CSE 105: Data Structures and Algorithms-I (Part 2)

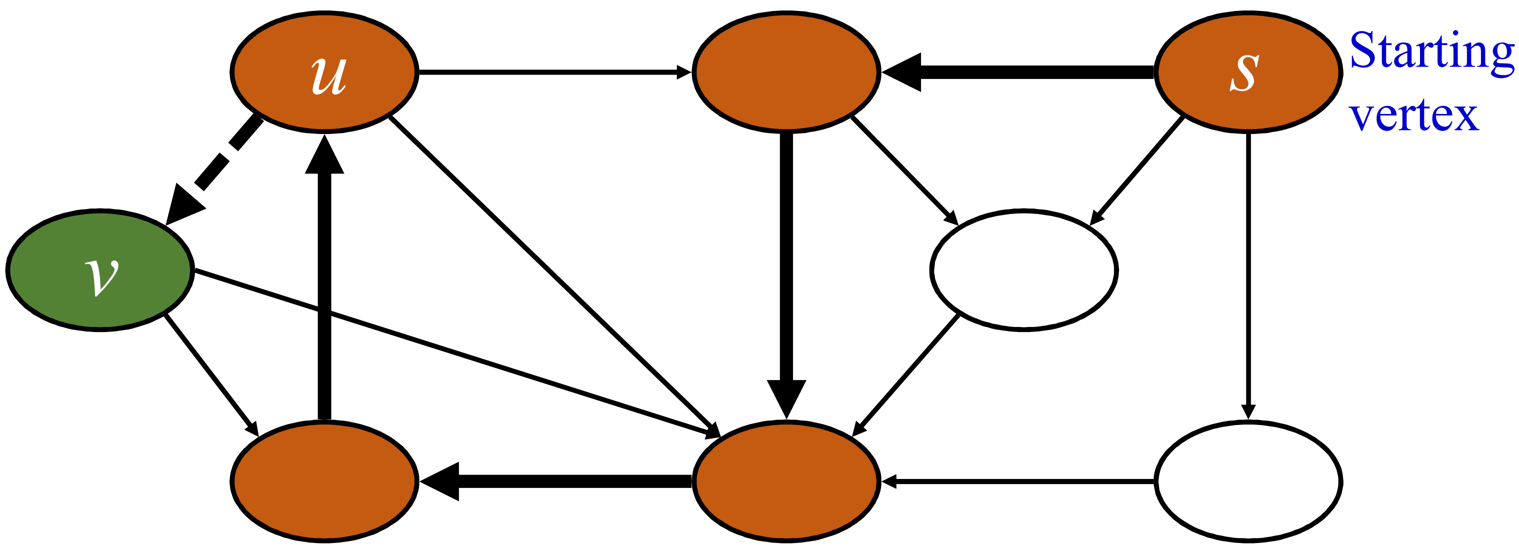
Instructor

Dr Md Monirul Islam

Graph Searching

Depth-First Search

- Explore "deeper" in the graph whenever possible



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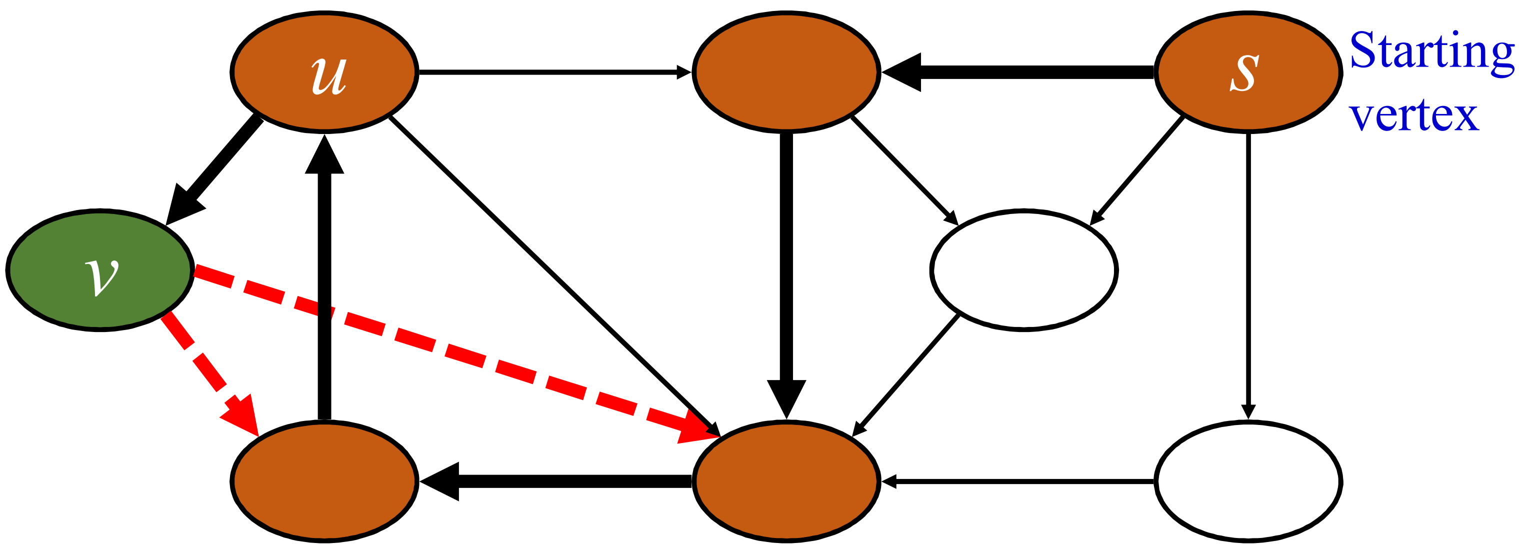
- Edges are explored out of the most recently discovered vertex v that still has unexplored edges



Depth-First Search

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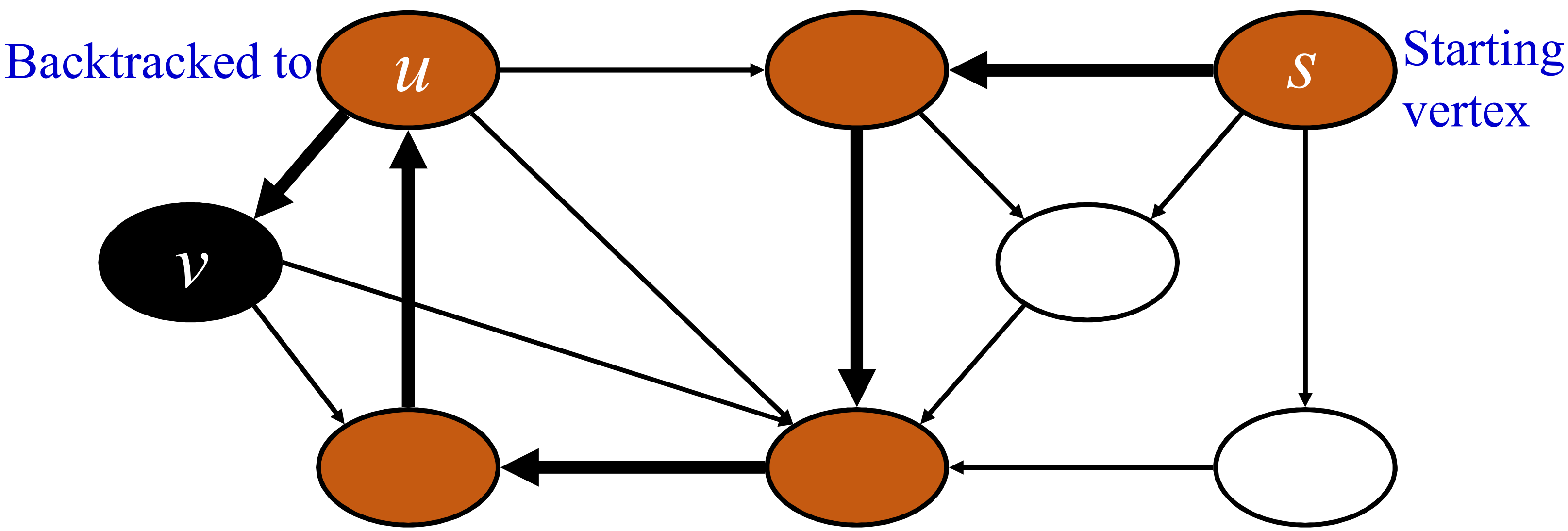
- When all of *v*'s edges have been explored, backtrack to the vertex from which *v* was discovered (i.e., its parent)



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- Edges are explored out of the most recently discovered vertex v that still has unexplored edges
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- Vertices initially colored white
- Then colored grey when discovered
- Then black when finished

```
DFS(G)
for each vertex u ∈ G.V
    u.color = WHITE
    u.π = NIL
time = 0
for each vertex u ∈ G.V
    if u.color == WHITE
        DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
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```

- records predecessors in π attributes
- Produces multiple trees

- we define the ***predecessor subgraph*** of G as $G_\pi = (V, E_\pi)$, where

  $E_\pi = \{(v.\pi, v) : v \in V \text{ and } v.\pi \neq NIL\}$

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time = time + 1
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```

- Records timestamps for each vertex, *v*
  - Discovery time, *d*: when *v* is discovered
  - Finishing time, *f* : when *v's adjacency list* is finished

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Θ(V)

Θ(V) EXCLUDING the time required for DFS-VISIT().

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$\Theta(V)$

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```

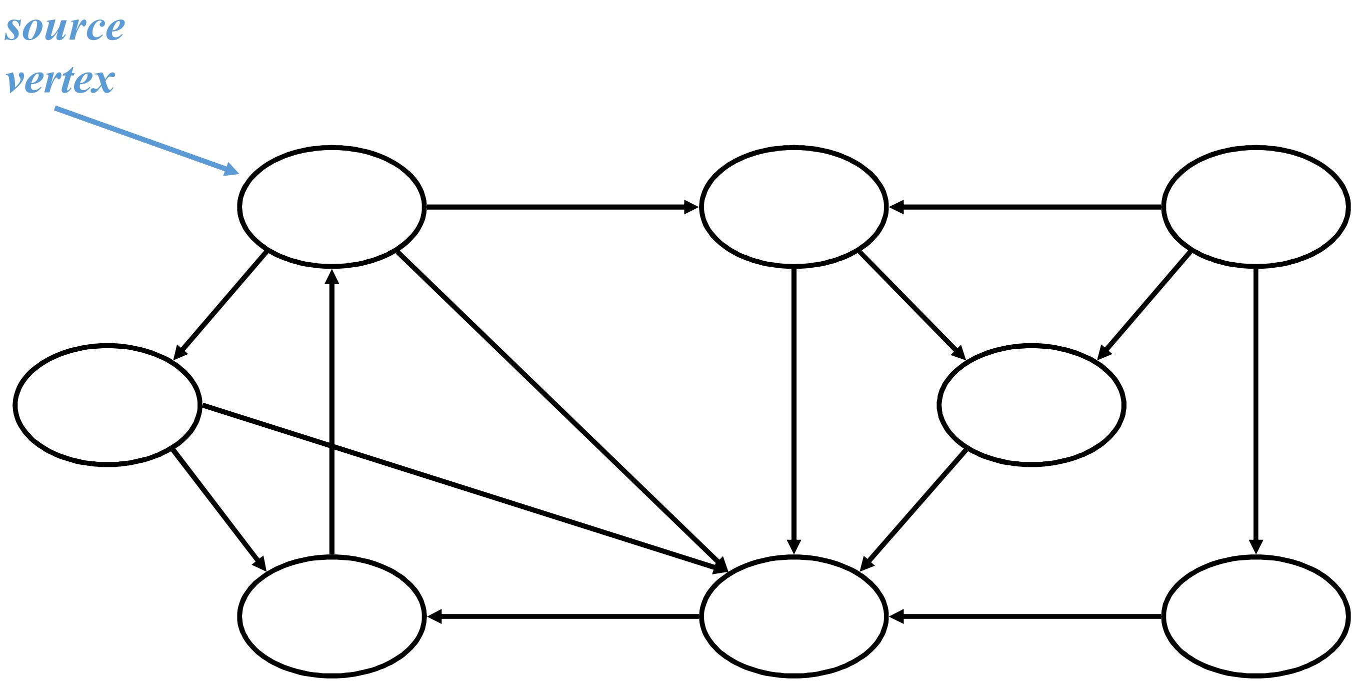
$$\sum_{v \in V} |Adj[v]| = \Theta(E)$$

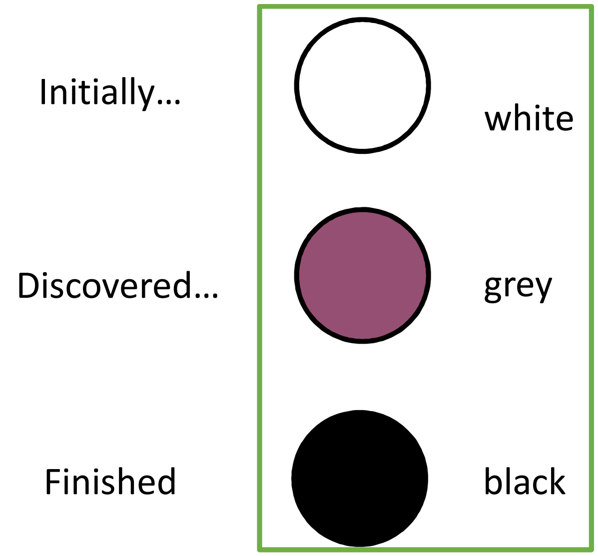
$\Theta(V + E)$

How many times DFS-Visit() is called?

- The procedure DFS-VISIT is called exactly once for each vertex since:
  - the vertex u on which DFS-VISIT() is invoked must be white
  - the first thing DFS-VISIT does is paint vertex u gray

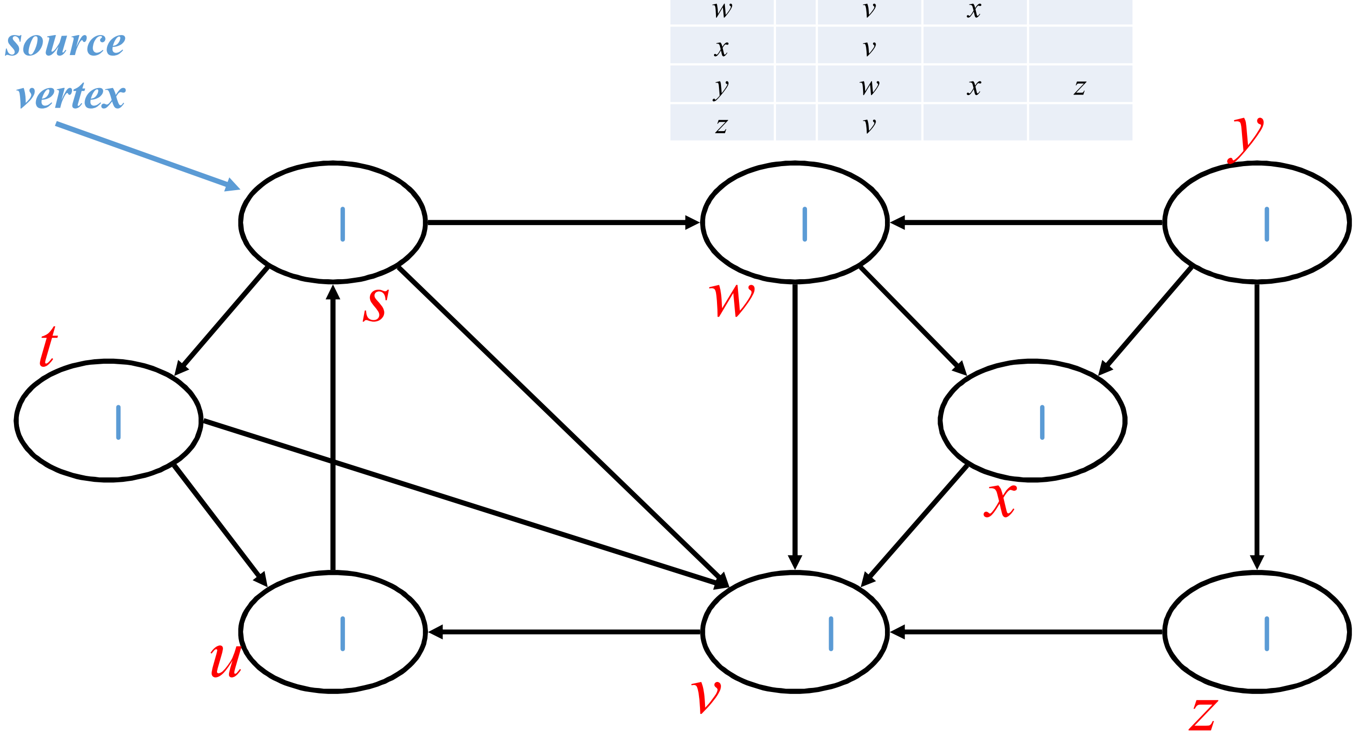
DFS Example

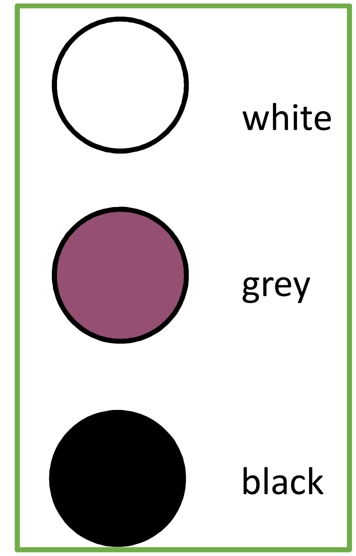

source
vertex


Initially…
white
Discovered…
grey
Finished
black

DFS Example

Vertices		Adjacency list		
s		*t*	*v*	*w*
t		*u*	*v*	
u		*s*		
v		*u*		
w		*v*	*x*	
x		*v*		
y		*w*	*x*	*z*
z		*v*		





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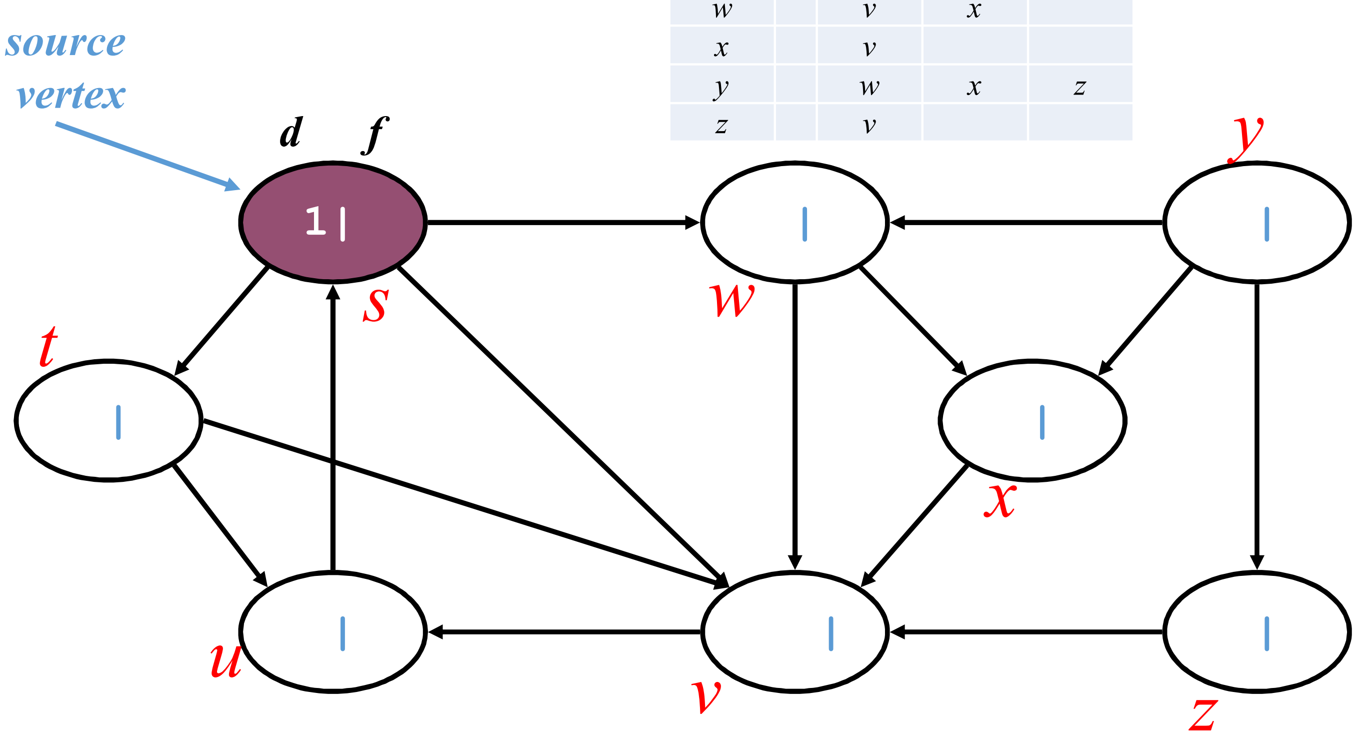

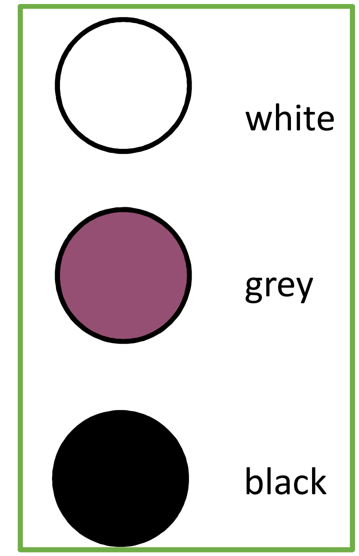

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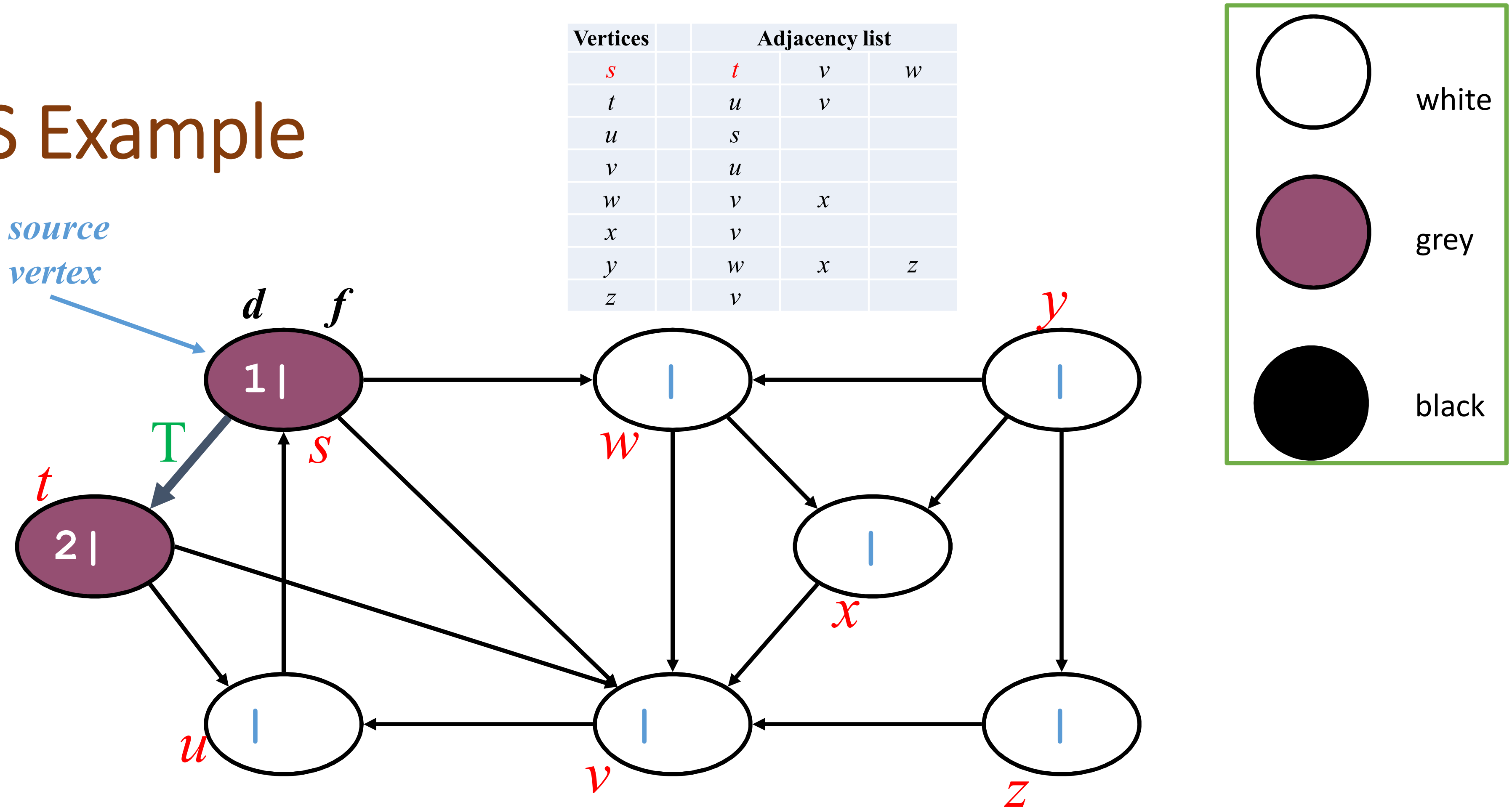
z		*v*		



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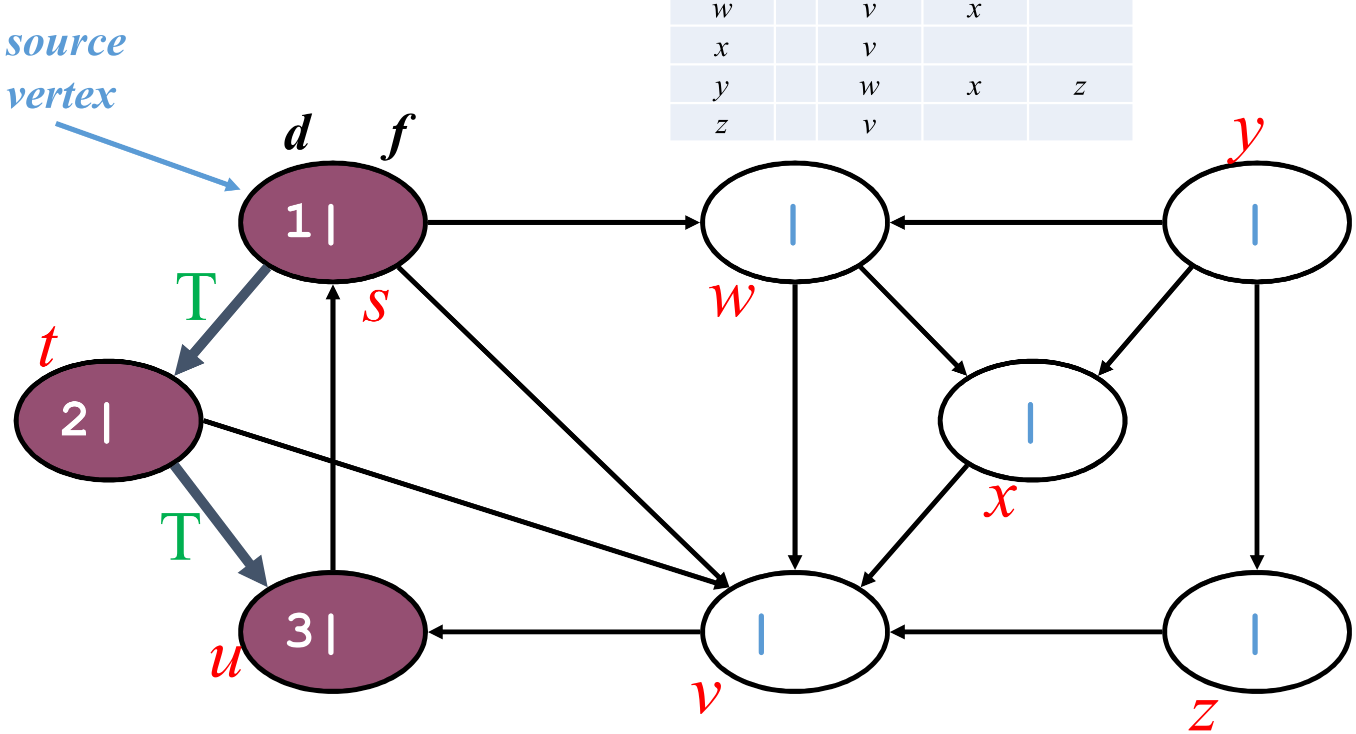

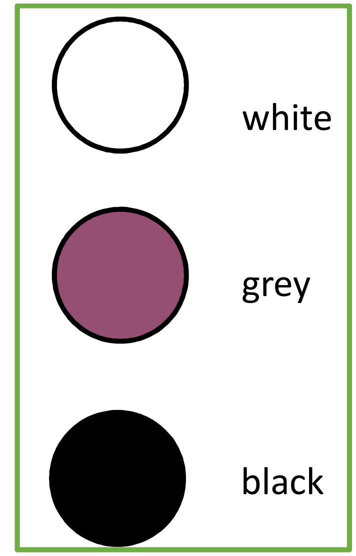

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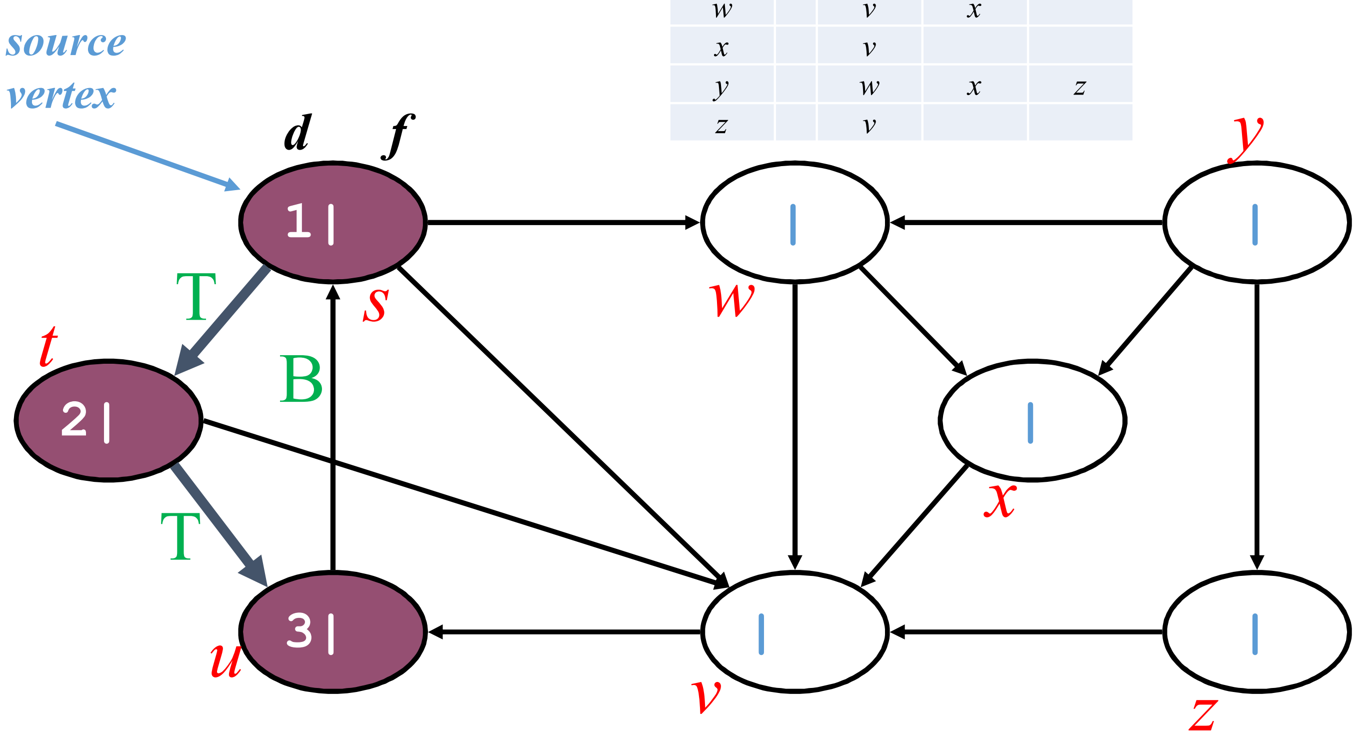

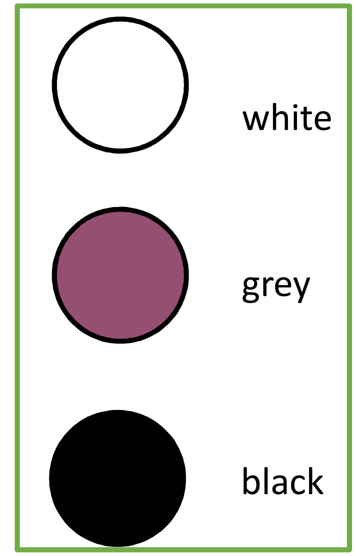

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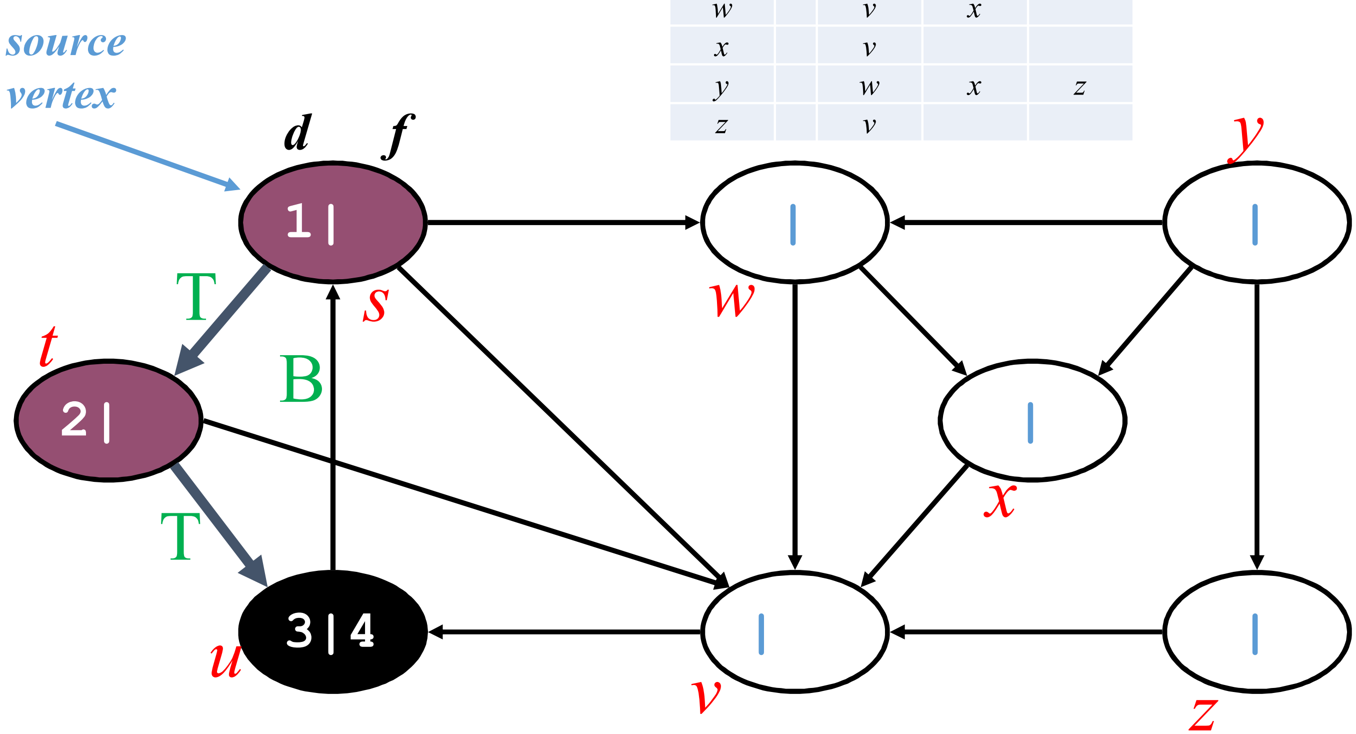

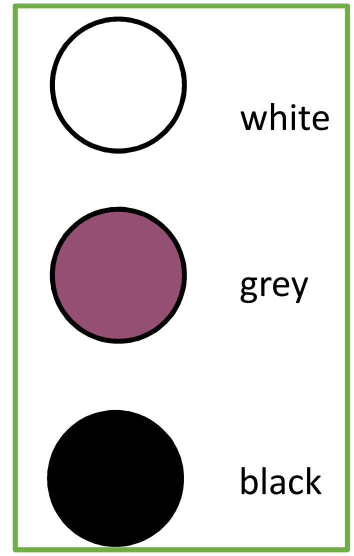

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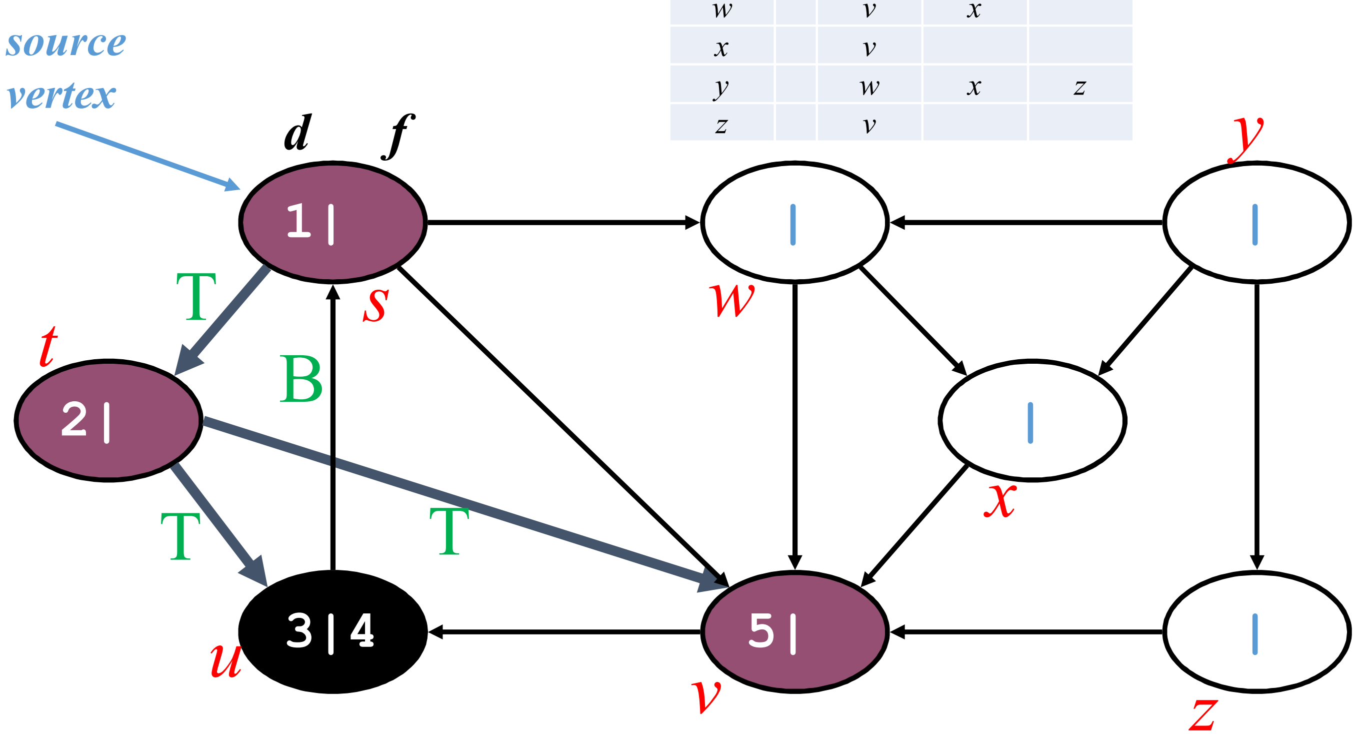

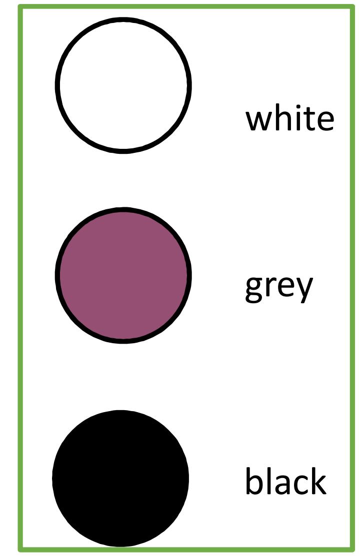

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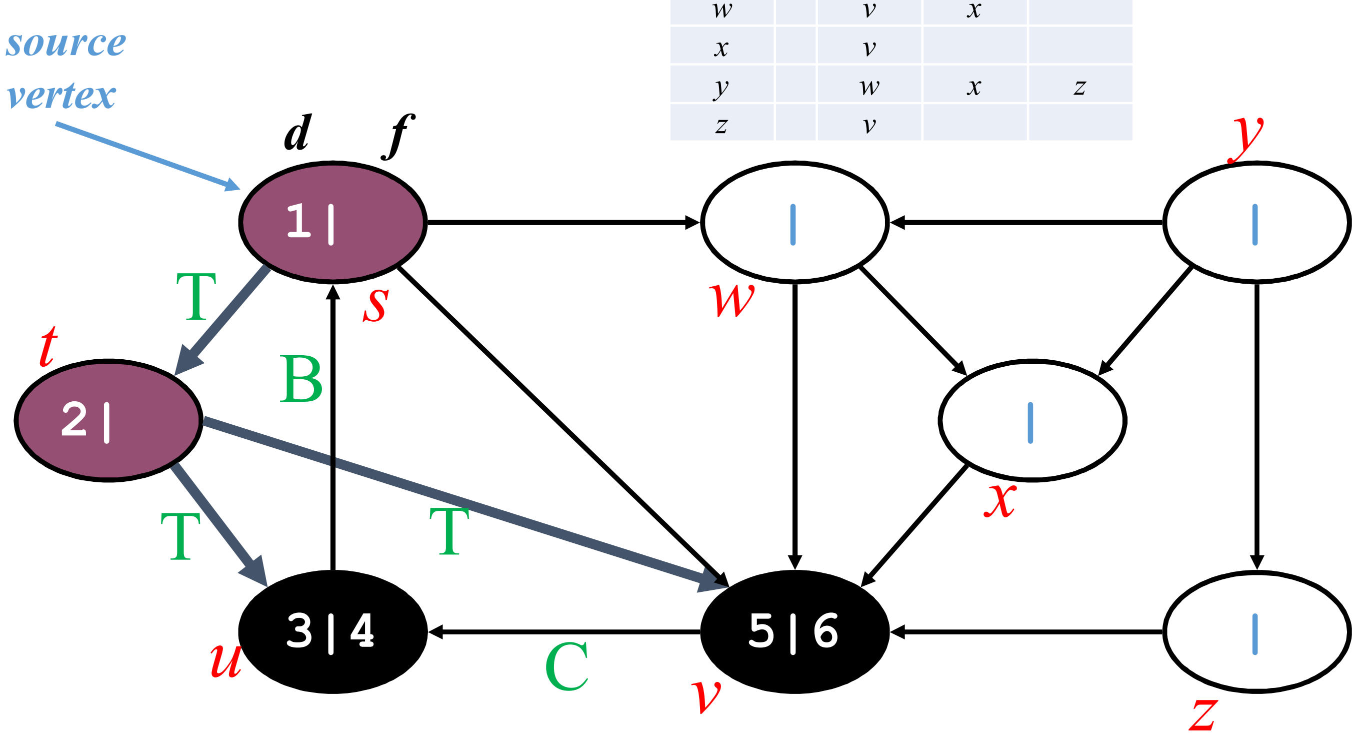

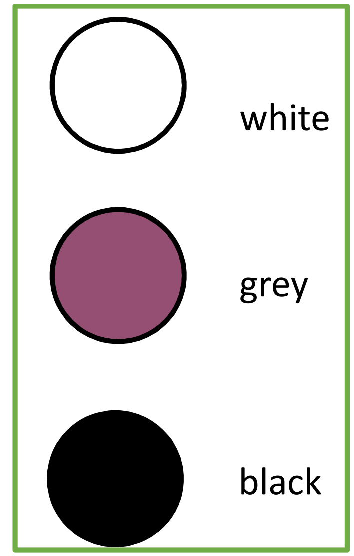

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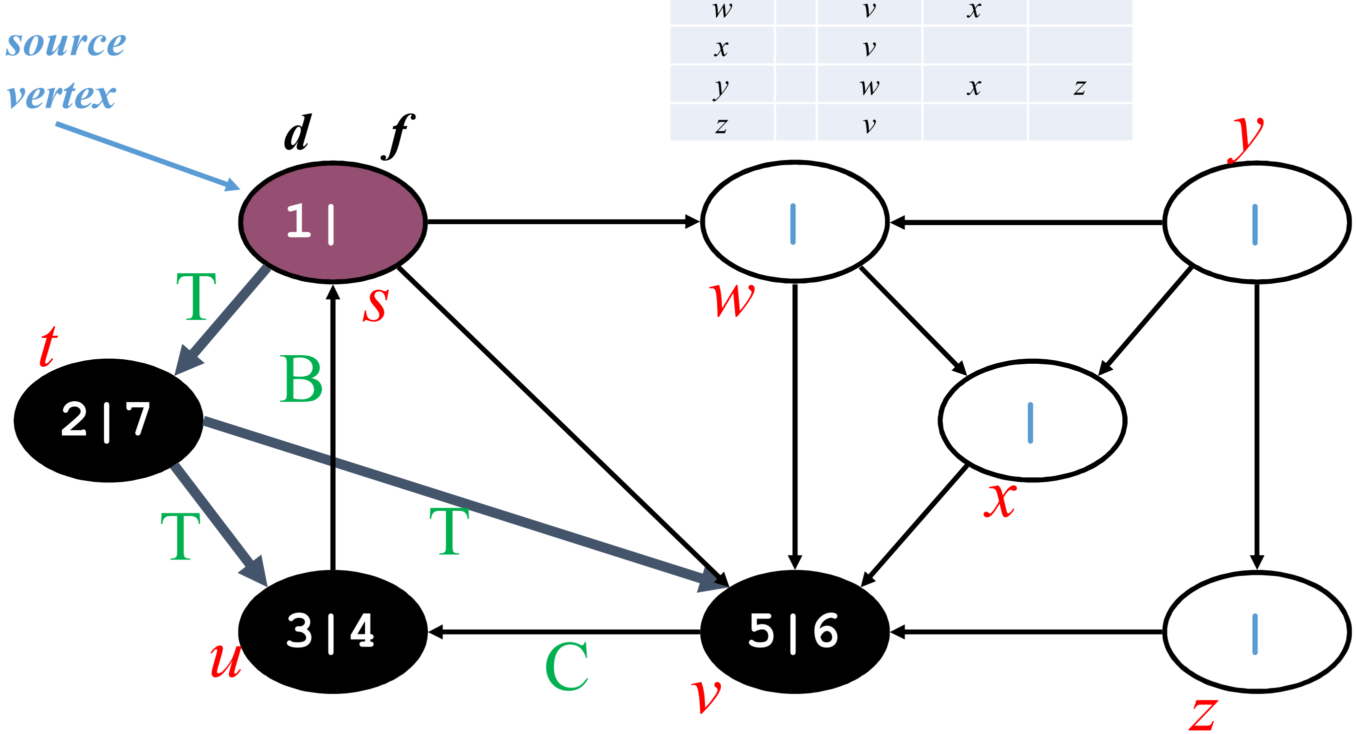

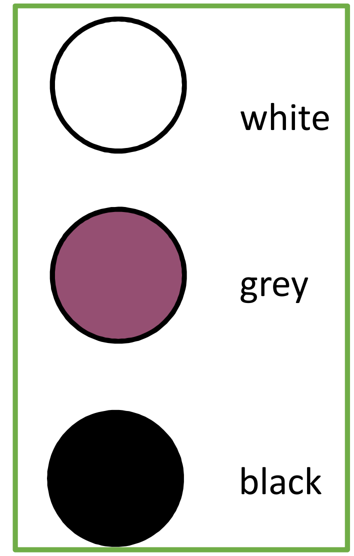

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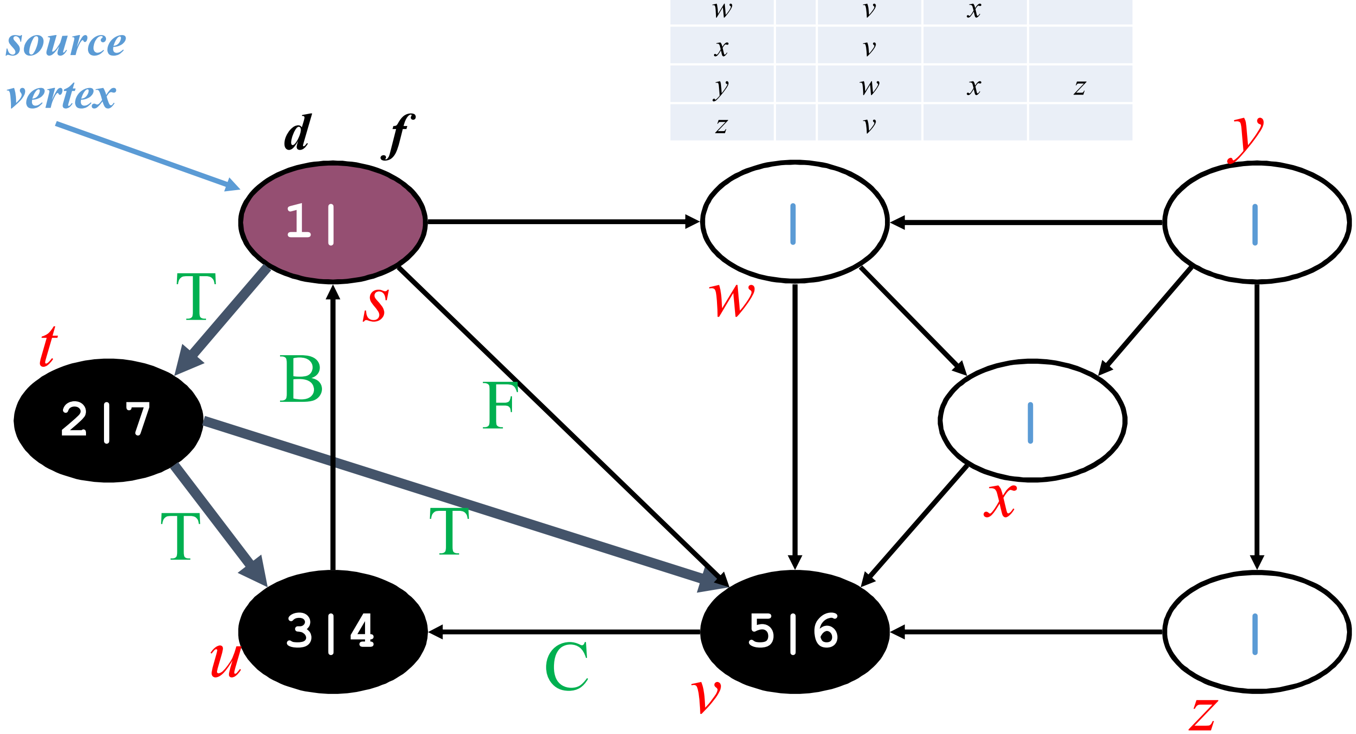

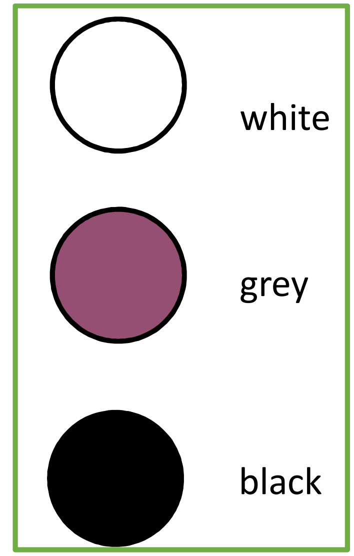

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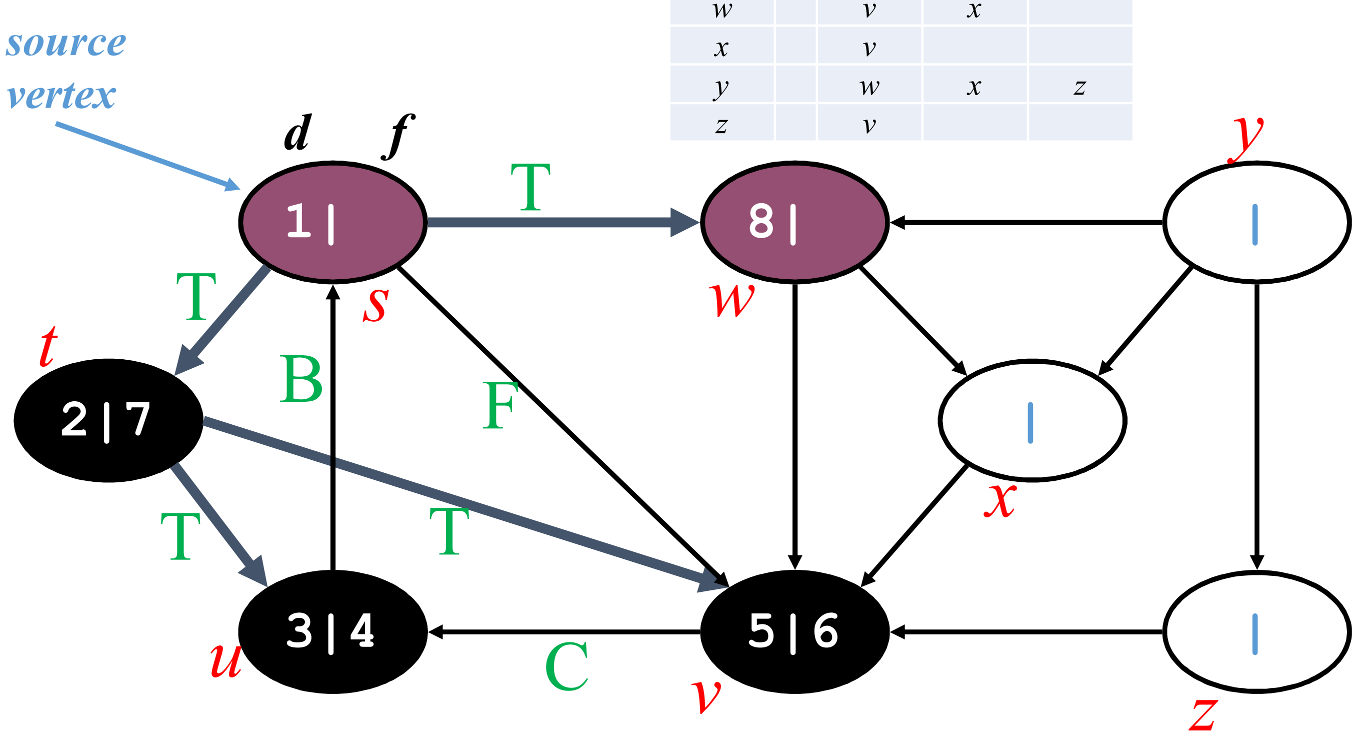

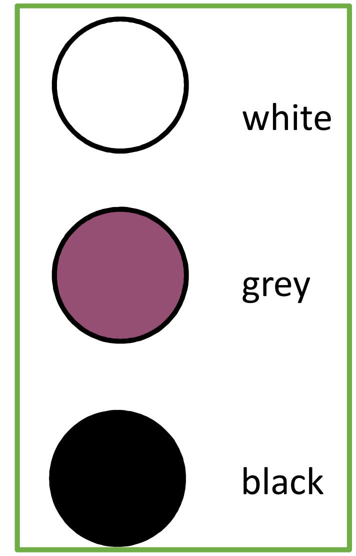

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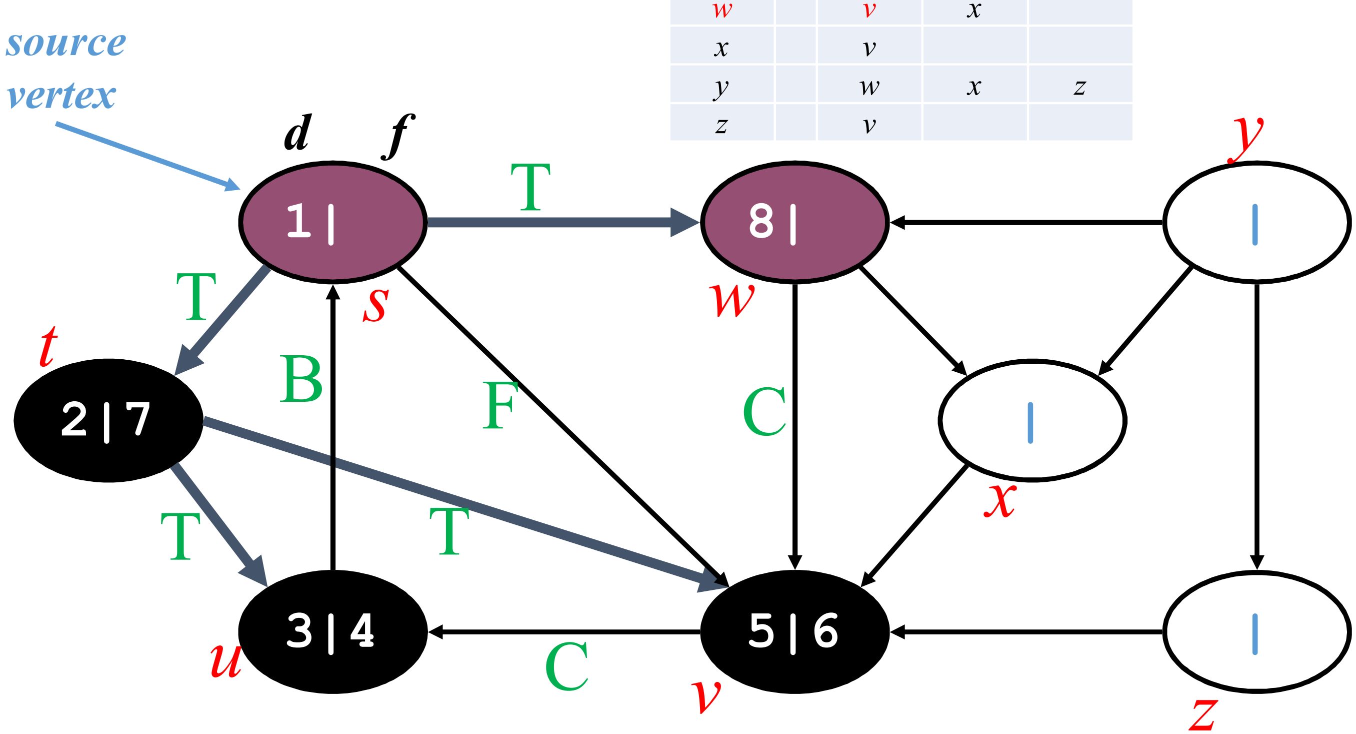

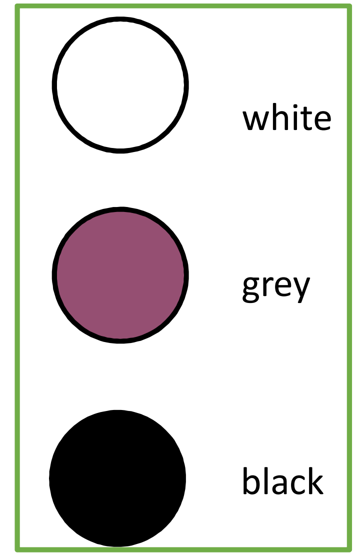

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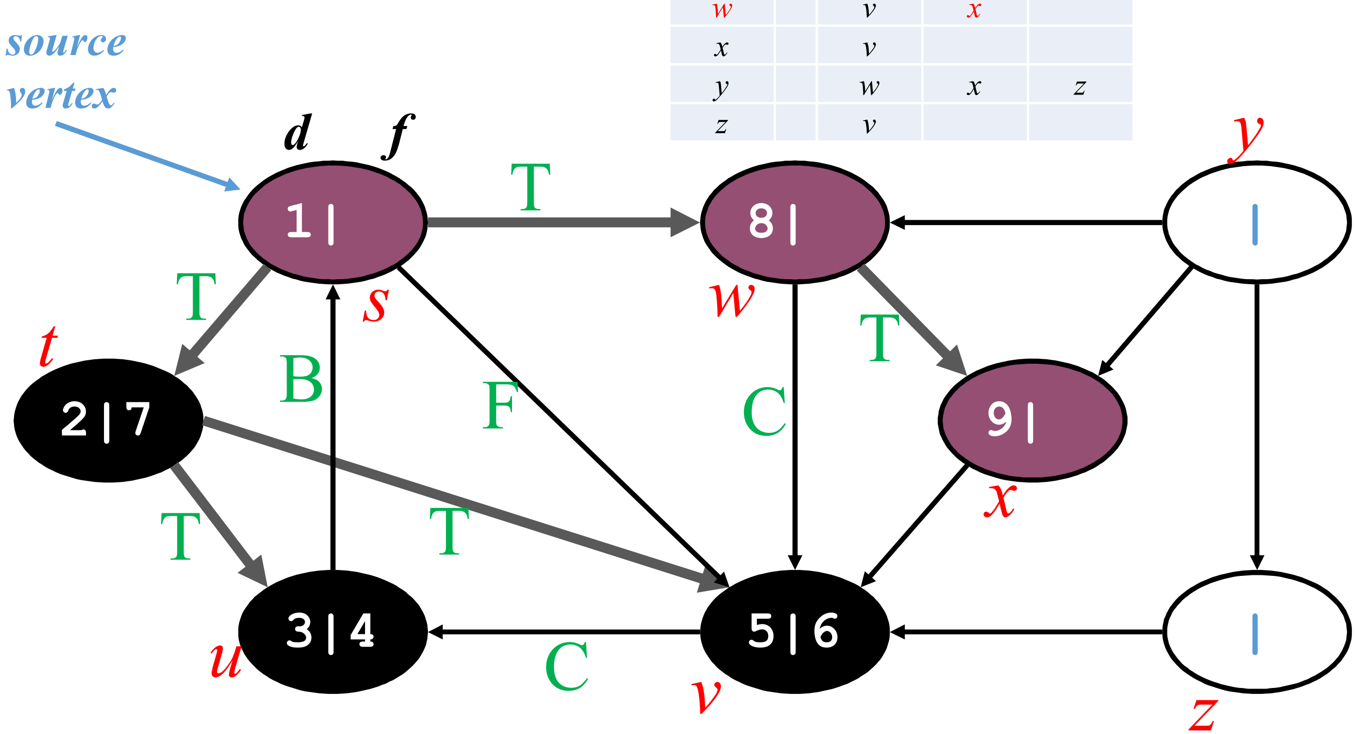

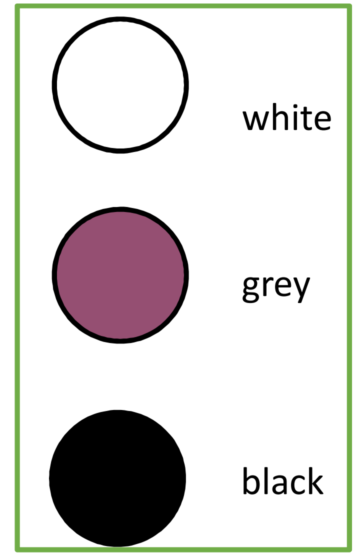

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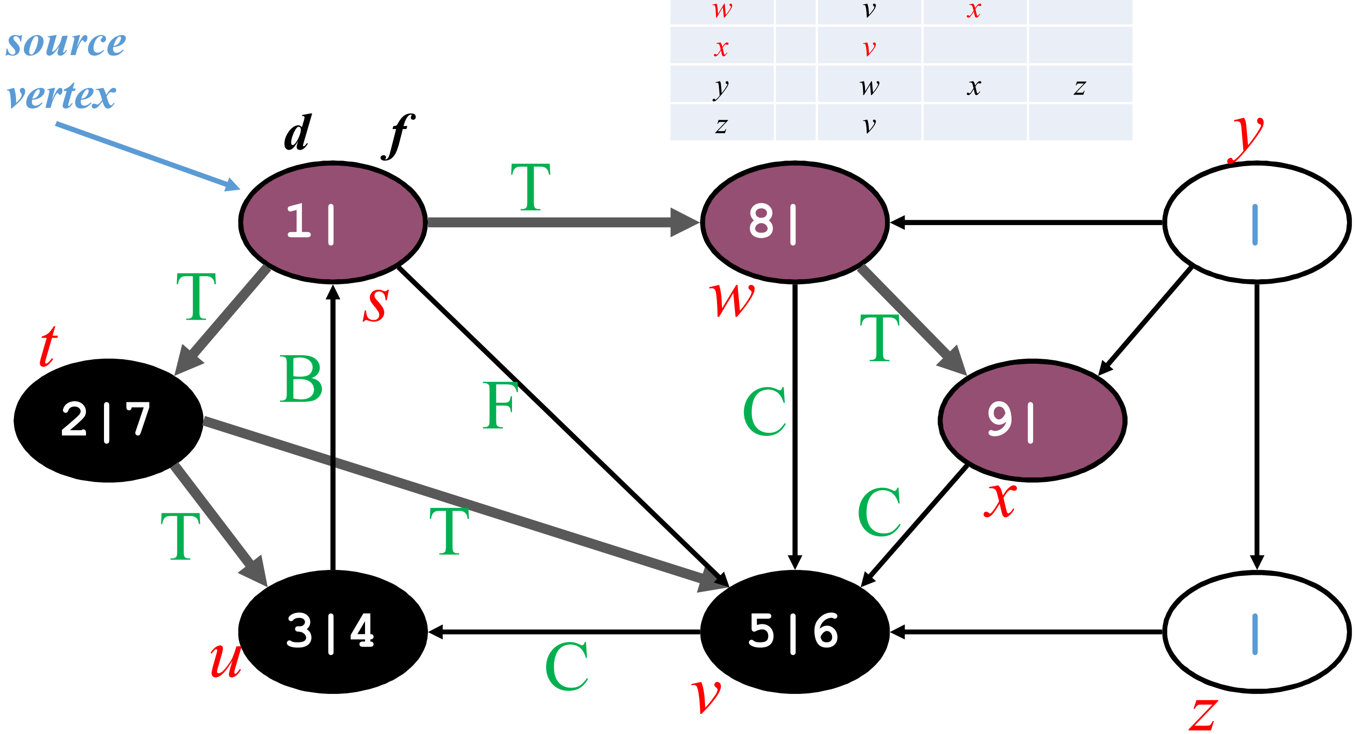

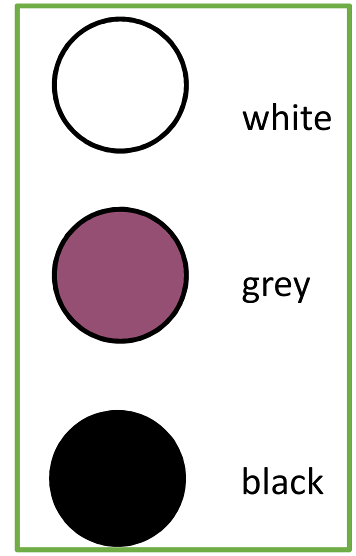

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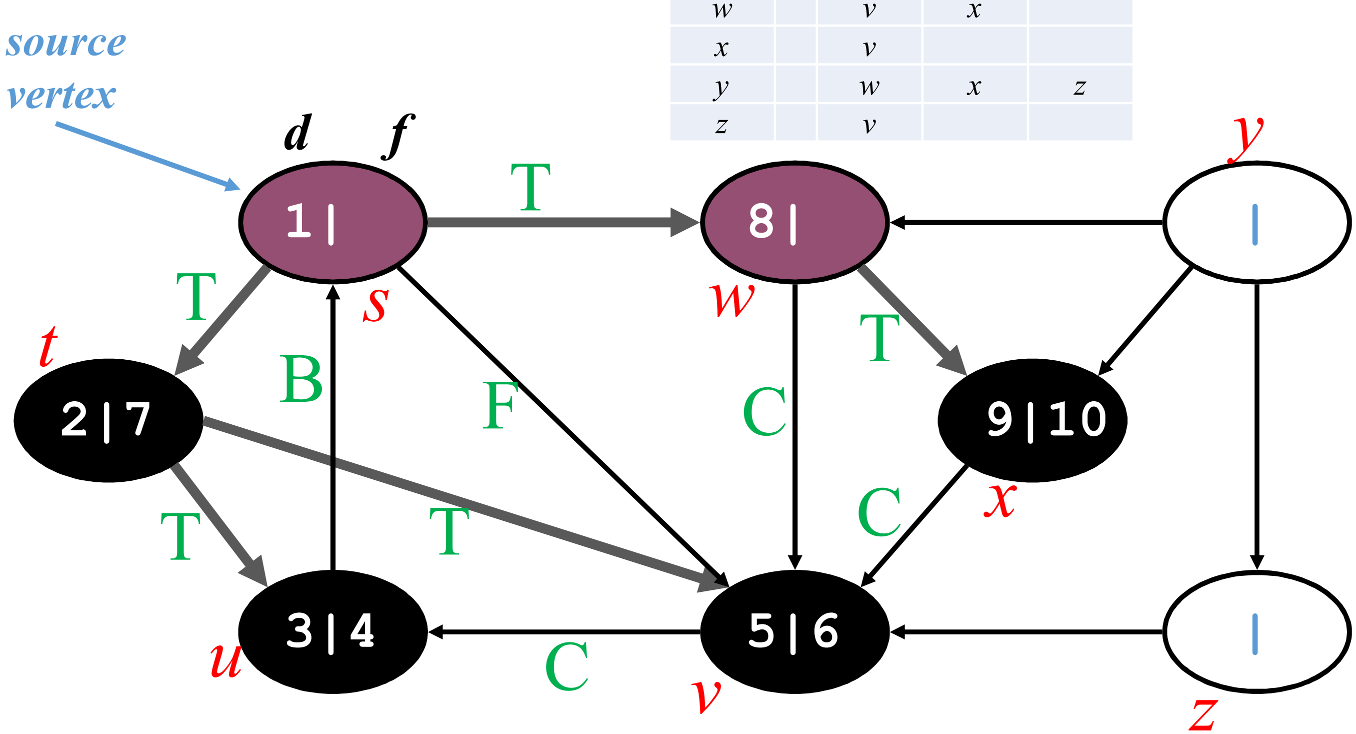

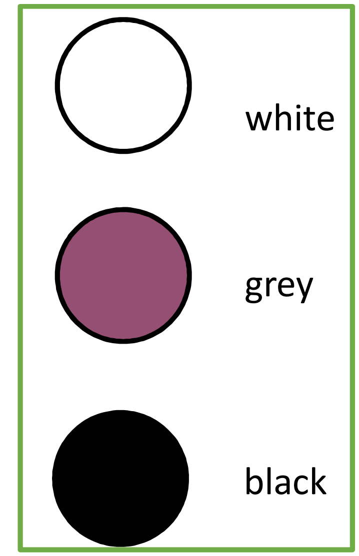

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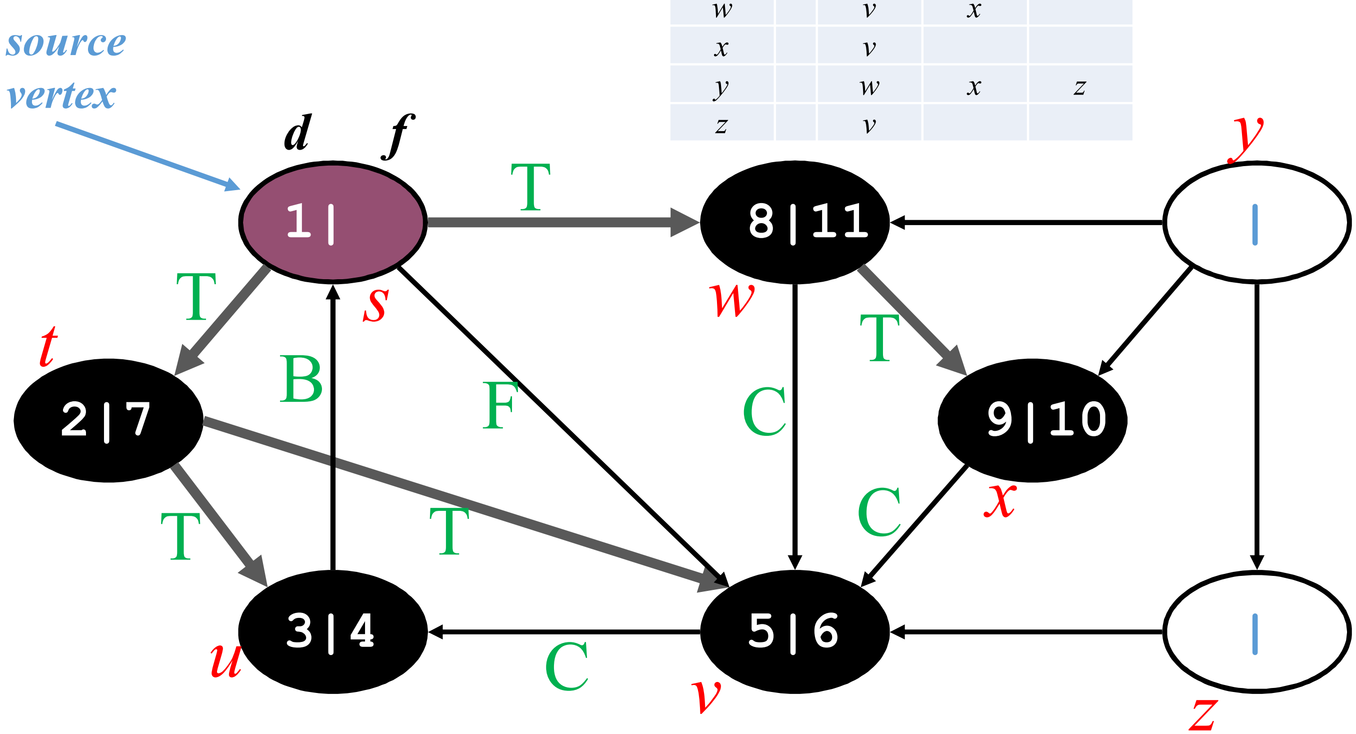

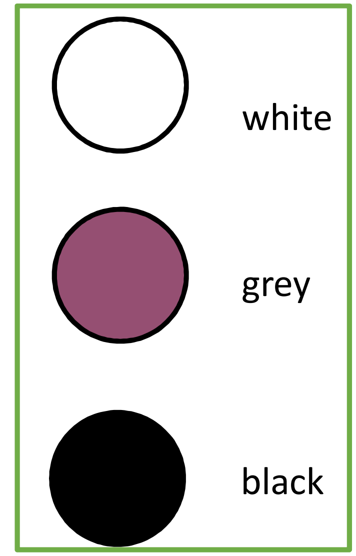

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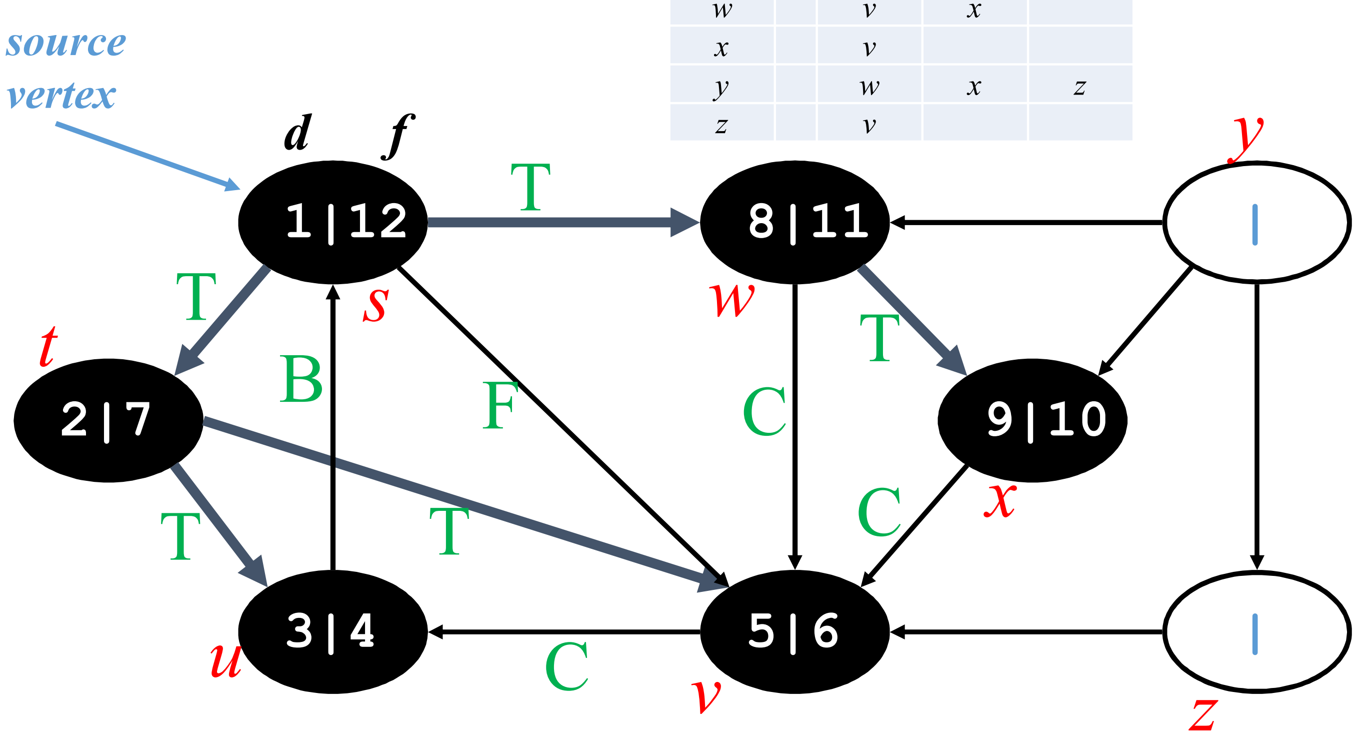

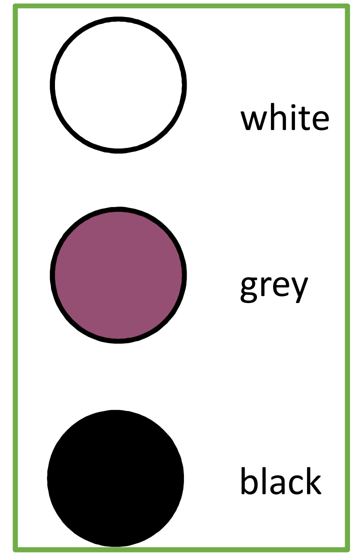

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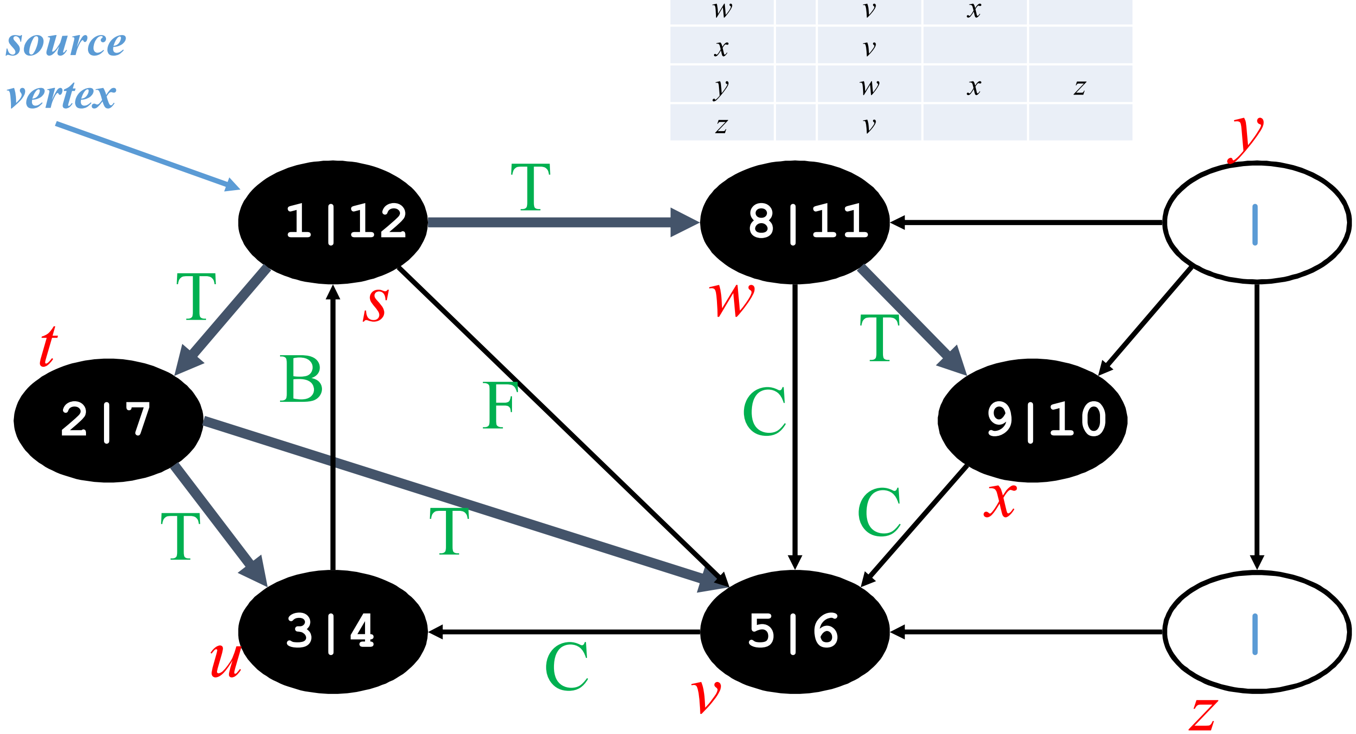

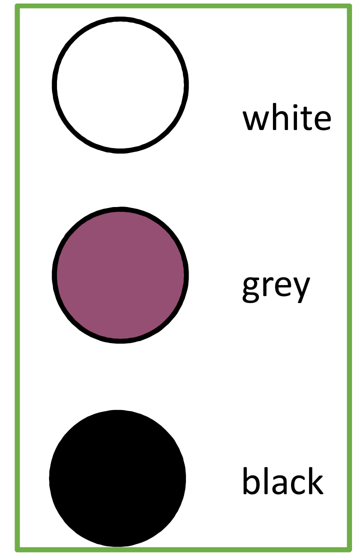

We have two WHITE vertices remaining.

They are unreachable from *s*

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Vertices		**Adjacency list**		
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t		u	v	
u		s		
v		u		
w		v	x	
x		v		
y		w	x	z
z		v		

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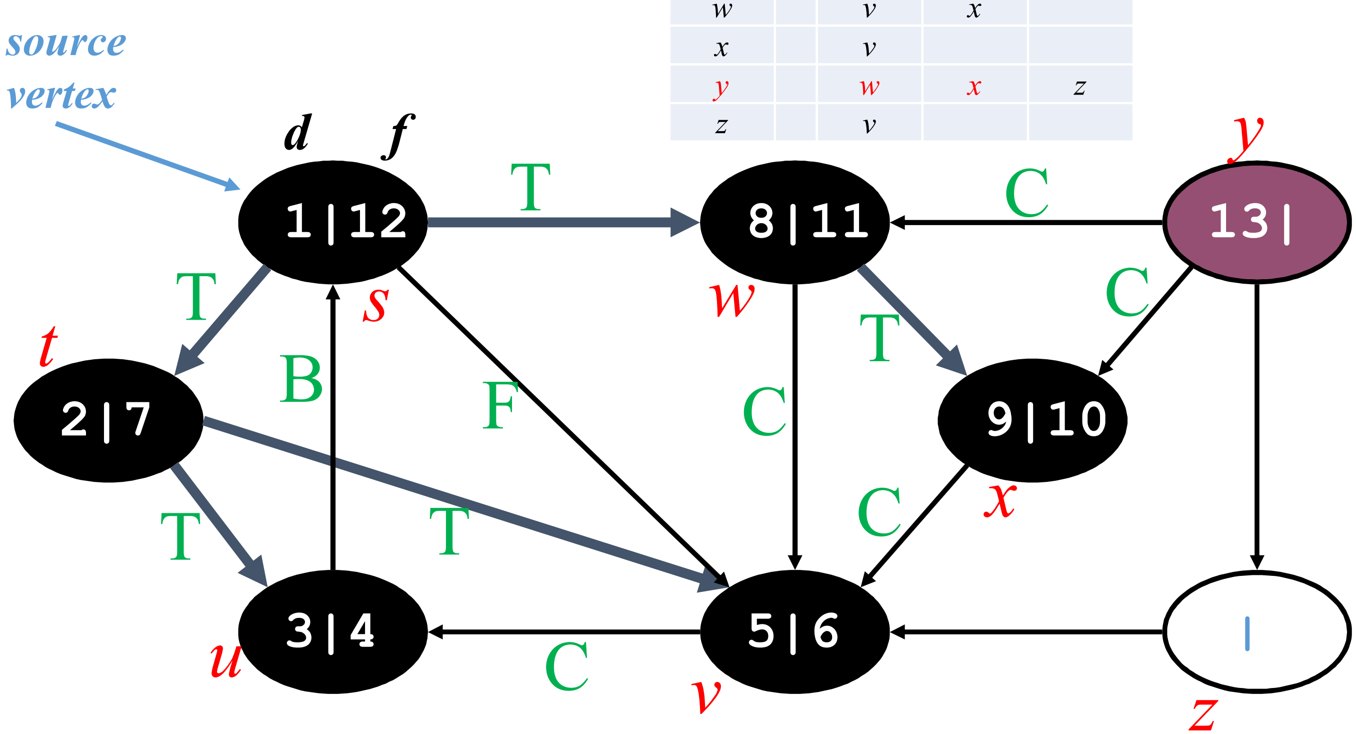
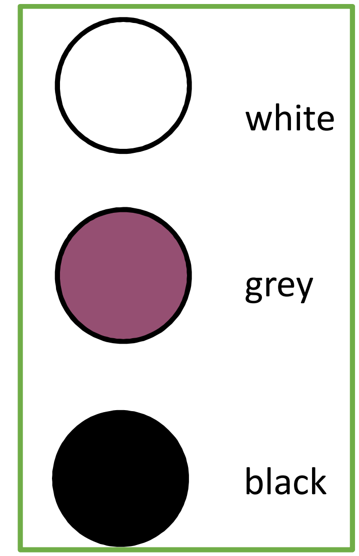
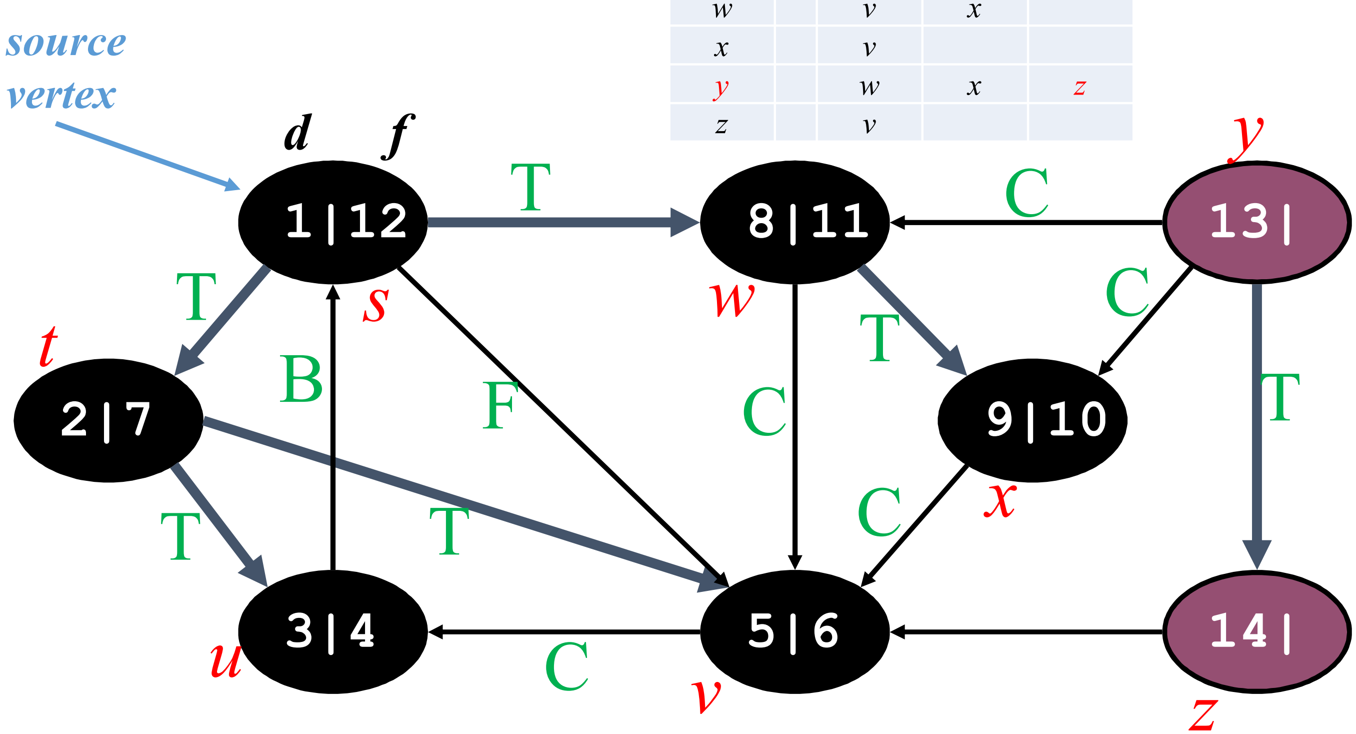
y		*w*	*x*	*z*
z		*v*		

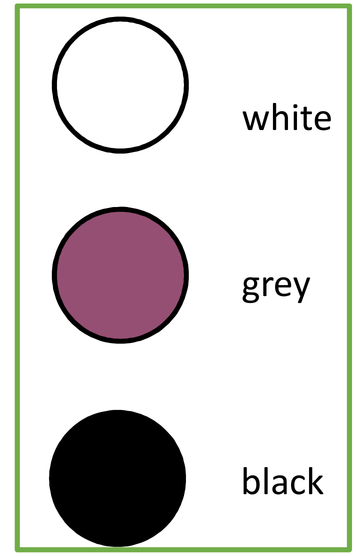




DFS Example

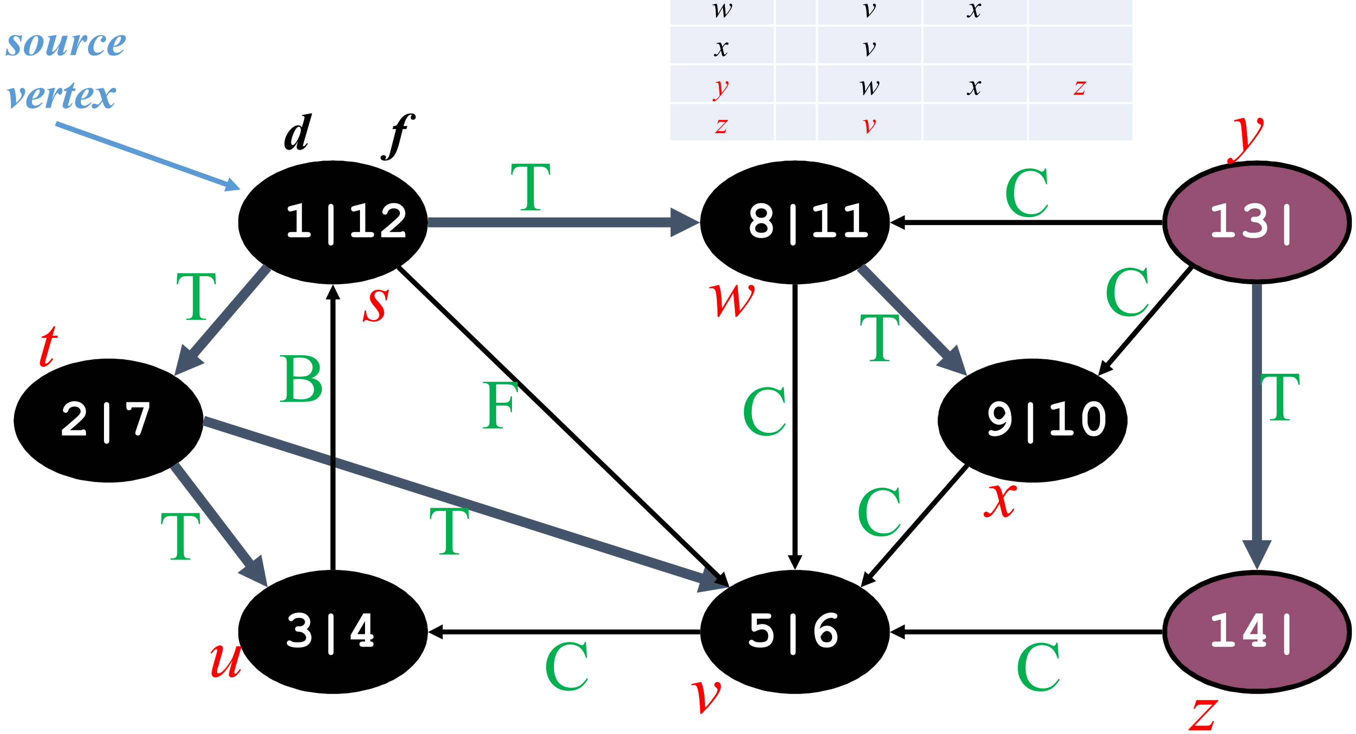
Vertices		Adjacency list		
s		*t*	*v*	*w*
t		*u*	*v*	
u		*s*		
v		*u*		
w		*v*	*x*	
x		*v*		
y		*w*	*x*	*z*
z		*v*		

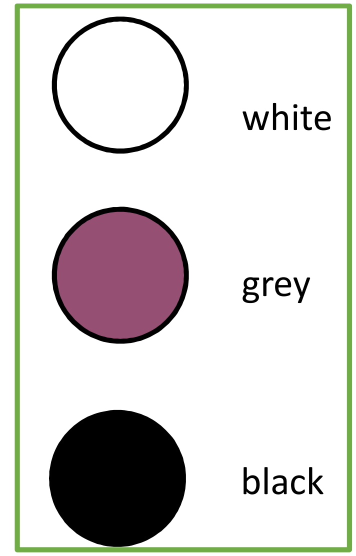




DFS Example

Nodes		Adjacency list		
s		*t*	*v*	*w*
t		*u*	*v*	
u		*s*		
v		*u*		
w		*v*	*x*	
x		*v*		
y		*w*	*x*	*z*
z		*v*		





DFS Example

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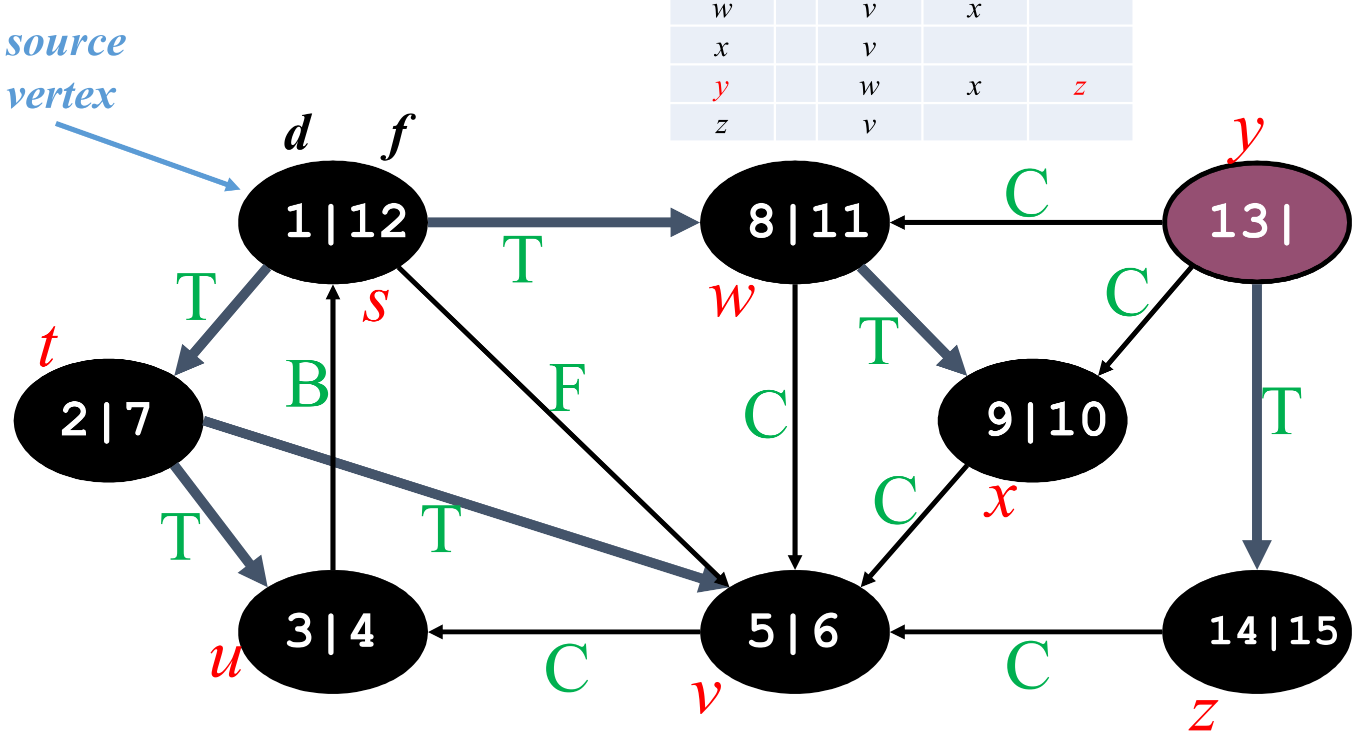
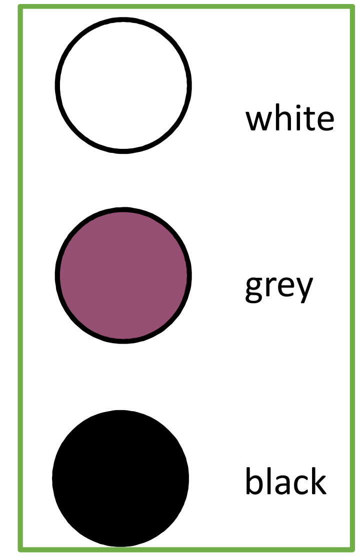
x		*v*		
y		*w*	*x*	*z*
z		*v*		





DFS Example

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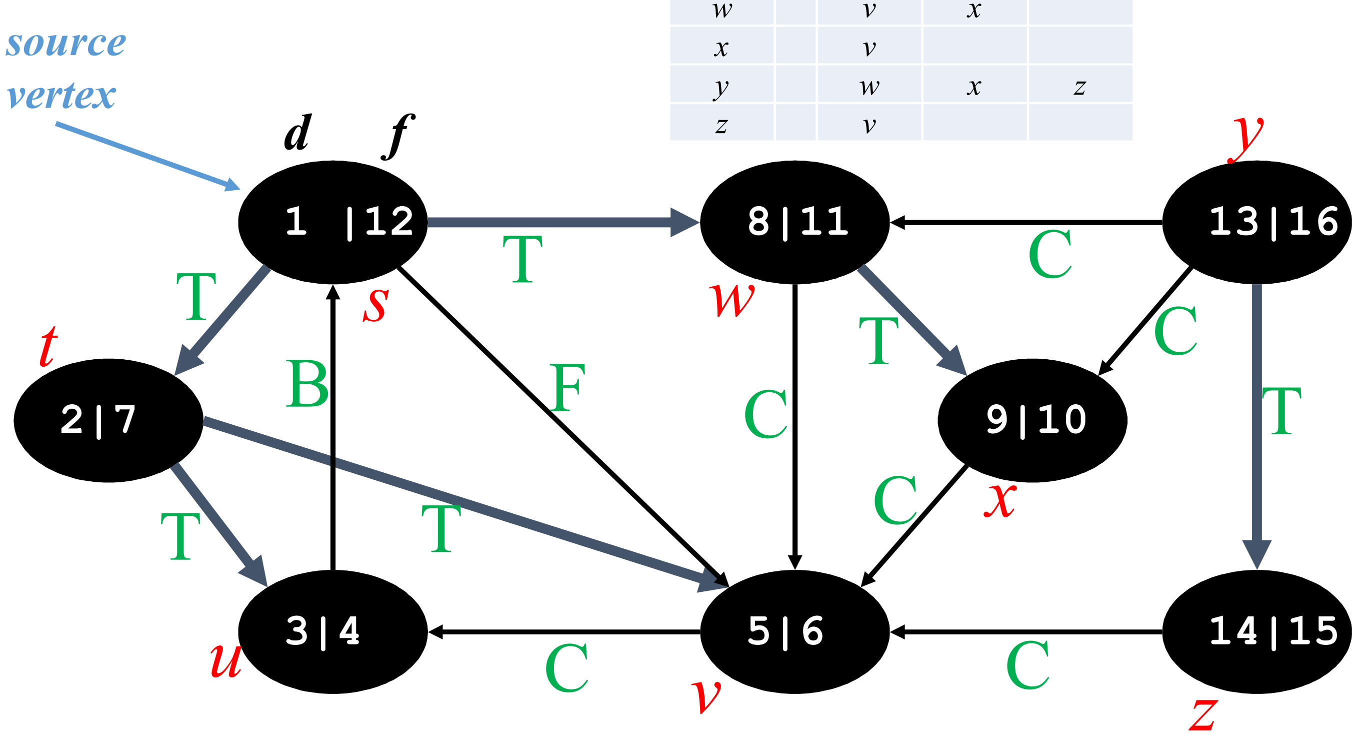
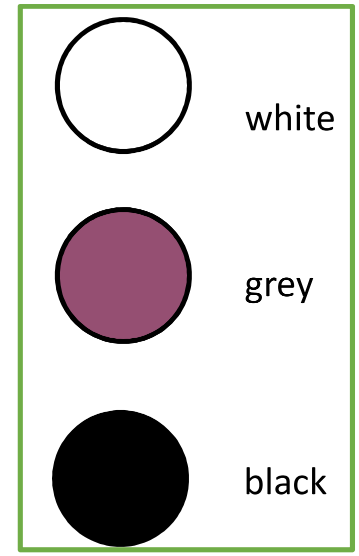
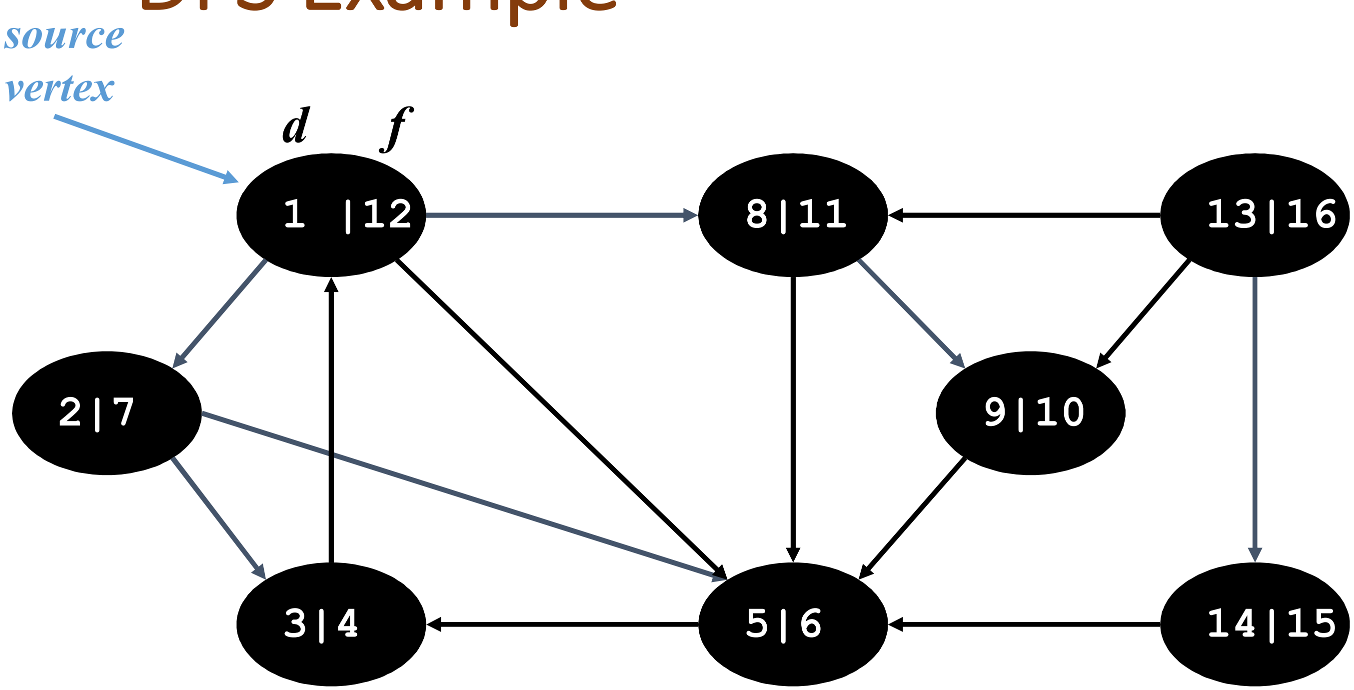
s		*t*	*v*	*w*
t		*u*	*v*	
u		*s*		
v		*u*		
w		*v*	*x*	
x		*v*		
y		*w*	*x*	*z*
z		*v*		





DFS Example



For every vertex u, we have: $u.d < u.f$ ---(22.2)

Interesting Facts

- $u.d$ records when vertex u is discovered
- $u.f$ *records* when the processing of vertex u is finished.
- These timestamps are integers between 1 and $2 \times |V|$.
  - Since there is one discovery event and one finishing event for each of the $|V|$ vertices

DFS: Properties

```
DFS(G)
for each vertex u ∈ G.V
    u.color = WHITE
    u.π = NIL
time = 0
for each vertex u ∈ G.V
    if u.color == WHITE
        DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

- $u = v.\pi$ if and only if DFS-VISIT (G, v) is called while searching u's adjacency list
- v is a descendent of u iff v is discovered WHITE while u is still grey

DFS: Properties

```
DFS(G)
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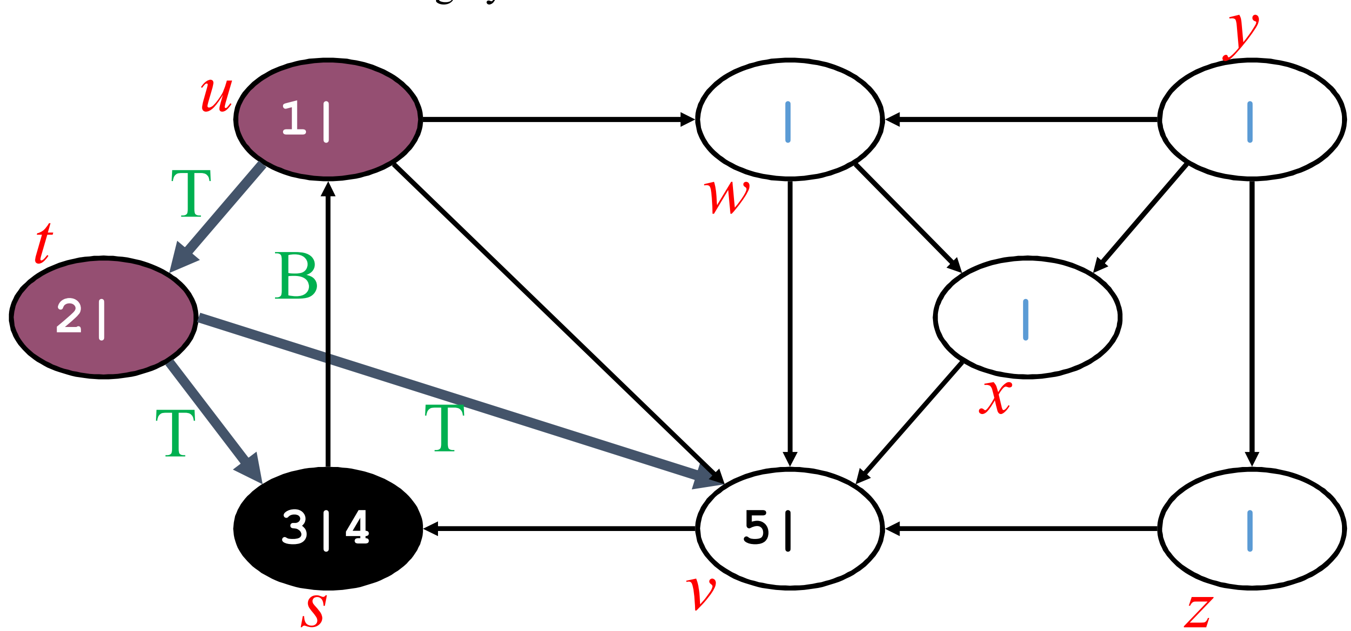
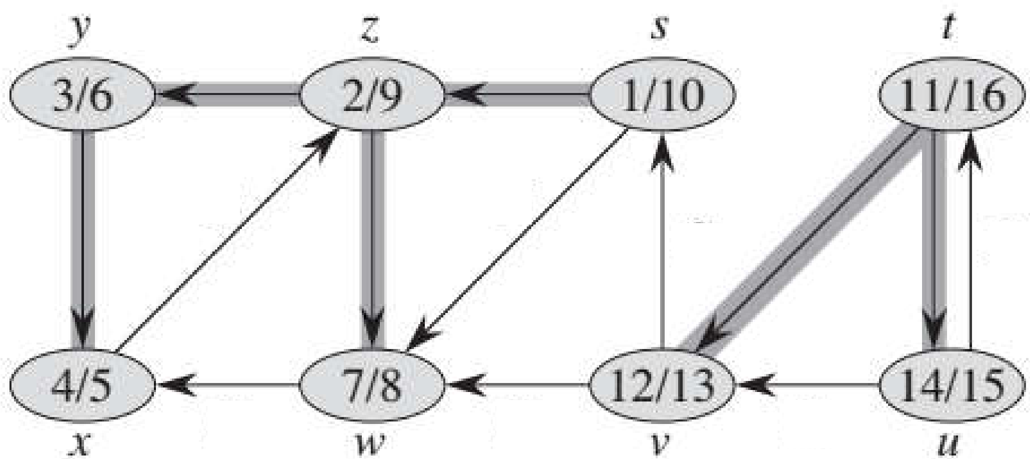
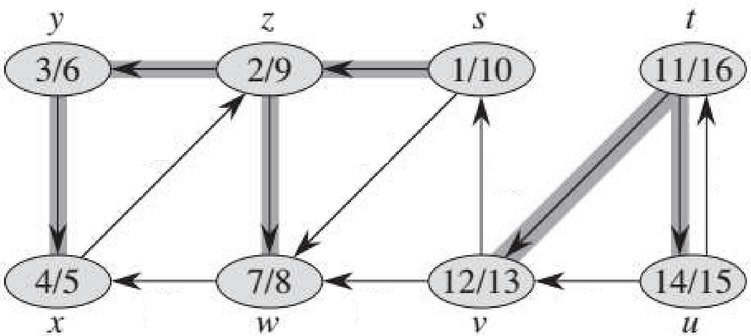


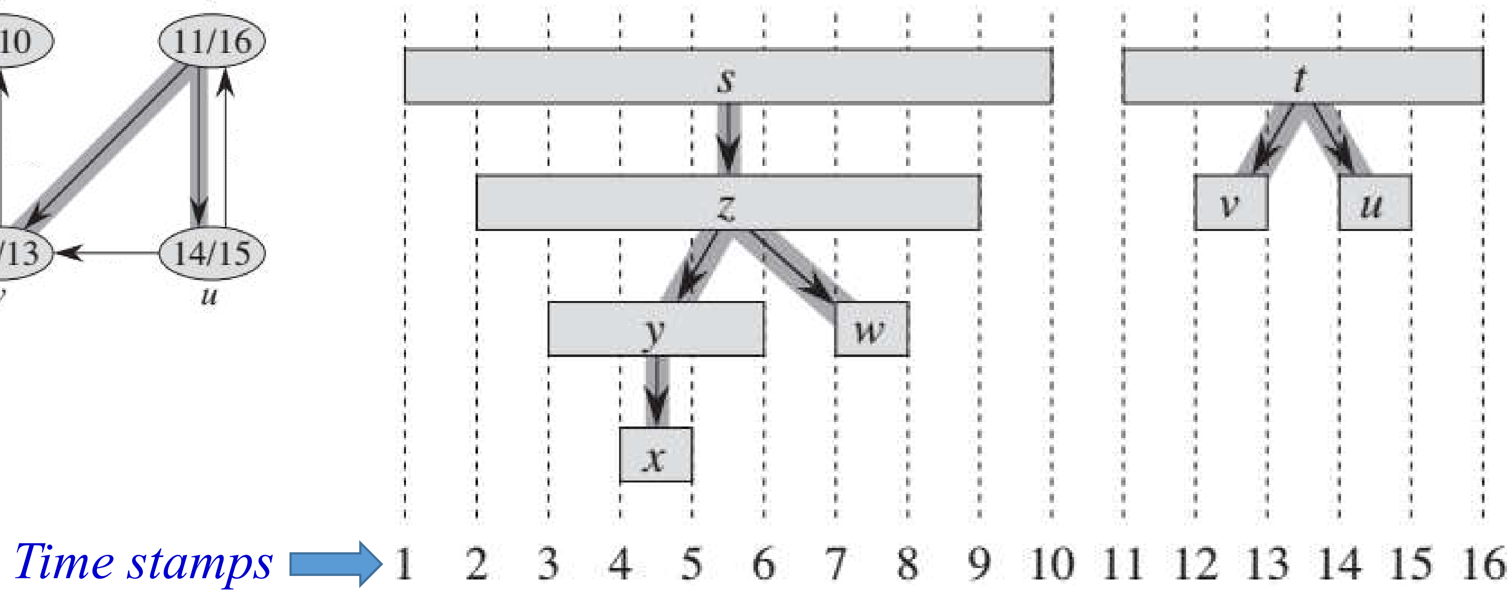
DFS: Properties

- *Parenthesis structure*

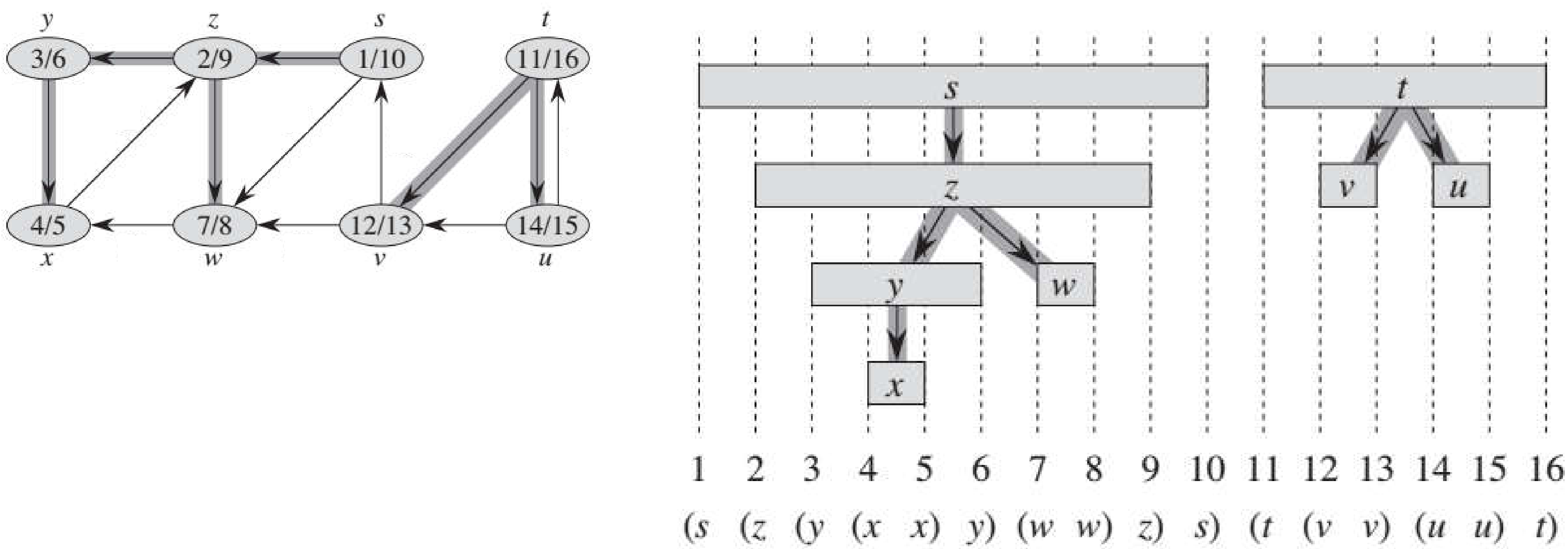


DFS: Parenthesis Structure

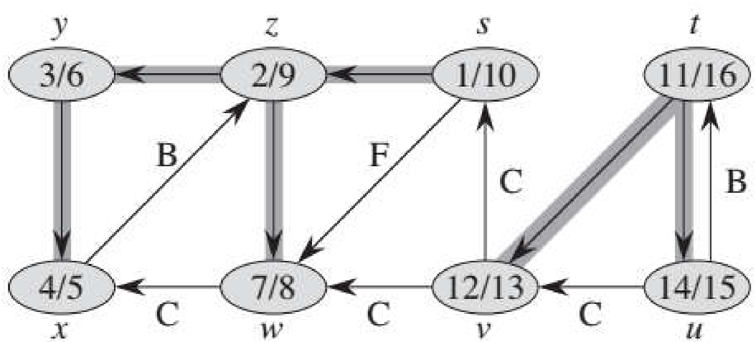


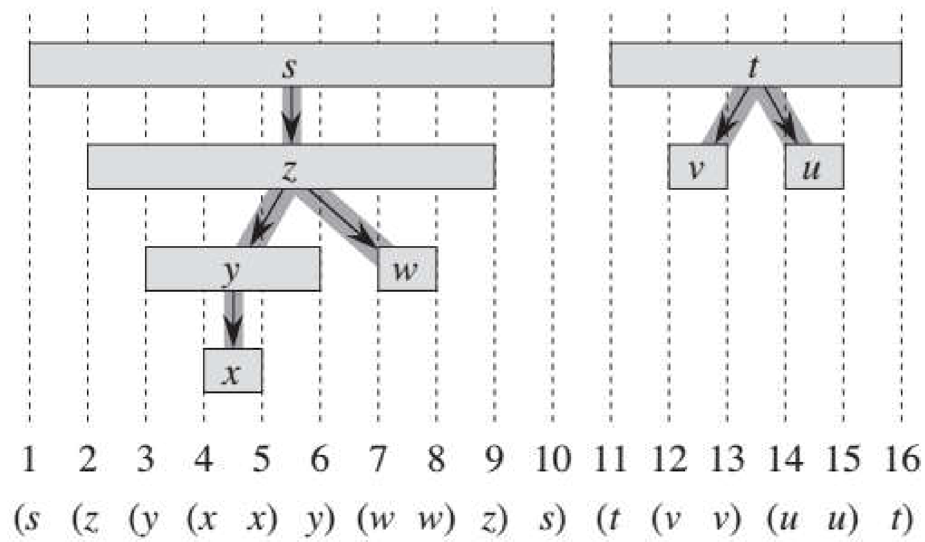


DFS: Parenthesis Structure



DFS: Parenthesis Structure





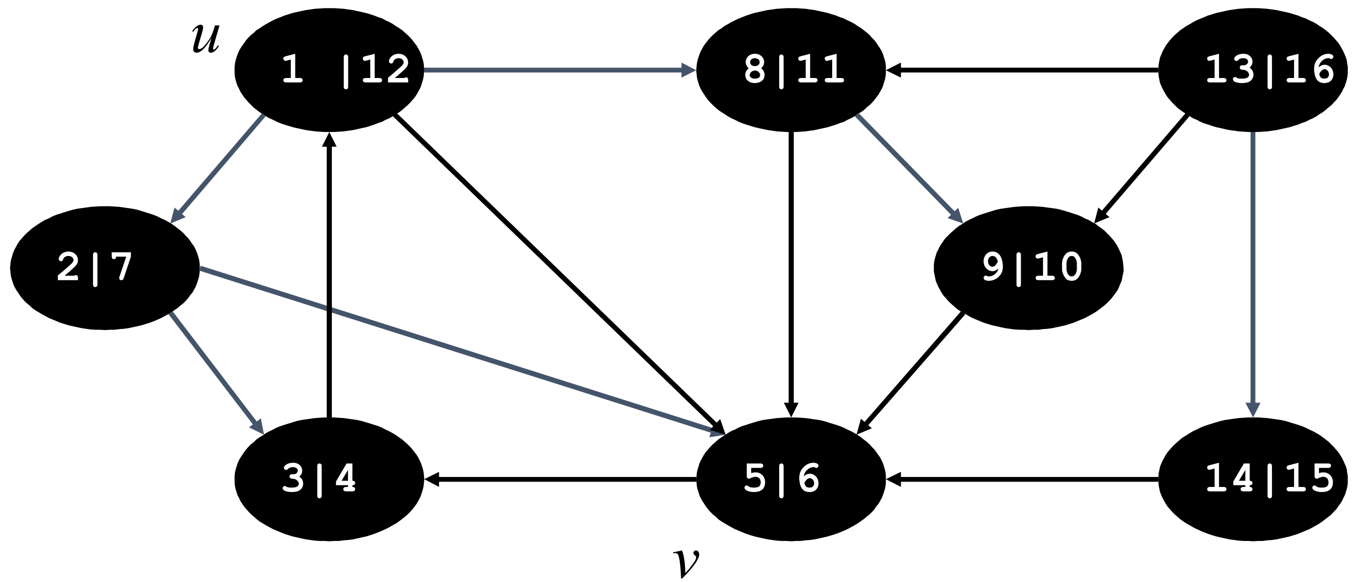
- ***parenthesis structure***
  - Represent discovery of vertex u with "(u"
  - represent finishing of vertex u with "u)"
  - Then the history of discoveries and finishings makes a well-formed expression in the sense that the parentheses are properly nested.

Theorem 22.7: Parenthesis Theorem

In any depth-first search of a (directed or undirected) graph G = (V, E), for any two vertices u and v, exactly one of the following three conditions holds:

- the intervals $[u.d, u.f]$ and $[v.d, v.f]$ are entirely disjoint, and neither u nor v is a descendant of the other in the depth-first forest,
- the interval $[u.d, u.f]$ is contained entirely within the interval $[v.d, v.f]$, and u is a descendant of v in a depth-first tree, or
- the interval $[v.d, v.f]$ is contained entirely within the interval $[u.d, u.f]$, and v is a descendant of u in a depth-first tree

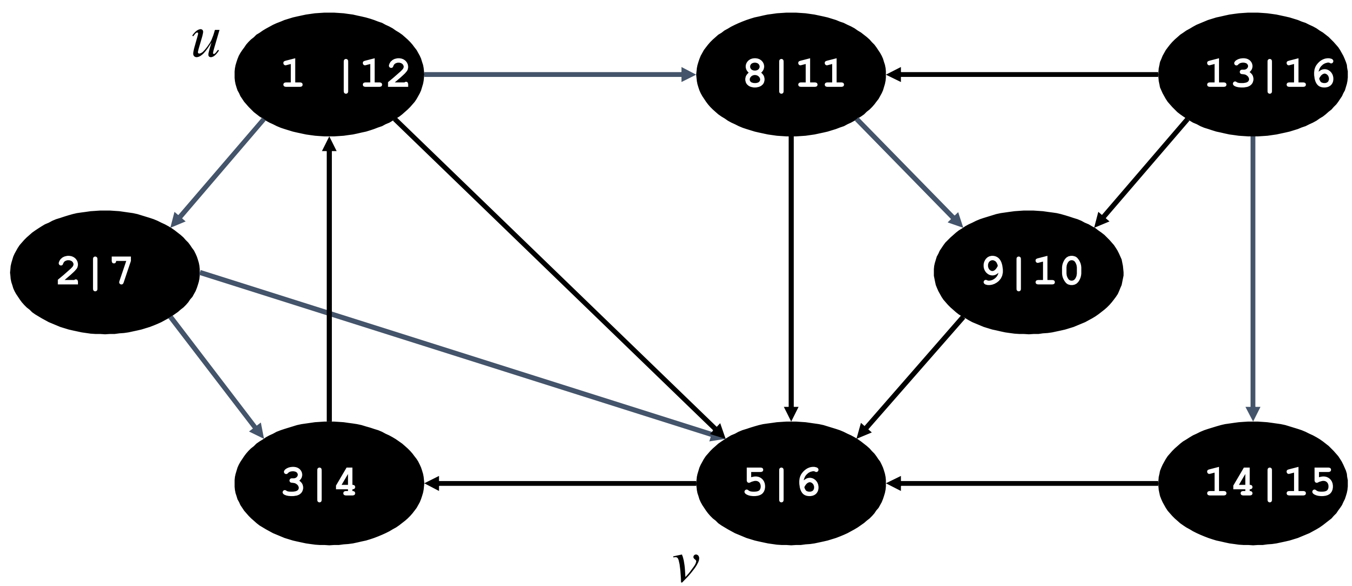
Case 1: $u.d < v.d$



Case 1: $u.d < v.d$

Sub-case 1A: $v.d < u.f$

Sub-case 1B: $v.d > u.f$



Case 1: $u.d < v.d$

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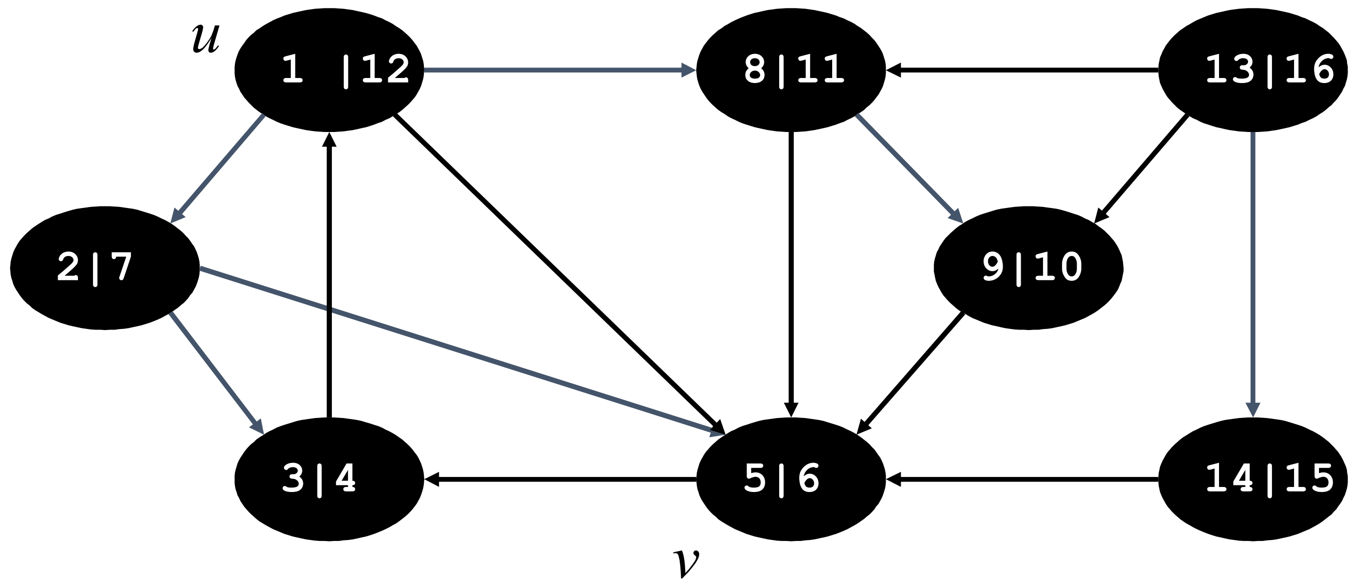
$\Rightarrow u.d < v.d < u.f$



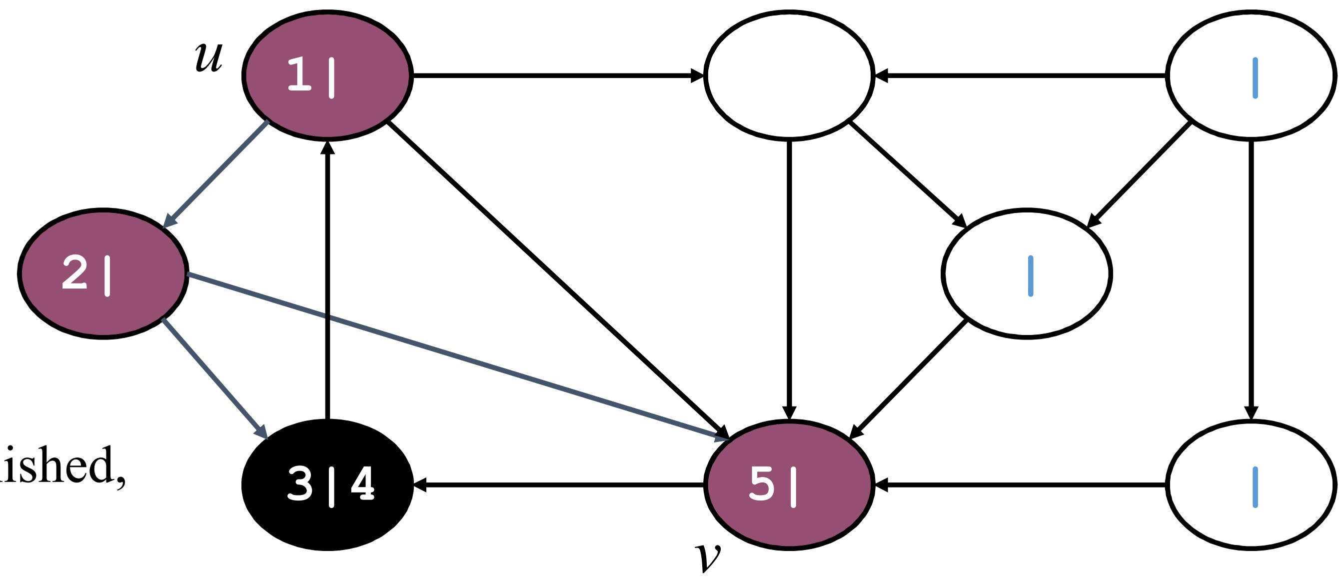
Case 1: $u.d < v.d$

Sub-case 1A: $v.d < u.f$

$\Rightarrow u.d < v.d < u.f$

$\Rightarrow$ v is discovered before u is finished,

$\Rightarrow$ i.e., u is gray.



<u>Case 1: $u.d < v.d$</u>

Sub-case 1A: $v.d < u.f$

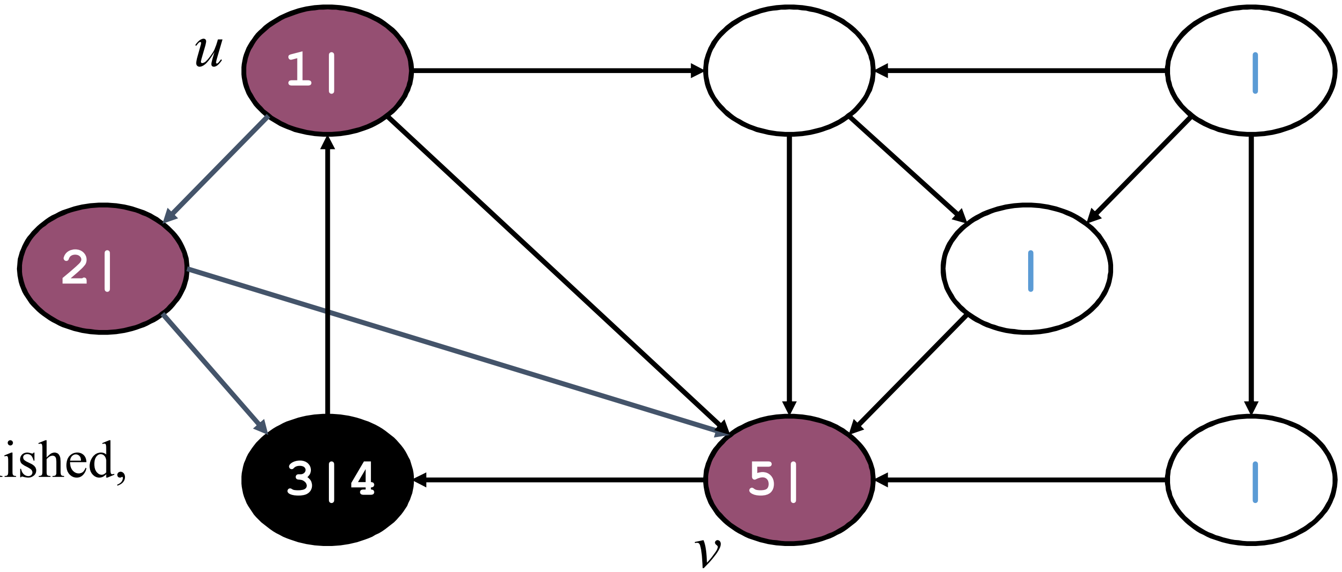
$\Rightarrow u.d < v.d < u.f$

$\Rightarrow$ v is discovered before u is finished,

$\Rightarrow$ i.e., u is gray.

v is a descendent of u.

v is discovered more recently than u



<u>Case 1: $u.d < v.d$</u>

Sub-case 1A: $v.d < u.f$

$\Rightarrow u.d < v.d < u.f$

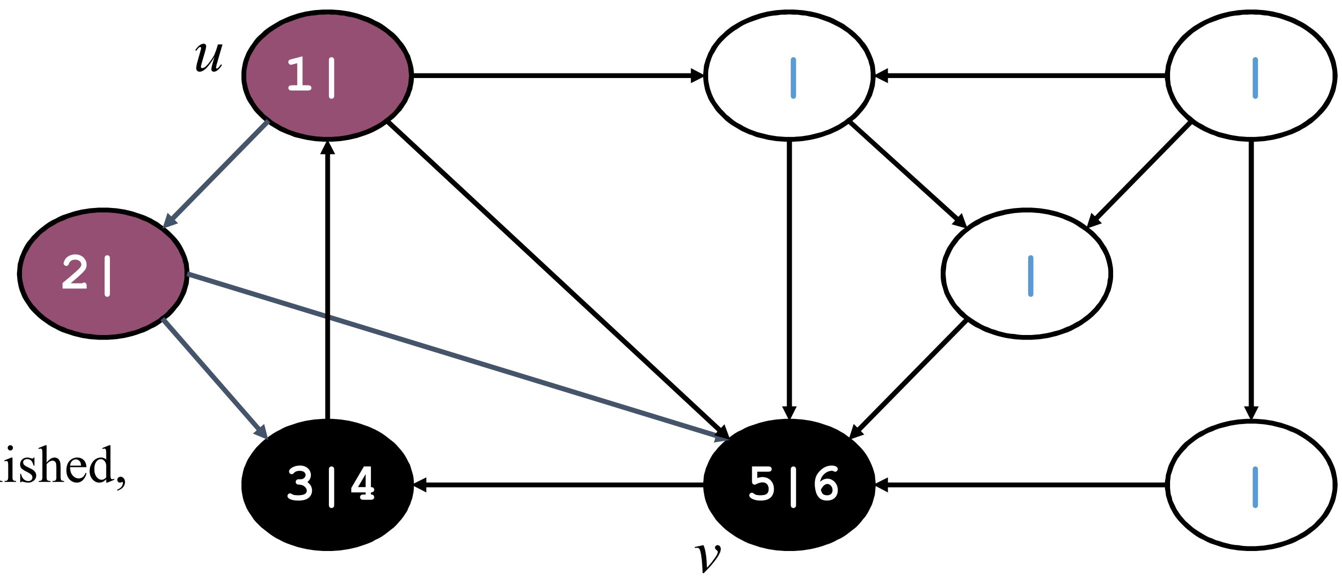
$\Rightarrow$ v is discovered before u is finished,

$\Rightarrow$ i.e., u is gray.

v is a descendent of u.

v is discovered more recently than u

=> v is finished before search returns to u.



Case 1: $u.d < v.d$

Sub-case 1A: $v.d < u.f$

$\Rightarrow u.d < v.d < u.f$

$\Rightarrow$ v is discovered before u is finished,

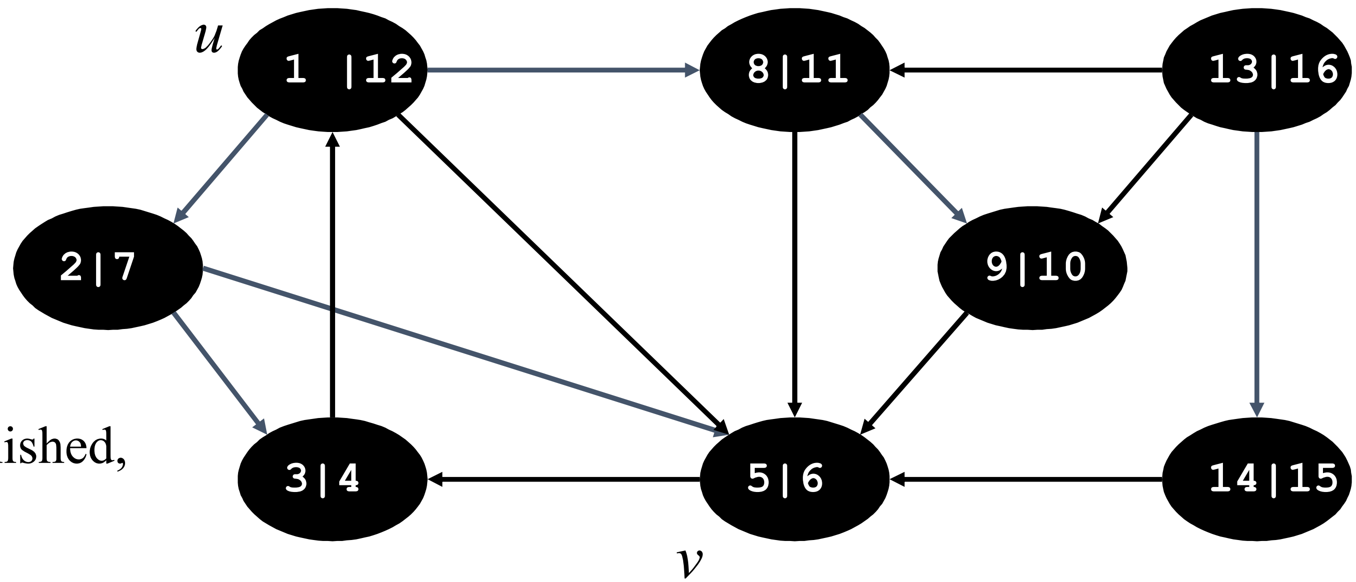
$\Rightarrow$ i.e., u is gray.

v is a descendent of u.

v is discovered more recently than u

=> v is finished before search returns to u.

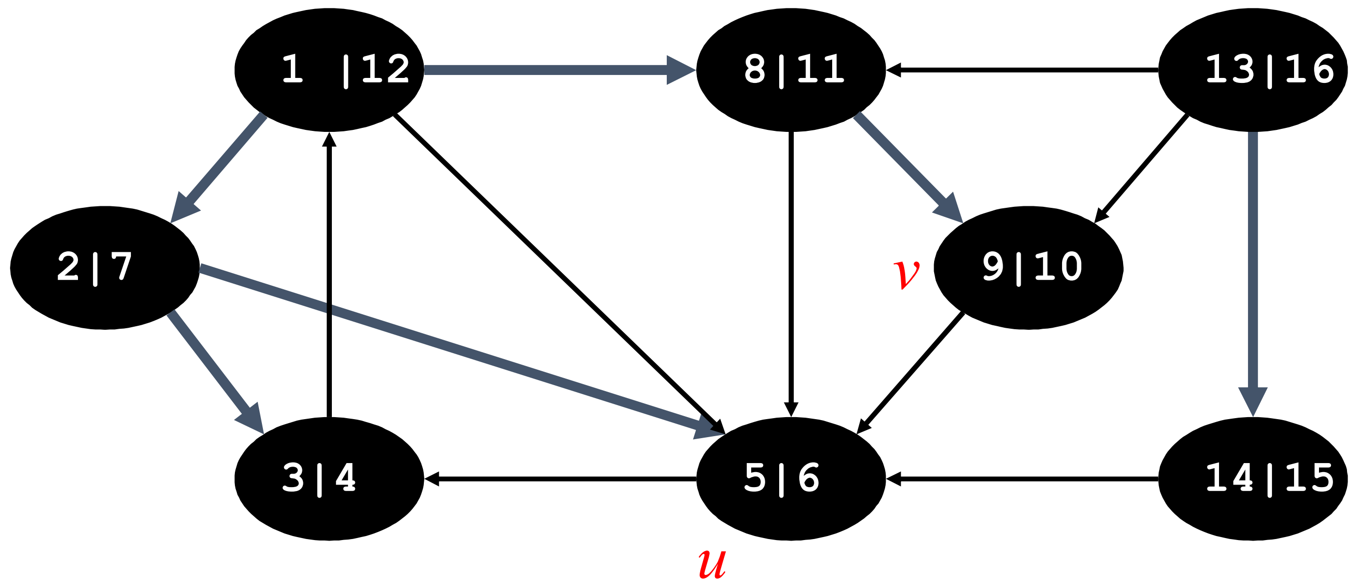
So, $[v.d, v.f]$ in $[u.d, u.f]$



Case 1: $u.d < v.d$

Sub-case 1B: $v.d > u.f$

$u.f < v.d \Rightarrow u.d < u.f < v.d$

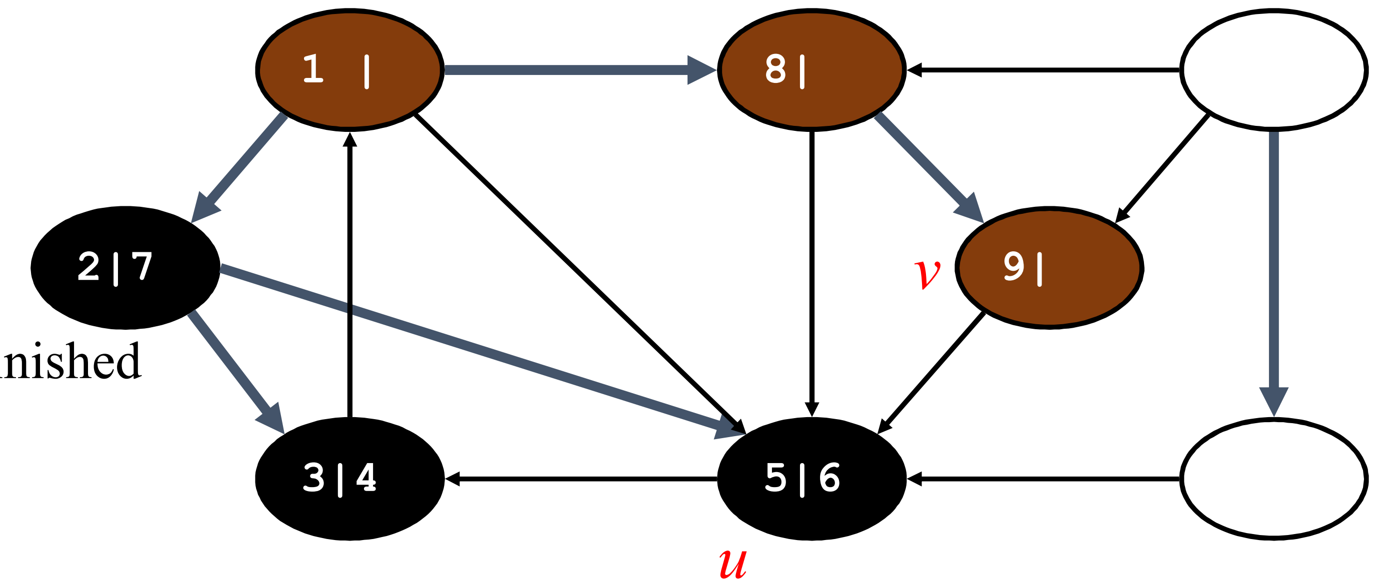


Case 1: $u.d < v.d$

Sub-case 1B: $v.d > u.f$

$u.f < v.d \Rightarrow u.d < u.f < v.d$

$\Rightarrow$ v is discovered AFTER u is finished
i.e., u is Black



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$\Rightarrow$ v is discovered AFTER u is finished
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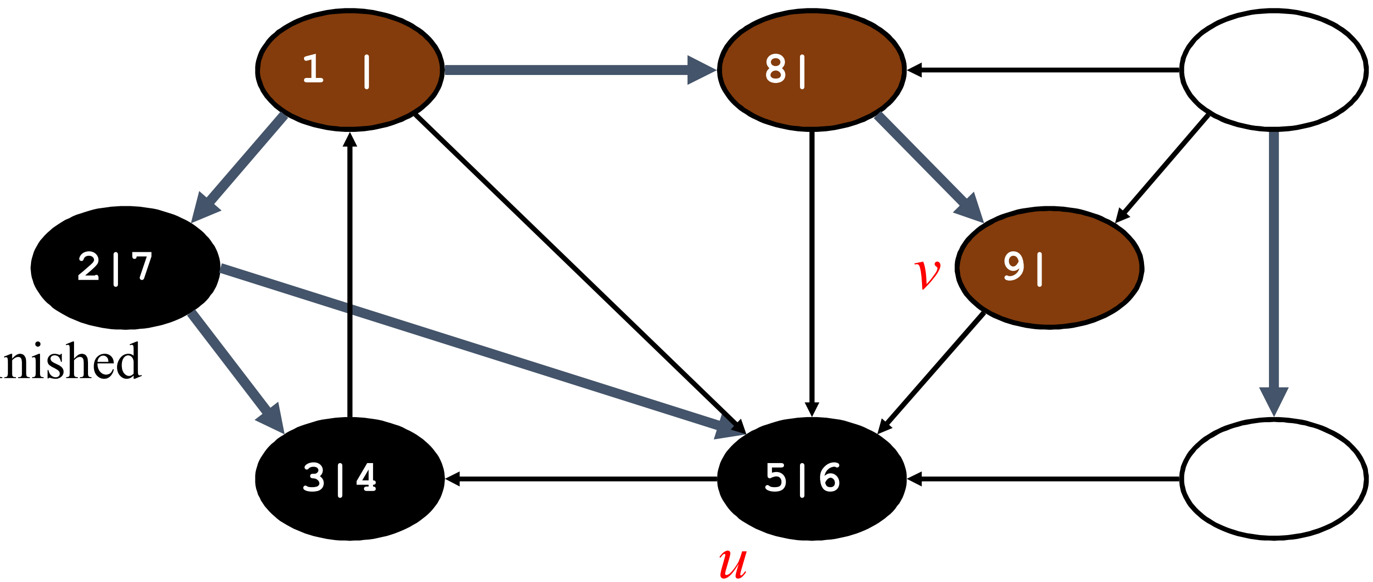
By Eq. (22.2) $\Rightarrow u.d < u.f < v.d < v.f$
$\Rightarrow [u.d, u.f]$ and $[v.d, v.f]$ are disjoint.



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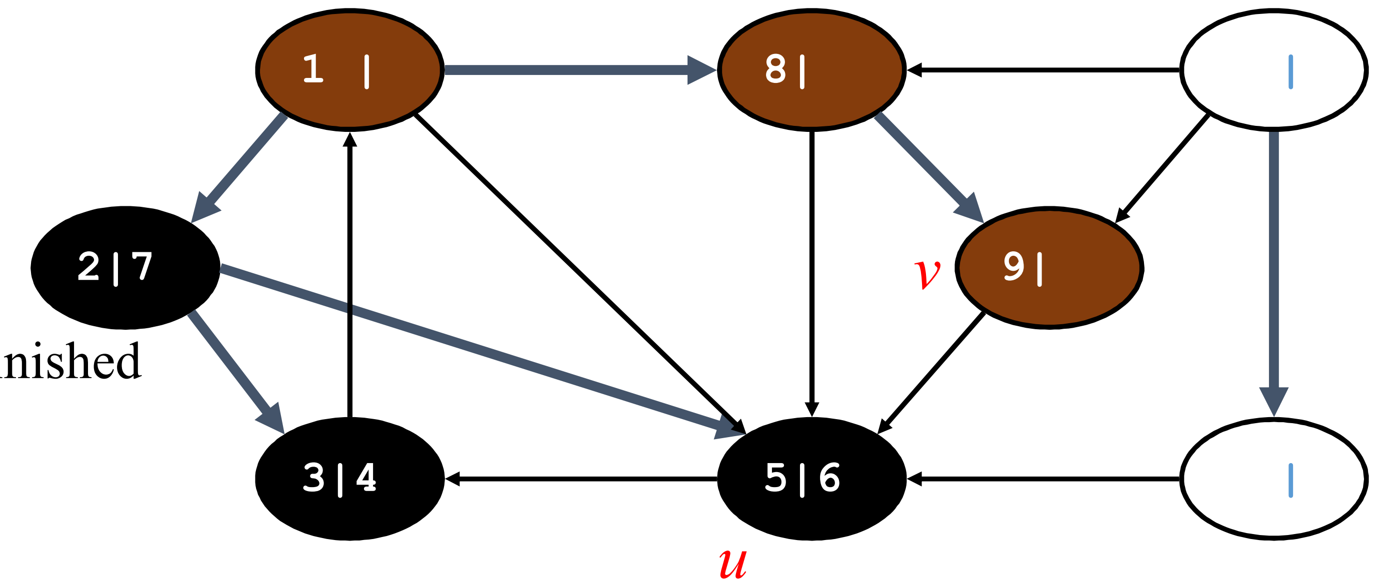


$\Rightarrow$ v is discovered AFTER u is finished
i.e., u is Black

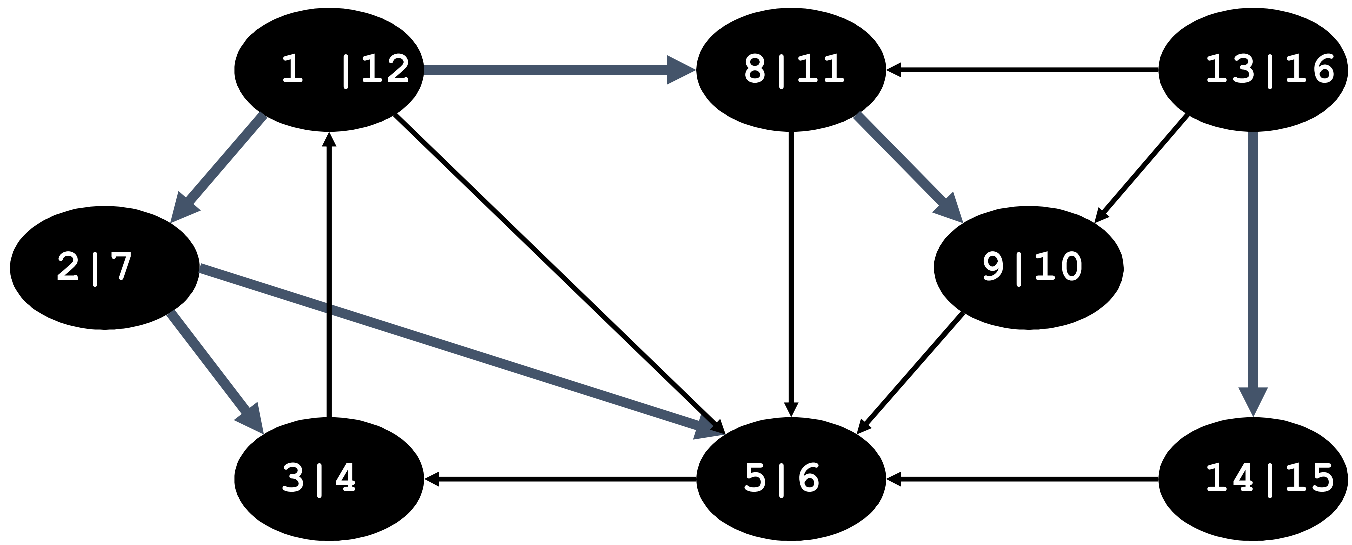
By Eq. (22.2) $\Rightarrow u.d < u.f < v.d < v.f$
$\Rightarrow [u.d, u.f]$ and $[v.d, v.f]$ are disjoint.

$\Rightarrow$ neither vertex was discovered when the other was gray
$\Rightarrow$ neither is a descendent of the other.

Case 1: $u.d < v.d$

Case 2: $v.d < u.d$

Exactly similar argument, with the roles of u and v reversed



Corollary 22.8 (Nesting of descendants' intervals)
Vertex v is a proper descendant of vertex u in the depth-first forest for a (directed or undirected) graph G if and only if $u.d < v.d < v.f < u.f$.

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Can be proved from Theorem 22.7

Theorem 22.7 (Parenthesis theorem)
In any depth-first search of a (directed or undirected) graph $G = (V, E)$, for any two vertices u and v, exactly one of the following three conditions holds:

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- the interval $[v.d, v.f]$ is contained entirely within the interval $[u.d, u.f]$, and v is a descendant of u in a depth-first tree.

P If, and only if Q

Theorem 22.9 (White-path theorem)

In a depth-first forest of a (directed or undirected) graph $G = (V, E)$, vertex v is a descendant of vertex u if and only if at the time $u.d$ that the search discovers u, there is a path from u to v consisting entirely of white vertices.

P If, and only if Q

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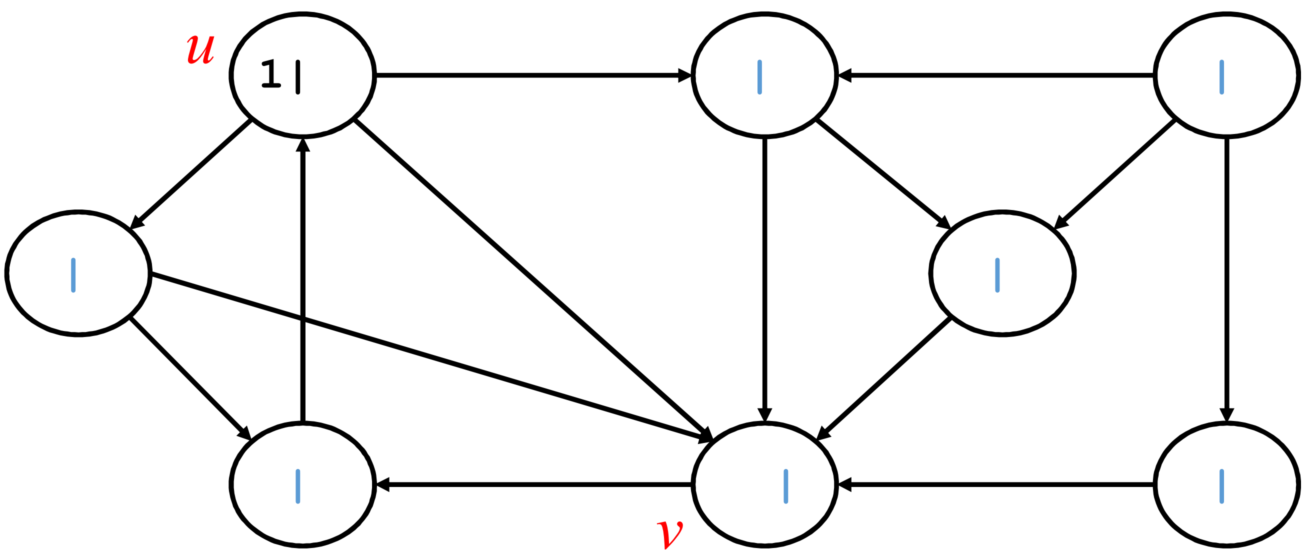

u is still white when $u.d$ is set.

DFS(G)
```
for each vertex u ∈ G.V
    u.color = WHITE
    u.π = NIL
time = 0
for each vertex u ∈ G.V
    if u.color == WHITE
        DFS-VISIT(G, u)
```

DFS-VISIT(G, u)
```
time = time + 1
u.d = time
u.color = GRAY
for each v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-VISIT(G, v)
u.color = BLACK
time = time + 1
u.f = time
```

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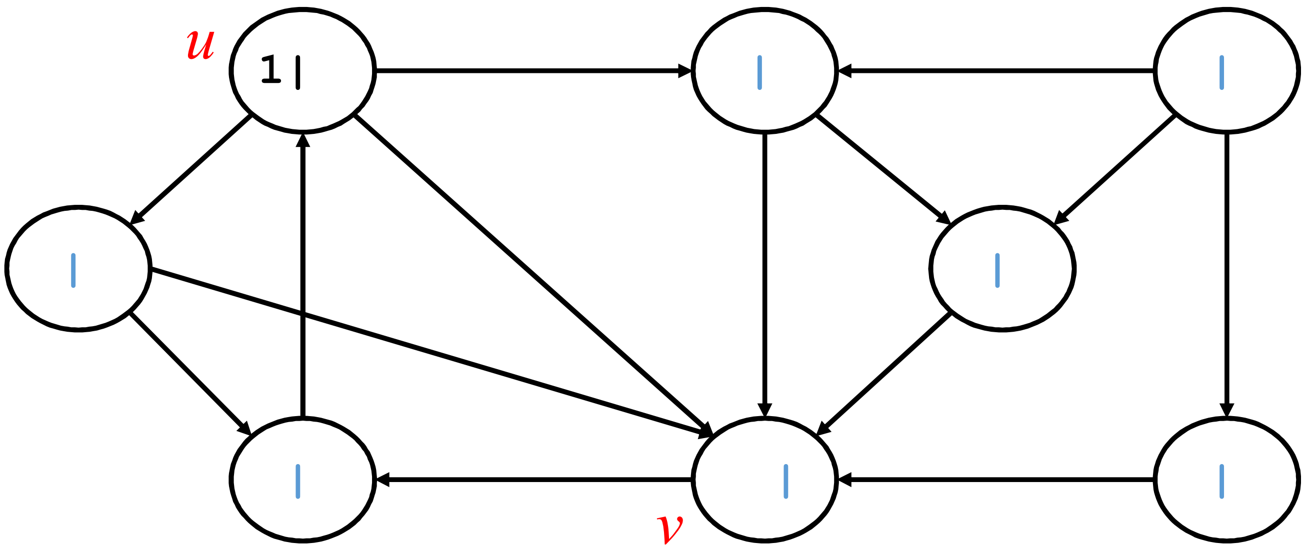

A. <u>If P then Q</u>

u is still white when $u.d$ is set.

Let v is a descendant of u

If $v = u$, we are done as both are white

P If, and only if Q

Theorem 22.9 (White-path theorem)
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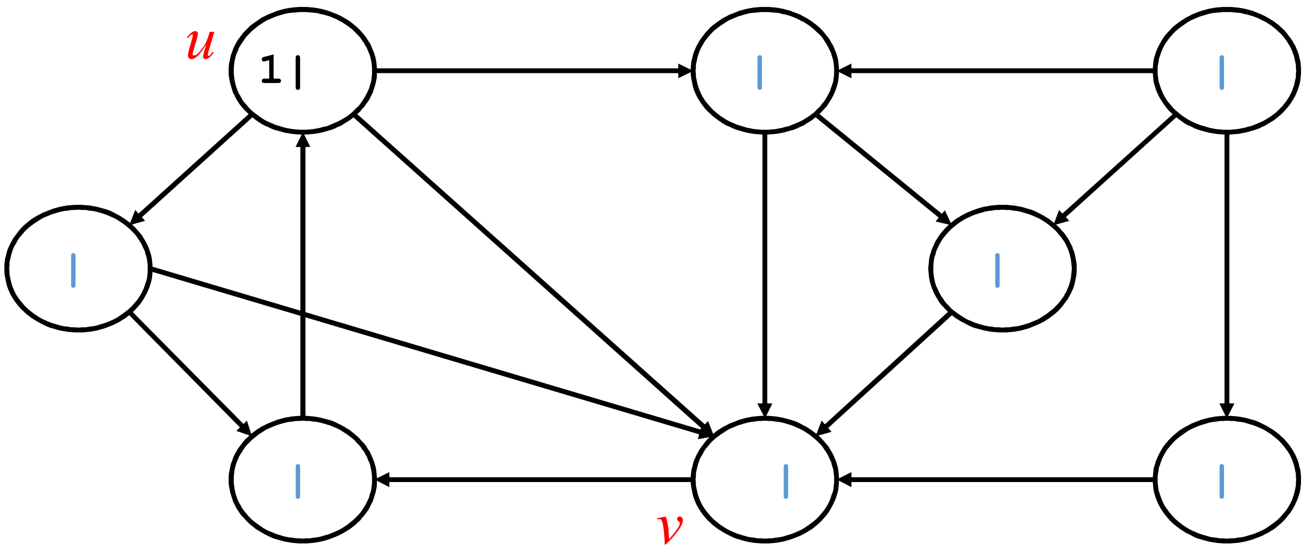


A. <u>If P then Q</u>
u is still white when $u.d$ is set.

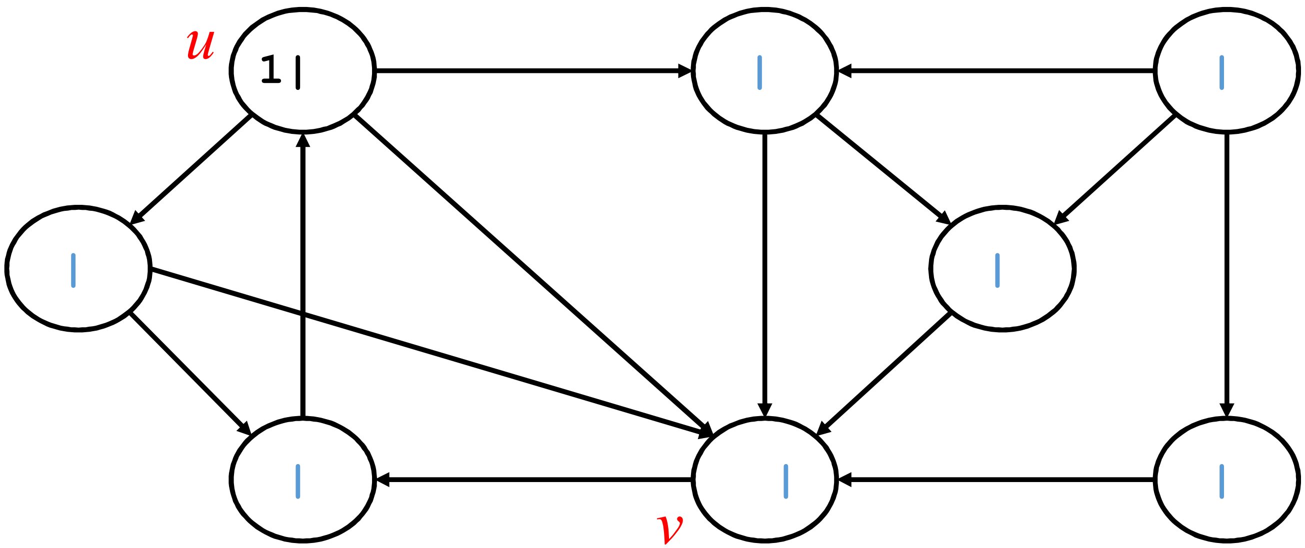
If v is a proper descendant of u,
$u.d < v.d$ [by Corollary 22.8]

=> v must be WHITE at time $u.d$

P If, and only if Q

Theorem 22.9 (White-path theorem)

In a depth-first forest of a (directed or undirected) graph $G = (V, E)$, vertex v is a descendant of vertex u if and only if at the time $u.d$ that the search discovers u, there is a path from u to v consisting entirely of white vertices.



A. <u>If P then Q</u>

u is still white when $u.d$ is set.

If v is a proper descendant of u,

$u.d < v.d$ [by Corollary 22.8]

$\Rightarrow$ v must be WHITE at time $u.d$

$\Rightarrow$ Other vertices in the path to v must be WHITE too.

P If, and only if Q

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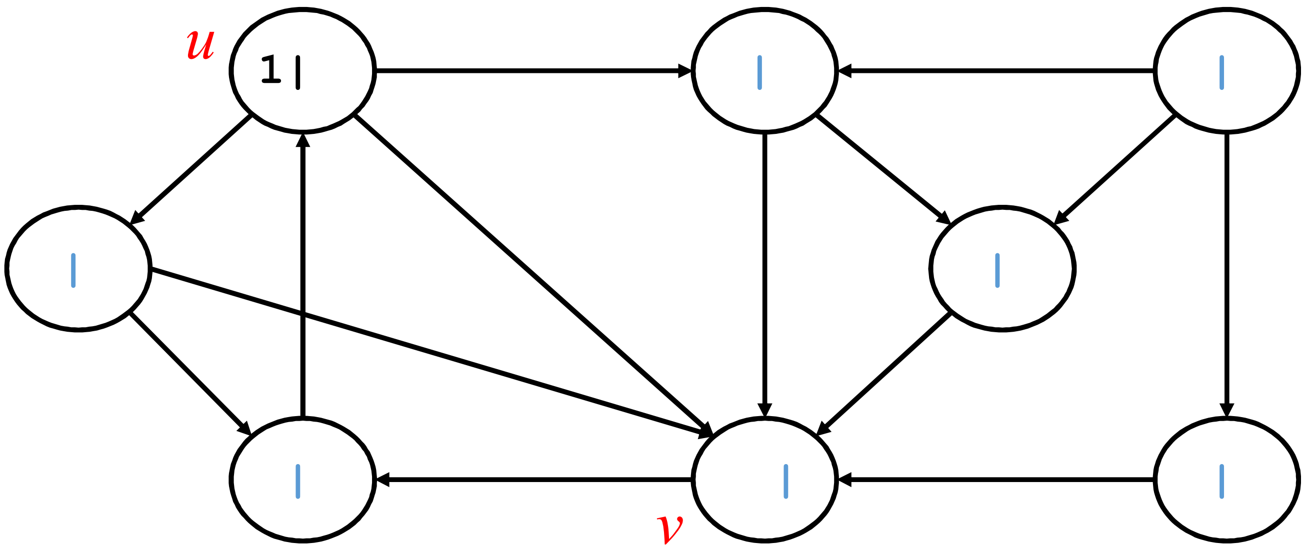

B. <u>If Q then P</u>
Let there is a path (Z) of white vertices from u to v at time $u.d$, but v is not a descendant of u in DFT.

P If, and only if Q

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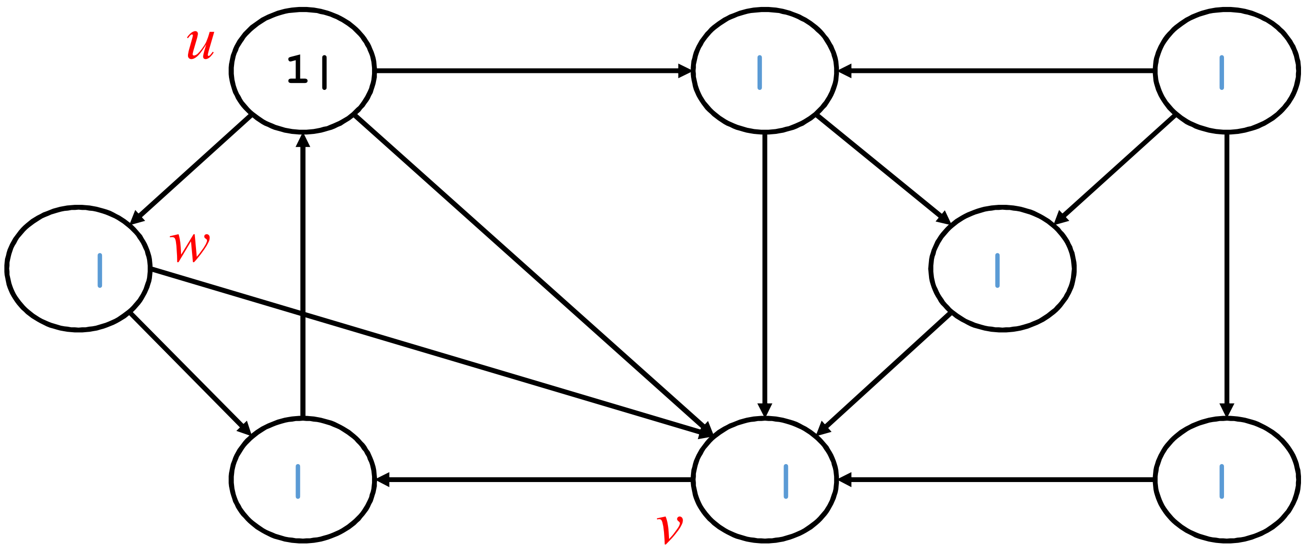

B. <u>If Q then P</u>

Let there is a path (Z) of white vertices from u to v at time $u.d$, but v is not a descendant of u in DFT.

Let w be a predecessor of v (in Z)

P If, and only if Q

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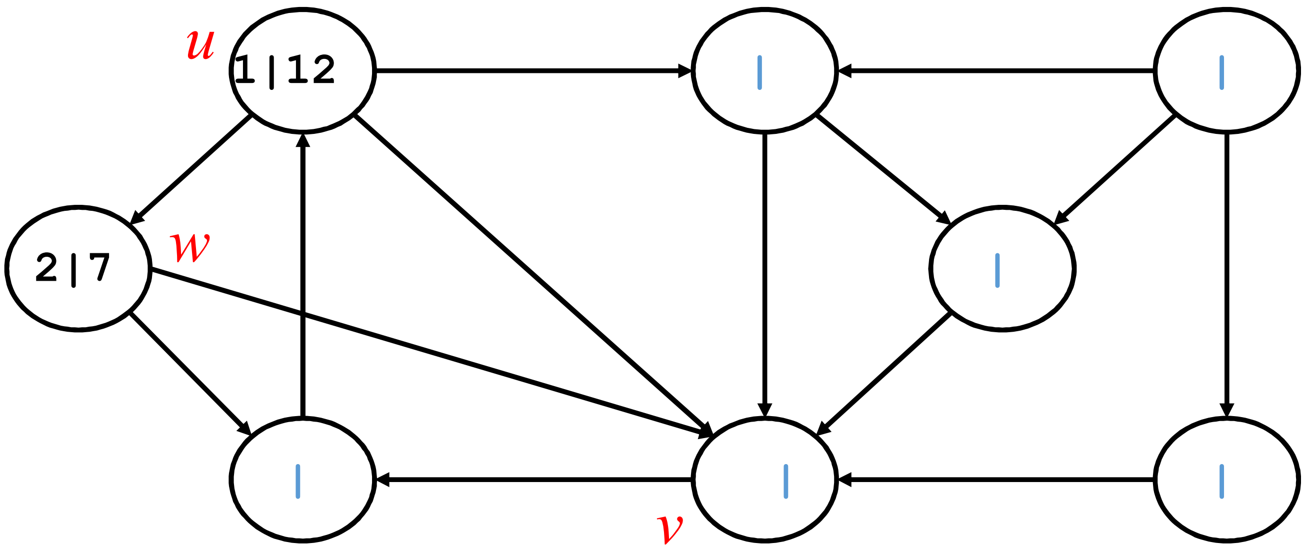

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Let there is a path (Z) of white vertices from u to v at time $u.d$, but v is not a descendant of u in DFT.
Let w be a predecessor of v (in Z)
w is a descendant of u.
$\Rightarrow w.f <= u.f$

P If, and only if Q

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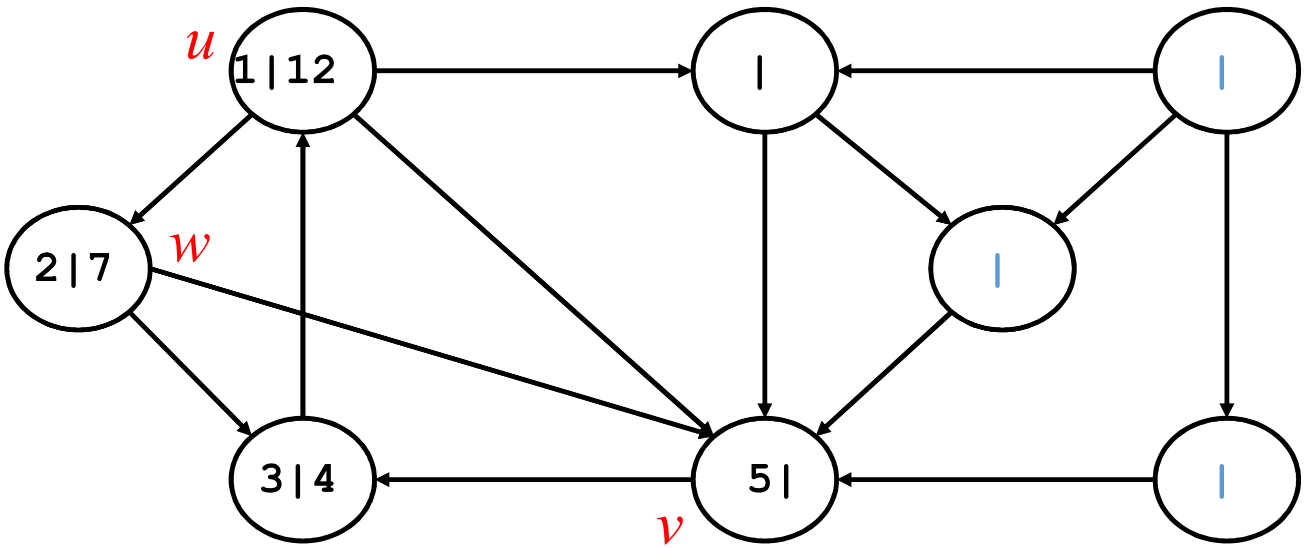
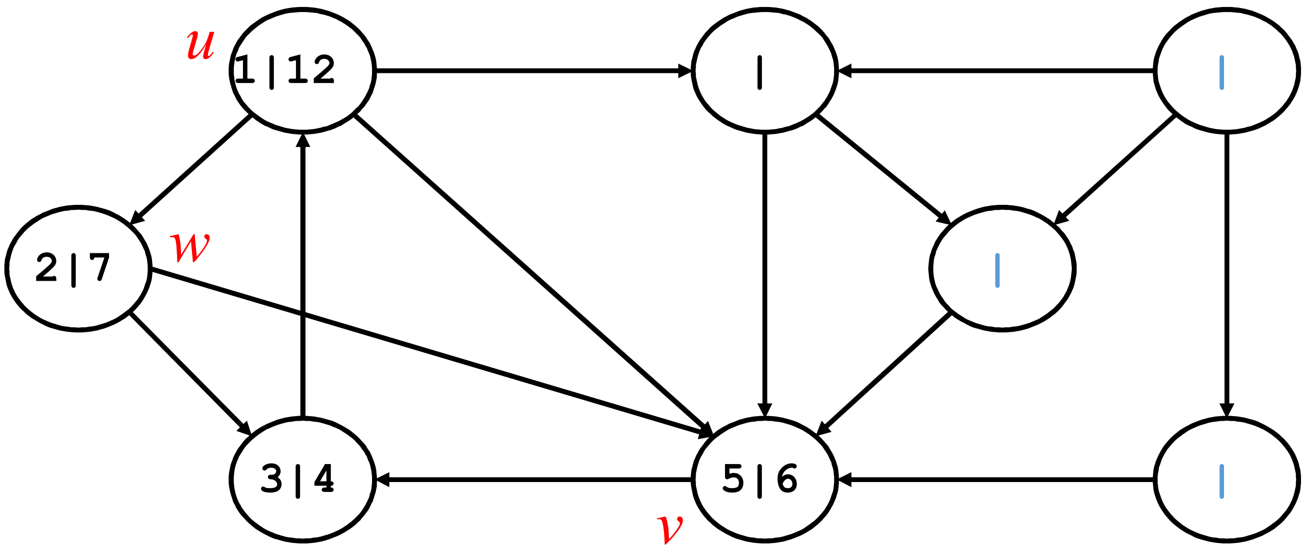
In a depth-first forest of a (directed or undirected) graph $G = (V, E)$, vertex v is a descendant of vertex u if and only if at the time $u.d$ that the search discovers u, there is a path from u to v consisting entirely of white vertices.



B. <u>If Q then P</u>
Let there is a path (Z) of white vertices from u to v at time $u.d$, but v is not a descendant of u in DFT.
Let w be a predecessor of v (in Z)
w is a descendant of u.
$\Rightarrow w.f <= u.f$
v must be discovered after u is discovered but before w is finished
$=> u.d < v.d < w.f$

P If, and only if Q

Theorem 22.9 (White-path theorem)
In a depth-first forest of a (directed or undirected) graph $G = (V, E)$, vertex v is a descendant of vertex u if and only if at the time $u.d$ that the search discovers u, there is a path from u to v consisting entirely of white vertices.



B. <u>If Q then P</u>
Let there is a path (Z) of white vertices from u to v at time $u.d$, but v is not a descendant of u in DFT.

$\Rightarrow w.f <= u.f$

$\Rightarrow u.d < v.d < w.f$

$\Rightarrow u.d < v.d < w.f <= u.f$

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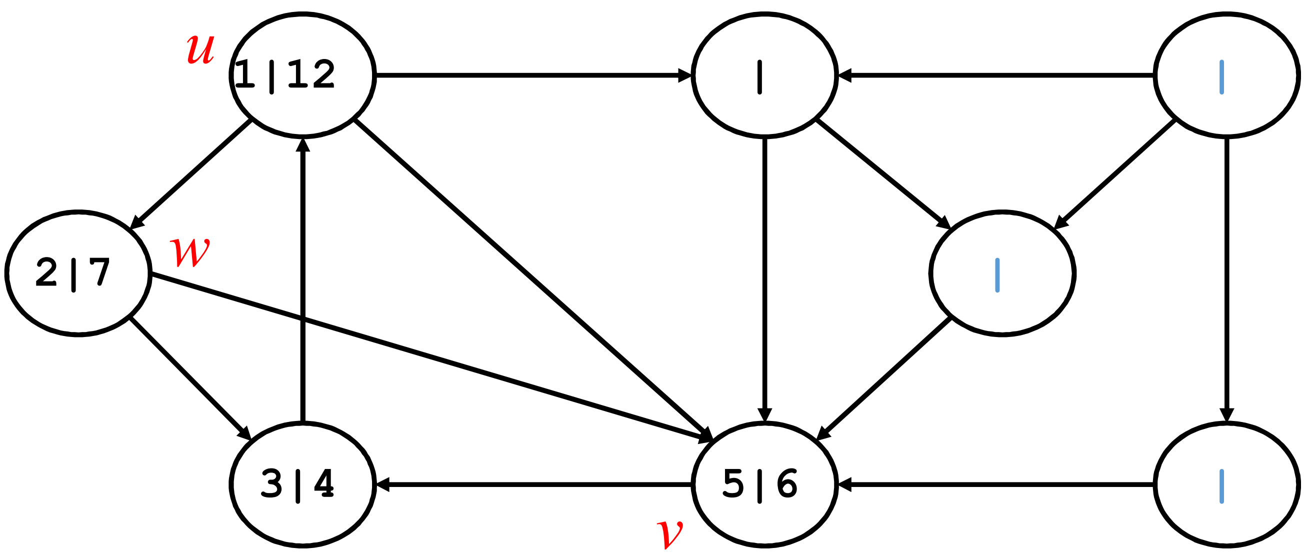

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$\Rightarrow$ Th. 22.7, $[v.d,\ v.f]$ must be contained within $[u.d,\ u.f]$

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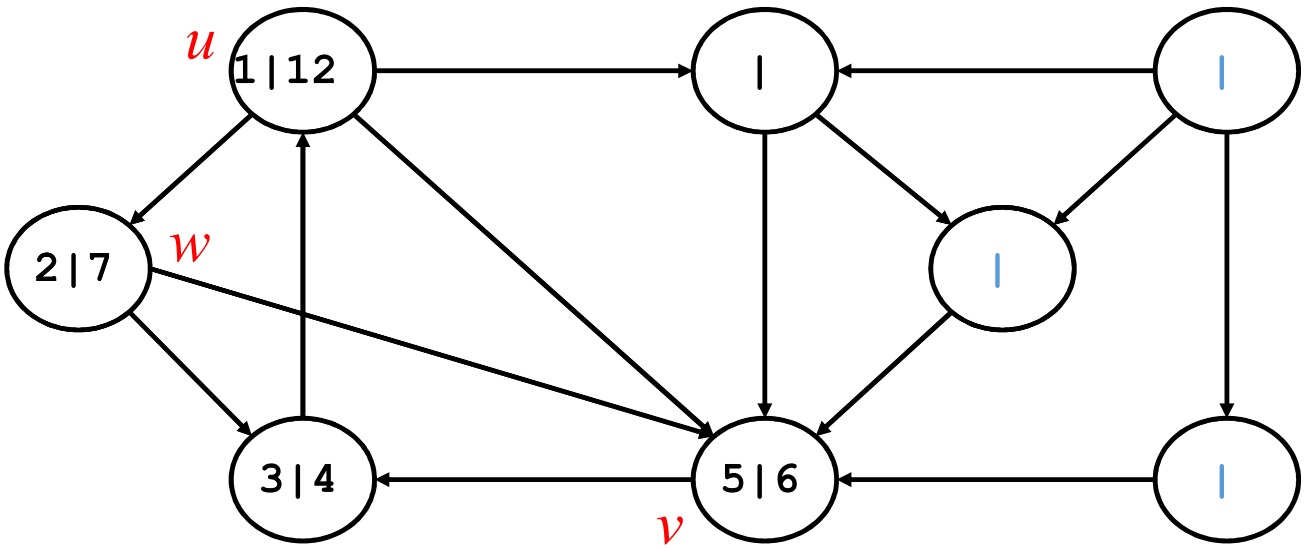

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$\Rightarrow w.f <= u.f$

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$\Rightarrow u.d < v.d < w.f <= u.f$

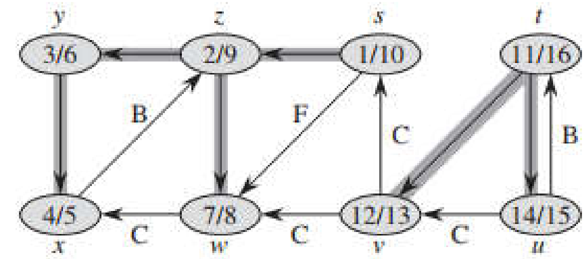
$\Rightarrow$ Th. 22.7, $[v.d, v.f]$ must be contained within $[u.d, u.f]$

$\Rightarrow$ v MUST BE a descendant of u in DFT

Depth-first forest

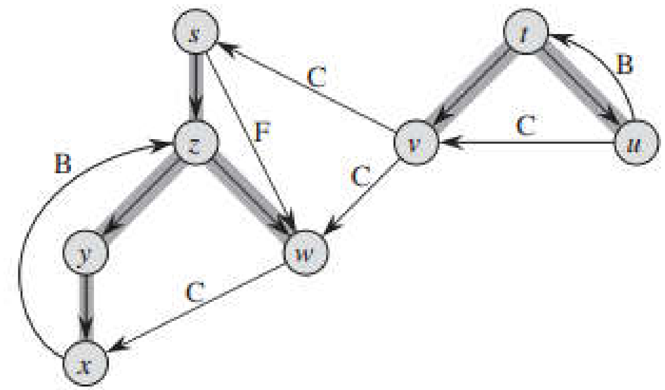
- The procedure DFS builds a depth-first forest comprising several depth-first trees as it searches the graph
- The forest/trees corresponds to the π attributes.
- More formally, for a graph $G = (V, E)$, we define the ***predecessor subgraph*** of G as $G_\pi = (V, E_\pi)$, where $E_\pi = \{(v.\pi, v) : v \in V \text{ and } v.\pi \neq \text{NIL}\}$
- The edges in E_π are called tree edges.

DFS: Types of Edges



- Tree Edges

- Forward Edges

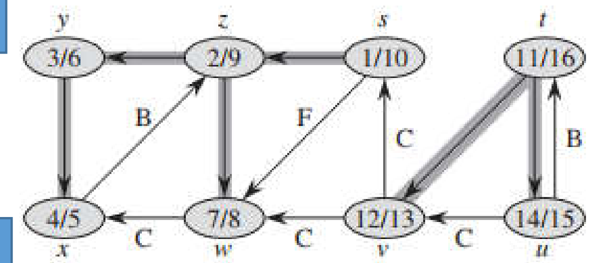


- Back Edges

- Cross Edges

DFS: Types of Edges

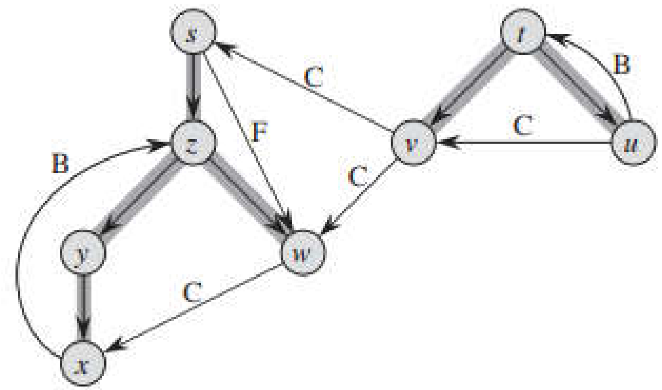
When we first explore (u, v), u is gray.
The color of v determines the edged type.



- Tree Edges are edges in the depth-first forest G_π. Edge (u, v) is a tree edge if it is first discovered by exploring edge (u, v)

encounters a WHITE vertex v

- Forward Edges

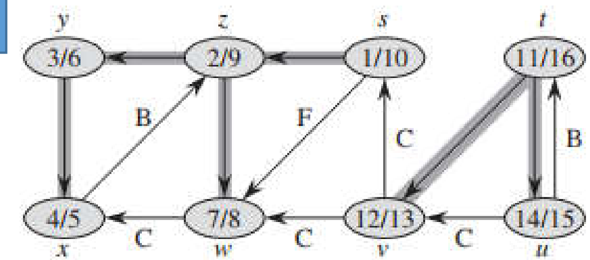


- Back Edges

- Cross Edges

When we first explore (u, v), u is gray.
The color of v determines the edged type.

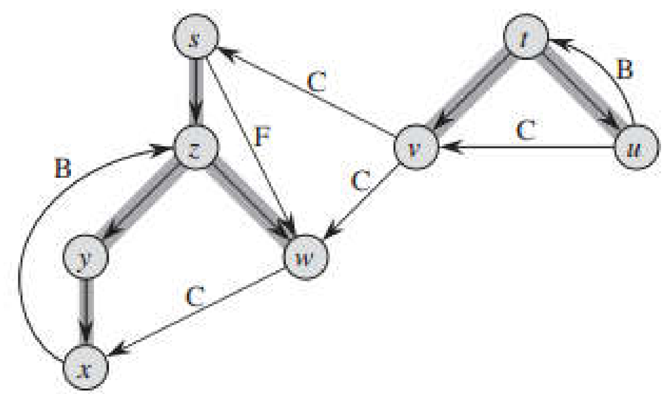
DFS: Types of Edges



- Tree Edges
- Forward Edges are those edges (u, v) connecting a vertex u to an descendant v in a depth-first tree.

Encounters a BLACK vertex, v

- Back Edges
- Cross Edges



DFS: Types of Edges

When we first explore (u, v), u is gray.
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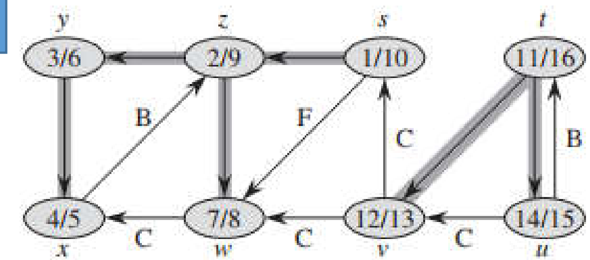


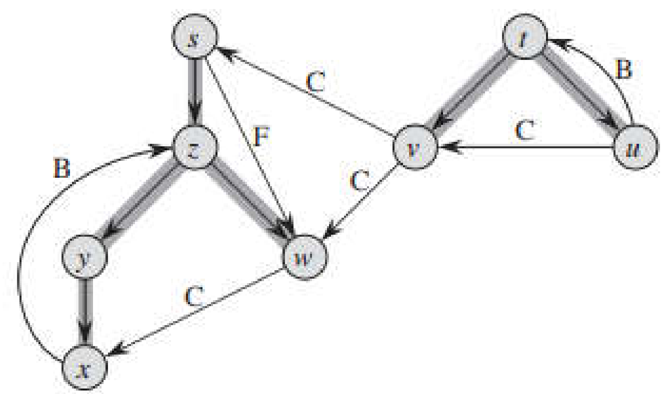
- Tree Edges

- Forward Edges

encounters a GRAY vertex, v

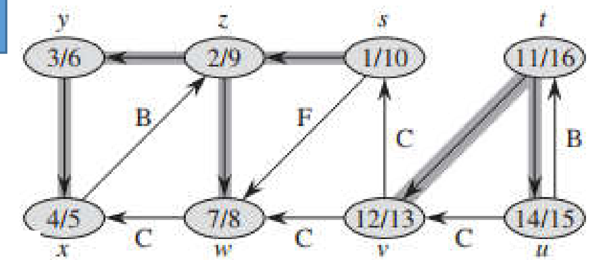
- Back Edges are those non-tree edges (u, v) connecting a vertex u to an ancestor v in a depth-first tree. We consider self-loops, which may occur in directed graphs, to be back edges.

- Cross Edges

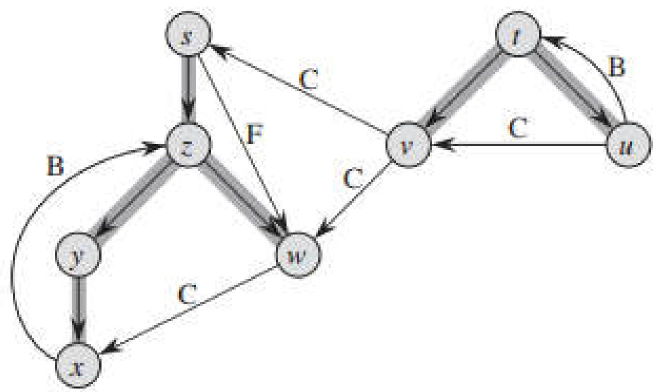


When we first explore (u, v), u is gray.
The color of v determines the edged type.

DFS: Types of Edges



- Tree Edges



- Forward Edges

- Back Edges

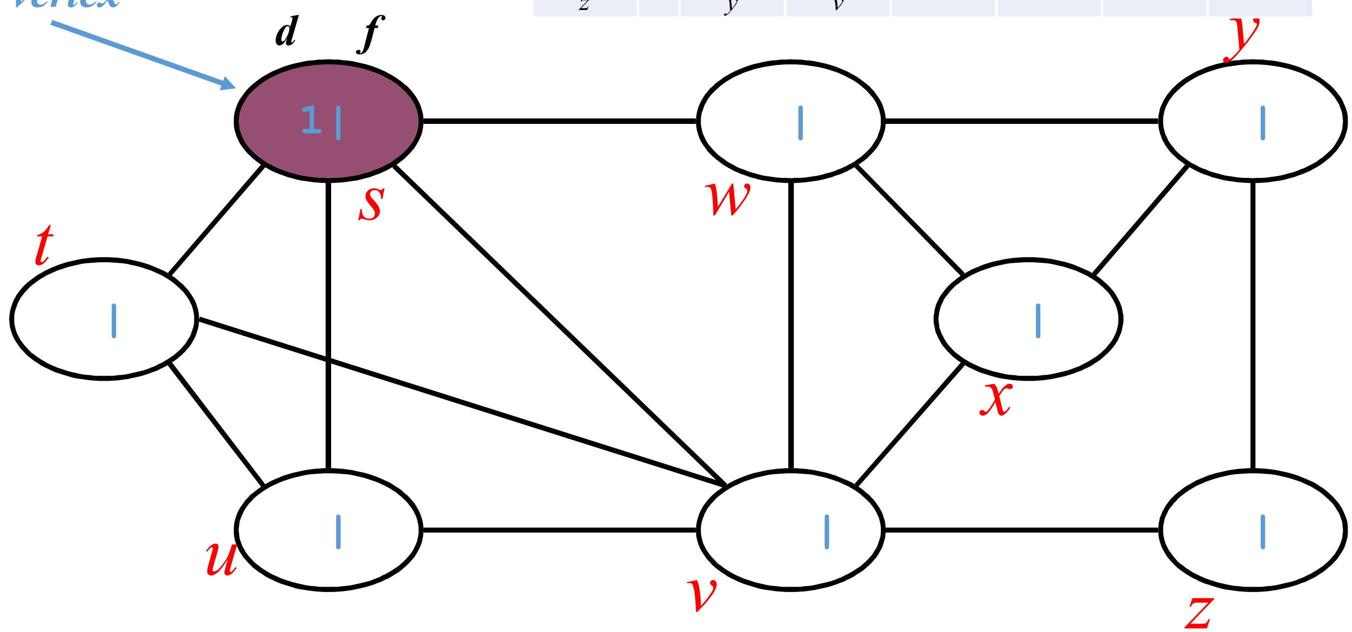
- Cross Edges are all other edges. They can go between vertices in the same depth-first tree, as long as one vertex is not an ancestor of the other, or they can go between vertices in different depth-first trees.

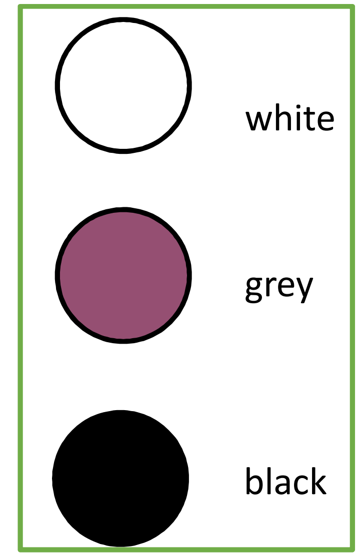
encounters a BLACK vertex, v

Undirected graph and edges

Nodes		Adjacency list					
s		*t*	*u*	*v*	*w*		
t		*s*	*u*	*v*			
u		*s*	*t*	*v*			
v		*s*	*t*	*u*	*w*	*x*	*z*
w		*s*	*v*	*x*	*y*		
x		*w*	*v*	*y*			
y		*w*	*x*	*z*			
z		*y*	*v*				

source vertex





Undirected graph and edges

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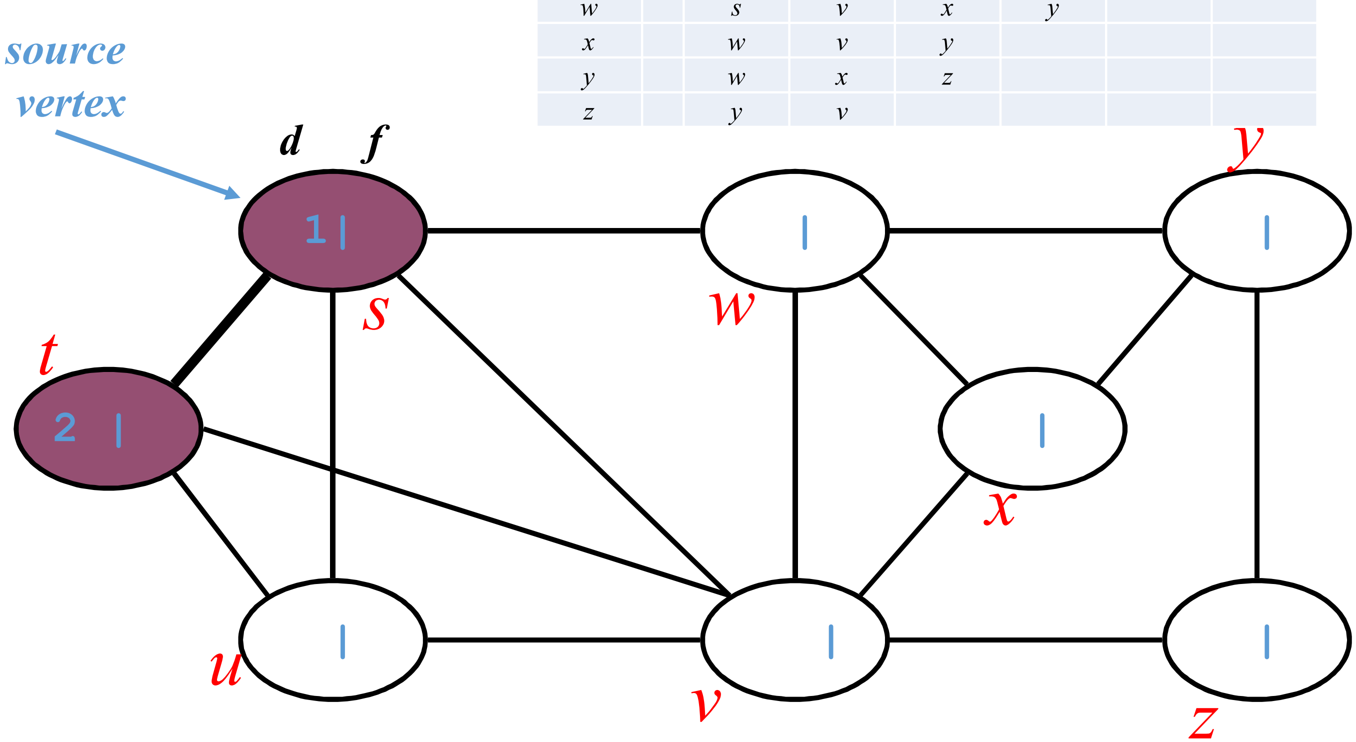
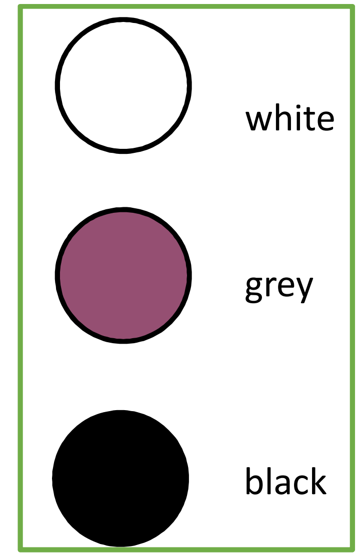
t		*s*	*u*	*v*			
u		*s*	*t*	*v*			
v		*s*	*t*	*u*	*w*	*x*	*z*
w		*s*	*v*	*x*	*y*		
x		*w*	*v*	*y*			
y		*w*	*x*	*z*			
z		*y*	*v*				





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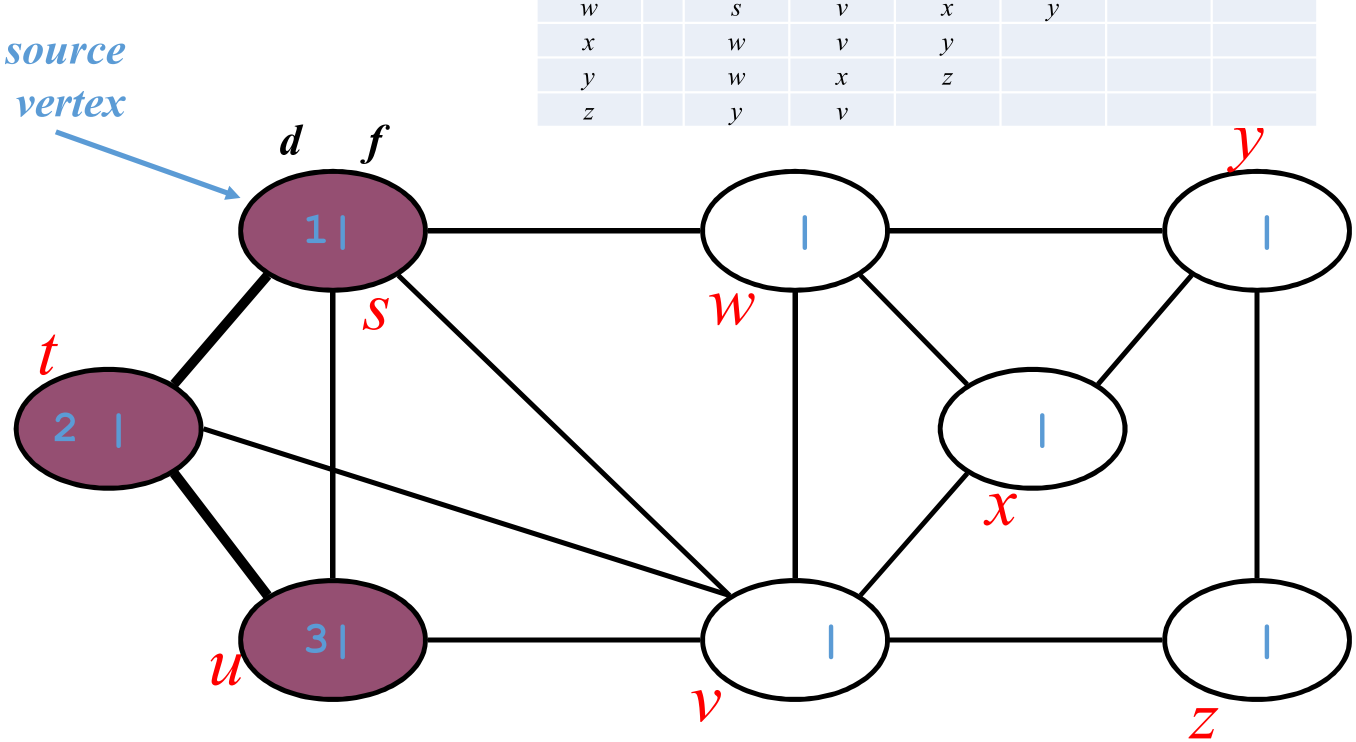
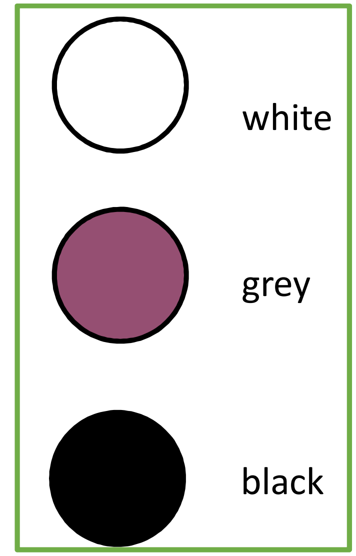
u		*s*	*t*	*v*			
v		*s*	*t*	*u*	*w*	*x*	*z*
w		*s*	*v*	*x*	*y*		
x		*w*	*v*	*y*			
y		*w*	*x*	*z*			
z		*y*	*v*				





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t		*s*	*u*	*v*			
u		*s*	*t*	*v*			
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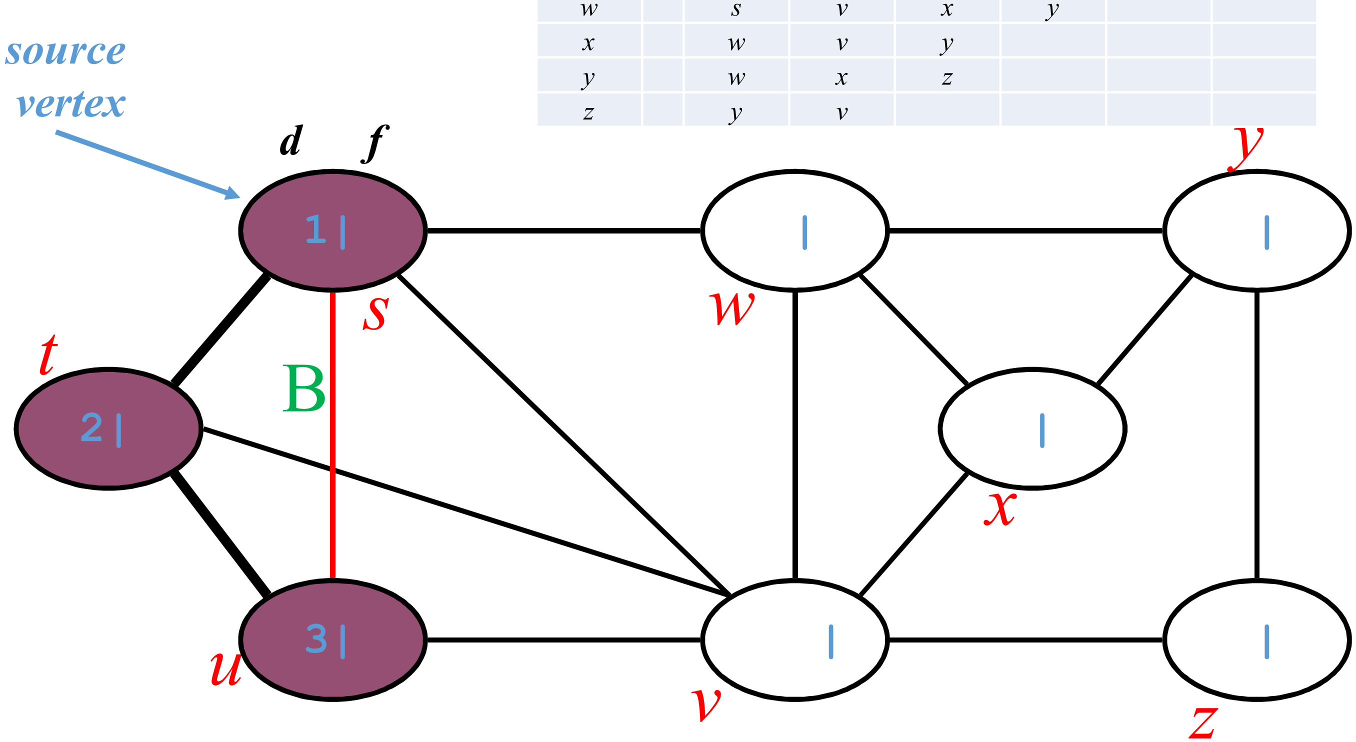
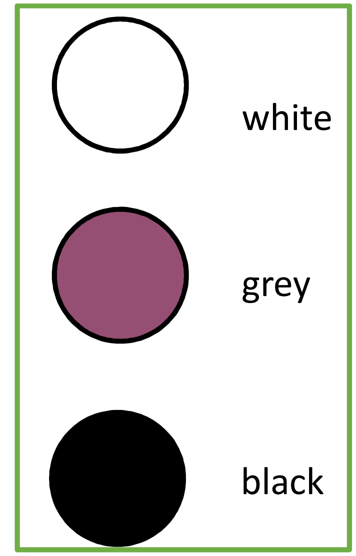
w		*s*	*v*	*x*	*y*		
x		*w*	*v*	*y*			
y		*w*	*x*	*z*			
z		*y*	*v*				





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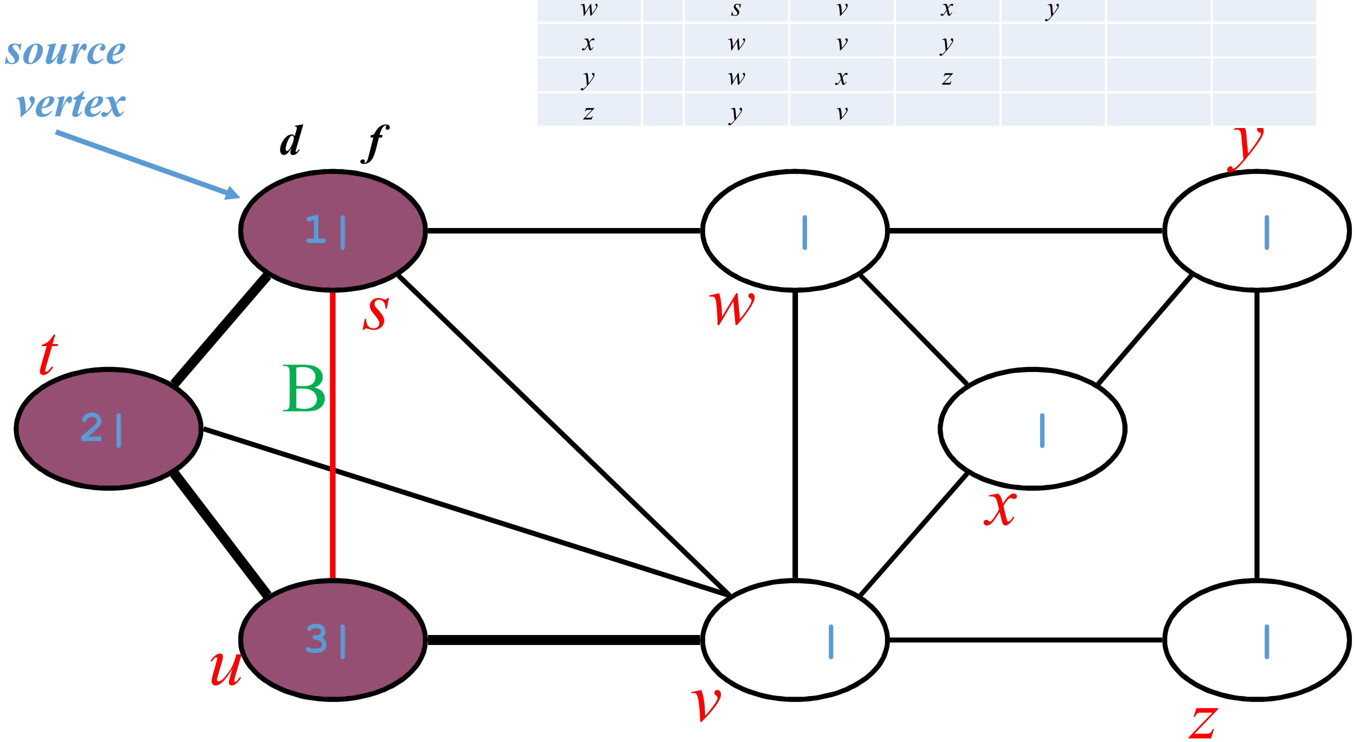
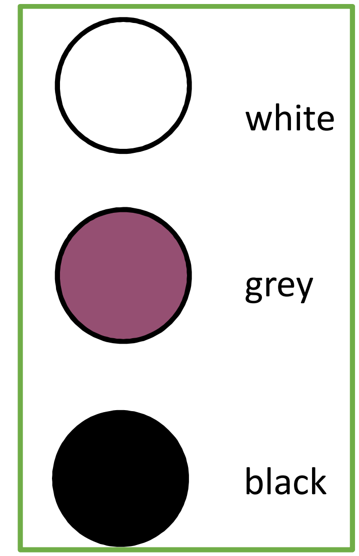
v		*s*	*t*	*u*	*w*	*x*	*z*
w		*s*	*v*	*x*	*y*		
x		*w*	*v*	*y*			
y		*w*	*x*	*z*			
z		*y*	*v*				





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w		*s*	*v*	*x*	*y*		
x		*w*	*v*	*y*			
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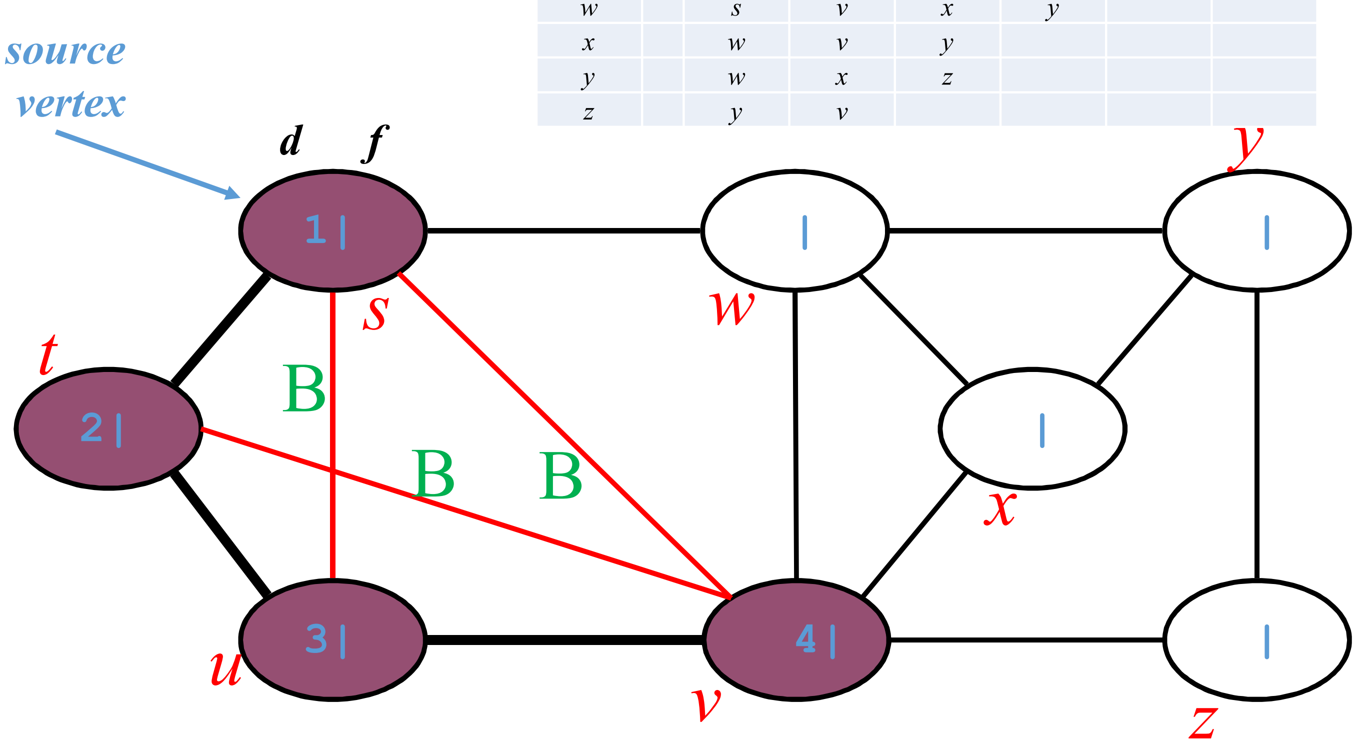
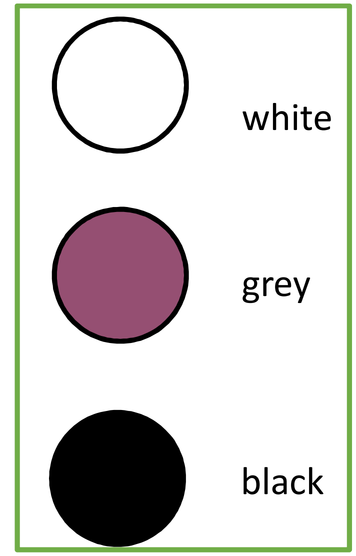
z		*y*	*v*				





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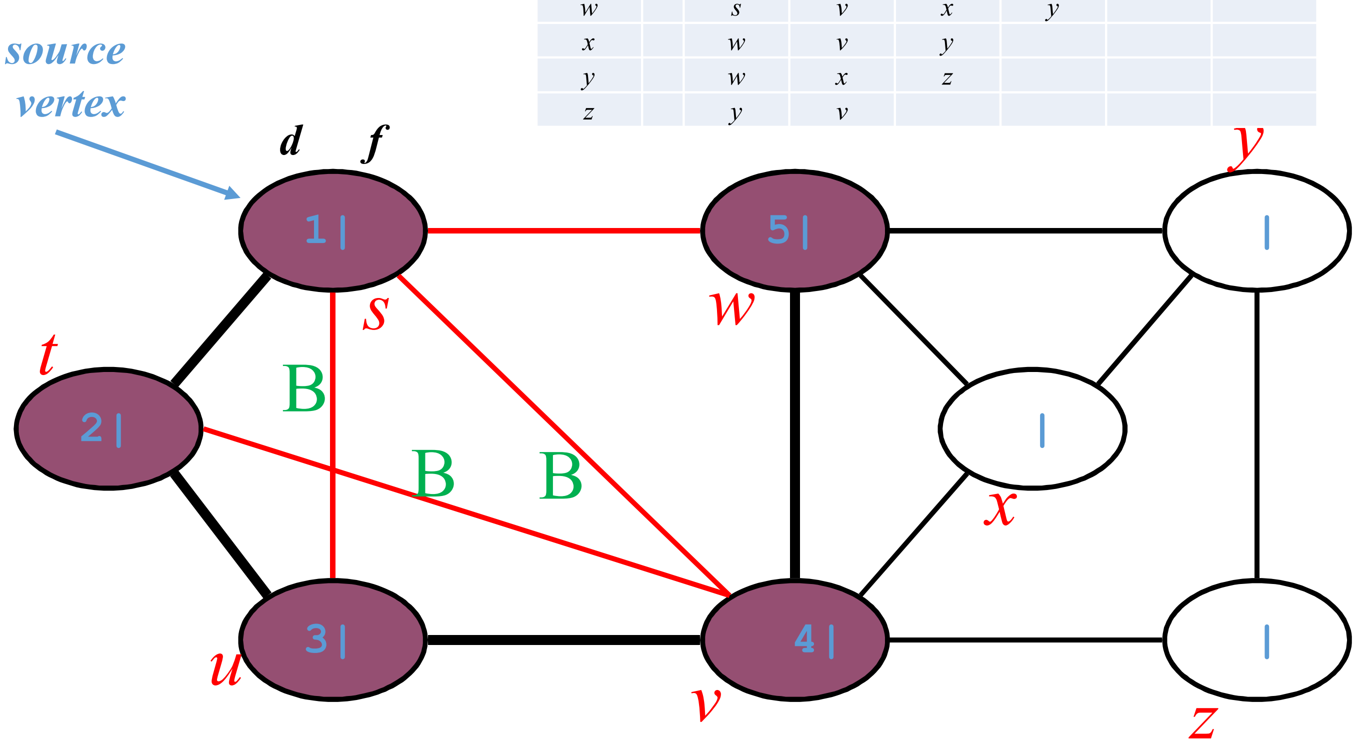
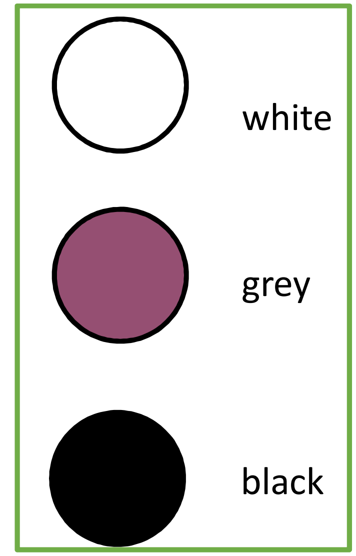
v		*s*	*t*	*u*	*w*	*x*	*z*
w		*s*	*v*	*x*	*y*		
x		*w*	*v*	*y*			
y		*w*	*x*	*z*			
z		*y*	*v*				





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w		*s*	*v*	*x*	*y*		
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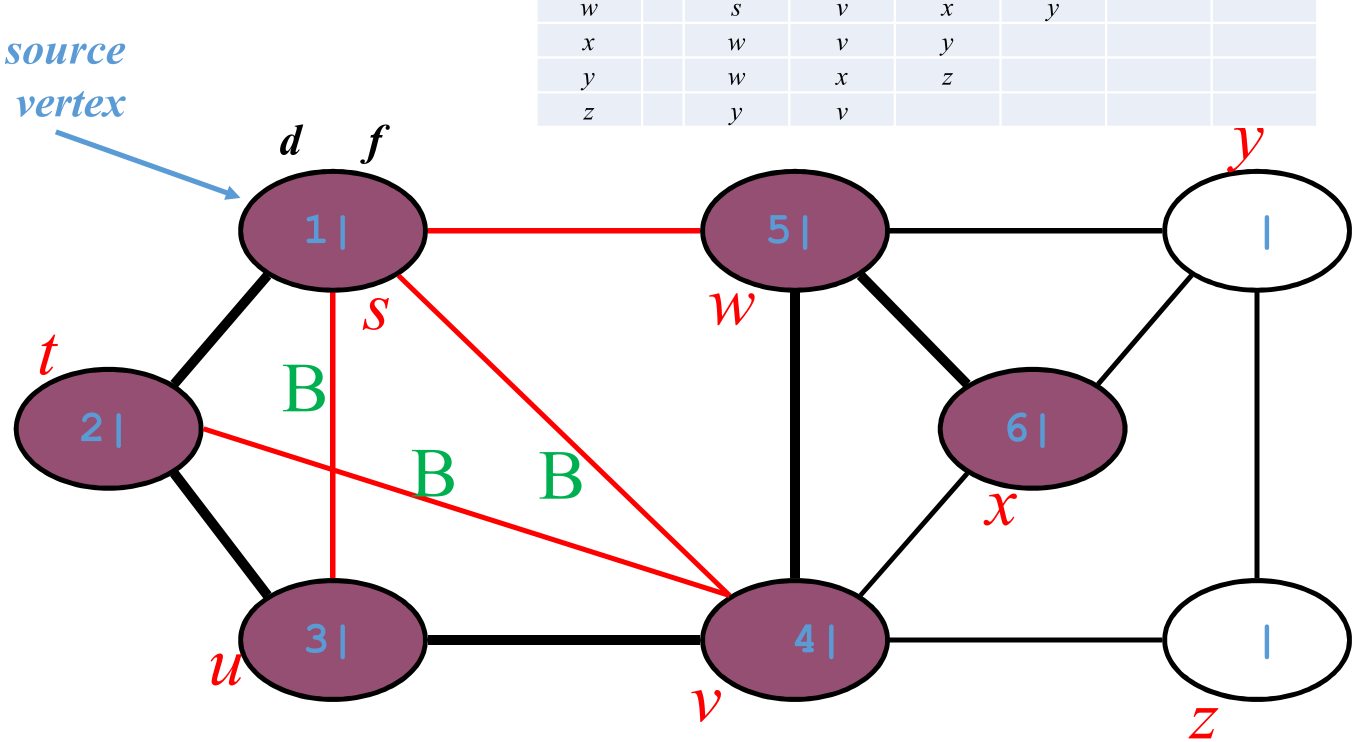
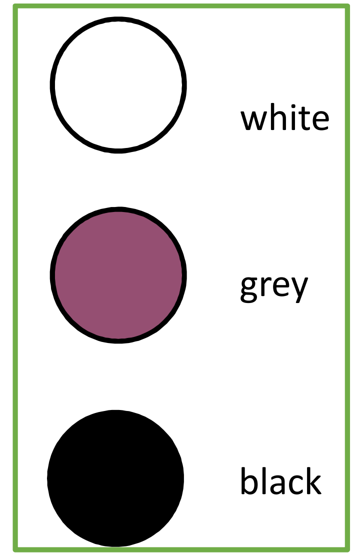
y		*w*	*x*	*z*			
z		*y*	*v*				





Undirected graph and edges

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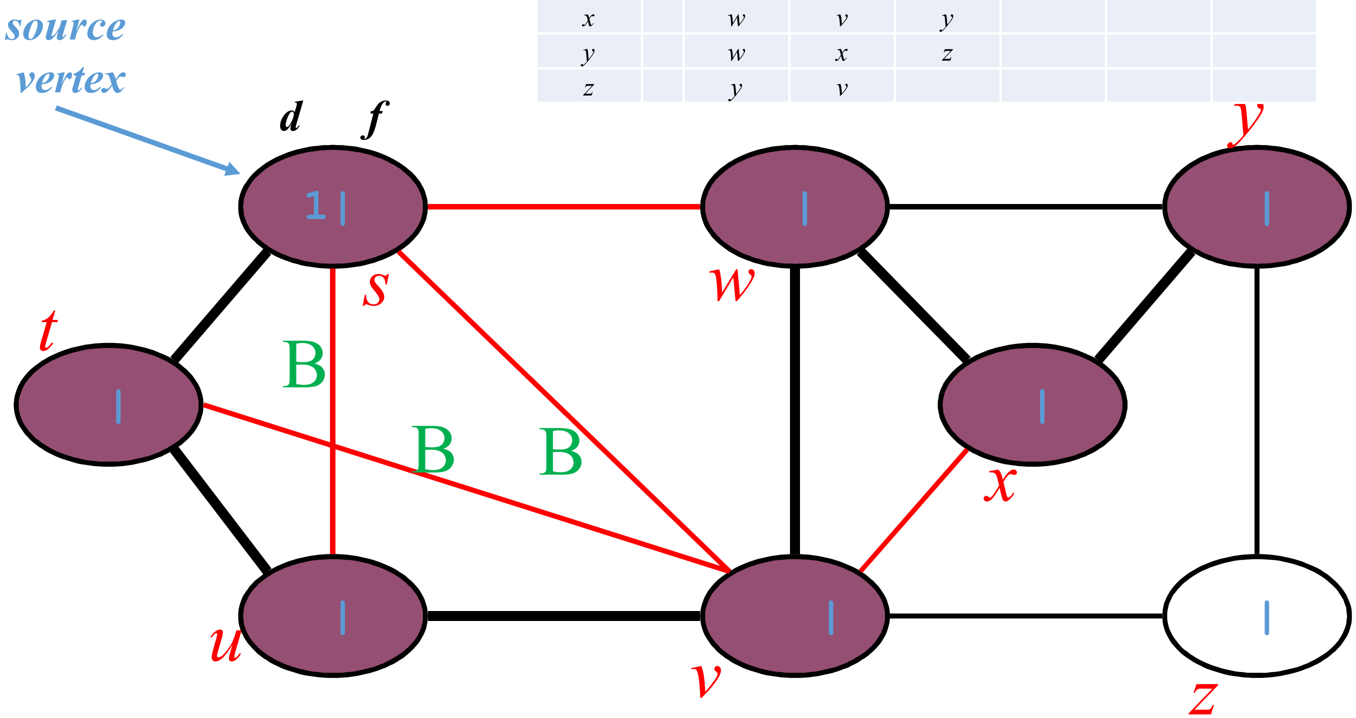

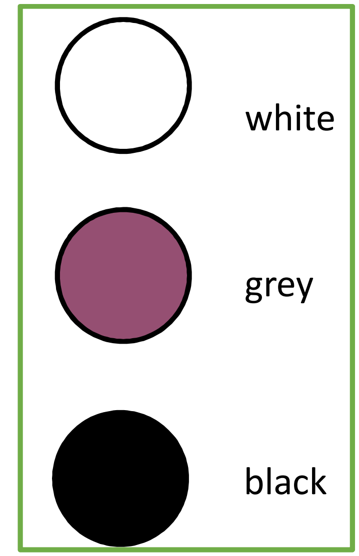

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w		*s*	*v*	*x*	*y*		
x		*w*	*v*	*y*			
y		*w*	*x*	*z*			
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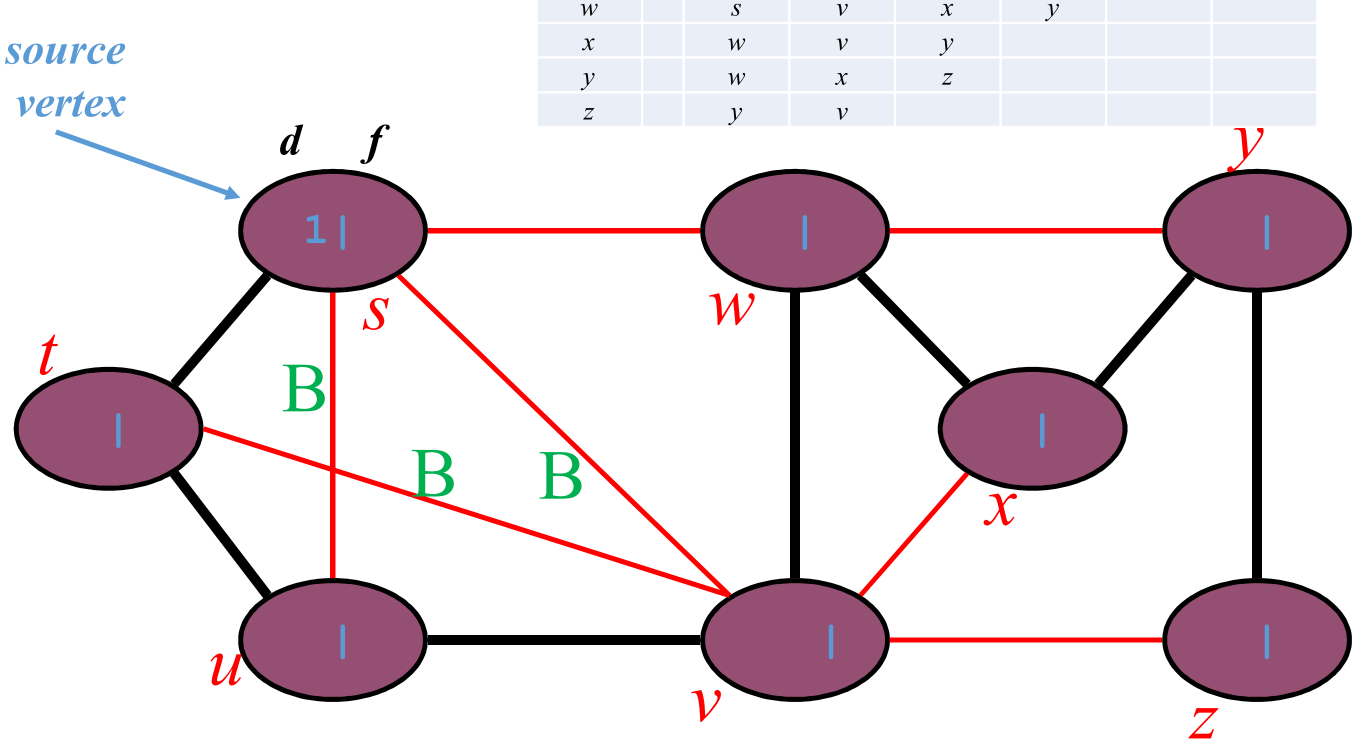
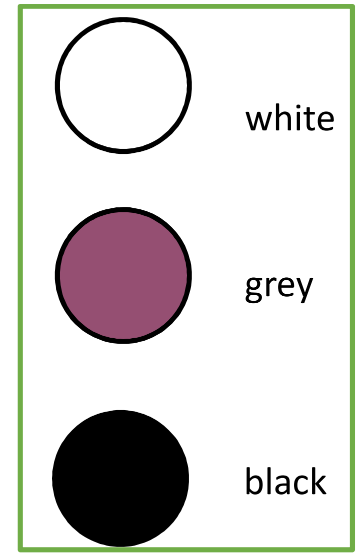
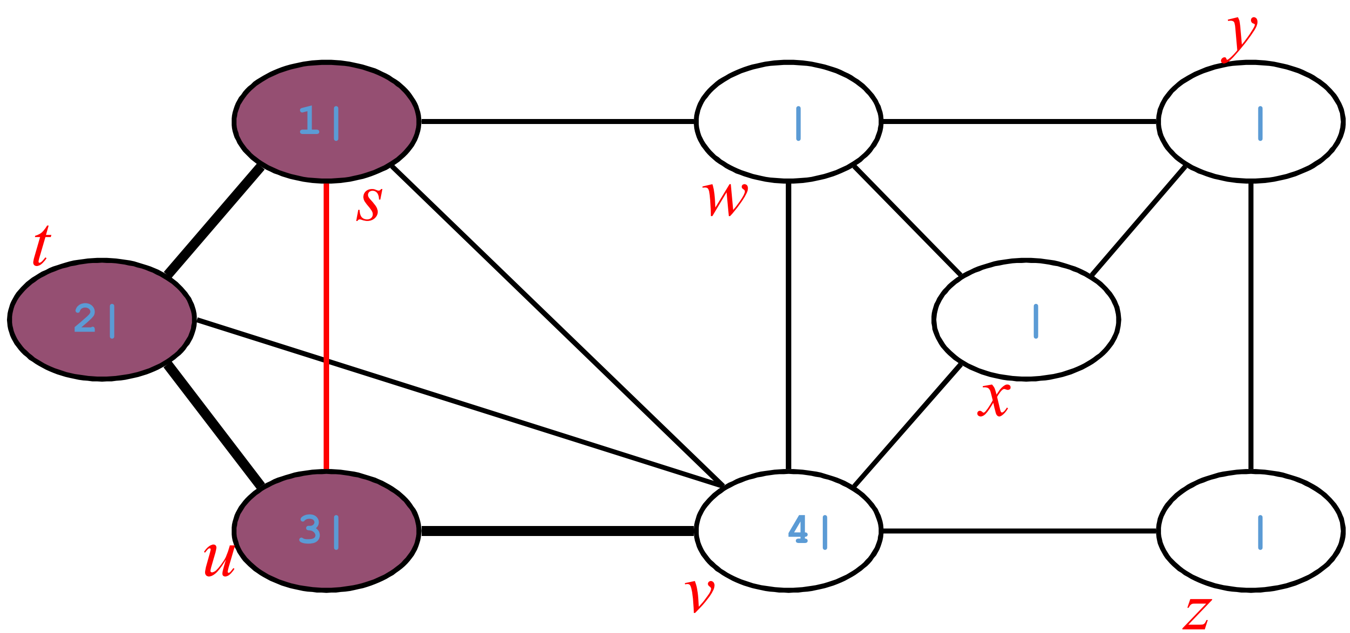




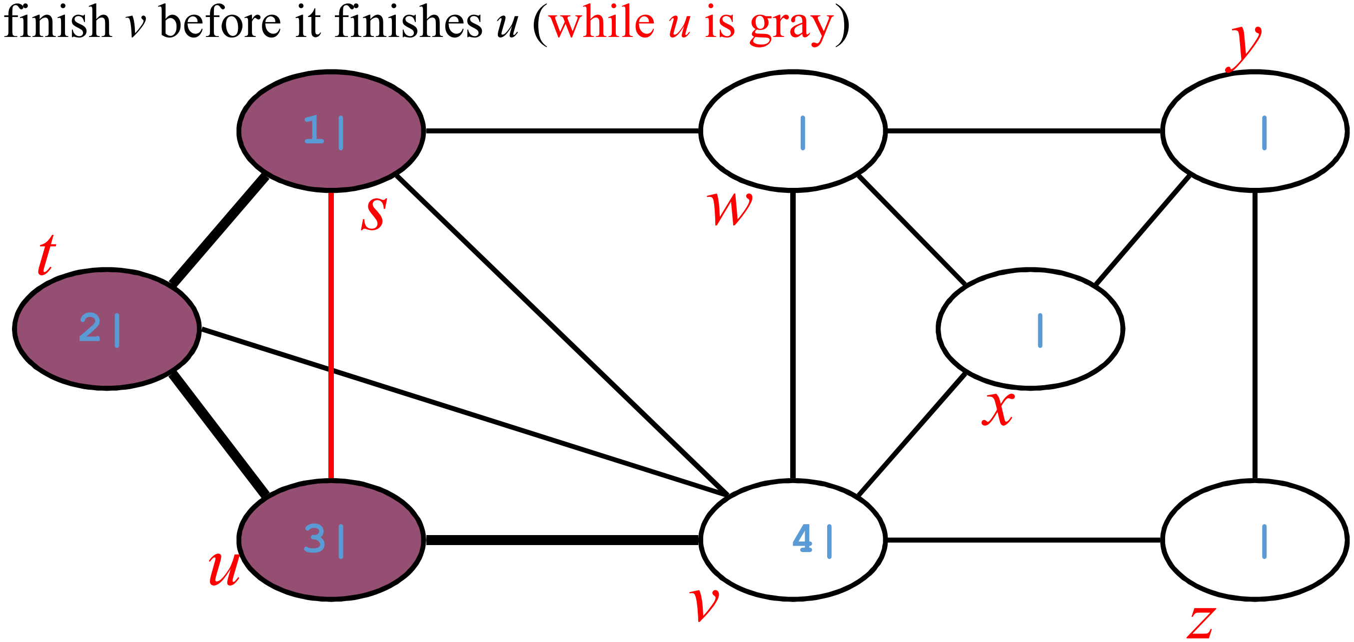
In a DFS of an undirected graph G, every edge of G is either a tree edge or a back edge.

- Let (u, v) be an arbitrary edge of G, and suppose without loss of generality that $u.d < v.d$.



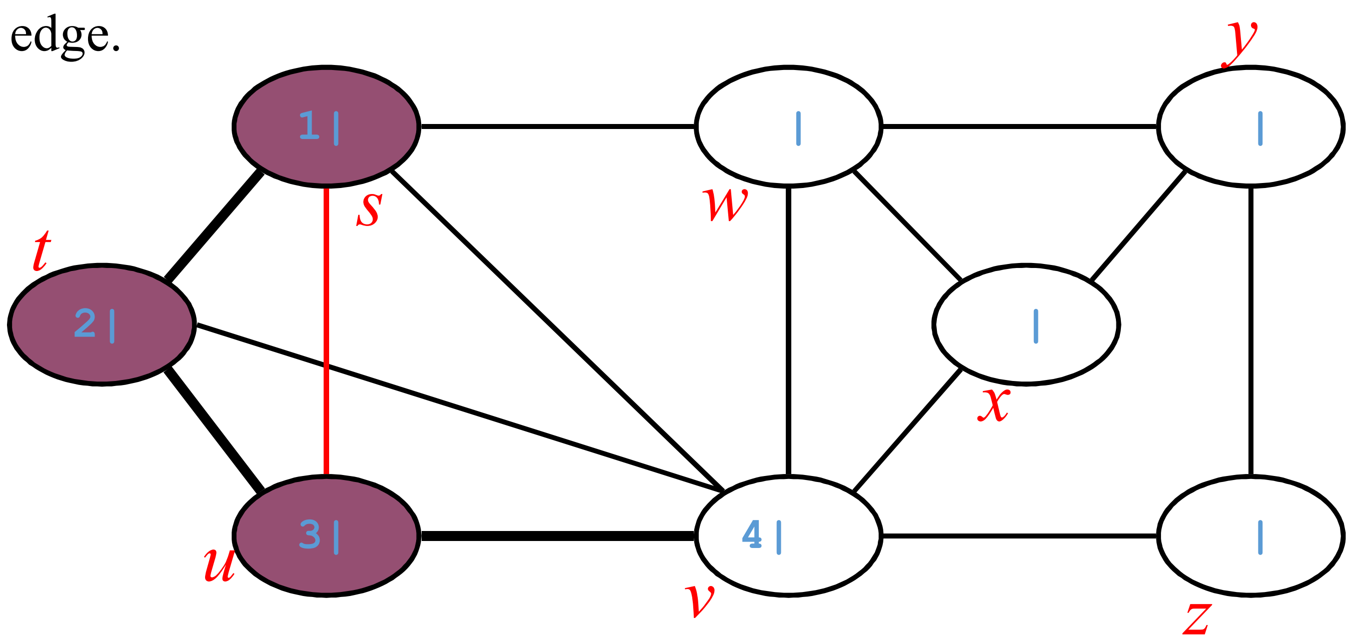
In a DFS of an undirected graph G, every edge of G is either a tree edge or a back edge.

- Let (u, v) be an arbitrary edge of G, and suppose without loss of generality that $u.d < v.d$.
  - v is on u's adjacency list.
  - the search must discover and finish v before it finishes u (while u is gray)



In a DFS of an undirected graph G, every edge of G is either a tree edge or a back edge.

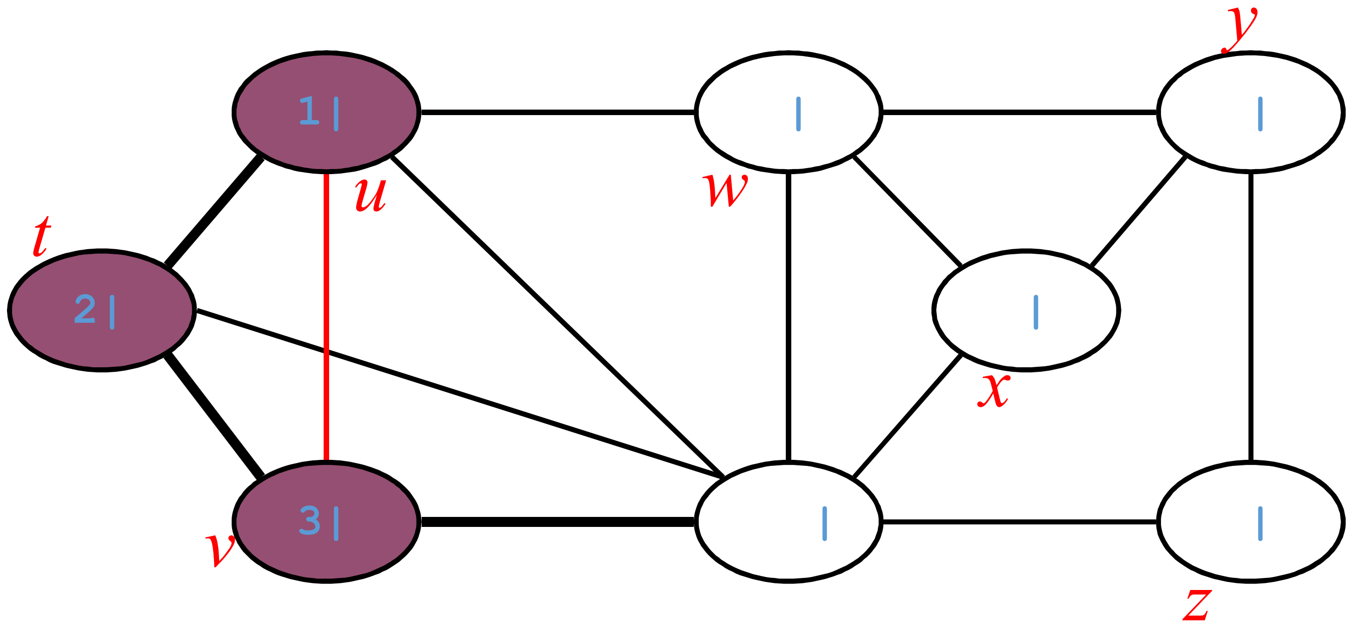
- Case A: The search explores edge (u, v) first in the direction from u to v:
  - then v is undiscovered (white) until that time ($u.d$)
  - Thus, (u, v) becomes a tree edge.



$u.d < v.d$

In a DFS of an undirected graph G, every edge of G is either a tree edge or a back edge.

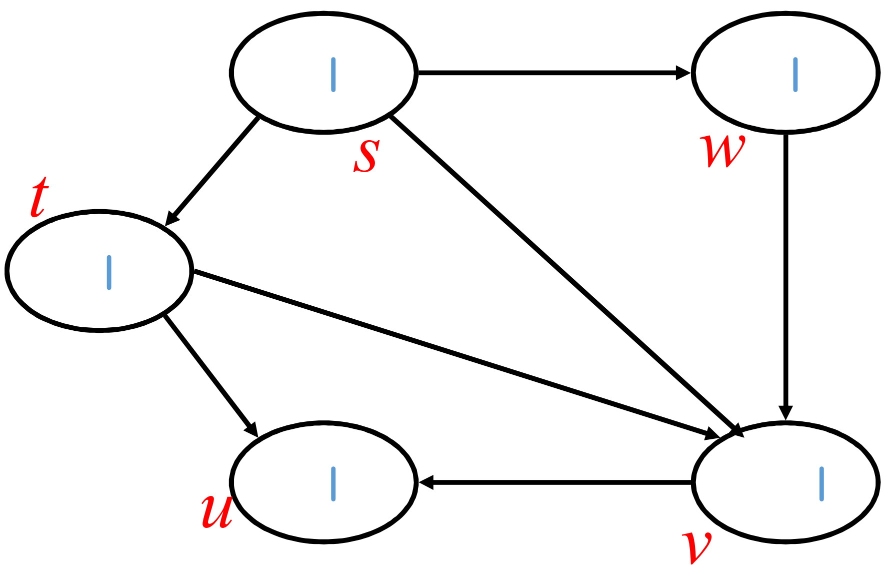
- Case B: The search explores (u, v) first in the direction from v to u:
  - u is still gray at the time the edge is first explored
  - then (u, v) is a back edge.



$u.d < v.d$

Topological sort

- Done on *directed acyclic graph* (*DAG*), $G = (V, E)$
  - makes a linear ordering of vertices: u appears before v if there is an edge (u, v)

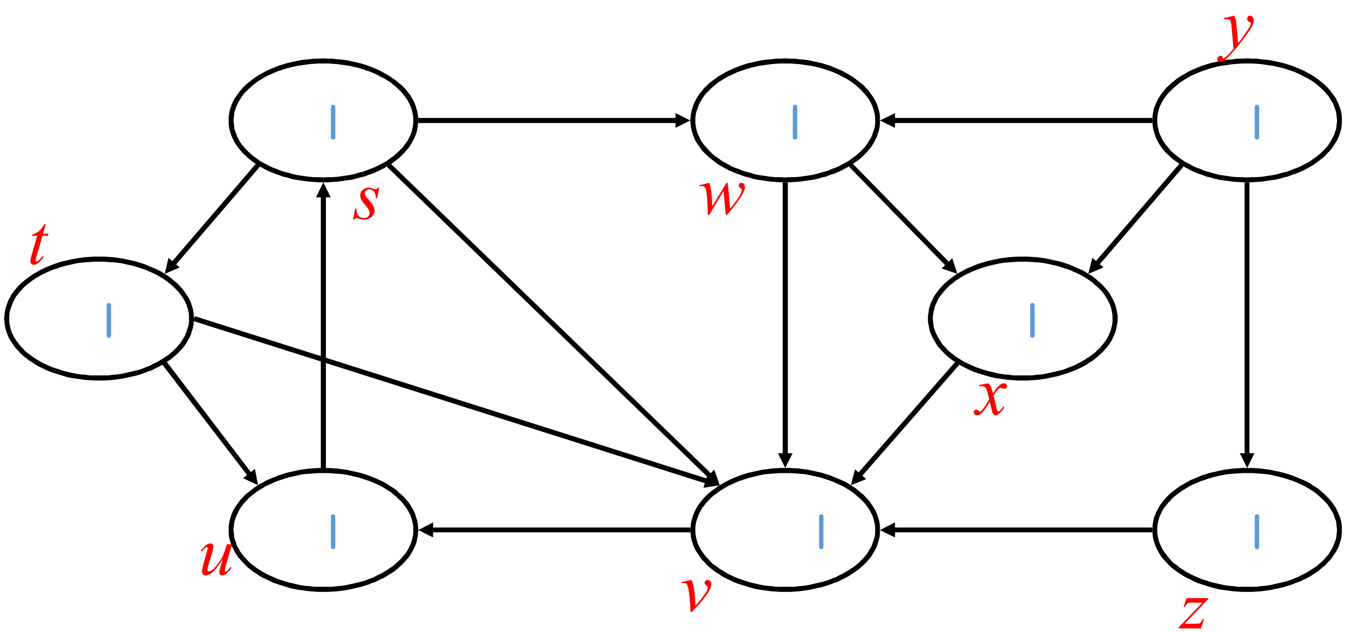


DAG

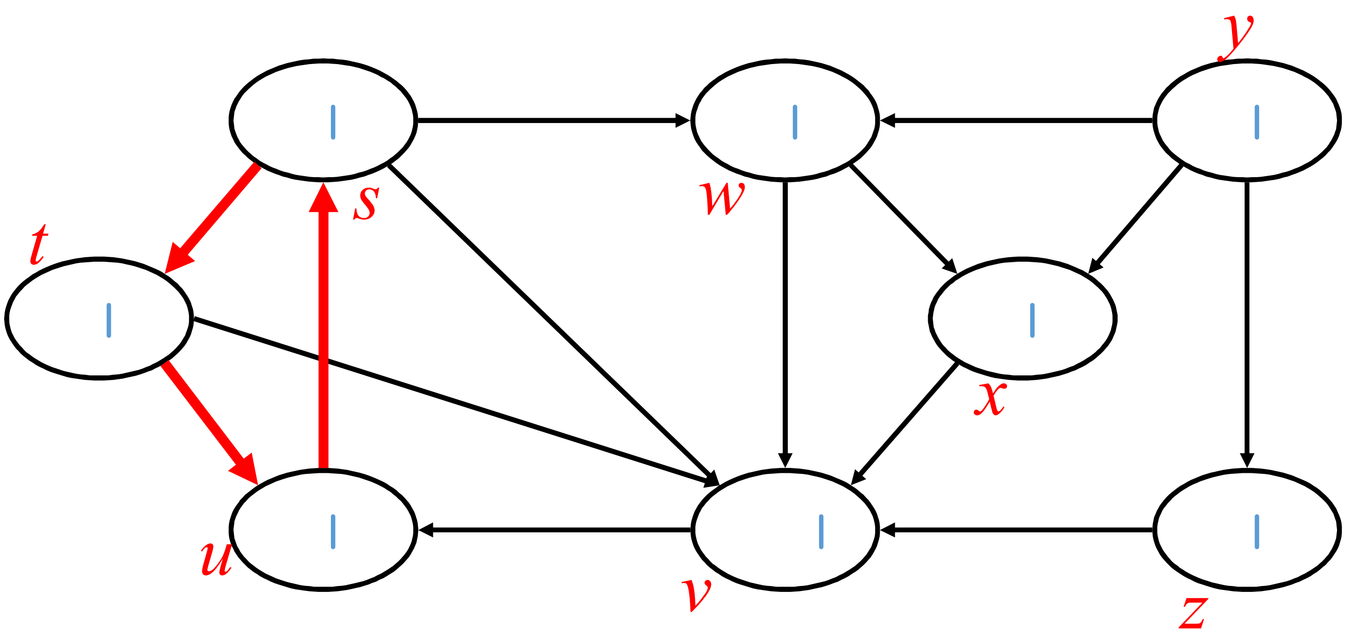
Linear ordering

$s,\ t,\ w,\ v,\ u$

IS it a DAG?

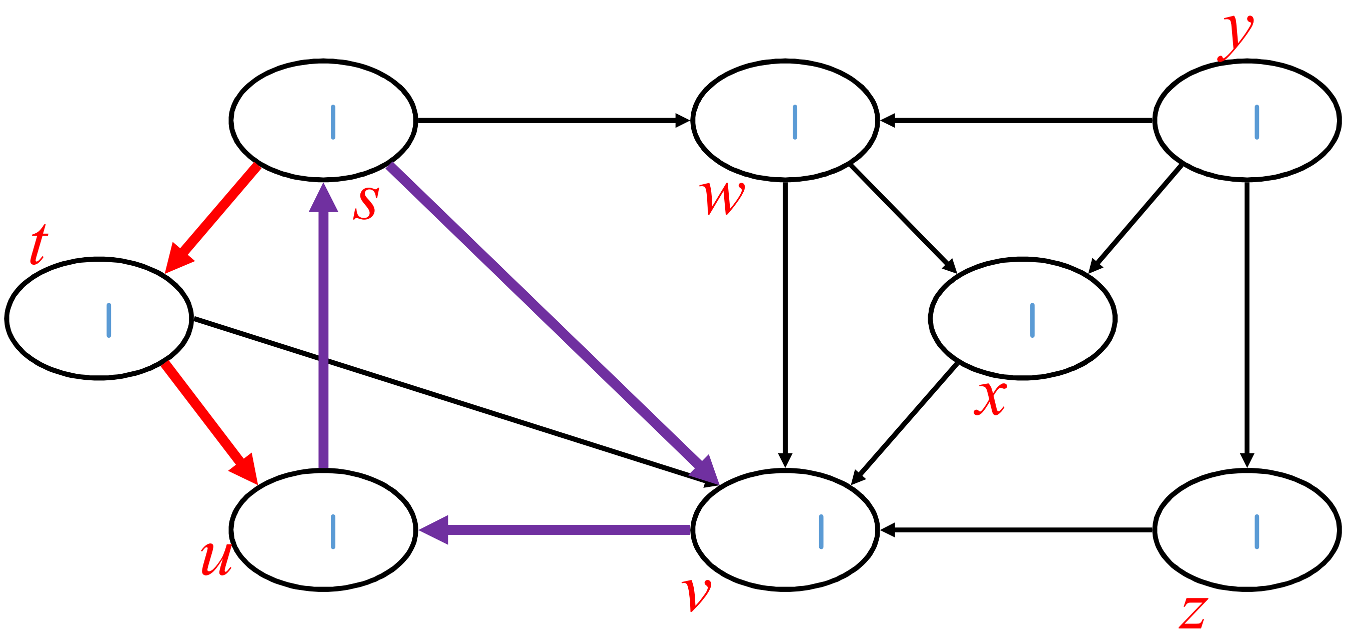


IS it a DAG?



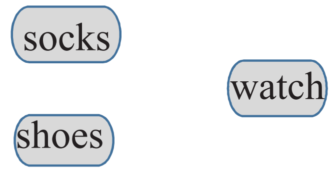
Cycle: s -> t ->, u ->s

IS it a DAG?



Cycle: s -> v ->, u ->s

Topological sort Example: dressing of a person



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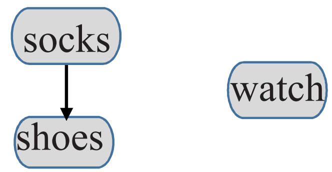

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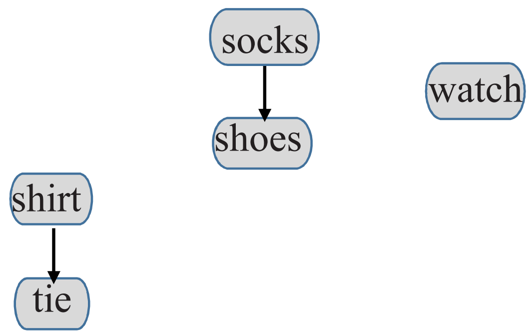
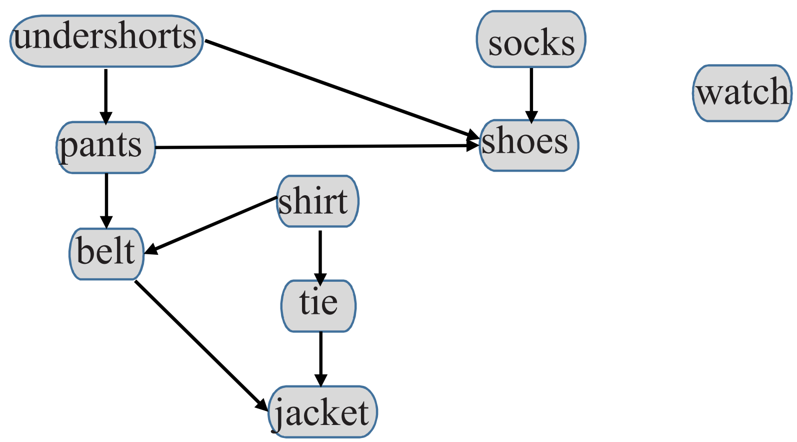
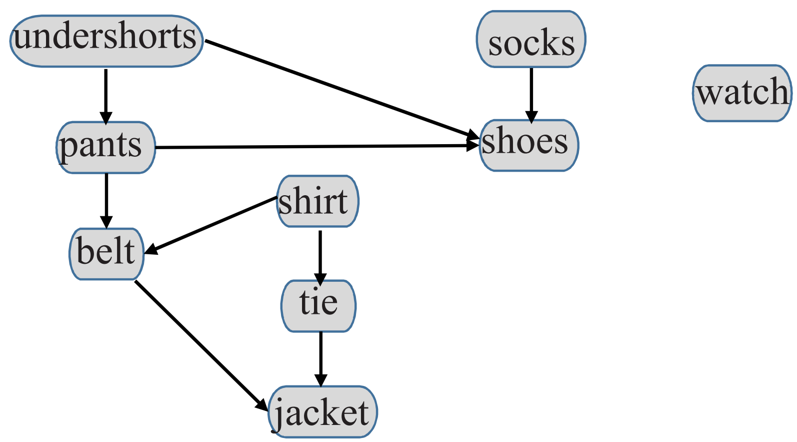


Topological sort Example: dressing of a person



DAG representation of dressing

Topological sort Example: dressing of a person



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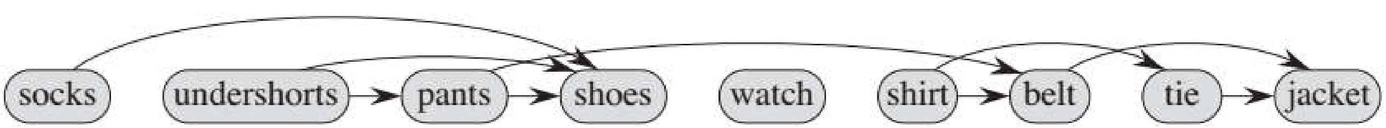
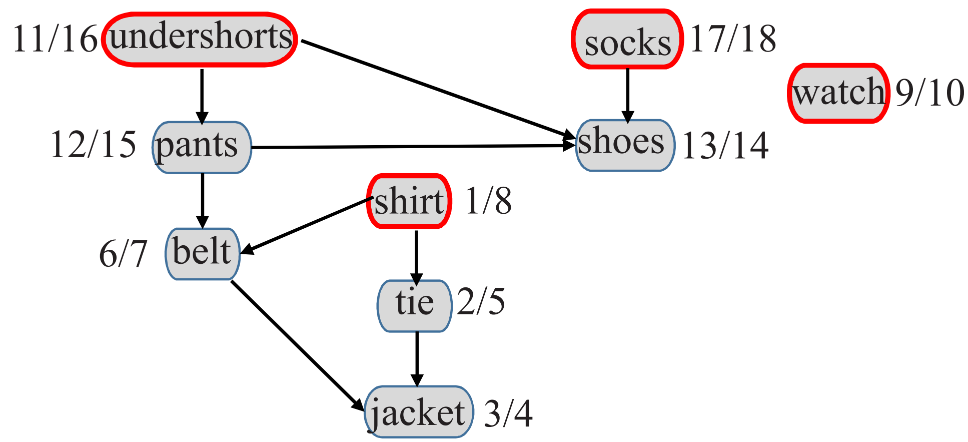


Topologically sorted actions

Topological sort Example: dressing of a person



Find finishing times by DFS of the DAG

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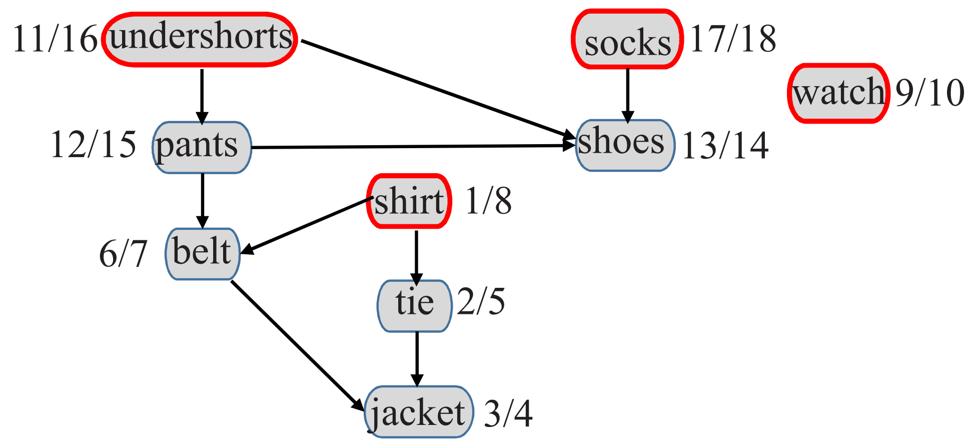

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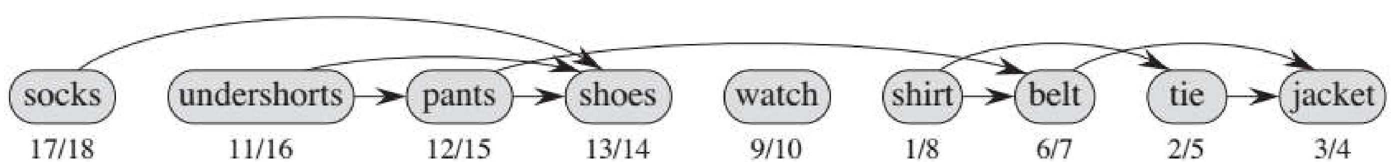

sorted by finishing times: use linked list

Topological sort Algorithm

TOPOLOGICAL-SORT(G)

call DFS(G) to compute finishing times $v.f$ for each vertex v
as each vertex is finished, insert it onto the front of a linked list
return the linked list of vertices

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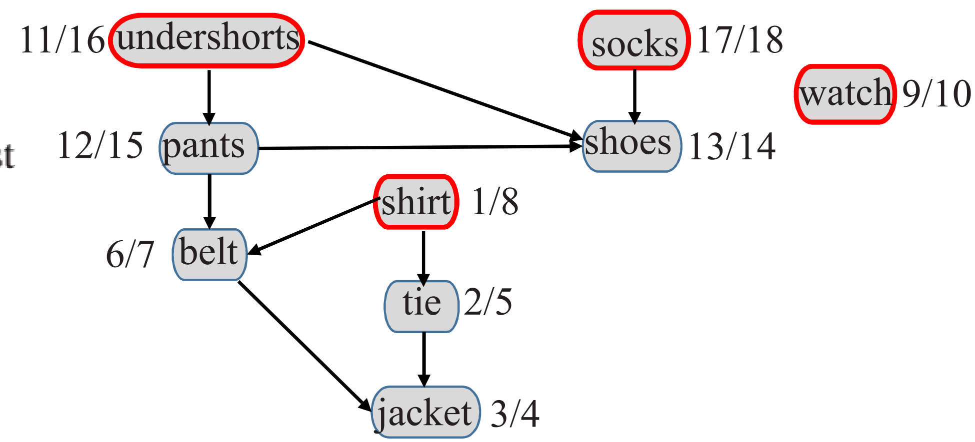

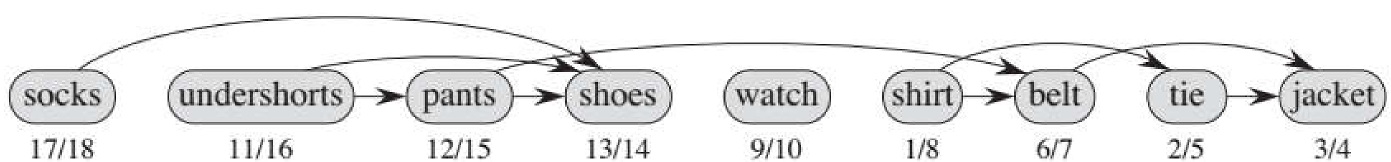

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Topological sort Algorithm: Complexity

TOPOLOGICAL-SORT(G)

call DFS(G) to compute finishing times $v.f$ for each vertex v $\longrightarrow O\,(V+E)$
as each vertex is finished, insert it onto the front of a linked list $\longrightarrow O\,(V)$
return the linked list of vertices

Lemma 22.11

A directed graph G is acyclic if and only if a depth-first search of G yields no back edges.

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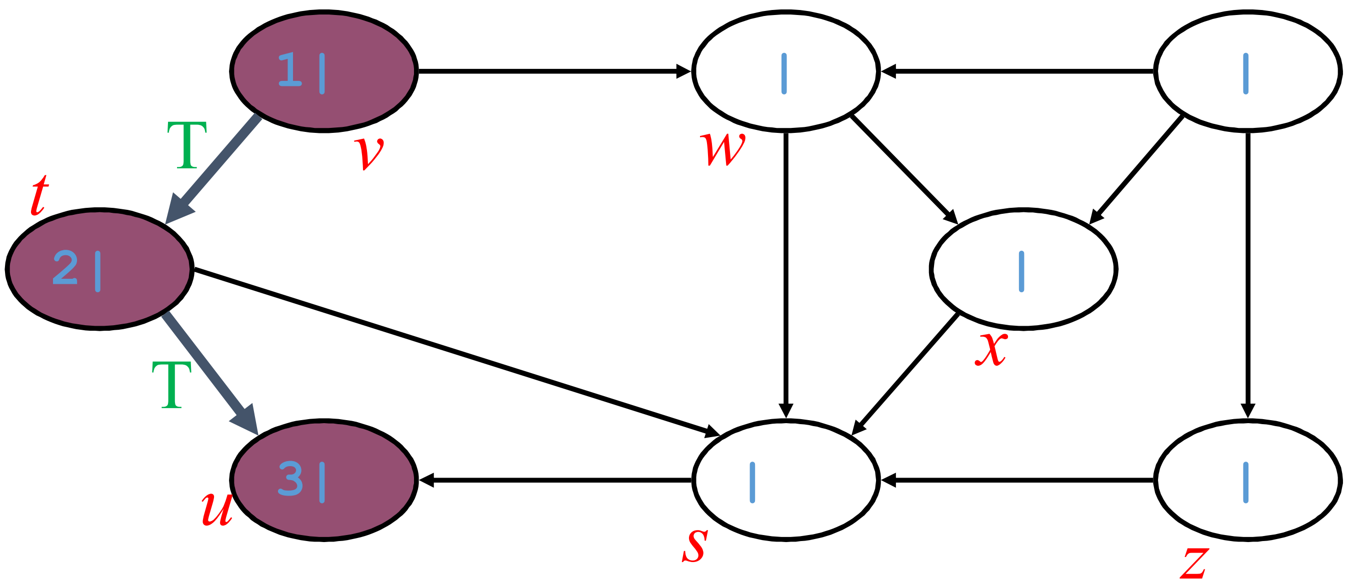
If P then Q:

Let G is a DAG. Prove that G has no back edge.



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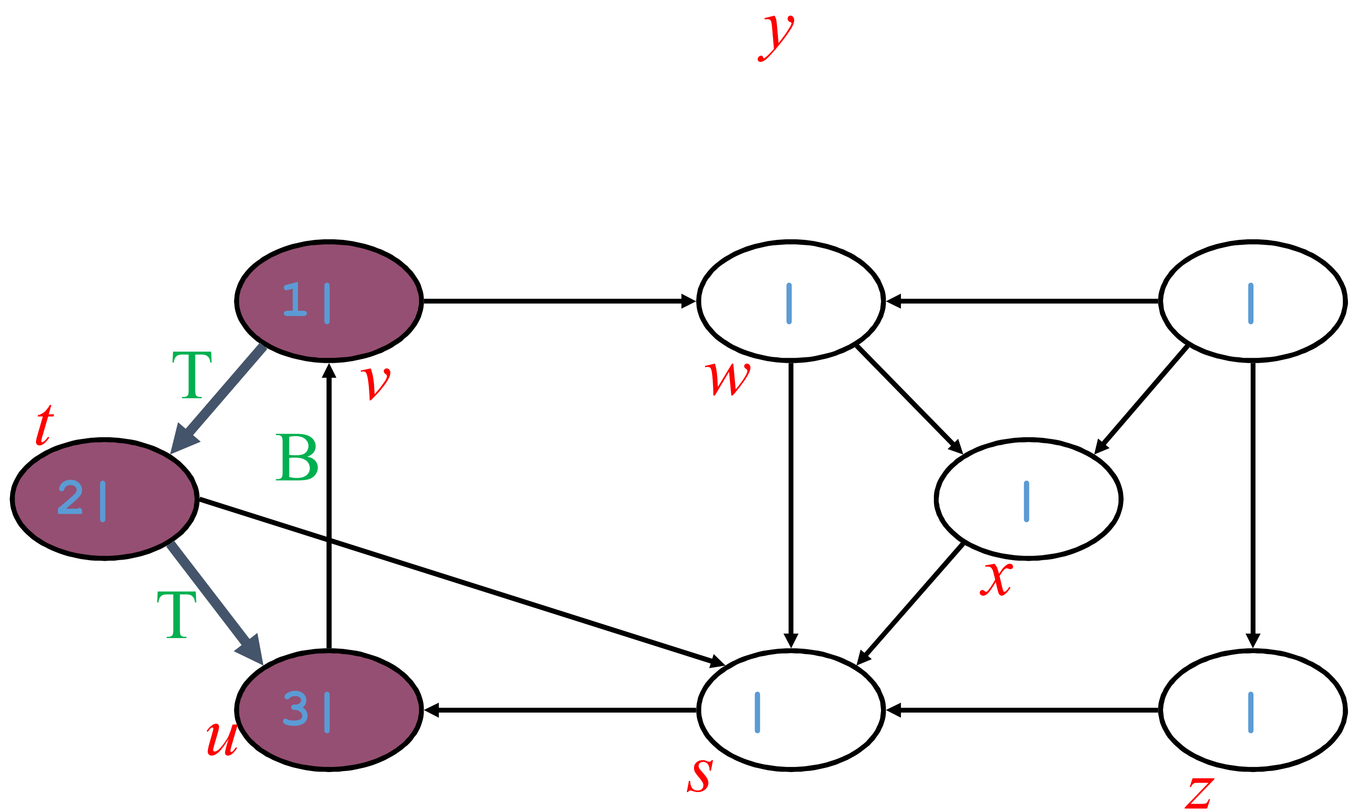

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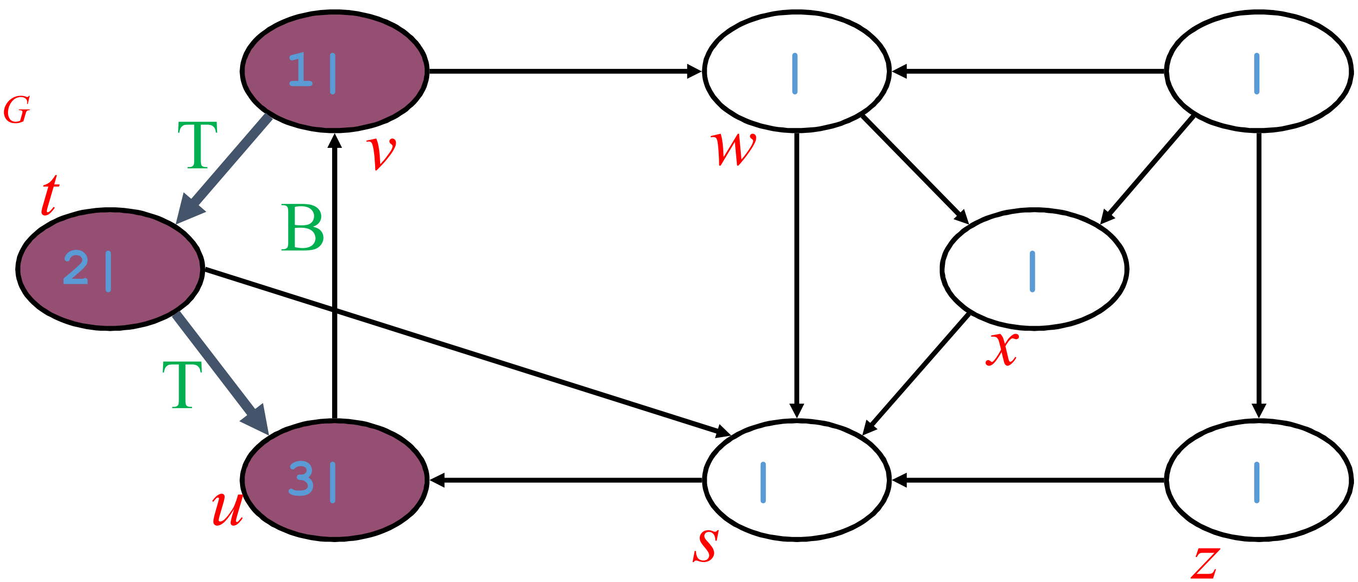
$\Rightarrow$ adding an edge (u, v) makes a cycle in G



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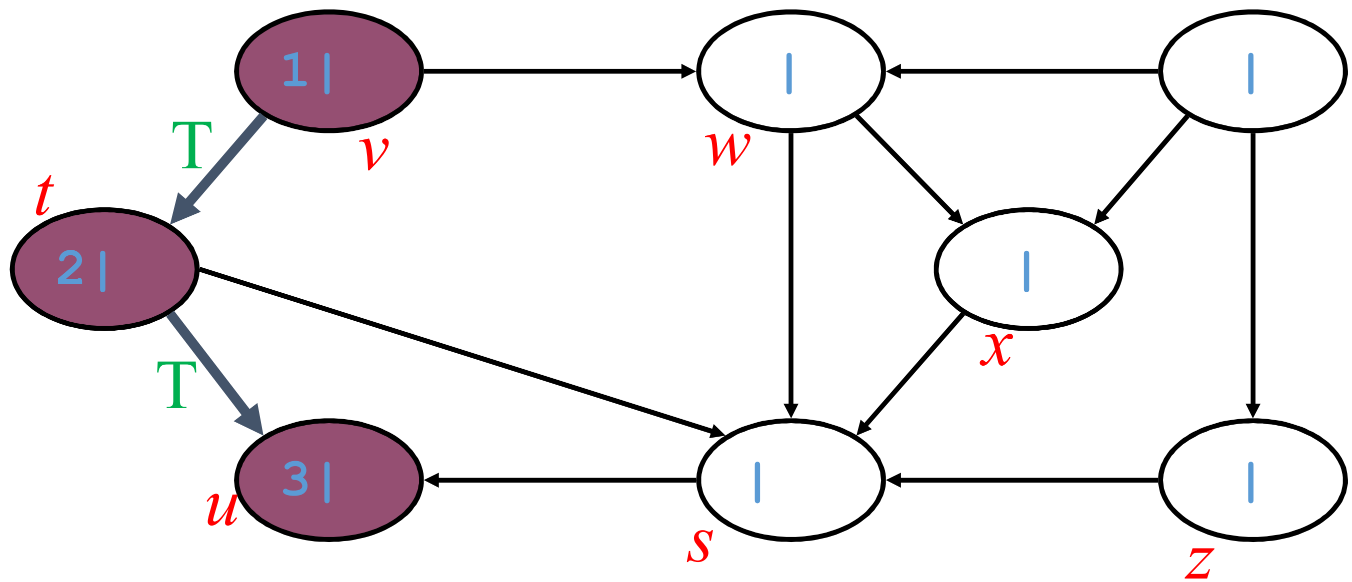

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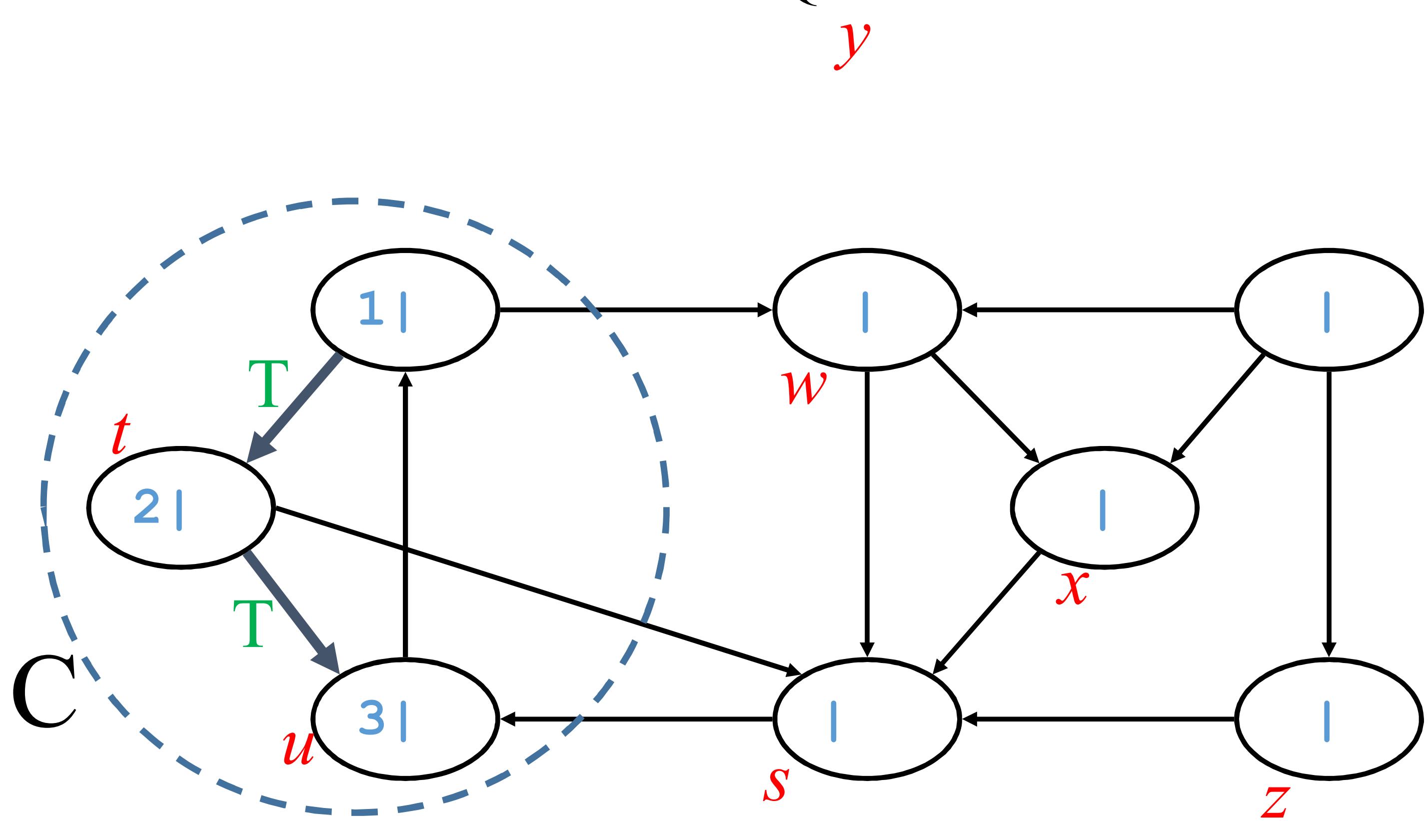

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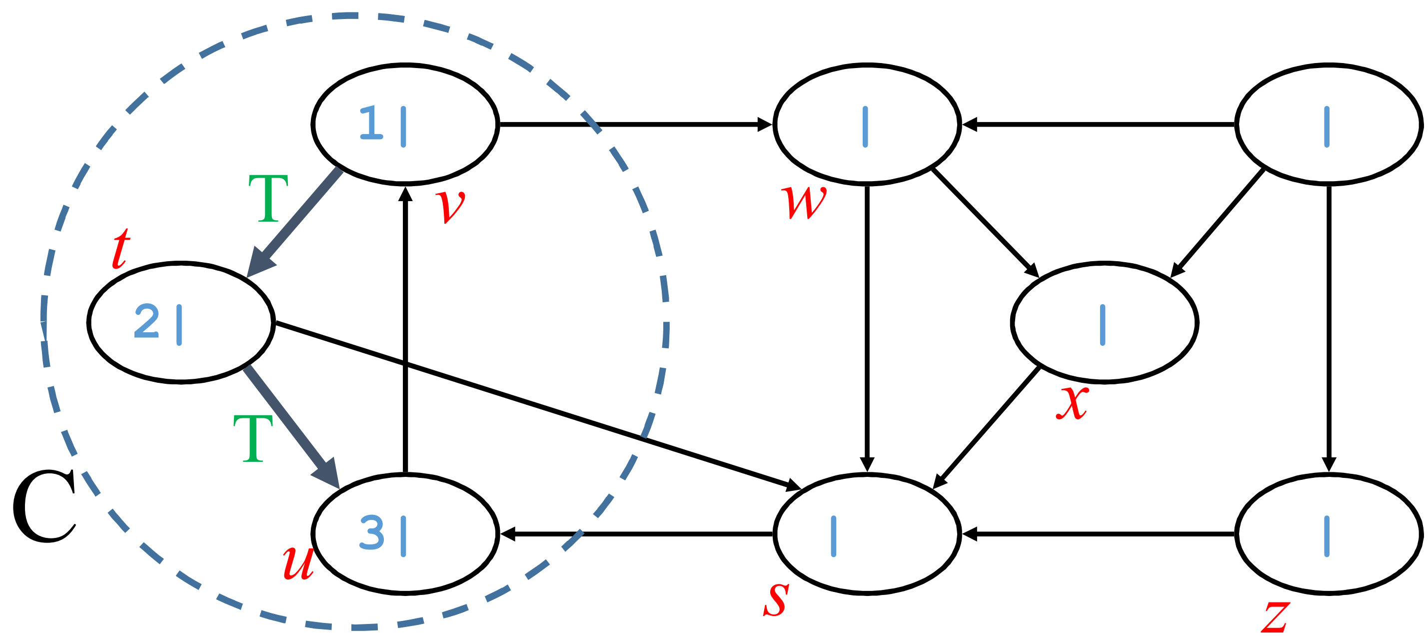

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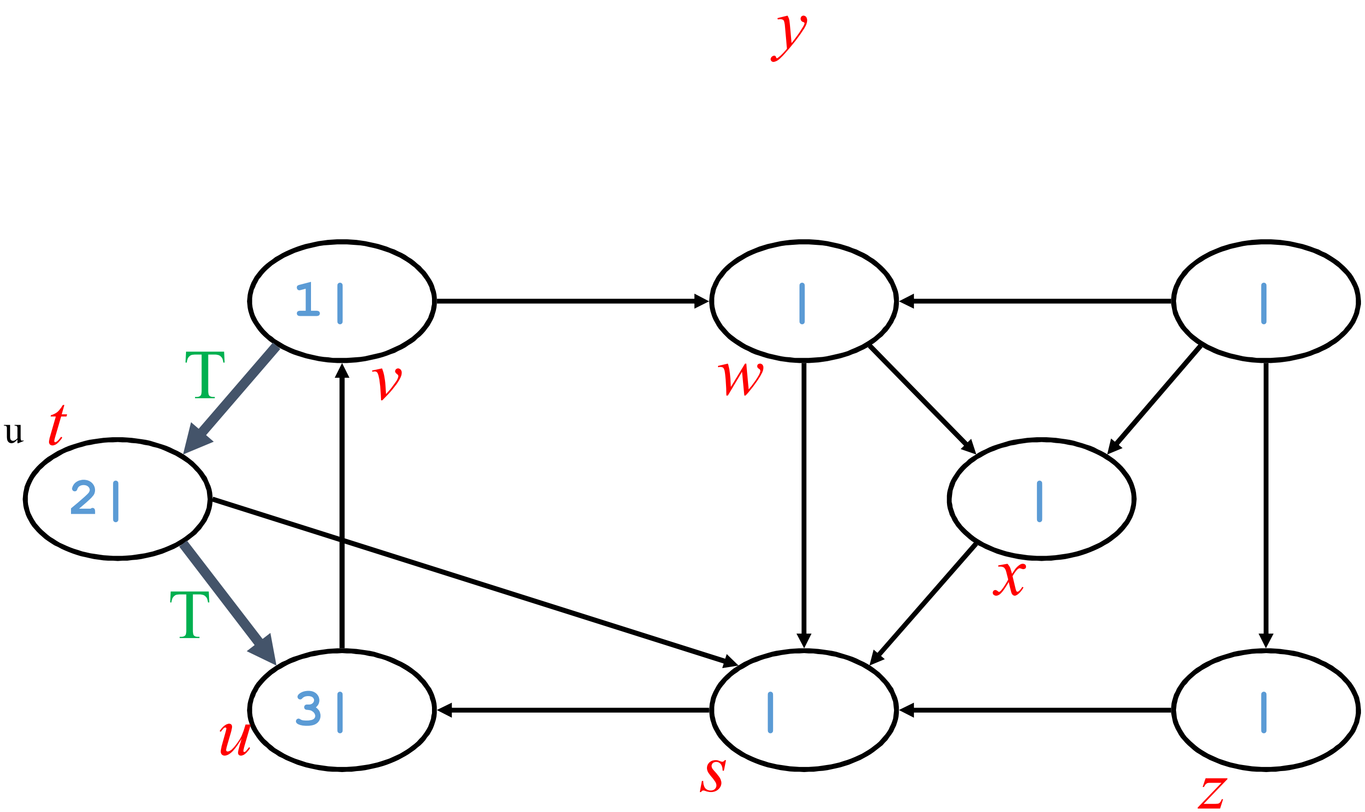
$\Rightarrow$ There is path of white vertices from v *to* u



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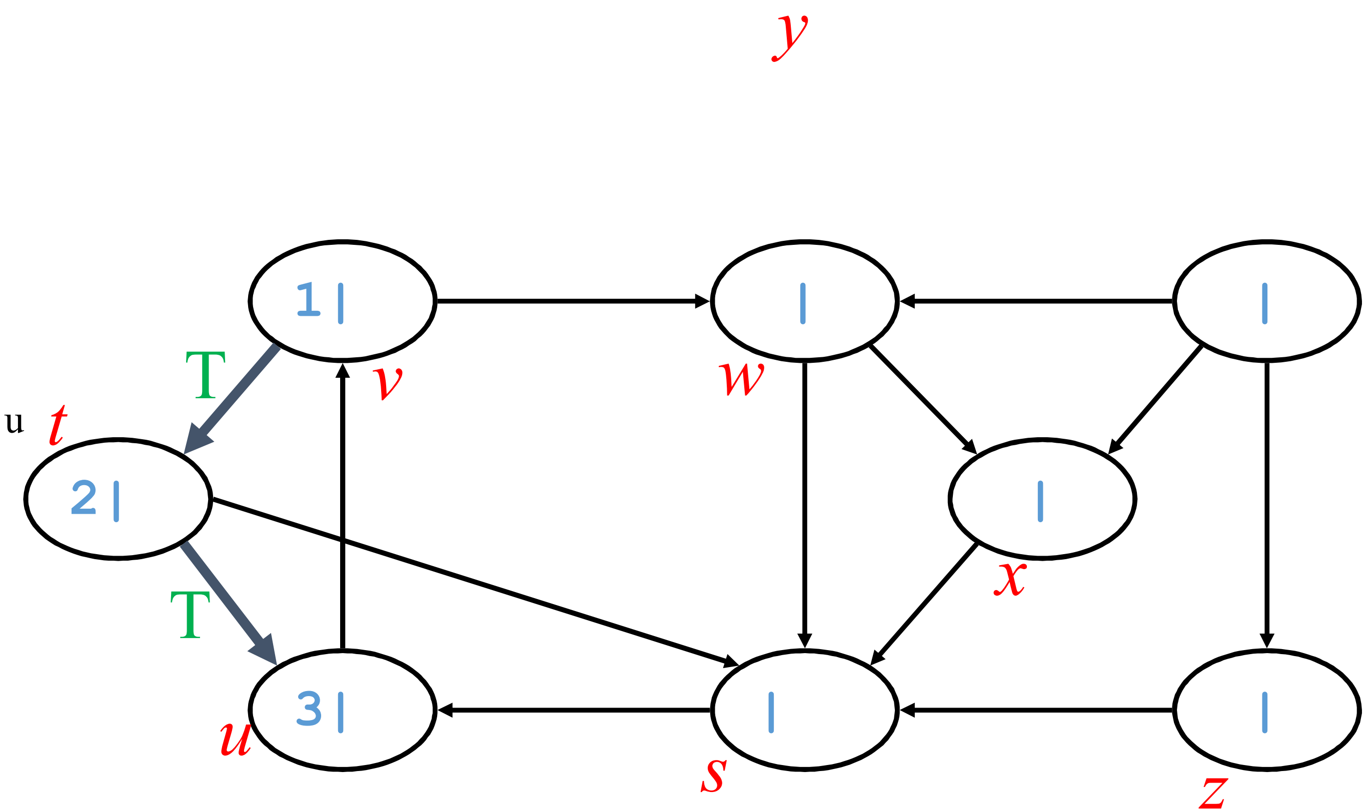

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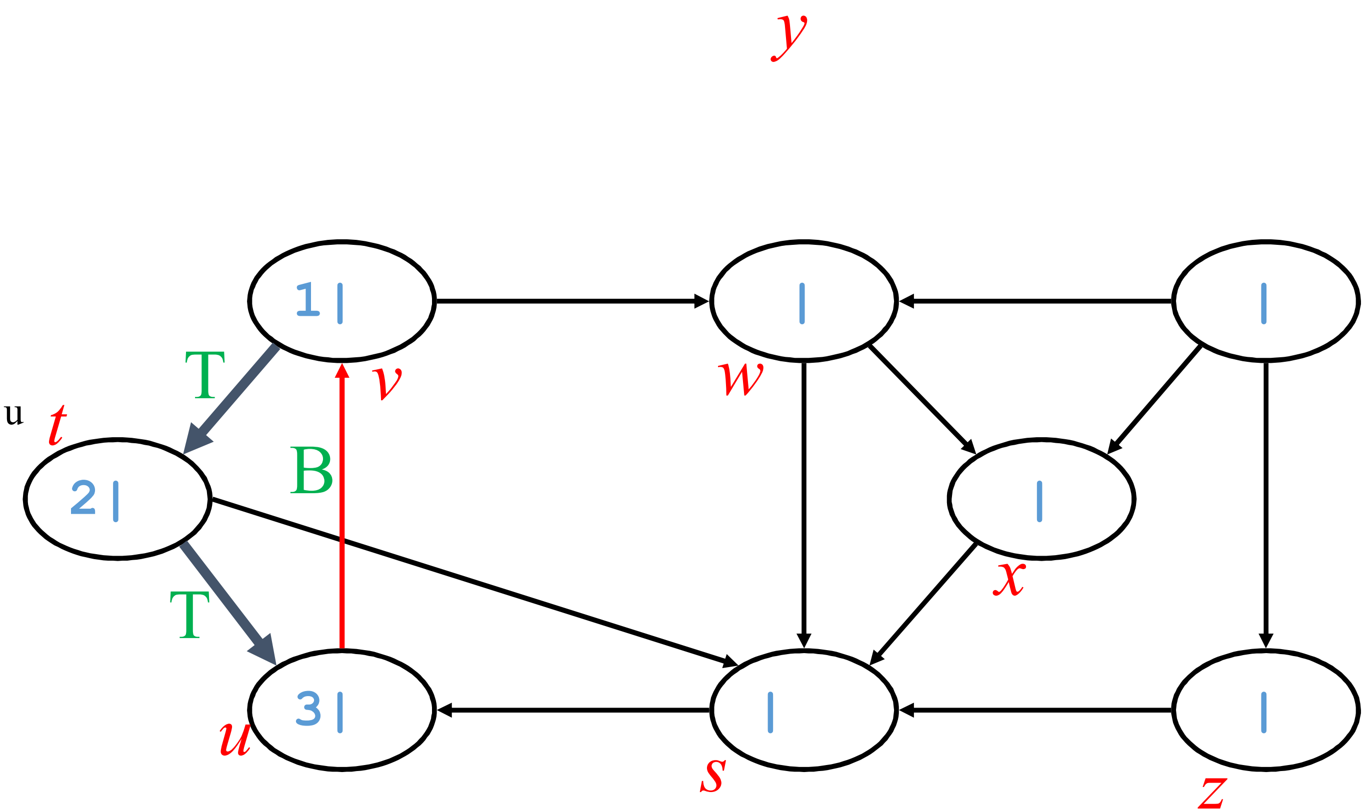

Theorem 22.12: Correctness of Topological sort

TOPOLOGICAL-SORT produces a topological sort of the directed acyclic graph provided as its input.

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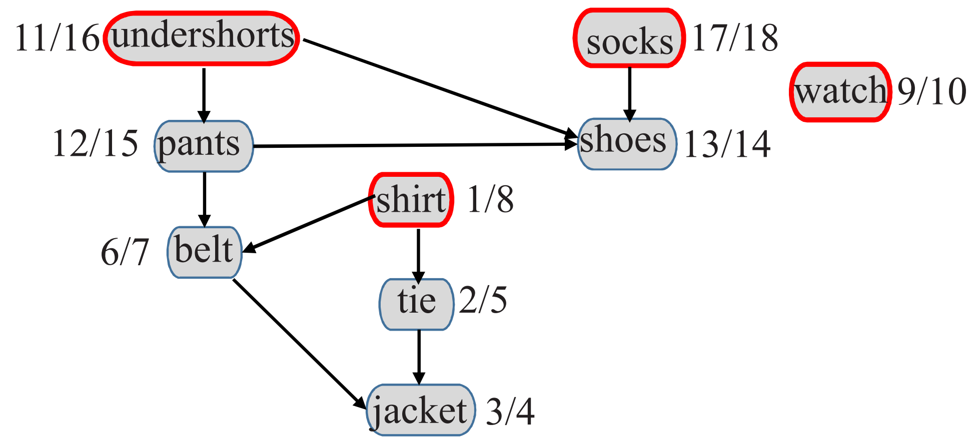

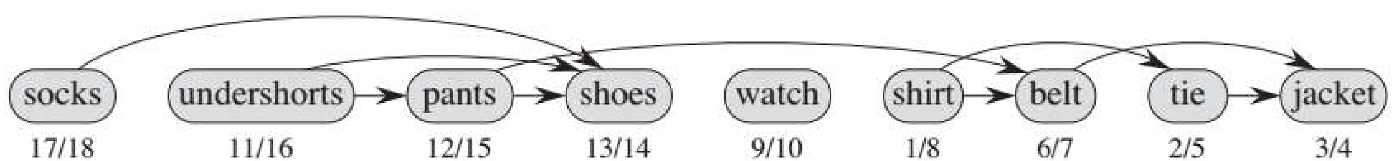

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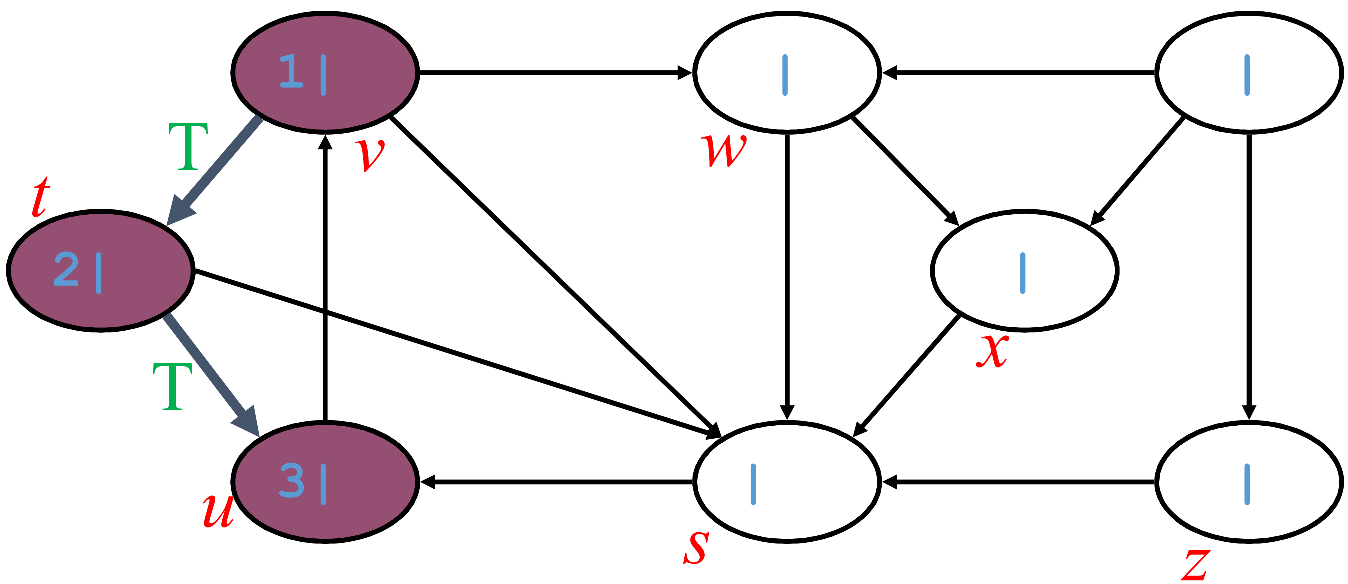
When (u, v) is explored, *v cannot be gray.*

If so, (u, v) will be a back edge. Contradiction!!



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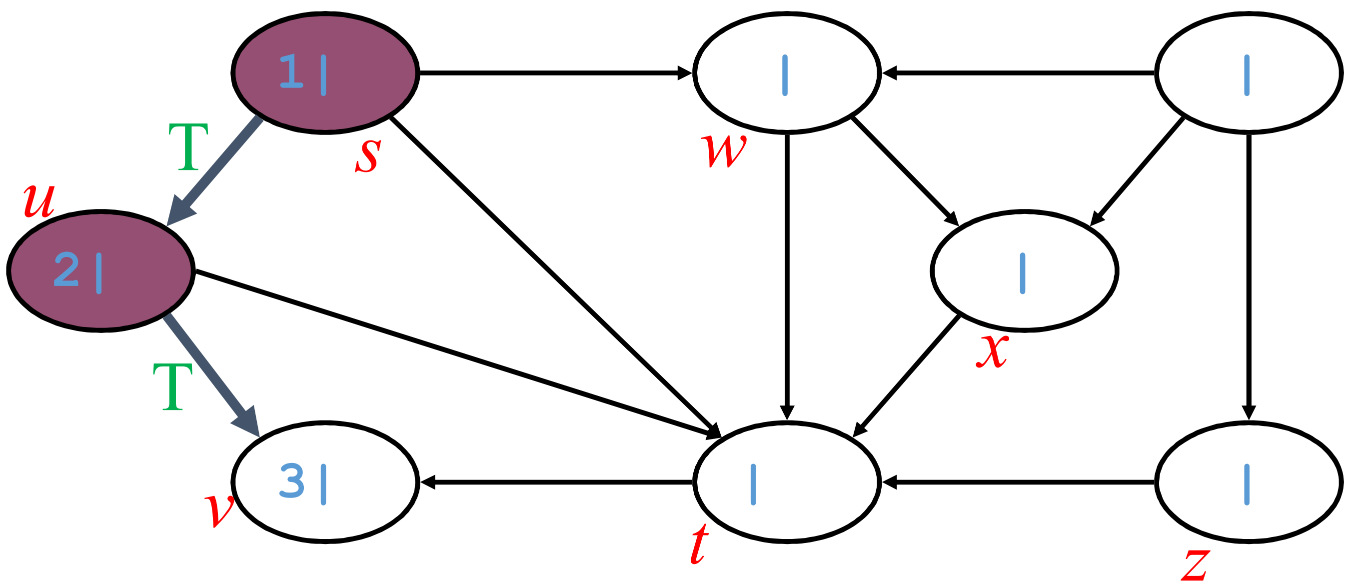

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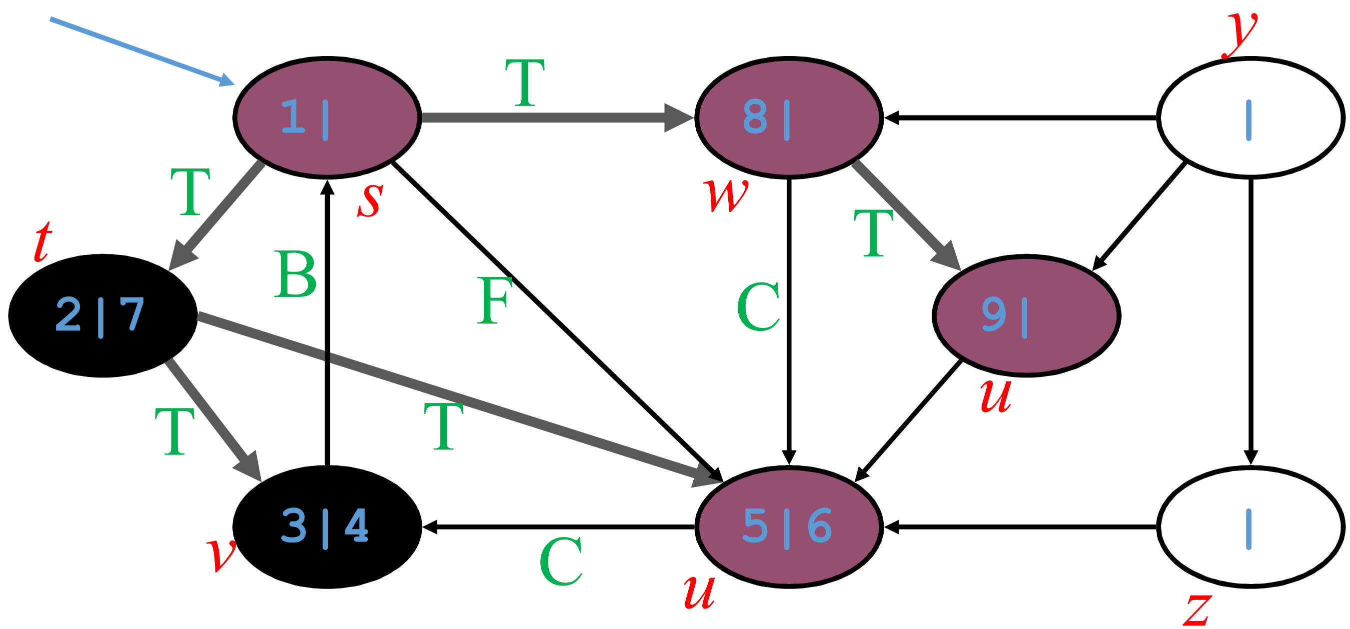

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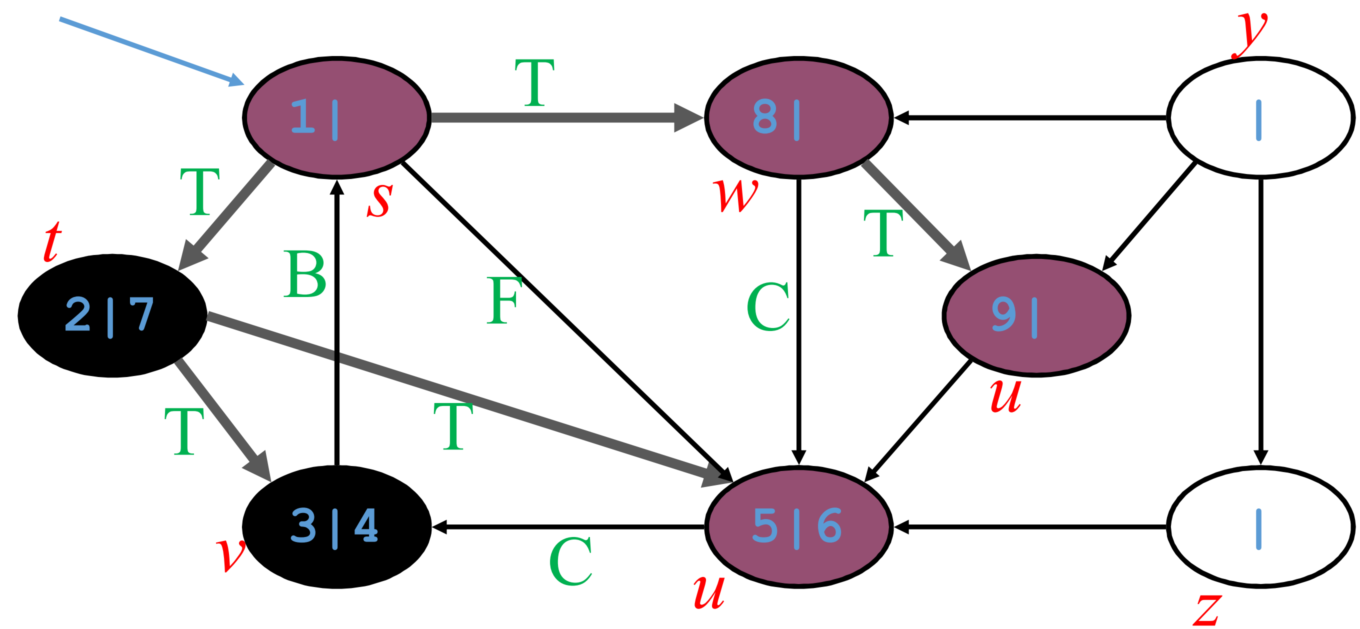

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