Numerical Methods

→ Algebraic Equations

An algebraic equation is an equation of the form:

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0$$

where, $a_0, a_1, a_2, \dots, a_n$ are real numbers, $n \geq 1$ are a positive integer.

→ Transcendental Equations

These are equations involving exponential, logarithmic, trigonometric or hyperbolic functions.

Eg: $f(x) = \ln x^3 - 0.7$

$\phi(x) = e^{-0.5x} - 5x$

$\psi(x) = \sin^2 x - x^2 - 2$ etc.

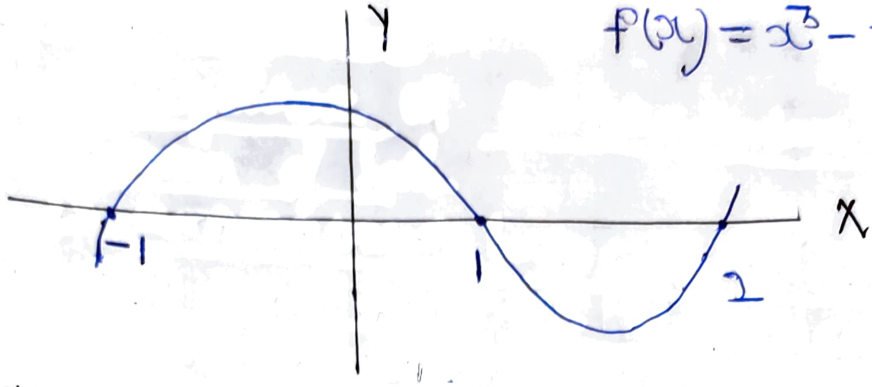
→ Every equation has a root, real or imaginary.

(i) Every algebraic equation of degree n has exactly n roots, real or imaginary

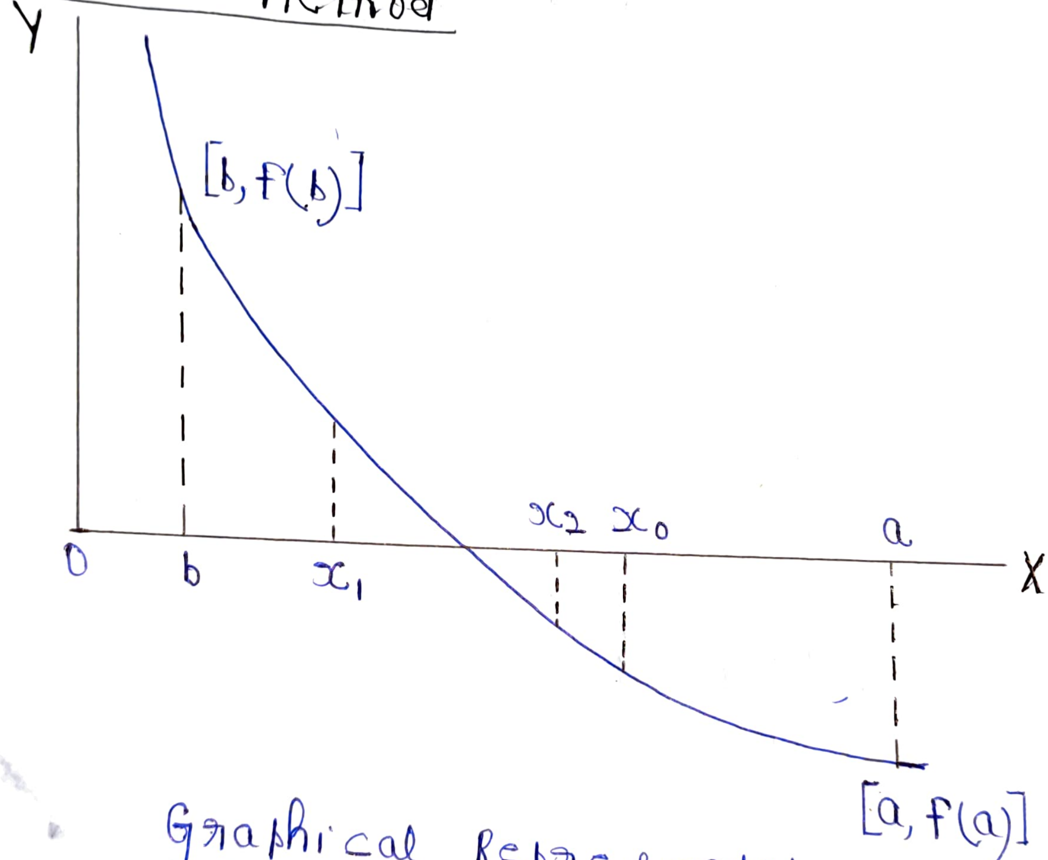
(ii) The number and nature of roots of a transcendental equation are not known.

(iii) If $f(x)$ is continuous between a and b, and $f(a)$ and $f(b)$ are of opposite signs, then there exists atleast one root between a and b.

→ $f(x) = x^3 - 2x^2 - x + 2$



→ Bisection Method



Graphical Representation of the bisection method

→ Steps

1. Choose two real numbers a and b such that $f(a) f(b) < 0$.
2. Set $x_r = (a+b)/2$.
3. (a) If $f(a) f(x_r) < 0$, the root lies in the interval (a, x_r). Then, set $b = x_r$ and go to step 2 above.

(b) If $f(a)f(x_r) > 0$, the root lies in the interval (x_r, b). Then, set $a = x_r$ and go to step (2).

(c) If $f(a)f(x_r) = 0$, it means that x_r is a root of the equation $f(x) = 0$ and the computation may be terminated.

→ In practical problems, the roots may not be exact so that condition (c) above is never satisfied.
In such a case, we need to adopt one of the following criterion for deciding when to terminate the computations-

(a) Maximum number of iterations may be specified in advance.

(b) Percentage Error,

$$E_r = \left| \frac{x_r' - x_r}{x_r'} \right| \times 100\%$$

where x_r is the new value of x_r. The computations can be terminated when E_r becomes less than a prescriped tolerance, say E_p.

Qu (1) Find the root of the equation $f(x) = x^3 - x - 1 = 0$.

(by Bisection method, upto 6 iteration)

Sol: $f(x) = x^3 - x - 1$

$f(1) = 1 - 1 - 1 = 1 - 2 = -1 < 0$

$f(2) = (2)^3 - 2 - 1 = 8 - 3 = 5 > 0$

$f(1) \cdot f(2) = (-1) \cdot (5) = -5 < 0$ $\quad [1, 2]$

$x_0 = \frac{1+2}{2} = \frac{3}{2} = 1.5$

$f(x_0) = \left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right) - 1$

$= \frac{27}{8} - \frac{3}{2} - 1$

$= \frac{27 - 12 - 8}{8} = \frac{27 - 20}{8} = \frac{7}{8} > 0$ $\quad [1, 1.5]$

(0.875)

$x_1 = \frac{1 + 3/2}{2} = \frac{5/2}{2} = \frac{5}{4} = 1.25$

$f(x_1) = \left(\frac{5}{4}\right)^3 - \left(\frac{5}{4}\right) - 1$

$= \frac{125}{64} - \frac{5}{4} - 1$

$= \frac{125 - 80 - 64}{64}$

$= \frac{125 - 144}{64} = -\frac{19}{64} < 0$ $\quad [1.25, 1.5]$

(-0.296875)

$x_2 = \frac{5/4 + 3/2}{2} = \frac{(5+6)/4}{2} = \frac{11}{8} = 1.375$

$$f(x_2) = \left(\frac{11}{8}\right)^3 - \left(\frac{11}{8}\right) - 1$$

$$= \frac{1331}{512} - \frac{11}{8} - 1$$

$$= \frac{1331 - 704 - 512}{512} = \frac{1331 - 1216}{512}$$

$$= \frac{115}{512} > 0 \qquad \underline{[1.25,\ 1.375]}$$

(0.224609375)

$$x_3 = \frac{(5/4) + (11/8)}{2} = \frac{(10+11)/8}{2} = \frac{21}{16}$$

$$= \underline{1.3125}$$

$$f(x_3) = \left(\frac{21}{16}\right)^3 - \left(\frac{21}{16}\right) - 1$$

$$= \frac{9261}{4096} - \frac{21}{16} - 1 = \frac{9261 - (256 \times 21) - 4096}{4096}$$

$$= \frac{9261 - 5376 - 4096}{4096} = \frac{9261 - 9472}{4096}$$

$$= \frac{-211}{4096} < 0 \qquad \underline{[1.3125,\ 1.375]}$$

(-0.0515136719)

$$x_4 = \frac{(21/16) + (11/8)}{2} = \frac{(21+22)/16}{2} = \frac{43}{32}$$

$$= \underline{1.34375}$$

$$f(x_4) = \left(\frac{43}{32}\right)^3 - \left(\frac{43}{32}\right) - 1$$

$$= \frac{79507}{32768} - \frac{43}{32} - 1$$

$$= \frac{79507 - 43 \times 1024 - 32768}{32768}$$

$$= \frac{79507 - 44032 - 32768}{32768}$$

$$= \frac{79507-76800}{32768}$$

$$= \frac{2707}{32768} > 0 \qquad \boxed{1.3125, 1.34375}$$

(0.082611084)

$$x_5 = \frac{(21/16)+(43/32)}{2}$$

$$= \frac{(42+43)/32}{2} = \frac{85}{64} = 1.328125$$

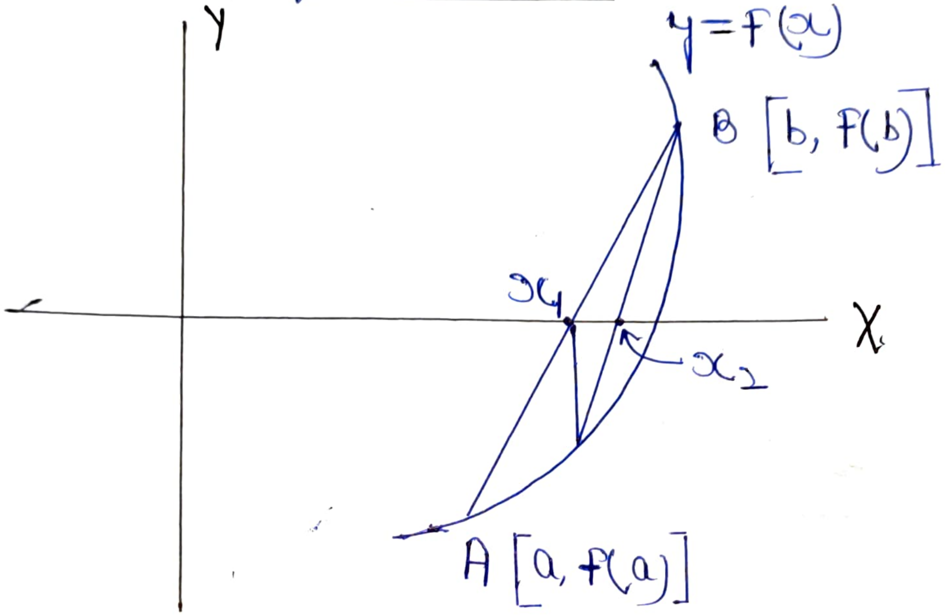
n	a	b	x	$f(x)$
1	1	2	1.5	0.875
2	1	1.5	1.25	−0.296875
3	1.25	1.5	1.375	0.224609375
4	1.25	1.375	1.3125	−0.0515136719
5	1.3125	1.375	1.34375	0.082611084
6	1.3125	1.34375	1.328125	—

H.W. Qu (2) Find a real root of the equation $x^3 - 2x - 5 = 0$.
(by Bisection method, upto 12 iteration)

② The Method of False Position

→ for finding the root of a nonlinear equation $f(x)=0$

→ also known as regula falsi or method of chords



Equation of the chord joining the two points $[a, f(a)]$ & $[b, f(b)]$ is given by

$$\frac{y-f(a)}{x-a} = \frac{f(b)-f(a)}{b-a}$$

or, $$(x-a) = \frac{(y-f(a))}{f(b)-f(a)}(b-a)$$

Put $y=0$, so,

$$x_1 = a - \frac{f(a)}{f(b)-f(a)}(b-a)$$

or, $$\boxed{x_1 = \frac{a\,f(b) - b\,f(a)}{f(b)-f(a)}}$$

x_1 is the first approximation to the root of $f(x)=0$