Added notes 3

Differential geometry and Lie groups

The purpose of this note is to discuss (in an informal way with no proofs) the relation between differential geometry and Lie groups. Lie groups are differential manifolds and they are very "nice" such manifolds, so many formulas from differential geometry take a simple form when expressed in Lie group and Lie algebra "language' when the manifold is a Lie group. The following note consists of two parts: the first introduces the notation of differential geometry. The original plan of the course was to combine differential geometry and the theory of Lie groups. However, it proved impossible to do in the 7 weeks assigned to the course. It takes 2 years (!!) to change the course description, so these notes can maybe serve as a little compensation for the present somewhat misleading course description (but they are of course in no way complete). The second part of the notes is intended to show how nice the differential geometry is on Lie groups. No mathematical rigour is attempted at any point, but I hope that at least the definitions are correct......

(1) Differential geometry

Basic concepts

A (C^∞) manifold can be defined in terms of so-called charts. The basic idea is that an n-dimensional manifold locally is just like $\mathbb{R}^n$

Given a (suitable) topological space M (see *addednotes2* for some definitions) a *chart* (U, φ) is a homeomorphism from an open set $U \subseteq M$ to an open set $\varphi(U) \subseteq \mathbb{R}^n$ for some n. For a given $p \in U$ we call $(x^1(p), \ldots, x^n(p)) = \varphi(p)$ *the coordinates of* p.

An *atlas* $\mathcal{A}$ on M is a family of charts $\{(\varphi_i, U_i)\}$ such that

M_1: $\forall i : \quad \varphi_i(U_i) \subseteq \mathbb{R}^n$

M_2: $M = \bigcup_i U_i$

M_3: if $U_i \cap U_j \neq \emptyset$ then $\varphi_i^j = \varphi_i \circ \varphi_j^{-1}$ is an C^∞ maps from $\varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$. The maps φ_i^j are called *transition maps*.

Two atlases $\mathcal{A}$ and $\mathcal{A}'$ are compatible if the union is still an atlas. This introduces an equivalence relation between atlases on M and we call M together with an equivalence class $\bar{\mathcal{A}}$ of atlases a smooth n-dimensional manifold. Note that in general M cannot be embedded in $\mathbb{R}^n$.

A manifold M is said to be *orientable* if there exists an atlas such that all mappings $\varphi_j \circ \varphi_i^{-1}$, have a strictly positive Jacobian determinant in its domain of definition $\varphi_i(U_i \cap U_j)$. The manifold M is said to be *oriented* if such an atlas is chosen.

We can now define the set $C^\infty(M)$ of smooth functions from $M \to \mathbb{R}$. $f \in C^\infty(M)$ if $f \circ \varphi_i^{-1}$ is an ordinary infinitely many times differentiable function from $\varphi_i(U_i) \to \mathbb{R}^n$ for all U_i in the atlas defining the differentiable structure on M. $C^\infty(M)$ is an associative algebra if we define multiplication of two functions as pointwise multiplication: $(f \cdot g)(p) = f(p)g(p)$. We say that $f \in C^\infty(p)$ if there exists a chart (U, φ) such that $f \circ \varphi^{-1}$ is a C^∞ function at $\varphi(p)$. Like $C^\infty(M)$ also $C^\infty(p)$ is an associative algebra. The set $C^\infty(M)$ has the following properties:

F_1: Let $\varphi_1, \dots, \varphi_r \in C^\infty(M)$ and let u be a differential function $\mathbb{R}^r \to \mathbb{R}$. Then $u(\varphi_1, \dots, \varphi_r) \in C^\infty(M)$.

F_2: Let f be a function $M \to \mathbb{R}$ such that for each $p \in M$ there exists a $g \in C^\infty$ which coincides with f in some neighborhood of p. Then $f \in C^\infty$.

F_3: For each $p \in M$ there exists n functions $\varphi_1, \dots, \varphi_n$ and an open neighborhood U of p such that the mapping $q \to (\varphi_1(q), \dots, \varphi_n(q))$, $q \in U$, is a homeomorphism of U onto an open subset of $\mathbb{R}^n$. U and $\varphi_1, \dots, \varphi_n$ can be chosen such that each $f \in C^\infty(M)$ coincides on U with a function of the form $u(\varphi_1, \dots, \varphi_n$ where u is a differential function on $\mathbb{R}^n$.

One can show that a set of functions with the properties F_1, F_2 and F_3 allows us to *define* a manifold structure on a suitable topological space M. It is a *dual* formulation, much in the same way as a finite dimensional vector space V is isomorphic to its dual space vector space V^*, which is the set of linear functions $f : V \to \mathbb{R}$. It follows from the following theorem:

Theorem *Assume M is a Hausdorff space*[1] *and $n \in \mathbb{N}$. Let $\mathcal{F}$ be a collection of functions on M with the properties $F_1 - F_3$. Then there exists a unique collection (U_i, φ_i) of open charts on M such that $M_1 - M_3$ are satisfied and such that the differentiable functions on the resulting manifold, i.e the functions in $C^\infty(M)$, are precisely the members of $\mathcal{F}$.*

[1] A Hausdorff space is a topological space where two distinct points have disjoint neighborhoods.

Given an algebra[2] A a *derivation* is a *linear map* $D : A \to A$ which satisfies the Leibniz rule

$$\forall f, g \in A: \quad D(f \cdot g) = D(f) \cdot g + f \cdot D(g). \tag{1}$$

One often writes Df rather than the more elaborate $D(f)$. A *vector field* X on a manifold is a derivation of the algebra $C^\infty(M)$. Let $\mathcal{D}^1(M)$ denote the set of all vector fields on M. If $f \in C^\infty(M)$ and $X, Y \in \mathcal{D}^1(M)$ then fX and $X + Y$ denote the vector fields

$$fX : g \to f(Xg), \quad X + Y : g \to Xg + Yg, \quad g \in C^\infty(M). \tag{2}$$

Technically speaking this makes $\mathcal{D}^1(M)$ a *module* over the *ring* $\mathbf{C}^\infty(M)^2$.

The product XY of two vector fields is not a derivation since it does not satisfy Leibniz rule, and thus it is not a vector field. However, $XY - YX := [X, Y]$ *does* satisfy the Leibniz rule as one can easily check, and is thus a vector field. It is called *the Lie derivative with respect to* X and often denoted $\mathcal{L}_X(Y)$.

If we have a vector field X on $C^\infty(M)$ one can show that X is also a vector field on $C^\infty(U)$, where U is any open subset of M. In particular, let (U, φ) be a chart and write $\varphi(q) = (x^1(q), \ldots, x^n(q))$ for all $q \in U$. The function $f^* = f \circ \varphi^{-1}$ is now an ordinary C^∞ function on $\varphi(U) \subseteq \mathbb{R}^n$ and we now have a vector field on U defined by the following map:

$$\frac{\partial}{\partial x^i} : f \to \frac{\partial f^*}{\partial x^i} \circ \varphi, \quad f \in C^\infty(U). \tag{3}$$

We write $\frac{\partial f}{\partial x_i}$ rather than the more correct $\frac{\partial}{\partial x_i}(f)$. Let now V be an open subset of U such that $\varphi(V)$ is an open ball in $\mathbb{R}^n$ with center $\varphi(p) = (a^1, \ldots, a^n)$. If

[2] A *ring* R is a structure (a set with some rules of composition) which is imore general than a *field* (like $\mathbb{R}$ and $\mathbb{C}$). It has to binary operators $+$ and $\cdot$ called addition and multiplication. R is an Abelian group with respect to $+$. It has *left and right distributivity*: $a\cdot(b+c) = (a\cdot b)+(a\cdot c)$ and $(b+c)\cdot a = (b\cdot a)+(c\cdot a)$. Finally it is usually assumed to be multiplicatively associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. A *module* is a generalization of a vector space. The same axioms are used to to define a module, except that a vector is no longer multiplied by number coming from a field, but by "numbers" belonging to a ring. A module taking its coefficients from a ring R is called a module over R or an R-module. An *algebra* A is a structure with compatible operations of addition $+$ and multiplication $*$ and where one in addition can multiply the elements by *scalars*, i.e. elements of a ring R (which can also be a field like $\mathbb{R}$ or $\mathbb{C}$). One usually assumes that multiplication is associative. If not obvious for some reasons one can talk about an associative algebra or a non-associative algebra. Similarly one talks about a non-commutative algebra if multiplication is not commutative. If one wants to emphasise that the scalars belong to a ring one can talk about an R-algebra. Thus an algebra has the structure of both a ring and an R-module and in addition $r \cdot (x * y) = (r \cdot x) * y = x * (r \cdot y)$ for all $r \in R$ and $x, y \in A$.

$(x^1, \dots, x^n) \in \varphi(V)$ then we have

$$\begin{aligned} f^*(x) &= f^*(a) + \int_0^1 dt \, \frac{\partial}{\partial t} f^*(a + t(x-a)) \\ &= f^*(a) + \sum_{j=1}^n (x^j - a^j) \int_0^1 dt \, \frac{\partial f^*(a + t(x-a))}{\partial x^j} \\ &= f^*(a) + \sum_{j=1}^n (x^j - a^j) \, g_j^*(x), \end{aligned} \tag{4}$$

If we transfer this relation back to M we obtain

$$f(q) = f(p) + \sum_{j=1}^n (x^j(q) - x^j(p)) g_j(q), \tag{5}$$

where $g_j = g_j^* \circ \varphi \in C^\infty(V)$ and where one can check that for $q = p$

$$g_i(p) = \left(\frac{\partial f^*}{\partial x^i} \right)_{\varphi(p)}. \tag{6}$$

If X is a vector field, i.e. a linear map with satisfies the Leibniz rule, the calculation of $(Xf)(q)$ using (5) with p kept fixed (i.e. X acting in the constant function $f(p)$ gives zero), and then taking the limit $q \to p$ and using (6) leads to

$$(Xf)(p) = \sum_{i=1}^n \left(\frac{\partial f^*}{\partial x^i} \right)_{\varphi(p)} (Xx^i)(p). \tag{7}$$

This shows that

$$X = \sum_{i=1}^n (Xx_i) \, \frac{\partial}{\partial x^i} \quad \text{on} \quad U \tag{8}$$

Thus $\frac{\partial}{\partial x^i}$*,* $i = 1, \dots, n$ *is a basis on* $\mathcal{D}^1(U)$*.*

For $p \in M$ and $X \in \mathcal{D}^1(M)$, let X_p denote the linear map

$$X_p : \; f \to (Xf)(p) \qquad \big(C^\infty(p) \to \mathbb{R}\big). \tag{9}$$

This map also satisfies the Leibniz rule and we call it a derivation from $C^\infty(p) \to \mathbb{R}$. *The set* $\{X_p \mid X \in \mathcal{D}^1(M)\}$ *is called the tangent space to* M *in* p *and is denoted* $\mathcal{D}^1(p)$ *or* T_p *(or* $T_p(M)$ *if any ambiguity).* Eq. (7) shows that T_p is a linear vector space over $\mathbb{R}$ spanned by the n independent vectors:

$$\left(\frac{\partial}{\partial x^i} \right)_p : f \to \left(\frac{\partial f^*}{\partial x^i} \right)_{\varphi(p)} \qquad f \in C^\infty(M). \tag{10}$$

Thus a vector field X on M can be identified with a collection X_p, $(p \in M)$ of tangent vectors to M, with the property that for each function $f \in C^\infty(M)$ the function $p \to X_p f$ belongs to $C^\infty(M)$.

One also denotes $X(f)(p)$ the *directional derivation* $D_X(f)$ of f at p. Let $f^* \in C^\infty(\mathbb{R}^n)$. Let x be the coordinates of a point in $\mathbb{R}^n$, let $\tilde{\gamma}(t)$ be a curve in $\mathbb{R}^n$ such that $\tilde{\gamma}(t=0) = x$ and $\tilde{\gamma}'(t=0) = \tilde{X}$ where $\tilde{X}$ a vector in $\mathbb{R}^n$ (the "velocity" at "time" $t = 0$). Then the directional derivative of f^* at x in direction $\tilde{X}$ is defined by

$$D_{\tilde{X}}(f^*)(x) = \frac{\mathrm{d}f^*(\tilde{\gamma}(t))}{\mathrm{d}t}\Big|_0 = \sum_{i=1}^n \tilde{X}^i \frac{\partial f^*}{\partial x^i}\Big|_x. \tag{11}$$

$X(f)(p)$ is the generalization of this to a manifold M. We consider a *curve* $\gamma(t)$ mapping $[-\epsilon, \epsilon]$, say, to M such that $\gamma(0) = p$ and $\tilde{X} = (\varphi \circ \gamma)'(0)$[3]. It is now seen from (7) that $X(f)(p)$ can be viewed as the directional derivation as defined in (11) for the function $f^* = f \circ \varphi^{-1}$ with the curve $\tilde{\gamma}(t) = \varphi \circ \gamma(t)$. We can thus identify the *vector* $\tilde{X} \in \mathbb{R}^n$, arising as the tangent vector for the curve $\varphi \circ \gamma(t)$ with the *vector* $X_p \in T_p$ which came from the *vector field* X via $\tilde{X}^i = (Xx^i)(p)$. In fact it is possible (and often done) to define the tangent space T_p as tangent vectors associated with the curves $\gamma(t)$ passing through p. Note also that we have

$$D_X(f) = X(f) = (f \circ \gamma)'(t=0) \tag{12}$$

Here $(f \circ \gamma)'$ denotes the ordinary derivative with respect to the parameter t, which is meaningful since $f \circ \gamma$ is an ordinary map from some interval I to $\mathbb{R}$.

Let A be a commutative ring with identity element, E a module over A. The *dual* of E is the set of all A linear maps of E into A. *The dual* of the $C^\infty(M)$-module $\mathcal{D}^1(M)$ is denoted $\mathcal{D}_1(M)$ and the elements are called *differential 1-forms (or just 1-forms) on* M. Thus $\omega \in \mathcal{D}_1(M)$ means that for $X \in D^1(M)$ we have that $\omega(X) \in C^\infty(M)$. In particular we can define the *differential* $\mathrm{d}f \in \mathcal{D}^1(M)$ by

$$\mathrm{d}f(X) := Xf, \qquad \mathrm{d}f(X)(p) = Xf(p) \tag{13}$$

One can view d as a map from $C^\infty(M) \to \mathcal{D}_1(M)$ which satisfies $\mathrm{d}(f \cdot g) = f\mathrm{d}g + g\mathrm{d}f$. Below, when we have introduced tensors, we will see that d is the so-called *exterior derivative*. Let (U, φ) be a chart. We have seen that $\frac{\partial}{\partial x^i}$ constitute a basis for $\mathcal{D}^1(U)$. The coordinate functions $x^i : p \to x^i(p)$, $p \in U$ belongs to $C^\infty(U)$ and thus $\mathrm{d}x^i \in \mathcal{D}_1(U)$. In fact they constitute basis for $\mathcal{D}_1(U)$ which is *dual* to $\frac{\partial}{\partial x^i}$, since the definition (13) implies that

$$\mathrm{d}x^i\Big(\frac{\partial}{\partial x^j}\Big) = \frac{\partial}{\partial x^i}(x^j) = \delta^i_j. \tag{14}$$

[3]We assume $p \in U$, where (U, φ) is a chart and that ϵ is chosen so small that $\gamma(t) \in U$ for all $t \in [-\epsilon, \epsilon]$. Then $\varphi \circ \gamma(t)$ is an ordinary curve in $\varphi(U) \subseteq \mathbb{R}^n$ and we assume that this curve is infinitely many times differentiable.

We can now expand any $\mathrm{d}f$ on the basis $\mathrm{d}x^i$

$$\mathrm{d}f = \sum_{i=1}^{n} \lambda^i(p)\, dx^i = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x^i}\right) \mathrm{d}x^i. \tag{15}$$

This formula is of course obvious when $M = \mathbb{R}^n$, but we have now generalized it to an n-dimensional manifold.

Starting with $X \in \mathcal{D}^1(M)$ we defined the tangent space T_p ("$\mathcal{D}^1(M)$ at p") as the linear maps $X_p : f \to (Xf)(p)$ $(\in \mathbb{R})$ from $C^\infty(p) \to \mathbb{R}$ (see eq. (9)). Similarly, starting with an $\omega \in \mathcal{D}_1(M)$ we can define a linear map ω_p from $T_p \to \mathbb{R}$ by $\omega_p(X_p) := \omega(X)(p)$. *The set* $\{\omega_p \mid \omega \in \mathcal{D}_1(M)\}$ *is a vector space called the cotangent space* $\mathcal{D}_1(p)$ *or* T_p^* *and it is the dual vector space of* T_p. Thus a 1-form ω on M can be viewed as a collection of cotangent vectors ω_p, $(p \in M)$ and $\mathcal{D}_1(M)$ can be viewed as the set of all $\mathcal{D}_1(p)$, $p \in M$ in the same way as $\mathcal{D}^1(M)$ can be viewed as set of all $\mathcal{D}^1(p)$, $p \in M$. Dual bases for T_p and T_p^* are

$$e_i = \left(\frac{\partial}{\partial x^i}\right)_p, \qquad f^j = \mathrm{d}x^j\big|_p, \tag{16}$$

Tensors

Given $\mathcal{D}^1(M)$ and $\mathcal{D}_1(M)$ we can form the direct product

$$\mathcal{D}_1 \times \cdots \times \mathcal{D}_1 \times \mathcal{D}^1 \times \cdots \times \mathcal{D}^1 \quad (\mathcal{D}_1\ r \text{ times},\ \mathcal{D}^1\ s \text{ times}). \tag{17}$$

We define $\mathcal{D}^r_s(M)$ as the set of all *multilinear maps*[4] from this direct product space into $C^\infty(M)$ and denote $\mathcal{D}^r_0 := \mathcal{D}^r$, $\mathcal{D}^0_s := \mathcal{D}_s$ and define $\mathcal{D}^0_0 := C^\infty(M)$. A *tensor field T on M of type (r,s) is by definition an element of* $\mathcal{D}^r_s(M)$. T is said to be *contravariant* of degree r and *covariant* of degree s. At a point $p \in M$ we define $\mathcal{D}^r_s(p)$ as the set of $\mathbb{R}$-multilinear mapppings of the direct product

$$T_p^* \times \cdots \times T_p^* \times T_p \times \cdots \times T_p \quad (T_p^*\ r \text{ times},\ T_p\ s \text{ times}) \tag{18}$$

into $\mathbb{R}$. The set $\mathcal{D}^r_s(p)$ is a vector space over $\mathbb{R}$ and is nothing but the standard tensor product

$$\mathcal{D}^r_s(p) = T_p \otimes \cdots \otimes T_p \otimes T_p^* \otimes \cdots \otimes T_p^* \quad (T_p\ r \text{ times},\ T_p^*\ s \text{ times}). \tag{19}$$

Finally we define $\mathcal{D}^0_0(p) := \mathbb{R}$. The shift between r and s in (18) and (19) is because $\mathcal{D}^r_s(p)$ is defined as a space of (multi)-linear maps, and the linear maps

[4] Remember that the word "linear" refers to the ring $C^\infty(M)$ related to the modules $\mathcal{D}_1(M)$ and $\mathcal{D}^1(M)$.

on T_p *is* T_p^* and vice versa. Thus tensor fields of type (1,0), (0,1) and (0,0) are just the vector fields, the 1-forms and the differentiable functions on M.

Let now ω_i, $i = 1, \ldots, r$ and X_j, $j = 1, s$ belong to $\mathcal{D}_1(M)$ and $\mathcal{D}^1(M)$, respectively. For a tensor $T \in \mathcal{D}_s^r$ we have

$$T(g_1\omega_1, .., g_r\omega_r, f_1 X_1, .., f_s X_s) = g_1..g_r f_1..f_r T(\omega_1, .., \omega_r, X_1, .., X_s) \tag{20}$$

for $f_j, g_i \in C^\infty(M)$. Also the relation between elements $T_p \in \mathcal{D}_s^r(p)$ and elements $T \in \mathcal{D}_s^r(M)$ is a simple generalization of the already mentioned relation valid for elements in $\mathcal{D}^1(p)$ and $\mathcal{D}^1(M)$ and for elements of $\mathcal{D}(p)$ and $\mathcal{D}_1(M)$ (see paragraph above eq. (16)):

$$T_p((\omega_1)_p, \ldots, (\omega_r)_p, (X_1)_p, \ldots, (X_s)_p) = T(\omega_1, \ldots, \omega_r, X_1, \ldots, X_s)(p). \tag{21}$$

Let now $\mathcal{D}(M)$ denote the direct sum of the $\mathbf{C}^\infty(M)$ modules $\mathcal{D}_s^r(M)$:

$$\mathcal{D}(M) := \sum_{r,s=0}^{\infty} \mathcal{D}_s^r(M). \tag{22}$$

Similarly, for $p \in M$:

$$\mathcal{D}(p) = \sum_{r,s=0}^{\infty} \mathcal{D}_s^r(p). \tag{23}$$

The *vector space* $\mathcal{D}(p)$ can be turned into an associative algebra over $\mathbb{R}$ as follows. Take two basis vectors from $\mathcal{D}_s^r(p)$ and $\mathcal{D}_w^t(p)$, i.e. using the notation in (16), and define the product of such two vectors in the rather obvious way:

$$\begin{aligned} v_s^r &= e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_r} \otimes f^{\beta_1} \otimes \cdots \otimes f^{\beta_s} \quad \in \mathcal{D}_s^r(p), \\ v_w^t &= e_{\gamma_1} \otimes \cdots \otimes e_{\gamma_t} \otimes f^{\delta_1} \otimes \cdots \otimes f^{\delta_w} \quad \in \mathcal{D}_w^t(p), \\ v_s^r \otimes v_w^t &= e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_r} \otimes e_{\gamma_1} \otimes \cdots \otimes e_{\gamma_t} \otimes \qquad (24) \\ & \quad f^{\beta_1} \otimes \cdots \otimes f^{\beta_s} \otimes f^{\delta_1} \otimes \cdots \otimes f^{\delta_w} \quad \in \mathcal{D}_{s+w}^{r+t}(p) \qquad (25) \end{aligned}$$

This is now extended by linearity from the basis vectors to the whole vector spaces and the product can be shown to be independent of the basis choice. After this we can now define the tensor product $\otimes$ on $\mathcal{D}(M)$ as the $\mathbf{C}^\infty(M)$-bilinear map $(S, T) \to S \otimes T$ such that

$$(S \otimes T)(p) = S_p \otimes T_p. \qquad S \in \mathcal{D}_s^r(M), \quad T \in \mathcal{D}_w^t(\mathcal{M}), \quad p \in M. \tag{26}$$

Thus $\mathcal{D}(M)$ becomes a *ring* satisfying

$$f(S \otimes T) = (fS) \otimes T = S \otimes (fT), \qquad f \in C^\infty(M), \quad S, T \in \mathcal{D}(M). \tag{27}$$

This implies that $\mathcal{D}(M)$ is an associative algebra over the ring $C^\infty(M)$. The algebras $\mathcal{D}(M)$ and $\mathcal{D}(p)$ are called *mixed tensor algebras* over M and T_p respectively. Finally,

$$\mathcal{D}_*(M) := \sum_{s=0}^{\infty} \mathcal{D}_s(M), \qquad \mathcal{D}_*(p) := \sum_{s=0}^{\infty} \mathcal{D}_s(p), \tag{28}$$

denotes subalgebras of $\mathcal{D}(M)$ and $\mathcal{D}(p)$.

A multilinear mapping $f \in \mathcal{D}_s(M)$ is called *alternate* if $f(X_1, \ldots, X_s) = 0$ whenever at least two X_i coincide. Let $\mathcal{U}_s(M)$ denote the set of alternate multilinear maps in $\mathcal{D}_s(M)$. $\mathcal{U}_s(M)$ is a submodule of $\mathcal{D}_s(M)$. We define $\mathcal{U}_0(M) := C^\infty(M)$. Define

$$\mathcal{U}(M) := \sum_{s=0}^{\infty} \mathcal{U}_s(M). \tag{29}$$

The elements of $\mathcal{U}(M)$ are called *exterior differential forms on M.* The elements of $\mathcal{U}_s(M)$ are called *differential s-forms of just s-forms.* The exists a *projection* $A_s : \mathcal{D}_s(M) \to \mathcal{U}_s(M)$ defined in the following way:

$$A_s(f)(X_1, \ldots, X_s) = \frac{1}{s!} \sum_{\sigma \in S_s} \epsilon(\sigma)\; f(X_{\sigma(1)}, \ldots, X_{\sigma(s)}). \tag{30}$$

Here S_s denote the symmetric group of s elements and $\epsilon(\sigma)$ the sign of the permutation σ. One easily checks that $A_s(f)$ is an alternation, that $A_s^2 = A_s$, i.e. that A_s is a projection and the $A_s(f) = f$ if f is an alternation. Thus $A_s(\mathcal{D}_s(M)) = \mathcal{U}_s(M)$. A_s can easily be extended to a map from $\mathcal{D}_*(M)$ to $\mathcal{U}(M)$ and this allows us to introduce a new product on $\mathcal{U}(M)$, called the *exterior product*:

$$\omega_1 \wedge \omega_2 = A(\omega_1 \otimes \omega_2), \qquad \omega_1, \omega_2 \in \mathcal{U}(M). \tag{31}$$

This turns the $C^\infty(M)$ module $\mathcal{U}$ into an associative algebra called the *Grassmann algebra* of the manifold M. As usual we also have a local statement of this: for each $p \in M$ we define the Grassmann algebra of T_p as the alternate, $\mathbb{R}$-multilinear real-valued functions on T_p and the product (also denoted $\wedge$) is

$$(\omega_1)_p \wedge (\omega_2)_p = (\omega_1 \wedge \omega_2)_p = (\omega_1 \wedge \omega_2)(p), \qquad \omega_1, \omega_2 \in \mathcal{U}(M). \tag{32}$$

This turns $\mathcal{U}(p)$ into an associative algebra containing the dual space T_p^*. *Exterior differentiation* d is described in the following theorem:

Theorem *The exists a unique $\mathbb{R}$-linear map $\mathrm{d} : \mathcal{U}(M) \to \mathcal{U}(M)$ such that*

$\mathrm{d}\mathcal{U}_s(M) \subset \mathcal{U}_{s+1}(M) \quad \forall\; s \geq 0.$

If $f \in \mathcal{U}_0$ $(= C^\infty(M))$ *then* $\mathrm{d}f(X) = X(f), \quad X \in \mathcal{D}^1(M)$.

$\mathrm{d} \circ \mathrm{d} = 0.$

$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge \mathrm{d}\omega_2, \quad \omega_1 \in \mathcal{U}_r(M),\; \omega_2 \in \mathcal{U}(M).$

Let us mention for completeness that exterior differentiation occurs naturally in connection with a general form of Stokes' theorem on a manifold. Consider a n-dimensional compact manifold M with boundary[5] ∂M. Let ω be an n-form on M and let V be some region contained in a chart (U, φ). One now defines

$$\int_V \omega = \int_{\varphi(V)} \omega_{1\dots n} dx^1 \cdots dx^n, \tag{33}$$

were one has the local coordinate expression $\omega = \omega_{1\dots n}(x) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. One can check that the expression is chart independent and thus it can be extended to an integral over M. Let now ω be a $(n-1)$-form on M. Thus $\mathrm{d}\omega$ is an n-form on M and the generalized Stokes theorem states:

$$\int_{\partial M} \omega = n \int_M \mathrm{d}\omega. \tag{34}$$

Mappings

Let M and N denote two manifolds and let Φ denote a mapping of M into N. We call the mapping differentiable at $p \in M$ if $g \circ \Phi \in C^\infty(p)$ $\forall$ $g \in C^\infty(\Phi(p))$ and it is called differentiable if it is differentiable at all $p \in M$. Let (U, ψ) be a chart on M and (U', ψ) be a chart on N and assume that $\psi(U) \subseteq U'$. The mapping $\varphi = \psi' \circ \Phi \circ \psi^{-1}$ of $\psi(U)$ into $\psi'(U')$ is a system of n functions:

$$y^j = \varphi^j(x^1, \dots, x^m), \qquad 1 \le j \le n. \tag{35}$$

is called *the expression* of Φ in coordinates. The mapping Φ is differentiable at p if and only if the functions φ^i have partial derivatives to all orders in some fixed neighborhood of $(x_1(p), \dots, x_n(p))$. The mapping Φ is called a *diffeomorphism* of M onto N if Φ is bijective differentiable mapping and Φ^{-1} is also differentiable.

Let Φ be differentiable in $p \in M$. Let X_p be a tangent vector to M at p, i.e. $X_p : C^\infty(p) \to \mathbb{R}$. Then $Y_{\Phi(p)}(g) = X_p(g \circ \Phi)$ for $g \in C^\infty(\Phi(p))$ is a tangent vector to N a $\Phi(p)$. The mapping $X_p \to Y_{\Phi(p)}$ of the tangent space T_p of M to the tangent space $T_{\Phi(p)}$ on N is denoted $d\Phi_p$ *and is called the differential of* Φ *at* p.

Given charts as described above (35), we can choose the standard basis e_i on T_p and $\bar{e}_j$ on $T_{\Phi(p)}$, respectively, defined by

$$e_i : f \to \left(\frac{\partial f^*}{\partial x^i}\right)_{\psi(p)}, \quad (1 \le i \le m) \quad f^* = f \circ \psi^{-1}, \tag{36}$$

$$\bar{e}_j : g \to \left(\frac{\partial g^*}{\partial y^j}\right)_{\psi'(\Phi(p))}, \quad (1 \le j \le n) \quad g^* = g \circ \psi'^{-1}, \tag{37}$$

[5]We will not give a rigorous definition of a n-dimensional manifold with boundary. It is sufficient here to note that the boundary ∂M is itself a manifold of dimension $n-1$.

Then a short calculation, using the definitions, leads to

$$d\Phi_p(e_i) = \sum_{j=1}^{n} \left(\frac{\partial \varphi^j}{\partial x^i}\right)_{\psi(p)} \bar{e_j}. \tag{38}$$

Written in matrix form this is just the Jacobian of the system (35). From the standard inverse function theorem for mapping from $\mathbb{R}^m \to \mathbb{R}^n$ we conclude

Theorem *If $d\Phi_p$ is an isomorphism of T_p onto $T_{\Phi(p)}$ then there exists open sets $U \subset M$ and $V \subset N$ such that $p \in U$ and Φ is a diffeomorphism of U on V.*

Let M and N be manifolds. A mapping $\Phi : M \to N$ is called *regular* at $p \in M$ if Φ is differentiable at p and $d\Phi_p$ is a one-to-one mapping of the tangent space T_p of M in p into $T_{\Phi(p)}$, the tangent space of N at $\Phi(p)$. M is called a *submanifold* of N if (i) $M \subset N$ (set theoretically) and (2) the identity map I of M into N is regular at each point $p \in M$.

Let X and Y be vector fields on M and N and Φ a differentiable mapping of M into N. We say that X and Y are Φ-*related* if

$$d\Phi_p(X_p) = Y_{\Phi(p)} \quad \forall\, p \in M \qquad \text{we write } d\Phi \cdot X = Y \text{ or } X^\Phi = Y. \tag{39}$$

Eq. (39) is equivalent to

$$(Yf) \circ \Phi = X(f \circ \Phi) \quad \forall\, f \in C^\infty(N). \tag{40}$$

One has the following

Theorem *Assume $X_i^\Phi = Y_i,\ i = 1, 2,$ then*

$$[X_1, X_2]^\Phi = [Y_1, Y_2] \quad \text{or} \quad d\Phi \cdot [X_1, X_2] = [d\Phi \cdot X_1, d\Phi \cdot X_2]. \tag{41}$$

Assume the Φ is a diffeomorphism $M \to M$ and define $f^\Phi := f \circ \Phi^{-1}$ for $f \in C^\infty(M)$. Then we have

$$(fX)^\Phi = f^\Phi X^\Phi, \quad (Xf)^\Phi = X^\Phi f^\Phi \qquad \forall X \in \mathcal{D}^1(M). \tag{42}$$

Let Φ be a diffeomorphism of M into M and A be a mapping of $C^\infty(M) \to C^\infty(M)$. Then the mapping A^Φ is defined by $A^\Phi(f) = (A(f^{\Phi^{-1}})^\Phi$, or $(A^\Phi f) \circ \Phi = A(f \circ \Phi)$ similar to (40). If Φ and Ψ are two diffeomorphisms on M we have the composition rule $f^{\Phi\Psi} = (f^\Psi)^\Phi$ and $A^{\Phi\Psi} = (A^\Psi)^\Phi$.

Finally, let the vector fields X_i and Y_i be Φ-related, $i = 1, \ldots, r$. Then there exists a unique r-form $\Phi^*\omega$ on M such that

$$\Phi^*\omega(X_1, \ldots, X_r) = \omega(Y_1, \ldots, Y_r) \circ \Phi. \tag{43}$$

In fact it is defined by

$$(\Phi^*\omega)_p(A_i,\dots,A_r) = \omega_{\Phi(p)}(d\Phi_p(A_1),\dots,d\Phi_p(A_r)) \tag{44}$$

for each $p \in M$ and $A_i \in T_p$. By linearity $\Phi^*\omega$, here defined on $\mathcal{U}_r(M)$, can be extended to the whole $\mathcal{U}(M)$. If we define $\Phi^* f := f \circ \Phi$ for $f \in C^\infty(N)$ the following formulas hold

$$\Phi^*(\omega_1 \wedge \omega_2) = \Phi^*(\omega_1) \wedge \Phi^*(\omega_2) \qquad \omega_1, \omega_2 \in \mathcal{U}(M), \tag{45}$$

$$\mathrm{d}(\Phi^*\omega) = \Phi^*(\mathrm{d}\omega) \tag{46}$$

The computation of $\Phi^*\omega$ in coordinates is simple. We still use the notation from (35) and find from the definitions that

$$\Phi^*(\mathrm{d}y^j) = \sum_{i=1}^{m} \Big(\frac{\partial\varphi^j}{\partial x^i} \circ \psi\Big)\, \mathrm{d}x^i \tag{47}$$

Now a general $\omega \in \mathcal{U}(N)$ can be written locally as

$$\omega = \sum g_{j_1\dots j_s} \mathrm{d}y^{j_1} \wedge \cdots \wedge \mathrm{d}y^{j_s} \tag{48}$$

and the form $\Phi^*\omega \in \mathcal{U}(M)$ has now the local expression

$$\Phi^*\omega = \sum f_{i_1\dots i_r} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_r} \tag{49}$$

where $f_{i_1\dots i_r}$ can be calculated from $g_{j_1\dots j_s}$ using (47) and (45).

For integrations of n-forms one has in the notation above (33) a mapping from a region $U \subset M$ to a region $\Phi(U) \subset N$ and the following formula

$$\int_U \Phi^*\omega = \int_{\Phi(U)} \omega, \qquad \omega \in \mathcal{U}_n(N). \tag{50}$$

Expressed in coordinates as in (33), (48) and (49) it is simply a charge of variables involving the appropriate Jabobian.

Affine connections and parallelism

An *affine connection* on a manifold M is a rule which assigns to each $X \in \mathcal{D}^1(M)$ a linear mapping $\nabla_X : \mathcal{D}^1(M) \to \mathcal{D}^1(M)$ which satisfies

$$\begin{aligned} \nabla_{fX+gY} &= f\nabla_X + g\nabla_Y && (51)\\ \nabla_X(fY) &= f\nabla_X(Y) + (Xf)Y && (52) \end{aligned}$$

Let (U, φ) be a chart. Thus we have a coordinate system $\varphi : p \to (x^1(p), \ldots, x^n(p))$ for $p \in U$. We have the following coordinate expression for the connection for $p \in U$:

$$\nabla_i := \nabla_{\partial/\partial x^i}, \qquad \nabla_i\Big(\frac{\partial}{\partial x^j}\Big) = \sum_k \Gamma_{ij}{}^k \, \frac{\partial}{\partial x^k} \tag{53}$$

Here $\Gamma_{ij}{}^k$ is a shorthand for $\Gamma_{ij}{}^k \circ \varphi^{-1}$. $\Gamma_{ij}{}^k$ depends on the chart (U, φ) and one can find the transformation properties of $\Gamma_{ij}{}^k$ when changing chart to (U, φ'), where $\varphi' : p \to (y^1(p), \ldots, y^n(p))$ by using (51) and (52):

$$\nabla_\alpha\Big(\frac{\partial}{\partial y^\beta}\Big) = \sum_\gamma \Gamma'_{\alpha\beta}{}^\gamma \, \frac{\partial}{\partial y^\gamma} \tag{54}$$

$$\Gamma'_{\alpha\beta}{}^\gamma = \sum_{i,j,k} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} \frac{\partial y^\gamma}{\partial x^k} \, \Gamma_{ij}{}^k + \sum_j \frac{\partial^2 x^j}{\partial y^\alpha \partial y^\beta} \, \frac{\partial y^\gamma}{\partial x^j} \tag{55}$$

Suppose ∇ is an affine connection on M and Φ is a diffeomorphism of M. We can now define a new affine connection ∇^Φ on M by

$$\nabla^\Phi_X(Y) := (\nabla_{X^\Phi}(Y^\Phi))^{\Phi^{-1}}, \qquad X, Y \in \mathcal{D}^1(M). \tag{56}$$

The affine connection ∇ is called *invariant* under Φ if $\nabla = \nabla^\Phi$. In this case Φ is called an *affine transformation* on M. Similarly one can define an affine transformation from one manifold onto another.

Let M be a manifold. A *curve* in M is a C^∞ mapping of an open interval I into M. The restriction of a curve to a closed subinterval of I is called a *curve segment*. Since I can be viewed as a manifold (with coordinate t) $\gamma : t \to \gamma(t) \in M,\ t \in I$ is a mapping from $I \to M$ and correspondingly the vector d/dt is mapped from T_t into $d\gamma_t\big((d/dt)_t\big) \in T_{\gamma(t)}$ by the differential $d\gamma$ of γ. We introduce the notation

$$\dot{\gamma}(t) := d\gamma_t \left(\Big(\frac{d}{dt}\Big)_t \right) \in T_{\gamma(t)}. \tag{57}$$

Let now J be a compact subinterval of I chosen so small that there are no double points in γ_J. Define $X(t) := \dot{\gamma}(t),\ (t \in I)$. One can show that there exist a vector field $X \in \mathcal{D}^1(M)$ such that $X_{\gamma(t)} = X(t)\ \forall t \in J$. The vector field X is in no way unique away from the points $p = \gamma(t)$. Let Y be another vector field in $\mathcal{D}^1(M)$. We define $Y(t) := Y_{\gamma(t)}\ \forall t \in J$. Again, there are many vector fields Y with the same family $Y(t),\ t \in J$. The family $Y(t),\ t \in J$ is said to be *parallel* with respect to γ_J (or parallel along γ_J) if

$$\nabla_X(Y)_{\gamma(t)} = 0 \quad \forall t \in J. \tag{58}$$

We can see that (58) is actually independent of the choice of X, Y, i.e. the arbitrariness of X, Y away from the points $\gamma(t)$, by expressing (58) in local coordinates. Using

$$X = \sum_i X^i \frac{\partial}{\partial x^i}, \qquad Y = \sum_i Y^i \frac{\partial}{\partial x^i} \tag{59}$$

we obtain

$$\nabla_X(Y) = \sum_k \left(\sum_i X^i \frac{\partial Y^k}{\partial x^i} + \sum_{i,j} X^i Y^j {\Gamma_{ij}}^k \right) \frac{\partial}{\partial x^k}. \tag{60}$$

If we write $x^i(t) := x^i(\gamma(t))$, $X^i(t) := X^i(\gamma(t))$ and $Y^i(t) := Y^i(\gamma(t))$ then from the definition $X(t) = \dot{\gamma}(t)$ we obtain $X^i(t) = \dot{x}^i(t)$ and from (58) and (60) we obtain

$$\frac{dY^k}{dt} + \sum_{i,j} {\Gamma_{ij}}^k \frac{dx^i}{dt} Y^j = 0 \tag{61}$$

This equation involves X, Y only through their values *on* the curve and eq. (58) for *parallelism* is independent of the choice of X, Y away from the curve. We can now extend the definition to any J and finally to I, i.e. to the entire curve γ.

Let γ be a curve passing though p and q. Let $Y_p \in T_p$. Solving (61) we parallel transport Y_p to a vector $Y_q \in T_q$. We write $Y_q = \tau_{qp} Y_p$. This induces a map of T_p into T_q. We have

Theorem *Let p and q be two points in M and γ a curve segment from p to q. The parallelism τ with respect to γ induces an isomorphism of T_p onto T_q.*

The curve γ is called a *geodesic* if the family of tangent vectors $\dot{\gamma}(t)$ is parallel with respect to γ. A geodesic is called *maximal* if it is not a proper restriction of any geodesic. In local coordinates we obtain from (61) the following equation for the geodesic:

$$\frac{d^2 x^k}{dt^2} + \sum_{i,j} {\Gamma_{ij}}^k \frac{dx^i}{dt} \frac{dx^j}{dt} = 0 \tag{62}$$

Note that if we change parameter on a geodesic and put $t = f(s)$ then we get a new curve $\tilde{\gamma}(s)$, but it is only a geodesic if f is a linear function. We have the following

Theorem *Let M be a manifold with an affine connection. Let p be a point in M and let $X \neq 0$ be in T_p. Then there exists a unique maximal geodesic $t \to \gamma(t)$ in M such that*

$$\gamma(0) = p, \qquad \dot{\gamma}(0) = X. \tag{63}$$

We denote this geodesic γ_X and we have the following theorem

Theorem *Let p be any point in M. Then there exists an open neighborhood N_0 of $0 \in T_p$ and an open neighborhood N_p of $p \in M$ such that the mapping $X \to \gamma_X(1)$ is a diffeomorphism of N_0 onto N_p.*

The mapping $X \to \gamma_X(1)$ is called the *Exponential mapping* at p and denoted Exp (or Exp_p).

An open neighborhood $N_0 \in T_p$ is called *normal* if (1) Exp is a diffeomorphism of N_0 onto an open neighborhood $N_p \in M$ and (2) $X \in N_0$ implies that $tX \in N_0,\ 0 \leq t \leq 1$ (N_0 is *star-shaped*). A neighborhood of N_p of $p \in M$ is called *normal* if $N_p = \mathrm{Exp}\, N_0$ where N_0 is a normal neighborhood of T_p. Let $X_1, \ldots, X_n$ be a basis of T_p and consider now the inverse mapping of $N_p \to \mathbb{R}^n$:

$$p = \mathrm{Exp}(x^1 X_1 + \cdots + x^n X_n) \to (x^1(p), \ldots, x_n(p)). \tag{64}$$

It is called *normal coordinates* at p. We have the following theorem

Theorem *Let M be a manifold with an affine connection. Then each point $p \in M$ has a normal neighborhood N_p which is a normal neighborhood of each of its points. In particular, two arbitrary points in N_p can be joined by exactly one[6] geodesic segment contained in N_p.*

We have defined parallelism of vector fields by means of covariant differentiation ∇_X. One can now use the concept of parallel transportation to define the covariant derivative of other tensor fields. Let $X \in \mathcal{D}^1(M)$. A curve $s \to \varphi(s),\ s \in I$ is called an *integral curve* of X if

$$\dot{\varphi}(s) = X_{\varphi(s)}, \qquad s \in I \tag{65}$$

Assuming $0 \in I$ and let $p = \varphi(0)$, writing (65) in local coordinates one obtains a standard system of first order differential equations which has a solution if $X_p \neq 0$. Thus there exists an integral curve of X through p and we have the following theorem

Theorem *Let M be a manifold with an affine connection. Let $p \in M$ and let $X, Y \in \mathcal{D}^1(M)$. Assume $X_p \neq 0$. Let $s \to \varphi(s)$ be an integral curve to X through $p = \varphi(0)$ and τ_t the parallel translation from p to $\varphi(t)$ with respect to φ. Then*

$$\big(\nabla_X(Y)\big)_p = \lim_{s \to 0} \frac{1}{s}\Big(\tau_s^{-1} Y_{\varphi(s)} - Y_p\Big) \tag{66}$$

We have define parallel transport $\tau \cdot Y$ of a vector field Y along the curve γ from p to q. If $F \in T_p^*$ we define $\tau \cdot F \in T_q^*$ by the formula

$$(\tau \cdot F)(A) := F(\tau^{-1} \cdot A) \qquad \forall\ A \in T_q \tag{67}$$

[6]Up to a linear change of parameter on the geodesic) geodesic segment contained in N_p.

and this generalizes to any tensor field $T \in \mathcal{D}^r_s(M)$. We define $\tau \cdot T_p \in \mathcal{D}^r_s(q)$ by

$$(\tau \cdot T_p)(F_1, \ldots, F_r, A_1, \ldots A_s) := T_p(\tau^{-1}F_1, \ldots, \tau^{-1}F_r, \tau^{-1}A_1, \ldots, \tau^{-1}A_s) \quad (68)$$

where $A_i \in T_q$ and $F_j \in T^*_q$. We then have the following:

$$\big(\nabla_X(T)\big)_p = \lim_{s\to 0} \frac{1}{s}\Big(\tau_s^{-1}T_{\varphi(s)} - T_p\Big) \quad (69)$$

If we finally define

$$\big(\nabla_X(f)\big)_p = \lim_{s\to 0} \frac{1}{s}\Big(f(\varphi(s)) - f(p)\Big), \qquad \text{(i.e. } \nabla_X f = Xf\text{)}, \quad (70)$$

then we have extended ∇_X to a linear mapping of $\mathcal{D}(M)$ into $\mathcal{D}(M)$:

Theorem *The operator ∇_X has the following properties:*
(1) ∇_X is a derivation of the mixed tensor algebra $\mathcal{D}(M)$ (considered as an algebra over $\mathbb{R}$).
(2) ∇_X preserves type of tensors.
(3) ∇_X commutes with contractions[7]

Let us end this section by defining two important tensor fields, the *torsion tensor field* $T(X,Y)$ and the *curvature tensor field* $R(X,Y)$ by

$$\begin{aligned} T(X,Y) &= \nabla_X(Y) - \nabla_Y(X) - [X,Y], & (71)\\ R(X,Y) &= \nabla_X\nabla_Y - \nabla_Y\nabla_Y - \nabla_{[X,Y]} & (72) \end{aligned}$$

The mappings

$$(\omega, X, Y) \to \omega(T(X,Y)), \quad (\omega, X, Y, Z) \to \omega(R(X,Y)Z) \quad (73)$$

are $C^\infty(M)$-multilinear maps from $\mathcal{D}_1 \times \mathcal{D}^1 \times \mathcal{D}^1 \times \mathcal{D}^1$ into $C^\infty(M)$ and from $\mathcal{D}_1 \times \mathcal{D}^1 \times \mathcal{D}^1 \times \mathcal{D}^1 \times \mathcal{D}^1$ into $C^\infty(M)$, respectively. Thus T and R belong to $\mathcal{D}^1_2(M)$ and $\mathcal{D}^1_3(M)$, respectively. Let $p \in M$ and let $X_1, \ldots, X_n$ be a basis for the vector fields in some neighborhood N_p of p. Thus $X = \sum_i f_i X_i$, $f_i \in C^\infty(N_p)$ and the functions ${\Gamma_{ij}}^k$, ${T^k}_{ij}$ and ${R^k}_{lij}$ are $C^\infty(N_p)$ functions defined by the formulas

$$\begin{aligned} \nabla_{X_i}(X_j) &= \sum_k {\Gamma_{ij}}^k X_k & (74)\\ T(X_i, X_j) &= \sum_k {T^k}_{ij} X_k & (75)\\ R(X_i, X_j)\, X_l &= \sum_k {R^k}_{lij} X_k. & (76) \end{aligned}$$

[7]We have strictly speaking not defined contractions.

Let ω^i be the 1-forms dual to X_j and let $\omega^i{}_j$ be the 1-forms on N_p determined by

$$\omega^i(X_j) = \delta^i{}_j \qquad \omega^i{}_j = \sum_k \Gamma_{kj}{}^i \, \omega^k. \tag{77}$$

Knowing $\omega^i{}_j$ in N_p determines $\Gamma_{ij}{}^k$ on N_p, and thus the connection ∇. The following theorem shows that $\omega^i{}_j$ is itself determined by T and R, which in this sense determine essential properties of the manifold:

Theorem *(the structural equations of Cartan)*

$$\mathrm{d}\omega^i = -\sum_k \omega^i{}_k \wedge \omega^k + \frac{1}{2}\sum_{j,k} T^i{}_{jk}\, \omega^j \wedge \omega^k \tag{78}$$

$$\mathrm{d}\omega^i{}_l = -\sum_k \omega^i{}_k \wedge \omega^k{}_l + \frac{1}{2}\sum_{j,k} R^i{}_{ljk}\, \omega^j \wedge \omega^k. \tag{79}$$

One geometry interpretation of the curvature tensor is linked to parallel transport of vector fields. In $\mathbb{R}^n$ parallel transport of a of a vector along a closed curve will not change the vector. It is different for a n-dimensional manifold where parallel transport is expressed via a non-trivial affine connection ∇. Let $p \in M$. Let us use the map Exp_p to create a small closed curve, starting and ending in p by creating a corresponding curve in T_p and map to back to M using Exp_p. Let $X, Y \in T_p$ and let the curve $\tilde{\gamma}$ in T_p be the boundary of the quadrilateral with sides tY, sX, $-tY$ and $-sX$. For sufficiently small t, s the map Exp_p will be well defined on this quadrilateral . The parallel transport of a vector $Z_p \in T_p$ along the curve $\gamma = \mathrm{Exp}_p \circ \tilde{\gamma}$ will result in a new vector $Z_p' \in T_p$. Let us first consider the curve segment $\mathrm{Exp}_p(t'Y)$, $0 \leq t' \leq t$ from p to $p_1 = \mathrm{Exp}_p(tY)$. A vector $Z_p \in T_p$ is parallel transported to a vector $Z_{p_1} = \tau_{pp_1} Z \in T_{p_1}$. To lowest order in t we have from (66) that $Z_{p_1} = Z_p + t\,\nabla_Y Z_p + O(t^2)$. Similar estimates can be made for the parallel transport along the other three curve segments, introducing ∇_X, ∇_{-Y} and ∇_{-X} . Applying the four parallel transportations successively to obtain the vector Z_p', the terms linear in t and s cancel and to quadratic order one obtains

$$\Delta Z_p := Z_p' - Z_p = st([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})_p Z_p = st\, R_p(X_p, Y_p) Z_p \tag{80}$$

Introducing a basis X_i on T_p, and writing $X = x^i X_i$, $Y = y^i X_i$ and $Z = z^i X_i$ (summation over repeated indices), as well as the coordinate expression (76) for the curvature tensor, (80) reads

$$\Delta z^k = st\, R^k{}_{lij} x^i y^j z^l = R^k{}_{lij} \Delta A^{ij}\, z^l, \tag{81}$$

where $\Delta A^{ij} = \frac{st}{2}\left(x^i y^j - x^j y^j\right)$ is the antisymmetric area element of the parallelogram spanned by the vectors sX and tY.

Let γ_p be a closed curve in M, starting and ending at p. Call the set of such curves C_p. Clearly parallel transportation $\tau(\gamma_p)$ along γ_p provides us with a linear map $T_p \to T_p$ and the set

$$H_p(\nabla) = \{\tau(\gamma_p) \,|\, \gamma_p \in C_p\} \subseteq \mathrm{Gl}(T_p) = \mathrm{Gl}(n, \mathbb{R}). \tag{82}$$

An element $\tau(\gamma_p)$ is called a *holonomy* at p and $H_p(\nabla)$ can be made a group, the *holonomy group* at p, by a proper composition of curves. We will not pursue this further, but close with the remark that the subgroup connected to the identity is a connected Lie subgroup of $\mathrm{Gl}(n, \mathbb{R})$.

Riemannian manifolds

Let M be a manifold. A *pseudo Riemannian structure* on M is a tensor field g of type (0,2) which satisfies $g(X,Y) = g(Y,X)\ \forall X, Y \in \mathcal{D}^1(M)$ and for each $p \in M$ g_p is a non-degenerate bilinear form on $T_p \times T_p$. One also calls the tensor field g *the metric tensor* for reasons which will soon be clear. If g_p is a positive definite bilinear form one drops the "pseudo" and talk about a *Riemannian structure* and a *Riemannian manifold*. The following theorem is valid:

Theorem *On a pseudo-Riemannian manifold there exists one and only one affine connection satisfying the following two conditions:*
(1) The torsion $T = 0$.
(2) The parallel displacement preserves the inner product on the tangent spaces.

The corresponding connection is called the *Riemannian connection or Levi-Civiti connection*. We can now introduce a Riemannian metric on a Riemannian manifold. Let $t \to \gamma(t),\ (\alpha \leq t \leq \beta)$ be a curve segment in M. The *arc length* of γ is defined by

$$L(\gamma) = \int_\alpha^\beta \left(g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))\right)^{1/2} dt. \tag{83}$$

The Riemannian structure g introduces a *real* scalar product $\langle \cdot | \cdot \rangle$ for the vectors $X, Y \in T_p$ by:

$$\langle X|Y\rangle := g_p(X,Y), \qquad ||X||^2 :=\rangle X|X\rangle = g_p(X,X), \tag{84}$$

and we have the following theorem:

Theorem *Let M be a Riemannian manifold and $p \in M$. Let N_0 be a normal neighborhood of 0 in T_p and put $N_p = \mathrm{Exp}\, N_0$. For each $q \in N_p$ let γ_{pq} denote* *the unique geodesic in N_p joining p and q. Then*

$$L(\gamma_{pq}) < L(\gamma) \tag{85}$$

for each curve segment $\gamma \neq \gamma_{pq}$ joining p and q. If the normal neighborhood N_0 is an open ball $0 \leq ||X|| < \delta$, the inequality (85) holds for each curve segment in M which joins p and q.

The Riemannian manifold can now be turned into a metric space. M is assumed connected so each pair of points $p, q \in M$ can be joined by an curve segment. The *distance* of p and q is now defined by

$$d(p,q) := \inf_{\gamma} L(\gamma) \tag{86}$$

where γ runs over all curve segments joining p and q. One can show that $d(p,q)$ satisfies the requirements of a metric: (1) $d(p,q) = d(q,p)$, (2) $d(p,q) \leq d(p,r) + d(r,q)$ and $d(p,q) = 0$ only if $p = q$. The distance function d turns M into a metric space. For $p \in M$ we now define the open "ball" around p as

$$B_r(p) = \{q \in M \,|\, d(p,q) < r\}, \qquad 0 \leq r \leq \infty \tag{87}$$

Theorem *If the open ball*

$$V_r(0) = \{X \in T_p \,|\, 0 \leq ||X|| < r\} \tag{88}$$

is a normal neigborhood of 0 in T_p then

$$B_r(p) = \mathrm{Exp} V_r(0). \tag{89}$$

One can show that the topology of M viewed as a metric space with metric $d(p,q)$ coincides with the original topology on M. One has the following theorem

Theorem *Let M be a Riemannian manifold with metric d. To each $p \in M$ corresponds a number $r(p) > 0$ such that if $0 < \rho \leq r$ then $B_\rho(p)$ has the properties*
(1) $B_\rho(p)$ is a normal neighborhood of each of its points.
(2) Let $a, b \in B_\rho(p)$ and let γ_{ab} be the unique geodesic in $B_\rho(p)$ joining a and b. The γ_{ab} is the only curve segment in M of length $d(a,b)$ which joins a and b.

Theorem *In the notation of the previous theorem let A and B be the unique points in T_p satisfying the relations*

$$\mathrm{Exp}_p A = a, \quad \mathrm{Exp}_p B = b, \quad ||A|| < r(p), \;\; ||B|| < r(p), \tag{90}$$

Then

$$\frac{||A-B||}{d(a,b)} \to 1 \quad \text{as} \quad (a,b) \to (p,q) \tag{91}$$

Recall that on a metric space with metric d a sequence (x_n) is called a Cauchy sequence if for all $\epsilon > 0$ there exists an integer N such that $d(x_m, x_n) < \epsilon$ for all $n, m > N$. A Riemannian manifold is said to be *complete* if every Cauchy sequence in M is convergent. The importance of the completeness condition is apparent in the following two theorems:

Theorem *Let M be a Riemannian manifold, The following conditions are equivalent:*
(1) M is complete.
(2) Each bounded closed subset of M is compact.
(3) Each maximal geodesic in M has the form $\gamma_X(t)$, $-\infty < t < \infty$.

Theorem *In a complete Riemanian manifold M with metric d each pair $p, q \in M$ can be joined by a geodesic of length $d(p, q)$.*

Let M be an oriented Riemannian manifold and let $\{x^1, \ldots, x^n\}$ be a local coordinate system on an open subset U and let

$$g_{ij} := g\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big), \qquad \bar{g}(x) := \det(g_{ij}). \tag{92}$$

Then $\bar{g} > 0$ and we consider the n-form on U defined by

$$\omega = \sqrt{\bar{g}(x)}\, dx^1 \wedge \cdots \wedge dx^n \tag{93}$$

If $\{y^1, \ldots, y^n\}$ denotes another coordinate system on U then it is easy to show that

$$\sqrt{\bar{g}(x)}\, dx^1 \wedge \cdots \wedge dx^n = \sqrt{\bar{g}(y)}\, dy^1 \wedge \cdots \wedge dy^n. \tag{94}$$

Thus ω is independent of the coordinate system and exists as an n-form on M, the so-called *volume element* corresponding to the Riemannian structure g on the oriented manifold M. We have the following

Theorem *Let ω be the volume element of an oriented Riemannian manifold M. Then*

$$\nabla_X(\omega) = 0 \tag{95}$$

for each vector field X on M

Let M and N be two two manifolds with pseudo-Riemannian structures g and h and let φ be a mapping of M into N. *We call φ an isometry if φ is a diffeomorphism of M onto N which satisfies $\varphi^* h = g$* . We have by definition

$$\varphi^* h = g \implies h_{\varphi(p)}(d\varphi_p(X), d\varphi_p(Y)) = g_p(X,Y) \implies ||d\varphi_p(X||_h = ||X||_g. \tag{96}$$

It is obvious that if φ is an isometry of a Riemannian manifold M on itself then φ preserves distances. However, the opposite is also true:

Theorem *Let M be a Riemannian manifold and φ a distance preserving mapping of M onto itself. Then φ is an isometry.*

Also, we have the following

Theorem *Let M be a Riemannian manifold, φ and ψ two isometries of M onto itself. Suppose there exists a point $p \in M$ such that $\varphi(p) = \psi(p)$ and $d\varphi_p = d\psi_p$. Then $\varphi = \psi$.*

Above we discussed the importance of Riemann curvature tensor for holonomies. Let us end this section with a discussion of the classical geometric significance of the curvature tensor and its relation to Gaussian curvature. As usual let $V_r(0)$ denote the open ball in T_p with center at 0 and radius r and $B_r(p) = \mathrm{Exp}_p(V_r(0))$. Again, as usual we assume r is small the Exp_p is a diffeomorphism of $V_r(0)$ onto $B_r(p)$. Let us now restrict ourselves to a *two-dimensional* manifold F and let $A_0(r)$ and $A(r)$ denote the volumes (i.e. now areas) of $V_r(0)$ and $B_r(p)$, respectively. The areas are calculated using the volume elements ω_V and ω_B on $V_r(0)$ and $B_r(p)$, respectively, as defined by (92)-(93). The *curvature* of F at p, also called the *Gaussian curvature* at p, is defined as the limit

$$K(p) = \lim_{r\to 0} 12\,\frac{A_0(r) - A(r)}{r^2 A_0(r)}. \tag{97}$$

We can define the function $f(X)$ on $V_r(0)$ by $\mathrm{Exp}_p^*(\omega_B)(X) = f(X)\omega_V(X)$. $f(X)$ is basically the density function betweens the measures on $B_r(p)$ and $V_r(0)$ and thus, according to (50),

$$A(r) = \int_{B_r(p)} \omega_B(p) = \int_{V_r(0)} \mathrm{Exp}_p^*(\omega_B)(X) = \int_{V_r(0)} f(X)\,dX \tag{98}$$

$$A_0(r) = \int_{V_r(0)} \omega_V(X) = \int_{V_r(0)} dX \tag{99}$$

and one can show (by Taylor expanding f to second order) that

$$K(p) = -\frac{3}{2}(\Delta f)(0) \tag{100}$$

where Δ is the Laplacian on V_p.

The Taylor expansion of f touches some general aspects also valid in higher dimensions so let us mention these. Let us consider a Riemannian manifold with a metric tensor g. As mentioned above the Exp-map of $V_r(0)$ onto $B_r(p)$ provides us with *Riemann normal coordinates.* Let us here define them more precisely.

First we identify T_p with $\mathbb{R}^n$ in an obvious way: We choose a basis $e_i(p)$ in T_p such that $g_p(e_i(p), e_j(p)) = \delta_{ij}$. Any vector $X \in T_p$ can be written as $X = x^i e_i(p)$ (Einstein's summation convention). Let $q \in B_r(p)$. The coordinate map φ is now $\varphi : q \to \mathrm{Exp}_p^{-1}(q) = (x^1(q), \ldots, x^n(q))$. This defines the normal coordinates and we have

$$\frac{\partial}{\partial x^i}\Big|_p = e_i, \quad g_{ij}(x) := g\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)_q, \quad x = \mathrm{Exp}_p^{-1}(q). \tag{101}$$

The geodesics passing through p, that is $x = 0$ to a point $(x_1, \ldots, x_n)$ are by the defintion of Exp_p just the straight line $t \to (tx_i, \ldots, tx_n)$. From this we obtain

$$\frac{\partial g_{ij}(x)}{\partial x^k}\Big|_{x=0} = 0, \quad \Gamma_{ij}{}^k(0) = 0, \quad g_{ij}(0) = \delta_{ij}. \tag{102}$$

Expanding $g_{ij}(x)$ to second order in x thus involves no linear terms and the second order derivatives of $g_{ij}(x)$ at $x = 0$ combine to the curvature tensor at $x = 0$:

$$g_{ij}(x) = g_{ij}(0) + \frac{1}{3} R_{ikjl}(0)\, x^k x^l + O(|x|^3). \tag{103}$$

It now follows that

$$\det g_{ij}(x) = 1 + \frac{1}{3} R_{ij}(0)\, x^i x^j + O(|x|^3). \quad \sqrt{\bar{g}(x)} = 1 + \frac{1}{6} R_{ij}(0)\, x^i x^j + O(|x|^3). \tag{104}$$

In (103) we have the following definition

$$R(X,Y,Z,V) := g(R(X,Y)Z, V), \quad \text{or in coordinates} \quad R_{ijkl} = g_{im} R^m{}_{jkl}, \tag{105}$$

while *Ricci tensor* is defined by

$$\mathrm{Ric}(X,Y) = \sum_{k=1}^n R(X, e_k, Y, e_k), \quad \text{or in coordinates} \quad R_{ij} = R^k{}_{ikj}, \tag{106}$$

Eq. (104) provides us with the expansion of f to second order used in (98) and in fact we can write

$$K(p) = -R(e_1, e_2, e_1, e_2)(p) \tag{107}$$

We will now see that in some sense this concept of two-dimensional curvature contains the full information about the higher dimensional curvature tensor.

Let M be a Riemannian manifold and $p \in M$. Let N_0 be a normal neighborhood of 0 in T_p and $N_p = \mathrm{Exp} N_0$. Let S be a two-dimensional vector subspace of T_p. Then $\mathrm{Exp}(N_0 \cap S)$ is a connected two-dimensional submanifold of M of dimension 2 and it has a Riemannian structure induced by that of M. The curvature of

$\mathrm{Exp}(N_0 \cap S)$ as defined in (97) is called the *sectional curvature* of M at p along the *plane section* S. One has

Theorem *Let M be a Riemannian manifold with curvature tensor field R and Riemannian structure g. Let p be a point in M and S a two-dimensional vector space of the tangent space T_p. The sectional curvature of M at p along the section S is then*

$$K(S) = -\frac{(R_p(Y,Z,Y,Z)}{|Y \vee Z|^2}. \tag{108}$$

Here Y and Z are any linearly independent vectors in S and $|Y \vee Z|$ denotes the area of the parallelogram spanned by the vectors Y and Z.

Note that the expression (109) is actually independent of the vectors Y, Z as long as they are chosen as linear independent vectors in the plane S (if that was not the case the notation $K(S)$ would of course not make any sense). By definition $|Y \vee Z|^2 = g_p(Y,Y)g_p(Z,Z) - g_p(Y,Z)^2$ and the symmetries of $R_p(Y,Z,Y,Z) = g_p(R(Y,Z)Y,Z)$ allow us to extract a similar factor from $R_p(Y,Z,Y,Z)$.

Let us choose a basis e_i in T_p such that $g_p(e_i,e_j) = \delta_{ij}$ and such that e_1, e_2 span S. Then (109) reads:

$$K(e_1,e_2) := K(S) = -R_p(e_1,e_2,e_1,e_2) \tag{109}$$

i.e. precisely the Gaussian curvature we met in (108).

As mentioned above the sectional curvature determines the complete curvature tensor because of the following theorem:

Theorem *Let M be a Riemannian manifold, p a point in M. Let g and g' be two Riemannian structures on M, R and R' the corresponding curvature tensors, $K(S)$ and $K'(S)$ the corresponding sectional curvatures at p along a plane section $S \subset T_p$. Suppose that $g_p = g'_p$. If $K(S) = K'(S)$ for all plane sections $S \subset T_p$. Then $R_p = R'_p$.*

In fact there exists an explicit formula which expresses $R_p(X,Y,Z,V)$ in terms of $R_p(X,Y,X,Y)$ and all possible combinations $R_p(X,Z,X,Z)$, $R_p(Y+V,Z,Y+V,Z)$ etc.

(2) Lie groups as analytic manifolds

As we saw the exponential map Exp played a crucial role when analysing Riemannian manifolds. However, we never really provided or used a Taylor expansion of the map, the reason being that the functions in the coordinate charts might not have such a Taylor expansion despite being infinitely many times differentiable. However we can make the situation nicer and the manifolds we consider correspondingly more special by defining so-called *analytic manifolds*. The are defined as smooth manifolds except that one requires that there exists an atlas such that the functions $\varphi_i \circ \varphi_j^{-1}$ are now *analytic* functions in $\varphi_j(U_i \cap U_j)$. Similarly we now define analytic tensor fields and analytic connections.

A *Lie group* is defined as a topological group G (see *addednotes2* for some definitions) which is also an analytic manifold such that the mapping $(g, \sigma) \to g\sigma^{-1}$ of $G \times G \to G$ is analytic.

A homomorphism of a Lie group into another which is also an analytic mapping is called an *analytic homomorphism*. Similarly an isomorphism of a Lie group into another which is also an analytic diffeomorphism is called an *analytic isomorphism*.

Left invariant connections and the exponential map

Let G be a Lie group, i.e. in particular an analytic manifold and let $\rho \in G$. Then *left translation* $L_\rho : g \to \rho g$ is an analytic diffeomorphism of G onto itself. The differential dL_ρ induces a mapping of $\mathcal{D}^1(G)$ into $\mathcal{D}^1(G)$ defined by (39)-(40):

$$(\mathrm{d}L_\rho \cdot Z)f = Z(f \circ L_\rho) \circ L_\rho^{-1} \qquad \forall Z \in \mathcal{D}^1(G) \quad \forall f \in C^\infty(G). \tag{110}$$

We call a vector field Z *left invariant* if

$$dL_\rho \cdot Z = Z \qquad \forall \rho \in G. \tag{111}$$

Given a tangent vector $X \in T_e$, where T_e denotes the tangent space at the identity element e of G, there exists exactly one left invariant vector field $\tilde{X}$ on G such that $\tilde{X}_e = X$. In fact $\tilde{X}$ is defined by

$$\big(\tilde{X}(f)\big)(\rho) = (X(f \circ L_\rho))(e) = X_e(f \circ L_\rho) \tag{112}$$

Let $\gamma(t)$ be any curve in G with tangent vector X for $t = 0$, i.e. $\dot{\gamma}(0) = X$. Then one can also write

$$\big(\tilde{X}(f)\big)(\rho) = \Big(\frac{d}{dt} f(\rho\gamma(t))\Big)_{t=0}. \tag{113}$$

It follows from the definition (57): $(X\tilde{f})(0) = (\dot{\gamma}(0)\tilde{f})(0) = \big(d/dt\,(\tilde{f}\circ\gamma)\big)_{t=0}$. In (113) this is applied for the function $\tilde{f} = f \circ L_\rho$. From the definition (113) and (111) one can check that (112) is fulfilled. Given $X, Y \in T_e$ we know that $[X, Y] \in T_e$, but in addition we have for the corresponding left invariant vector fields $\tilde{X}, \tilde{Y}$ that also $[\tilde{X}, \tilde{Y}]$ is an invariant vector field due to (41), and obviously $[\tilde{X}, \tilde{Y}]_e = [X, Y]$.

The vector space T_e of G, with the composition rule $(X, Y) \to [X, Y]$ is called the Lie algebra of G. We call it $\mathfrak{g}$ or $\mathcal{L}(G)$. It can also be identified as the correspondingly defined algebra of left invariant vector fields on G which we will denote $\mathfrak{g}^L$.

One can show that $\mathfrak{g}$ indeed satisfies the axioms of a real Lie algebra. Let ∇ be an affine connection on G. We say that ∇ is a left invariant connection if each L_ρ, $\rho \in G$ is an affine transformation of G. According to the definition of an affine transformation below eq. (56) this means that the affine connection $\nabla^{L_\rho} = \nabla$ for all $\rho \in G$. One can show that this is equivalent to the requirement that $\nabla_{\tilde{X}}(\tilde{Y})$ is a left invariant vector field for any two left invariant vector fields $\tilde{X}, \tilde{Y} \in \mathfrak{g}^L$. One has the following theorem

Theorem *There is a one-to-one correspondence between the set of left invariant affine connections on G and the set of bilinear functions α on $\mathfrak{g} \times \mathfrak{g}$ with values in $\mathfrak{g}$, given by:*

$$\alpha(X, Y) = (\nabla_{\tilde{X}}(\tilde{Y}))(e). \tag{114}$$

Let $X \in \mathfrak{g}$. The following statements are equivalent:
(1) $\alpha(X, X) = 0$
(2) The geodesic $t \to \gamma_X(t)$ is an analytic homomorphism of $\mathbb{R} \to G$.

In (2) above $\mathbb{R}$ is viewed as a Lie group. If θ is any analytic homomorphism of $\mathbb{R}$ into G such that $\dot{\theta}(0) = X$ then it follows from $\theta(s+t) = \theta(s)\theta(t)$ that

$$\theta(0) = e, \quad \dot{\theta}(t) = \tilde{X}_{\theta(t)} \qquad \forall t \in \mathbb{R}. \tag{115}$$

If γ_X is an analytic homomorphism we have $\nabla_{\tilde{X}}(\tilde{X}) = 0$ on the curve γ_X. Thus $\alpha(X, X) = \nabla_{\tilde{X}}(\tilde{X}) = 0$ and we have a corollary to the above theorem:

Corollary *Let $X \in \mathfrak{g}$. Then there exists a unique analytic homomorphism θ of $\mathbb{R} \to G$ such that $\dot{\theta}(0) = X$.*

For each $X \in \mathfrak{g}$ we put $\exp X := \theta(1)$ if θ is the homomorphism in the corollary. The mapping $X \to \exp X$ of $\mathfrak{g} \to G$ is called *the exponential mapping*. It satisfies

$$\exp(s+t) = \exp(s)\exp(t) \qquad \forall s, t \in \mathbb{R}, \quad \forall X \in \mathfrak{g}. \tag{116}$$

This follows from the fact that if $\alpha(X, X) = 0$ then $\theta(t) = \gamma_X(t) = \gamma_{tX}(t) = \exp tX$.

We define a *one-parameter subgroup* of a Lie group G to be an analytic homomorphism of $\mathbb{R} \to G$ and the above results imply that the one-parameter subgroups are precisely the mappings $t \to \exp tX$, $X \in \mathfrak{g}$. It is important to realise that these one-parameter subgroups are "internal" Lie group properties independent of the actual left invariant connection we used. For a given left invariant connection the geodesic $\gamma_X(t)$ defined by the connection might or might not agree with one-parameter groups $t \to \exp(tX)$. If the α corresponding to the given connection satisfies $\alpha(X,X) = 0$ the one-parameter group $t \to \exp tX$ *is* a geodesic with respect to the given connection. One can now ask if it is possible to find left invariant connections such that this automatically the case for all $X \in \mathfrak{g}$. Since α is bilinear the answer is: *any skew-symmetric α, i.e. an α which satisfies $\alpha(X,Y) = -\alpha(Y,X)$, corresponds to a left invariant affine connection such that $t \to \exp(tX)$ is a geodesic with respect to that connection for all $X \in \mathfrak{g}$.* If we further demand that the connection is torsion free, i.e. torsion field defined by (71) is zero we see that it implies

$$\alpha(X,Y) - \alpha(Y,X) = [X,Y], \quad \text{or} \quad \alpha(X,Y) = \frac{1}{2}[X,Y]. \tag{117}$$

This choice is called the *Cartan connection.*

For such a connection it follows that exp agrees with the map Exp_e defined defined earlier for an arbitrary connection, and since the map exp is actually independent of the specific connection we immediately have from the corresponding Exp_e theorem:

Theorem *There exists an open neighborhood N_0 of 0 in $\mathfrak{g}$ and an open neighborhood N_e of e in G such that* exp *is an analytic diffeomorphism of N_0 onto N_e.*

Let us mention some properties of the map exp and some theorems. Let $X \in \mathfrak{g}$, $g \in G$ and $f \in C^\infty(G)$. The homomorphism $\theta(t) = \exp tX$ satisfies $\dot{\theta}(0) = X$ and from (113) and (114) we obtain

$$\tilde{X}_g f = X(f \circ L_g) = \left(\frac{d}{dt}\, f(g \exp tX\right)_{t=0}. \tag{118}$$

It follows that the value of $\tilde{X}f$ at $g \exp uX$ is

$$[\tilde{X}f](g \exp uX) = \left(\frac{d}{dt}\, f(g \exp uX \exp tX)\right)_{t=0}. = \frac{d}{du}\, f(g \exp uX), \tag{119}$$

and by induction one obtains

$$[\tilde{X}^n f](g \exp uX) = \frac{d^n}{du^n}\, f(g \exp uX). \tag{120}$$

Assuming that f is *analytic* at g, i.e. that it in local coordinates $(x^1, \dots, x^n)$ has a convergent power series one then obtains "Taylors formula":

$$f(g \exp X) = \sum_{n=0}^{\infty} \frac{1}{n!} \, [\tilde{X}^n f](g). \tag{121}$$

Using this formula one can now prove

Theorem *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\exp$ be the exponential mapping of $\mathfrak{g}$ into G. Then, if $X, Y \in \mathfrak{g}$:*

$$\text{(i)} \qquad \exp tX \exp tY = \exp\Big(t(X+Y) + \frac{t^2}{2}[X,Y] + O(t^3)\Big) \tag{122}$$

$$\text{(ii)} \qquad \exp(-tX)\exp(-tY)\exp tX \exp tY = \exp\Big(t^2[X,Y] + O(t^3)\Big) \tag{123}$$

$$\text{(iii)} \qquad \exp tX \exp(tY)\exp(-tX) = \exp\Big(tY + t^2[X,Y] + O(t^3)\Big) \tag{124}$$

Let $X \in \mathfrak{g}$. Let us define the map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ by

$$\mathrm{ad}_X : Y \to [X, Y] \qquad \forall\, Y \in \mathfrak{g}. \tag{125}$$

It is now possible to express the differential of the exponential mapping as a power series in ad_X:

Theorem *Let G be a Lie group with Lie algebra $\mathfrak{g}$. The map $X \to \exp X$ of the manifold $\mathfrak{g}$ into G has the differential (mapping $T_X \to T_{\exp X}$)*

$$(d\exp)_X = d(L_{\exp X})_e \circ \frac{1 - e^{-\mathrm{ad}_X}}{\mathrm{ad}_X} \qquad X \in \mathfrak{g}. \tag{126}$$

In eq. (127) the fraction should be understood as the power series $\sum_{n=1}^{\infty} \frac{1}{n!}(-\mathrm{ad}_X)^{n-1}$.

We have the following

Theorem *Let G and G' be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. Let φ be an analytic homomorphism of G into G'. Then $d\varphi_e$ is a homomorphism of $\mathfrak{g}$ into $\mathfrak{g}'$ and*

$$\varphi(\exp X) = \exp d\varphi_e(X), \qquad \forall\; X \in \mathfrak{g}. \tag{127}$$

Let G and G' be two Lie groups. The groups are said to be *isomorphic* if there is an analytic isomorphism of G onto G'. The groups are said to be *locally isomorphic* if there exists open neighborhoods U and U' of $e \in G$ and $e' \in G'$ and an analytic diffeomorphism of U onto U' satisfying

(1) If $x, y, xy \in U$ then $f(xy) = f(x)f(y)$.
(2) If $x', y', x'y' \in U'$ then $f^{-1}(x'y') = f^{-1}(x')f^{-1}(y')$.

Theorem *Two Lie groups are locally isomorphic if and only if their Lie algebras are isomorphic.*

Let us finally define a Lie subgroup of a Lie group G. A submanifold H of G is called a *Lie subgroup* if (i) H is a subgroup of the (abstract) group G and (ii) H is a topological group.

One can show that a Lie subgroup is itself a Lie group. A connected Lie subgroup is often called an *analytic subgroup*. One has the following theorem

Theorem *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If H is a Lie subgroup of G then the Lie algebra $\mathfrak{h}$ of H is a subalgebra of $\mathfrak{g}$. Each Lie subalgebra of $\mathfrak{g}$ is the Lie algebra of exactly one connected Lie subgroup of G.*

The adjoint group

Let $\mathfrak{a}$ be a Lie algebra over $\mathbb{R}$. Recall that an endomophism of a vector space V (the Lie algebra $\mathfrak{a}$ is also a vector space) is a linear mapping of $V \to V$. The group $\mathrm{Gl}(\mathfrak{a})$ is the group of non-singular endomorphisms of $\mathfrak{a}$. It can be shown that the group $\mathrm{Gl}(\mathfrak{a})$ a Lie group. The Lie algebra $\mathfrak{gl}(\mathfrak{a})$ of $\mathrm{Gl}(\mathfrak{a})$ consists of the vector space of all endomorphisms of $\mathfrak{a}$ with the bracket operation $[A, B] := AB - BA,\ A, B \in \mathfrak{gl}(\mathfrak{a})$. For a given $X \in \mathfrak{a}$ the map ad_X as defined by (126)[8] is an endomorphism of $\mathfrak{a}$, i.e. $\mathrm{ad}_X \in \mathfrak{gl}(\mathfrak{a})$. But in addition the set $\{\mathrm{ad}_X \,|\, X \in \mathfrak{a}\}$ is by itself a Lie algebra since it follows from the Jacobi identity for $\mathfrak{a}$:

$$[X, [Y, Z]] - [Y, [X, Z]] - [[X, Y], Z] = 0 \quad \text{or} \quad [\mathrm{ad}_X, \mathrm{ad}_Y](Z) = \mathrm{ad}_{[X,Y]}(Z). \tag{128}$$

(again here we are abusing notation, using the same bracket for the Lie algebra bracket in $\mathfrak{a}$ and the commutator bracket for endomorphisms ad_X acting on the vector space $\mathfrak{a}$.) Let us denote the set $\{\mathrm{ad}_X \,|\, X \in \mathfrak{a}\}$ by $\mathrm{ad}(\mathfrak{a})$. We have by definition $\mathrm{ad}(\mathfrak{a}) \subset \mathfrak{gl}(\mathfrak{a})$ Denote by $\mathrm{Int}(\mathfrak{a})$ the connected Lie subgroup of $\mathrm{Gl}(\mathfrak{a})$ whose Lie algebra is $\mathrm{ad}(\mathfrak{a})$. $\mathrm{Int}(\mathfrak{a})$ *is called the adjoint group of* $\mathfrak{a}$. The uniqueness of $\mathrm{Int}(\mathfrak{a})$ follows from the last theorem in the section above.

An *automorphism* of a Lie algeba $\mathfrak{a}$ is a non-singular endomorphism of $\mathfrak{a}$ which also respect the bracket structure of $\mathfrak{a}$. The set $\mathrm{Aut}(\mathfrak{a})$ of automorphisms forms a group and it is a Lie subgroup of $\mathrm{Gl}(\mathfrak{a})$. Let $\mathfrak{d}(\mathfrak{a})$ denote the Lie algebra of

[8] To avoid misunderstanding the bracket use in (126) is the Lie algebra bracket belonging to $\mathfrak{a}$ and it is not related to the bracket just mentioned, which is just commutator of two endomorphisms of $\mathfrak{a}$, viewed as a vector space.

Aut($\mathfrak{a}$). Let $D \in \mathfrak{d}(\mathfrak{a})$. Thus we know from above that $e^{tD} \in \mathrm{Aut}(\mathfrak{a})\ \forall\, t \in \mathbb{R}$. Let $X, Y \in \mathfrak{a}$. Since $e^{tD} \in \mathrm{Aut}(\mathfrak{a})$ it respects the Lie algebra structure of $\mathfrak{a}$:

$$\mathrm{e}^{tD}[X,Y] = [\mathrm{e}^{tD}X, \mathrm{e}^{t}DY] \qquad \forall\, t \in \mathbb{R}. \tag{129}$$

From this it follows, differentiating after t and taking $t \to 0$, that

$$D[X,Y] = [DX,Y] + [X,DY] \qquad \forall\, X,Y \in \mathfrak{a}. \tag{130}$$

An endomorphism D of a Lie algebra $\mathfrak{a}$ which satisfies (131) is called a *derivation of* $\mathfrak{a}$. Applying D successively to eq. (131) one obtains a formula for D^k acting on X, Y which allows one to prove that if D satisfies (131) then e^{tD} will satisfy (130): Thus

Theorem $\mathfrak{d}(\mathfrak{a})$ *consists of all derivations of* $\mathfrak{a}$.

It is seen that the Jabobi identity (129) can be written as (131) with $D = \mathrm{ad}_Z$. Thus ad_Z, $Z \in \mathfrak{a}$ is a derivation and $\mathrm{ad}(\mathfrak{a}) \subset \mathrm{Aut}(\mathfrak{a})$ and therefore $\mathrm{Int}(\mathfrak{a}) \subset \mathrm{Aut}(\mathfrak{a})$. The elements of $\mathrm{ad}(\mathfrak{a})$ and $\mathrm{Int}(\mathfrak{a})$, respectively, are called *inner derivations* and *inner automorphisms* of $\mathfrak{a}$. One can prove that $\mathrm{Int}(\mathfrak{a})$ is a Lie subgroup of $\mathrm{Aut}(\mathfrak{a})$. In fact it is an invariant subgroup[9] of $\mathrm{Aut}(\mathfrak{a})$. We have to prove that $s\big(\mathrm{Int}(\mathfrak{a})\big)s^{-1} = \mathrm{Int}(\mathfrak{a})\ \forall\ s \in \mathrm{Aut}(\mathfrak{a})$. The group $\mathrm{Int}(\mathfrak{a})$ is connected (being an analytic subgroup of $\mathfrak{gl}(\mathfrak{a})$) and thus it is generated by the elements of its Lie algebra, i.e. the elements e^{ad_X} . The statement now follows from

$$s\,\mathrm{e}^{\mathrm{ad}_X} s^{-1} = \mathrm{e}^{s(\mathrm{ad}_X)s^{-1}} = \mathrm{e}^{\mathrm{ad}_{sX}} \in \mathrm{Int}(\mathfrak{a}). \tag{131}$$

The first equality is true in general for any endomorphisms A, B of a vector space V, where A is non-singular: $Ae^{B}A^{-1} = e^{ABA^{-1}}$. The next equality follows because s is an automorphism of $\mathfrak{a}$, i.e. $s[X,Y] = [sX, sY]$ or $s\,\mathrm{ad}_X = \mathrm{ad}_{sX} s$.

Let us now generalize this to an arbitrary Lie group G. If $\sigma \in G$ the mapping $I(\sigma) : g \to \sigma g \sigma^{-1}$ is an analytic isomorphism of G onto itself. We put $\mathrm{Ad}(\sigma) = dI(\sigma)_e$. The mapping $\mathrm{Ad}(\sigma)$ is an automorphism of $\mathfrak{g}$, the Lie algebra of G, i.e. $\mathrm{Ad}(\sigma) \in \mathrm{Aut}(\mathfrak{g}) \subset \mathrm{Gl}(\mathfrak{g})$. We have from (128):

$$\exp\big(\mathrm{Ad}(\sigma)X\big) = \sigma \exp X \sigma^{-1} \qquad \forall\ \sigma \in G,\ X \in \mathfrak{g}. \tag{132}$$

The mapping $\sigma \to \mathrm{Ad}(\sigma)$ is a homomorphism of $G \to \mathrm{Gl}(\mathfrak{g})$ and it is called the *adjoint representation* of G.

Next, let X and Y be arbitrary vectors in $\mathfrak{g}$. From (125) we have

$$\exp\big(\mathrm{Ad}(\exp tX)tY\big) = \exp\big(tY + t^2[X,Y] + O(t^3)\big) \tag{133}$$

[9]Recall the H is an invariant subgroup of G is $gHg^{-1} = H\ \forall\ g \in G$. If H is invariant the coset space $\{gH\} = G/H$ is a group, the so-called quotient or factor group.

and thus, expanding in t

$$\mathrm{Ad}(\exp tX)Y = Y + t\,[X,Y] + O(t^2). \tag{134}$$

The differential $d\mathrm{Ad}_e$ is a homomorphism of $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ and from (135) we obtain

$$d\mathrm{Ad}_e(X) = \mathrm{ad}_X \qquad X \in \mathfrak{g}. \tag{135}$$

Finally, applying the exponential map according to (128) one has

$$\mathrm{Ad}(\exp X) = \mathrm{e}^{\mathrm{ad}_X}, \qquad X \in \mathfrak{g}. \tag{136}$$

From this equation it follows that the image of G in $\mathrm{Gl}(\mathfrak{g})$ by Ad is $\mathrm{Int}(\mathfrak{g})$: $\mathrm{Ad}(G) = \mathrm{Int}(\mathfrak{g})$. Further, from (133) it follows that the center [10] Z of G is the kernel of Ad: $Z = \mathrm{Ad}^{-1}(e)$. Thus we have

Theorem *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let Z denote the center of G. Then*
(i) Ad *is an analytic homomorphism of G onto* $\mathrm{Int}(\mathfrak{g})$ *with kernel Z.*
(ii) The mapping $gZ \to \mathrm{Ad}(g)$ is an analytic isomorphism of G/Z onto $\mathrm{Int}(\mathfrak{g})$.

Riemannian structures on Lie groups

We saw above that there was a one-to-maping between left invariant connections on a Lie group G and bilinear mapping of $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. If we in addition have a Riemannian structure g on the manifold G, it is natural to demand that the correpsonding metric d and the group operations are compatible in the following sense: left translations are analytic isomophisms of G onto itself. We will be interested in metrics d which make left translations isometries of G. If that is the case, the distance between points will be preserved by left translations.

Let $L_\sigma : h \to \sigma h.\ \forall\, h \in G$ and $R_\sigma : h \to h\sigma\ \forall\, h \in G$ denote left and right translations, respectively. Let the differential dL_σ denote the differential of L_σ. From (96) we know that L_σ is an isometry if

$$g_{\sigma h}\big((dL_\sigma)_h X, (dL_\sigma)_h Y\big) = g_h(X,Y), \quad X, Y \in T_h, \quad h \in G. \tag{137}$$

We say that the Riemannian structure (or metric tensor) g (and the associated *metric d*) is *left invariant* if (138) is valid for all L_σ, $\sigma \in G$. Similarly we say that g is *right invariant* if

$$g_{h\sigma}((dR_\sigma)_h X, (dRL_\sigma)_h Y) = g_h(X,Y), \quad X, Y \in T_h, \quad \sigma,\, h \in G. \tag{138}$$

[10] Recall that the center of a group G is the elements which commute with all elements of the group and that this is an invariant subgroup. Thus the coset space $\{gZ\}$ forms the factor group G/Z.

Finally we say the g *is bi-invariant* if it is both left and right invariant.

Via left translations L_σ the group G acts *transitively* on itself: $L_{\sigma_1} \circ L_{\sigma_2} = L_{\sigma_1\sigma_2}$ and for all $h_1, h_2 \in G$ there exists a σ such that $h_2 = L_\sigma h_1$. If G has a left invariant metric (resp. right invariant metric), then left translations are isometries and this makes G, viewed as the manifold on which G itself acts via left translations, a so-called *homogeneous Riemannian manifold*. This is reflected in the structure of geodesics. What are the geodesics in a Lie group equipped with a bi-invariant metric? The answer is simple: they are the integral curves of left-invariant vector fields. Also, since left and right translations are isometries and since isometries map geodesics to geodesics, the geodesics through any point $\sigma \in G$ are the left (or right) translates of the geodesics through e, and thus are expressed in terms of the group exponential. Therefore, the geodesics through $\sigma \in G$ are of the form

$$\gamma(t) = L_\sigma(\exp(tX)), \quad \text{where} \quad \dot\gamma(0) = (dL_\sigma)_e(X). \tag{139}$$

Another consequence is that

theorem *A Lie group with a left invariant (respect. right invariant) metric is a complete manifold.*

We have seen that the T_e, the tangent space at the identity e of G can be identified with the Lie algebra $\mathfrak{g}$ of G. Given a Riemannian structure g we have the real scalar product $\langle X|Y\rangle = g_e(X,Y),\ X, Y \in T_e\,(=\mathfrak{g})$. Conversely, any scalar product om $\mathfrak{g}$ can be turned into left invariant Riemannian structure g and a corresponding left invariant metric d on G by defining g via left translations using (138)

$$g(X,Y)(h) := \langle (dL_{h^{-1}})_h X | (dL_{h^{-1}})_h Y\rangle, \quad \forall h \in G,\ \forall X, Y \in \mathcal{D}^1(G). \tag{140}$$

Using $(dL_{(\sigma h)^{-1}})_{\sigma h}(dL_\sigma)_h = (dL_{h^{-1}})_h$ one can verify that (138) indeed is satisfied for the g defined by (141). We thus have the following (rather obvious)

Theorem *There is a bijective correspondence between left-invariant (resp. right invariant) metrics on a Lie group G, and scalar products on the Lie algebra $\mathfrak{g}$ of G.*

Let us now turn to the situation where the Riemannian structure g and the corresponding metric d are *bi-invariant*. Recall that the adjoint representation Ad of G was defined starting from the analytic isomorphism $I(\sigma) : h \to \sigma h \sigma^{-1}$ of G onto G. We defined $\mathrm{Ad}(\sigma) = dI(\sigma)_e$ and $\mathrm{Ad}(\sigma) \in \mathfrak{gl}(\mathfrak{g})$, i.e. $\mathrm{Ad}(\sigma)$ is a non-singular linear map of $\mathfrak{g}$ into $\mathfrak{g}$, and $\sigma \to \mathrm{Ad}(\sigma)$ provided us with the adjoint representation of G. We can write

$$I(\sigma) = R_{\sigma^{-1}} \circ L_\sigma, \qquad \mathrm{Ad}(\sigma) = dI(\sigma)_e = (dR_{\sigma^{-1}})_\sigma (dL_\sigma)_e \tag{141}$$

Let $\langle\cdot|\cdot\rangle$ be a scalar product on the vector space $\mathfrak{g}$. We say that Ad is an *isometry* on $\mathfrak{g}$ with respect to the scalar product if

$$\langle Ad(\sigma)X|\mathrm{Ad}(\sigma)Y\rangle = \langle X|Y\rangle \qquad \forall\, X, Y \in \mathfrak{g},\ \forall\, \sigma \in G. \tag{142}$$

and we call the scalar product Ad-*invariant* if (143) is satisfied.

If g is bi-invariant then clearly $I(\sigma)$ becomes an isometry and from (142) it is clear that Ad is an isometry on $\mathfrak{g}$ provided with the scalar product from g. On the other hand, we can define a left invariant Riemannian structure g by the following definition

$$g(X, Y)(h) = \langle (dL_{h^{-1}})_h X | (dL_{h^{-1}})_h Y\rangle. \tag{143}$$

If Ad is a $\mathfrak{g}$ isometry a little algebra shows that g is also righ invariant and we have the following

Theorem *There is a bijective correspondence between bi-invariant Riemannian structures g on a Lie group G and* Ad*-invariant scalar products on the Lie algebra $\mathfrak{g}$ of G.*

This theorem can also be formulated using (136) and (137). We call $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ *skew-adjoint* if

$$\langle \mathrm{ad}_X Y|Z\rangle = -\langle Y|\mathrm{ad}_X Z\rangle, \quad \forall\, X, Y, Z \in \mathfrak{g}. \tag{144}$$

This is equivalent to

$$\langle [X, Y]|Z\rangle = \langle Y|[Z, X]\rangle, \quad \forall\, X, Y, Z \in \mathfrak{g}. \tag{145}$$

A scalar product on $\mathfrak{g}$ with this property is called *associative* and the above theorem can be formulated as

Theorem *Let G be a connected Lie group. A scalar product on the Lie algebra $\mathfrak{g}$ induces a bi-invariant Riemannian structure g on G if and only if it is associative.*

The theorem above implies that if G possesses a bi-invariant g, then every $\mathrm{Ad}(\sigma)$ is an orthogonal transformation of $\mathfrak{g}$. It follows that in this case $\mathrm{Int}(\mathfrak{g}) = \mathrm{Ad}(G)$ is a subgroup of the orthogonal group of $\mathfrak{g}$. The orthogonal group is compact and that implies that the closure $\bar{\mathrm{Ad}(G)}$ is compact. It can be proven that the opposite is also true:

Theorem *For any Lie group G, a scalar product on $\mathfrak{g}$ induces a bi- invariant Riemannian structure g on G if and only if the closure of $Ad(G)$ is compact. In particular, every compact Lie group has a bi-invariant metric.*

A final theorem in this direction is the following:

Theorem *A connected Lie group G admits a bi-invariant Riemannian structure g if and only if it is isomorphic to the cartesian product of a compact group and a vector space ($\mathbb{R}^m$, for some $m \geq 0$).*

Connections and curvature for bi-invariant structures

For Lie groups equipped with a bi-invariant structure one can obtain simple expressions for the Riemannian connection and for the associated curvature tensor, expressed entirely by Lie algebra quantities:

Theorem *For any Lie group G equipped with a bi-invariant metric, the following properties hold:*

(1) The connection $\nabla_X Y$ is given by

$$\nabla_X Y = \frac{1}{2}[X, Y], \quad \forall\, X, Y \in \mathfrak{g}^L. \tag{146}$$

(2) The curvature tensor $R(X, Y)$ is given by

$$R(X, Y) = -\frac{1}{4}\mathrm{ad}_{[X,Y]}, \quad \text{or} \quad R(X, Y)Z = -\frac{1}{4}[[X, Y], Z] \tag{147}$$

(3) The sectional curvature $K(e_1, e_2)$ is given by

$$K(S) = K(e_1, e_2) = -\frac{1}{4}\langle [e_1, e_2], [e_1, e_2]\rangle, \tag{148}$$

for all pairs of orthonormal vectors $e_1, e_2 \in \mathfrak{g}$ and where S is the two-dimensional plane in $\mathfrak{g}$ spanned by e_1, e_2.

(4) The Ricci curvature $\mathrm{Ric}(X, Y)$ is given by

$$\mathrm{Ric}(X, Y) = -\frac{1}{4}B(X, Y), \quad \forall\, X, Y \in \mathfrak{g}, \tag{149}$$

where $B(X, Y) := \mathrm{tr}\,(\mathrm{ad}_X \mathrm{ad}_Y)$ is the Killing form of $\mathfrak{g}$.

An *ideal* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a subalgebra which satisfies $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$. A Lie algebra is called *simple* if its dimension is larger than one and if it has no non-trivial ideals (the trivial ideals are $\{0\}$ and $\mathfrak{g}$). The corresponding definition for Lie groups is that a Lie group G is *simple* if it has no non-trivial invariant subgroups H. A *semi-simple Lie algebra* is a direct sum of simple Lie algebras and a Lie group is called semi-simple if its Lie algebra is semi-simple. The Killing form characterises the the semi-simple Lie algebras due to a theorem by Cartan:

Theorem *A Lie algebra* $\mathfrak{g}$ *is semi-simple if and only if its Killing form* $B(X,Y)$ *is non-degenerate on* $\mathfrak{g}$.

and we have in addition

Theorem *A connected Lie group is compact and semisimple if and only if its Killing form is negative definite.*

It is seen that these theorems have consequences for the Ricci tensor because of the relation (150) between the Killing form and the curvature.

The Killing form has a number of important features, some of which are summarised in the following

Theorem *The Killing form* B *of a Lie algebra* $\mathfrak{g}$ *has the following properties:*

(1) It is a symmetric bilinear form invariant under all automorphisms of $\mathfrak{g}$. *In particular, if* $\mathfrak{g}$ *is the Lie algebra of a Lie group* G*, then* B *is* $\mathrm{Ad}(\sigma)$*-invariant, since* $\mathrm{Ad}(\sigma) \in \mathrm{Int}(\mathfrak{g}) \subset \mathrm{Aut}(\mathfrak{g})$ *for all* $\sigma \in G$.

(2) The linear map ad_X *is skew-adjoint w.r.t* B *for all* $X \in \mathfrak{g}$*, i.e.*

$$B(\mathrm{ad}_X Y, Z) = -B(Y, \mathrm{ad}_X Z), \quad \text{or} \quad B([X,Y],Z) = B(X,[Y,Z]), \tag{150}$$

that is: B *is associative.*

We have the following uniqueness property of associative forms:

Theorem *On a simple Lie algebra all associative bilinear forms* $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ *are proportional to the Killing form.*

This implies, in view of the other theorems above:

Theorem *If* G *is any compact, simple, Lie group, then the bi-invariant Riemannian structure* g *is unique up to a constant, and the corresponding Ricci curvature is constant.*

Symmetric spaces and group theory

In many situations a group G is be defined as a transformation group acting transitively on a manifold M, i.e. for each $g \in G$ we have a bijection $p \to g(p)$ from M onto M, such that $g_1(g_2(p)) = (g_1 g_2)(p)$. If H_p denotes the *stability group* at $p \in M$, i.e. $H_p = \{g \in G \,|\, g(p) = p\}$ then under suitable assumptions M can be identified with the coset space G/H_p. This opens up for the possibility to apply group theoretical tools to analyze properties of the manifold M. A standard example is $M = S^n$, the n-sphere. The group $SO(n+1)$ acts transitively on S^n. If $p \in S^n$ is the "north pole" then $H_p = SO(n)$, the rotations which leave the north pole fixed, and one can write $S^n = SO(n+1)/SO(n)$ as topological spaces and even as manifolds when appropriate definitions are made. We have no space for entering into details with the general description $M \cong G/H$, but let us briefly discuss a set of manifolds M where group theory have played a very important role and where one has been able to express curvature and covariant differentiation along the lines we have indicated above when M itself was a Lie group. This class of manifolds are called symmetric spaces and the S^n example belongs to this class.

The starting point of symmetric spaces is the following observation: consider a C^∞ manifold M with affine connection ∇. Let $p \in M$ and consider a normal neighborhood $N_0 \in T_p$ and put $N_p = \mathrm{Exp}_p N_0$. Let $q \in N_p$ and consider the geodesic $t \to \gamma(t)$ such that $q = \gamma(1)$ and $p = \gamma(0)$. Put $q' = \gamma(-1)$. This is a map s_p of N_p onto itself and is called the geodesic symmetry with respect to p. In normal coordinates we have $(x^1, \ldots, x^n) \to (-x^1, \ldots, -x^n)$. s_p is a diffeomorphism of N_p onto itself such that $(ds_p)_p = -I$, I denoting the identity map. We call M *affine locally symmetric* if each $p \in M$ has an open neighborhood N_p on which the geodesic symmetry s_p is an affine transformation (see below (56) for definition). One has

Theorem *A manifold M is affine locally symmetric if and only if the torsion $T = 0$ and $\nabla_Z R = 0$ for all $Z \in \mathcal{D}^1(M)$.*

A Riemannian manifold M is called a *Riemannian locally symmetric space* if for each $p \in M$ there exists an open normal neighborhood on which the geodesic symmetry with respect to p is an isometry. One has

Theorem *Let M be a Riemannian manifold. Then M is a Riemannian locally symmetric space if and only if the sectional curvature is invariant under parallel translations.*

Let M a Riemannian manifold. A mapping s_p on M is called *involutive* if $s_p^2 = I$, but $s_p \neq I$. M is called *Riemannian globally symmetric* if each point $p \in M$ is an isolated fixed point for an involutive isometry s_p of M. One can show that there is only one such s_p and that it agrees with the geodesic symmetry with respect to p.

At this point we have not identified any group acting on M. However, if M is a Riemannian manifold we can consider the set of all isotropies, $I(M)$, on M. Composition of maps makes $I(M)$ a transformation group acting on M and one can show that if M is a Riemannian globally symmetric space then one can give $I(M)$ an analytic structure such that it becomes a Lie transformation group of M. Let $I_0(M)$ denote the connect subgroup of $I(M)$ which contains the identity element. One has the

Theorem *Let M be a Riemannian globally symmetric space and $p_0 \in M$. if $G = I_0(M)$ and H is the subgroup of G which leaves p invariant then*

(1) G/H is analytically diffeomorphic to M under the mapping $gH \to g(p)$, $g \in G$.

(2) The mapping $\sigma : g \to s_p \sigma s_p$ is an involutive automorphism of G such that all elements $h \in H$ are fixed points[11]*for σ.*

(3) Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively. Then $\mathfrak{h} = \{X \in \mathfrak{g} \,|\, (d\sigma)_e X = X\}$ and if $\mathfrak{p} = \{X \in \mathfrak{g} \,|\, (d\sigma)_e X = -X\}$ then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. Let π denote the natural mapping $g \to g(p)$ of G onto M. Then $(d\pi)_e$ maps $\mathfrak{h}$ into $\{0\}$ and $\mathfrak{p}$ isomorphically into T_p. If $X \in \mathfrak{p}$ then the geodesic emanating from p with tangent vector $(d\pi)_e X$ is given by

$$\gamma_{(d\pi)_e X}(t) = \exp tX(p). \tag{151}$$

If $Y \in T_p$ then $(d \exp tX)_p(Y)$ is the parallel transport of Y along the geodesic.

From this theorem we see that a Riemannian globally symmetry space gives rise to a pair $(\mathfrak{g}, s)$, $(s = (d\sigma)_e)$ such that s is an involutive automorphism of $\mathfrak{g}$ and s decompose $\mathfrak{g}$ in two subspaces $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, according to $s(X) = X,\ X \in \mathfrak{h}$ and $s(X) = -X,\ X \in \mathfrak{p}$. One has $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$. We call $(\mathfrak{g}, s)$ an *orthogonal symmetric Lie algebra* and the pair (G, H) of the theorem *associated* with $(\mathfrak{g}, s)$[12]. Now let π be the natural mapping of $G \to G/H$ and put $o = \pi(e)$. For $g \in G$ let $\tau(g)$ denote the mapping $\tau(g) : hH \to ghH$ of G/H onto itself. We can identify $\mathfrak{p}$ with the tangent space $\big(G/K\big)_o$ by means of $d\pi$.

Let now $\mathfrak{Q}$ be any G-invariant Riemannian structure on G/H (i.e. $\mathfrak{Q}$ is invariant under $\tau(g)$ for all g). Then G/H is complete and locally symmetric. *We can now describe the geometric concepts Exponential mapping and curvature for G/H in group theoretical terms.*. For $X \in \mathfrak{p}$, let T_X denote the restriction of $(\mathrm{ad}_X)^2$ to $\mathfrak{p}$. Then $T_X \mathfrak{p} \subseteq \mathfrak{p}$ (since $[[\mathfrak{p}, \mathfrak{p}], \mathfrak{p}] \subseteq \mathfrak{p}$) and we have the following two theorems:

Theorem *The Exponential map of $\mathfrak{p}$ into G/H is independent of the choice of $\mathfrak{Q}$. Its differential is given by*

$$d\mathrm{Exp}_X = d\tau(\exp X)_o \circ \sum_{n=0}^{\infty} \frac{\big(T_X\big)^n}{(2n+1)!}, \qquad X \in \mathfrak{p}. \tag{152}$$

Here $\mathfrak{p}$ is considered as a manifold in an obvious way and its tangent space at each point is identified with $\mathfrak{p}$ itself.

[11] Things are slightly complicated: let H_σ denote the fixed points of σ and $(H_\sigma)_0$ the connected component of H_σ which contains the identity. Then $(H_\sigma)_0 \subseteq H \subseteq H_\sigma$

[12] As in the footnote of the theorem, there are also technical issues if we want to start out with an orthogonal symmetric Lie algebra and then associate a pair (G, H) to this algebra. We will ignore these complications here

Theorem *Let R denote the curvature tensor of the space G/H corresponding to the Riemannian structure $\mathfrak{Q}$. then, at the point $o \in G/H$ we have:*

$$R_o(X,Y)Z = -[[X,Y],Z], \qquad X,Y,Z \in \mathfrak{p}. \tag{153}$$

As a consequence we have

Corollary *the Riemannian connection on G/H is the same for all G-invariant Riemannian structures $\mathfrak{Q}$ on G/K.*

A lot more can be said about symmetry spaces using Lie group theory. However, not in this note........