Exam MATF3 June 2018

4 hour exam with all usual help items. The exam may be answered using pencil and English or Danish language can be used. The exam includes 10 sub-questions which are weighted equally in the evaluation.

Problem 1

Let G be a finite group and K a subset of G. The *centralizer* $C(K)$ of K is defined by

$$C(K) = \{c \in G \mid ck = kc \; \forall \; k \in K\}.$$

(1) Prove that $C(K)$ is a subgroup of G.

(2) Prove that if K is an abelian subgroup of G then K is an invariant subgroup of $C(K)$.

Problem 2

There exists an non-abelian group of order 8, which has the following presentation

$$G = \langle x, y | x^4 = e, \quad y^2 = e, \quad yx = x^3 y \rangle,$$

where e is the identity element.

(1) Show that any element of the form $x^k y^l x^m y^n$ can be written as one of the 8 elements $e, x, x^2, x^3, y, xy, x^2y, x^3y$.

This can easily be turned into a proof that any element of G can be written as one of the 8 mentioned elements.

(2) Fill in the entries marked with $\star$ in the group multiplication table below, expressed in terms of the eight mentioned elements.

	e	x	x^2	x^3	y	xy	x^2y	x^3y
e	$\star$							
x		$\star$	$\star$					
x^2		$\star$	$\star$	$\star$				
x^3			$\star$	$\star$	$\star$			
y				$\star$	$\star$	$\star$		
xy					$\star$	$\star$	$\star$	
x^2y						$\star$	$\star$	
x^3y								$\star$

Problem 3

Consider the group $SO(3)$. Its irreducible representations $D^{(J)}(g)$, $g \in SO(3)$, are labeled by J, $J = 0, 1, 2, \ldots$. For a given J the eigenvectors of J_3 are denoted $|JM\rangle$ and constitute a normalized basis for the $2J+1$ dimensional vectorspace where the matrices $D^{(J)}_{MM'}(g)$ act.

(1) Express the $|J = 4, M = 2\rangle$ vector in terms of the vectors $|j = 2, m\rangle|j = 2, m'\rangle$ constituting a basis for the reducible tensor product representation $D^{(j=2)} \otimes D^{(j=2)}$.

Problem 4

We consider $SU(2)$. We denote the irreducible representations by $D^{(J)}(g)$, $g \in SU(2)$, where $J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. For a given J the normalized eigenvectors $|JM\rangle$ of J_3 form the basis of the $2J+1$ dimensional vectorspace in which $D^{(J)}_{MM'}(g)$ acts. Let $d\mu(g)$ be the Haar measure of $SU(2)$. It is invariant under left and right translations: $d\mu(g_0 g) = d\mu(g g_0) = d\mu(g)$ for all $g_0 \in SU(2)$.

(1) Show that

$$(*) : \qquad \int d\mu(g)\, D^{(J)}_{MM'}(g) = 0 \quad \text{unless} \quad J = 0.$$

(Hint 1: Denote the integral $(*)$ by $V_M(M')$ and use the group properties as well as the invariance of the Haar measure to prove that for all $g_0 \in SU(2)$ we have $\sum_M D^{(J)}_{M_1 M}(g_0) V_M(M') = V_{M_1}(M')$. Use this to conclude that either $J = 0$ or $V_M(M') = 0$ for all M, M'. Instead of hint 1 you can also use hint 2: the "Great Orthogonality Theorem" for continuous compact groups (the Peter-Weil theorem mentioned in addednotes2).)

For $SU(2)$ we have the Clebsch-Gordan tensor product decomposition

$$D^{(j_1)}(g) \otimes D^{(j_2)}(g) = \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} D^{(J)}.$$

(2) Use the result $(*)$ above to prove that

$$(**): \qquad \int d\mu(g)\; D^{(j_1)}_{m_1,m_1'}(g) D^{(j_2)}_{m_2 m_2'}(g) = 0 \quad \text{unless} \quad j_1 = j_2.$$

Let $\chi^{(j)}(g)$ denote the character corresponding to the irreducible representation $D^{(j)}(g)$.

(3) Use $(**)$ above to prove that

$$\int d\mu(g)\; \chi^{(j_1)}(g)\chi^{(j_2)}(g) = 0 \quad \text{unless} \quad j_1 = j_2.$$

Problem 5

Let S^{ij} and T^{kl} be symmetric, traceless tensors corresponding to irreducible representations of $SO(3)$. The tensor product $S^{ij}T^{kl}$ forms a reducible representation of $SO(3)$ which can decomposed into a sum of irreducible representations of $SO(3)$.

(1) Determine the dimensions of these irreducible representations.

(2) One of the irreducible components is a symmetric, traceless tensor W^{ijkl}. Give an explicit expression of W^{ijkl} in terms of S^{ij} and T^{kl}.