Unit - II

Linear and Ordinary Simultaneous Differential Equation

Linear Differential Equation of Second Order

Differential Equation of Second Order of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

where P, Q and R are the function of independent variable x.

Complementary function can be found by inspection,

- $y = e^x$ is a part of C.f if,
  $1 + P + Q = 0$

- $y = e^{-x}$ is a part of C.f if,
  $1 - P + Q = 0$

- $y = e^{mx}$ is a part of C.f if,
  $m^2 + mP + Q = 0$

- $y = x$ is a part of C.f if,
  $P + Qx = 0$

- $y = x^2$ is a part of C.f if,
  $2 + 2Px + Qx^2 = 0$

- $y = x^m$ is a part of C.f if,
  $m(m-1) + Pmx + Qx^2 = 0$

Ques. Solve : $x\dfrac{d^2y}{dx^2} - (2x-1)\dfrac{dy}{dx} + (x-1)y = 0$

Solⁿ :-

Given differential equation can be written as,

$$\frac{d^2y}{dx^2} - \frac{(2x-1)}{x}\frac{dy}{dx} + \frac{(x-1)}{x}y = 0 \quad \text{--- (1)}$$

Here, $P = -\dfrac{(2x-1)}{x}$, $Q = \dfrac{x-1}{x}$, $R = 0$

$$\Rightarrow \quad 1 + P + Q = 1 - \frac{(2x-1)}{x} + \frac{x-1}{x} = 0$$

So,

$y_1 = e^x$ is a part of Complementary function

Let $y = vy_1$ be the solution of equation (1)

$$\frac{dy}{dx} = \frac{v\,dy_1}{dx} + y_1\frac{dv}{dx}$$

$$= vy_1 + y_1\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = vy_1 + y_1\frac{dv}{dx} + y_1\frac{d^2v}{dx^2} + y_1\frac{dv}{dx}$$

And $y = vy_1$

Substituting these value in equation (1)

$$vy_1 + y_1\frac{dv}{dx} + y_1\frac{d^2v}{dx^2} + y_1\frac{dv}{dx} - P\left[vy_1 + y_1\frac{dv}{dx}\right] + Qvy_1 = 0$$

$$v + \frac{dv}{dx} + \frac{d^2v}{dx^2} + \frac{dv}{dx} + Pv + P\frac{dv}{dx} + Qv = 0$$

$$\frac{d^2v}{dx^2} + 2\frac{dv}{dx} + P\left\{\frac{dv}{dx} + v + Pv + Qv = 0\right.$$

$$\frac{d^2v}{dx^2} + \left[2 + P\right]\frac{dv}{dx} + (1 + P + Q)v = 0$$

$$\frac{d^2v}{dx^2} + \left[2 + \left(-\frac{(2x-1)}{x}\right)\right]\frac{dv}{dx} = 0$$

$$\frac{d^2v}{dx^2} + \left(2 - \frac{(2x-1)}{x}\right)\frac{dv}{dx} = 0$$

$$\frac{d^2v}{dx^2} + \left(\frac{2x - 2x + 1}{x}\right)\frac{dv}{dx} = 0$$

$$\frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx} = 0$$

or $$x\frac{d^2v}{dx^2} + \frac{dv}{dx} = 0 \quad \text{--- (2)}$$

On putting $p = \frac{dv}{dx}$, then $\frac{d^2v}{dx^2} = \frac{dp}{dx}$

From equation ② we get

$$x\frac{dp}{dx} + p = 0$$

$$\frac{dp}{p} + \frac{dx}{x} = 0$$

On integrating,

$$\int \frac{dp}{p} + \int \frac{dx}{x} = 0$$

$$\log p + \log x = \log c_1$$

$$px = c_1 \quad \text{or} \quad p = c_1/x$$

$$\frac{dv}{dx} = p = \frac{c_1}{x}$$

$$dv = \frac{c_1}{x}\,dx$$

On Integration, $v = c_1 \log x + c_2$

And, $y = vy_1 \Rightarrow y = ve^x$

$$y = (c_1 \log x + c_2)e^x \quad \text{Ans}$$

Hence Complete solution of given D.E.

Ques Solve : $x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$

Sol^n Given differential equation can be written as,

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(1+x)}{x^2}y = x \quad \text{— ①}$$

Here, $P = -\frac{2(1+x)}{x}$, $Q = \frac{2(1+x)}{x^2}$, $R = x$

Now, $\Rightarrow P + Qx = -\frac{2(1+x)}{x} + \frac{2(1+x)}{x^2}x = 0$

$\therefore$ $y_1 = x$ is a part of C.F

Let $y = vy_1$ be the solution of eqn ①

$$\cancel{\frac{dy}{dx}} = \cancel{vy_1} + \cancel{y_1\frac{dv}{dx}}$$

$$\cancel{\frac{d^2y}{dx^2}} = \cancel{vy_1} + \cancel{y_1\frac{dv}{dx}} + \cancel{y_1\frac{d^2v}{dx^2}} + \cancel{y_1\frac{dv}{dx}}$$

And, $y = vy_1$

On substituting above value in eqn ① we get

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{y_1}\frac{dy_1}{dx}\right)\frac{dv}{dx} = \frac{R}{y_1}$$

where, $y_1 = x$

$$\frac{d^2v}{dx^2} + \left(-\frac{2(1+x)}{x} + \frac{2}{x}\right)\frac{dv}{dx} = \frac{x}{x}$$

$$\frac{d^2v}{dx^2} + \left(\frac{-2-2x+2}{x}\right)\frac{dv}{dx} = 1$$

$$\frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 1 \quad \text{— (2)}$$

Putting $\frac{dv}{dx} = p$ then $\frac{d^2v}{dx^2} = \frac{dp}{dx}$

from eqn (2)

$$\frac{dp}{dx} - 2p = 1 \quad \text{— (3)}$$

which is linear D.E in p.

$$I.F = e^{\int -2\,dx} = e^{-2x}$$

Solution of equation (3).

$$p.(I.F) = \int 1.(I.F)\,dx + c_1$$

$$p.e^{-2x} = \int e^{-2x}\,dx + c_1$$

$$pe^{-2x} = -\frac{e^{-2x}}{2} + c_1$$

$$\frac{dv}{dx} = p = -\frac{1}{2} + c_1 e^{2x}$$

$$dv = \left(-\frac{1}{2} + c_1 e^{2x}\right)dx$$

On integration

$$v = -\frac{1}{2}x + c_1\frac{e^{2x}}{2} + c_2$$

Hence the solution of given differential equation,

$$y = vy_1 \Rightarrow y = vx$$

$$y = \left(-\frac{1}{2}x + c_1\frac{e^{2x}}{2} + c_2\right)x$$

$$y = -\frac{1}{2}x^2 + c_1 x\frac{e^{2x}}{2} + c_2 x$$

Que Solve :- $\frac{d^2y}{dx^2} - \cot x\frac{dy}{dx} - (1-\cot x)y = e^x \sin x$

Solⁿ: Given differential Equation,

$$\frac{d^2y}{dx^2} - \cot x\frac{dy}{dx} - (1-\cot x)y = e^x \sin x \quad \text{— (1)}$$

Here,

$$P = -\cot x, \quad Q = -(1-\cot x), \quad R = e^x \sin x$$

$$\Rightarrow 1 + P + Q = 1 - \cot x - 1 + \cot x$$
$$= 0$$

$\therefore$ $y = e^x$ is a part of C.F

Let $y = vy_1$ be the solution of eqn (1)

On substituting $y = vy_1$ in eqn (1) we get

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{y_1}\frac{dy_1}{dx}\right)\frac{dv}{dx} = \frac{R}{y_1}$$

$$\frac{d^2v}{dx^2} + \left(-\cot x + \frac{2}{e^x} \cdot e^x\right)\frac{dv}{dx} = \frac{e^x \sin x}{e^x}$$

$\{ y_1 = e^x$

$$\frac{d^2v}{dx^2} + (2 - \cot x)\frac{dv}{dx} = \sin x$$

Putting $P = \frac{dv}{dx}$ then $\frac{d^2v}{dx^2} = \frac{dP}{dx}$

$$\frac{dP}{dx} + (2 - \cot x) P = \sin x$$

It is linear differential equation in P, Hence

$$I.F = e^{\int P dx}$$
$$= e^{\int 2 - \cot x \, dx} = e^{2x - \log \sin x}$$
$$= \frac{e^{2x}}{\sin x}$$

Solution of the equation,

$$P \times (I.F) = \int \sin x . (I.F)\, dx + c_1$$

$$P . \frac{e^{2x}}{\sin x} = \int \sin x . \frac{e^{2x}}{\sin x} dx + c_1$$

$$P \frac{e^{2x}}{\sin x} = \frac{e^{2x}}{2} + c_1$$

$$P = \frac{\sin x}{2} + c_1 e^{-2x} \sin x$$

$$\frac{dv}{dx} = \frac{\sin x}{2} + c_1 e^{-2x} \sin x$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$$

$$dv = \left(\frac{\sin x}{2} + c_1 e^{-2x} \sin x\right) dx$$

On integration, we get

$$v = -\frac{\cos x}{2} + c_1 \frac{e^{-2x}}{5}(-2\sin x - \cos x) + c_2$$

Hence the solution of equation (1)

$$y = v y_1$$

$$y = \left[\frac{-\cos x}{2} + c_1 \frac{e^{-2x}}{5}(-2\sin x - \cos x) + c_2\right] e^x$$

$$y = \frac{-e^x \cos x}{2} - c_1 \frac{e^{-x}}{5}(2\sin x + \cos x) + c_2 e^x$$

Ans

(#) Removal of the first Derivative
(Reduction to Normal form)

Differential Equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$$y_1 = e^{-1/2\int P dx} \quad , \quad Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$$

$$R_1 = Re^{1/2\int P dx} \text{ or } \frac{R}{y_1}$$

Normal form

$$\frac{d^2v}{dx^2} + Q_1 v = R_1$$

<u>Ques</u> Solve : $x^2\frac{d^2y}{dx^2} - 2(x^2+x)\frac{dy}{dx} + (x^2+2x+2)y = 0$

Solⁿ:- Given differential equation can be written as

$$\frac{d^2y}{dx^2} - 2\left(1+\frac{1}{x}\right)\frac{dy}{dx} + \left(1+\frac{2}{x}+\frac{2}{x^2}\right)y = 0 \quad \text{— (1)}$$

Here, $P = -2\left(1+\frac{1}{x}\right)$, $Q = \left(1+\frac{2}{x}+\frac{2}{x^2}\right)$, $R = 0$

Substituting $y = vy_1$ in eqn (1)
Normal form is given by

$$\frac{d^2v}{dx^2} + Q_1 v = R_1$$

$$y_1 = e^{-1/2\int P dx} = e^{\int\left(1+\frac{1}{x}\right)dx}$$

$$= e^{x+\log x}$$

$$= xe^x$$

$$Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$$

$$= 1+\frac{2}{x}+\frac{2}{x^2} - \frac{1}{4}\cdot 4\left(1+\frac{1}{x}\right)^2 - \frac{1}{2}\cdot\frac{2}{x^2}$$

$$= 0$$

$$R_1 = Re^{1/2\int P dx} = 0$$

Hence Normal form of eqn (1)

$$\frac{d^2v}{dx^2} = 0$$

Integrating twice w.r.t. x we get

$$v = C_1x + C_2$$

Solution of the given equation,

$$y = vy_1$$

$$y = (C_1x + C_2)xe^x \quad \text{Ans}$$

Que Solve : $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-3)y = e^{x^2}$

Solⁿ: Given differential Equation,

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-3)y = e^{x^2} \quad \text{— (1)}$$

Here, $P = -4x$, $Q = 4x^2 - 3$, $R = e^{x^2}$

On substituting $y = vy_1$ in eqn (1)

Normal form, $\frac{d^2v}{dx^2} + Q_1 v = R_1$

$$y_1 = e^{-\int \frac{1}{2} P dx} = e^{x^2}$$

$$Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$$

$$= 4x^2 - 3 - \frac{1}{4}(16x^2) + \frac{1}{2} \times 4 \quad = -3 + 2$$

$$= -1$$

$$R_1 = Re^{\int \frac{1}{2} P dx} = e^{x^2} \cdot e^{x^2}$$

$$= 1$$

Normal form of equation (1)

$$\frac{d^2v}{dx^2} - v = 1$$

$$C.F = C_1 e^{x} + C_2 e^{-x}$$

$$P.I = \frac{1}{D^2 - 1} 1 = -(1 - D^2)^{-1} \cdot 1$$

$$= -1$$

$$v = C_1 e^{x} + C_2 e^{-x} - 1$$

Required Solution

$$y = vy_1$$

$$y = (C_1 e^{x} + C_2 e^{-x} - 1) e^{x^2} \quad \text{Ans}$$

Que Solve : $\frac{d}{dx}\left(\cos^2 x \frac{dy}{dx}\right) + y\cos^2 x = 0$

Solⁿ: Given differential Equation,

$$\frac{d}{dx}\left(\cos^2 x \frac{dy}{dx}\right) + y\cos^2 x = 0$$

$$\cos^2 x \frac{d^2y}{dx^2} - 2\cos x \sin x \frac{dy}{dx} + y\cos^2 x = 0$$

$$\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + y = 0 \quad \text{— (1)}$$

Here, $P = -2\tan x$, $Q = 1$ and $R = 0$

Substituting $y = vy_1$ in eqn (1)

Normal form, $\frac{d^2v}{dx^2} + Q_1 v = R_1$

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where, $y_1 = e^{-\int \frac{1}{2} P dx} = e^{\int \tan x \, dx} = e^{\log \sec x}$

$= \sec x$

$Q_1 = Q - \frac{1}{4} P^2 - \frac{1}{2} \frac{dP}{dx} = 1 - \frac{1}{4} \times 4 \tan^2 x - \frac{1}{2} \times -2 \sec^2 x$

$= 1 - \tan^2 x + \sec^2 x = 2$

$R_1 = R e^{\frac{1}{2} \int P dx} = 0$

Reduced Normal form of equation (1) is

$$\frac{d^2 v}{dx^2} + 2v = 0$$

$$(D^2 + 2) v = 0$$

A.E, $m^2 + 2 = 0$, $m = \pm \sqrt{2} i$

C.F $= c_1 \cos \sqrt{2} x + c_2 \sin \sqrt{2} x = v$

P.I $= \frac{1}{(D^2 + 2)} 0 = 0$

Hence, Required Solution of given differential Equation

$y = v y_1$

$y = (c_1 \cos \sqrt{2} x + c_2 \sin \sqrt{2} x) \sec x$

Q. Solve: $\frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + (4x^2 - 1) y = -3 e^{x^2} \sin 2x$

Sol^n. Given differential Equation,

$$\frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + (4x^2 - 1) y = -3 e^{x^2} \sin 2x$$

Here,

$P = -4x$, $Q = 4x^2 - 1$, $R = -3 e^{x^2} \sin 2x$

Substituting $y = v y_1$ in given equation,

Normal form, $\frac{d^2 v}{dx^2} + Q_1 v = R_1$

where, $y_1 = e^{-\int \frac{1}{2} P dx} = e^{+\int \frac{1}{2} 4x \, dx}$

$= e^{x^2}$

$Q_1 = Q - \frac{1}{4} P^2 - \frac{1}{2} \frac{dP}{dx} = 4x^2 - 1 - \frac{1}{4} \times 16x^2 - \frac{1}{2} \times -4$

$= 4x^2 - 1 - 4x^2 + 2 = 1$

$R_1 = R e^{\frac{1}{2} \int P dx} = -3 e^{x^2} \sin 2x \cdot e^{-x^2}$

$= -3 \sin 2x$

Reduced Normal form,

$$\frac{d^2 v}{dx^2} + v = -3 \sin 2x$$

OR, $(D^2 + 1) v = -3 \sin 2x$

A.E, $m^2 + 1 = 0 \Rightarrow m = \pm i$

$$C.F = c_1 \cos x + c_2 \sin x$$

$$P.I = \frac{1}{D^2+1}(-3\sin 2x) = \frac{-3}{-2^2+1}\sin 2x$$

$$= \sin 2x$$

$$\therefore \quad v = c_1 \cos x + c_2 \sin x + \sin 2x$$

Hence, Required Solution of given equation

$$y = v y_1$$

$$y = (c_1 \cos x + c_2 \sin x + \sin 2x)\, e^{x^2} \qquad \underline{\text{Ans}}$$

(2) Transformation of The Equation By changing the Independent Variable

Differential Equation,

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad - (i)$$

Let $z = f(x)$

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + Q_1 y = R_1 \qquad - (ii)$$

where

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Case I - $P_1 = 0 \Rightarrow \frac{d^2z}{dx^2} + P\frac{dz}{dx} = 0$

$$\Rightarrow z = \int e^{-\int P\,dx}\,dx$$

Thus Equation (2) becomes

$$\frac{d^2y}{dz^2} + Q_1 y = R_1$$

Case II - $Q_1 = a^2 \Rightarrow \frac{Q}{(dz/dx)^2} = a^2$ or $a\frac{dz}{dx} = \sqrt{\pm Q}$

$$az = \int \sqrt{\pm Q}\,dx$$

Equation (2) becomes $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + a^2 y = R_1$

Ques Solve : $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \frac{a^2}{x^4}y = 0$

Solⁿ:- Given differential Equation,

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \frac{a^2}{x^4}y = 0$$

Here, $P = \frac{2}{x}$, $Q = \frac{a^2}{x^4}$ and $R = 0$

Changing the independent variable from x to z equation becomes

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \text{ and } R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Now, let $Q_1 = a^2$

$$\therefore \quad a^2 = \frac{a^2}{x^4\left(\frac{dz}{dx}\right)^2} \Rightarrow \frac{dz}{dx} = \frac{1}{x^2} \Rightarrow \frac{d^2z}{dx^2} = \frac{-2}{x^3}$$

$$z = -\frac{1}{x}$$

Then, $P_1 = \frac{-2/x^3 + 2/x \cdot 1/x^2}{(1/x^2)^2} = 0$

$Q_1 = a^2$ and $R_1 = 0$

$\therefore$ Transformed Equation,

$$\frac{d^2y}{dz^2} + a^2y = 0$$

or $\quad (D^2 + a^2)y = 0$

A.E, $\quad m^2 + a^2 = 0 \Rightarrow m = \pm ai$

$$y = c_1\cos(az + c_2)$$

$$y = c_1\cos\left(c_2 - \frac{a}{x}\right) \qquad \left\{ z = -\frac{1}{x} \right\}$$

Ques Solve : $\frac{d^2y}{dx^2} + \cot x\frac{dy}{dx} + 4y\,\mathrm{cosec}^2x = 0$

Solⁿ:- Given differential Equation,

$$\frac{d^2y}{dx^2} + \cot x\frac{dy}{dx} + 4y\,\mathrm{cosec}^2x = 0$$

Here, $P = \cot x$, $Q = 4\,\mathrm{cosec}^2x$, $R = 0$

Changing independent variable from x to z,

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \text{ and } R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Let $Q_1 = a^2 = 1$ (say) then

$$\left(\frac{dz}{dx}\right)^2 = 4\operatorname{cosec}^2 x$$

$$\frac{dz}{dx} = 2\operatorname{cosec} x \Rightarrow \frac{d^2z}{dx^2} = -2\operatorname{cosec} x \cot x$$

$$\int dz = \int 2\operatorname{cosec} x \, dx$$

$$z = 2\log(\operatorname{cosec} x - \cot x)$$

$$z = 2\log\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$

$$z = 2\log\left(\frac{1-\cos x}{\sin x}\right) = 2\log\left(\frac{2\sin^2 x/2}{2\sin x/2 \cos x/2}\right)$$

$$z = 2\log\tan\frac{x}{2}$$

$$P_1 = \frac{-2\operatorname{cosec} x \cot x + 2\cot x \operatorname{cosec} x}{(2\operatorname{cosec} x)^2} = 0$$

$Q_1 = 1$ and $R_1 = 0$

Transformed Equation

$$\frac{d^2y}{dz^2} + y = 0$$

or $(D^2 + 1)y = 0$

A. E., $m^2 + 1 = 0 \Rightarrow m = \pm i$

$$y = c_1 \cos z + c_2 \sin z$$

$$y = c_1 \cos(z + c_2)$$

$$y = c_1 \cos\left(2\log\tan\frac{x}{2} + c_2\right) \qquad \left\{ z = 2\log\tan\frac{x}{2} \right\}$$

Que Solve : $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3\sin x^2$

Solⁿ: Given differential Equation can be written as,

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = 8x^2\sin x^2$$

Here, $P = -\frac{1}{x}$, $Q = -4x^2$, $R = 8x^2\sin x^2$

Changing the independent variable from x to z

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \text{ and } R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Let, $Q_1 = -1$ (say)

$$\left(\frac{dz}{dx}\right)^2 = 4x^2 \Rightarrow \frac{dz}{dx} = 2x \text{ or } \frac{d^2z}{dx^2} = 2$$

$$\Rightarrow z = x^2$$

Now, $P_1 = \dfrac{2 - \frac{1}{x}\cdot 2x}{4x^2} = 0$, $Q_1 = -1$ and $R_1 = \dfrac{8x^2 \sin x^2}{4x^2}$

$R_1 = 2\sin z$

Transformed Equation,

$$\frac{d^2y}{dz^2} - y = 2\sin z$$

or $(D^2 - 1)y = 2\sin z$

A.E, $m^2 - 1 = 0 \Rightarrow m = \pm 1$

C.F, $= c_1 e^{z} + c_2 e^{-z}$

P.I, $= \dfrac{1}{D^2 - 1} 2\sin z = \dfrac{1}{-1^2 - 1} 2\sin z = -\sin z$

$$y = c_1 e^{z} + c_2 e^{-z} - \sin z$$

Required Solution,

$$y = c_1 e^{x^2} + c_2 e^{-x^2} - \sin x^2$$ Ans

Ques Solve :- $\cos x \dfrac{d^2y}{dx^2} + \sin x \dfrac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x$

Solⁿ :- Given differential equation can be written as,

$$\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} - 2y\cos^2 x = 2\cos^4 x$$

Here, $P = \tan x$, $Q = -2\cos^2 x$, $R = 2\cos^4 x$

changing the Independent variable from x to z

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

$P_1 = \dfrac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$, $Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$ and $R_1 = \dfrac{R}{\left(\frac{dz}{dx}\right)^2}$

Let $Q_1 = -2$ (say)

$-2\left(\dfrac{dz}{dx}\right)^2 = -2\cos^2 x \Rightarrow \dfrac{dz}{dx} = \cos x$

$\therefore z = \sin x$ $\qquad \dfrac{d^2z}{dx^2} = -\sin x$

Now, $P_1 = \dfrac{-\sin x + \tan x \cos x}{\cos^2 x} = 0$

$Q_1 = -2$

$R_1 = \dfrac{2\cos^4 x}{\cos^2 x} = 2\cos^2 x = 2(1 - z^2)$

Transformed Equation,

$$\frac{d^2y}{dz^2} - 2y = 2(1 - z^2)$$

or $(D^2 - 2)y = 2(1 - z^2)$

A.E, $m^2 - 2 = 0 \Rightarrow m = \pm\sqrt{2}$

C.F $= c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{2}z}$

$$P.I = \frac{1}{D^2-2}(1-z^2) = -\left(1 - \frac{D^2}{2}\right)^{-1}(1-z^2)$$

$$= -\left[1 + \frac{D^2}{2} + \frac{D^4}{4} + \dots\right](1-z^2)$$

$$= -(1-z^2) + \frac{D^2(1-z^2)}{2}$$

$$= -1 + z^2 + \frac{1}{2} \cdot 2$$

$$= z^2$$

$$y = c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{2}z} + z^2$$

Required Solution,

$$y = c_1 e^{\sqrt{2}\sin x} + c_2 e^{-\sqrt{2}\sin x} + \sin^2 x$$ Ans

(ii) Method of Variation of Parameters

$$c_1'(x) = \frac{-y_2 x}{w(y_1, y_2)}, \quad c_2'(x) = \frac{-y_1 x}{w(y_1, y_2)}$$

Ques Solve :- $\frac{d^2y}{dx^2} + y = x$ or $(D^2+1)y = x$

Soln: $(D^2+1)y = x$

Auxiliary Equation, $m^2 + 1 = 0$

$m = \pm i$

Complementary function, $y_c = c_1 \cos x + c_2 \sin x$

c_1 and c_2 are constant

Let Particular Integral be

$$y_p = c_1(x)\cos x + c_2(x)\sin x \quad - (1)$$

Here, $y_1 = \cos x \qquad y_2 = \sin x$

$y_1' = -\sin x \qquad y_2' = \cos x$

$c_1(x)$ and $c_2(x)$ satisfies the following equation,

$$c_1'(x)\cos x + c_2'(x)\sin x = 0$$

$$c_1'(x)(-\sin x) + c_2'(x)\cos x\} = x$$

By Wronskian determinant

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$\therefore \quad c_1'(x) = \frac{-y_2 X}{W(y_1, y_2)} = \frac{-\sin x \cdot x}{1}$$

$$= -x \sin x$$

$$c_2'(x) = \frac{y_1 X}{W(y_1, y_2)} = \frac{\cos x \cdot x}{1}$$

$$= x \cos x$$

On integration,

$$c_1(x) = -\int x \sin x \, dx$$

$$= -\left[x \int \sin x \, dx - \int \left(\frac{dx}{dx} \int \sin x \, dx \right) \right]$$

$$= -\left[-x \cos x + \int \cos x \, dx \right]$$

$$= x \cos x - \sin x$$

$$c_2(x) = \int x \cos x \, dx$$

$$= \left[x \int \cos x \, dx - \int \frac{dx}{dx} \int \cos x \, dx \right]$$

$$= \left[x \sin x + - \int \sin x \, dx \right]$$

$$= x \sin x + \cos x$$

Substituting the value in eqn ①

$$y_p = (x \cos x - \sin x) \cos x + (x \sin x + \cos x) \sin x$$

$$= x \cos^2 x - \sin x \cos x + x \sin^2 x + \sin x \cos x$$

$$= x \cos^2 x + x \sin^2 x$$

$$= x$$

Complete Solution of equation,

$$y = y_c + y_p$$

$$y = c_1 \cos x + c_2 \sin x + x \qquad \underline{\text{Ans}}$$

<u>Que</u> Solve : $\frac{d^2y}{dx^2} + 4y = 4 \tan 2x$

<u>Sol^n</u>: Given equation can be written as,

$$(D^2 + 4)y = 4 \tan 2x$$

Auxiliary Equation, $m^2 + 4 = 0$

$$m = \pm 2i$$

Complementary function, $y_c = c_1 \cos 2x + c_2 \sin 2x$

Let, $y_p = c_1(x) \cos 2x + c_2(x) \sin 2x$

Here $y_1 = \cos 2x$ and $y_2 = \sin 2x$

$y_1' = -2 \sin 2x$ and $y_2' = 2 \cos 2x$

$C_1(x)$ and $C_2(x)$ satisfy the following equation

$$C_1'(x)\cos 2x + C_2'(x)\sin 2x = 0$$
$$C_1'(x)(-2\sin 2x) + C_2'(x)\,2\cos 2x = 4\tan 2x$$

By Wronskian determinant

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x = 2$$

$$\therefore \quad C_1'(x) = \frac{-y_2 X}{w(y_1, y_2)} = \frac{-\sin 2x \cdot 4\tan 2x}{2}$$
$$= -2\sin 2x \cdot \tan 2x$$

$$C_2'(x) = \frac{y_1 X}{w(y_1, y_2)} = \frac{\cos 2x \cdot 4\tan 2x}{2} = \frac{\cancel{\cos 2x}\cdot \cancel{4}^{2}\sin 2x}{\cancel{2\cos 2x}}$$
$$= 2\sin 2x$$

On integration,

$$C_1(x) = -\int 2\sin 2x \cdot \tan 2x\, dx = -2\int \frac{\sin^2 2x}{\cos 2x}\, dx$$
$$= -2\int \frac{(1-\cos^2 2x)}{\cos 2x}\, dx$$
$$= -2\int \sec 2x\, dx + 2\int \cos 2x\, dx$$
$$= -2\cdot\frac{1}{2}\log(\sec 2x + \tan 2x) + 2\cdot\frac{1}{2}\sin 2x$$

$$= \sin 2x - \log(\sec 2x + \tan 2x)$$

$$C_2(x) = \int 2\sin 2x = \cancel{2}\cdot\frac{-\cos 2x}{\cancel{2}}$$
$$= -\cos 2x$$

Thus Particular integral of given equation,

$$y_p = C_1(x)\cos 2x + C_2(x)\sin 2x$$
$$= [\sin 2x - \log(\sec 2x + \tan 2x)]\cos 2x + (-\cos 2x)(\sin 2x)$$
$$= \sin 2x\cos 2x - \log(\sec 2x + \tan 2x)\cos 2x - \sin 2x\cos 2x$$
$$= -\cos 2x \log(\sec 2x + \tan 2x)$$

Complete Solution of given differential equation is,

$$y = y_c + y_p$$

$$y = c_1\cos 2x + c_2\sin 2x \cancel{+} - \cos 2x\log(\sec 2x + \tan 2x)$$

Ans

Ques. Solve : $(D^3 + D)y = \tan x$

Sol^n :-

Auxiliary Equation,

$$m^3 + m = 0$$
$$m(m^2+1) = 0$$
$$m = 0 , \quad m = \pm i$$

Complementary function, $y_c = c_1(1) + c_2 \cos x + c_3 \sin x$

Let $y_p = c_1(x) + c_2(x)\cos x + c_3(x)\sin x$

Here $y_1 = 1$, $y_2 = \cos x$ and $y_3 = \sin x$

$y_1' = 0$, $y_2' = -\sin x$ and $y_3' = \cos x$

$y_1'' = 0$, $y_2'' = -\cos x$ and $y_3'' = -\sin x$

$c_1(x)$, $c_2(x)$ and $c_3(x)$ satisfy the following system of eqn.

$$c_1'(x) + c_2'(x)\cos x + c_3'(x)\sin x = 0$$
$$c_1'(x).0 + c_2'(x)(-\sin x) + c_3'(x)\cos x = 0$$
$$c_1'(x).0 + c_2'(x).(-\cos x) + c_3'(x)(-\sin x) = -\tan x$$

By Wronskian determinant,

$$W(y_1, y_2, y_3) = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

$$= \sin^2 x + \cos^2 x = 1$$

Now, $c_1'(x) = \dfrac{X.\,W(y_2, y_3)}{W(y_1, y_2, y_3)} = \tan x \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$$= \tan x(\cos^2 x + \sin^2 x)$$
$$= \tan x$$

$c_2'(x) = \dfrac{-X\,W(y_1, y_3)}{W(y_1, y_2, y_3)} = -\tan x \begin{vmatrix} 1 & \sin x \\ 0 & \cos x \end{vmatrix}$

$$= -\tan x(\cos x - 0)$$
$$= -\sin x$$

$c_3'(x) = \dfrac{-X\,W(y_1, y_2)}{W(y_1, y_2, y_3)} = -\tan x \begin{vmatrix} 1 & \cos x \\ 0 & -\sin x \end{vmatrix}$

$$= -\tan x(-\sin x - 0)$$
$$= \frac{\sin^2 x}{\cos x}$$

On integration,

$$c_1(x) = \int \tan x \, dx$$
$$= -\log(\cos x)$$

$$c_2(x) = -\int \sin x \, dx = -(-\cos x)$$
$$= \cos x$$

$$c_3(x) = \int \frac{\sin^2 x}{\cos x} dx = \int \frac{\cos^2 x - 1}{\cos x} dx$$
$$= \int (\cos x - \sec x) \, dx$$
$$= \sin x - \log(\sec x + \tan x)$$

Thus Particular integral of given equation

$$y_p = c_1(x) + c_2(x)\cos x + c_3(x)\sin x$$
$$= \cancel{-\log(\sec x)}$$
$$= -\log\cos x + \cos x \cdot \cos x + \left[\frac{\sin x - \log(\sec x + \tan x)}{\sin x}\right]$$
$$= -\log\cos x + \cos^2 x + \sin^2 x - \log(\sec x + \tan x)$$
$$= \log\sec x + 1 - \log(\sec x + \tan x)$$

Complete Solution of given equation.

$$y = y_c + y_p$$
$$y = c_1 + c_2\cos x + c_3\sin x + \log\sec x + 1 - \log(\sec x + \tan x)$$

Ans

Ques Solve: $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ or $(D^2+1)y = \operatorname{cosec} x$

Solⁿ:- Auxiliary Equation,

$$m^2 + 1 = 0$$
$$m = \pm i$$

Complementary function, $y_c = c_1\cos x + c_2\sin x$

let $y_p = c_1(x)\cos x + c_2(x)\sin x$

Here, $y_1 = \cos x$ and $y_2 = \sin x$

$y_1' = -\sin x$ and $y_2' = \cos x$

$c_1(x)$ and $c_2(x)$ satisfy the following system of equation

$$c_1'(x)\cos x + c_2'(x)\sin x = 0$$
$$c_1'(x)(-\sin x) + c_2'(x)\cos x = \operatorname{cosec} x$$

By Wronskian Determinant

$$w(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$\therefore \quad c_1'(x) = \frac{-y_2 X}{w(y_1, y_2)} = \frac{-\sin x \operatorname{cosec} x}{1} = -1$$

$$c_2'(x) = \frac{y_1 X}{w(y_1, y_2)} = \frac{\cos x \cdot \operatorname{cosec} x}{1} = \cot x$$

On integration,

$$c_1(x) = -\int dx$$
$$= -x$$

$$c_2(x) = \int \cot x \, dx$$
$$= \log(\sin x)$$

Thus Particular integral,

$$y_p = -x\cos x + \log(\sin x)\sin x$$

Complete Solution of Equation,

$$y = c_1\cos x + c_2\sin x + \log(\sin x)\cdot\sin x - x\cos x$$

Ans

15 Ordinary Simultaneous Differential Equation

$$\frac{d}{dt} = D$$

Ques Solve the simultaneous equation

$$\frac{dx}{dt} - 7x + y = 0 \; , \; \frac{dy}{dt} - 2x - 5y = 0$$

Sol^n:-

Writing D for $\frac{d}{dt}$

from the given equation,

$$(D-7)x + y = 0 \quad - (i)$$
$$-2x + (D-5)y = 0 \quad - (ii)$$

from eqn (i) and (ii)

$$y = -(D-7)x$$

$$-2x + (D-5).-(D-7)x = 0$$
$$-x\{2 + (D-5)(D-7)\} = 0$$
$$\{D^2 - 7D - 5D + 35 + 2\}x = 0$$
$$(D^2 - 12D + 37)x = 0$$

This is linear differential equation of second order in x and t

Auxiliary equation,

$$m^2 - 12m + 37 = 0$$

$$m = 6 \pm i$$

Solution. $x = e^{6t}(c_1 \cos t + c_2 \sin t)$

$$\frac{dx}{dt} = 6e^{6t}(c_1 \cos t + c_2 \sin t) + e^{6t}(-c_1 \sin t + c_2 \cos t)$$

from equation (i),

$$(D-7)x + y = 0$$
$$y = 7x - Dx$$
$$y = 7x - \frac{dx}{dt}$$

$$y = 7e^{6t}(c_1 \cos t + c_2 \sin t) - 6e^{6t}(c_1 \cos t + c_2 \sin t) - e^{6t}(-c_1 \sin t + c_2 \cos t)$$

$$y = e^{6t}(c_1 \cos t + c_2 \sin t) - e^{6t}(-c_1 \sin t + c_2 \cos t)$$

$$y = e^{6t}[(c_1 - c_2)\cos t + (c_1 + c_2)\sin t]$$

Ques Solve : $\frac{dx}{dt} + wy = 0 \; , \; \frac{dy}{dt} - wx = 0$

Sol^n:-

Writing D for $\frac{d}{dt}$

The given equation,

$$Dx + wy = 0 \quad - (i)$$
$$-wx + Dy = 0 \quad - (ii)$$

Eliminating y from eqn (i) & (ii)

$$(D^2 + w^2)x = 0$$

Hence its solution is

$$x = A\cos wt + B\sin wt$$

$$\frac{dx}{dt} = -A\omega\sin\omega t + B\omega\cos\omega t$$

from equation (i)

$$Dx + \omega y = 0$$

$$\omega y = -Dx$$

$$y = \frac{-1}{\omega}\frac{dx}{dt}$$

$$y = \frac{-1}{\omega}(-A\omega\sin\omega t + B\omega\cos\omega t)$$

$$y = A\sin\omega t - B\cos\omega t$$

Required Solution

Q. Solve the following simultaneous differential equation

$$\frac{dx}{dt} = x - 2y, \quad \frac{dy}{dt} = 5x + 3y$$

Solⁿ: The given equation,

$$\frac{dx}{dt} = x - 2y \quad \text{— (i)}$$

$$\frac{dy}{dt} = 5x + 3y \quad \text{— (ii)}$$

Differentiating eqn (i) w.r.t. 't',

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} - 2\frac{dy}{dt} \quad \text{— (iii)}$$

from eqn (ii) & (iii)

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} - 2(5x + 3y)$$

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} - 10x + 6y \quad \text{— (iv)}$$

Again from (i) $\quad 2y = x - \frac{dx}{dt}$

from equation (iv)

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} - 10x - 3\left(x - \frac{dx}{dt}\right)$$

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 13x = 0$$

$$(D^2 - 4D + 13)x = 0 \quad \text{where } D = \frac{d}{dt}$$

Now, A.E $= m^2 - 4m + 13 = 0$

$$m = \frac{4 \pm \sqrt{16-52}}{2}$$

$$m = 2 \pm 3i$$

Solution,

$$x = e^{2t}(c_1 \cos 3t + c_2 \sin 3t)$$

$$\therefore \frac{dx}{dt} = 2e^{2t}(c_1 \cos 3t + c_2 \sin 3t) + e^{2t}(-3c_1 \sin 3t + 3c_2 \cos 3t)$$

Substituting above value in eqn (1)

$$2y = e^{2t}(c_1 \cos 3t + c_2 \sin 3t) - 2e^{2t}(c_1 \cos 3t + c_2 \sin 3t) - e^{2t}(-3c_1 \sin 3t + 3c_2 \cos 3t)$$

$$2y = -e^{2t}(c_1 \cos 3t + c_2 \sin 3t) - e^{2t}(-3c_1 \sin 3t + 3c_2 \cos 3t)$$

$$2y = e^{2t} \cos 3t (c_1 + 3c_2) - e^{2t} \sin 3t (3c_1 - c_2)$$

$$y = \frac{1}{2}\left[e^{2t} \cos 3t (c_1 + 3c_2) - e^{2t} \sin 3t (3c_1 - c_2)\right]$$

Required Solution

Solve : $t\,dx = (t - 2x)\,dt$

$t\,dy = (tx + ty + 2x - t)\,dt$

The given equation,

$$t\,dx = (t - 2x)\,dt \quad \text{— (i)}$$

$$t\,dy = (tx + ty + 2x - t)\,dt \quad \text{— (ii)}$$

from (i),

$$\frac{dx}{dt} + \frac{2}{t}x = 1$$

$$\text{I.F.} = e^{\int 2/t\,dt} = e^{2\log t} = e^{\log t^2} = t^2$$

$$xt^2 = \int t^2 \cdot 1\,dt + c_1$$

$$xt^2 = \frac{t^3}{3} + c_1 \Rightarrow x = \frac{t}{3} + c_1 t^{-2} \quad \text{— (iii)}$$

Adding eqn (i) and (ii)

$$t(dx + dy) = (t - 2x + tx + ty + 2x - t)\,dt$$

$$t(dx + dy) = t(x + y)\,dt$$

$$\frac{dx + dy}{x + y} = dt$$

On integration, $\log(x + y) = t + \log c_2$

$$x + y = c_2 e^t \Rightarrow y = c_2 e^t - x$$

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from eqⁿ (iii)

$$y = c_2 e^{t} - \frac{1}{3}t - c_1 t^{-2}$$

Required Solution

Que Solve : $\frac{dx}{dt} + 4x + 3y = t$, $\frac{dy}{dt} + 2x + 5y = e^t$

Solⁿ: Given equation can be written as,

$$(D+4)x + 3y = t \quad \text{— (i)}$$
$$2x + (D+5)y = e^t \quad \text{— (ii)}$$

Eliminating y

$$(D+5)(D+4)x - 6x = (D+5)t - 3e^t$$
$$(D^2 + 9D + 14)x = 1 + 5t - 3e^t$$

Auxiliary equation,

$$m^2 + 9m + 14 = 0$$
$$m^2 + 7m + 2m + 14 = 0$$
$$m(m+7) + 2(m+7) = 0$$
$$(m+2)(m+7) = 0$$
$$m = -2, -4$$

Complementary function, $C.F = c_1 e^{-2t} + c_2 e^{-7t}$

Particular Integral, $P.I = \frac{1}{D^2 + 9D + 14}(1 + 5t - 3e^t)$

$$= \frac{1}{D^2 + 9D + 14}(1 + 5t) - 3\frac{1}{D^2 + 9D + 14}e^t$$

$$= \frac{1}{14}\left(1 - \frac{9}{14}D + \frac{D^2}{14}\right)(1 + 5t) - \frac{3e^t}{1 + 9 + 14}$$

$$= \frac{1}{14}\left(1 + 5t - \frac{9}{14}\cdot 5\right) - \frac{e^t}{8}$$

$$= \frac{1}{14}\left(5t - \frac{31}{14}\right) - \frac{e^t}{8}$$

Solution, $x = C.F + P.I$

$$x = c_1 e^{-2t} + c_2 e^{-7t} + \frac{1}{14}\left(5t - \frac{31}{14}\right) - \frac{e^t}{8}$$

$$\therefore \frac{dx}{dt} = -2c_1 e^{-2t} - 7c_2 e^{-7t} + \frac{5}{14} - \frac{e^t}{8}$$

Substituting above value in eqⁿ (1) we get

$$3y = 2c_1 e^{-2t} + 7c_2 e^{-7t} - \frac{5}{14} + \frac{e^t}{8} - 4c_1 e^{-2t} - 4c_2 e^{-7t}$$
$$- \frac{20}{14}t + \frac{124}{196} + \frac{e^t}{2} + t$$

$$y = \frac{1}{3}\left[-2c_1 e^{-2t} + 3c_2 e^{-7t} - \frac{3}{7}t + \frac{5}{8}e^t + \frac{27}{98}\right]$$

Required Solution

~~Ques. Solve:~~ $\cancel{t\,dx = (t - 2x)\,dt}$

$\cancel{t\,dy = (tx + ty + 2x - t)\,dt}$

~~Solⁿ: Given equation~~

$\cancel{t\,dx = (t - 2x)\,dt}$

$\cancel{t\,dy = (tx + ty + 2x - t)\,dt}$

→ Simultaneous Equation of the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

(#) Geometrical Interpretation of D.E

Ques. Solve : $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$

Solⁿ:- Given differential Equation

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Taking first two ratio,

$$\frac{dx}{x} = \frac{dy}{y}$$

On integration,

$$\log x = \log y + \log a$$
$$\log x = \log ay$$
$$x = ay$$

Again from second and third ratio,

$$\frac{dy}{y} = \frac{dz}{z}$$

On integration,

$$\log y = \log z + \log b$$
$$\log y = \log bz$$
$$y = bz$$

Hence Required solution,

$$x = ay \quad \text{and} \quad y = bz$$

Ques. Solve : $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$

Solⁿ:- Given differential equation

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

from first and second ratio,

$$\frac{dx}{yz} = \frac{dy}{zx} \Rightarrow x\,dx = y\,dy$$

On integration,

$$\frac{x^2}{2} = \frac{y^2}{2} + a$$

$$x^2 - y^2 = 2a$$

$$x^2 - y^2 = c_1 \qquad \{2a = c_1\}$$

from second and third Ratio,

$$\frac{dy}{zx} = \frac{dz}{xy} \Rightarrow y\,dy = z\,dz$$

On integration,

$$\frac{y^2}{2} = \frac{z^2}{2} + b$$

$$y^2 - z^2 = 2b$$

$$y^2 - z^2 = c_2 \qquad \{ c_2 = 2b \}$$

Required Solution of given D.E

$$x^2 - y^2 = c_1 \quad \text{and} \quad y^2 - z^2 = c_2$$

Ques Solve : $\frac{dx}{1+y} = \frac{dy}{1+x} = \frac{dz}{z}$

Solⁿ:- Given equation,

$$\frac{dx}{1+y} = \frac{dy}{1+x} = \frac{dz}{z}$$

$$\frac{dx}{1+y} = \frac{dy}{1+x} = \frac{dz}{z} = \frac{dx+dy}{(1+y)+(1+x)} = \frac{dx-dy}{(1+y)-(1+x)}$$

$$\frac{dx}{1+y} = \frac{dy}{1+x} = \frac{dz}{z} = \frac{dx+dy}{2+x+y} = \frac{dx-dy}{-(x-y)}$$

Taking Third and fourth ratio of equation,

$$\frac{dz}{z} = \frac{dx+dy}{2+x+y}$$

On integration, $\log z = \log(2+x+y) + \log a$

$$\log z = \log a(2+x+y)$$

$$z = a(2+x+y)$$

Taking third and fifth equation

$$\frac{dz}{z} = \frac{dx-dy}{-(x-y)}$$

On integration, $\log z = -\log(x-y) + \log b$

$$\log z + \log(x-y) = \log b$$

$$z(x-y) = b$$

Hence Required Solution of given differential equation

$$z = a(2+x+y) \quad \text{and} \quad z(x-y) = b$$

Ques Solve : $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{nxy}$

Solⁿ:- Given equation,

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{nxy}$$

Taking first and second ratio,

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

On integration,

$$\frac{-1}{x} = \frac{-1}{y} + c_1$$

$$\frac{1}{y} - \frac{1}{x} = c_1$$

$$\cancel{x - y = c_1 xy}$$

$$y - x = c_1 xy$$

$$y = c_1 xy + x \quad -(1)$$

using $\frac{1}{x}$, $-\frac{1}{y}$, $\frac{c_1}{n}$ as multipliers,

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{nxy} = \frac{\frac{1}{x}dx - \frac{1}{y}dy + \frac{c_1}{n}dz}{x - y + c_1 xy} = \frac{\frac{1}{x}dx - \frac{1}{y}dy + \frac{c_1}{n}dz}{0}$$

{from (1)}

$$\therefore \quad \frac{1}{x}dx - \frac{1}{y}dy + \frac{c_1}{n}dz = 0$$

On Integration,

$$\log x - \log y + \frac{c_1}{n}z = \cancel{\log}\ c$$

$$\log\frac{x}{y} + \frac{c_1}{n}z = c$$

$$\frac{c_1}{n}z = -\log\frac{x}{y} + c$$

$$\frac{c_1}{n}z = \log\frac{y}{x} + c$$

$$z = \frac{n}{c_1}\log\frac{y}{x} + \frac{cn}{c_1}$$

$$z = \frac{n}{c_1}\log\frac{y}{x} + c_2$$

$\left\{ c_2 = \frac{cn}{c_1} \right\}$

Hence Required Solution,

$$y - x = c_1 xy \quad \text{and} \quad z = \frac{n}{c_1}\log\frac{y}{x} + c_2$$

Ques Solve : $\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$

Solⁿ :- Given Equation,

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

$$\therefore \quad \frac{dx + dy + dz}{xy - xz + yz - xy + zx - yz} = \frac{dx + dy + dz}{0}$$

$$dx + dy + dz = 0$$

On Integration, $x + y + z = c_1$

using $\frac{1}{x}$, $\frac{1}{y}$ and $\frac{1}{z}$, we get

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y - z + z - x + x - y} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \quad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

On integration, $\log x + \log y + \log z = \log c_2$

$$\log xyz = \log c_2$$

$$xyz = c_2$$

Required Solution,

$$x + y + z = c_1, \quad xyz = c_2$$

Ques Solve : $\dfrac{dx}{mz - ny} = \dfrac{dy}{nx - lz} = \dfrac{dz}{ly - mx}$

Sol^n:

Given equation,

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

using l, m, n multiplier, we have

$$\frac{l\,dx + m\,dy + n\,dz}{lmz - lny + mnx - mlz + nly - nmx} = \frac{l\,dx + m\,dy + n\,dz}{0}$$

$$\therefore \quad l\,dx + m\,dy + n\,dz = 0$$

On Integration, $lx + my + nz = c_1$

Again using x, y, z multiplier, we have

$$\frac{x\,dx + y\,dy + z\,dz}{xmz - xny + ynx - ylz + zly - zmx} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\therefore \quad x\,dx + y\,dy + z\,dz = 0$$

On integration, $\dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{z^2}{2} = c$

$$x^2 + y^2 + z^2 = 2c$$

$$x^2 + y^2 + z^2 = c_2 \qquad \{2c = c_2\}$$

Required Solution,

$$lx + my + nz = c_1 \quad \text{and} \quad x^2 + y^2 + z^2 = c_2$$

Ques Solve : $\dfrac{dx}{y^2 + yz + z^2} = \dfrac{dy}{z^2 + zx + x^2} = \dfrac{dz}{x^2 + xy + y^2}$

Sol^n: Given equation,

$$\frac{dx}{y^2 + yz + z^2} = \frac{dy}{z^2 + zx + x^2} = \frac{dz}{x^2 + xy + y^2}$$

$$\frac{dx - dy}{(y^2 - x^2) + z(y - x)} = \frac{dy - dz}{(z^2 - y^2) + x(z - y)} = \frac{dx - dz}{(z^2 - x^2) + y(z - x)}$$

$\Rightarrow \quad (y^2 - x^2) + z(y - x)$

$(y - x)(y + x) + z(y - x)$

$(y - x)(y + x + z)$

$|||^{ly}$, $(z - y)(z + y + x)$

$(z - x)(z + x + y)$

$$\frac{dx - dy}{(y - x)(x + y + z)} = \frac{dy - dz}{(z - y)(x + y + z)} = \frac{dx - dz}{(z - x)(x + y + z)}$$

Taking first two Ratio,

$$\frac{dx - dy}{(y - x)(x + y + z)} = \frac{dy - dz}{(z - y)(x + y + z)}$$

$$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}$$

On Integration, $\log(x - y) = \log(y - z) + \log c_1$

$$(x - y) = c_1 (y - z)$$

or, $$\frac{x - y}{y - z} = c_1$$

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Taking first and third ratio,

$$\frac{dx - dy}{x - y} = \frac{dx - dz}{x - z}$$

On integration, $\log(x-y) = \log(x-z) + \log c_2$

$$x - y = c_2(x-z)$$

or $$\frac{x-y}{x-z} = c_2$$

Required Solution, $(x-y) = c_1(y-z)$

and $(x-y) = c_2(x-z)$