Added notes 4

From the Cartan-Killing classification of complex Lie algebras to real Lie groups and globally symmetry spaces

From Lie algebras to Lie groups: general remarks

We have seen in the Zee-book that it is possible to classify all simple *complex* Lie algebras, the so-called Cartan-Killing classificatioin. However, our starting point was that we wanted to classify all simple Lie groups, and from first principles, the associated Lie algebras were *real*. We extended these real Lie algebras to complex Lie algebras in order to simplify the classification, but we now need to have a procedure to get back to the real Lie algebras. Secondly, in order to get from the Lie algebra to the Lie group we need the exponential map or, if that does not work, some theorems , and there are a few subtleties since several Lie groups have isomorphic Lie algebras. Let us address the last question first. First a theorem

> *Theorem:* Let L be a d-dimensional abstract real Lie algebra. Then there exists a d-dimensional *simply connected* Lie group $\bar{G}$, uniquely determined up to isomorphism, with a Lie algebra isomorphic to L.

The simply connected group is called the *universal covering group* and all Lie groups G with the same Lie algebra can be obtained from $\bar{G}$ in the following way: Let D be a discrete, invariant subgroup of $\bar{G}$. Then the factor group

$$G = \bar{G}/D \tag{1}$$

has the same Lie algebra as $\bar{G}$ (the groups are locally isomorphic). One can show that every Lie group with the same Lie algebra as $\bar{G}$ is isomorphic to a group of the form (1). Further, since $\bar{G}$ is simply connected D has to be a subgroup of the center group Z of $\bar{G}$, i.e. the elements in $\bar{G}$ which commute with all elements in $\bar{G}$. The proof is easy: D discrete and invariant imply

$$\forall\ d_i \in D\ \ \exists\ d_j \in D : g d_i g^{-1} = d_j\ \ \forall\ g \in G, \tag{2}$$

but because we can continuously deform g to the identity (since $\bar{G}$ is simply connected) $d_i = d_j$.

Thus we obtain all Lie groups with a given Lie algebra from $\bar{G}$ by first determining the center group Z and then its discrete subgroups D. For the simple Lie groups the center group itself *is* by definition discrete.

Let G a simple group and consider an N dimensional irreducible representation of G. By Schur's lemma the invariant discrete subgroups have to be subsets of those multiples of identity matrix which are also group elements, i.e. we have

$$d_i = c_i I_{N\times N}, \quad \forall d_i \in D. \tag{3}$$

Thus the center elements are often not difficult to determine.

The *fundamental group* or the *first homotopy group* $\pi_1(X)$ of a topological space X basically describes how different non-contractable loops combine. If $\pi_1(X)$ is trivial, X is said to be simply connected: all loops can be continuously deformed to a point. Thus $\pi_1(\bar{G})$ is trivial. Let D be a discrete subgroup of the center group Z of $\bar{G}$, then

$$G = \bar{G}/D \quad \Rightarrow \quad \pi_1(G) = D. \tag{4}$$

This is a surprising relation between algebraic and topological properties of G. Note that while D determines G from $\bar{G}$, we can have a situation where D_1 and D_2 are isomorphic but *distinct* subgroups of the center group Z of $\bar{G}$, and then $G_1 = \bar{G}/D_1$ is *not* isomorphic to $G_2 = \bar{G}/D_2$, but $\pi_1(G_1) = D_1$ is isomorpic to $\pi_1(G_2) = D_2$.

The above discussion tells us to what extent a Lie group is determined by its Lie algebra. A theorem (*Ado's theorem plus extensions*) tells us that a any finite dimensional complex or real Lie algebra has a faithful finite dimensional representation, i.e. is isomorphic to some d-dimensional matrix Lie algebra. For such a matrix Lie algebra the exponential map can be defined simply by the matrix power series. Does the exponential map corresponding to this matrix algebra define the Lie group? In general the answer is no. A counter example is $Sl(2,\mathbb{R})$. As discussed in the Zee-book, the exponential map of the Lie algebra $sl(2,\mathbb{R})$ into $Sl(2,\mathbb{R})$ is not surjective. The group $Sl(2,\mathbb{R})$ is connected but non-compact. On the other hand $Gl(n,\mathbb{C})$ is also connected and non-compact and in both cases the fundamental group is $\mathbb{Z}$, but the exponential map of the Lie algebra $gl(n,\mathbb{C})$ *is* $Gl(n,\mathbb{C})$. Things are relatively complicated if the Lie groups are non-compact. If G a connected *and* compact then the exponential map is surjective, when understood correctly as will be explained now.

Let $D'(g')$ be a representation of G'. Let Π be a homomorphism $G' \mapsto G$ and let $D(g)$ be a representation of G. $D \circ \Pi$ is a representation of G'. $\Pi^{-1}(e)$ is an invariant subgroup of G' and $G \cong G'/\Pi^{-1}(e)$. The map $D'|_G$, where G is viewed as $G'/\Pi^{-1}(e)$ is not necessarily a representation of G. This is true if and only if $D'(\Pi^{-1}(e)) = \{e'\}$. The case of $G' = SU(2)$ and $G = SO(3) \cong SU(2)/Z_2$ illustrates this. Π is the homomorphism

$SU(2) \mapsto SU(2)/Z_2$. The representations $D'(g')$ of $SU(2)$ are the representations $D'^{(j)}(g')$, $j = n/2$. The representations of $SO(3)$ are $D^{(j)}(g)$, $j = n$. We use the convention $g' \in SU(2)$ and $g \in SO(3)$. Thus for $j = n$ we have indeed that $D^{(j)}(\Pi(g')) = D^{(j)}(g)$ for $g = \Pi(g')$ and all irreducible representations of $SO(3)$ are automatically representations of $SU(2)$ in agreement with the general statement above. If we label the elements in $SU(2)$ by their 2×2 unitary matrices in the fundamental representation and the elements in $SO(3)$ by their 3×3 orthogonal matrices we have $\Pi^{-1}(I_{3\times3}) = \{I_{2\times2}, -I_{2\times2}\}$. Let now $D'^{(j)}$, $j = n$ be an $SU(2)$ representation. Then $D'^{(j)}(I_{2\times2}) = D'^{(j)}(-I_{2\times2})(= I_{(2j+1)\times(2j+1)})$. Thus it can indeed be restricted to $SU(2)/Z_2$ and serve as a representation for $SO(3)$. However, if j is an half integer we have $D'^{(j)}(I_{2\times2}) = -D'^{(j)}(-I_{2\times2})$ and indeed, the half integer j representations of $SU(2)$ are not representations of $SO(3)$. We now apply this insight to the situation where $G' = \bar{G}$, the covering group corresponding to the Lie algebra and where $G = \bar{G}/D$. Any matrix representation of G/D can be viewed as a matrix representation of $\bar{G}$. Now consider a representation $D^{(r)}$ of $\bar{G}$. The exponentiation of the corresponding matrix Lie algebra will lead to the group $\bar{G}/D_{max}$, where D_{max} is the largest center group of $\bar{G}$ where $D^{(r)}$ restricted to $\bar{G}/D_{max}$ *is* a representation (as described above). This settles the question to which extent exponentiation of a matrix Lie algebra leads back to the matrix Lie group in the case of compact groups.

One final remark about the exponential map. It allows us in principle to construct the Lie groups from the Lie algebras for a certain class of Lie algebras. However, given a representation of a Lie algebra it is usually quite difficult to explicit perform the exponentiation. To find the representation of Lie group one will either use tensor-product methods or use the exponentiation via so-called Hausdorff-Campell-Baker (BCH) formulas. We will see examples of this last method when we discuss globally symmetric spaces.

From complex Lie algebras to real Lie algebras

Let us now turn to the second question mentioned above: how to return from the complex simple Lie algebra $L_{\mathbb{C}}$ found in the Cartan-Killing classification to a real simple Lie algebra $L_{\mathbb{R}}$. Implicit, in the Zee-book, the starting point was the *real Lie algebras* corresponding to the *compact* classical groups $SU(n)$, $SO(n)$ and $USp(2n)$ (this last group is isomorphic to $U(2n) \cap Sp(2n, \mathbb{C})$. This group is a *real* Lie group with *real* dimension $n(2n+1)$. It is compact, connected and simply connected. This is in contrast to $Sp(2n, \mathbb{R})$ which is also $n(2n+1)$ dimensional, but non-compact, connected and with fundamental group isomorphic to $\mathbb{Z}$). Let X_a be a set of generators for the Lie algebra in the adjoint representation. The Cartan-Killing form was defined as

$$g_{ab} = \mathrm{tr}\, X_a X_b \tag{5}$$

For these compact classical groups the Cartan-Killing form on the real Lie algebra could, by a suitable orthogonal transformation $X'_a = O_{ab} X_b$ be chosen as a diagonal matrix with *positive diagonal elements* if we us the convention of the book (which is the physics convention) and represent the X_a's as Hermitian matrices. If we instead choose the convention to represent the generators by anti-Hermitian matrices (i.e. multiply the X_a's by i, which is the convention of the mathematicians), the Killing form g_{ab} have purely negative eigenvalues. We will here follow the latter convention. First two theorems

Theorem: (Cartan's criterion) A Lie algebra is semisimple if and only if the corresponding Cartan-Killing form is non-degenerate.

This is assumed without much discussion in the Zee-book. It allows us to skip any discussion about the so-called solvable Lie algebras which are not of that great importance in physics. The semisimple Lie algebras are direct sums of simple Lie algebras and in the following we will consider mainly simple Lie algebras which are the Lie algebras which contain no invariant subalgebras. One reason the *regular representation* plays such an important role when discussing the simple (and semisimple) algebras is the following theorem

Theorem: the regular representation of a semisimple Lie algebra is faithful.

Another important theorem is

Theorem: A simple (or just semisimple) Lie group is compact if and only if the Cartan-Killing form on its Lie algebra is negative definite.

This last theorem tells us why we indirectly assumed as a starting point that we were dealing with compact Lie groups, when it in Zee-book was stated that the Cartan-Killing form was negative definite. Yet another theorem relates compactness of the group to that of its covering group

Theorem If a Lie group is (semi)simple and compact, its universal covering group is compact.

Let us now look at the complex Lie algebra generated by the H_i and E_α where we have

$$[H_i, E_\alpha] = \alpha_i E_\alpha, \quad [E_\alpha, E_{-\alpha}] = \alpha_i H_i, \quad [E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}, \tag{6}$$

where H_i, E_α are in some representation which we assume is faithful and the scalar product is the Cartan-Killing form

$$\langle X|Y\rangle = \mathrm{tr}\, ad_X \, \mathrm{ad}_Y, \qquad \mathrm{ad}_X : L \mapsto L, \quad \mathrm{ad}_X(V) := [X, V], \tag{7}$$

and where

$$\langle H_i|H_j\rangle = \delta_{ij}, \quad \langle E_\alpha|E_{-\alpha}\rangle = 1, \quad \langle E_\alpha|E_\beta\rangle = 0, \; \beta \neq -\alpha. \tag{8}$$

The choice of normalization of H_i in (8), a choice which implies that the α_i's in (6) are real, is only possible in the complex Lie algebra $L_\mathbb{C}$. If the starting point had been a compact real Lie algebra the Cartan-Killing form would be negative definite in contradiction with $\langle H_i|H_j\rangle = \delta_{ij}$ in (8). Thus the original H_i's from the compact real Lie algebra have been multiplied by i, which has turned them from antisymmetric real matrices to Hermitian matrices. Such a multiplication with i is legal if the Lie algebra is over the complex field $\mathbb{C}$. The H_i's being Hermitian imply that the eigenvalues α_i's in (6) are real. The reality of α_i and $N_{\alpha,\beta}$ is important because α_i and $N_{\alpha,\beta}$ *are* the structure constants for the Lie algebra in the chosen basis H_i, E_α.

The complex Lie algebra is now

$$L_\mathbb{C} = c_i H_i + c_\alpha E_\alpha, \quad c_i, c_\alpha \in \mathbb{C}, \tag{9}$$

and from this complex algebra we can easily extract two real Lie algebras, two so-called *real forms* corresponding to the complex Lie algebra. We denote these *the normal form* , $L_\mathbb{R}^{(nor)}$ and *the compact form* $L_\mathbb{R}^{(com)}$. The normal form is obtained from $L_\mathbb{C}$ by restricting the coefficients c_i, c_α to be real:

$$L_\mathbb{R} = c_i H_i + c_\alpha E_\alpha, \quad c_i, c_\alpha \in \mathbb{R}. \tag{10}$$

The reality α_i and $N_{\alpha,\beta}$ ensures that the restriction trivially defines a real Lie algebra. However, this Lie algebra is not compact. In fact it is maximally non-compact. Recall that a real Lie group (or the corresponding Lie algebra) was compact if and only if its Cartan-Killing form was negative definite. Make the following basis change for $L_\mathbb{R}^{(nor)}$:

$$E_\alpha, E_{-\alpha} \to X_\alpha, Y_\alpha, \quad X_\alpha = \frac{E_\alpha + E_{-\alpha}}{2}, \quad Y_\alpha = \frac{E_\alpha - E_{-\alpha}}{2}, \quad \alpha > 0. \tag{11}$$

With this choice of basis we have in stead of (8)

$$\langle H_i | H_j \rangle = \delta_{ij}, \quad \langle X_\alpha | X_\beta \rangle = \delta_{\alpha\beta}, \quad \langle Y_\alpha | Y_\beta \rangle = -\delta_{\alpha\beta}, \tag{12}$$

while the H's, X_α's and Y_α's are mutual orthogonal. Thus the number of negative eigenvalues of the Cartan-Killing form is $(n-l)/2$ while the number of positive eigenvalues is $l+(n-l)/2$, i.e. $(n+l)/2$, where n is the dimension of the Lie algebra and l the rank. However, another choose of basis in $L_{\mathbb{C}}$ is

$$\tilde{H}_i = iH_i, \quad \tilde{X}_\alpha = iX_\alpha, \quad \tilde{Y}_\alpha = Y_\alpha. \tag{13}$$

This is a perfectly legal change of basis in $L_{\mathbb{C}}$ and we now define

$$L_{\mathbb{R}}^{(com)} = c_i \tilde{H}_i + c_\alpha \tilde{X}_\alpha + c'_\alpha \tilde{Y}_\alpha, \quad c_i, c_\alpha, c'_\alpha \in \mathbb{R}. \tag{14}$$

One can rather easily show, using properties of $N_{\alpha,\beta}$, that the commutation relations indeed closes, i.e. that commutators of $\tilde{H}$'s, $\tilde{X}$'s and $\tilde{Y}$'s can be expressed in terms of the $\tilde{H}$'s, $\tilde{X}$'s and $\tilde{Y}$'s. Thus $L_{\mathbb{R}}^{(com)}$ *is* a real Lie algebra. Further, one obtains from (12)

$$\langle \tilde{H}_i | \tilde{H}_j \rangle = -\delta_{ij}, \quad \langle \tilde{X}_\alpha | \tilde{X}_\beta \rangle = -\delta_{\alpha\beta}, \quad \langle \tilde{Y}_\alpha | \tilde{Y}_\beta \rangle = -\delta_{\alpha\beta}, \tag{15}$$

i.e. the Cartan-Killing form is negative definite and the corresponding Lie group is thus *compact*.

The above constructions where we move between $L_{\mathbb{R}}^{(com)}$ and $L_{\mathbb{R}}^{(nor)}$ by multiplying some of the basis vectors by i is a special case of the following. Let us start from a compact real Lie algebra and consider a linear map $S : L_{\mathbb{R}}^{(com)} \mapsto L_{\mathbb{R}}^{(com)}$ defined by $M \mapsto SMS^{-1}$, which also satisfies $S^2 = I$. We say that S is an *involutive automorphism*. S has eigenvalues ± 1 and we can decompose $L_{\mathbb{R}}^{(com)}$ into the corresponding eigenspaces

$$L_{\mathbb{R}}^{(com)} = T \oplus P, \qquad S(T) = (+)T, \quad S(P) = (-)P. \tag{16}$$

The subspaces T and P are orthogonal (with respect to the Cartan-Killing form) and one can show

$$[T,T] \subseteq T, \quad [T,P] = P, \quad [P,P] \subseteq T. \tag{17}$$

Thus T forms a subalgebra and P its orthogonal complementary subspace. The decomposition (17) is called a *Cartan decomposition*.

We can now perform the so-called *Weil unitary trick* on the subspace P and construct what turns out to be a new real Lie algebra $L_{\mathbb{R}}^*$ by multiplying all vectors in P with i:

$$L_{\mathbb{R}}^* = T \oplus iP. \tag{18}$$

From (17) it follows that $[L_{\mathbb{R}}^*, L_{\mathbb{R}}^*] = L_{\mathbb{R}}^*$, i.e. $L_{\mathbb{R}}^*$ *is* a real Lie algebra. Thus, starting from a compact simple Lie algebra we have constructed a new Lie algebra which will be non-compact if P is different from zero.

Theorem: As S goes though all involutive automorphisms, $L_{\mathbb{R}}^*$ runs through all real forms of the complex simple Lie algebra $L_{\mathbb{C}}$ of which $L_{\mathbb{R}}^{(com)}$ is the known compact real form

Theorem The involutive automorphisms of the simple Lie algebras is one of the three following types: $S = K$ (complex conjugation), $S = I_{p,q}$ or $S = J_{p,p}$.

In the theorem complex conjugation refers to multiplying the vectors in the vectorspace P by i and the matrices $I_{p,q}$ and $J_{p,p}$ are defined by

$$I_{p,q} = \begin{pmatrix} I_{p\times p} & 0 \\ 0 & -I_{q\times q} \end{pmatrix}, \quad J_{p,p} = \begin{pmatrix} 0 & I_{p\times p} \\ -I_{p\times p} & 0 \end{pmatrix}, \tag{19}$$

It is beyond these notes to discuss how to apply systematically the involutive automorphisms to the type A_n, B_n, C_n, D_n Lie algebras as well as the exceptional Lie algebras. We will confine ourselves to 3 examples.

(1) The first examples is from root space A_{n-1}. The corresponding complex Lie algebra is $sl(n,\mathbb{C})$ and in the defining representation this Lie algebra is represented by $n \times n$, traces matrices:

$$L_{\mathbb{C}} = sl(n,\mathbb{C}) = \{M_{n\times n} | \mathrm{tr}\, M = 0,\ M \text{ complex}\} \tag{20}$$

Restricting ourselves to real matrices we obtain the normal real form

$$L_{\mathbb{R}}^{(nor)} = sl(n,\mathbb{R}) = \{M_{n\times n} | \mathrm{tr}\, M = 0,\ M \text{ real}\} \tag{21}$$

A Cartan decomposition is now

$$\begin{aligned} sl(n,\mathbb{R}) &= T \oplus iP, & (22)\\ A \in T &: A \text{ real antisymmetric}, & (23)\\ B \in iP &: B \text{ real symmetric}, \quad \mathrm{tr}\, B = 0. & (24) \end{aligned}$$

By using Weil's unitary trick we obtain $su(n,\mathbb{C})$

$$L_{\mathbb{R}}^{(com)} = su(n,\mathbb{C}) = \{M_{n\times n} | \mathrm{tr}\, M = 0,\ M^\dagger = -M\} \tag{25}$$

$$
\begin{aligned}
su(n,\mathbb{C}) &= T\oplus P, && (26)\\
A\in T &: A \text{ real antisymmetric}, && (27)\\
iB\in P &: B \text{ real symmetric}, \quad \operatorname{tr} B = 0. && (28)
\end{aligned}
$$

This is an example of the complex operation K in the A_{n-1} case.

(2) For A_{n-1} all three type of involutive automorphisms can act. Let us illustrate the action of the type $I_{p,q}$. The starting point is $su(n,\mathbb{C})$, the compact Lie algebra, where a matrix in the defining representation can be written as

$$
\begin{pmatrix} A_{p\times p} & B_{p\times q} \\ -B^\dagger_{q\times p} & C_{q\times q}\end{pmatrix}, \quad A^\dagger = -A,\ C^\dagger = -C,\ \operatorname{tr} A + \operatorname{tr} C = 0,\ B \text{ arbitrary}. \tag{29}
$$

We now have the Cartan decomposition

$$
su(n,\mathbb{C}) = T\oplus P, \quad T = \begin{pmatrix} A & 0 \\ 0 & C, \end{pmatrix} \quad P = \begin{pmatrix} 0 & B \\ -B^\dagger & \end{pmatrix} \tag{30}
$$

The action of S on an arbitary element M was defined above as $M \to SMS^{-1}$. Let us now apply this with M given by (29) and $S = I_{p,q}$:

$$
\begin{pmatrix} I_p & 0 \\ 0 & -I_q\end{pmatrix}, \begin{pmatrix} A & B \\ -B^\dagger & C\end{pmatrix}, \begin{pmatrix} I_p & 0 \\ 0 & -I_q\end{pmatrix}, = \begin{pmatrix} A & -B \\ B^\dagger & C\end{pmatrix}, \tag{31}
$$

Thus we see that a matrix in T is unchanged, while a matrix in P changes sign, i.e. T and P are the eigenspaces of $I_{p,q}$ corresponding to eigenvalues ± 1, respectively. The algebra

$$
L^*_{\mathbb{R}} = T\oplus iP := su(p,q;\mathbb{C}) \tag{32}
$$

is non-compact.

(3) For the root space B_n we have the complex Lie algebra $so(2n+1,\mathbb{C})$, consisting of all the complex antisymmetric matrices (in the defining representation). The compact real form is the Lie algebra $so(2n+1,\mathbb{R})$ consisting of the real, antisymmetric matrices and the corresponding compact Lie group is $SO(2n+1,\mathbb{R})$. In this case the only involutive automorphisms which act non-trivially on $so(2n+1,\mathbb{R})$ are of the type $I_{p,q}$, $p+q = 2n+1$ and one can repeat the arguments of example 2 by noting that an arbitrary element can be written as (29) with some obvious changes:

$$
\begin{pmatrix} A_{p\times p} & B_{p\times q} \\ -B^t_{q\times p} & C_{q\times q}\end{pmatrix}, \quad A^t = -A, \quad C^t = -C, \quad A, C, B \text{ real}. \tag{33}
$$

$$so(n,\mathbb{R}) = L_{\mathbb{R}}^{(com)} = T \oplus P, \quad T = \begin{pmatrix} A & 0 \\ 0 & C, \end{pmatrix} \quad P = \begin{pmatrix} 0 & B \\ -B^t & \end{pmatrix} \quad (34)$$

$$so(p,q,\mathbb{R}) = L_{\mathbb{R}}^{*} = T + iP = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}. \quad (35)$$

Since $SO(p,q;\mathbb{R}) \cong SO(q,p;\mathbb{R})$ all inequivalent real forms corresponding to B_n are given by

$$SO(p+q,\mathbb{R}) \to SO(p,q;\mathbb{R}), \quad p > q, \quad p+q = 2n+1. \quad (36)$$

Riemannian globally symmetric spaces

We have seen that there is a one-one correspondence between homogenous topological spaces and cosets G/K, where G is the transformation group which acts on X and K a stability group. Let us now consider a real form $L_{\mathbb{R}}^{(com)}$ associated with a Cartan decomposition $T \oplus P$. Exponentiation of $L_{\mathbb{R}}^{(com)}$ provide us with a Lie group G and exponentiation of the subalgebra T results in a compact subgroup K in G, and we thus have the identification

$$G/K = \exp P. \quad (37)$$

Similarly, for $L_{\mathbb{R}}^{*}$ we have a decomposion $T \oplus iP$ and an identification

$$G^*/K = \exp iP. \quad (38)$$

G/K and G^*/K are called a globally symmetric spaces because any point and its neighborhood can be moved to any other point and a corresponding neighborhood by a suitable group operation (we have no space here for discussing the details, but in *Addednotes3* there is a slightly longer discussion of globally symmetric spaces.).

Let us illustrate the situation in the case where the involutive automorphisms have the block-diagonal form (34). Firstly we have that

$$K = \exp T = SO(p) \otimes SO(q) \quad (39)$$

and further

$$G/K = SO(p+q)/SO(p) \otimes SO(q) = \exp P \quad (40)$$

where

$$\exp P = \exp \begin{pmatrix} 0 & B \\ -B^\dagger & 0 \end{pmatrix} = \begin{pmatrix} \cos\sqrt{BB^\dagger} & B\dfrac{\sin\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}} \\ -\dfrac{\sin\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}}B^\dagger & \cos\sqrt{B^\dagger B} \end{pmatrix} \quad (41)$$

This coset space is compact. Its *dual*, non-compact cousin is

$$G^*/K = SO(p,q)/SO(p) \otimes SO(q) = \exp iP, \tag{42}$$

where

$$\exp iP = \exp \begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix} = \begin{pmatrix} \cosh\sqrt{BB^\dagger} & B\dfrac{\sinh\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}} \\ -\dfrac{\sinh\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}}B^\dagger & \cosh\sqrt{B^\dagger B} \end{pmatrix} \tag{43}$$

The expressions can be rewritten

$$X = B\frac{\sin\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}}, \qquad \exp P = \begin{pmatrix} (I - XX^\dagger)^{1/2} & X \\ -X^\dagger & (I - X^\dagger X)^{1/2} \end{pmatrix}. \tag{44}$$

$$Y = B\frac{\sinh\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}}, \qquad \exp iP = \begin{pmatrix} (I + YY^\dagger)^{1/2} & Y \\ Y^\dagger & (I + Y^\dagger Y)^{1/2} \end{pmatrix}. \tag{45}$$

The dimensions of these coset spaces are pq (the number of independent entries in B).

The spaces described here are called *Riemannian globally symmetric space.* The reason is the folllowing: we started out with the compact G and its corresponding compact real Lie algebra. By construction the "metric" on P (the Cartan-Killing form restricted to P) was negative definite. By exponentiation this metric becomes the metric around the "identity" element of G/K and thus around any element by suitable group operations. This G/K, viewed as a differential manifold, inherits a *Riemannian metric* from P. We call it Riemannian because it has a definite signature which we can take to be positive when exponentiating. The same is now true for G^*/K since also on iP the Cartan-Killing form has a definite sign (positive). Again, by exponentiation this allows us to definite a positive definite metric on G^*/K.

The very simplest example is $p+q=3$. In this case $p=2$ and $q=1$ and $SO(2)\otimes SO(1) \cong SO(2)$. Thus we obtain two Riemannian spaces

$$SO(3)/SO(2) \cong S^2, \qquad SO(2,1)/SO(2) \cong H^2 \tag{46}$$

where S^2 (the sphere) is a Riemannian symmetric space with constant positive curvature, while H^2 (the hyperbolic plane, or the pseudo-sphere) is a Riemannian non-compact space with constant negative curvature. The construction can be generalized to any $p=n$, $q=1$ and we obtain

$$SO(n+1)/SO(n) \cong S^n, \qquad SO(n,1)/SO(n) \cong H^n, \tag{47}$$

the so-called *maximally symmetric spaces* with positive and negative curvature.

Let us describe part of the geometry on these globally symmetry spaces. They are parametrized by P which in this case is a n-dimensional vector B_i:

$$P_{(n+1)\times(n+1)} = \begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \\ \cdot \\ \cdot \\ B_n \end{pmatrix} \tag{48}$$

$B = 0$ corresponds to the point IK in G/K (or G^*/K), where I is the identity matrix in G. We can now define a curve in G/K from $B = 0$ to B by the map

$$t \mapsto \exp\big[tP(B)\big], \quad (\text{or } \exp\big[itP(B)\big]), \quad t \in [0,1] \tag{49}$$

This curve is in addition *a geodesic*, i.e. the shortest possible path from point 0 to B, where the metric on the spaces G/K or G^*/K are defined as decribed above (but not in detail, for a more detailed description see *Addednotes3*.) from the Cartan-Killing form restricted to P. It turns out that the distance from 0 to $P(B)$ simply *is* this norm, which is just the ordinary norm of vector B:

$$\mathrm{Dist}(0, tP(B)) = t||B||, \quad ||B|| = \sqrt{B_i B_i}. \tag{50}$$

Let us now change parametrization of G/K or G^*/K from B to the X and Y used in (44) and (45). We have

$$\exp P = \begin{pmatrix} (I - XX^t)^{1/2} & X \\ -X^t & x_{n+1} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \quad x_1^2 + \cdots + x_n^2 + x_{n+1}^2 = 1. \tag{51}$$

Thus the x_i, $i = 1, \ldots, n+1$ provide a parametrization like to sphere S^n, but most importantly, the distance assignment (50) shows that the distance between the point $B = 0$, i.e. $x_i = 0$ $i = 1, \ldots, n$, $x_{n+1} = 1$ (the "north pole") and the point with coordinates B or x_i are precisely the distance one would obtain by using the distance measure from $\mathbb{R}^{n+1}$, and following the "arc" on S^n rather than the straight line from the north pole to point x_i: G/K *is* S^n also when it comes to geometry.

We can repeat the same for G^*/K and we have

$$\exp iP = \begin{pmatrix} (I - YY^t)^{1/2} & Y \\ -Y^t & y_{n+1} \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_n \end{pmatrix}, \quad -y_1^2 - \cdots - y_n^2 + y_{n+1}^2 = 1. \tag{52}$$

which illustrates the non-compactness of the space G^*/K. Again the intrinsic distance between the point $B = 0$, i.e. $y_i = 0$ $i = 1, \ldots, n$, $y_{n+1} = 1$ and the point with B or y_i is given by (50). However, in this case *it does not* correspond to the distance one obtains by following the "hyperbola" (52) embedded in $\mathbb{R}^{n+1}$. It is only the case if one insist on counting the square distances in directions $i = 1, \ldots, n$ negative (thus the name "pseudo sphere"). Returning to (46), the intrinsic geometries of both S^2 and H^2 are non-Euclidean, corresponding to positive and negative curvature, respectively, but historically it was much more difficult to understand the intrinsic geometry of H^2 because it could not be embedded in R^3 in a way which reflected this intrinsic geometry faithfully.