac{\partial z}{\partial x}\right)^2 - \left(\sqrt{z}\frac{\partial z}{\partial y}\right)^2 = x - y \quad \text{— (1)}$$

Putting $\sqrt{z}\,\partial z = \partial Z$ i.e. $Z = \frac{2}{3}z^{3/2}$

from eqⁿ (1)

$$\left(\frac{\partial Z}{\partial x}\right)^2 - \left(\frac{\partial Z}{\partial y}\right)^2 = x - y$$

$$P^2 - Q^2 = x - y \quad \text{where } \frac{\partial Z}{\partial x} = P,\; \frac{\partial Z}{\partial y} = Q$$

$$P^2 - x = Q^2 - y$$

Equation is standard form III

Let $P^2 - x = Q^2 - y = a$

$P^2 - x = a$ and $Q^2 - y = a$

$P = \sqrt{x+a}$ and $Q = \sqrt{y+a}$

Now, $dZ = P\,dx + Q\,dy$

$$dZ = \sqrt{x+a}\,dx + \sqrt{y+a}\,dy$$

On integrating, $Z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2} + b$

$$\frac{2}{3}z^{3/2} = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2} + b$$

$$z^{3/2} = 8(x+a)^{3/2} + (y+a)^{3/2} + c$$

where $c = \frac{2}{3}b$

Required Complete integral

* Standard form IV

$$z = px + qy + f(p, q)$$

$$p = \frac{\partial z}{\partial x} = a \quad , \quad q = \frac{\partial z}{\partial y} = b$$

Complete Integral :- Replace p ad q by a and b where a and b are arbitrary constant

$$z = ax + by + f(a, b) \quad - (i)$$

General Integral :- $b = \phi(a)$

$$z = ax + \phi(a)\, y + f(a, b) \quad - (ii)$$

Differentiating w.r.t 'a'

$$0 = x + \phi'(a) + f'(a, b) \quad - (iii)$$

G.I is obtained by eliminating a in (ii) & (iii)

Singular Solution. Diff. (i) part. w.r.t a and b

$$0 = x \;,\; 0 = y \quad - (IV) \;\&\; (V)$$

Eliminating a & b from (i), (IV) & (V) we get Singular Solution

Que. find Complete, General and Singular Solution
$$z = px + qy + pq$$

Solⁿ. Given eqn,

$$z = px + qy + pq$$

Complete integral is given by

$$z = ax + by + ab \quad - (1)$$

General integral, $b = \phi(a)$, where ϕ is an arbitrary function

from eqn (1), $z = ax + \phi(a)y + a\phi(a) \quad - (2)$

Differentiating (2) partially w.r.t 'a'

$$0 = x + \phi'(a)y + a\phi'(a) \quad - (3)$$

The general integral is obtained by eliminating 'a' from (2) & (3)

Singular Solution :-

Diff. (1) partially w.r.t 'a' and 'b'

$$0 = x + b \Rightarrow b = -x$$
$$0 = y + a \Rightarrow a = -y$$

Singular Integral is given by, from eqn (1)

$$z = (-y)x + (-x)y + (-y)\cdot(-x)$$
$$z = -xy$$

Ques Solve :- Find Complete, General and Singular Solution

$$z = px + qy + p^2 + q^2$$

Solⁿ :- Given Equation,

$$z = px + qy + p^2 + q^2$$

which is standard form IV

Complete Integral is given by

$$z = ax + by + a^2 + b^2 \quad - (1)$$

Singular Integral :-

Diff. (1) w.r.t 'a' and 'b'

$$0 = x + 2a \Rightarrow a = -x/2$$

$$0 = y + 2b \Rightarrow b = -y/2$$

Substituting value of a and b in eqn (1), we get

$$z = -\frac{x^2}{2} - \frac{y^2}{2} + \frac{x^2}{4} + \frac{y^2}{4}$$

$$x^2 + y^2 + 4z = 0$$

General Solution :- Putting $b = \phi(a)$ in eqn (1)

$$z = ax + \phi(a)y + a^2 + [\phi(a)]^2 \quad - (2)$$

Diff. (2) partially w.r.t 'a' we get

$$0 = x + \phi'(a)y + 2a + 2\phi(a) \cdot \phi'(a) \quad - (3)$$

By eliminating a from equation (2) and (3) we obtain the general solution

(#) Charpit's General Method of Solution

$$f(x, y, z, p, q) = 0$$

Auxiliary equation of charpit's

$$\frac{dp}{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p} = \frac{dq}{\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{\partial F}{0}$$

Ques 1 Solve by charpit's method

$$px + qy = pq$$

Solⁿ Given differential Equation

$$f \equiv px + qy - pq = 0 \quad - (1)$$

$$\therefore \frac{\partial f}{\partial x} = p, \quad \frac{\partial f}{\partial y} = q, \quad \frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial p} = x - q, \quad \frac{\partial f}{\partial q} = y - p$$

Auxiliary equation of charpit's,

$$\frac{dp}{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p} = \frac{dq}{\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial q}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{\partial F}{0}$$

$$\frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-p(x-q)-q(y-p)} = \frac{dx}{q-x} = \frac{dy}{p-y} = \frac{dF}{0}$$

Taking first two ratios, we have

$$\frac{dp}{p} = \frac{dq}{q}$$

On integration, we get

$$\log p = \log q + \log a$$

$$p = aq \text{ , where } a \text{ is constant}$$

Substituting the value of p in eqn (1)

$$aqx + qy \mp aq^2 = 0$$

$$q(ax+y) = aq^2$$

$$q = \frac{ax+y}{a}$$

$\Rightarrow \quad p = aq$

$$p = a \cdot \frac{ax+y}{a}$$

$$p = ax + y$$

Hence $\quad dz = p\,dx + q\,dy$

$$dz = (ax+y)\,dx + \frac{ax+y}{a}\,dy$$

$$a\,dz = a(ax+y)\,dx + (ax+y)\,dy$$

$$a\,dz = (ax+y)(a\,dx + dy)$$

$$a\,dz = (ax+y)\,d(ax+y)$$

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On integration, we get

$$az = \frac{1}{2}(ax+y)^2 + b$$

where b is arbitrary constant

Required Complete integral

$$az = \frac{1}{2}(ax+y)^2 + b \quad \text{Ans}$$

Ques Solve by Charpit's method

$$z = px + qy + p^2 + q^2$$

Soln

Given equation.

$$f = px + qy + p^2 + q^2 - z = 0$$

$$\therefore \frac{\partial f}{\partial x} = p, \quad \frac{\partial f}{\partial y} = q, \quad \frac{\partial f}{\partial z} = -1, \quad \frac{\partial f}{\partial p} = x + 2p, \quad \frac{\partial f}{\partial q} = y + 2q$$

By Charpit's auxiliary equation

$$\frac{dp}{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p} = \frac{dq}{\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}$$

$$\frac{dp}{p-p} = \frac{dq}{q-q} = \frac{dz}{-p(x+2p)-q(y+2q)} = \frac{dx}{-(x+2p)} = \frac{dy}{-(y+2q)} = \frac{dF}{0}$$

$$\Rightarrow \frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-p(x+2p)-q(y+2q)} = \frac{dx}{-(x+2p)} = \frac{dy}{-(y+2q)} = \frac{df}{0}$$

Taking 1^{st} and 3^{rd} ratio,

$$dp = 0$$

On integration, $p = a$

Taking 2^{nd} and 3^{rd} ratio

$$dq = 0$$

On integration, $q = b$

Substituting the value of p and q in given equation

$$z = ax + by + a^2 + b^2$$

where a & b are arbitrary constant

Required Complete Integral.

Ques Solve by charpit's method

$$p^2 + q^2 - 2px + 2qy + 1 = 0$$

Soln:- Given equation,

$$f = p^2 + q^2 - 2px - 2qy + 1 = 0 \quad -(1)$$

$$\therefore \frac{\partial f}{\partial x} = -2p, \quad \frac{\partial f}{\partial y} = -2q, \quad \frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial p} = 2p - 2x, \quad \frac{\partial f}{\partial q} = 2q - 2y$$

By charpit's auxiliary equation

$$\frac{dp}{-2p} = \frac{dq}{-2q} = \frac{dz}{-p(2p-2x)-q(2q-2y)} = \frac{dx}{-2p+2x} = \frac{dy}{-2q+2y}$$

from first two ratio, we get

$$\frac{dp}{-2p} = \frac{dq}{-2q} \Rightarrow \frac{dp}{p} = \frac{dq}{q}$$

On integration, $\log p = \log q + \log a$

$$p = aq \quad -(2)$$

Putting above value in eqn (1), we get

$$a^2q^2 + q^2 - 2aqx - 2qy + 1 = 0$$

$$(a^2+1)q^2 - 2(ax+y)q + 1 = 0$$

$$\therefore \quad q = \frac{2(ax+y) \pm \sqrt{4(ax+y)^2 - 4(a^2+1)}}{2(a^2+1)}$$

$$q = \frac{ax+y \pm \sqrt{(ax+y)^2 - (a^2+1)}}{a^2+1}$$

Putting the value of (q) in eqn (2), we get

$$p = \frac{a\left[ax+y \pm \sqrt{(ax+y)^2-(a^2+1)}\right]}{a^2+1}$$

Now, $dz = p\,dx + q\,dy$

Putting value of p and q in above equation

$$dz = \frac{a\left[ax + y + \sqrt{(ax+y)^2 - (a^2+1)}\right]}{a^2+1}\,dx + \frac{\left[ax + y + \sqrt{(ax+y)^2 - (a^2+1)}\right]}{a^2+1}\,dy$$

$$dz = \frac{(ax+y) + \sqrt{(ax+y)^2 - (a^2+1)}}{a^2+1}\,(a\,dx + dy)$$

Let $ax + y = t \Rightarrow a\,dx + dy = dt$

$$dz = \frac{t + \sqrt{t^2 - (a^2+1)}}{a^2+1}\,dt$$

$$(a^2+1)\,dz = \left[t + \sqrt{t^2 - (a^2+1)}\right]dt$$

On integration,

$$(a^2+1)z = \frac{t^2}{2} + \frac{t}{2}\sqrt{t^2 - (a^2+1)} - \frac{(a^2+1)}{2}\log\left[t + \sqrt{t^2-(a^2+1)}\right] + b$$

$t = ax + y$

$$(a^2+1)z = \frac{(ax+y)^2}{2} + \frac{(ax+y)}{2}\sqrt{(ax+y)^2 - (a^2+1)} - \frac{(a^2+1)}{2}\log\left[(ax+y) + \sqrt{(ax+y)^2-(a^2+1)}\right] + b$$

where a, b are arbitrary constant

Required Complete Integral

Que. Solve by charpit Method

$$p = (qy + z)^2$$

Sol^n: Given D.E,

$$f = p - (qy+z)^2$$

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = -2q(qy+z), \quad \frac{\partial f}{\partial z} = -2(qy+z), \quad \frac{\partial f}{\partial p} = 1$$

$$\frac{\partial f}{\partial q} = -2y(qy+z)$$

By charpit auxiliary equation,

$$\frac{dp}{-2p(qy+z)} = \frac{dq}{-4q(qy+z)} = \frac{dz}{-p + 2qy(qy+z)} = \frac{dx}{-1} = \frac{dy}{2y(qy+z)}$$

from first and last ratio, we have

$$\frac{dp}{p} = \frac{dy}{-y} \Rightarrow \frac{dp}{p} + \frac{dy}{y} = 0$$

On integration, we get

$$\log p + \log y = \log a$$

$$py = a \qquad \therefore \quad p = a/y$$

Putting above value in given equation

$$qy + z = \sqrt{\frac{a}{y}}$$

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$$\Rightarrow \quad q = \frac{1}{y}\left(\sqrt{\frac{a}{y}} - z\right)$$

Now, $\quad dz = p\,dx + q\,dy$

Putting value of p and q in above equation,

$$dz = \frac{a}{y}\,dx + \frac{1}{y}\left(\sqrt{\frac{a}{y}} - z\right)dy$$

$$y\,dz = a\,dx + \sqrt{\frac{a}{y}}\,dy - z\,dy$$

$$y\,dz + z\,dy = a\,dx + \sqrt{\frac{a}{y}}\,dy$$

$$d(yz) = d\left(ax + \sqrt{(ay)}\right)$$

On integration, we get

$$yz = ax + 2\sqrt{ay} + b \qquad \underline{\underline{\text{Ans}}}$$

where a and b are arbitrary constant

Required Complete Integral