Added notes 1

Let X be the set $\{x_1, \dots, x_d\}$. *The group G is said to act on X* if

(1) for all $g \in G$ there exists a bijective map $g : X \mapsto X$ (which we also denote g)

(2) for all $g_1, g_2 \in G$ and for all $x \in X$ we have $(g_1 g_2)(x) = g_1(g_2(x))$.

(2) expresses that the group composition of the abstract elements g in G is compatible with the the ordinary composition of maps of X to X when we consider maps g associated with the abstract elements g. We often encounter situations where the group G is *defined* by its action on a set X, i.e. by the maps $g : X \mapsto X$. In this case one calls G a *transformation group* defined on X.

If for any order pair $\{x_1, x_2\}$ there is a g such that $g(x_1) = x_2$ then G is said to act *transitively* on X.

<ins>Ex 1</ins> $G = S_d$, the permutation group of d elements and $X = \{x_1, \dots, x_d\}$. Let g be a permutation, i.e. an interchange of the numbers $1, 2, \dots, d$ to $g(1), g(2), \dots, g(d)$. G will now act on X by the assignment $x_i \mapsto g(x_i) = x_{g(i)}$, where as above we use the same g for the element in S_d and for the map $g : X \mapsto X$ which permutes the elements in X according to the presciption $g(x_i) = x_{g(i)}$. G clearly acts transitively on X.

<ins>Ex 2</ins> Let H be a subgroup of G and $X = G/H$, the coset. If G has n elements and H m elements then there exists $d = n/m$ group elements $g_1, \dots, g_d$ such that we can write $X = G/H = \{g_1 H, \dots, g_d H\}$. The action of G on X is now defined by $g(g_i H) = (g g_i)H$. G acts transitively on X.

<ins>Ex 3</ins> The definition of G acting on a space X is not restricted to finite groups and spaces X consisting of a finite number of elements. Let $X = \mathbb{R}^n$. A translation T_a, $a \in \mathbb{R}^n$ is a map $X \mapsto X$ defined by $T_a : v \mapsto v + a$ for all $v \in \mathbb{R}^n$. $G = \{T_a | a \in \mathbb{R}^n\}$ is then in a natural way defined as the transformation group of translations on $X = \mathbb{R}^n$. G acts transitively on X.

<ins>Ex 4</ins> Let $X = \mathbb{R}^3$. Write an $v \in X$ as $v_i \hat{e}_i$ where $\hat{e}_i$ are three orthonormal vectors in X. v_i are the coordinates of the vector v. Let R_{ij} be an

orthonormal matrix with determinant 1. The set of these matrices forms a group G under matrix multiplication (called $SO(3)$) and they *act* on X by rotating the vectors in X: $v \mapsto v' = Rv$, or in coordinates $v'_i = R_{ij}v_j$. Note that in this case G *does not* act transitively on X, since two vectors with different length cannot be rotated into each other. If we restrict X to vectors v with norm 1, these vectors can be uniquely identified with points on S^2, the unit sphere in $\mathbb{R}^3$. Under rotation around the origin in the coordinate system defined by the vectors $\hat{e}_i$, S^2 is mapped onto S^2 and $G = SO(3)$ acts transitively on S^2.

One might think that Ex 2 above was a little special, but in fact the contrary is the case: *whenever G acts transitively on X there exists a subgroup H such that X can be identified with G/H.* Let G act on X and choose a point x_0 in X. Define the set

$$H_{x_0} = \{g \in G | g(x_0) = x_0\}. \tag{1}$$

It is easy to prove that H_{x_0} is a subgroup of G. It is called *the stability group* or *the isotropy group* of G at x_0. One can now define a map

$$\phi : G/H_{x_0} \mapsto X, \qquad \phi(gH_{x_0}) = g(x_0). \tag{2}$$

One can prove that this map is well defined and is a map of G/H_{x_0} into X. If G acts *transitively* on X the map is bijective and $H_{x_0} = \hat{g}^{-1}H_{x_1}\hat{g}$ where $x_1 = \hat{g}(x_0)$, i.e. the various stability groups are related by conjugation, which implies that the corresponding spaces G/H_{x_0} and G/H_{x_1} also are related by conjugation. Thus there is essentially only one coset space. For continuous groups one can prove that the map ϕ defined in (2) is a homeomorphism and we can thus identify X and G/H_{x_0} as topological spaces.

Let us now consider the *vector space* $\mathcal{F}(X)$ of functions $X \to \mathbb{C}$. If the set $X = \{x_1, \ldots, x_d\}$ then $\mathcal{F}(X)$ is a d-dimensional vector space. We will usually assume we have defined a scalar product $\langle \cdot | \cdot \rangle$ on $\mathcal{F}(X)$ by

$$\langle f | h \rangle = \sum_{x \in X} f^*(x)h(x). \tag{3}$$

An orthonormal set of functions $e_n(x)$ will satisfy

$$\langle e_n | e_m \rangle = \delta_{nm}, \qquad n, m = 1, \ldots, d. \tag{4}$$

Any function $f \in \mathcal{F}(X)$ can now be expanded as

$$f(x) = \sum_{n=1}^{d} c_n(f)\, e_n(x), \qquad c_n(f) = \langle e_n | f \rangle, \tag{5}$$

and we have

$$\langle f|h\rangle = \sum_{x\in X} f^*(x)\,h(x) = \sum_{n=1}^{d} c_n^*(f)\,c_n(h). \tag{6}$$

In Ex 3 and Ex 4 $\mathcal{F}(X)$ will be infinite dimensional. In these cases one can choose $\mathcal{F}(X) = L^2(\mathbb{R}^d)$, $\mathcal{F}(X) = L^2(\mathbb{R}^3)$ and $\mathcal{F}(X) = L^2(S^2)$, respectively. This is the natural choice in quantum mechanical applications. In pure mathematics other choices are sometimes of interest.

In the case where G and X are finite a particular simple set basis functions for $\mathcal{F}(X)$ is

$$e_n(x) = \delta_{x_n x}, \quad x_n, x \in X. \tag{7}$$

With this choice of basis functions one obtains

$$f(x) = \sum_{n=1}^{d} c_n(f)\,e_n(x), \quad c_n(f) = \langle e_n|f\rangle = f(x_n). \tag{8}$$

In the case where X is infinite, i.e. $X = \mathbb{R}$, one can also choose a bases $e_y(x) = \delta(x-y)$. In fact this basis e_y is precisely denoted $|y\rangle$ in the usual physics notation, and eq. (8) reads in this notation, using $\psi(x)$ instead of $f(x)$:

$$\psi(x) = \langle x|\psi\rangle = \int dy\,c_y(\psi)e_y(x), \quad e_y(x) = \langle x|y\rangle = \delta(x-y), \quad c_y(\psi) = \psi(y). \tag{9}$$

The disadvantage of this choice of basis is that the functions $e_y(x)$ do not belong to the space $L^2(\mathbb{R})$ where the functions $\psi(x)$ usually recides. Nevertheless it is used all the time in physics when convenient.

The advantage of introducing the vector space $\mathcal{F}(X)$ is that it offers a generic representation of the group G if G acts on X. This representation associates to every $g \in G$ a linear map $D(g) : \mathcal{F}(X) \mapsto \mathcal{F}(X)$ by the following prescription:

$$\boxed{(D(g)f)(x) = f(g^{-1}(x))} \quad x \in X, \quad f \in \mathcal{F}(X), \quad g \in G. \tag{10}$$

Since G acts on X we can view g as a bijective map $X \mapsto X$ and eq. (10) then tells us that $D(g)$ maps the function f to the function $f \circ g^{-1}$. It is easy to check that $D(g)$ acts as a linear map $\mathcal{F}(X) \mapsto \mathcal{F}(X)$. Let us now verify that $D(g)$ *is* a representation of G, i.e. most importantly that $D(g_1 g_2) = D(g_1)D(g_2)$:

$$(D(g_1g_2)f)(x) = f((g_1g_2)^{-1}(x)) = f(g_2^{-1}(g_1^{-1}(x))).$$
$$((D(g_1)D(g_2))f)(x) = (D(g_1)(D(g_2)f))(x) = (D(g_2)f)(g_1^{-1}(x)) = f(g_2^{-1}(g_1^{-1}(x))).$$

Further, this representation of G is a unitary representation, i.e. the linear operator $D(g) : \mathcal{F}(X) \mapsto \mathcal{F}(X)$ satisfies $D(g)^\dagger = D(g)^{-1}$. Recall that the linear map $M^\dagger$, adjoint to the linear map M is defined by

$$\langle M^\dagger f_1 | f_2 \rangle = \langle f_1 | M f_2 \rangle \quad \forall f_1, f_2 \in \mathcal{F}(X). \tag{11}$$

We have

$$\begin{aligned} \langle f_1 | D(g) f_2 \rangle &= \sum_{x \in X} f_1^*(x)\, f_2(g^{-1}(x)) = \sum_{x \in X} f_1(g(x))^* f_2(x) && (12) \\ &= \langle D(g^{-1}) f_1 | f_2 \rangle = \langle D(g)^{-1} f_1 | f_2 \rangle. && (13) \end{aligned}$$

For a given choice of basis $e_n(x)$ in $\mathcal{F}(X)$ $D(g)$ is represented by a matrix $D_{nm}(g)$ which is determined by expanding $f(x)$ in terms of $e_n(x)$. We write:

$$h = D(g)f, \quad h = \sum_{n=1}^{d} c_n(h)\, e_n, \quad f = \sum_n c_n(f)\, e_n, \tag{14}$$

and we obtain

$$c_n(h) = \langle e_n | h \rangle = \langle e_n | D(g) f \rangle = \sum_{m=1}^{d} \langle e_n | D(g) e_m \rangle c_m(f), \tag{15}$$

$$\boxed{c_n(h) = D_{nm}(g) c_m(f), \quad D_{nm}(g) = \langle e_n | D(g) e_m \rangle = \langle e_n | e_m \circ g^{-1} \rangle} \tag{16}$$

where we have used the convention that repeated indices imply summation. In the case where $e_n(x)$ is given by (7) we have

$$D_{nm}(g) = \langle e_n | e_m \circ g^{-1} \rangle = \sum_{x \in X} \delta_{x_n x} \delta_{x_m g^{-1}(x)} = \delta_{x_n g(x_m)}. \tag{17}$$

Relationen $\boxed{D_{nm}(g) = \delta_{x_n g(x_m)}}$ er precisely the one we used in the book to construct the *defining* representation for the permutation group S_d of d elements. Here we have shown that it is part of a much more general story. Note also that while the order of S_d is $d!$ the dimension of the representation $D(g)$ constructed by letting $G = S_d$ act on $X = \{x_1, \ldots, x_d\}$ according to $x_i \to g(x_i) = x_{g(i)}$ is much smaller, here equal to d. Nevertheless it is seen that there are precisely $d!$ different matrices defined by eq. (17) when g runs through all permutations of the numbers $1, \ldots, d$.

Let G act on X. A point $x_0 \in X$ is called a *fixed point* of the map $g : X \mapsto X$ if $g(x_0) = x_0$. Let $\chi(g)$ be the character function of the represention $D(g)$, i.e. $\chi(g) = \operatorname{tr} D(g)$. The choice of basis used in (17) allows us to give a *geometric interpretation to* $\chi(g)$. We have from (17)

$$\begin{aligned}\chi(g) &= \operatorname{tr} D(g) = \sum_n D_{nn}(g) = \sum_n \delta_{x_n g(x_n)} \Rightarrow \\ \chi(g) &= \#\{\text{fixed points of the map } g : X \mapsto X\}. \end{aligned} \tag{18}$$

We have seen that for any subgroup H one can view $X = G/H$ as a space on which G acts. In particular we can take $H = \{e\}$, the identity element. In this case $X = G$ and the action of $g \in G$ on X simply becomes the group multiplication itself: $g(x) = g\,x$, $x, g \in G$. Denote the order of G by $N(G)$. We can now use our general construction to find a representation of G on $\mathcal{F}(X) = \mathcal{F}(G)$. The dimension of this representation is $N(G)$. Thus, in the case where $G = S_d$ the dimension will be $d!$, in contrast to the defining representation which had dimension d. The representation exists for all finite groups (and as we shall see later for all continuous compact groups) and it is called the *regular representation of* G. If we choose as basis functions for $\mathcal{F}(G)$ the functions $e_g(h) = \delta_{gh}$ we have as before

$$\boxed{(D^{(reg)}(g))_{h_1 h_2} = \delta_{h_1\, g h_2}.} \tag{19}$$

Let us now use (18) to calculate $\chi^{(reg)}(g)$. For the identity element e of G every point is a fixed point ($eh = h\ \forall h \in G$ and thus $\chi^{(reg)}(e) = N(G)$, which of course is always true for any representation: $\chi(e)$ is equal to the order of the group G. However, no other g has a fixed point since $gh = h$ implies $g = e$. Thus

$$\chi^{(reg)}(g) = N(G)\delta_{e,g}. \tag{20}$$

Let us now return to the case of infinite X as in Ex 3 and 4 above. Let us first consider translations and let us take $X = \mathbb{R}$ for simplicity, rather than $X = \mathbb{R}^n$. Everything is trivially generalized to $n > 1$. We want to understand what kind of representation eq. (10) gives of the translation group. $G = \{T_a\}$ and it acts on X by $T_a : x \to x + a$. It translates then point labelled $x \in X$ to the point labelled $x + a$, i.e. a translation by a. The prescription (10) tells us how G acts on functions $f \in \mathcal{F}(X)$:

$$(D(T_a)f)(x) = f(T_a^{-1}x) = f(x - a). \tag{21}$$

It is seen (see Fig. 1) that this is precisely a translation of the function $f(x)$ by a. Note that the argument $x - a$, with a minus sign, corresponds to a

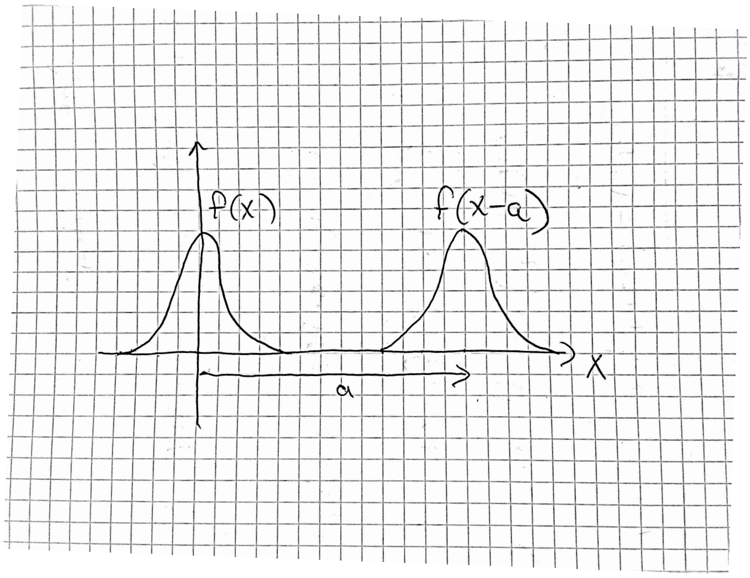

Figure 1: The translation of a function $f(x)$ a distance a.

translation of f with a, not $-a$. In a quantum mechanical setting it is natural to choose $\mathcal{F}(X) = L^2(X)$, the Hilbert space of square integrable functions, and while T_a represents a translation of a in X, $D(T_a)$ will now represents the same translation on wave functions $f \in L^2(X)$.

Let us now understand how we can represent$D(T_a)$ as a linear map from $L^2(X) \to L^2(X)$, i.e. like a linear operator on $L^2(X)$. Let us assume first that $f(x)$ is a nice function, such that we can Taylor expand it. We can then write

$$\begin{aligned} f(x-a) &= f(x) - a\frac{df(x)}{dx} + \frac{a^2}{2!}\frac{d^2f(x)}{dx^2} - \cdots & (22) \\ & (1 - a\frac{d}{dx} + \frac{a^2}{2!}\frac{d^2}{dx^2} - \cdots)f(x) = \left(\mathrm{e}^{-a\frac{d}{dx}}\right)f(x). & (23) \end{aligned}$$

Thus we see that we have

$$\boxed{D(T_a) = \mathrm{e}^{-a\frac{d}{dx}}}. \tag{24}$$

This operator can be shown to be a unitary linear operator on $L^2(X)$ and the operators $D(T_a)$ offer an unitary representation of the translation group on $L^2(X)$. It is clear that this is the natural way we encounter the translation group in quantum mechanics, since we work with wave functions $\psi(x) \in L^2(X)$. Note that the momentum operator $P_x = -i\hbar\frac{d}{dx}$ and we can thus write

$$D(T_a) = \mathrm{e}^{-iaP_x/\hbar}, \tag{25}$$

which shows that the *momentum operator is the generator of translations.* These words will be made precise when we discuss continuous groups and the so-called *generators* of these groups. We can find similar representations for the rotation group, the rotations being represented by unitary operators acting on $L^2(X)$, X being either $\mathbb{R}^3$ or S^2 and where the angular momentum will now act as generators for the rotation group. Like P they will be certain differential operators acting on the wave functions. Details will be provided when we discuss continuous groups.