Opgave IV.3

opgave 1

Basis formel IV.2 (6):

$$J_- |jm\rangle = \underbrace{\sqrt{(j+1-m)(j+m)}}_{C(j,m)} |j\, m-1\rangle \quad (*)$$

~~Eksplicit for $j=2$, $j=1$~~

Formler: IV.2 (7) – (9) for $j=2$, $j=1$.

Start $|\overset{J}{3};3\rangle = |\overset{j_2}{2},2\rangle \otimes |\overset{j_1}{1},1\rangle$

$$J_- |3,3\rangle = C(3,3)|3;2\rangle$$

$$J_-\left(|2,2\rangle \otimes |1,1\rangle\right) = \left(J_-|2,2\rangle\right)\otimes|1,1\rangle + |2,2\rangle \otimes J_-|1,1\rangle$$

Vi får da:

$$C(3,3)\,|3,2\rangle = C(2,2)\,|2,1\rangle \otimes |1,1\rangle + C(1,1)\,|2,2\rangle|1,0\rangle$$

$$|3,2\rangle = a_1\,|2,2\rangle|1,0\rangle + a_2\,|2,1\rangle\,|1,1\rangle$$

Ved at anvende J_-: får vi hele serien:

$$|3,1\rangle,\ |3,0\rangle,\ |3,-1\rangle,\ |3,-2\rangle,\ |3,-3\rangle$$

$$|3,1\rangle = b_1\,|2,2\rangle|1,-1\rangle + b_2\,|2,1\rangle|1,0\rangle + b_3\,|2,0\rangle|1,+1\rangle$$

$$|3,0\rangle = c_1\,|2,1\rangle|1,-1\rangle + c_2\,|2,0\rangle|1,0\rangle + c_3\,|2,-1\rangle|1,1\rangle$$

$\vdots$

opgave $\overline{IV}$.3

opgave 1 (fortsat).

Vi bestemmer nu $|J=2,2\rangle = d_1 \overset{j_2}{|2,2\rangle}\overset{j_1}{|1,0\rangle} + d_2 |2,1\rangle|1,1\rangle$

ved at kræve at $|J=2,2\rangle \perp |J=3,2\rangle$

dvs vi finder: $d_1 = a_2$, $d_2 = -a_1$

Vi kan nu anvende J_- på $|J=2,2\rangle$ og får:

$$|J=2,1\rangle = e_1 |2,2\rangle|1,-1\rangle + e_2 |2,1\rangle|1,0\rangle + e_3 |2,0\rangle|1,+1\rangle$$

$$|J=2,0\rangle = f_1 |2,1\rangle|1,-1\rangle + f_2 |2,0\rangle|1,0\rangle + f_3 |2,-1\rangle|1,1\rangle$$

$$\vdots$$

$$|J=2,-2\rangle = h_1 |2,-1\rangle|1,-1\rangle + h_2 |2,-2\rangle|1,0\rangle$$

Vi bestemmer nu $|j=1,1\rangle$

$$|J=1,1\rangle = g_1 |2,2\rangle|1,-1\rangle + g_2 |2,1\rangle|1,0\rangle + g_3 |2,0\rangle|1,1\rangle$$

ved at kræve at $|j=1,1\rangle$ er $\perp$ $|J=2,1\rangle$ og $\perp$ $|J=3,1\rangle$

Ved at kræve at $g_1 > 0$ er g_1, g_2, g_3 entydigt bestemt da vi kender: b_1, b_2, b_3 og e_1, e_2, e_3

(og $\sum g_i^2 = 1$)

Vi anvender nu J_- på $|J=1,1\rangle$ og får

$$|J=1,0\rangle = h_1 |2,1\rangle|1,-1\rangle + h_2 |2,0\rangle|1,0\rangle + h_3 |2,-1\rangle|1,1\rangle$$

(Alternativt kan vi bestemme h_1, h_2, h_3 ved at kræve at $|J=1,0\rangle \perp |J=2,0\rangle$ og $\perp$ $|J=3,0\rangle$)

Vi kan nu bestemme $|J=1,-1\rangle$ etc.

Opgave IV.3

Opg 2

Vi bruger igen basis formlen:

$$J_-|j\,m\rangle = \sqrt{(j+1-m)(j+m)}\;|j\;m-1\rangle$$

og starter med

$$\left|\tfrac{3}{2},\tfrac{3}{2}\right\rangle = |1,1\rangle \otimes \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

$$J_-\left|\tfrac{3}{2},\tfrac{3}{2}\right\rangle = \sqrt{3}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle$$

$$J_-\left(|1,1\rangle \otimes \left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\right) = \left((J_-|1,1\rangle)\otimes\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + |1,1\rangle\otimes J_-\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle\right)$$

$$= \sqrt{2}\,|1,0\rangle\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle + |1,1\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$$

altså

$$\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\,|1,0\rangle\otimes\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{1}{3}}\,|1,1\rangle\otimes\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$$

Vi anvender nu igen J_- på begge sider og får:

$$\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\,|1,-1\rangle\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\,|1,0\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$$

og en gang til:

$$\left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle = |1,-1\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle.$$

Opgave IV. 3

Opg 2 (fortsat)

Vi konstruerer nu $|(\tfrac{1}{2}),\tfrac{1}{2}\rangle \perp |(\tfrac{3}{2}),\tfrac{1}{2}\rangle$:
$j=\tfrac{1}{2}$

$$|(\tfrac{1}{2}),\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}\,|1,0\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

Vi kan nu anvende J_- på begge side og finde at.

$$|(\tfrac{1}{2}),-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\,|1,0\rangle|\tfrac{1}{2}-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|1,-1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

Som også kunne bestemmes ~~Ba~~ ved at bruge at

$$|(\tfrac{1}{2}),-\tfrac{1}{2}\rangle \perp |(\tfrac{3}{2}),-\tfrac{1}{2}\rangle$$

(det er lettere).

opgave IV.3

opgave 3

Vi ved at $\langle j'm' | \mathcal{J}_{J,M} | jm \rangle = 0$

med mindre

$|J - j| \leq j' \leq J + j$ og $m' = M + m$

(A) : $\Delta m = m' - m = M \leq J$ klart.

(B) $|\Delta j| = |j' - j| \leq J$.

(B1): $J \geq j$:

$j' = J - j, J - j + 1, \dots, J + j$

$j' - j = J - 2j, J - 2j + 1, \dots, J$, $|J - 2j| \leq J$ da $J \geq j$

(B2) $J \leq j$

$j' = j - J, j - J + 1, \dots, j + J$.

$j' - j = -J, -J + 1, \dots, J$: $|j' - j| \leq J$.

Opgave IV.3

opg 4

Igen bruges $J_-|j\,m\rangle = \sqrt{(j+1-m)(j+m)}\;|j\;m-1\rangle$

$$(*)\;|J\,M\rangle = \sum_{\tilde{m}=-j}^{j}\sum_{\tilde{m}'=-j'}^{j'} |j\,j',\,\tilde{m},\tilde{m}'\rangle\langle j\,j',\,\tilde{m}\tilde{m}'|J\,M\rangle$$

$$J_-|J\,M\rangle = \sqrt{(J+1-M)(J+M)}\;|J\;M-1\rangle$$

$$J_-\left(|j\,\tilde{m}\rangle\otimes|j'\tilde{m}'\rangle\right) = \left(J_-|j\tilde{m}\rangle\otimes|j'\tilde{m}'\rangle + |j\tilde{m}\rangle\otimes J_-|j'\tilde{m}'\rangle\right) =$$

$$\sqrt{(j+1-\tilde{m})(j+\tilde{m})}\;|j\;\tilde{m}-1\rangle|j'\tilde{m}'\rangle + \sqrt{(j'+1-\tilde{m}')(j'+\tilde{m}')}\;|j\tilde{m}\rangle|j'\tilde{m}'-1\rangle$$

Betragt nu

$$\langle j\,m|\langle j'm'|\;J_-\;|J\,M\rangle = \sqrt{(J+1-M)(J+M)}\;\langle jm|\langle j'm'|J\;M-1\rangle$$

På højre siden af $(*)$ bidrager 2 led:

$$\tilde{m},\tilde{m}' = m+1,\,m' \quad \text{og} \quad \tilde{m},\tilde{m}' = m,\,m'+1$$

$$\sqrt{(j+1-(m+1))(j+m+1)} \qquad \sqrt{(j'+1-(m'+1))(j'+m'+1)}$$

Dvs formel (18) i opg 4

IV.3 opgaver

opg 5

$$|l+\tfrac{1}{2}, l+\tfrac{1}{2}\rangle = |l,l\rangle\,|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$|l+\tfrac{1}{2}, m\rangle = C_1(m)\,|l, m-\tfrac{1}{2}\rangle\,|\tfrac{1}{2},\tfrac{1}{2}\rangle + C_2(m)\,|l, m+\tfrac{1}{2}\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$C_1(l+\tfrac{1}{2}) = 1, \quad C_2(l+\tfrac{1}{2}) = 0$$

Anvend J_- på begge sider for at få en recursion formel for $C_1(m)$ og $C_2(m)$ for $m = l-\tfrac{1}{2}, l-\tfrac{3}{2}, \ldots$

$$\sqrt{(l+\tfrac{3}{2}-m)(l+\tfrac{1}{2}+m)}\;|l+\tfrac{1}{2}, m-1\rangle =$$

$$C_1(m)\sqrt{(l+1-(m-\tfrac{1}{2}))(l+m-\tfrac{1}{2})}\;|l, m-\tfrac{3}{2}\rangle\,|\tfrac{1}{2}\,\tfrac{1}{2}\rangle \;+$$

$$C_1(m)\;|l, m-\tfrac{1}{2}\rangle\,|\tfrac{1}{2}, -\tfrac{1}{2}\rangle \;+$$

$$C_2(m)\sqrt{(l+1-(m+\tfrac{1}{2}))(l+(m+\tfrac{1}{2}))}\;|l, m-\tfrac{1}{2}\rangle\;|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$$|l+\tfrac{1}{2}, m-1\rangle = C_1(m)\sqrt{\frac{l+m-\frac{1}{2}}{l+m+\frac{1}{2}}}\;|l, m-\tfrac{3}{2}\rangle\,|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

$$+\left(C_1(m)\frac{1}{\sqrt{(l+\frac{3}{2}-m)(l+m+\frac{1}{2})}} + C_2(m)\sqrt{\frac{l+\frac{1}{2}-m}{l+\frac{3}{2}-m}}\right)|l, m-\tfrac{1}{2}\rangle\,|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$\boxed{IV}$ 3. opgave opg 5 fortsat

altså $\quad C_1(m-1) = \sqrt{\dfrac{l+m-\frac12}{l+m+\frac12}}\, C_1(m)$,

$$C_1(l+\tfrac12) = 1$$

$$C_1(m) = \sqrt{\frac{l+m+\frac12}{l+m+\frac32}}\sqrt{\frac{l+m+\frac32}{l+m+\frac52}} \cdots \sqrt{\frac{2l}{2l+1}} \cdot 1 = \sqrt{\frac{l+m+\frac12}{2l+1}}$$

$$C_2(m-1) = C_1(m)\frac{1}{\sqrt{(l+\frac32-m)(l+\frac12+m)}} + C_2(m)\sqrt{\frac{l+\frac12-m}{l+\frac32-m}}$$

$$= \frac{1}{\sqrt{2l+1}}\,\frac{1}{\sqrt{l+\frac32-m}} + C_2(m)\sqrt{\frac{l+\frac12-m}{l+\frac32-m}}$$

$$\underbrace{\sqrt{l+\tfrac12-(m-1)}\,\sqrt{2l+1}\,C_2(m-1)}_{f(m-1)} = 1 + \underbrace{\sqrt{l+\tfrac12-m}\,\sqrt{2l+1}\,C_2(m)}_{f(m)}$$

$$f(m-1) = 1 + f(m)$$

$$f(l+\tfrac12) = 0 \;:\; \Rightarrow\; f(m) = (l+\tfrac12-m)$$

$$C_2(m) = \sqrt{\frac{(l+\frac12-m)}{2l+1}}$$

$$|l-\tfrac12, m\rangle = d_1(m)\,|l, m-\tfrac12\rangle|\tfrac12\tfrac12\rangle + d_2(m)\,|l, m+\tfrac12\rangle|\tfrac12 -\tfrac12\rangle$$

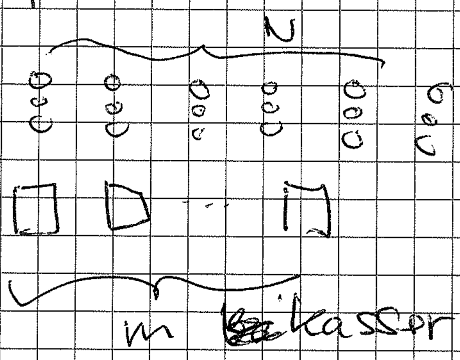
$$|l-\tfrac12, m\rangle \perp |l+\tfrac12, m\rangle \Rightarrow \begin{aligned} d_2(m) &= C_1(m) \\ d_1(m) &= -C_2(m)\end{aligned}$$

Opgave IV.4

Opgave 1:

Dimension af symmetrisk tensor

$\phi^{i_1 \cdots i_m}$ for $SU(N)$



1 kugle i hver kasse

Man kan vælge frit rækkefølge af kasser ikke gyldig

Antal $\binom{N+m-1}{m} = \binom{N+m-1}{N-1}$

$$\frac{(N+m-1)(N+m-2)\cdots N}{m!} = \frac{(N+m-1)\cdots(m+1)}{(N-1)!}$$

Eks: $SU(3)$: $\frac{(m+2)(m+1)}{2}$

opgave IV.5

$\boxed{\text{opg } 1}$

Vi kan antage at X^i er en enhedsvektor $X^i = n^i$

$$\cancel{\text{σ}}\; n^i \sigma^i = U\, n^{\prime j} \sigma^{\prime j}\, U^\dagger$$

under rotation U. Vi vælger rotation således at $n'^i = (0,0,1)$.

$$\sigma^3 = U\, n^j \sigma^j\, U^\dagger \text{ eller}$$

$$\boxed{U^\dagger \sigma^3 U = n^i \sigma^i}$$

$$\mathrm{Tr}\,(n^i\sigma^i)^k = \mathrm{Tr}\,(U^\dagger \sigma^3 U)^k = \mathrm{Tr}\,\sigma^{3k}$$

$\sigma_3{}^k = \underline{I}$ for k lige, σ_3 for k ulige

dvs

$$\frac{1}{2}\mathrm{Tr}\,(n^i\sigma^i)^k = 1 = \left(\frac{1}{2}\mathrm{Tr}\,(\sigma^i n^i)^2\right)^k = 1$$

for lige k og ~~for~~

vi får $\mathrm{Tr}\,(n^i\sigma^i)^k = 0$ for ulige k

opgave IV.7

opgave 1 og opgave 2

Vi så i første forelæsning, baseret på geometri, at en rotation med vinkel ϕ om aksen n^i var givet ved:

$$X'^i = \cos\phi \, X^i + (1-\cos\phi)\, n^i (n\cdot X) + \sin\phi \,(n \times X)^i$$

eller

$$X'^i = R^{ij}(\phi, n)\, X^j$$

$$R^{ij}(\phi,n) = \cos\phi \,\delta^{ij} + (1-\cos\phi)\, n^i n^j + \sin\phi \,\varepsilon^{ikj} n^k$$

Heraf:

$$\mathrm{Tr}\, R = R^{ii} = 2\cos\phi + 1 \quad : \text{ bestem } \phi \in [0,\pi]$$

$$\varepsilon^{ijk} R^{jk} = \sin\phi \, n^i \quad : \text{ bestem } n^i$$

(virker ikke hvis $\phi = 0, \pi$)

opgave IV.7

opgave 1 og 2 fortsat

Lad os for fuldstændigheds skyld bestemme $R^{(ij)}$ udfra

$$X = X_i \sigma_i$$

$$X' = U X U^{+}, \quad U = \cos\frac{\theta}{2} I + i \sin\frac{\theta}{2} n_i \sigma_i$$

Vi har $\mathrm{Tr}(\sigma_i \sigma_j) = 2\delta_{ij}$ $\quad \mathrm{Tr}\,\sigma_i = 0$ (*)

og $\sigma_i \sigma_j = \delta_{ij} \cdot I + i \varepsilon_{ijk} \sigma_k$ (**)

$[\sigma_i, \sigma_j] = i \varepsilon_{ijk} \sigma_k$ (xxx)

Definer : $U \sigma_j U^{+} = \sigma_i R_{ij}$

($U \sigma_j U^{+}$ er Hermitisk sporløs matrice så den kan skrives som linearkombination af σ_i's dvs som $\sigma_i R_{ij}$.)

Vi viser nu at R_{ij} er rotationsmatrice:

$$X' = X_i' \sigma_i = U X_j \sigma_j U^{+} = (U \sigma_j U^{+}) X_j$$

$$= (R_{ij} X_j) \sigma_i$$

altså

$$X_i' = R_{ij} X_j \,.$$

Opgave IV.7

Opgave 1 og 2 fortsat 2

Fra $\sigma_i R_{ij} = U \sigma_j U^{\dagger}$ får vi

$$\mathrm{Tr}(\sigma_{i_0} \sigma_i) R_{ij} = \mathrm{Tr}\, \sigma_{i_0} U \sigma_j U^{\dagger}$$

$$\underset{2\delta_{i_0 i} R_{ij}}{\parallel}$$

Altså $\boxed{R_{ij} = \frac{1}{2} \mathrm{Tr}\, \sigma_i U \sigma_j U^{\dagger}}$

Lad os nu udregne højre side ved brug af (*) og (**), (***)

$$\frac{1}{2} \mathrm{Tr}\, \sigma_i \left(\cos \frac{\varphi}{2} I + i \sin \frac{\varphi}{2} n_k \sigma_k \right) \sigma_j \left(\cos \frac{\varphi}{2} I - i \sin \frac{\varphi}{2} n_\ell \sigma_\ell \right)$$

$$= \frac{1}{2} \cos^2 \frac{\varphi}{2} \mathrm{Tr}\, \sigma_i \sigma_j + \frac{1}{2} \cos \frac{\varphi}{2} \sin \frac{\varphi}{2}\, i \,\mathrm{Tr}\, \sigma_i \underset{i \varepsilon_{kj\ell} \sigma_\ell}{\underset{\parallel}{[\sigma_k, \sigma_j]}} n_k$$

$$+ \frac{1}{2} \sin^2 \frac{\varphi}{2} \mathrm{Tr}\, \sigma_i \sigma_k \sigma_j \sigma_\ell \, n_k n_\ell$$

$(n_k n_\ell)$

$$\frac{1}{2} \mathrm{Tr}(\sigma_i \sigma_k \sigma_j \sigma_\ell)\, n_k n_\ell = \frac{1}{2} \mathrm{Tr}\, (\delta_{ik} I + i \varepsilon_{ikm} \sigma_m)(\delta_{j\ell} + i \varepsilon_{j\ell n} \sigma_n)$$

$$= 2 n^i n^j - \delta^{ij} n^2$$

(når man bruger: $\varepsilon_{ikm} \varepsilon_{j\ell m} = \delta_{ij} \delta_{k\ell} - \delta_{i\ell} \delta_{kj}$)
(Se IV.1 (31))

Opgave IV.7

Opgave 1 og 2 fortsat 3

Det kan nu samles til

$$\left(\cos^2 \tfrac{\phi}{2} - \sin^2 \tfrac{\phi}{2}\right) \delta^{ij} + 2 \sin^2 \tfrac{\phi}{2}\, n^i n^j$$

$$+ 2 \sin \tfrac{\phi}{2} \cos \tfrac{\phi}{2}\, \varepsilon_{ikj} n_k \quad =$$

$$\cos\phi\, \delta_{ij} + (1 - \cos\phi)\, n_i n_j + \sin\phi\, \varepsilon_{ikj} n_k$$

Som er udtrykket for R_{ij}

Det udledes lettere ved en ren gloende betragtning, som nævnt, men ved at bruge (*), (**) og (***) følger det også fra formlen

$$\boxed{R_{ij} = \frac{1}{2} \operatorname{Tr} \sigma_i U \sigma_j U^{+}}$$

Denne formel er generel og giver (også for $SO(N)$), den adjungerede representations matrix i termer af den fundamentale U og de tilsvarende hvadsiger ; $\operatorname{Tr} T^a T^b \sim \delta^{ab}$