Exam matf3 June 2017

4 hour exam with all usual help items. The exam may be answered using pencil and English or Danish language can be used. The exam includes 11 sub-questions which are weighted equally in the evaluation.

Problem 1

Let K be a finite group of order 4 with elements e, x, y and z, where e is the identity element. Complete the following group multiplication table

	e	x	y	z
e				
x		e		
y			e	
z				

Problem 2

Let G be the group $Z_2 \times Z_2$.

(1) Prove that there are four irreducible representations of G.

(2) Write down the corresponding character functions $\chi^{(r)}(g)$, $g \in G$, $r = 1, 2, 3$ and 4.

Problem 3

Let G be a cyclic group of order n. By definition this means that there exists an $x \in G$ such that $G = \{x^0, x^1, \dots, x^{n-1}\}$.

(1) Prove that G has a cyclic subgroup of order p for any p that divides n.

One can prove that any subgroup of a cyclic group is a cyclic group of this kind. Assuming this is true then

(2) prove that there is at most one subgroup of G of any given order less than or equal n.

Problem 4

Let R^{ij} be matrices in the defining representation of $SO(N)$, i.e. components V^i of vectors in $\mathbb{R}^N$ transform as $V'^i = R^{ij}V^j$. $R^{ik}R^{jl}$ are components of the tensor product representation $R\otimes R$ of two defining representations of $SO(N)$. In this representation tensor components T^{ij}, i.e. components of vectors T in $\mathbb{R}^N\otimes\mathbb{R}^N$, transform as $T'^{i,j} = R^{ik}R^{jl}T^{kl}$. The tensor product representation $R\otimes R$ decomposes into irreducible representations, thereby defining orthogonal subspaces of $\mathbb{R}^N\otimes\mathbb{R}^N$.

(1) Find the dimensions of these vector subspaces.

(2) Decompose explicitly a vector $T \in \mathbb{R}^N\otimes\mathbb{R}^N$ with components T^{ij} in vectors lying in the vector subspaces corresponding to the irreducible representations of the tensor product representation $R\otimes R$, i.e. express the components of vectors in these subspaces in terms of T^{ij}.

Problem 5

Let U^{ij} be matrices in the defining representation of $SU(3)$, i.e. components V^i of vectors in $\mathbb{C}^3$ transform as $V'^i = U^{ij}V^j$. $U^{ik}U^{jl}$ are components of the tensor product representation $U\otimes U$ of two defining representations of $SU(3)$. In this representation tensor components T^{ij}, i.e. components of vectors T in $\mathbb{C}^3\otimes\mathbb{C}^3$, transform as $T'^{i,j} = U^{ik}U^{jl}T^{kl}$. The tensor product representation $U\otimes U$ decomposes into irreducible representations, thereby defining orthogonal subspaces of $\mathbb{C}^3\otimes\mathbb{C}^3$.

(1) Find the dimensions of these vector subspaces.

(2) Decompose explicitly a vector $T \in \mathbb{C}^3\otimes\mathbb{C}^3$ with components T^{ij} in vectors lying in the vector subspaces corresponding to the irreducible representations of the tensor product representation $U\otimes U$, i.e. express the components of vectors in these subspaces in terms of T^{ij}.

Problem 6

(1) Use the raising and lowering operators $J_\pm$ to construct the matrices J_x and J_y for the irreducible representation of $SU(2)$ corresponding to $J = 3/2$.

(2) Find the 4×4 matrix $M = \exp(-i\pi J_x/2)\, J_y \exp(i\pi J_x/2)$.