Added notes 2

Part 1: Continuous groups and the Haar measure

Let us provide some definitions which allow us to talk about continous groups in a (relatively) well defined way.

A topological space $(X, \mathcal{O})$ can be defined as a set X and a collection of subsets $\mathcal{O} = \{O_i\}$ which are called open sets. They satisfy

$$\emptyset \in \mathcal{O}, \qquad \bigcup_{i=1}^{\alpha} O_i \in \mathcal{O}, \qquad \bigcap_{i=1}^{n} O_i \in \mathcal{O}, \qquad \alpha = n \text{ or } \infty. \tag{1}$$

One can now define continuous functions between topological spaces X and Y: $f : X \mapsto Y$ is continuous if the preimage $f^{-1}(O_i) \in \mathcal{O}(X)$ for all $O_i \in \mathcal{O}(Y)$.

When X and Y are topological spaces one defines in a natural way a topology on the product space $X \times Y$, and one can now define a *topological group* G as a group which is also a topological space and where the map between the topological spaces $G \times G$ and G defined by group multiplication: $(g_1, g_2) \mapsto g_1 g_2$ is continuous.

A topological space is *compact* if every collection of open sets such that $X = \bigcup_{i=1}^{\alpha} O_i$ can be refined to a finite set such that $X = \bigcup_{i_k=1}^{n} O_{i_k}$. A topological group is compact if G is compact viewed as a topological space. Finally a topological space is *locally compact* if all $g \in G$ has a *compact neighborhood* (where a neighborhood of a point p is a subset V of X that includes an open set O, i.e. $p \in O \subseteq V$). As an example $(\mathbb{R}, +)$, the real numbers as an additive group is locally compact , but not compact.

On a locally compact space X we can define a *measure*. Let $C_0(X)$ denote the continuous functions on X with compact support, i.e. a function $f \in C_0(X)$ is only different from zero on a compact subset of X. $C_0(X)$ is a vector space and it can be made a *topological vector space* too, i.e. a topological space where the vector operations are continuous. A *(radon) measure* on X is a continuous function $\mu_X : C_0(X) \mapsto \mathbb{C}$ which satisfies

$$\mu_X(f) \geq 0 \qquad \forall f \geq 0. \tag{2}$$

Here the meaning of $f \geq 0$ is that $f(x) \geq 0$ for all $x \in X$. We write

$$\mu_X(f) = \int_X d\mu(x)\, f(x) \tag{3}$$

and call $\mu_X(f)$ the integral of f with respect to μ_X. μ_X can be extended to act on a much larger class of functions than functions belonging to $C_0(X)$, and we talk about the *integrable functions* with respect to the measure μ_X. We can then also introduce the space $L^2(X)$ of square integrable functions on X and the scalar product

$$\langle f_1|f_2\rangle = \int d\mu(x)\, f_1^*(x)f_2(x), \quad f_1, f_2 \in L^2(X), \tag{4}$$

which makes $L^2(X)$ a Hilbert space.

Let E be a compact subset of X. For the so-called measurable sets the function $1_E(x)$, which is 1 for $x \in E$ and zero for $x \notin E$, can be integrated and we write

$$\mu_X(E) = \int_X d\mu(x)\, 1_E(x). \tag{5}$$

One calls $\mu_X(E)$ the volume of E with respect to μ_X.

If a group G acts on X we will be particularly interested in measures which are invariant under the action of G. Let E be a measurable subset of X. We define

$$g(E) = \{x \in X | x = g(y), y \in E\}. \tag{6}$$

μ_X is said to be invariant under action of G if

$$\mu_X(g(E)) = \mu_X(E) \quad \forall\; g, E. \tag{7}$$

Alternatively, we can express the action of G on X via the functions $f \in C_0(X)$. As usual we define $(D(g)f)(x) = f(g^{-1}(x))$ and the requirement (7) can be expressed as

$$\mu_X(D(g)f) = \mu_X(f \circ g^{-1}) = \mu_X(f), \quad \forall g \in G,\ \forall f \in C_0(X). \tag{8}$$

It is seen that (8) implies (7) when one uses (8) on $f = 1_E$. (8) can be written in the more conventional form

$$\int_X d\mu(x)\, f(g^{-1}(x)) = \int_X d\mu(x)\, f(x), \quad \text{or} \quad d\mu(g(x)) = d\mu(x). \tag{9}$$

If $X = G$, G acts on itself by left multiplication $h \mapsto gh$. A measure on G which is invariant under this action is called a *left invariant Haar measure.* G also acts on itself by right multiplication $h \mapsto hg^{-1}$. A measure invariant under this action is called a *right invariant Haar measure.* If there exists a measure which is invariant under both left and right multiplication we just talk about the Haar measure and for such a measure we have for integrable functions $f : G \mapsto \mathbb{C}$

$$\int_G d\mu_G(g)\, f(hg) = \int_G d\mu_G(g)\, f(gh) = \int_G d\mu_G(g)\, f(g) \quad \forall h \in G. \tag{10}$$

$$d\mu_G(hg) = d\mu_G(gh) = d\mu_G(g) \quad \forall h, g \in G. \tag{11}$$

For compact groups G there exists a Haar measure, uniquely determined up to a constant of proportionality. Often one fixes this constant by demanding that $\int_G d\mu_G = 1$. However, sometimes one has a natural definition of the volume of some group G and one then uses this as the normalization:

$$\mathrm{Vol}(G) = \int d\mu_G(g). \tag{12}$$

Note that for compact groups we always have $\int_G d\mu_G < \infty$. For locally compact groups which are not compact there always exist left and right invariant Haar measures, but they might not be identical. If they are (which is often the case for the groups which have the interest of physicists) the group G is called *unimodular*. Note that for the locally compact, but not compact groups one has $\int_G d\mu_G(g) = \infty$. Note also that for our finite groups the Haar measure is defined by

$$\mu_G(f) = \sum_{g \in G} f(g). \tag{13}$$

and we have indeed many times used the identity (10), which here reads

$$\sum_{g \in G} f(hg) = \sum_{g \in G} f(gh) = \sum_{g \in G} f(g) \quad \forall h \in G. \tag{14}$$

Let us list a some Haar measures for various G and some invariant measures on topological spaces X where G acts.

Ex 1 The Haar measure on the additive group $G = (\mathbb{R}, +)$ is the usual Lebesgue measure dx (we have $d(x + x_0) = dx$). G is locally compact but not compact. The group is abelian and thus unimodular.

Ex 2 The Haar measure of the multiplicative group $G = (\mathbb{R}, \cdot)$ is $dx/|x|$, where dx is the Lebesgue measure. Again the group is not compact but unimodular since it is abelian.

Ex 3 The general linear group $G = Gl(n,\mathbb{R})$ of real, invertible $n\times n$ matrices. The Haar measure is

$$d\mu_G(g) = |\det X(g)|^{-n} dx_{11}dx_{12}\cdots dx_{n,n-1}dx_{n,n}, \tag{15}$$

where $dx_{11}\cdots dx_{nn}$ is the Lebesgue measure on $\mathbb{R}^{n^2}$ and the element g is represented by the $n\times n$ matrix X with entries x_{ij}. The x_{ij} are n^2 coordinates for the group elements. The group is not compact but unimodular. Similarly the general linear group $G = Gl(n,\mathbb{C})$ of complex , invertible $n\times n$ matrices have the Haar measure

$$d\mu_G(g) = |\det Z(g)|^{-2n} dx_{11}dy_{11}dx_{12}\cdots dx_{nn}dy_{nn} \tag{16}$$

where $Z(g)$ is a $n\times n$ matrix with entries $z_{ij} = x_{ij} + iy_{ij}$. The group is also unimodular.

Ex 4 Next an example of a matrix group which is not unimodular.

$$G = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \quad x,y\in\mathbb{R}, \quad x\neq 0. \tag{17}$$

The left Haar measure is $dxdy/x^2$ while the right Haar measure is $dxdy/|x|$.

Ex 5 $SU(2)$ is a subgroup of $Gl(2,\mathbb{C})$ defined by

$$g = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}, \quad g^\dagger = g^{-1}, \quad \det g = 1. \tag{18}$$

This implies that

$$z_{22} = \bar{z}_{11}, \quad z_{21} = -\bar{z}_{12}, \quad |z_{11}|^2 + |z_{12}|^2 = 1. \tag{19}$$

We can now obtain the Haar measure from that of $Gl(2,\mathbb{C})$ by multiplying with the delta-functions

$$\delta(x_{22}-x_{11})\delta(y_{22}+y_{11})\delta(x_{21}+x_{12})\delta(y_{21}-y_{12})\delta(x_{11}^2+y_{11}^2+x_{12}^2+y_{12}^2-1). \tag{20}$$

Thus we obtain, introducing first a simplifying notation:

$$\begin{matrix} x_1 = x_{11}, & x_2 = y_{11} \\ x_3 = x_{12}, & x_4 = y_{12} \end{matrix} \quad g = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}, \quad \sum_{i=1}^4 x_i^2 = 1. \tag{21}$$

the following measure for $SU(2)$

$$d\mu_G(g) \propto \delta(\sum_{i=1}^{4} x_i^2 - 1) dx_1 dx_2 dx_3 dx_4 \propto \frac{dx_1 dx_2 dx_3}{\sqrt{1 - x_1^2 - x_2^2 - x_3^2}} \tag{22}$$

The condition $\sum_{i=1}^{4} x_i^2 = 1$ shows that $SU(2)$ topologically can be identified with the three-sphere S^3. The expression (22) for the measure shows that representing $SU(2)$ as S^3, the measure is precisely the uniform measure on S^3, which it naturally inherits from being embedded in Euclidean space $\mathbb{R}^4$. If we introduce spherical coordinates for S^3 we can rewrite $d\mu_G(g)$ as follows

$$\begin{aligned} x_4 &= \cos\psi \\ x_3 &= \cos\theta \sin\psi \\ x_2 &= \cos\phi \sin\theta \sin\psi \\ x_1 &= \sin\phi \sin\theta \sin\psi \end{aligned} \qquad d\mu_G(g) \propto \sin^2\psi \sin\theta d\psi d\theta d\phi, \tag{23}$$

where $\phi \in [0, 2\pi]$, $\theta, \psi \in [0, \pi]$. For a fixed value of ψ one generates an ordinary sphere S^2 of radius $\sin\psi$ when varying θ and ϕ.

Ex 6 The group $SO(3)$ is locally identical to $SU(2)$, but globally antipodal points on S^3 correspond to the same group element in $SO(3)$ ($SO(3) = SU(2)/Z_2$) as we will now explain. A given point on S^3, with polar coordinates (ψ, θ, ϕ) corresponds to the following ordinary rotation of vectors in $\mathbb{R}^3$: a rotation around the axis $\hat{n}(\theta, \phi)$ with rotation angle $\tilde{\psi} = 2\psi$, where (θ, ϕ) are ordinary spherical coordinates in the space $\mathbb{R}^3$ where the rotations are performed. Thus, when ψ changes from 0 to π, i.e. from the north pole to the south pole of S^3 in the notation of Ex. 5, $\tilde{\psi}$ changes from 0 to 2π. However, a rotation around a axis $\hat{n}$ with angle $\tilde{\psi}$ is identical to a rotation around the axis $-\hat{n}$ with angle $2\pi - \tilde{\psi}$. If $\hat{n}$ corresponds to polar coordinates (θ, ϕ) then $-\hat{n}$ corresponds to polar coordinates $(\pi - \theta, 2\pi - \phi)$. Thus altogether the first rotation can by identified with the S^3 point (ψ, θ, ϕ) while the second rotation (which is identical to the first) can by identified with the point $(\pi - \psi, \pi - \theta, 2\pi - \phi)$, i.e. the antipodal point on S^3. Since the physical rotation angle of the $SO(3)$ rotation is $\tilde{\psi}$ rather than the ψ used in Ex. 5, we will us it in the Haar measure for $SO(3)$ and thus write

$$d\mu_G(g) \propto \sin^2\frac{\tilde{\psi}}{2} \sin\theta d\tilde{\psi} d\theta d\phi, \tag{24}$$

where where $\phi \in [0, 2\pi]$, $\theta, \tilde{\psi} \in [0, \pi]$. We here constrain $\tilde{\psi}$ to $[0, \pi]$ since we allow all directions of $\hat{n}(\theta, \phi)$ by allowing $\phi \in [0, 2\pi]$ and $\theta \in [0, \pi]$. In this way we avoid the double counting described above.

Let us now return to the topological spaces X where G acts transitivily, where we assume that G is a continuous group. We call such spaces *globally symmetric spaces* and we have seen in "added-notes-1" that such spaces can by identified with coset spaces G/H where H is a subgroup of G. If we have a Haar measure on G, the subgroup H will inherit its own Haar measure from that of G and similarly G/H and thus X will inherit (under certain technical assumptions) a measure which is invariant under the action of G on X of G/H. We can then write

$$\int_G d\mu(g) = \int_X d\mu_X(x) \int_H d\mu_H, \qquad X \text{ homeomorphic to } G/H \tag{25}$$

where $d\mu_X(g(x)) = d\mu(x)$ for all g.

Rather than discussing the general situation, let us illustrate it in the case where $G = SO(3)$, $H = SO(2)$ and $X = SO(3)/SO(2) = S^2$. First the general picture: G acts on S^2 by standard rotation: we identify the unit vectors in $\mathbb{R}^3$ with S^2, and a unit vector $\hat{e}$ is rotated into another unit vector $\hat{e}'$ by the standard rotation. This action is clearly transitive: any unit vector can be rotated to any other unit vector by a suitable rotation. Let us for instance consider $\hat{e}_z$, the unit vector along the $\hat{z}$-axis. The stability group $H_{\hat{e}_z}$ consists of all rotations with leave $\hat{e}_z$ invariant, i.e. all rotations around the $\hat{z}$-axis. Let $R \in H_{\hat{e}_z}$ be such a rotation, with rotation angle $\tilde{\psi}$. We have:

$$R(\hat{e}_z, \tilde{\psi}) = \begin{pmatrix} \cos\tilde{\psi} & -\sin\tilde{\psi} & 0 \\ \sin\tilde{\psi} & \cos\tilde{\psi} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \tilde{\psi} \in [0, 2\pi]. \tag{26}$$

Thus the subgroup $H_{\hat{e}_z}$ is clearly isomorphic to the rotation group $SO(2)$ and we can write

$$S^2 = SO(3)/H_{\hat{e}_z} = SO(3)/SO(2), \qquad \Big(S^n = SO(n+1)/SO(n)\Big), \tag{27}$$

where "=" means "homeomorphic to" and the last equation in (27) indicates that the construction in an obvious way generalises from $SO(3)$ to $SO(n+1)$.

We can obtain a very direct realisation of the coset structure by using the so-called *Euler angles* α, β, γ to describe the rotations. Using these angles any rotation has a unique decomposition as

$$R(\alpha, \beta, \gamma) = R(\hat{z}, \alpha) R(\hat{y}, \beta) R(\hat{z}, \gamma), \tag{28}$$

where $0 < \alpha < \pi$, $0 \leq \beta \leq \pi$ and $-\pi \leq \gamma \leq \pi$. The rotation $R(\alpha, \beta) = R(\hat{z}, \alpha) R(\hat{y}, \beta)$ is seen to do the following

$$R(\alpha, \beta) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha \sin\beta \\ \sin\alpha \sin\beta \\ \cos\beta \end{pmatrix} \tag{29}$$

Thus $R(\alpha, \beta)$ rotates $\hat{e}_z$ to the unit vector $\hat{e}(\alpha, \beta)$ with spherical coordinates (α, β), and we can from a topological point of view identify the set of rotation matrices $R(\alpha, \beta)$ with the points on S^2. with spherical coordinates (α, β). As before the stability group $H_{\hat{e}_z}$ of the point $\hat{e}_z$ can be identified with the subgroup $R(\hat{z}, \gamma)$ and in this way the decomposition

$$R(\alpha, \beta, \gamma) = R(\alpha, \beta) R(\hat{z}, \gamma) \tag{30}$$

provides us with an explicit realisation of the formal equations $G = G/H \times H$, $X = G/H$ in the case $G = SO(3)$, $H = SO(2)$ and $X = S^2$. Finally, when we express the Haar measure on $SO(3)$ in terms of the Euler angles one can show that it is (normalized to 1)

$$d\mu_G(g) = \frac{1}{8\pi^2} \sin \beta d\beta d\alpha d\gamma \tag{31}$$

and we see that it is precisely the split alluded to in eq. (25), $\sin \beta d\beta d\alpha$ being the measure on S^2 invariant under the action of G and $d\gamma$ being the measure on a $SO(2)$ subgroup.

Part 2: Representations of continuous groups.

As already emphasized, the way we often meet continuous groups in physics is as transformation groups acting on a continuous topological space X. Generic examples are the group of translations acting on our usual physical space $\mathbb{R}^3$ and the group of rotations acting on $\mathbb{R}^3$ or on S^2. In quantum mechanics this action manifests itself via the equivalent action on the wave functions $\psi(x)$ These wave functions belong to $L^2(X)$, where, for a topological space X, $L^2(X)$ is defined via a measure $d\mu_X(x)$ which usually is invariant under the action of G.

The action of G on X is translated to an action of G on $L^2(X)$ via the generic representation of G on $L^2(X)$ defined by

$$(D(g)f)(x) = f(g^{-1}(x)), \quad g \in G, \quad x \in X, \quad f \in L^2(X). \tag{32}$$

This generic representation has the virtue that it precisely captures what we mean by translations and rotations of functions: if g is a translation of points in $X = \mathbb{R}$ by a, then $f(g^{-1}(x))$ is the function $f(x)$ translated by a, i.e. the function $f(x - a)$ as described in "addednotes1". In the same way, if g represents a rotation around an axis $\hat{n}$ with angle φ then $f(g^{-1}(x))$ will be the function $f(x)$ rotated this angle around the axis $\hat{n}$. $\{D(g)|g \in G\}$

provides us with a representation of G: For all $g \in G$ $D(g)$ is a linear map $L^2(X) \mapsto L^2(X)$, i.e. a linear operator on the infinite dimensional Hilbert space $L^2(X)$. We found explicit expressions for these operators in the case of translations and noticed that they were unitary operators, and we will later find expressions in the case of the rotation group. However, the point we want the emphasize here is that the representations which naturally appear in quantum mechanics are infinite dimensional unitary representations on Hilbert spaces and we can then ask the question: how many of the beautiful results valid for finite groups are still valid for the continuous groups. The answer is the following: almost all of them in the case of compact groups. Below we will outline (some of) the results known for continuous compact groups.

First let us show that if the measure on X is invariant under the action of G then (32) is a unitary representation. The proof is identical to the proof provided for finite groups: we show that $(D(g))^{-1} = (D(g))^\dagger$:

$$\begin{aligned} \langle f_1 | D(g) f_2 \rangle &= \int_X d\mu(x) f_1^*(x) f_2(g^{-1}(x)) = \int_X d\mu(y) f_1^*(g(y)) f_2(y) \\ &= \langle D(g^{-1}) f_1 | f_2 \rangle = \langle (D(g))^{-1} f_1 | f_2 \rangle.. \end{aligned} \tag{33}$$

Here we have used that $d\mu(g(y)) = d\mu(y)$ and eq. (33) shows the desired result since this is the defining equation for $(D(g))^\dagger$.

We now assume G is a compact continuous group and let V denote the Hilbert space where $D(g)$ acts. The definition of *equivalent representations* is identical to the one for finite groups. One can also prove that *any representation is equivalent to a a unitary representation* (under suitable assumptions). Like for finite groups we say that D is an *irreducible representation* if for all subspaces $K \subseteq V$ we have

$$D(g)K \subseteq K \ \ \forall g \in G \quad \Rightarrow \quad K = \{0\} \ \vee \ K = V. \tag{34}$$

We now have the following

Theorem: Every continuous, unitary representation of a compact group G is a direct sum of *finite dimensional* irreducible components. Every irreducible representation can occur several times (with multiplicity $n_r \leq \infty$)

$$V = \bigoplus_r \bigoplus_{\alpha_r=1}^{n_r} V_{d_r}^{(r,\alpha_r)}, \quad D(g) = \bigoplus_r \bigoplus_{\alpha_r=1}^{n_r} D_{d_r}^{(r,\alpha_r)}(g) \tag{35}$$

where the $V_{d_r}^{(r,\alpha_r)}$ are subspaces of dimensions $d_r < \infty$, all mutually orthogonal: $V^{(r,\alpha)} \perp V^{(r',\alpha')}$, and where $D^{(r,\alpha)}$ is equivalent to $D^{(r,\beta)}$ for $\alpha, \beta = 1, \ldots, n_r$.

Thus all irreducible representations of G are finite dimensional. What about the “great othogonality theorem?. It is also valid!. Let us consider a compact group and normalize the Haar measure such that $\int_G d\mu(g) = 1$. We consider representations on $L^2(G)$ and we have the following theorem, called the *Peter-Weil theorem*

Theorem: Let $(D_{d_r}^{(r)}(g), V_{d_r}^{(r)})$ denote the irreducible representations and the corresponding d_r dimensional vector spaces of the compact group G (we choose a representative for each equivalence class of irreducible representations). Now choose an orthonormal basis $e_i^{(r)}$, $i = 1, \ldots, d_r$ in each $V_{d_r}^{(r)}$. $D_{d_r}^{(r)}(g)$ then has matrix elements $(D_{d_r}^{(r)}(g))_{ij} = \langle e_i^{(r)} | D(g) e_j^{(r)} \rangle$, $i, j = 1, \ldots, d_r$. These matrix elements can be viewed as functions $G \mapsto \mathbb{C}$, i.e. functions in $L^2(G)$. The functions $f_{r,i,j}(g) = \sqrt{d_r}\, (D_{d_r}^{(r)}(g))_{ij}$ constitute a complete orthonormal set of functions on $L^2(G)$:

$$\boxed{\int_G d\mu(g)\, f_{r,i,j}^*(g) f_{r',i',j'}(g) = \delta_{rr'} \delta_{ii'} \delta_{jj'}} \tag{36}$$

and the completeness relation reads:

$$\boxed{\sum_{r,i,j} f_{r,i,j}^*(g')\, f_{r,i,j}(g) = \delta(g, g')} \tag{37}$$

In eq. (37) the delta-function $\delta(g, g')$ is defined by

$$\int_G d\mu(g) f(g)\, \delta(g, g') = f(g') \tag{38}$$

for all suitable nice functions $f(g)$. Thus any function $f \in L^2(G)$ can be expanded as

$$f(g) = \sum_{r,i,j} c_{r,i,j} f_{r,i,j}(g), \qquad c_{r,i,j} = \int_G d\mu(g) f_{r,i,j}^*(g)\, f(g). \tag{39}$$

The group G acts on itself (i.e. $X = G$ in the notation surrounding eq. (32)) by left multiplication: $h \mapsto g\, h$, $\forall h \in G$. The representation defined by eq. (32) is then called *the regular representation of G*, as for finite groups:

$$(D^{(reg)}(g) f)(h) = f(g^{-1} h), \quad g, h \in G, \quad \forall f \in L^2(G), \tag{40}$$

Like for finite groups the representation $D^{(reg)}$ is highly reducible. We can now decompose $D^{(reg)}$ into irreducible representations and we have (as for finite groups)

> **Theorem:** Let G be compact. In the decomposition of the regular representation $D^{(reg)}(g)$ into irreducible representations $D_{d_r}^{(r)}(g)$, every irreducible representation will be present with multiplicity $n_r = d_r$.

Let us finally turn to the characters $\chi(g)$ of a representation. For finite groups we defined the character of a representation D as $\chi(g) = \operatorname{tr} D(g)$. For the infinite dimensional representations $\operatorname{tr} D(g)$ can be infinite. However, for the irreducible representations of compact groups we can still use this definition since these representations are finite dimensional. For any irreducible representation $D^{(r)}$ of a compact group we thus define

$$\chi^{(r)}(g) \equiv \operatorname{tr} D^{(r)}(g). \tag{41}$$

We have as for finite groups

$$\chi^{(r)}(g) = \chi^{(r)}(h^{-1}g\,h) \quad \forall g, h \in G, \quad \text{i.e.} \quad \chi^{(r)}(h\,g) = \chi^{(r)}(g\,h) \quad \forall g, h \in G. \tag{42}$$

Functions $f : G \mapsto \mathbb{C}$ which satisfy (42) are called *central functions*. The square integrable central functions constitute a Hilbert space $L_o^2(G)$ (as usual refering to the Haar measure which we in the sentence below assume are normalized to 1).

> **Theorem:** For a compact group G the characters $\chi^{(r)}(g)$ of the irreducible representations constitute a complete orthonormal set of central functions on $L_o^2(G)$:

$$\boxed{\int_G d\mu(g)(\chi^{(r)}(g))^*\,\chi^{(r')}(g) = \delta^{rr'} \qquad \sum_r (\chi^{(r)}(g))^*\,\chi^{(r)}(g') = \delta(g, g').} \tag{43}$$

Thus any central function $f \in L_o^2(G)$ can be expanded as

$$f(g) = \sum_r c_r \chi^{(r)}(g), \quad c_r = \int_G d\mu(g)\,(\chi^{(r)}(g))^* f(g). \tag{44}$$

The characters distinguish the irreducible representations:

Theorem: Let G be a compact group. Two irreducible representations $D^{(r)}$ and $D^{(r')}$ are equivalent if and only if they have the same character function $\chi^{(r)}$.

Finally we notice that if D is a representation of a compact group which has a finite character function $\chi(g)$ then D is finite dimensional and the multiplicity n_r of the irreducible representation $D^{(r)}$, i.e. the number of times $D^{(r)}$ occurs when D is decomped into irreducible representations, is given by

$$n_r = \langle \chi^{(r)} | \chi \rangle = \int_G d\mu(g)\, (\chi^{(r)}(g))^* \, \chi(g) \tag{45}$$

Part 3: SO(3) as an example

We define $SO(3)$ as the transformation group which acts on $\mathbb{R}^3$ by rotating vectors $X = x_i \hat{e}_i$ into new vectors $X' = x'_i \hat{e}_i$ where $\hat{e}_i$ are orthogonal unit vectors, usually identified with the unit vectors $\hat{x}, \hat{y}, \hat{z}$ in the directions of the coordinate axises. The rotation is now given by

$$x'_i = R_{ij} x_j, \quad R^T = R^{-1}, \quad \det R = 1. \tag{46}$$

The 3×3 matrix R_{ij} depends by the axis $\hat{n}$ around which the rotation is performed, and the angle $\varphi \in [0, 2\pi[$ of rotation. It is a two-to-one labelling of the rotations since $R(\hat{n}, \varphi) = R(-\hat{n}, 2\pi - \varphi)$. As is shown in the Zee-book we can write R as

$$R(\hat{n}, \varphi) = \mathrm{e}^{\varphi n_k I_k}, \quad \hat{n} = n_k \hat{e}_k, \quad (I_k)_{ij} = -\varepsilon_{ijk} \tag{47}$$

where the matrix I_i defines the rotation around $\hat{e}_i$-axis and where the anti-symmetric I_i matrices form a basis for the Lie algebra $so(3)$ of the Lie group $SO(3)$. The matrices I_i are the *generators* of infinitesimal rotations around the $\hat{e}_i$-axis:

$$R(\hat{n}, \varphi) = I_{3\times 3} + \varphi n_k I_k + o(\varphi^2) \tag{48}$$

and the group structure ensures we can obtain (47) from (48) by exponentiation. The structure constants of $so(3)$ are given by

$$[I_i, I_j] = \varepsilon_{ijk} I_k. \tag{49}$$

As explained in the Zee-book, we were led to the matrices I_i by considering infinitesimal rotations around the $\hat{e}_i$ axis. Since $SO(3)$ acts as a transformation group on $\mathbb{R}^3$ we know we have the generic representation of $SO(3)$ on $L^2(\mathbb{R}^3)$ defined by

$$(D(R)f)(x_i) = f((R^{-1}_{ij} x_j) = f(x_j R_{ji}). \tag{50}$$

$D(R)$ is an *operator* acting on $L^2(\mathbb{R}^3)$ and we now want to find the generators of infinitesimal rotations represented by $D(R)$. From the group properties we then know that we can obtain $D(R)$ for non-infinitesimal rotations by exponentiation, in the same way as we can obtain (47) from (48). Let now φ be infinitesimal. From (50) we obtain, using (48) and Taylor expansion of the function $f(x_i)$ of the three variables x_i to first order

$$(D(R)f)(x_i) = f(x_i+\varphi x_j(-n_k\varepsilon_{jik})+o(\varphi^2)) = (I+\varphi n_k\mathcal{I}_k)f(x_i)+o(\varphi^2), \quad (51)$$

where

$$\mathcal{I}_k = -\varepsilon_{ijk}x_i\frac{\partial}{\partial x_j}, \qquad [\mathcal{I}_i,\mathcal{I}_j] = \varepsilon_{ijk}\mathcal{I}_k. \quad (52)$$

Thus we obtain by exponentiation when φ is not infinitesimal

$$R(\hat{n},\varphi) = \mathrm{e}^{\varphi n_k I_k} \quad \Rightarrow \quad D(R(\hat{n},\varphi)) = \mathrm{e}^{\varphi n_k \mathcal{I}_k}. \quad (53)$$

In the defining representation of $SO(3)$ both R and the generators I_i act on the ordinary 3-dimensional vector space $X = \mathbb{R}^3$ and are thus represented by 3×3 matrices. However, when $SO(3)$ acts on the Hilbert space $L^2(X)$ of functions $f : X \mapsto \mathbb{C}$, which is infinite dimensional, both $D(R)$ and its generators $\mathcal{I}_i$ are represented by infinite dimensional matrices, i.e. operators. *The group properties imply that the operators representing the generators are differential operators acting on functions belonging to* $L^2(X)$ because we can Taylor expand around the identity. The algebra of the generators $\mathcal{I}_i$ has to be identical to the algebra of I_i, again because it is defined by the group property, but the representation of the Lie algebra now lives in $L^2(X)$ rather than in X.

In physics it is customary to use the operators $J_k = i\mathcal{I}_k$, rather than $\mathcal{I}_k$ because J_k is an Hermitian operator and because the J_k's precisely are the components of the angular momentum operator (divided by $\hbar$, recall that the momentum $p_k = \frac{-i}{\hbar}\frac{\partial}{\partial x_k}$ and the angular momentum is defined by $J_k = \varepsilon_{kij}x_ip_j$.). Thus we can write

$$D(R(\hat{n},\varphi)) = \mathrm{e}^{-i\varphi n_k J_k}, \qquad [J_i,J_k] = i\varepsilon_{ijk}J_k. \quad (54)$$

The angular momentum operators J_k are the generators of rotation in quantum mechanics.

We now have the following scenario in quantum mechanics: ket vectors (i.e. basically wave functions in $L^2(X)$) will transform as $|\psi\rangle \mapsto |\psi'\rangle = D(R)|\psi\rangle$ under rotations. Correspondingly operators $\hat{A}$ (i.e. linear maps $L^2(X) \to L^2(X)$) transform as $\hat{A} \mapsto \hat{A}' = D(R)\hat{A}D(R)^{-1}$. An operator is said to be invariant under rotation if $\hat{A}' = \hat{A}\ \forall R$, i.e. if $[D(R), \hat{A}] = 0\ \forall R$. Since J_k are the generators for rotations the statement is equivalent to the statement that $[J_k, \hat{A}] = 0$, $k = 1, 2, 3$. In particular, a quantum mechanical system is said to be invariant under rotation if the Hamiltonian $\hat{H}$ is invariant under rotation:

$$\hat{H} \text{ invariant under rotation} \quad \Leftrightarrow \quad [J_k, \hat{H}] = 0, \quad k = 1, 2, 3. \tag{55}$$

Thus we can diagonalize any of the J_i and $\hat{H}$ simultanously, but since the J_k's do not commute we have to choose one of them, J_z say. From the algebra of the J_k's it follows, as is well know, that $J^2 \equiv J_x^2 + J_y^2 + J_z^2$ commutes with all J_k (and of course with $\hat{H}$ if the J_k do). Thus we can choose simultanously eigenfunctions for $\hat{H}$, J_z and J^2 if (55) is valid.

Let us now understand the Hilbert space $L^2(\mathbb{R}^3)$ from the viewpoint of $SO(3)$. If we use spherical coordinates, i.e. write $f(x_i)$ as $\psi(r, \theta, \varphi)$, then it is clear from the very definition (50) that $D(R)$ does not act on r, since rotations do not change r. Thus $D(R)$ and its generators J_k depend only on θ and φ. This might not be immediately obvious from the explicit expressions (52), but is true. Let us here only give the expressions for J_z and J^2 in spherical coordinates:

$$J_z = -i \frac{\partial}{\partial \varphi}, \qquad J^2 = -\Big(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\Big(\sin\theta \frac{\partial}{\partial \theta}\Big) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \varphi^2}\Big). \tag{56}$$

They are differential operators acting on functions $\psi(\theta, \varphi)$ belonging to the Hilbert space $L^2(S^2)$, defined from the measure $\mathrm{d}\mu(\theta, \varphi) = \sin\theta d\theta \mathrm{d}\varphi$. This measure (which just gives the area element of S^2 in spherical coordinates) is invariant under rotations, and was exactly the measure we meet before, when we considered viewed $S^2 = SO(3)/SO(2)$. From the general theorems mentioned above we know that the representation $D(R)$ on $L^2(S^2)$ can be decomposed in irreducible finite dimensional representations. From elsewhere we know that these irreducible representations are $2j + 1$ dimensional where the eigenvalue of J^2 is $j(j + 1)$. Let us we denote them $D^{(j)}(R)$. One can prove that each $D^{(j)}(R)$ occurs with multiplicity $n_j = 1$ in the decomposition of the representation $D(R)$, i.e. we have

$$L^2(S^2) = \bigoplus_{j=0}^{\infty} V_{2j+1}, \qquad D(R) = \bigoplus_{j=0}^{\infty} D^{(j)}(R). \tag{57}$$

On the $2j + 1$ dimensional vector space V_{2j+1} the (differential) operator J^2 acts as

$$J^2|_{V_{2j+1}} = j(j + 1)\, I_{(2j+1)\times(2j+1)}, \tag{58}$$

simply because it is the eigenspace of J^2 corresponding to the eigenvalue $j(j+1)$. On this eigenspace J_z can take values $m = -j, -j+1, \dots, j-1, j$, i.e. we have on V_{2j+1}:

$$J^2|jm\rangle = j(j+1)|jm\rangle, \qquad J_z|jm\rangle = m|jm\rangle, \tag{59}$$

where the common eigenvectors of J^2 and J_z are denoted $|jm\rangle$. As shown in the Zee-book one can in this basis find the matrices $\langle jm'|J_k|jm\rangle$ entirely from the properties of the Lie algebra $so(3)$. and then by exponentiation the matrices $\langle jm'|D^{(j)}(R)|jm\rangle$. If one wants explicit expressions for the eigenvectors $|jm\rangle$ as functions $\langle\theta,\varphi|jm\rangle$ on $L^2(S^2)$ one can choose to solve (59) as differential equations, using (56). One finds that $\langle\theta,\varphi|jm\rangle = Y_j^m(\theta,\varphi)$, the so-called spherical harmonics. However, one can also easily find $Y_j^m(\theta,\varphi)$ from our standard Lie algebra construction of $|jm\rangle$ starting from $|jj\rangle$ and successively applying the lowering operator J_- to create $|jm\rangle \propto J_-^{j-m}|jj\rangle$. One has explicitly from (52) and $J_k = i\mathcal{I}_k$ that

$$J_\pm = \mp(x \pm iy)\frac{\partial}{\partial z} \pm iz\left(\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}\right) \tag{60}$$

from which it follows

$$J_\pm(x \pm iy) = 0, \quad J_- z = i(x - iy), \quad J_-(x+iy) = -2iz \tag{61}$$

or, using $x \pm iy = r\,\mathrm{e}^{\pm i\varphi}\sin\theta$, $z = r\cos\theta$ and $J_\pm r^2 = J_\pm(x^2+y^2+z^2) = 0$,

$$J_\pm(e^{\pm i\varphi}\sin\theta) = 0, \quad J_-\cos\theta = ie^{-i\varphi}\sin\theta, \quad J_-(e^{i\varphi}\sin\theta) = -2i\cos\theta. \tag{62}$$

From these relations it follows easily that if we identify

$$\langle\theta,\varphi|jj\rangle = \left(\mathrm{e}^{i\varphi}\sin\theta\right)^j := Y_j^j(\theta,\varphi) \tag{63}$$

then repeated application of J_- leads to

$$J_-^{j-m}Y_j^j(\theta,\varphi) = F_{j-|m|}(\cos\theta)\,\mathrm{e}^{im\varphi}\big(\sin\theta\big)^{|m|}, \quad m = j, j-1, \dots, -j, \tag{64}$$

where $F_k(x)$ is a polynomial of order k. Also, we see that (from (56) for J_z)

$$J_+Y_j^j(\theta,\varphi) = 0, \quad J_-Y_j^{-j}(\theta,\varphi) = 0, \quad J_zY_j^j(\theta,\varphi) = jY_j^j(\theta,\varphi), \tag{65}$$

i.e. Y_j^j is indeed a higest weight state and the repeated application of J_- comes to an end when we have $2j+1$ states $|jm\rangle$. The functions $J_-^{j-m}Y_j^j(\theta,\varphi)$ are precisely the spherical harmonics $Y_j^m(\theta,\varphi)$ up to normalization.

When we express the rotation R in terms of the Euler angles α, β, γ the corresponding matrix $D^{(j)}(R)$ is called the *Wigner D-matrix*. According to (28) and (54) we can then write

$$D^{(j)}_{m'm}(\alpha,\beta,\gamma) \equiv \langle jm'|\mathrm{e}^{-i\alpha J_z}\mathrm{e}^{-i\beta J_y}\mathrm{e}^{-i\gamma J_z}|jm\rangle = \mathrm{e}^{i(m'\alpha - m\gamma)}\langle jm'|\mathrm{e}^{-i\beta J_y}|jm\rangle. \tag{66}$$

Thus we know $D^{(j)}_{m'm}(\alpha,\beta,\gamma)$ if we know the so-called *small-d Wigner matrix* $d^{j}_{m'm}(\beta) = \langle jm'|\mathrm{e}^{-i\beta J_y}|jm\rangle$, which describes rotations around the $\hat{y}$-axis. Finally, using (31) and (37) we have the following

$$\int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta d\beta \int_{-\pi}^{\pi} d\gamma\, D^{(j')}_{m'k'}(\alpha,\beta,\gamma)^* D^{(j)}_{mk}(\alpha,\beta,\gamma) = \frac{8\pi^2}{2j+1}\delta_{m'm}\delta_{k'k}\delta_{j'j} \tag{67}$$

as an explicit realization of the orthogonality theorem for compact groups.