CSE 105: Data Structures and Algorithms-I (Part 2)

Instructor
Dr Md Monirul Islam

Graph Searching

Breadth-First Search

```
BFS(G, s)

1 for each vertex u \in G. V - \{s\}

2 u.color = WHITE

3 u.d = \infty
```

 $u.\pi = NIL$

5 s.color = GRAY

 $6 \, s.d = 0$

7 $s.\pi = NIL$

 $8 \quad Q = \emptyset$

9 ENQUEUE(Q, s)

10 while $Q \neq \emptyset$

11 u = DEQUEUE(Q)

12 **for** each $v \in G.Adj[u]$

if v.color == WHITE

v.color = GRAY

v.d = u.d + 1

16 $v.\pi = u$

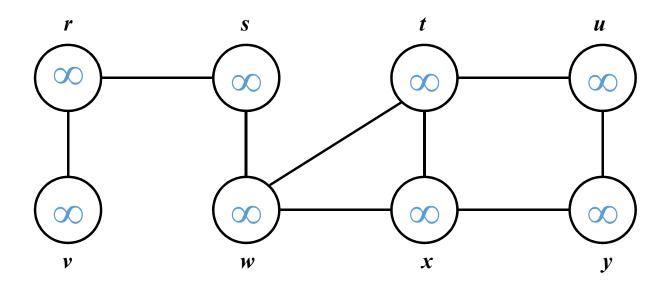
17 $ENQUEUE(Q, \nu)$

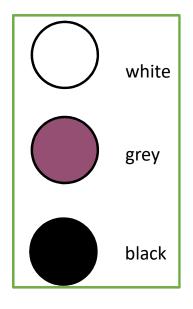
18 u.color = BLACK

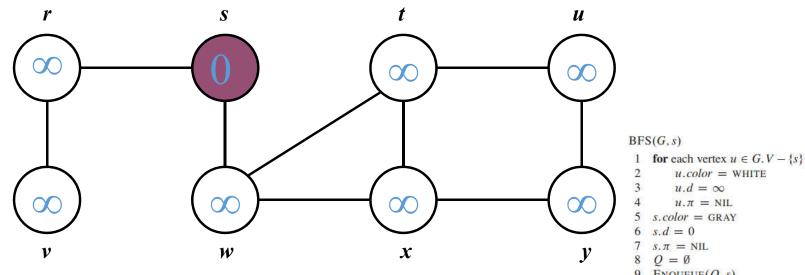


Enqueue the root

runs until queue is empty

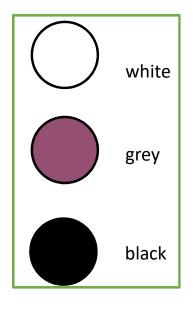


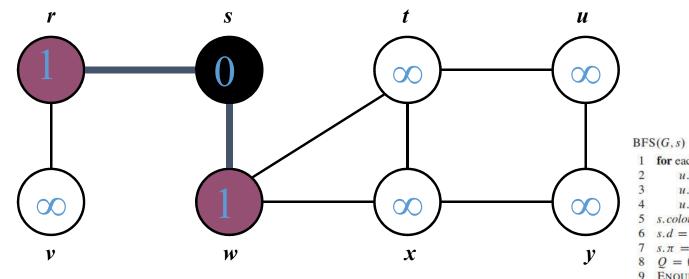


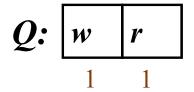


Q: s

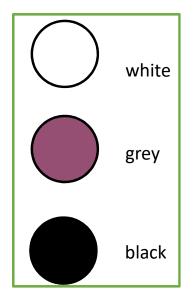
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        for each v \in G.Adj[u]
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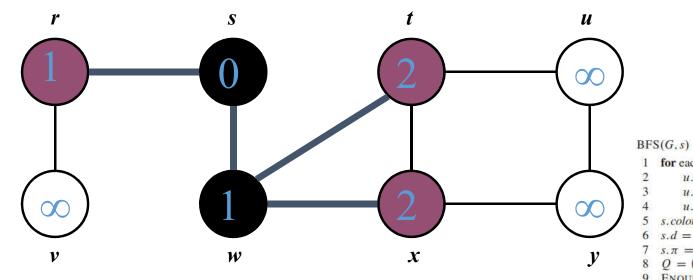






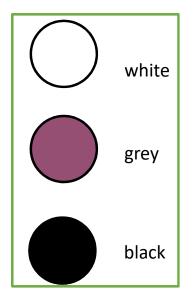
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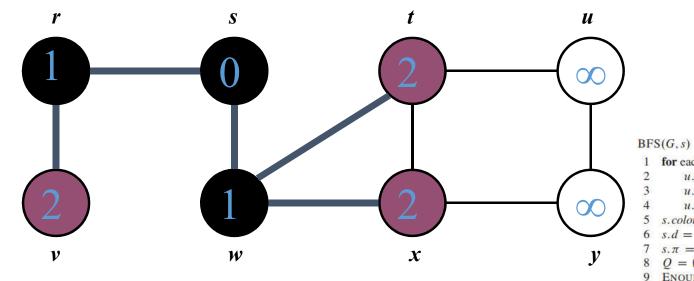


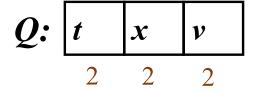


```
Q: \begin{bmatrix} r & t & x \\ 1 & 2 & 2 \end{bmatrix}
```

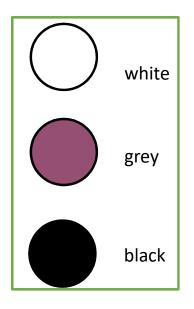
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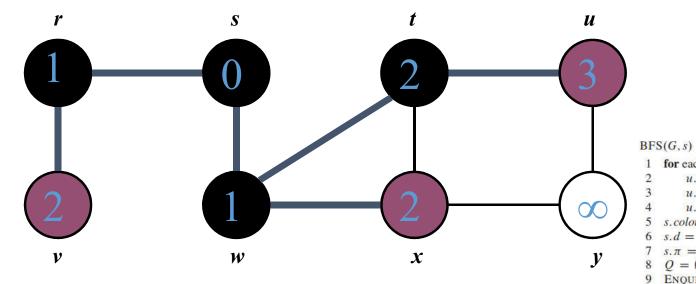


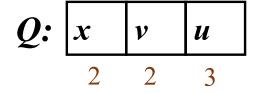




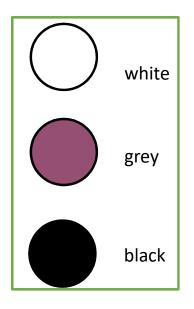
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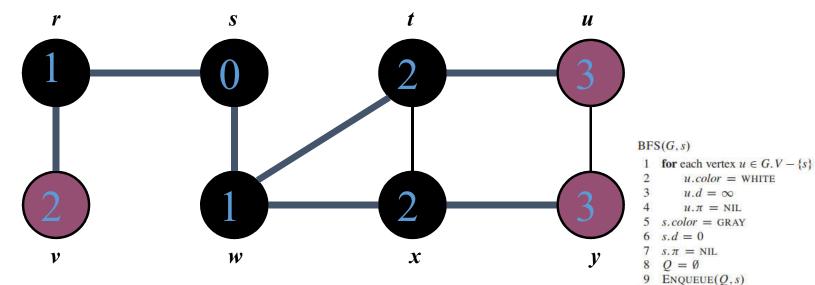




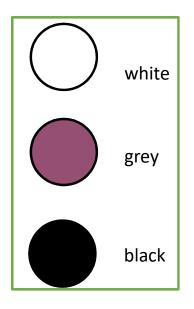


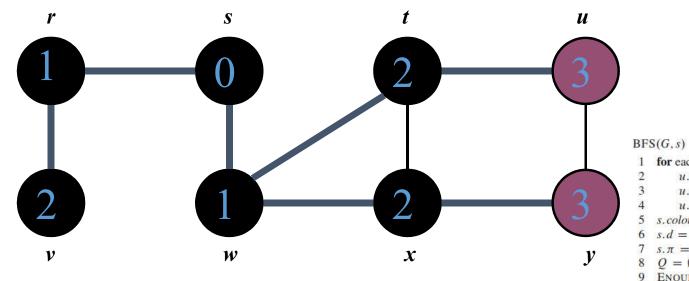
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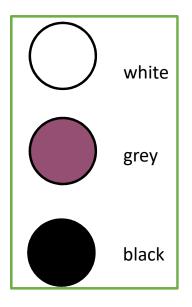
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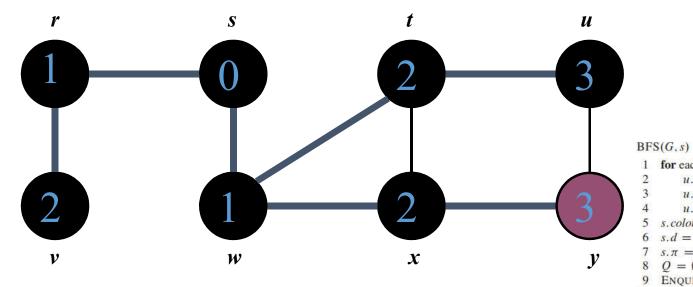




```
Q: u y 3
```

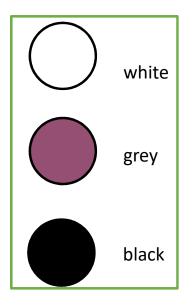
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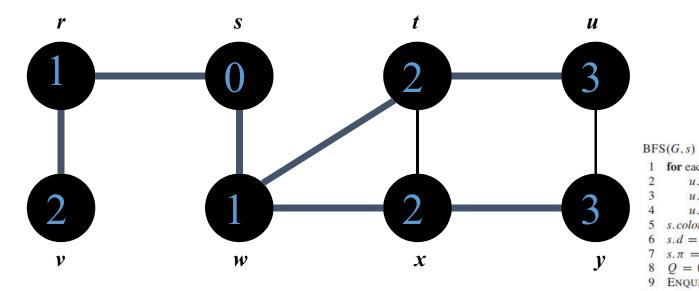




 $Q: \boxed{y}$

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```
BFS(G,s)
    for each vertex u \in G.V - \{s\}
        u.color = WHITE
                                          O(V)
       u.d = \infty
        u.\pi = NIL
   s.color = GRAY
 6 \, s.d = 0
                                          O(1)
    s.\pi = NIL
    Q = \emptyset
    ENQUEUE(Q, s)
   while Q \neq \emptyset
10
        u = \text{DEQUEUE}(Q)
11
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BFS: Analysis

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BFS: Analysis

O(1)

O(V)

Each vertex is enqueued/dequeued once: O(V) in total

Once dequeued, the adjacency list of a vertex is explored: O(E) in total

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O(V)

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Once dequeued, the adjacency list of a vertex is explored: O(E) in total

BFS: Analysis

O(V+E)

Breadth-First Search: Properties

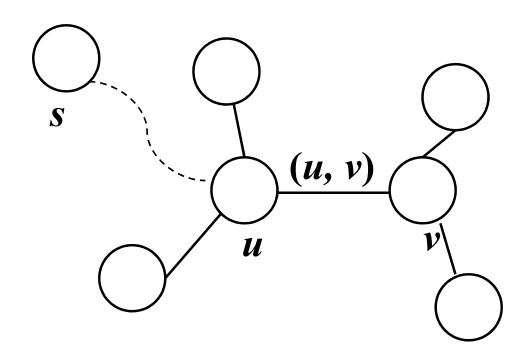
- BFS calculates the *shortest-path distance* from the source node
 - Shortest-path distance $\delta(s, v)$
 - = minimum number of edges from s to v, OR
 - $= \infty$, if v is NOT reachable from s

Breadth-First Search: Properties

- BFS calculates the *shortest-path distance* from the source node
 - Shortest-path distance $\delta(s, v)$
 - = minimum number of edges from s to v, OR
 - $= \infty$, if v is NOT reachable from s
- BFS builds *breadth-first tree*, in which paths from root represent shortest paths in *G*
 - Thus can use BFS to calculate shortest path from one vertex to another in O(V+E) time in an unweighted graph

Let G = (V, E) be a directed or undirected graph, and let $s \in V$ be an arbitrary vertex. Then, for any edge $(u, v) \in E$,

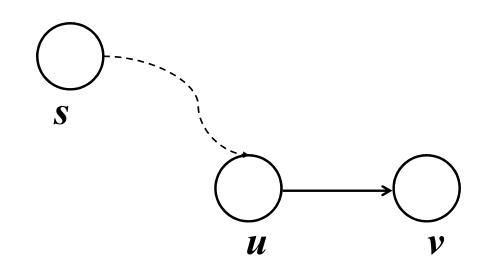
$$\delta(s, v) \le \delta(s, u) + 1$$



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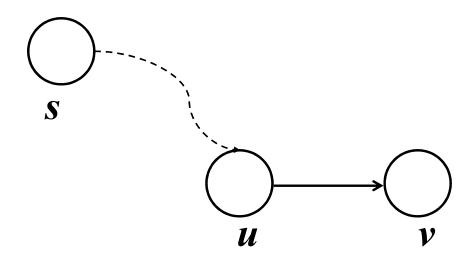
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If u is reachable from s, so is v



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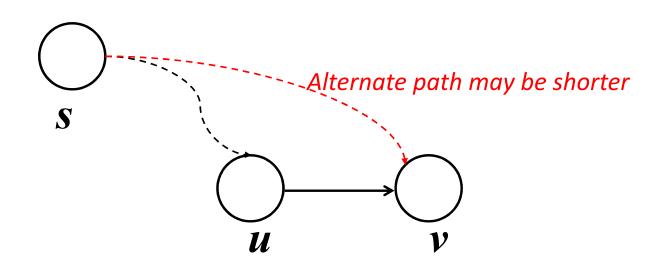
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Shortest-path to v cannot be longer than shortest path to u plus edge (u, v)

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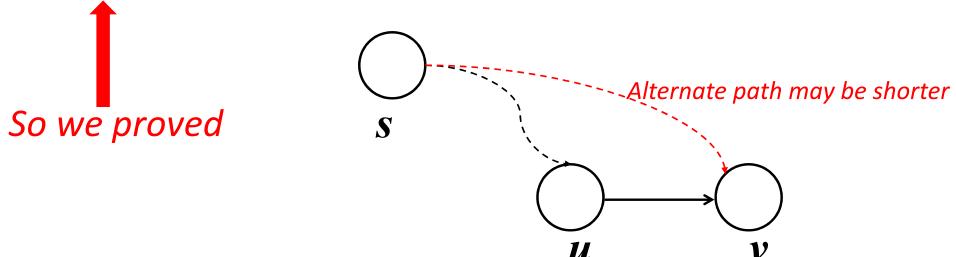
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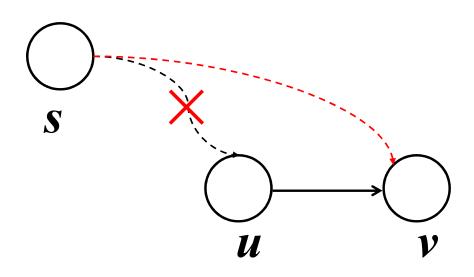


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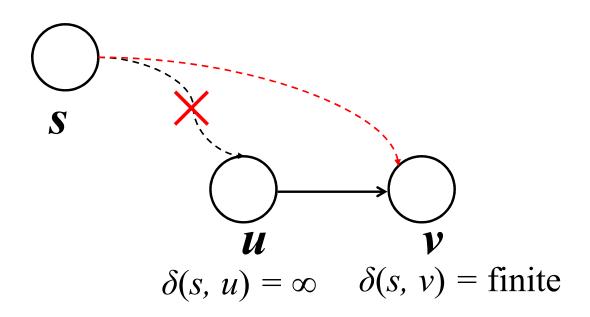
If *u* is NOT reachable from *s*



Let G = (V, E) be a directed or undirected graph, and let $s \in V$ be an arbitrary vertex. Then, for any edge $(u, v) \in E$,

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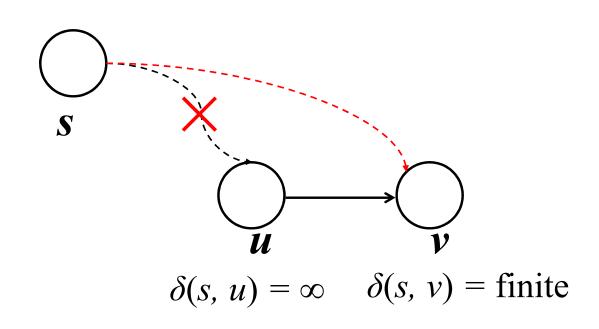


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$$\delta(s, v) \le \delta(s, u) + 1$$

we proved again

If *u* is NOT reachable from *s*



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     O = \emptyset
    ENQUEUE(O,s)
    while Q \neq \emptyset
        u = \text{DEQUEUE}(O)
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        for each v \in G.Adi[u]
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Let G = (V, E) be a directed or undirected graph, and suppose that BFS is run on G from a given source vertex $s \in V$. Then upon termination, for each vertex $v \in V$, the value v.d computed by BFS satisfies $v.d \geq \delta(s, v)$

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Assume, before an EnQ, P holds

Then show, after the next EnQ, P still holds

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Then show, after the next EnQ, P still holds

Lemma 22.2

Then upon termination, for each vertex $v \in V$, the value v.d computed by BFS satisfies $v.d \ge \delta(s, v)$

Basis:

$$s.d = 0 = \delta(s, s)$$
 and $s.d \ge \delta(s, s)$ and $v.d = \infty \ge \delta(s, v)$ for all other vertices v

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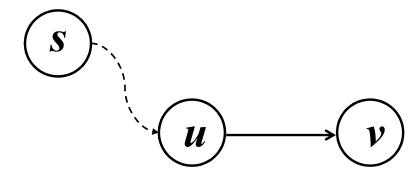
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Induction:

Let, white vertex *v* is discovered from *u*



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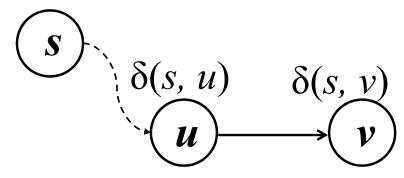
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Induction:

Let, white vertex v is discovered from u

Now, by induction: $u.d \ge \delta(s, u)$ By lemma 22.1: $\delta(s, v) \le \delta(s, u) + 1$



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Then upon termination, for each vertex $v \in V$, the value v.d computed by BFS satisfies $v.d \ge \delta(s, v)$

Induction:

Let, white vertex *v* is discovered from *u*

Now, by induction: $u.d \ge \delta(s, u)$ By lemma 22.1: $\delta(s, v) \le \delta(s, u) + 1$

Now,
$$v.d = u.d + 1$$
 (s) $\delta(s, u)$ $\delta(s, v)$

```
BFS(G,s)
    for each vertex u \in G.V - \{s\}
        u.color = WHITE
        u.d = \infty
        u.\pi = NIL
    s.color = GRAY
   s.d = 0
    s.\pi = NIL
     O = \emptyset
    ENQUEUE(O,s)
    while Q \neq \emptyset
        u = \text{DEQUEUE}(O)
11
        for each v \in G.Adi[u]
12
             if v.color == WHITE
13
                 v.color = GRAY
14
                 v.d = u.d + 1
15
16
                 \nu.\pi = u
      Next
                 ENQUEUE(Q, v)
17
      EnOs
18
         u.color = BLACK
```

Then show, after the next EnQ, P still holds

Lemma 22.2

Then upon termination, for each vertex $v \in V$, the value v.d computed by BFS satisfies $v.d \ge \delta(s, v)$

Induction:

Let, white vertex v is discovered from u

Now, by induction: $u.d \ge \delta(s, u)$ By lemma 22.1: $\delta(s, v) \le \delta(s, u) + 1$

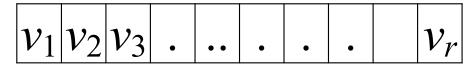
Now,
$$v.d = u.d + 1$$

 $\geq \delta(s, u) + 1$
 $\geq \delta(s, v)$

BFS(G,s)1 for each vertex $u \in G.V - \{s\}$ u.color = WHITE $u.d = \infty$ $u.\pi = NIL$ s.color = GRAYs.d = 07 $s.\pi = NIL$ $O = \emptyset$ ENQUEUE(O,s)while $Q \neq \emptyset$ 11 u = DEQUEUE(Q)for each $v \in G.Adj[u]$ 12 13 if v.color == WHITEv.color = GRAY14 v.d = u.d + 115 16 $\nu.\pi = u$ 17 ENQUEUE(O, v)18 u.color = BLACK

Lemma 22.3

Suppose that during the execution of BFS on a graph G = (V, E), the queue Q contains the vertices $\langle v_1, v_2, \dots, v_r \rangle$ where v_1 is the head of Q and v_r is the tail. Then (1) $v_r.d \leq v_1.d + 1$ and (2) $v_i.d \leq v_{i+1}.d$ for $i = 1, 2, \dots, r-1$



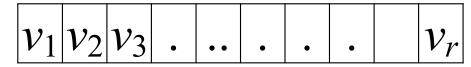
Vertices in Queue

1 **for** each vertex $u \in G.V - \{s\}$ u.color = WHITE $u.d = \infty$ $u.\pi = NIL$ 5 s.color = GRAY $6 \, s.d = 0$ 7 $s.\pi = NIL$ $8 \quad O = \emptyset$ 9 ENQUEUE(O,s)10 while $Q \neq \emptyset$ 11 u = DEQUEUE(Q)12 for each $v \in G.Adj[u]$ 13 if v.color == WHITEv.color = GRAY14 15 v.d = u.d + 116 $\nu.\pi = u$ 17 ENQUEUE(O, v)18 u.color = BLACK

BFS(G,s)

Lemma 22.3

during the execution, the queue $Q = \langle v_1, v_2, \dots, v_r \rangle$ where $v_1 =$ head and $v_r =$ tail Then (1) $v_r.d \le v_1.d + 1$ and (2) $v_i.d \le v_{i+1}.d$ for $i = 1, 2, \dots, r-1$



Vertices in Queue

We have to 2. $v_1.d \le v_2.d \le v_3.d ... \le v_r.d$ **prove:**

1.
$$v_r d \leq v_1 d + 1$$

BFS(G,s)1 **for** each vertex $u \in G.V - \{s\}$ u.color = WHITE $u.d = \infty$ $u.\pi = NIL$ s.color = GRAY $6 \ s.d = 0$ 7 $s.\pi = NIL$ $8 \quad Q = \emptyset$ 9 ENQUEUE(O,s)while $Q \neq \emptyset$ 11 u = DEQUEUE(Q)12 for each $v \in G.Adj[u]$ 13 if v.color == WHITE14 v.color = GRAYv.d = u.d + 115 16 $\nu.\pi = u$ $ENQUEUE(Q, \nu)$ 17

u.color = BLACK

18

Lemma 22.3

```
Q = \langle v_1, v_2, \dots, v_r \rangle where v_1 = \text{head and } v_r = \text{tail}

Prove: (1) v_r.d \le v_1.d + 1

(2) v_i.d \le v_{i+1}.d for i = 1, 2, \dots, r-1

v_1 v_2 v_3 \dots v_r
```

Basis:

It is true., as queue contains only s.

BFS(G,s)1 **for** each vertex $u \in G.V - \{s\}$ u.color = WHITE $u.d = \infty$ $u.\pi = NIL$ 5 s.color = GRAY $6 \, s.d = 0$ 7 $s.\pi = NIL$ $O = \emptyset$ 9 ENQUEUE(O,s)while $Q \neq \emptyset$ 11 u = DEQUEUE(Q)12 for each $v \in G.Adj[u]$ 13 if v.color == WHITE14 v.color = GRAY15 v.d = u.d + 116 $\nu.\pi = u$ ENQUEUE(O, v)17 u.color = BLACK18

Lemma 22.3

We will prove
$$1. v_1.d \le v_2.d \le v_3.d ... \le v_r.d$$
 both for $2. v_r.d \le v_1.d + 1$ Enqueue operations

```
BFS(G,s)
 1 for each vertex u \in G.V - \{s\}
        u.color = WHITE
        u.d = \infty
        u.\pi = NIL
   s.color = GRAY
 6 \, s.d = 0
 7 s.\pi = NIL
 8 \quad Q = \emptyset
9 ENQUEUE(O,s)
    while Q \neq \emptyset
11
         u = \text{DEQUEUE}(Q)
12
         for each v \in G.Adj[u]
13
             if v.color == WHITE
14
                 v.color = GRAY
                 v.d = u.d + 1
15
16
                 \nu.\pi = u
17
                 ENQUEUE(Q, v)
         u.color = BLACK
18
```

```
Prove: (1) v_r.d \le v_1.d + 1
(2) v_1.d \le v_2.d \le v_3.d \le \cdots \le v_r.d
```

$ v_1 v_2 v_3 $		v_1	v_2	<i>V</i> 3	•	••	•	•	•		v_r	Before DEQUEUE
-----------------	--	-------	-------	------------	---	----	---	---	---	--	-------	----------------

Induction

Before DEQUEUE, IH holds:

$$v_1.d \le v_2.d \le v_3.d... \le v_r.d$$

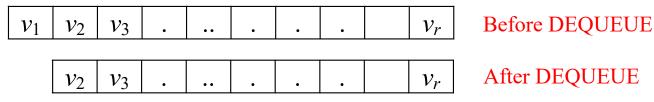
 $v_r.d \le v_1.d + 1$

```
for each vertex u \in G.V - \{s\}
         u.color = WHITE
        u.d = \infty
         u.\pi = NIL
   s.color = GRAY
 6 \, s.d = 0
 7 s.\pi = NIL
 8 \quad Q = \emptyset
 9 ENQUEUE(O, s)
    while Q \neq \emptyset
11
         u = \text{DEQUEUE}(Q)
12
         for each v \in G.Adj[u]
13
             if v.color == WHITE
14
                  v.color = GRAY
                 v.d = u.d + 1
15
16
                  \nu.\pi = u
                  ENQUEUE(O, v)
17
18
         u.color = BLACK
```

BFS(G,s)

Lemma 22.3

Prove: (1) $v_r.d \le v_1.d + 1$ (2) $v_1.d \le v_2.d \le v_3.d \le \cdots \le v_r.d$



Induction

Before DEQUEUE, IH holds:

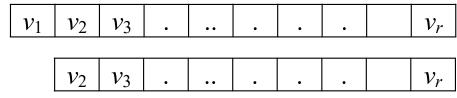
$$v_1.d \le v_2.d \le v_3.d... \le v_rd$$

 $v_rd \le v_1.d+1$

After DEQUEUE (of v1): $v_2.d \le v_3.d... \le v_r.d$ (Okay)

```
BFS(G,s)
 1 for each vertex u \in G.V - \{s\}
        u.color = WHITE
        u.d = \infty
        u.\pi = NIL
   s.color = GRAY
 6 \ s.d = 0
 7 s.\pi = NIL
 8 \quad Q = \emptyset
 9 ENQUEUE(O,s)
    while Q \neq \emptyset
11
         u = \text{DEQUEUE}(Q)
12
         for each v \in G.Adj[u]
13
             if v.color == WHITE
14
                 v.color = GRAY
15
                 v.d = u.d + 1
16
                 \nu.\pi = u
                 ENQUEUE(O, v)
17
18
         u.color = BLACK
```

Prove: (1) $v_r.d \le v_1.d + 1$ (2) $v_1.d \le v_2.d \le v_3.d \le \cdots \le v_r.d$



Before DEQUEUE

After DEQUEUE

Induction

Before DEQUEUE, IH holds:

$$v_1.d \le v_2.d \le v_3.d... \le v_rd$$

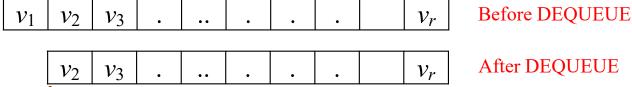
 $v_rd \le v_1.d+1$

After DEQUEUE (of v1):

$$v_2.d \le v_3.d... \le v_r.d$$
 (Okay)
 $v_r.d \le v_1.d+1$ (from previous relation)

```
BFS(G,s)
    for each vertex u \in G.V - \{s\}
         u.color = WHITE
        u.d = \infty
         u.\pi = NIL
    s.color = GRAY
 6 \, s.d = 0
 7 s.\pi = NIL
 8 \quad O = \emptyset
 9 ENQUEUE(O,s)
    while Q \neq \emptyset
11
         u = \text{DEQUEUE}(Q)
12
         for each v \in G.Adj[u]
13
             if v.color == WHITE
                 v.color = GRAY
14
                 v.d = u.d + 1
15
16
                 \nu.\pi = u
17
                 ENQUEUE(O, v)
18
         u.color = BLACK
```

Prove: (1) $v_r d \le v_1 d + 1$ (2) $v_1.d \le v_2.d \le v_3.d \le \cdots \le v_r.d$



After DEQUEUE

Induction

Before DEQUEUE, IH holds:

$$v_1.d \le v_2.d \le v_3.d... \le v_rd$$

 $v_rd \le v_1.d + 1$

After DIQUEUE (of v1):

$$v_2.d \le v_3 d... \le v_r d \text{ (Okay)}$$

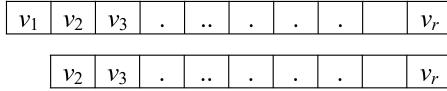
 $v_r d \le v_1.d + 1 \le v_2.d + 1$

```
for each vertex u \in G.V - \{s\}
         u.color = WHITE
        u.d = \infty
         u.\pi = NIL
   s.color = GRAY
 6 \, s.d = 0
 7 s.\pi = NIL
 8 \quad O = \emptyset
9 ENQUEUE(O, s)
    while Q \neq \emptyset
11
         u = \text{DEQUEUE}(Q)
12
         for each v \in G.Adj[u]
13
             if v.color == WHITE
14
                  v.color = GRAY
                 v.d = u.d + 1
15
16
                  \nu.\pi = u
17
                  ENQUEUE(O, v)
         u.color = BLACK
18
```

BFS(G,s)

Lemma 22.3

Prove: (1) $v_r.d \le v_1.d + 1$ (2) $v_1.d \le v_2.d \le v_3.d \le \cdots \le v_r.d$



Before DEQUEUE

After DEQUEUE

Induction

Before DEQUEUE, IH holds:

$$v_1.d \le v_2.d \le v_3.d... \le v_r.d$$

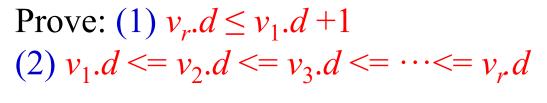
 $v_r.d \le v_1.d+1$

After DEQUEUE (of v1):

$$v_2.d \le v_3.d... \le v_r.d \text{ (Okay)}$$

 $v_r.d \le v_1.d + 1 \le v_2.d + 1$ (Okay)

```
BFS(G,s)
 1 for each vertex u \in G.V - \{s\}
        u.color = WHITE
        u.d = \infty
        u.\pi = NIL
    s.color = GRAY
 6 \, s.d = 0
 7 s.\pi = NIL
    Q = \emptyset
 9 ENQUEUE(O, s)
    while Q \neq \emptyset
11
        u = \text{DEQUEUE}(Q)
12
        for each v \in G.Adj[u]
13
             if v.color == WHITE
14
                 v.color = GRAY
15
                 v.d = u.d + 1
16
                 \nu.\pi = u
17
                 ENQUEUE(Q, v)
        u.color = BLACK
18
```



V	1ν	v_2 v_3	•	••	•	•			v_r
---	---------	-------------	---	----	---	---	--	--	-------

Before ENQUEUE

Induction

Before ENQUEUE, IH holds:

$$v_1.d \le v_2.d \le v_3.d... \le v_r.d$$

 $v_r.d \le v_1.d + 1$

BFS(G,s)**for** each vertex $u \in G.V - \{s\}$ u.color = WHITE $u.d = \infty$ $u.\pi = NIL$ s.color = GRAYs.d = 07 $s.\pi = NIL$ $O = \emptyset$ ENQUEUE(O, s)while $Q \neq \emptyset$ 11 u = DEQUEUE(Q)12 for each $v \in G.Adj[u]$ 13 if v.color == WHITEv.color = GRAY14 15 v.d = u.d + 116 $\nu.\pi = u$ ENQUEUE(O, v)17 18 u.color = BLACK

Lemma 22.3

Prove: (1) $v_r d \le v_1 d + 1$

(2)
$$v_1.d \le v_2.d \le v_3.d \le \cdots \le v_r.d$$

v_1	v_2	<i>v</i> ₃	•	• •	•	•	•	v_r

Before ENQUEUE

 $v_1 \mid v_2 \mid v_3 \mid$ $v_r \mid v_{r+1}$ After ENQUEUE

Induction

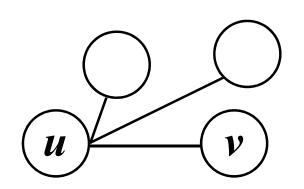
Before ENQUEUE, IH holds:

$$v_1.d \le v_2.d \le v_3.d... \le v_rd$$

 $v_rd \le v_1.d+1$

After ENQUEUE (of v): Let, we enqueue v from u

v becomes v_{r+1} .



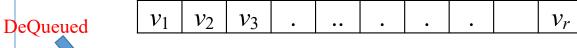
BFS(G,s)**for** each vertex $u \in G.V - \{s\}$ u.color = WHITE $u.d = \infty$ $u.\pi = NIL$ s.color = GRAYs.d = 0 $s.\pi = NIL$ $O = \emptyset$ ENQUEUE(O,s)while $Q \neq \emptyset$ 11 u = DEQUEUE(Q)12 for each $v \in G.Adj[u]$ 13 if v.color == WHITEv.color = GRAY14 15 v.d = u.d + 116 $\nu.\pi = u$

u.color = BLACK

17 18 ENQUEUE(O, v)

Lemma 22.3

Prove: (1) $v_r.d \le v_1.d + 1$ (2) $v_1.d \le v_2.d \le v_3.d \le \cdots \le v_r.d$



Before ENQUEUE

 v_{r+1} After ENQUEUE

Induction

 ν_2

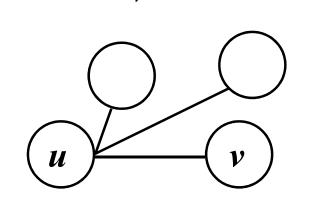
Before ENQUEUE, IH holds:

 V_3

$$v_1.d \le v_2.d \le v_3.d... \le v_rd$$

 $v_rd \le v_1.d+1$

After ENQUEUE (of v): u was is IN queue but dequeued $u.d \le v_1.d$



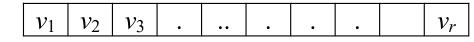
```
BFS(G,s)
    for each vertex u \in G.V - \{s\}
         u.color = WHITE
        u.d = \infty
         u.\pi = NIL
    s.color = GRAY
    s.d = 0
 7 s.\pi = NIL
    O = \emptyset
    ENQUEUE(O, s)
    while Q \neq \emptyset
11
         u = \text{DEQUEUE}(Q)
12
         for each v \in G.Adj[u]
13
             if v.color == WHITE
                  v.color = GRAY
14
15
                 v.d = u.d + 1
16
                 \nu.\pi = u
17
                 ENQUEUE(O, v)
18
         u.color = BLACK
```

Prove: (1) $v_r . d \le v_1 . d + 1$

 v_2

(2)
$$v_1.d \le v_2.d \le v_3.d \le \cdots \le v_r.d$$

DeQueued



Before ENQUEUE

 v_{r+1} After ENQUEUE

Induction

Before ENQUEUE, IH holds:

 V_3

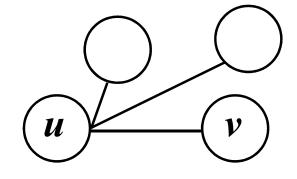
$$v_1.d \le v_2.d \le v_3.d... \le v_rd$$

 $v_rd \le v_1.d+1$



 \mathcal{V}_r

After ENQUEUE (of v): u was is IN queue but dequeued $u.d \le v_1.d$ $v_{r+1}.d=v.d=u.d+1$



BFS(G,s)**for** each vertex $u \in G.V - \{s\}$ u.color = WHITE $u.d = \infty$ $u.\pi = NIL$ s.color = GRAYs.d = 0 $s.\pi = NIL$ $O = \emptyset$ ENQUEUE(O,s)while $Q \neq \emptyset$ 11 u = DEQUEUE(Q)for each $v \in G.Adj[u]$ 12 13 if v.color == WHITEv.color = GRAY14 15 v.d = u.d + 116 $\nu.\pi = u$

u.color = BLACK

ENQUEUE(O, v)

17

18

Lemma 22.3

Prove: (1) $v_r d \le v_1 d + 1$

(2) $v_1.d \le v_2.d \le v_3.d \le \cdots \le v_r.d$

DeQueued

v_1 v_2 v_3 .			V_{I}
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Before ENQUEUE

Induction

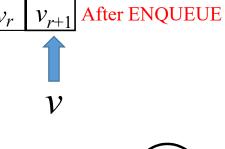
Before ENQUEUE, IH holds:

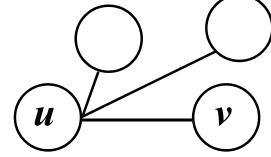
 V_3

$$v_1.d \le v_2.d \le v_3.d... \le v_rd$$

 $v_rd \le v_1.d+1$

After ENQUEUE (of v): u was is IN queue but dequeued $u.d \leq v_1.d$ $v_{r+1}.d=v.d=u.d+1 \le v_1.d+1$





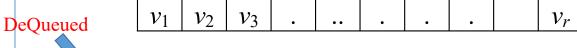
BFS(G,s)**for** each vertex $u \in G.V - \{s\}$ u.color = WHITE $u.d = \infty$ $u.\pi = NIL$ s.color = GRAYs.d = 07 $s.\pi = NIL$ $O = \emptyset$ ENQUEUE(O, s)while $Q \neq \emptyset$ 11 u = DEQUEUE(Q)12 for each $v \in G.Adj[u]$ 13 if v.color == WHITEv.color = GRAY14 15 v.d = u.d + 116 $\nu.\pi = u$ ENQUEUE(O, v)17

u.color = BLACK

18

Lemma 22.3

Prove: (1) $v_r.d \le v_1.d + 1$ (2) $v_1.d \le v_2.d \le v_3.d \le \cdots \le v_r.d$



Before ENQUEUE

 v_{r+1} After ENQUEUE

Induction

 v_2

Before ENQUEUE, IH holds:

 V_3

$$v_1.d \le v_2.d \le v_3.d... \le v_rd$$

 $v_rd \le v_1.d+1$

 \mathcal{V}_r

After ENQUEUE (of v): u was is IN queue but dequeued $u.d \le v_1.d$ $v_{r+1}.d \le v_1.d+1$

BFS(G,s)**for** each vertex $u \in G.V - \{s\}$ u.color = WHITE $u.d = \infty$ $u.\pi = NIL$ s.color = GRAYs.d = 0 $s.\pi = NIL$ $O = \emptyset$ ENQUEUE(O, s)while $Q \neq \emptyset$ 11 u = DEQUEUE(Q)12 for each $v \in G.Adj[u]$ 13 if v.color == WHITEv.color = GRAY14 15 v.d = u.d + 116 $\nu.\pi = u$

u.color = BLACK

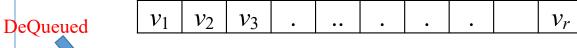
ENQUEUE(O, v)

17

18

Lemma 22.3

Prove: (1) $v_r d \le v_1 d + 1$ (2) $v_1.d \le v_2.d \le v_3.d \le \cdots \le v_r.d$



Before ENQUEUE

Induction

 ν_2

Before ENQUEUE, IH holds:

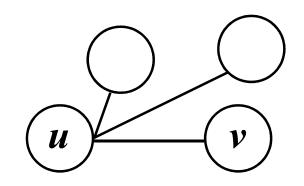
 V_3

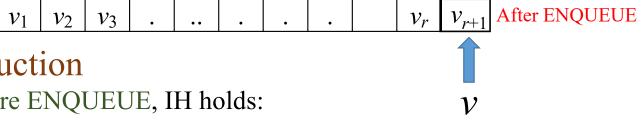
$$v_1.d \le v_2.d \le v_3.d... \le v_rd$$

 $v_rd \le v_1.d+1$

After ENQUEUE (of v):

By induction $v_r d \le u \cdot d + 1$





BFS(G,s)**for** each vertex $u \in G.V - \{s\}$ u.color = WHITE $u.d = \infty$ $u.\pi = NIL$ s.color = GRAYs.d = 07 $s.\pi = NIL$ $O = \emptyset$ ENQUEUE(O, s)while $Q \neq \emptyset$ 11 u = DEQUEUE(Q)12 for each $v \in G.Adj[u]$ 13 if v.color == WHITEv.color = GRAY14 15 v.d = u.d + 116 $\nu.\pi = u$

u.color = BLACK

ENQUEUE(O, v)

17

18

Lemma 22.3

Prove: (1) $v_r.d \le v_1.d + 1$ (2) $v_1.d \le v_2.d \le v_3.d \le \cdots \le v_r.d$



 v_1 v_2 v_3 v_r

Before ENQUEUE

Induction

 ν_2

Before ENQUEUE, IH holds:

 V_3

$$v_1.d \le v_2.d \le v_3.d... \le v_rd$$

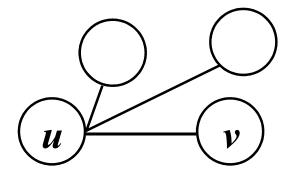
 $v_rd \le v_1.d+1$

After ENQUEUE (of v):

By induction

$$v_r d \le u.d + 1 = v.d = v_{r+1}.d$$





BFS(G,s)**for** each vertex $u \in G.V - \{s\}$ u.color = WHITE $u.d = \infty$ $u.\pi = NIL$ s.color = GRAYs.d = 0 $s.\pi = NIL$ $O = \emptyset$ ENQUEUE(O,s)while $Q \neq \emptyset$ 11 u = DEQUEUE(Q)12 for each $v \in G.Adj[u]$ 13 if v.color == WHITEv.color = GRAY14 15 v.d = u.d + 116 $\nu.\pi = u$ 17 ENQUEUE(Q, v)

u.color = BLACK

18

Lemma 22.3

Prove: (1) $v_r d \le v_1 d + 1$

 V_3

(2)
$$v_1.d \le v_2.d \le v_3.d \le \cdots \le v_r.d$$



|--|

Before ENQUEUE

 v_{r+1} After ENQUEUE

Induction

Before ENQUEUE, IH holds:

 ν_2

$$v_1.d \le v_2.d \le v_3.d... \le v_rd$$

 $v_rd \le v_1.d+1$

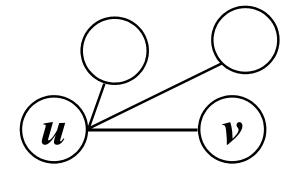
l v

 \mathcal{V}_r

After ENQUEUE (of v):

By induction
$$v_r d \le u . d + 1 = v . d = v_{r+1} . d$$

That means, $v_r d \le v_{r+1} d$



```
BFS(G,s)
 1 for each vertex u \in G.V - \{s\}
         u.color = WHITE
        u.d = \infty
         u.\pi = NIL
 5 \quad s. color = GRAY
 6 \, s.d = 0
 7 s.\pi = NIL
 8 \quad O = \emptyset
 9 ENQUEUE(O, s)
    while 0 \neq \emptyset
         u = \text{DEQUEUE}(Q)
11
12
         for each v \in G. Adj[u]
13
             if v.color == WHITE
14
15
                  v.d = u.d + 1
16
                  \nu.\pi = u
                  ENQUEUE(Q, v)
17
         u.color = BLACK
18
```

```
Prove: (1) v_r.d \le v_1.d + 1
(2) v_1.d \le v_2.d \le v_3.d \le \cdots \le v_r.d
```

v.color = GRAY Corollary 22.4

Suppose that vertices v_i and v_j are enqueued during the execution of BFS, and that v_i is enqueued before v_i . Then $v_i \cdot d \le v_j \cdot d$ at the time that v_i is enqueued.

$ v_1 v_2 v_3 . v_i v_j . v_r$
