Exam MATF3 June 2018

4 hour exam with all usual help items. The exam may be answered using pencil and English or Danish language can be used. The exam includes 10 sub-questions which are weighted equally in the evaluation.

Problem 1

Let G be a finite group and K a subset of G. The *centralizer* $C(K)$ of K is defined by

$$C(K) = \{c \in G \mid ck = kc \ \forall \ k \in K\}.$$

(1) Prove that $C(K)$ is a subgroup of G.

Proof

$e \in C(K)$. If $c \in C(K)$ then $ck = kc$, i.e. $kc^{-1} = c^{-1}k$ for all $k \in K$. Thus $c^{-1} \in C(K)$. Finally, let $c_1, c_2 \in C(K)$. $c_1c_2k = c_1kc_2 = kc_1c_2$. Thus $c_1c_2 \in C(K)$.

(2) Prove that if K is an abelian subgroup of G then K is an invariant subgroup of $C(K)$.

Proof

If K is an abelian subgroup then each $k \in K$ belongs to $C(K)$. Thus K is a subgroup of $C(K)$. Let now $c \in C(K)$. $\forall \ k \in K : ck = kc$, i.e. $c^{-1}kc = k \in K$. Thus K is an invariant subgroup of $C(K)$.

Problem 2

There exists an non-abelian group of order 8, which has the following presentation

$$G = \langle x, y | x^4 = e, \ \ y^2 = e, \ \ yx = x^3y \rangle,$$

where e is the identity element.

(1) Show that any element of the form $x^k y^l x^m y^n$ can be written as one of the 8 elements $e, x, x^2, x^3, y, xy, x^2y, x^3y$.

Proof ——————————————————————————-

Using $x^4 = e$ and $y^2 = e$ it follows that any element of the form $x^m y^n$ can be written as mentioned. Next, from $yx = x^3y$ we have $y^l x^m = y^{l-1}x^{3m}y = x^{3^l m}y^l$ and thus $x^k y^l x^m y^n = x^{k+3^l m}y^{l+n}$.

This can easily be turned into a proof that any element of G can be written as one of the 8 mentioned elements.

(2) Fill in the entries marked with $\star$ in the group multiplication table below, expressed in terms of the eight mentioned elements.

	e	x	x^2	x^3	y	xy	x^2y	x^3y
e	$\star$							
x		$\star$	$\star$					
x^2		$\star$	$\star$	$\star$				
x^3			$\star$	$\star$	$\star$			
y				$\star$	$\star$	$\star$		
xy					$\star$	$\star$	$\star$	
x^2y						$\star$	$\star$	
x^3y								$\star$

Proof ————————————————————

We give only the non-trivial $\star$-elements:

$$(y)(x^3) = x^{3\cdot 3}y = xy, \qquad (y)(xy) = x^3y^2 = x^3$$

$$(xy)(xy) = xx^3y^2 = e, \qquad (xy)(x^2y) = xx^6y^2 = x^3$$

$$(x^2y)(xy) = x^2x^3y^2 = x, \qquad (x^2y)(x^2y) = x^2x^6y^2 = e$$

$$(x^3y)(x^3y) = x^3x^9y^2 = e$$

Problem 3

Consider the group $SO(3)$. Its irreducible representations $D^{(J)}(g)$, $g \in SO(3)$ are labeled by J, $J = 0, 1, 2, \ldots$, and for a given J the eigenvectors of J_3 are denoted $|JM\rangle$ and constitute a normalized basis for the $2J+1$ dimensional vectorspace where the matrices $D^{(J)}_{MM'}(g)$ act.

(1) Express the $|J = 4, M = 2\rangle$ vector in terms of the vectors $|j = 2, m\rangle|j = 2, m'\rangle$ constituting a basis for the reducible tensor product representation $D^{(j=2)} \otimes D^{(j=2)}$.

Proof

Starting point $|J = 4, M = 4\rangle = |j = 2, m = 2\rangle|j' = 2, m' = 2\rangle$. Apply J_- on both sides twice using

$$|JM\rangle = |jm\rangle|j'm'\rangle \Rightarrow J_-|JM\rangle = (j_-|jm\rangle)|j'm'\rangle + |jm\rangle(j_-|j'm'\rangle)$$

together with

$$J_-|JM\rangle = \sqrt{(J+1-M)(J+M)}\;|JM-1\rangle.$$

We obtain

$$|42\rangle = \sqrt{\frac{3}{14}}\;|22\rangle|20\rangle + \sqrt{\frac{8}{14}}\;|21\rangle|21\rangle + \sqrt{\frac{3}{14}}\;|20\rangle|22\rangle$$

Problem 4

We consider $SU(2)$. We denote the irreducible representations by $D^{(J)}(g)$, $g \in SU(2)$, where J are half-integers. For a given J the normalized eigenvectors $|JM\rangle$ of J_3 form the basis of the $2J+1$ dimensional vectorspace in which $D^{(J)}_{MM'}(g)$ acts. Let $d\mu(g)$ be the Haar measure of $SU(2)$. It is invariant under left and right translations: $d\mu(g_0 g) = d\mu(g g_0) = d\mu(g)$ for all $g_0 \in SU(2)$.

(1) Show that

$$(*): \qquad \int d\mu(g)\; D^{(J)}_{MM'}(g) = 0 \quad \text{unless} \quad J = 0.$$

(Hint 1: Denote the integral $(*)$ by $V_M(M')$ and use the group properties as well as the invariance of the Haar measure to prove that for all $g_0 \in SU(2)$ we have $\sum_M D^{(J)}_{M_1 M}(g_0) V_M(M') = V_{M_1}(M')$. Use this to conclude that either $J = 0$ or $V_M(M') = 0$ for all M, M'. Instead of hint 1 you can also use hint 2: the "Great Orthogonality Theorem" for continuous compact groups (the Peter-Weil theorem mentioned in addednotes2).)

Proof ————————————————————

Following hint 2, the "Great Orthogonality Theorem", when used for two irrreducible representations corresponding to $J_1 = 0$ and $J_2 = J$ immediately gives the formula, since the $D^{J=0}$ matrix is just a number different from 0.

Following hint 1 we have, using the group property (summation over repeated indices):

$$D^{(J)}_{M_1 M}(g_0) V_M(M') = \int d\mu(g)\, D^{(J)}(g_0)_{M_1 M} D^{(J)}_{M M'}(g) = \int d\mu(g)\, D^{(J)}_{M_1 M'}(g_0 g)$$

as well as, using the invariance of the Haar measure

$$\int d\mu(g)\, D^{(J)}_{M_1 M'}(g_0 g) = \int d\mu(g_0 g)\, D^{(J)}_{M_1 M'}(g_0 g) = V_{M_1}(M').$$

Thus we have $D^{(J)}_{M_1 M}(g_0) V_M(M') = V_{M_1}(M')$ for all $g_0 \in SU(2)$. This implies that if there is a value of M' where the vector $V_M(M') \neq 0$ then it defines a one-dimensional invariant vector space under the action of $D^{(J)}$, and thus $J = 0$.

For $SU(2)$ we have the Clebsch-Gordan tensor product decomposition

$$D^{(j_1)}(g) \otimes D^{(j_2)}(g) = \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} D^{(J)}.$$

(2) Use the result $(*)$ above to prove that

$$(**): \qquad \int d\mu(g)\, D^{(j_1)}_{m_1, m_1'}(g) D^{(j_2)}_{m_2 m_2'}(g) = 0 \quad \text{unless} \quad j_1 = j_2.$$

Proof

$D^{(j_1)}_{m_1,m_1'}(g) D^{(j_2)}_{m_2 m_2'}(g)$ is a matrix element of $D^{(j_1)}(g) \otimes D^{(j_2)}(g)$. From question $(*)$ in this problem we know that the tensor decomposition in irreducible representations has to contain $J = 0$ in order to obtain an integral different from zero. This only occurs if $j_1 = j_2$.

Let $\chi^{(j)}(g)$ denote the character corresponding to the irreducible representation $D^{(j)}(g)$.

(3) Use $(**)$ above to prove that

$$\int d\mu(g)\, \chi^{(j_1)}(g) \chi^{(j_2)}(g) = 0 \quad \text{unless} \quad j_1 = j_2.$$

Proof

$\chi^{(j)}(g) = \operatorname{tr} D^{(j)}(g) = \sum_m D^{(j)}_{mm}(g)$. Thus the result follows from the formula $(**)$ provided in question (2) of this problem.

Problem 5

Let S^{ij} and T^{kl} be symmetric, traceless tensors corresponding to irreducible representations of $SO(3)$. The tensor product $S^{ij}T^{kl}$ forms a reducible representation of $SO(3)$ which can decomposed into a sum of irreducible representations of $SO(3)$.

(1) Determine the dimensions of these irreducible representations.

Proof

The representation of symmetric traceless tensors has dimension 5. Thus it corresponds to $j = 2$. The tensor product of two such representations is a reducible representation with dimension 25. From the Clebsch-Gordan decomposition we have $j_1 \otimes j_2 = |j_1 - j_2| \oplus \cdots \oplus (j_1 + j_2)$, which in our case reads $2 \otimes 2 = 0 \oplus 1 \oplus 2 \oplus 3 \oplus 4$. The corresponding dimensions are 1,3,5,7 and 9 (which add to 25).

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(2) One of the irreducible components is a symmetric, traceless tensor W^{ijkl}. Give an explicit expression of W^{ijkl} in terms of S^{ij} and T^{kl}.

Proof ---

The symmetric traceless tensor W^{ijkl} corresponds to $j = 4$ in the above decomposition and it is thus the 9-dimensional representation (a symmetric 4-tensor V^{ijkl} of $SO(3)$ has 15 independent components, but the trace condition $V^{iikl} = 0$ subtracts 6 components).

Let us first symmetrize $S^{ij}T^{kl}$:

$$6V^{ijkl} = S^{ij}T^{kl} + S^{ik}T^{jl} + S^{il}T^{jk} + S^{jk}T^{il} + S^{jl}T^{ki} + S^{kl}T^{ij}. \tag{1}$$

In the subspace of symmetric tensors V^{ijkl} we can enforce tracelessness by writing

$$\begin{aligned}\tilde{V}^{ijkl} &= V^{ijkl} - c_1\Big(\delta^{ij}V^{mmkl} + \delta^{ik}V^{mmjl} + \delta^{il}V^{mmjk} + \delta^{jk}V^{mmil} + \\ &\qquad \delta^{jl}V^{mmki} + \delta^{kl}V^{mmij}\Big) + c_2\Big(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\Big)\, V^{mmnn}.\end{aligned}$$

Demanding $\tilde{V}^{iikl} = 0$ for all k, l determines $c_1 = 1/7$ and $c_2 = c_1/5$ and $W^{ijkl} = \tilde{V}^{ijkl}$ with these values of c_1, c_2. Using (1) we can write W^{ijkl} in terms of S, T, using in addition that $S^{ii} = T^{ii} = 0$, i.e.

$$6\tilde{V}^{mmkl} = 2(S^{mk}T^{ml} + S^{ml}T^{mk}), \qquad 6\tilde{V}^{mmnn} = 4S^{mn}T^{mn}$$

$$\begin{aligned}6W^{ijkl} &= \Big(S^{ij}T^{kl} + S^{ik}T^{jl} + S^{il}T^{jl} + S^{jk}T^{il} + S^{jl}T^{ki} + S^{kl}T^{ij}\Big) - \\ &\quad \frac{2}{7}\Big(\delta^{ij}(S^{mk}T^{ml} + S^{ml}T^{mk}) + \delta^{ik}(S^{mj}T^{ml} + S^{ml}T^{mj}) + \\ &\qquad \delta^{il}(S^{mj}T^{mk} + S^{mk}T^{mj}) + \delta^{jk}(S^{mi}T^{ml} + S^{ml}T^{mi}) + \\ &\qquad \delta^{jl}(S^{mi}T^{mk} + S^{mk}T^{mi}) + \delta^{kl}(S^{mi}T^{mj} + S^{mj}T^{mi})\Big) + \\ &\quad \frac{4}{5 \cdot 7}\Big(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\Big)\, S^{mn}T^{mn}.\end{aligned}$$
