Added notes 5

A few facts from representation theory of compact Lie groups

Representations of Lie algebras

For each representation of a compact Lie group we have a corresponding representation of its Lie algebra. Similarly we can obtain a representation of the group by exponentiation if we know the representation of the Lie algebra (except for the subtleties discussed in "addednotes4", subtleties we will ignore here). We know from the Great Orthogonality Theorem for compact Lie groups (see "addednotes2") that any representation can be decomposed in a (possible infinite) sum of finite dimensional unitary irreducible representations. We will thus only consider the finite dimensional unitary representations. Further, we have seen that semisimple Lie algebras decompose into a direct sum simple Lie algebras and we will consequently only consider the representations of simple Lie algebras.

Recall that in the classification of the simple Lie algebras we saw that the Lie algebra was determined entirely by its root system. Let L be a simple Lie algebra of dimension n and rank ℓ. Recall that a root system Φ could be viewed as vectors in the Euclidean space $\mathbb{E} = \mathbb{R}^\ell$, satisfying the following axioms

(R1) Φ is finite, span $\mathbb{E}$, and does not contain 0.

(R2) If $\alpha \in \Phi$, the only multiples of α in Φ is $\pm\alpha$.

(R3) If $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant.

(R4) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$

The notations in the axioms are as follows.We have the standard symmetric real scalar product (v_1, v_2) on $\mathbb{E}$. The **hyperplane** in $\mathbb{E}$ orthogonal to α is $P_\alpha = \{\beta \in \mathbb{E} | \, (\beta, \alpha) = 0\}$. The **reflection** $\sigma_\alpha(\beta)$ of the vector β in P_α is the vector

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\, \alpha = \beta - \langle \beta, \alpha \rangle\, \alpha, \tag{1}$$

where

$$\langle \beta, \alpha \rangle := \frac{2(\beta, \alpha)}{(\alpha, \alpha)}. \tag{2}$$

Let $\mathcal{W}$ denote the subgroup of the general linear group $Gl(\mathbb{E})$ generated by the reflections $\sigma_\alpha,\ \alpha \in \Phi$. $\mathcal{W}$ permutes the set Φ and is thus a subgroup of the symmetric group acting on the set Φ. It is called the **Weyl group** of Φ. A subset Δ of Φ is called a **base** if

(B1) Δ is a basis of $\mathbb{E}$.

(B2) For each root β we have $\beta = \sum_{\alpha\in\Delta} k_\alpha\, \alpha$ where the k_α are integers, all nonnegative or all nonpositive.

It follows that the number of element in Δ is ℓ. The roots in Δ are called **simple** roots and from (B1) it follows that the expression for β in (B2) is unique. We can now define the **height** of a root β (relative to Δ) by

$$\mathrm{ht}(\beta) = \sum_{\alpha\in\Delta} k_\alpha \quad \text{where} \quad \beta = \sum_{\alpha\in\Delta} k_\alpha\, \alpha. \tag{3}$$

If all $k_\alpha \geq 0$ (resp. $k_\alpha \leq 0$) we say that β is **positive** (resp. **negative**) and write $\beta \succ 0$ (resp. $\beta \prec 0$). This induces a partial ordering on Φ (and Δ): we say that $\beta \prec \alpha$ if and only if $\alpha - \beta$ is a sum of positive roots or $\beta = \alpha$. Finally Φ is called **irreducible** if it cannot be partitioned into a union of two proper subsets where each root in one set is orthogonal to each root in the other set. One can show that Φ is irreducible if and only if Δ cannot be partitioned in this way, and further that if Φ is irreducible then there exists a unique **maximal** root β such that $\alpha \neq \beta$ implies $\mathrm{ht}(\alpha) < \mathrm{ht}(\beta)$ and for this β one has $\beta = \sum_{\alpha\in\Delta} k_\alpha\ \alpha$ where all $k_\alpha > 0$.

As shown in the Zee book all irreducible root systems can be classified and this becomes a classification of all **simple** Lie algebras as one has the theorem

Theorem. *Let L be a simple Lie algebra of rank ℓ with Cartan subalgebra H. That defines an irreducible root system Φ and a base Δ (as described in the Zee book). Conversely, given an irreducible root system Φ and a choice of base Δ we can construct a simple Lie algebra with a Cartan subalgebra which has Φ as its root system and Δ as base.*

Let now X_a be generators of an irreducible d-dimensional representation of the simple (real) Lie algebra L of dimension n. The X_a, $a = 1, \ldots, n$ are thus Hermitian matrices satisfying

$$[X_a, X_b] = i f_{abc} X_c. \tag{4}$$

Let H_i, $i = 1, \ldots, \ell$ be a linear combinations of the X_a which form a Cartan subalgebra H. The H_i commute

$$[H_i, H_j] = 0. \tag{5}$$

From the d-dimensional matrices X we can form the adjoint representation of L. These are $n \times n$ matrices ad_X acting on the elements of L viewed as an n-dimensional vector space by

$$\mathrm{ad}_X Z = [X, Z], \quad [\mathrm{ad}_X, \mathrm{ad}_Y] = \mathrm{ad}_{[X,Y]} \tag{6}$$

In particular ad_{H_i} form a Cartan subalgebra in the adjoint representation.

The so-called *Killing form*

$$K(X, Y) := \mathrm{tr}\, \mathrm{ad}_X \mathrm{ad}_Y \tag{7}$$

is a symmetric, bilinear and associative form on L, where associativity of a bilinear symmetric form on L means

$$K([X, Y], Z) = K(X, [Y, Z]). \tag{8}$$

Cartan's theorem states that a Lie algebra is (semi)-simple if and only its Killing form is non-degenerate. As mentioned above, a semi-simple Lie algebra decomposes into a direct sum of simple Lie algebras and here we will only consider simple Lie algebras. For these we have the following lemmas

Lemma. *Let L be a simple Lie algebra. If $f(X, Y)$ and $g(X, Y)$ are non-degenerate symmetric, associative bilinear forms on L, then there is a nonzero scalar a such that $f(X, Y) = a \cdot g(X, Y)$ for all $X, Y \in L$.*

Lemma. *Let $\phi : L \to \mathrm{gl}(V)$ be a representation of L. Then $\mathrm{tr}\, \phi(X)\phi(Y)$ is a symmetric, bilinear and associative form on L. If L is simple and ϕ is faithful then there exists a nonzero constant a such that*

$$\mathrm{tr}\, \phi(X)\phi(Y) = a\, K(X, Y) \quad \forall\, X, Y \in L \tag{9}$$

In the following we will study the faithful representations $\phi : L \to \mathrm{gl}(V)$. For an $X \in L$ we thus have a corresponding $\phi(X) \in \mathrm{gl}(V)$. We will usually consider Lie algebras coming from unitary representations of a compact real Lie group. In this setting V is a d dimensional vector space and the matrices $\phi(X)$ are d dimensional Hermitian matrices which provide us with a representation of L. With this understanding we will in the following drop the notation $\phi(X)$ and talk about $d \times d$-dimensional Hermitian matrices X and say that $X \in L$. In the same way we will call the $d \times d$-dimensional unitary matrices $D = \mathrm{e}^{iX}$ group elements rather matrices in a d-dimensional representation of the group. While the vector space V sometimes can be chosen as a real vector space (e.g. in the case of the group $\mathrm{SO}(N)$), we will in the following consider V extended to a complex vector space on which the unitary matrices D and the Hermitian matrices X acts by trivial extension of the action on the real vector space.

Since the Cartan matrices H_i in our representation now are Hermitian, commuting matrices we can find simultaneous eigenvectors in the *complex* vector space V, which span the whole V

$$H_i|v(k)\rangle = w_i(k)|v(k)\rangle, \quad k = 1, \ldots, d, \quad i = 1, \ldots, \ell. \tag{10}$$

For each k we call the ℓ numbers $w_i(k)$ a **weight** of the Cartan subalgebra H on the vector space V. We can view the d weights as vectors in $\mathbb{E}$ since they have ℓ real components. Note that some of the d weights can be identical. This happens if two or more of the eigenvectors $|v(k)\rangle$ have the same eigenvalues for all H_i. We will often call the eigenvectors $|v(k)\rangle$ for **weight states**, where "states" refer to the fact in physical applications they will often be degenerate energy eigenstates (or related to eigenstates) of a Hamiltonian which is invariant under the action of the Lie group G related to L.

Let us now consider the adjoint representation. The Cartan subalgebra consists of the ℓ matrices ad_{H_i} which, as mentioned, act on L viewed as a vector space. Viewing L as a vector space the lemmas mentioned above provides us with a natural scalar product on L:

$$\langle X|Y\rangle := \mathrm{tr}\, X^\dagger Y, \quad \langle Y|[Z, X]\rangle = \langle [Z^\dagger, Y]|X\rangle \tag{11}$$

At this point the Hermitian conjugate in (11) is of no importance since the matrices are Hermitian, except for commutator in the associativity rule, which is anti-Hermitian and thus results in a minus sign compared to (6) Given a basis X_a we find from eq. (6) the matrix elements

$$\langle X_a|\mathrm{ad}_{H_i}|X_b\rangle = \langle X_a|[H_i, X_b]\rangle = i f_{ibc}\langle X_a|X_c\rangle. \tag{12}$$

In order to find common eigenvectors for ad_{H_i} we need in general a vector space over the complex numbers. From the start L itself is a real vector space, i.e L consists of vectors of the form $X = c_a X_a$, where c_a are real numbers (else X will not be Hermitian). However, we now extend L to be a complex vector space simply by allowing the constants c_a to be complex numbers. From the point of view of vector spaces this extension is trivial and similar to what we just did above if the vector space V started out being a real vector space. But in this way we also extend our real Lie algebra to a complex Lie algebra and the classification of the Lie algebras as described in the Zee book is the classification of the complex Lie algebras. How to get back from the complex Lie algebras to real Lie algebras and the corresponding Lie groups is described in "addednotes4".

On this complex vector space we keep the definition (11) of the scalar product. The only difference is now that $X^\dagger$ is no longer equal to X if some of the coefficients multiplying the basis vectors X_a (of Hermitian matrices) are complex. One can check from (12) that ad_H is an Hermitian matrix with this scalar product. It also follows directly from the associativity of the scalar product as formulated in (11): the definition of a Hermitian conjugated operator is

$$\langle Y|\hat{O}(X)\rangle = \langle \hat{O}^\dagger(Y)|X\rangle \quad \forall\, X, Y \tag{13}$$

and we have for $\hat{O} = \mathrm{ad}_H$ from (11) since H is Hermitian

$$\langle Y|\mathrm{ad}_H(X)\rangle = \langle Y|[H,X]\rangle = \langle [H,Y]|X\rangle, \quad \text{i.e.} \quad (\mathrm{ad}_H)^\dagger(Y) = \mathrm{ad}_H(Y). \tag{14}$$

Once we have extended our L to a complex Lie algebra, the vector space on which the ad_{H_i} acts is complex, and we can find the common eigenvectors for ad_{H_i}. Let us call the eigenvectors with nonzero weights $|E_\alpha\rangle$ or just E_α, E_α denoting a complex vector in the L. Thus eq. (10) reads in the adjoint representation

$$\mathrm{ad}_{H_i}|E_\alpha\rangle = \mathrm{ad}_{H_i} E_\alpha = [H_i, E_\alpha] = \alpha_i E_\alpha \tag{15}$$

We thus see that the roots α of the Lie algebra L are just the nonzero weights in the adjoint representation. We also know that each of the eigenvector spaces corresponding to a root α is one-dimensional, just spanned by the vector E_α, as explained in the Zee book. However, 0 is also a weight in the adjoint representation and the corresponding eigenvector space is ℓ dimensional as is clear from eq. (5).

Recall from the Zee book that the whole classification of simple complex Lie algebras was based on the following properties of the roots

$$2\,\frac{(\alpha,\beta)}{(\alpha,\alpha)} = q - p \tag{16}$$

where p, q are nonnegative integers such that $\gamma(k) = \beta + k\alpha$ is also a root for integer $k \in [-q, p]$ and not a root for any other integer k. However, the proof of this relation is valid not only if β is a root, but also if it is a weight w in a representation of the Lie algebra (recall that both β and w are ℓ-dimensional real vectors in $\mathbb{E}$, so it makes sense to replace β by w in (16)). We have

Theorem. *If w is a weight and α a root there exists nonnegative integers q, p such that $w(k) = w + k\alpha$ is a weight if the integer $k \in [-q, p]$ and not a* *weight for any other integers k and*

$$2\,\frac{(\alpha,w)}{(\alpha,\alpha)}=q-p \tag{17}$$

We call the set of weights $w(k)=w+k\alpha$, $k\in[-q,p]$ the α-string of weights through w.

The reason that everything is valid with β replaced by w is that E_α acts as lowering and raising operators in the same way in the representation given by X_a as do ad_{E_α} in the adjoint representation: assume that w is a weight, i.e. there exists a vector (or several vectors if the weight w is degenerate, in which case we choose one of them) v such that $H_i|v\rangle = w_i|v\rangle$, i.e. $|v\rangle$ is a weight state. Then $E_\alpha|v\rangle$ is a weight state with eigenvalues $w_i+\alpha_i$ (or zero) since

$$H_i(E_\alpha|v\rangle)=([H_i,E_\alpha]+E_\alpha H_i)|v\rangle=(\alpha_i E_\alpha+E_\alpha H_i)|v\rangle=(\alpha_i+w_i)(E_\alpha|v\rangle).$$

If we introduce the somewhat sloppy notation $|v\rangle = |w\rangle$, again with the understanding that there can be several vectors $|w\rangle$ for the same weight w if w is a degenerate set of eigenvalues, we can write

$$E_\alpha|w\rangle=|w+\alpha\rangle,\quad E_\alpha|w\rangle=0\quad\text{if}\quad w+\alpha\quad\text{is not a weight.} \tag{18}$$

Recall that while the weight w is a collection of ℓ eigenvalues of the H_i, and can be viewed as a vector in $\mathbb{E}$ in the same way as the roots α, the corresponding common eigenvector $|w\rangle$ lives in the vector space of the representation where H_i and E_α act.

We say that w is a **highest weight** in the representation if $w+\alpha$ is *not* a weight for all $\alpha\in\Delta$, i.e.

$$E_\alpha|w\rangle=0\quad\forall\,\alpha\in\Delta, \tag{19}$$

The former introduced so-called maximal root is an example of a highest weight in the adjoint representation. One can show that for a highest weight w, the eigenvalue space is non-degenerate, i.e. $|w\rangle$ is unique (up to a constant of proportionality). One calls $|w\rangle$ a **maximal vector** or a **highest weight state**. By repeated action with lowering operators $E_{-\alpha^{(i)}}$ on the state $|w\rangle$, where w is a highest weight, we can now construct the entire representation. It is seen that this is a generalization of the construction of the spin j representation in the case of $SU(2)$, starting from the state $|j,j\rangle$ and acting with the lowering operators J_- to create $|j,j-1\rangle,\ldots|j,-j\rangle$. It is however quite a lot more complicated in general since we have now ℓ independent lowering operators $E_{-\alpha^{(i)}}$. In the Zee book there is a number of examples of this procedure in the next simplest case, namely $SU(3)$, where one has two independent lowering operators $E_{\alpha^{(i)}}$. Since one can obtain all the other weights of the representation by repeated application of the $E_{\alpha^{(i)}}$, $\alpha^{(i)} \in \Delta$, all weights μ in the representation has the form

$$\mu = w - \sum_{\alpha^{(i)} \in \Delta} k_i \alpha^{(i)}, \quad k_i \text{ nonnegative integers.} \tag{20}$$

Denote the set of weights obtained in this way by $\Pi(w)$. Thus we have a **partial ordering of the set $\mathbf{\Pi(w)}$** in the same way as we had a partial ordering of the roots $\alpha \in \Phi$, by saying that $\mu \succ \mu'$ is $\mu - \mu'$ is a sum of simple roots with positive coefficients. With this partial ordering the term highest weight for w makes sense since we have

$$\mu \in \Pi(w) \;\wedge\; \mu \neq w \implies w \succ \mu. \tag{21}$$

Eq. (19) implies that if w is a highest weight then the $p = 0$ in (16) for all $\alpha \in \Delta$ and we can write

$$w \text{ highest weight} \implies \frac{2(\alpha^{(i)}, w)}{(\alpha^{(i)}, \alpha^{(i)})} = q^{(i)} \geq 0 \quad \alpha^{(i)} \in \Delta, \quad i = 1, \ldots, \ell. \tag{22}$$

Since the simple roots $\alpha^{(i)}$ are linear independent and span $\mathbb{E}$ a choice of $q^{(i)}$ completely determines w and every choice $q^{(i)}$ (not identical zero) corresponds to a highest weight w in a representation. One has the following fundamental theorem of representation theory

Theorem *There is a one to one correspondence between highest weights (characterized by the set of ℓ nonnegative integers $q^{(i)}$), and the irreducible representations of the simple Lie algebra L (characterized by the root system Φ with base Δ).*

It is now natural to introduce special highest weights $w^{(i)}$ where $q^{(j)} = 0$ except for $j = i$, i.e. where

$$\frac{2(\alpha^{(j)}, w^{(i)})}{(\alpha^{(j)}, \alpha^{(j)})} = \delta_{ij}. \tag{23}$$

It is seen that if we view $\alpha^{(j)}$ as ℓ basis vectors on $\mathbb{E}$ then $w^{(i)}$ are ℓ **dual** basis vectors on $\mathbb{E}$ (except for normalization). The weights $w^{(i)}$ are called the **fundamental weights** relative to Φ, Δ, and correspond to ℓ irrreducible representations of L which are called the **fundamental representations**. Now any highest weight w can be written as a unique sum of $w^{(i)}$:

$$w = \sum_{i=1}^{\ell} q^{(i)} w^{(i)}. \tag{24}$$

We can now build the representation corresponding to the highest weight w from the tensor product of $q^{(1)}$ representations with highest weight $w^{(1)}$, $q^{(2)}$ representations with highest weight $w^{(2)}$ etc, again just as we built the spin $n/2$ representation of $SU(2)$ from the tensor product of n spin $1/2$ representation. As already mentioned, the construction is more complicated since we have many different lowering operators $E_{\alpha^{(i)}}$.

To be slightly more explicit about the tensor product construction, recall that if we have representations $X_a^{(1)}$ and $X_a^{(2)}$ of the Lie algebra, acting on vector spaces $V^{(1)}$ and $V^{(2)}$, then the tensor product representation acts on $V^{(1)} \otimes V^{(2)}$ in the following way

$$X_a := X_a^{(1)} \otimes I^{(2)} + I^{(1)} \otimes X_a^{(2)}, \tag{25}$$

$$X_a(|v^{(1)}\rangle \otimes |v^{(2)}\rangle = (X_a^{(1)}|v^{(1)}\rangle) \otimes |v^{(2)}\rangle + |v^{(1)}\rangle \otimes (X_a^{(2)}|v^{(2)}\rangle). \tag{26}$$

In particular, the Cartan subalgebra on the tensor product will be

$$H_i = H_i^{(1)} \otimes I^{(2)} + I^{(1)} \otimes H_i^{(2)} \tag{27}$$

and *if $w(1)$ is a weight from the representation on $V^{(1)}$ and $w(2)$ is a weight from the representation on $V^{(2)}$ then $w(1)+w(2)$ will be a weight in the tensor product representation* since

$$H_i(|w(1)\rangle \otimes |w(2)\rangle) = (w_i(1) + w_i(2))(|w(1)\rangle \otimes |w(2)\rangle. \tag{28}$$

It now follows immediately that if $w(1)$ is a highest weight in the representation on $V^{(1)}$ and $w(2)$ is a highest weight in the representation on $V^{(2)}$ then $w(1) + w(2)$ is a highest weight for the tensor product representation, since all E_{α^i} defined on the tensor product will annihilate $|w(1)\rangle \otimes |w(2)\rangle$. However, even if the representations on $V^{(1)}$ and $V^{(2)}$ are the irreducible representations corresponding to the weights $w(1)$ and $w(2)$, the tensor product representation will in general not be irreducible, but it can be decomposed in a direct sum of irreducible representations, and one (and only one) of these irreducible representations will have highest weight $w(1) + w(2)$. These considerations can clearly be generalized to the more complicated tensor product mentioned below eq. (24), where the vector space is

$$V = \underbrace{V^{(1)} \otimes \cdots \otimes V^{(1)}}_{q^{(1)} \text{ factors}} \otimes \cdots \otimes \underbrace{V^{(\ell)} \otimes \cdots \otimes V^{(\ell)}}_{q^{(\ell)} \text{ factors}} \tag{29}$$

and where we have the highest weight $q^{(1)}w^{(1)}+\cdots+q^{(\ell)}w^{(\ell)}$, but as remarked above, the irreducible representation corresponding to this highest weight is only one of many irreducible components of the reducible tensor product representation. It is (in principle) explicitly constructed by acting with the various $E_{-\alpha^{(i)}}$ on the highest weight vector corresponding to weight w in (24), which in the tensor product is

$$|w\rangle = \underbrace{|w^{(1)}\rangle \otimes \cdots \otimes |w^{(1)}\rangle}_{q^{(1)} \text{ factors}} \otimes \cdots \otimes \underbrace{|w^{(\ell)}\rangle \otimes \cdots \otimes |w^{(\ell)}\rangle}_{q^{(\ell)} \text{ factors}} \tag{30}$$

A few general properties of representations

We have seen that we can associate the representations of a Lie algebra with highest weights. Given a highest weight w and acting with $E_{-\alpha^{(i)}}$, $\alpha^{(i)} \in \Delta$ we create additional weights $w - \alpha^{(i)}$ if $E_{-\alpha^{(i)}}|w\rangle \neq 0$. As stated above the repeated action with the $E_{-\alpha^{(i)}}$ led to the complete set of weights associated with the representation corresponding to w, and we denoted this set by $\Pi(w)$. This set is important also in physics since the actual values of the weights often correspond to conserved quantities of the given physical system for which the Lie group in question is a symmetry group.

Given a highest weight w (i.e. an irreducible representation) we are (among other things) interested in *(1) finding the set* $\Pi(w)$ and *(2) finding the dimension* $d(w)$ *of the representation corresponding to* w. If any of the weights $\mu \in \Pi(w)$ are degenerate then $\mathrm{card}(\Pi(w)) < d(w)$. Thus we are also interested in *(3) finding the degeneracy of a given* $\mu \in \Pi(w)$. If we denote the degeneracy of μ by $m_w(\mu)$, $m_w(\mu)$ is by definition the dimension of the vector space of the common eigenvectors of H_i corresponding the eigenvalues μ_i. Since all the common eigenvectors of the H_i span the whole vector space we have

$$d(w) = \sum_{\mu \in \Pi(w)} m_w(\mu). \tag{31}$$

Finally, since every highest weight can be built from the fundamental weights and the corresponding representation appears in a suitable tensor product of fundamental representations we are also interested in *(4) understanding the fundamental representations* as well as *(5) the decomposition of tensor productions in irreducible representations.*

There are some general statements about (1)-(5) which we will mention below for completeness. But in general these formulas which cover all cases are quite difficult to use and in practise one uses more specialized techniques for the various set of groups like $SU(N)$ and $SO(N)$. In the next section we will look at the $SU(N)$ group, which is the easiest general set of groups to deal with, and also discuss some special representations (the so-callled **spinor representations**) of $SO(N)$.

(1) *Some general features of* $\Pi(w)$.

First we choose a w. The possible w are determined from the simple roots $\alpha^{(i)}$ from (23) and (24). The general form of $\mu \in \Pi(w)$ is given by (20), but we need to determine which k_i appear. These are in principle determined by using that for each $\mu \in \Pi(w)$ and each $\alpha \in \Phi$ there exists according (17) an α string of weights through μ of a certain length (which can be zero). In particular, since the Weyl group $\mathcal{W}$ of reflections in the hyper-planes orthogonal to $\alpha \in \Phi$ acts as follows on μ (see (2))

$$\sigma_\alpha(\mu) = \mu - \frac{2(\mu, \alpha)}{(\alpha, \alpha)}\, \alpha = \mu - \langle \mu, \alpha \rangle\, \alpha \tag{32}$$

it follows from (17) that the action of the Weyl group will leave $\Pi(w)$ invariant (in the same way as we mentioned earlier that it leave Φ invariant). In fact the Weyl group do not only permute the different weights $\mu \in \Phi(w)$, but in addition it preserves the degeneracy of the weights:

$$m_w(\mu) = m_w(\sigma_\alpha(\mu)). \tag{33}$$

The geometric interpretation of the Weyl group action as well as (33) can sometimes be useful when actually trying to determine $\Pi(w)$, starting from w.

(2) *Determination of* $d(w)$.

There exists a general formula, due to **Weyl**, for $d(w)$. Define

$$\delta = \sum_{i=1}^{\ell} w^{(i)} \tag{34}$$

From the definition of the fundamental weight one has $\sigma_{\alpha^{(i)}}(\delta) = \delta - \alpha^{(i)}$ and one can use this to prove that the following inequality (which we will need below)

$$(\mu + \delta, \mu + \delta) < (w + \delta, w + \delta), \quad \mu \in \Pi(w) \ \wedge\ \mu \neq w. \tag{35}$$

One now has for the highest weight w:

$$d(w) = \frac{\prod_{\alpha \succ 0}(w + \delta, \alpha)}{\prod_{\alpha \succ 0}(\delta, \alpha)} = \frac{\prod_{\alpha \succ 0} \langle w + \delta, \alpha \rangle}{\prod_{\alpha \succ 0} \langle \delta, \alpha \rangle} \tag{36}$$

where the last equality is obtained by multiplying both denominator and numerator by $\prod_{\alpha\succ 0} 2/(\alpha,\alpha)$ and using the definition (2) of $\langle\mu,\alpha\rangle$. By (17) the last formula has the virtue of only involving integers. Finally, recall that the positive roots can be written as $\alpha = \sum_{i=1}^{\ell} k_i \alpha^{(i)}$, where k_i are nonnegative integers and that $w = \sum_{i=1}^{\ell} q^{(i)} w^{(i)}$. Thus we have

$$\langle w+\delta,\alpha\rangle = \sum_{i=1}^{\ell} k_i(q^{(i)}+1), \qquad \langle\delta,\alpha\rangle = \sum_{i=1}^{\ell} k_i, \tag{37}$$

and in order to apply (36) we have in principle only to calculate the integers k_i for the positive roots. We will state the explicit form of $d(w)$ below (formula (64)) in the case of an irreducible $SU(N)$ representation.

(3) *The multiplicity* $m_w(\mu)$

We know already that $m_w(w) = 1$ and it is not difficult to prove that

$$m_w(w - k\alpha) = 1, \quad \alpha \in \Delta, \quad 0 \le k \le \langle w,\alpha\rangle. \tag{38}$$

The general multiplicity can be found recursively by **Freudenthal's formula**:

$$\big((w+\delta, w+\delta) - (\mu+\delta,\mu+\delta)\big)\; m_w(\mu) = 2\sum_{\alpha\succ 0}\sum_{i=1}^{\infty} m_w(\mu+i\alpha)\,(\mu+i\alpha,\alpha). \tag{39}$$

Due to formula (35) one can show that this is actually a well defined recursive formula, allowing us to recursively determine $m_w(\mu)$, starting out with $m_w(w) = 1$. From the point of efficiency it is convenient to make use of (33), and one can make computer programs which generate the number $m_w(\mu)$.

There also exists a closed formula due to **Konstant**. In order to present the formula we introduce the so-called *Konstant function* $p(\mu)$, where the argument μ can be any weight, belonging to any representation. $p(\mu)$ is the number of sets of nonnegative integers $\{k_\alpha, \alpha \succ 0\}$ for which $\mu + \sum_{\alpha\succ 0} k_\alpha \alpha = 0$. $p(\mu)$ is clearly zero unless $\mu \in \Phi$. Finally let us introduce *the length* $n(\sigma)$ of a reflection $\sigma \in \mathcal{W}$. Every $\sigma \in \mathcal{W}$ can be written as products of simple reflections, i.e. reflections $\sigma_\alpha, \alpha \in \Delta$. If we write $\sigma = \sigma_{\alpha^{(i_1)}} \cdots \sigma_{\alpha^{(i_t)}}$, where t is minimal, we define $n(\sigma) = t$. One can show that $n(\sigma) =$ number of positive roots α for which $\sigma(\alpha) \prec 0$. With these definitions we can state Konstant's formula for the multiplicity

$$m_w(\mu) = \sum_{\sigma\in\mathcal{W}} (-1)^{n(\sigma)} p\big(\mu+\delta-\sigma(w+\delta)\big). \tag{40}$$

It is not easy to use.

(4) *Understanding the fundamental representations corresponding to* $w^{(i)}$.

We will postpone this to the next two sections for the specific cases of the Lie algebras of $SU(N)$ and $SO(N)$.

(5) *Understanding the tensor decomposition of two irreducible representations*

Let w_1 and w_2 be two highest weights and denote the vector spaces corresponding to the irreducible representations by V_{w_1} and V_{w_2}. The tensor product representation acts on $V_{w_1} \otimes V_{w_2}$ and can be decomposed in a direct sum of irreducible representations, characterized by highest weights ν and we have

$$V_{w_1} \otimes V_{w_2} = \oplus_\nu V_\nu \tag{41}$$

Which highest weights ν appear in the sum and how many times? Again there exists a general formula which in principle answers this equation, but which, like Konstant's formula, is not easy to use. It is called **Steinberg's formula** and it states that the number of times, $N(\nu)$, that ν appears in the decomposition (41) is

$$N(\nu) = \sum_{\sigma \in \mathcal{W}} \sum_{\tau \in \mathcal{W}} (-1)^{n(\sigma\tau)} p\big(\nu + 2\delta - \sigma(w_1 + \delta) - \tau(w_2 + \delta)\big). \tag{42}$$

A simpler result, also of interest, is the following. We have already seen that the possible weights μ of the (reducible) tensor product representation are related to the weights $\mu_1 \in \Pi(w_1)$ and $\mu_2 \in \Pi(w_2)$ by

$$\mu = \mu_1 + \mu_2. \tag{43}$$

Furthermore we showed that $\mu = w_1 + w_2$ was a highest weight of multiplicity 1, so $V_{w_1+w_2}$ will appear in the sum (41) exactly one time (i.e. $N(w_1+w_2) = 1$ in (42)). An arbitrary weight μ of the tensor product representation will in general be degenerate and the dimension $m(\mu)$ of the eigenspace corresponding to μ will be given by

$$m(\mu) = \sum_{\mu_1+\mu_2=\mu} m_{w_1}(\mu_1) m_{w_2}(\mu_2) \tag{44}$$

where the summation is over $\mu_1 \in \Pi(w_1)$ and $\mu_2 \in \Pi(w_2)$. It is a good exercise to check and understand (44) in the case of $SU(2)$ where $w_1 = j_1$ and $w_2 = j_2$ and weights μ_1, μ_2 will then be the m_1, m_2 in the allowed range for j_1 and j_2. In this case $m_{j_1}(m_1) = 1$ when m_1 is in the allowed range and zero outside and similarly for m_2, j_2.

The representations of the Lie algebra of SU(N)

Let us illustrate the general concepts outlined above for $G = SU(N)$. In chapter VI in the Zee book we have explicit expressions for the Cartan matrices $H_i^{(d)}$ in the N-dimensional defining representation of $SU(N)$ (see VI (25)). Here the superscript d stands for "defining". The $\ell = N-1$ Cartan matrices are already in diagonal form, and thus we can directly read off the N weights $w^{(d)}(k)$, $k = 1, \ldots, N$ of the defining representation from the diagonal elements (VI (26)):

$$w_i^{(d)}(k) = (H_i^{(d)})_{kk} \tag{45}$$

The roots connect different weights by application of lowering and raising operators. Thus the $\alpha(k,l)$ are found as the difference between the weights

$$\alpha(k,l) = w^{(d)}(k) - w^{(d)}(l), \quad k,l = 1, \ldots, N, \quad k \neq l. \tag{46}$$

This gives the $N(N-1)$ roots $\alpha_j(k,l)$, $j = 1, \ldots, \ell$ which can formally be viewed as vectors in $\mathbb{E} = \mathbb{R}^\ell$. They constitute Φ. The number is correct since for a Lie algebra of dimension n, the number of roots are $n - \ell$ and for the group $SU(N)$ we have $n = N^2 - 1$ and $\ell = N - 1$. The simple roots are

$$\alpha^{(i)} = \alpha(i, i+1), \quad i = 1, \ldots \ell \tag{47}$$

and they constitute Δ. The defining representation is irreducible and we will show below that the corresponding highest weight (relative to Δ) is $w^{(d)}(1)$. In accordance with the general notation we then call the set of weights of the defining representation $\Pi(w^{(d)}(1))$. The elements are vectors in $\mathbb{E}$. We now the standard partial ordering on $\Pi(w^{(d)}(1))$ defined by Δ, the simple roots. In fact it is in this case a complete ordering

$$w^{(d)}(1) \succ w^{(d)}(2) \succ \cdots \succ w^{(d)}(N) \tag{48}$$

since we have

$$w^{(d)}(k) - w^{(d)}(l) = \sum_{i=k}^{l-1} \alpha^{(i)} \succ 0 \quad \text{for } l > k. \tag{49}$$

We can now define the fundamental weights $w^{(j)}$ by

$$\frac{2(w^{(j)}, \alpha^{(i)})}{(\alpha^{(i)}, \alpha^{(i)})} = \delta_{ij} \tag{50}$$

It is easy to check that

$$w^{(j)} = \sum_{i=1}^{j} w^{(d)}(i). \tag{51}$$

Thus in particular $w^{(d)}(1) = w^{(1)}$ from which we conclude that the defining representation is one of fundamental representations and of its N weights $w^{(d)}(k)$, $k = 1, \ldots, N$, the highest weight is $w^{(d)}(1)$.

Which representations correspond to the other $\ell - 1$ fundamental weights $w^{(j)}$, $j = 2, \ldots, \ell$? It turns out that they can be represented as certain tensor products of the defining representation, i.e. the representation with highest weight $w^{(1)}$. Let us consider the tensor product of of k versions of the defining representation. For a moment we work with the group rather than the algebra. The tensor product space $W(k) = V \otimes \cdots \otimes V$ (k times, $V = \mathbb{C}^N$) has vectors (tensors) with $k \cdot N$ components $T^{i_1 \cdots i_k}$ (i_j takes values $1, \ldots, N$ for $j = 1, \ldots, k$). If $U \in SU(N)$ is a $N \times N$ matrix in the defining representation, then the action of $SU(N)$ on vectors in W is

$$T'^{i_1 \cdots i_k} = U^{i_1}_{j_1} \cdots U^{i_k}_{j_k} \, T^{j_1 \cdots j_k} \tag{52}$$

We have not specified the basis of V in which the matrix elements U^i_j are calculated. It is convenient below to use as basis the N eigenvectors of the commuting Cartan matrices H_i, i.e. the vectors we have denoted $|w^{(d)}(j)\rangle$ satisfying $H_i|w^{(d)}(j)\rangle = w^{(d)}_i(j)|w^{(d)}(j)\rangle$. Thus a basis for $W(k)$ will be the $k \times N$ vectors

$$|w^{(d)}(j_1)\rangle \otimes |w^{(d)}(j_2)\rangle \otimes \cdots \otimes |w^{(d)}(j_k)\rangle \tag{53}$$

As we have already noted these are precisely the weight states of the tensor product and the state (53) has weight

$$w = w^{(d)}(j_1) + \cdots w^{(d)}(j_k). \tag{54}$$

Thus in general there will be many states with the same weight since any permutation of the indices j_i in formula (53) will result the same weight w in (54) but in general in different vectors in $W(k)$. A tensor is really a vector in $W(k)$:

$$T = \sum_{j_1, \ldots, j_k} T^{j_1 \cdots j_k} |w^{(d)}(j_1)\rangle \otimes |w^{(d)}(j_2)\rangle \otimes \cdots \otimes |w^{(d)}(j_k)\rangle. \tag{55}$$

and in general T will not be a vector with a well defined weight. However, *if it is*, then any additional symmetry requirement on the indices will *not* change the weight since it only involves permutations of the basic vectors and does not affect the sum (54).

This tensor product provides us with a representation of $SU(N)$ which is reducible, but as explained in the Zee book any symmetry we impose on the indices commutes with the group operation. Thus, if the symmetry imposed on the indices is such that that the corresponding tensors form a subspace of $W(k)$ then (52) applied to this subspace of $W(k)$ also provides us with a representation of $SU(N)$. We now consider the subspace $A(W(k))$ consisting of tensors which are antisymmetric in all indices (it *is* clearly a subspace: the sum of two antisymmetric tensors is still antisymmetric). The dimension of this subspace is

$$\mathrm{dim} A(W(k)) = \binom{N}{k}, \tag{56}$$

the number of antisymmetric tensors with k indices where the indices can take N different values. Recall from the Zee book that for $k = 2, 3$ these tensors are

$$\frac{1}{2}\Big(T^{ij} - T^{ji}\Big), \quad \frac{1}{6}\Big(T^{ijk} - T^{ikj} + T^{kij} - T^{kji} + T^{jki} - T^{jik}\Big). \tag{57}$$

The corresponding representation is irreducible with highest weight $w^{(k)}$. The reason for this is that (as we have seen) the weights are additive on tensor products. Thus the weights which can occur in the representation corresponding to $A(W(k))$ are the sum of k of the N weights $w^{(d)}(i)$, $i = 1, \ldots, N$. Because of the antisymmetry only different $w^{(d)}(i)$ can occur in the sum, and it follows from (48) that the highest such weight is the $w^{(k)}$ defined by (51).

As we have seen, the general irreducible representation is contained in the tensor product of the fundamental representations as described below eq. (24). We will now outline how this works in the case of $SU(N)$, where we have just determined the fundamental representations. Not surprisingly it involves the representations of the symmetric group S_N since the various irreducible components of a tensor representation involves imposing symmetry restrictions on the indices. The representations of the symmetric group is described by so-called Young tableaux. We have no space here to enter into the general description of the this theory and will only mention what is need for the representations of $SU(N)$.

Consider a tensor like in (55) with k indices which can take values $1, \ldots, N$. We have considered until now the situation where we either symmetrize or antisymmetrize with respect to the indices, but one can clearly have more complicated mixed symmetries. We can describe these as follows. Divide k in l groups $l < N$, of length $a_1 \geq a_2 \geq \cdots \geq a_l$. Associated with these numbers is a Young tableau, a collection k of boxes arranged in l rows, where the top row has a_1 boxes, the second row from the top has a_2 boxes, etc, and

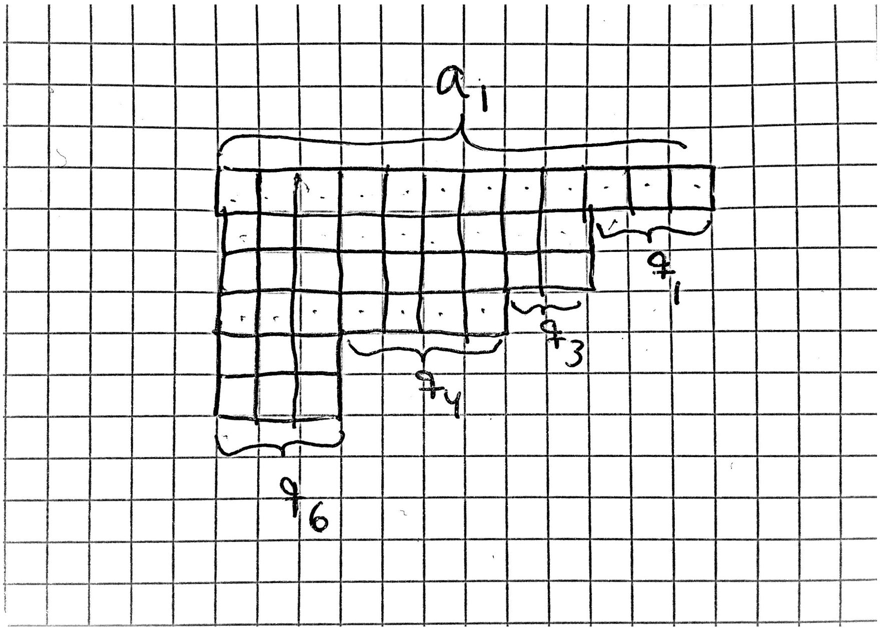

Figure 1: A Young tableau with $a_1 = 12, a_2 = 9, a_3 = 9, a_4 = 7, a_5 = 3, a_6 = 3$ and $q_1 = 3, q_2 = 0, q_3 = 2, q_4 = 4, q_5 = 0, q_6 = 3$.

the l^{th} row has a_l boxes. If we align the rows to the left (see fig. 1), the first column from the left will contain l boxes and the number will never increase moving right. In total there are then a_1 columns. Rather than labelling the Young tableau by the integers $a_1, \ldots, a_l$, it is often convenient to use

$$q^{(i)} = a_i - a_{i+1}, \quad i = 1, \ldots l, \quad a_{l+1} = 0, \tag{58}$$

The Young tableau will have $q^{(1)}$ columns to the very right with only one box, next $q^{(2)}$ columns with two boxes etc, ending with $q^{(l)}$ columns with l boxes to the very left. Some of the $q^{(i)}$ can of course be zero (see fig. (1)).

Corresponding to the Young tableau we now impose the following symmetry structure on the k indices of a tensor like (55): we place the indices in the Young table. The first a_1 indices are symmetrized, the next a_2 are symmetrized independently etc.. After that, we antisymmetrize the indices in the first column, the indices in the second column etc.. Note that in this way we partially spoil the symmetrization already performed. These symmetry constraints on the tensor structure commute with the group operators and define a subspace in $W(k)$ invariant under the action of $SU(N)$. We thus have a representation of $SU(N)$ on this subspace. The simplest example is when we have a single column with k boxes. The vector space will then be $A(W(k))$ and the representation will precisely be the k^{th} fundamental representation corresponding to highest weight $w^{(k)}$, $1 \leq k < N$. For $k = N$ we have to antisymmetrize $T^{i_1 \cdots i_N}$, but since the indices i only can take values up to N for $SU(N)$

$$T^{i_1 \cdots i_N} \propto \epsilon^{i_1 \cdots i_N}, \tag{59}$$

and is thus a singlet which we will ignore. If the column has more than N boxes we will get zero. We can thus ignore Young tableaux with columns with more than $N - 1 = \ell$ boxes.

At this point the reader might wonder what has happened to the tensors with lower indices introduced in the Zee book. In fact they *are* included in this treatment. Recall that we can always use the $\epsilon^{i_1 \dots i_N}$ symbol to raise lower indices and in this way we do not really need lower indices, although they can be very convenient. As an example consider the tensor T_i, the cousin to T^i. By using $\epsilon^{i_1 \dots i_N}$ we see that T_i corresponds to a the totally antisymmetric tensor $T^{i_1 \dots i_{N-1}}$ in $A(W(N-1))$. The representation T_i is thus also a fundamental representation and its highest weight is $w^{(N-1)}$ and actually we have $w^{(N-1)} = -w^{(1)}$, the reason being that the Lie algebra of $SU(N)$ consists of Hermitian *traceless* matrices, and thus

$$\operatorname{tr} H_i = \sum_{j=1}^{N} w_i^{(d)}(j) = 0 \quad \forall\, i, \tag{60}$$

which implies $w^{(N-1)} = -w^{(1)}$ by using (51). The relation $w^{(N-1)} = -w^{(1)}$ extends to all the other weights of the representations: $\Pi(w^{(N-1)}) = -\Pi(w^{(1)})$.

If we now consider a Young tableau defined by the numbers $q^{(i)}$ in (58) it is seen that the symmetry imposed on the tensors matches precisely what is expected from (24) with the same $q^{(i)}$. According to (24) we have $q^{(i)}$ copies of the fundamental representation with weight $w^{(i)}$, but this is precisely the $q^{(i)}$ columns of length i in the Young tableau. The symmetrization performed in the row direction is needed in order to obtain a well defined subspace in $W(k)$. One can also check that the highest weight of the representation comes out right: since the weights are additive on the tensor products, they are just certain sums of the weights in the defining representation. Consider the vector defined by (53). It is an eigenvector of the H_i in the tensor product representation and it has weight

$$w = w^{(d)}(j_1) + \cdots w^{(d)}(j_k). \tag{61}$$

The weights of the defining representation are ordered according to (48). Let us now take an arbitrary set $\{j_1, \dots, j_k\}$ and form the vector (55) with no summation. This is a state T_0 with weight (61). Now put the j_k into the Young tableau in question and perform the required symmetrization. This results in a new state vector T which might very well be zero, but if it is not it will still have the same weight (61) since we are just permuting the basic vectors, as already discussed. What is the highest weight we can obtain for the given Young tableau? Since $w^{(d)}(1)$ is the highest weight, we want to have as many $j_i = 1$ as possible. We thus fill the top row with 1s, since the top row is the longest. After that we can not put more $j_i = 1$ in the tableau because of the antisymmetrization of the columns. The next highest weight is $w^{(d)}(2)$ so we fill row two which is now the longest row left with $j_i = 2$, etc. The corresponding T will then be a vector in $W(K)$

$$T = \underbrace{|w^{(d)}(1)\rangle \otimes |w^{(d)}(1)\rangle}_{a_1 \text{ factors}} \otimes \cdots \otimes \underbrace{|w^{(d)}(l)\rangle \otimes \cdots \otimes |w^{(d)}(l)\rangle}_{a_l \text{ factors}}, \tag{62}$$

which according to (61) has the weight

$$w = \sum_{j=1}^{l} a_j w^{(d)}(j) = \sum_{j=1}^{l} q^{(i)} w^{(i)} \quad l \leq \ell \tag{63}$$

Of course T does not have the correct symmetry. It is by construction symmetric in the row indices, but we have to antisymmetrize with respect to the columns in the Young tableu, but this antisymmetrizing will not change the weight as already discussed. We have thus constructed the highest state vector corresponding to the irreducible representation characterized by the numbers $q^{(i)}$, $i = 1, \ldots, \ell$.

We have seen that the irreducible representations of $SU(N)$, characterized by the integers $q^{(i)}$ in (24), can be associated with a Young tableau defined by the same integers. It turns out that one can use the Young tableaux to answer in an easier way a number of the questions for which we gave painfully complicated general answered above: (1) what is the set of weights $\Pi(w)$ corresponding to the irreducible $SU(N)$ with highest weight $w = \sum_i q^{(i)} w^{(i)}$, (2) what is the dimension an irreducible representation of $SU(N)$ corresponding to the integer $q^{(i)}$, (3) what are the multiplicities of the weights in $\Pi(w)$, (4) what is the decomposition of the tensor product of two irreducible $SU(N)$ representations in irreducible representations and (5) how do we find the $SU(N-1)$ subgroups of $SU(N)$. Unfortunate we have no space for going into a description of the methods and formulas, so let us just finish giving the explicit formula for the dimension of the irreducible $SU(N)$ representation corresponding to the highest weight $w = \sum_i q^{(i)} w^{(i)}$:

$$d(w) = \prod_{i=1}^{\ell} \frac{1}{i!} \prod_{j=i}^{\ell} \Big(\sum_{k=i}^{\ell} (q^{(k)} + 1) \Big) \tag{64}$$

This is actually Weyl's formula (36) in the case of $SU(N)$, but it is easier to prove in this case using Young tableaux.

Some remarks about SO(N) representations

The representations of $SO(N)$ contain the so-called **spinor representation** which is an interesting generalization of the fundamental spin 1/2 representation of $SU(2)$. For $SO(3)$, the spin 1/2 representation is strictly speaking not a representation of the $SO(3)$ group but of its *covering group*, which happens to be $SU(2)$, but if we only discuss the representations of the complexified Lie algebras of the simple Lie group in question and its covering group, there will be no difference (see the discussion in addednotes4). Likewise the spinor representation of $SO(N)$ is not really a representation of the group $SO(N)$ but of its covering group. The covering group of $SO(N)$ is denoted **spin(N)**. The purpose of this section is to describe how the spinor representation for $SO(N)$ appears in the general description of representations of Lie algebras outlined above. There are small differences between N even and N odd and due to lack of space we here restrict ourselves to the case of odd N. We define

$$N = 2\ell + 1, \quad \ell = 1, 2, \ldots \tag{65}$$

The *defining* representation of (the Lie algebra of) $SO(2\ell+1)$, as well as the corresponding weights and the roots are described in the Zee-book chapter VI.2, where also the Cartan subalgebra is provided. The dimension of the Cartan subalgegra is ℓ, while the dimension $d(w^{(d)})$ of the defining representation is $2\ell+1$. The weights are thus vectors in $\mathbb{E}^\ell$ and they are all different, i.e. there are $2\ell+1$ orthogonal weight states $|w^{(d)}(k)\rangle$ in the vector space $\mathbb{C}^{2\ell+1}$ of the defining representation . These weight states satisfy

$$H_i^{(d)} |w^{(d)}(k)\rangle = w_i(k) |w^{(d)}(k)\rangle, \quad i = 1, \ldots, \ell, \quad k = 1, \ldots, 2\ell+1. \tag{66}$$

The superscript "d" refers to the *defining* representation. The explicit form of $w_i^{(d)}(k)$ is (Zee-book VI.2 (20) and later discussion)

$$\begin{aligned} w_i^{(d)}(k) &= \delta_{ik}, & i = 1, \ldots \ell \quad k = 1, \ldots, \ell \\ w_i^{(d)}(k) &= -\delta_{i,k-\ell}, & i = 1, \ldots, \ell, \quad k = \ell+1, \ldots 2\ell \\ w_i^{(d)}(k) &= 0, & i = 1 \ldots \ell \quad k = 2\ell+1, \end{aligned} \tag{67}$$

It is convenient to introduce the following notation

$$\begin{array}{rclll} w_i^{(d)}(k) & = & +e_i(k), & i=1,\dots\ell & k=1,\dots,\ell. \\ w_i^{(d)}(k) & = & -e_i(k-\ell), & i=1,\dots\ell & k=\ell+1,\dots,2\ell \\ w_i^{(d)}(k) & = & 0 & i=1,\dots\ell & k=2\ell+1. \end{array} \tag{68}$$

The ℓ vectors $e(k)$ are thus the standard unit basis vectors for $\mathbb{E}^\ell$ and as described in the Zee-book the roots for $SO(2\ell+1)$ are given by

$$\alpha = \pm e(i) \pm e(j), \quad i \neq j, \qquad \alpha = \pm e(i), \quad i,j = 1,\dots,\ell. \tag{69}$$

Note that the roots have different lengths, either $\sqrt{2}$ or 1. A choice of simple roots, Δ, is

$$\alpha^{(i)} = e(i) - e(i+1), \quad i = 1, \ell-1, \qquad \alpha^{(\ell)} = e(\ell). \tag{70}$$

After we have made the choice of Δ we can check that the weights in the defining representation satisfy an ordering similar to (48)

$$e(1) \succ \cdots \succ e(\ell) \succ 0 \succ -e(\ell) \succ \cdots \succ -e(1), \tag{71}$$

We can also find the ℓ fundamental highest weights $w^{(j)}$, i.e. the weights which satisfy

$$\frac{2(w^{(j)}, \alpha^{(i)})}{(\alpha^{(i)}, \alpha^{(i)})} = \delta_{ij}, \tag{72}$$

and for $i = 1,\dots,\ell-1$ the construction is exactly the same as for $SU(N)$ since the roots have the same form and relation to the fundamental representation as was the case for $SU(N)$. Thus we have in analogy with (51)

$$w^{(j)} = \sum_{i=1}^{j} e(i), \quad j = 1,\dots,\ell-1. \tag{73}$$

In particular the defining representation is a fundamental highest weight representation: $w^{(d)} = w^{(1)}$. However, we cannot extend (73) to $j = \ell$. The reason is that the root $\alpha^{(\ell)}$ is of length 1 while the rest of the roots $\alpha^{(i)}$ have length $\sqrt{2}$. Instead we find

$$w^{(\ell)} = \frac{1}{2}\left(e^{(1)} + \cdots + e^{(\ell)}\right) \tag{74}$$

As for $SU(N)$ the first $\ell-1$ fundamental representations, corresponding to the highest weights $w^{(j)}$, $j = 1,\dots,\ell-1$ can be build from tensor products of the defining representation (which is the first fundamental representation corresponding to the highest weight $w^{(1)}$). We can use the same notation: call the vector space $V(w^{(d)})$ of the defining representation V (which is just $\mathbb{C}^{2\ell+1}$). Then the vector space corresponding to $w^{(k)}$ is just $A(W(k))$, the space of antisymmetric $SO(2\ell+1)$ tensors with k indices. The arguments are exactly the same as for $SU(N)$ because we have the ordering (71) of the weights in the defining representation. One can associate a corresponding Young tableau consisting of one column with k boxes to the corresponding representation, but in general it is somewhat more complicated to use the young tableaux in the case of $SO(N)$ representations with mixed symmetries because one also has to ensure that the tensors are traceless and we will not discuss this further (for the one-column tableaux it is automatically satisfied by the antisymmetry). In particular the fundamental representation corresponding to $w^{(2)} = e^{(1)} + e^{(2)}$ is the representation provided by the antisymmetric tensors with two indices, i.e. the adjoint representation (recall the Zee-book Sec. IV.1 : *The adjoint representation of* $SO(N)$). Indeed, it is easy to see, by reflecting the highest weight $e^{(1)} + e^{(2)}$ in the roots α that one generates all the roots themselves, and the roots *are* precisely the non-zero weights of the adjoint representation.

However, the representation corresponding to highest weight $w^{(\ell)}$ given by (74) is different from the other fundamental representations because of the factor one-half. Let us now find the lower weights in this representation. Recall that for $\mu \in \Pi(w)$ we have $\sigma_\alpha(\mu) \in \Pi(w)$ for all $\alpha \in \Phi$. All reflections corresponding to roots $e^{(i)} - e^{(j)}$ leave $w^{(\ell)}$ invariant since the two vectors are orthogonal in $\mathbb{E}^\ell$. On the other hand reflections in hyperplanes othogonal to the roots $e^{(i)}$ will change $w^{(\ell)}$ to $w^{(\ell)} - e^{(i)}$. Successively reflections in the various hyperplanes orthogonal to the $e^{(k)}$'s will thus generate the weights

$$w(x_1, \ldots, x_\ell) = \sum_{i=1}^{\ell} x_i e^{(i)}, \quad x_i = \pm \frac{1}{2}. \tag{75}$$

Recall that we have in general for the H_i and the raising and lowing operators E_α and weight states $|\mu\rangle$

$$H_i|\mu\rangle = \mu_i|\mu\rangle, \quad E_\alpha|\mu\rangle = N_{\alpha,\mu}|\mu+\alpha\rangle, \quad \forall\alpha \in \Phi, \quad \forall\mu \in \Pi(w), \tag{76}$$

where w is a highest weight for the given representation and $N_{\alpha,\mu}$ is non-zero if and only if $\mu + \alpha \in \Pi(w)$. In the case of $SO(2\ell+1)$ we can normalize the $E_{\pm e^{(i)}}$ such that

$$[E_{\eta e^{(i)}}, E_{\eta' e^{(j)}}] = i\, E_{\eta e^{(i)} + \eta' e^{(j)}}, \quad [E_{e^{(i)}}, E_{-e^{(i)}}] = H_i. \tag{77}$$

Using the fact that $w^{(\ell)} = w(\frac{1}{2}, \ldots, \frac{1}{2})$ is the highest weight of the representation and eqs. (76) and (77), it is not difficult to show that the weights $w(x_1, \ldots, x_\ell)$ from (75) are in fact *all* the weights in $\Pi(w^{(\ell)})$ and that

$$\begin{aligned} H_i|w(x_1, \ldots, x_\ell)\rangle &= x_i|w(x_1, \ldots, x_\ell)\rangle, & (78) \\ E_{\eta e^{(i)}}|w(..., x_i, ...)\rangle &= \frac{\delta_{-\eta, 2x_i}}{\sqrt{2}} \, \mathrm{e}^{i\phi(x_1, \ldots, x_\ell; \eta; i)} \, |w(..., \eta x_i, ...)\rangle, & (79) \end{aligned}$$

as well as

$$E_{\eta e^{(i)}} E_{\eta' e^{(j)}} + E_{\eta e^{(j)}} E_{\eta' e^{(i)}} = 0. \tag{80}$$

The phase $\mathrm{e}^{i\phi(x_1, \ldots, x_\ell; \eta, i)}$ in (79) is not determined since we have not yet fixed the relative phase between the state vectors $|w(x_1, \ldots, x_\ell)\rangle$.

The dimension of the vector space $V(w^{(\ell)})$ of the representation with highest weight $w^{(\ell)}$ is thus $d(w^{(\ell)}) = 2^\ell$, and the states $|w(x_1, \ldots, x_\ell)\rangle$ constitute an orthonormal basis for $V(w^{(\ell)})$. Let us assume for a moment that the phase $\phi(x_1, \ldots, x_\ell; \eta; i) = 0$ in (78). Then we have

$$H_i|w(..x_i..)\rangle = x_i|w(..x_i..)\rangle, \quad E_{\eta e^{(i)}}|w(..x_i..)\rangle = \frac{\delta_{-\eta, 2x_i}}{\sqrt{2}} \, |w(..\eta x_i..)\rangle, \tag{81}$$

as well as

$$[H_i, E_{\pm e^{(i)}}] = \pm E_{\pm e^{(i)}}, \quad E^2_{\pm e^{(i)}} = 0, \quad H_i^2 = \frac{1}{4} I \tag{82}$$

On the two-dimensional subspace spanned by $|w(..., x_i, ...)\rangle$ the $H_i, E_{\pm e^{(i)}}$ thus behave (except for a possible complication with $\phi(x_1, \ldots, x_\ell; \eta; i)$) exactly as the standard spin 1/2 operators on the spin states $| \pm \frac{1}{2}\rangle$:

$$s_a = \frac{1}{2}\sigma_a \quad a = 1, 2, 3, \quad s_\pm = \frac{1}{\sqrt{2}}(s_1 \pm i s_2), \tag{83}$$

where σ_a are the Pauli matrices. The $s_3, s_\pm$ precisely satisfy (82). This suggests that we view the 2^ℓ-dimensional vector space $V(w^{(\ell)})$ as a tensor product of ℓ two-dimensional spin 1/2 vector spaces which we denote $|x_i e^{(i)}\rangle$:

$$|w(x_1, \ldots, x_\ell)\rangle = |x_1 e^{(1)}\rangle \otimes \cdots \otimes |x_\ell e^{(\ell)}\rangle. \tag{84}$$

If we define $\sigma_0 = I_{2\times 2}$ a tensor product of Pauli matrices will now act on the vectors (84) as follows

$$\sigma_{a(1)} \otimes \cdots \otimes \sigma_{a(\ell)} \, |x_1 e^{(1)}\rangle \otimes \cdots \otimes |x_\ell e^{(\ell)}\rangle = \sigma_{a(1)}|x_1 e^{(1)}\rangle \otimes \cdots \otimes \sigma_{a(\ell)}|x_\ell e^{(\ell)}\rangle. \tag{85}$$

Let us introduce a final notation:

$$\sigma_a^{(j)} = \otimes_{i=1}^{j-1} \sigma_0 \otimes \sigma_a \otimes_{i=j+1}^{\ell} \sigma_0. \tag{86}$$

Our task is now to construct the $H_i, E_{\pm e^{(i)}}$ in terms of the $\sigma_a^{(j)}$. A first guess would be $H_i = \frac{1}{2}\sigma_3^{(i)}$, $E_{\pm e^{(i)}} = \frac{1}{2}\sigma_\pm^{(i)}$. However, it is not quite right since the $E_{\pm e^{(i)}}$ are not independent for different values of i as is seen explicitly in eq. (80). We can explicitly implement the conditions (80) by the following choice

$$H_i = \frac{1}{2}\sigma_3^{(i)}, \quad E_{\pm e^{(i)}} = \frac{1}{2}\,\sigma_3^{(1)} \cdots \sigma_3^{(i-1)} \sigma_\pm^{(i)}. \tag{87}$$

This turns out to be all there is! First, the phase $\phi(x_1, \ldots, x_\ell; \eta; i)$ in (78) is now fixed:

$$\mathrm{e}^{i\phi(x_1,\ldots,x_\ell;\eta;i)} = \prod_{j=1}^{i-1}(2x_j) = \pm 1. \tag{88}$$

Secondly, all the other lowering and raising operators can be constructed from (77).

It is now possible to return to the original Hermitian generators M_{ab} for $SO(2\ell+1)$. Recall that they satisfy the algebra

$$[M_{ab}, M_{cd}] = -i\big(\delta_{bc}M_{ad} - \delta_{ac}M_{bd} - \delta_{bd}M_{ac} + \delta_{ad}M_{bc}\big). \tag{89}$$

and one has

$$M_{2i-1,2\ell+1} = \frac{1}{\sqrt{2}}\big(E_{e^{(i)}} + E_{-e^{(i)}}\big) \tag{90}$$

$$M_{2i,2\ell+1} = \frac{-i}{\sqrt{2}}\big(E_{e^{(i)}} - E_{-e^{(i)}}\big) \tag{91}$$

from which we obtain

$$M_{2i-1,2\ell+1} = \frac{1}{2}\sigma_3^{(1)} \cdots \sigma_3^{(i-1)} \sigma_1^{(i)} \tag{92}$$

$$M_{2i,2\ell+1} = \frac{1}{2}\sigma_3^{(1)} \cdots \sigma_3^{(i-1)} \sigma_1^{(i)}. \tag{93}$$

Finally we can obtain the rest of the M_{ab} from (89):

$$M_{ab} = -i[M_{a,2\ell+1}, M_{b,2\ell+1}], \quad a, b \neq 2\ell+1. \tag{94}$$

Thus each M_{ab} is just $\pm 1/2$ times a product of Pauli matrices in a rather natural generalization of the spin 1/2 representation of $SO(3)$.

Here we have shown how one encounters the spinor representation of $SO(N)$ as one of the fundamental representations which has special properties. It is possible to use another perspective based a generalization of Pauli matrices to so-called γ-matrices which are 2^ℓ dimensional matrices satisfying certain commutation relations such that they form a so-called Clifford algebra. These gamma matrices can be expressed as certain tensor products of Pauli matrices, like described about. More information about this can be found in the Zee-book, chapter VII.1.