Problem 1

Let K be a finite group of order 4 with elements e, x, y and z, where e is the identity element. Complete the following group multiplication table

	e	x	y	z
e				
x		e		
y			e	
z				

We can immediately fill the first column and the first row to obtain

	e	x	y	z
e	e	x	y	z
x	x	e		
y	y		e	
z	z			

The rest of the table is now constructed by using the "once and only once" rule for the appearance of a group element in a row and a column, and we obtain

	e	x	y	z
e	e	x	y	z
x	x	e	z	y
y	y	z	e	x
z	z	y	x	e

Problem 2

Let G be the group $Z_2 \times Z_2$.

(1) Prove that there are four irreducible representations of G.

The group G is abelian. Hence the irreducible representations are all one-dimensional and the number of different irreducible representations is equal to the order of the group, i.e. here 4.

(2) Write down the corresponding character functions $\chi^{(r)}(g)$, $g \in G$, $r = 1, 2, 3$ and 4.

For abelian groups the characters are just the one-dimensional representations. Denote the group elements $g \in Z_2$ as 0 and 1. We know the one-dimensional irreducible representations of Z_2 are $D^{(j)}(g) = e^{2\pi i j g/2}$,$j = 0, 1$. If we write a group element of G as (g_1, g_2), where both g_1 and g_2 belongs to Z_2, we have the following four representations of $G = Z_2 \times Z_2$:

$$D^{(j_1,j_2)}((g_1, g_2)) = D^{(j_1)}(g_1)D^{(j_2)}(g_2).$$

It is easily checked that they are representations of G and that they are orthogonal. They are thus the four irreducible representations of G, and the same is then true for the characters $\chi^{(j_1,j_2)}((g_1, g_2)) = D^{(j_1,j_2)}((g_1, g_2))$.

Problem 3

Let G be a cyclic group of order n. By definition this means that there exists an $x \in G$ such that $G = \{x^0, x^1, \ldots, x^{n-1}\}$.

(1) Prove that G has a cyclic subgroup of order p for any p that divides n.

Let $m = n/p$ and $y = x^m$. Then it is trivially shown that $\{y^0, y^1, \ldots, y^{p-1}\}$ constitutes a cyclic subgroup of order p in G.

One can prove that any subgroup of a cyclic group is a cyclic group of this kind. Assuming this is true then

(2) prove that there is at most one subgroup of G of any given order less than or equal n.

Assume there exists a subgroup of order p. By assumption p divides n. Let $n/p = m$ and $z = x^m$. Then we know that $H_z = \{z^0, z^1, \ldots, z^{p-1}\}$ is a subgroup of order p. Let now H be a subgroup of order p. By assumption it has to have the form $H = \{y^0, y^1, \ldots, y^{p-1}\}$ for some $y \in G$. Since $y \in G$, $y = x^{m'}$ for some m'. $y^p = e$, i.e. $m'p = kn$ where $k \geq 1$. This implies that $m' = km$. Thus H is a subgroup of H_z but since they have the same order $H = H_z$.

Problem 4

Let R^{ij} be matrices in the defining representation of $SO(N)$, i.e. components V^i of vectors in $\mathbb{R}^N$ transform as $V'^i = R^{ij}V^j$. $R^{ik}R^{jl}$ are components of the tensor product representation $R \otimes R$ of two defining representations of $SO(N)$. In this representation tensor components T^{ij}, i.e. components of vectors T in $\mathbb{R}^N \otimes \mathbb{R}^N$, transform as $T'^{i,j} = R^{ik}R^{jl}T^{kl}$. The tensor product representation $R \otimes R$ decomposes into irreducible representations, thereby defining orthogonal subspaces of $\mathbb{R}^N \otimes \mathbb{R}^N$.

(1) Find the dimensions of these vector subspaces.

We know that the tensor product can be decomposed into a symmetric part with $N(N+1)/2$ components and an antisymmetric part with $N(N-1)/2$ components. The antisymmetric part constitutes an irreducible representation while for the symmetric part we have to subtract the trace, which corresponds to a one-dimensional irreducible representation. Thus the dimensions of the subspaces are (a) N(N+1)/2-1 (symmetric traceless), (b) N(N-1)/2 (antisymmetric) and (c) 1 (the identity (times the trace)).

(2) Decompose explicitly a vector $T \in \mathbb{R}^N \otimes \mathbb{R}^N$ with components T^{ij} in vectors lying in the vector subspaces corresponding to the irreducible representations of the tensor product representation $R \otimes R$, i.e. express the components of vectors in these subspaces in terms of T^{ij}.

$$T^{ij} = \left(\frac{1}{2}(T^{ij} + T^{ji}) - \frac{1}{N}\delta^{ij}T^{kk}\right) \oplus \left(\frac{1}{2}(T^{ij} - T^{ji})\right) \oplus \frac{1}{N}\delta^{ij}T^{kk}$$

Problem 5

Let U^{ij} be matrices in the defining representation of $SU(3)$, i.e. components V^i of vectors in $\mathbb{C}^3$ transform as $V'^i = U^{ij}V^j$. $U^{ik}U^{jl}$ are components of the tensor product representation $U\otimes U$ of two defining representations of $SU(3)$. In this representation tensor components T^{ij}, i.e. components of vectors T in $\mathbb{C}^3\otimes\mathbb{C}^3$, transform as $T'^{i,j} = U^{ik}U^{jl}T^{kl}$. The tensor product representation $U\otimes U$ decomposes into irreducible representations, thereby defining orthogonal subspaces of $\mathbb{C}^3\otimes\mathbb{C}^3$.

(1) Find the dimensions of these vector subspaces.

Symmetric tensors provide an irreducible representation. The dimension of the space of symmetric $SU(3)$ tensors is $(N+1)N/2$, $N=3$, i.e. 6. The antisymmetric tensors likewise provide an irreducible representation. The dimension of the space of antisymmetric tensors is $(N-1)N/2$, $N=3$, i.e. 3. By use of the ε_{ijk} symbol the space of the antisymmetric tensors of $SU(3)$ can be identified with the 3-dimensional vector space of vectors with a lower index.

(2) Decompose explicitly a vector $T \in \mathbb{C}^3\otimes\mathbb{C}^3$ with components T^{ij} in vectors lying in the vector subspaces corresponding to the irreducible representations of the tensor product representation $U\otimes U$, i.e. express the components of vectors in these subspaces in terms of T^{ij}.

$$T^{ij} = \Big(\frac{1}{2}(T^{ij}+T^{ji})\Big) \oplus \Big(\frac{1}{2}(T^{ij}-T^{ji})\Big)$$

or, introducing $T_k = \frac{1}{2}\varepsilon_{ijk}(T^{ij}-T^{ji})$

$$T^{ij} = \Big(\frac{1}{2}(T^{ij}+T^{ji})\Big) \oplus \Big(\varepsilon^{ijk}T_k\Big).$$

Problem 6

(1) Use the raising and lowering operators $J_\pm$ to construct the matrices J_x and J_y for the irreducible representation of $SU(2)$ corresponding to $J = 3/2$.

By using $J_-|jm\rangle = \sqrt{(j+m)(j-m+1)}|jm-1\rangle$, $J_+ = J_-^\dagger$, $J_x = \frac{1}{2}(J_+ + J_-)$ and $J_y = \frac{i}{2}(J_- - J_+)$ one obtains

$$J_- = \begin{array}{c|c|c|c|c|} & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \\ \hline \frac{3}{2} & 0 & 0 & 0 & 0 \\ \hline \frac{1}{2} & \sqrt{3} & 0 & 0 & 0 \\ \hline -\frac{1}{2} & 0 & 2 & 0 & 0 \\ \hline -\frac{3}{2} & 0 & 0 & \sqrt{3} & 0 \\ \hline \end{array} \qquad J_+ = \begin{array}{c|c|c|c|c|} & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \\ \hline \frac{3}{2} & 0 & \sqrt{3} & 0 & 0 \\ \hline \frac{1}{2} & 0 & 0 & 2 & 0 \\ \hline -\frac{1}{2} & 0 & 0 & 0 & \sqrt{3} \\ \hline -\frac{3}{2} & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

$$2J_x = \begin{array}{c|c|c|c|c|} & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \\ \hline \frac{3}{2} & 0 & \sqrt{3} & 0 & 0 \\ \hline \frac{1}{2} & \sqrt{3} & 0 & 2 & 0 \\ \hline -\frac{1}{2} & 0 & 2 & 0 & \sqrt{3} \\ \hline -\frac{3}{2} & 0 & 0 & \sqrt{3} & 0 \\ \hline \end{array} \qquad 2J_y = \begin{array}{c|c|c|c|c|} & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \\ \hline \frac{3}{2} & 0 & -i\sqrt{3} & 0 & 0 \\ \hline \frac{1}{2} & i\sqrt{3} & 0 & -i2 & 0 \\ \hline -\frac{1}{2} & 0 & i2 & 0 & -i\sqrt{3} \\ \hline -\frac{3}{2} & 0 & 0 & i\sqrt{3} & 0 \\ \hline \end{array}$$

(2) Find the 4×4 matrix $M = \exp(-i\pi J_x/2)\, J_y \exp(i\pi J_x/2)$.

For any irreducible representation U of $SU(2)$ with $j > 0$ J_x, J_y and J_z transform like ordinary unit vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$ under the action $UJ_iU^\dagger$, $i = x, y, z$. Thus for $j = 3/2$ with J_x and J_y calculated in (1) we have

$$J_z = \exp(-i\pi J_x/2)\, J_y \exp(i\pi J_x/2)$$

since $U = \exp(-i\pi J_x/2)$ for $j = 3/2$ describes a rotation of vectors in the $j = 3/2$ representation by $\pi/2$ around the $\hat{x}$ axis. Such a rotation will rotate vector $\hat{y}$ into $\hat{z}$. The matrix J_z is given by $\langle jm'|J_z|jm\rangle = m\,\delta_{m'm}$.