Opg IV – 1

Opg 1 :

$S^{ij} = T^{ij} + T^{ji}$ er en tensor:

T^{ji} er en tensor:

kalder i for j og j for i

$T'^{ij} = R^{ik} R^{jl} T^{kl} \Rightarrow T'^{ji} = R^{jk} R^{il} T^{kl}$

$= R^{il} R^{jk} T^{kl}$

$\tilde{T}^{ij} = T^{ji}$ $\quad \underline{\tilde{T}'^{ij} = R^{il} R^{jk} \tilde{T}^{lk}}$

Altså er $T^{ij} + \tilde{T}^{ij}$ en tensor

Opg 4

A^{ijk} antisymmetrisk for $SO(N)$, $i = 1, \ldots, N$

Vælg i: N, vælg j: $N-1$, vælg k: $N-2$

Men antisym: $A^{\sigma(i)\sigma(j)\sigma(k)} = \text{sign}\,\sigma\, A^{ijk}$

antal permutationer $3!$

antal uafhængige komponenter: $\frac{1}{3!} N(N-1)(N-2)$.

~~Opg 5~~ Opg 5

For $N=3$ har vi $\frac{1}{3!} 3 \cdot 2 \cdot 1 = 1$ komponent

Dvs $T^{ijk} = \varepsilon^{ijk} \cdot S$ hvor S er invariant:

$T'^{ijk} = R^{ii'} R^{jj'} R^{kk'} T^{i'j'k'}$; ε^{ijk} er inv $\Rightarrow$

$S' \underbrace{\varepsilon^{ijk} = R^{ii'} R^{jj'} R^{kk'} \varepsilon^{i'j'k'}}_{\text{Formel (12) i bog}} S \Rightarrow S' = S$

Opg IV − 1

Opg 12

T^{ii} invariant : rep. af dimension $\underline{1}$

T^{i} : vektor repr. af dimension N for $SO(N)$

$\tilde{S}^{ij}$: sporløs tensor : $\frac{1}{2}N(N+1)-1$ for $SO(N)$

A^{ij} : antisymmetrisk tens. $\frac{1}{2}N(N-1)$ for $SO(N)$

For $SO(5)$: $1, 5, 14$ og 10

Lad os nu tensorer med 3 indices:

A^{ijk} dual til $A^{kl} = \varepsilon^{kl\,ijl} A^{ijl}$

Deler ikke noget nyt.

Symmetrisk, & sporløs:

$\tilde{S}^{ijk}$:

Symmetrisk $SO(N)$: $S^{i_1 \cdots i_r}$

N farver

$\underbrace{\square\ \square\ \square\ \cdots\ \square}_{r}$

Vi har bolde i N farver

Vi putter 1 bold i hver boks (r bokse)

Hvor mange måder kan det gøres på når rækkefølgen er ligegyldig : $\boxed{1}\ \boxed{2}\ \boxed{4} = \boxed{2}\ \boxed{1}\ \boxed{4}$ ...

$$\boxed{\binom{N+r-1}{r}}$$

Opg IV – 1.

opg. 12 fortsat

For $r=3$, $N=5$: $\binom{5+3-1}{3} = \binom{7}{3} = 35$

$\dim S^{ijk} = 35$

Men $\tilde{S}^{ijk}$ sporløs : $S^{iik} = 0$

$k = 1, \dots, 5$

$\boxed{\text{Dvs } \tilde{S}^{ijk} \text{ er 30-dimensional}}$

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opgave 13

$\tilde{S}^{i_1 \cdots i_n}$ for $SO(4)$.

$S^{i_1 \cdots i_n}$ har dimension : $\binom{N+n-1}{n}$, $N=4$

$= \binom{n+3}{n} = \frac{(n+3)(n+2)(n+1)}{3 \cdot 2 \cdot 1}$

Men $S^{ii\, i_1 \cdots i_{n-2}} = 0$: $\binom{(n-2)+3}{n-2}$ betingelser

dvs: $\frac{(n+1)n(n-1)}{3 \cdot 2 \cdot 1}$ betingelser.

I alt : $\frac{(n+3)(n+2)(n+1)}{6} - \frac{(n+1)n(n-1)}{6} = (n+1)^2$.

Opgave IV. 2

Opg 1 :

$$J_x = \frac{1}{2}(J_+ + J_-) \qquad J_\pm = J_x \pm iJ_y$$

$$J_y = -\frac{i}{2}(J_+ - J_-) \qquad J_- = (J_+)^\dagger$$

$\boxed{J=\frac{1}{2}}$ Vi ved: $\langle \frac{1}{2} | J_+ | -\frac{1}{2} \rangle = 1$, $\langle -\frac{1}{2} | J_+ | -\frac{1}{2} \rangle = 0$

$\langle +\frac{1}{2} | J_- | +\frac{1}{2} \rangle = 0$ $\langle -\frac{1}{2} | J_- | +\frac{1}{2} \rangle = 1$

(og $J_+ | \frac{1}{2} \rangle = J_- | -\frac{1}{2} \rangle = 0$)

$$J_+ = \begin{array}{c|cc} & \frac{1}{2} & -\frac{1}{2} \\ \hline \frac{1}{2} & \langle \frac{1}{2} | J_+ | \frac{1}{2} \rangle & \langle \frac{1}{2} | J_+ | -\frac{1}{2} \rangle \\ -\frac{1}{2} & \langle -\frac{1}{2} | J_+ | \frac{1}{2} \rangle & \langle -\frac{1}{2} | J_+ | -\frac{1}{2} \rangle \end{array} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$J_- = (J_+)^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$$J_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$J_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$: Tjek $[J_i, J_j] = i\varepsilon_{ijk} J_k$.

Opgave IV. 2

opg 1 (fortsat)

$\boxed{J=1}$ $\quad \langle 0 | J_+ | -1 \rangle = \sqrt{2}, \quad \langle 1 | J_+ | 0 \rangle = \sqrt{2}$

$$J_+ = \begin{array}{c} \\ 1 \\ 0 \\ -1 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 0 & -1 \end{array} \\ \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \end{array}, \quad J_- = (J_+)^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$J_x = \begin{pmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & -i/\sqrt{2} & 0 \\ i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & i/\sqrt{2} & 0 \end{pmatrix}$$

$$J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \text{Tjek } [J_i, J_j] = i\varepsilon_{ijk} J_k$$

$\boxed{J=2}$ $\quad \langle -1 | J_+ | -2 \rangle = 2, \quad \langle 0 | J_+ | -1 \rangle = \sqrt{6}$

$\langle 1 | J_+ | 0 \rangle = \sqrt{6}, \quad \langle 2 | J_+ | 1 \rangle = 2.$

$$J_+ = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad J_- = (J_+)^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

$$J_x = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & -2i & 0 & 0 & 0 \\ 2i & 0 & -\sqrt{6}i & 0 & 0 \\ 0 & \sqrt{6}i & 0 & -\sqrt{6}i & 0 \\ 0 & 0 & \sqrt{6}i & 0 & -2i \\ 0 & 0 & 0 & 2i & 0 \end{pmatrix}$$

$$J_z = \begin{pmatrix} 2 & & & & 0 \\ & 1 & & & \\ & & 0 & & \\ & & & -1 & \\ 0 & & & & 2 \end{pmatrix}, \quad [J_i, J_j] = i\varepsilon_{ijk} J_k$$

Opgave IV.2

Opg 2

For $SO(N)$ har vi

$$\hat{S}^{ijkl} = S^{ijkl} - C_1\left[\delta^{ij} S^{hhkl} + \delta^{ik} S^{hhjl} + \delta^{il} S^{hhjk} + \delta^{jk} S^{hhil} + \delta^{jl} S^{hhik} + \delta^{kl} S^{hhij}\right]$$

$$+ C_2\left[\delta^{ij}\delta^{kl} S^{hhff} + \delta^{ik}\delta^{jl} S^{hhff} + \delta^{il}\delta^{jk} S^{hhff}\right]$$

$$\hat{S}^{iikl} = 0 \Rightarrow$$

$$S^{iikl} - C_1\left[(N+4) S^{hhkl} + \delta^{kl} S^{hhii}\right] + C_2\left[(N+2) S^{hhff}\right] = 0 \Rightarrow$$

$$C_1 = \frac{1}{N+4}, \quad C_2 = \frac{1}{(N+4)(N+2)}$$

For $N=3$ får vi $C_1 = \frac{1}{7}$, $C_2 = \frac{1}{35}$

For $S^{ijkl} = n^i n^j n^k n^l$, n^i enhedsvektor

$$\hat{S}^{ijkl} = n^i n^j n^k n^l - \frac{1}{7}\left[\delta^{ij} n^k n^l + \delta^{ik} n^j n^l + \delta^{il} n^j n^k + \delta^{jk} n^i n^l + \delta^{jl} n^i n^k + \delta^{kl} n^i n^j\right]$$

$$+ \frac{1}{35}\left[\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right]$$

Opgave IV. 2

Opg 2 fortsat

Vi har nu, for $n^3 = \cos\theta$ $\left(\begin{matrix} n^1 = \cos\varphi \sin\theta \\ n^2 = \sin\varphi \sin\theta \end{matrix}\right)$

$$\hat{g}^{3333} = (n^3)^4 - \frac{6}{7}(n^3)^2 + \frac{3}{35}$$

$$= (\cos\theta)^4 - \frac{6}{7}\cos^2\theta + \frac{3}{35} = 35 \cdot P_4(\cos\theta)$$

---

opg 3

Lad n^i være en enhedsvektor $\left(\begin{matrix} n^1 = \cos\varphi \sin\theta \\ n^2 = \sin\varphi \sin\theta \\ n^3 = \cos\theta \end{matrix}\right)$

Under en rotation transformer

n^i som en vektor og derfor:

$n^{i_1} n^{i_2} \cdots n^{i_n}$ som en tensor:

$$n'^{i_1} \cdots n'^{i_k} = R^{i'_1 i_1} \cdots R^{i'_k i_k} \, n^{i_1} \cdots n^{i_k}$$

Lad nu f være vilkårlig funktion:

$f : n^i \to \mathbb{C}$. (f.ex: $f(n^i) = n^{i_1} \cdots n^{i_n}$)

$f : S^2 \to \mathbb{C}$

Opgave IV.2

Opg 3 (fortsat).

$d\Omega = \frac{1}{4\pi} \sin\theta \, d\theta \, d\phi$; $\int_{S^2} d\Omega \cdot 1 = 1$

$$\int_{S^2} d\Omega \, f(R^{ij} n^j) = \int_{S^2} d\Omega \, f(n^i)$$

Det er fordi $d\Omega$ er invariant under rotationer (det er et area-element).

Betragt nu:

$$\int_{S^2} d\Omega \, n^{i_1} n^{i_2} \cdots n^{i_k} = S^{i_1 \cdots i_k}$$

$S^{i_1 \cdots i_k}$ er en tensor, men invariant under rotationer fordi:

$$R^{i_1 i_1'} \cdots R^{i_k i_k'} S^{i_1' \cdots i_k'} = \int_{S^2} d\Omega \, (R^{i_1 i_1'} n^{i_1'}) \cdots (R^{i_k i_k'} n^{i_k'})$$

$$= \int_{S^2} d\Omega \, f(Rn) = \int_{S^2} d\Omega \, f(n) = S^{i_1 \cdots i_k}$$

Opgave IV.2

Opg 3 (fortsat 2)

f. eks.

$$\int_{S^2} d\Omega\, n^i n^j = c \cdot \delta^{ij}$$

Fordi δ^{ij} er den eneste invariante symmetriske tensor med 2 indices.

Men vi ved også at

$\hat{T}^{ij} = n^i n^j - \frac{1}{3}\delta^{ij}$ er sporløs og "ortogonal" på den invariante tensor, idet udgør en 1-dimensional irreducibel rep af SO(3). $\hat{T}^{ij}$ gav anledning til en 5-dimensional rep af SO(3) der netop ikke indeholder den invariante tensor. Det samme vil derfor også gælde for integralet af $\hat{T}^{ij}$:

$$\int_{S^2} d\Omega \left(n^i n^j - \frac{1}{3}\delta^{ij}\right) = c' \delta^{ij} = 0$$

Opgave IV.2

Opg 3 (fortsat 3)

Det samme gælder nu for:

$$\hat{S}^{i_1 i_2 i_3 i_4} = n^{i_1} n^{i_2} n^{i_3} n^{i_4} - \frac{1}{7}\left(\delta^{i_1 i_2} n^{i_3} n^{i_4} + \dots\right) + \frac{1}{35}\left(\delta^{i_1 i_2}\delta^{i_3 i_4} + \dots\right)$$

Som vi betragtede i opg. 2:

$$\int_{S^2} d\Omega \, \hat{S}^{i_1 \cdots i_4} = c\left(\delta^{i_1 i_2}\delta^{i_3 i_4} + \delta^{i_1 i_3}\delta^{i_2 i_4} + \delta^{i_1 i_4}\delta^{i_2 i_3}\right)$$
$$= 0 \quad (c = 0)$$

Fordi $\hat{S}^{i_1 \cdots i_4}$ ikke indeholder den trivielle invariante repræsentation.

Så vi har generelt:

$$\boxed{\int_{S^2} d\Omega \, \hat{S}^{i_1 \cdots i_n} = 0}$$

opgave IV.2

opg 3 (fortsat 4)

Hvad hvis vi integrerer produktet af to $\tilde{S}^{i_1 \cdots i_{j_1}}$, $\tilde{S}^{i_1 \cdots i_{j_2}}$

$$\boxed{\int_{S^2} d\Omega\, \tilde{S}^{i_1 \cdots i_{j_1}} \cdot \tilde{S}^{i'_1 \cdots i'_{j_2}} = 0 \quad \text{for } j_1 \neq j_2}$$

Dette er fordi vi kan skrive (Clebsch-Gordan dekompositionen) $\tilde{S}^{i_1 \cdots i_{j_1}} \tilde{S}^{i'_1 \cdots i'_{j_2}}$ som sum af komponenter $\tilde{S}^{k_1 \cdots k_j} \delta^{l_1 l_2} \delta^{l_3 l_4} \cdots$ for

$|j_1 - j_2| \leq j \leq j_1 + j_2$.

og integralet af disse er nul med mindre $j = 0$, dvs $j_1 = j_2$.

specielt er dette rigtigt for

$$\tilde{S}^{\overbrace{3 \cdots 3}^{l_1}} \tilde{S}^{\overbrace{3 \cdots 3}^{l_2}} \sim P_{l_1}(\cos\theta) P_{l_2}(\cos\theta).$$