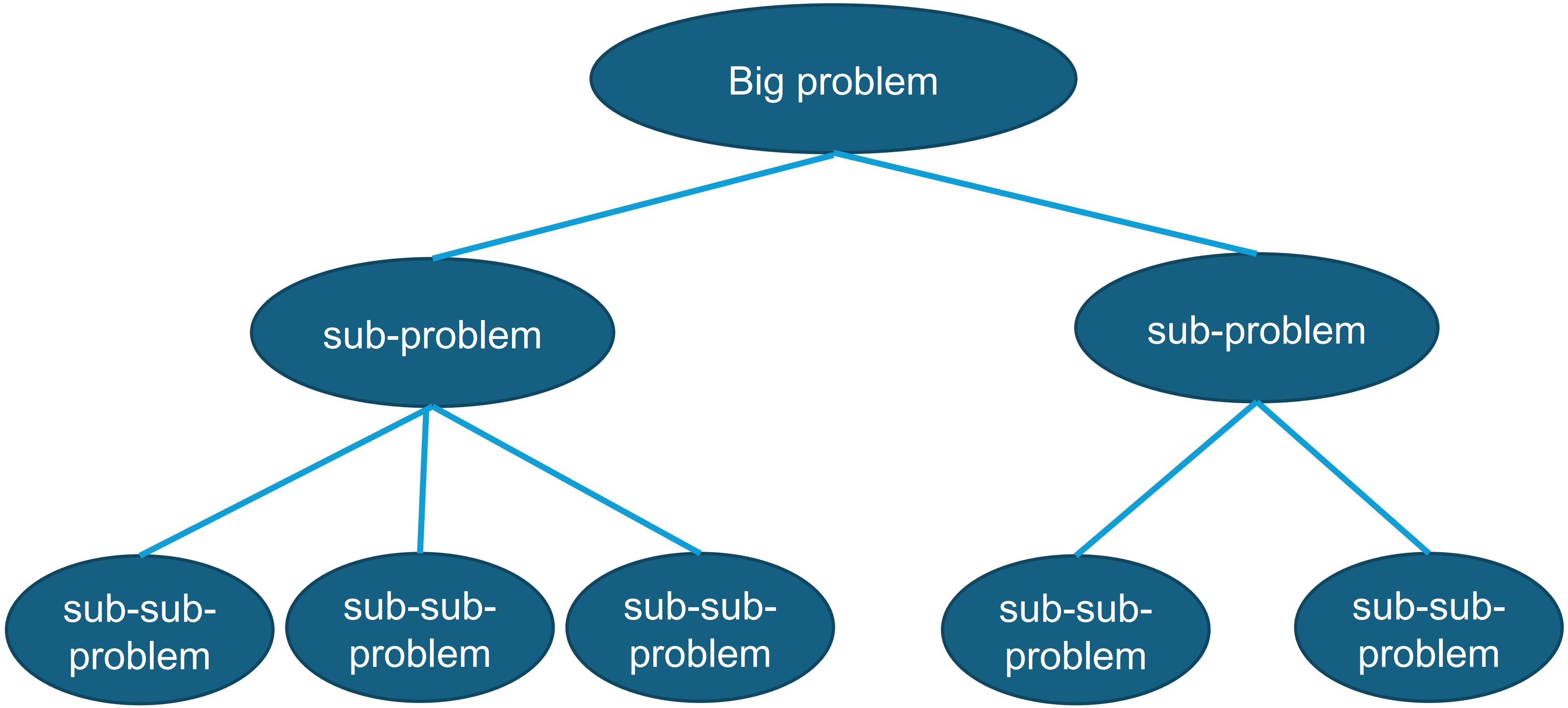
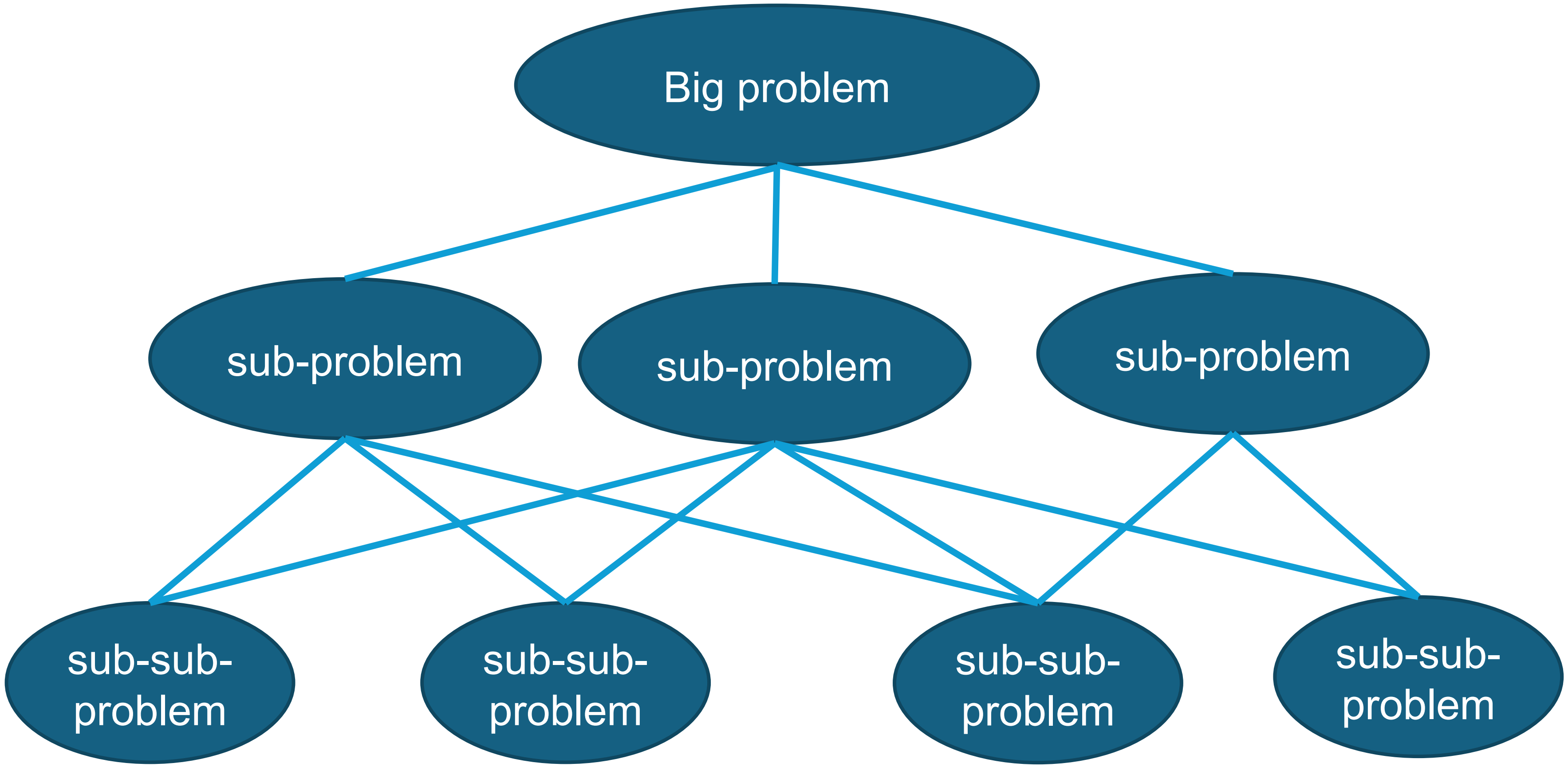
Greedy Algorithms

Divide and Conquer (Recap)



Dynamic Programming (Recap)



Greedy Algorithms

- Make greedy choices one-at-a-time.
- Never look back.
- Hope (prove) that the greedy choice leads to optimal solution.

Activity Selection Problem

- Activities are competing for an exclusive access to a common resource, i.e., conference room
  - A set $S = \{a_1, a_2, \dots, a_n\}$ of n activities requires the same resource
  - The resource can serve only one activity at a time
  - Each activity a_i has a start time s_i and a finish time f_i

- a_i and a_j are compatible if $s_i \geq f_j$ or $s_j \geq f_i$
  - Starts after the other finishes

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  - The resource can serve only one activity at a time
  - Each activity a_i has a start time s_i and a finish time f_i

- Goal:
  - Schedule maximum-size subset of mutually compatible activities

Activity Selection Problem

- Assumption: The activities are sorted by their finish times

$$f_1 \le f_2 \le f_3 \le \ \ldots \le f_n$$

- If not, sort with $O(n \ lgn)$

Activity Selection Problem

- Does this have optimal substructure property?

- Let S_{ij} be the set of activities that
  - Start after activity a_i finishes and that finish before activity a_j starts.

- If $a_k \in S_{ij}$ is included in the optimal solution,
  - Solve activity selection problem on S_{ik} and S_{kj}

Activity Selection Problem

- Does this have optimal substructure property?

- If A_{ij} is the maximum-size compatible subset of activities in S_{ij}

$$A_{ij} = A_{ik} \cup \{a_k\} \cup A_{kj}$$
$$|A_{ij}| = |A_{ik}| + 1 + |A_{kj}|$$

- Let, $c[i, j]$ denote the size of an optimal solution for the set S_{ij}

$$c[i,j] = \begin{cases} 0 & \text{if } S_{ij} = \emptyset \\ \max\{c[i,k] + c[k,j] + 1 : a_k \in S_{ij}\} & \text{if } S_{ij} \neq \emptyset \end{cases}$$

Activity Selection Problem

- Does this have optimal substructure property?
  - Yes!!!
- Can also use cut-and-paste argument for proof
- DP??

Activity Selection Problem

- Let’s make a greedy choice
- Choose the activity that leaves the resource available for as many other activities as possible
- Any optimal solution has an activity that finishes first
- Here, choose the activity in S with the earliest finish time,
  - Leave the resource available for maximum number of other activities

Activity Selection Problem

- Let’s make a greedy choice
  - Choose the activity with earliest finish time

- Once a greedy choice is made,
  - There is only one remaining subproblem to solve
  - Solve for the activities that starts after the first-choice finishes

Activity Selection Problem

- Let $S_k = \{a_i \in S; s_i \geq f_k\}$

- Once we have picked a_k, all we need to solve is S_k
  - Optimal Substructure Property

Activity Selection Problem

- The greedy choice
  - Choose the activity with earliest finish time
  - Is it optimal?

Theorem 15.1
Consider any nonempty subproblem S_k, and let a_m be an activity in S_k with the earliest finish time. Then a_m is included in some maximum-size subset of mutually compatible activities of S_k.

Activity Selection Problem

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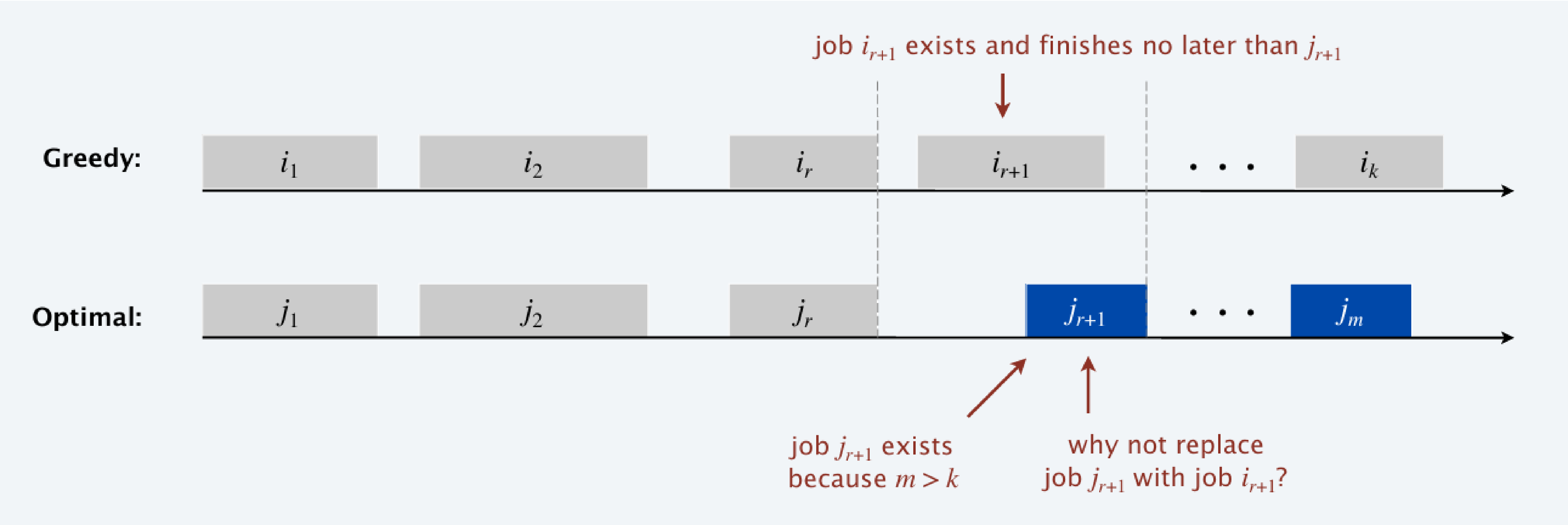
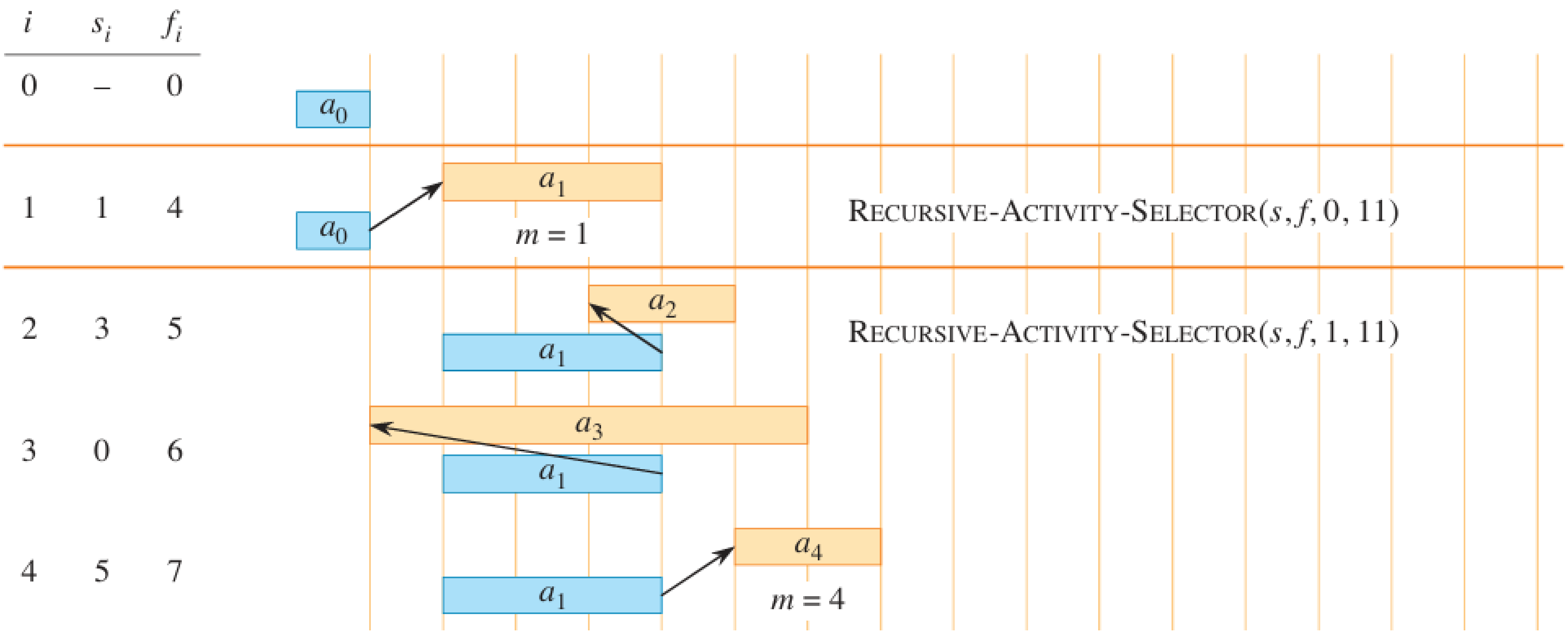


Activity Selection Problem

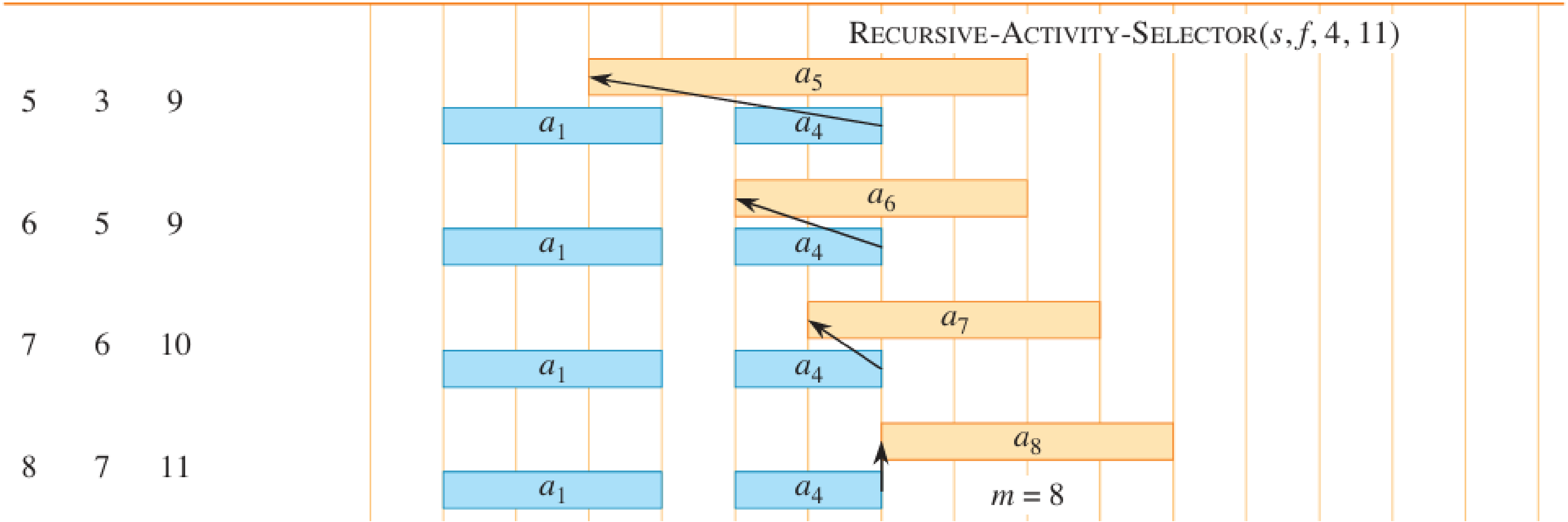
GREEDY-ACTIVITY-SELECTOR(s, f, n)

```
A = {a_1}
k = 1
for m = 2 to n
        if s[m] ≥ f[k]              // is a_m in S_k?
                A = A ∪ {a_m}       // yes, so choose it
                k = m               // and continue from there
return A
```

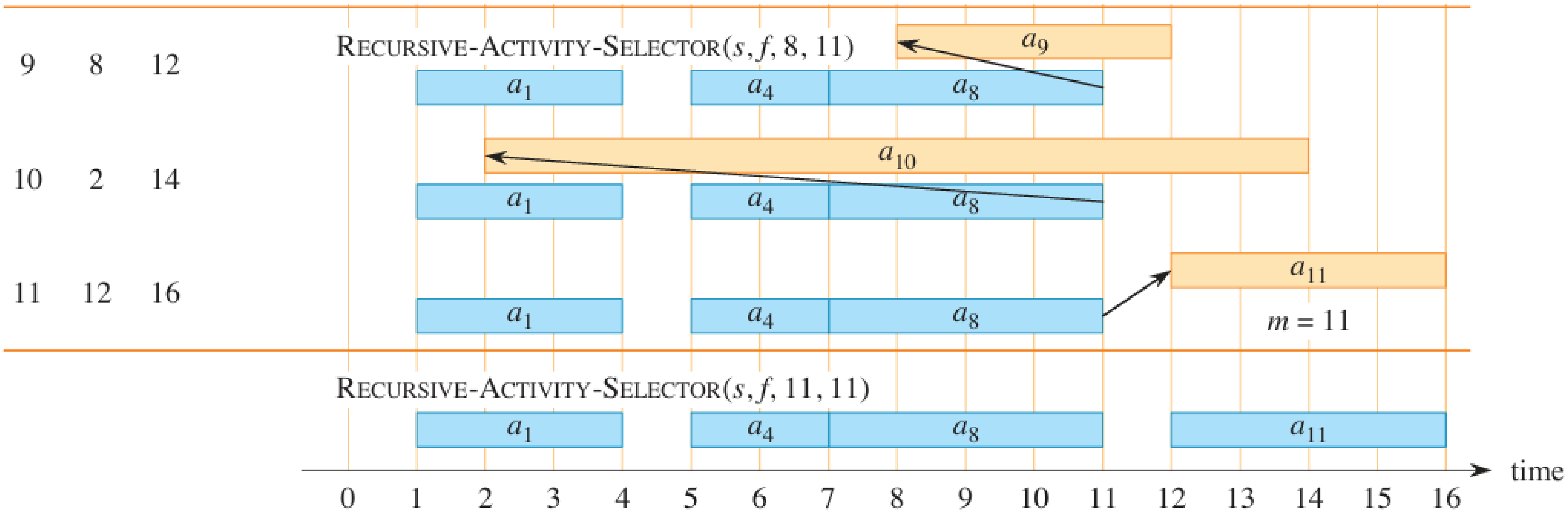
Activity Selection Problem



Activity Selection Problem



Activity Selection Problem



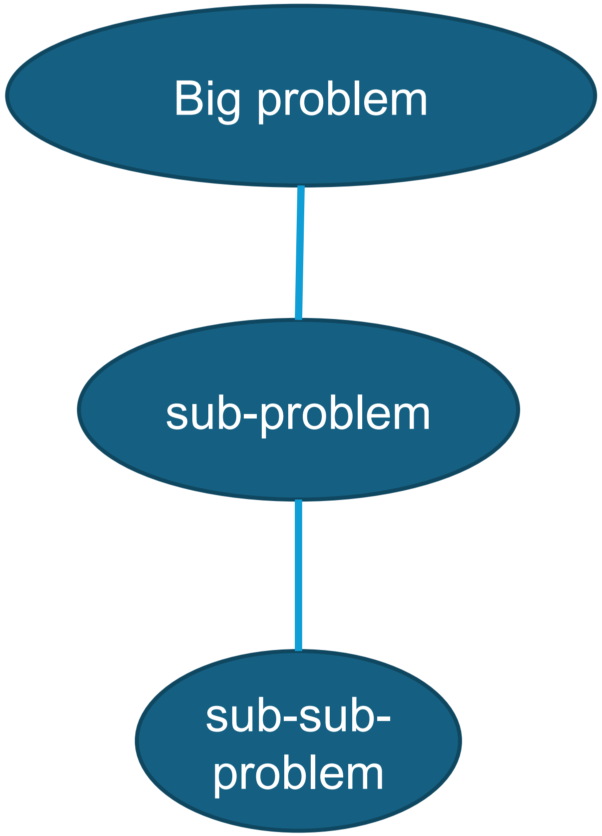
Elements of Greedy Strategy

- Optimal Substructure property

- Greedy Choice Property
  - Assemble a globally optimal solution by making locally optimal (greedy) choices.

  - Make the choice that looks best in the current problem, without considering results from subproblems.

Elements of Greedy Strategy



Knapsack Problems

- 0-1 Knapsack Problem
  - Choice at each step
  - The choice usually depends on the solutions to subproblems.

- Fractional Knapsack Problem
  - Pick the item with maximum $\mathrm{v_i}/w_i$ ratio
  - Repeat as long as
    - The supply is not exhausted
    - The thief can still carry more,

Knapsack Problems

- W = 50
- w = [10, 20, 30], v = [60, 100, 120]

- 0-1 Knapsack Problem
  - Pick item 2 and 3

- Fractional Knapsack Problem
  - Pick item 1, 2 and portion of 3

Single Source Shortest Path

- Given
  - A weighted and directed graph, $G = (V, E)$ What about unweighted graphs?
  - A source, s

- Goal: Find out the shortest path weight from the source to a given node / other nodes.

Single Source Shortest Path

- Subpaths of shortest paths are shortest paths
- Proof:
  - Decompose the shortest path into smaller sub-paths

$$\mathrm{p} = v_1 \sim v_i \sim v_j \sim v_k$$

  - The subpaths are p_{0i}, p_{ij}, p_{jk}
  - Assume there exists p_{ij}' such that $w(p_{ij}') < w(p_{ij})$
  - Then replacing p_{ij} with p_{ij}' gives a shorter path

- Optimal Substructure Property

Single Source Shortest Path

- Negative-weight Edges
  - Graph may contain negative weight cycles reachable from source s
  - Shortest path problem is not well-defined
- Cycle
  - Shortest path cannot contain cycle
  - Just dropping the cycle gives a lower cost path

Dijkstra's Algorithm

- Given
  - A weighted and directed graph, $G = (V, E)$
  - A source, s
  - Non-negative weights on edges, $w(u, v) \geq 0$

- Goal: Find out the shortest path weight from the source to a given node / other nodes.

Dijkstra’s Algorithm

- A path weight of a, $p = < v_1, v_2, \ldots, v_k >$, is

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

- Shortest path weight is defined as,

$$\delta(u, v) = \begin{cases} \min\{w(p) : u \overset{p}{\leadsto} v\} & \text{if there is a path from } u \text{ to } v \\ \infty & \text{otherwise .} \end{cases}$$

Dijkstra’s Algorithm

- Relaxation
  - $x.d$ best known estimate of the shortest distance from s to x

v.d = min(v.d , u.d + w(u,v))

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$\textsc{Relax}(u, v, w)$

```
if v.d > u.d + w(u,v)
    v.d = u.d + w(u,v)
    v.π = u
```

- Also keep track of the predecessor
  - The node immediately before v on the shortest path from s to v

Dijkstra’s Algorithm

- Initialization

```
INITIALIZE-SINGLE-SOURCE(G, s)
for each vertex v ∈ G.V
    v.d = ∞
    v.π = NIL
s.d = 0
```

Dijkstra’s Algorithm

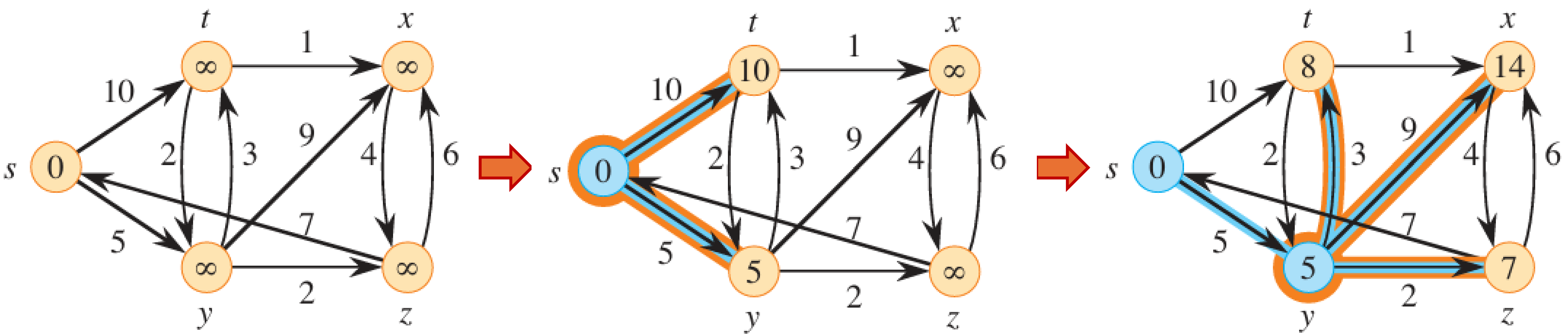
- Remember BFS?
  - At each step, pick the next node from discovered nodes in the queue
- Dijkstra’s Algorithm
  - Similar to BFS
  - Except that at each step, pick the next node with the minimum estimated shortest-path weight
  - Replace the FIFO queue of BFS with a minimum priority-queue

The greedy choice!!!!

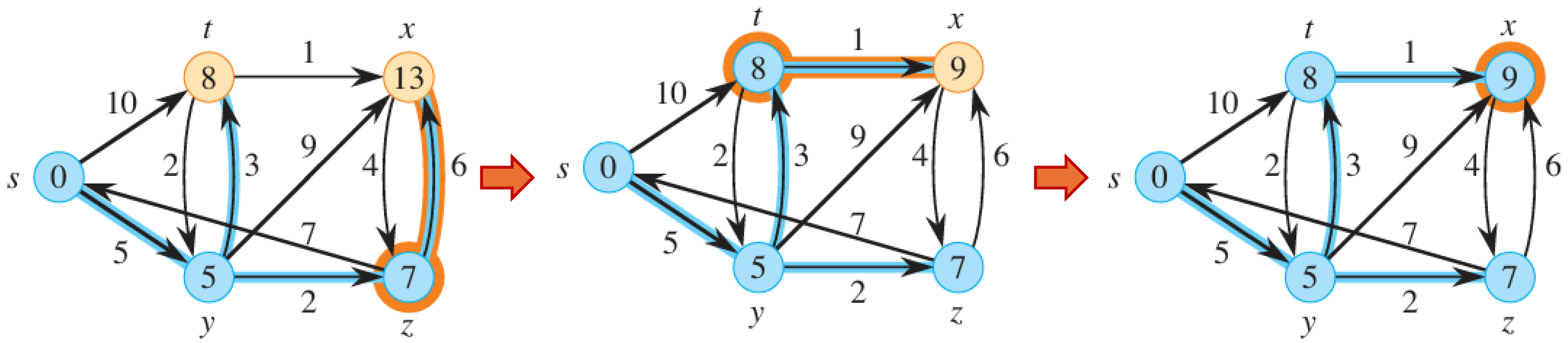
Dijkstra's Algorithm

```
DIJKSTRA(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
S = ∅
Q = ∅
for each vertex u ∈ G.V
    INSERT(Q, u)
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    S = S ∪ {u}
    for each vertex v in G.Adj[u]
        RELAX(u, v, w)
        if the call of RELAX decreased v.d
            DECREASE-KEY(Q, v, v.d)
```

Dijkstra’s Algorithm



Dijkstra’s Algorithm



Dijkstra’s Algorithm

- Correctness of Dijkstra’s Algorithm
  - S is the set of visited nodes
  - The algorithm terminates when $S = V$

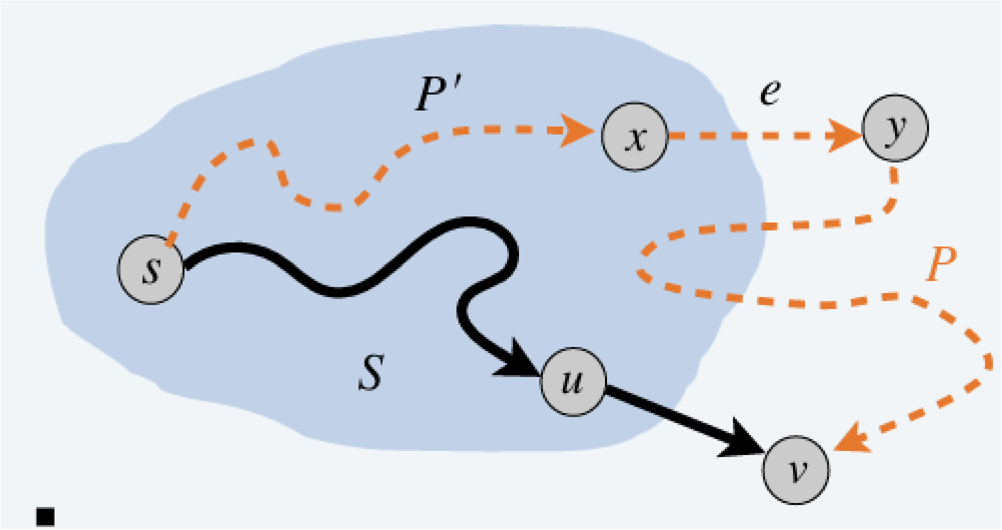
  - Inductive Hypothesis:
    - At the start of each iteration, $v = \delta(s, v)$ for all $v \in S$

Dijkstra’s Algorithm

- Correctness of Dijkstra’s Algorithm
  - At some iteration, v is extracted from the priority queue
  - y first node not in S on the shortest path P to u
  - x predecessor of y on P

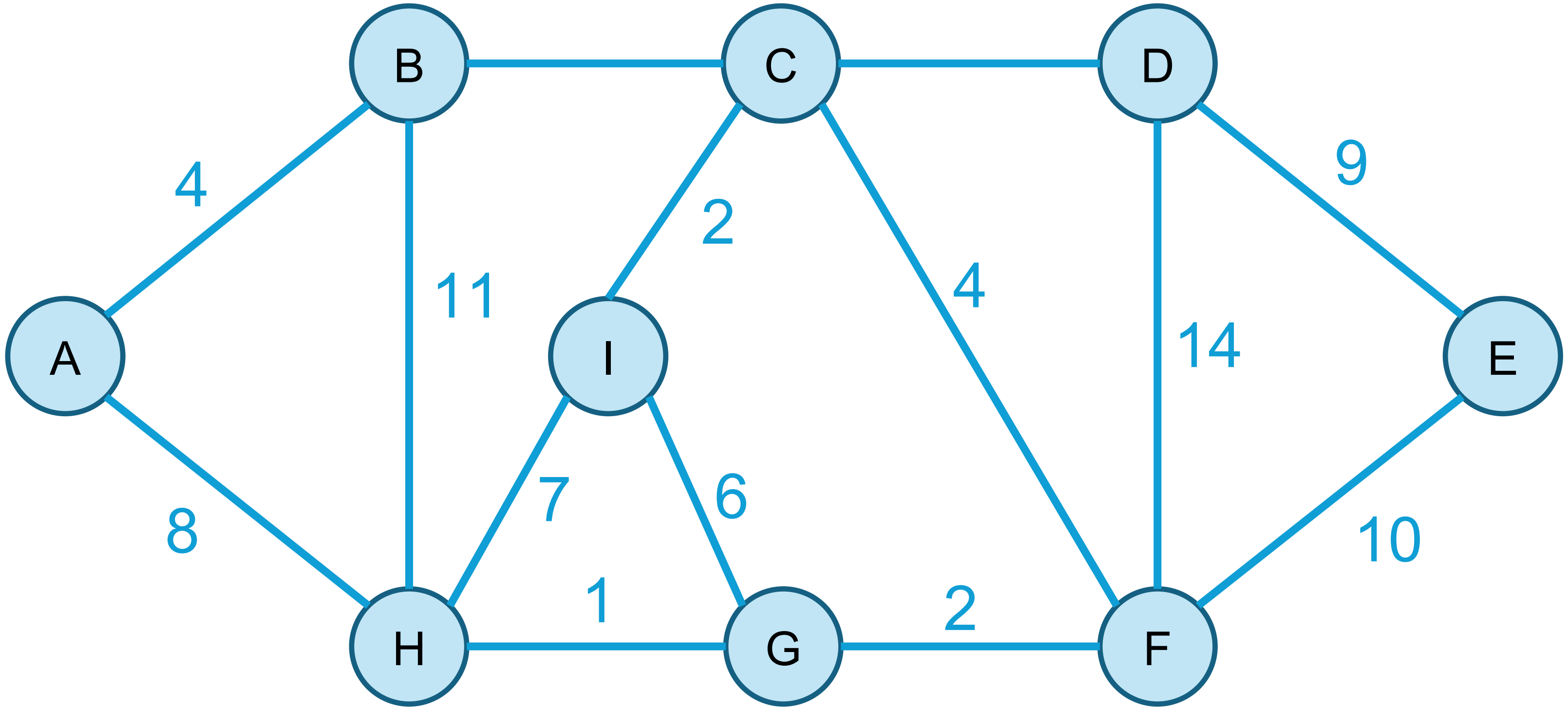
$$y.d \geq v.d$$
$$\delta(s,x) + w(x,y) \geq \delta(s,u) + w(u,v)$$

$$w(P) \geq w(s \sim v)$$



Minimum Spanning Tree

- A **spanning tree** is a **tree** that connects all of the vertices.
- The **cost** of a spanning tree is the sum of the weights on the edges.



Minimum Spanning Tree

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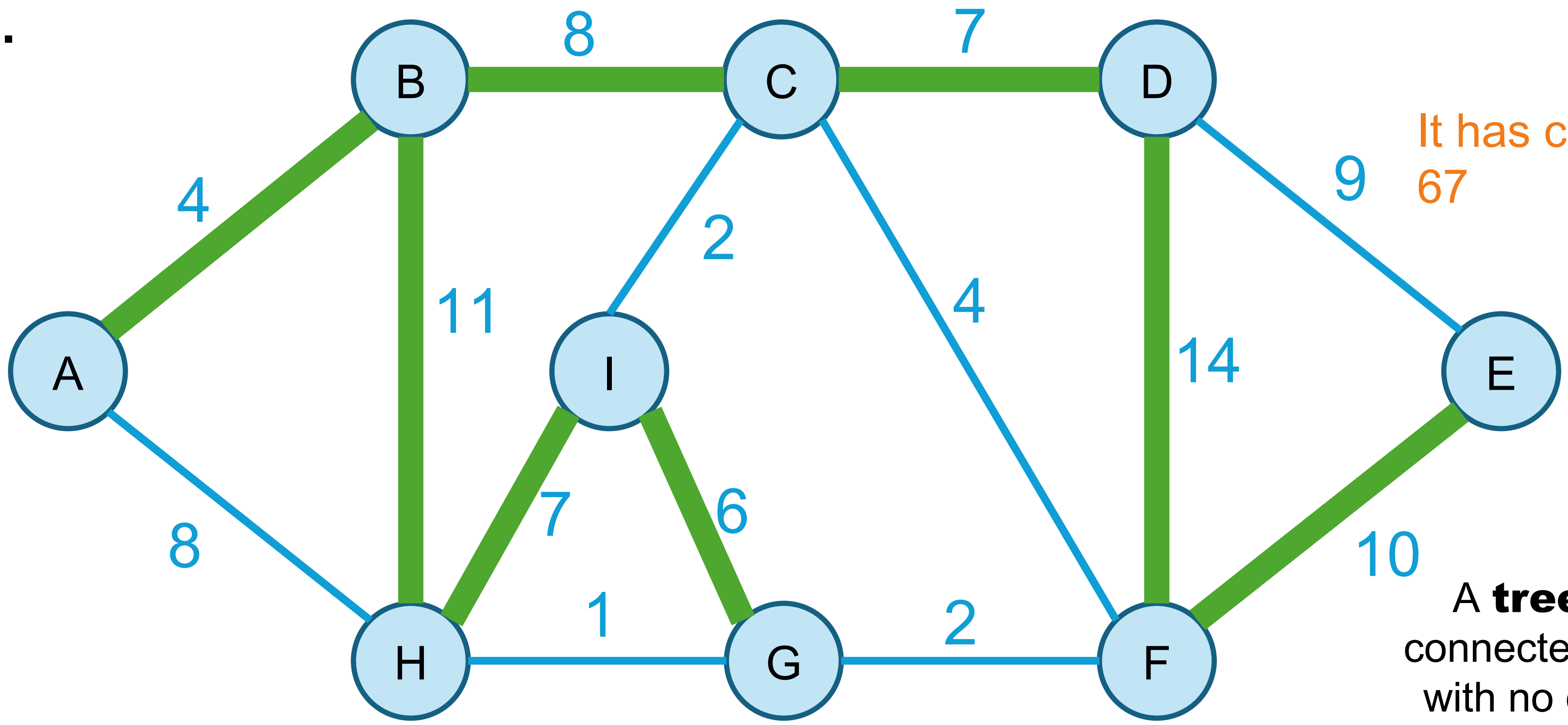
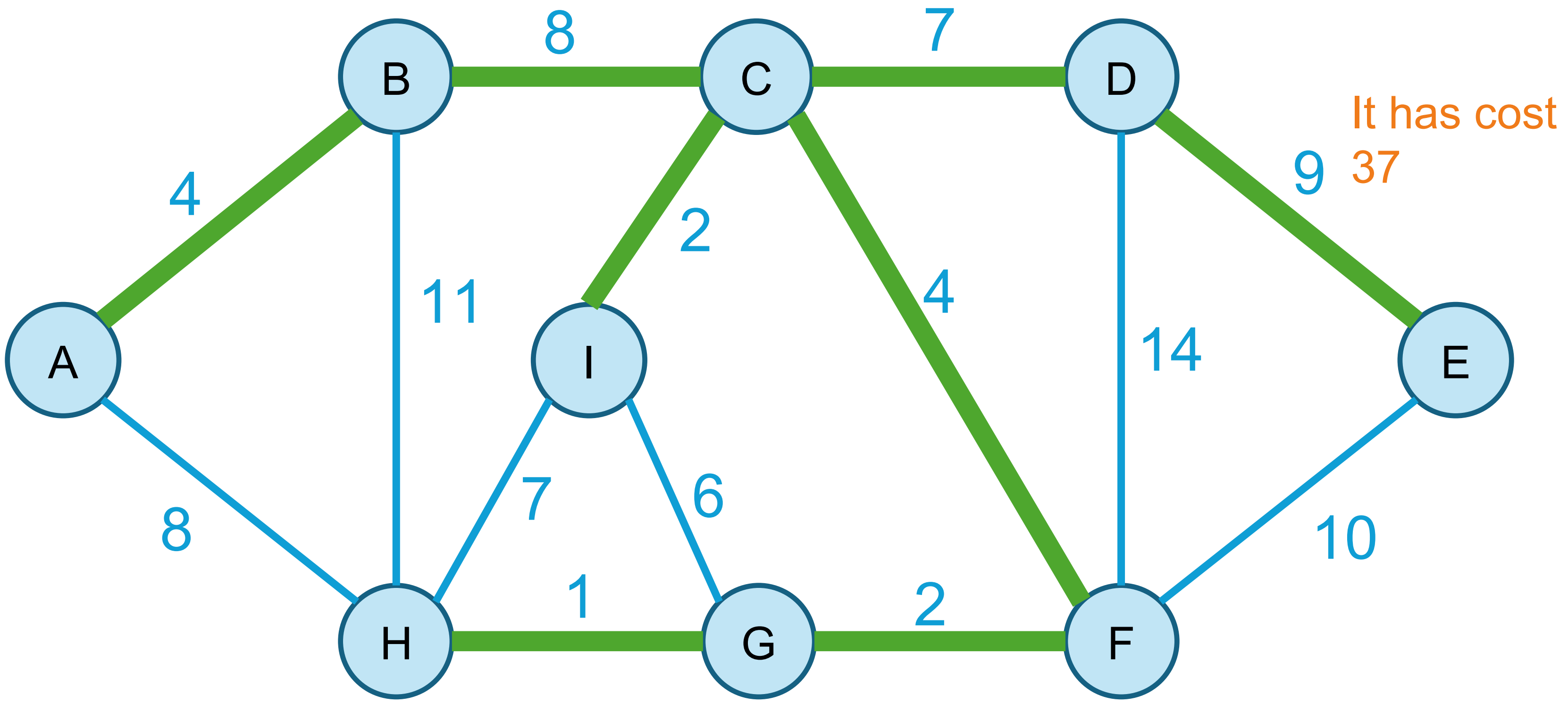


It has cost 67

A **tree** is a connected graph with no cycles!

Minimum Spanning Tree

- A **minimum** spanning tree is a tree with **minimum cost** that connects all of the vertices.



Some Definitions

- An $(S, V - S)$ partition is a cut of V of an undirected graph $G = (V, E)$
- Edge (u, v) crosses a cut if $u \in S$ and $v \in V - S$
- An edge is a light edge crossing a cut if its weight is the minimum of any edge crossing the cut.

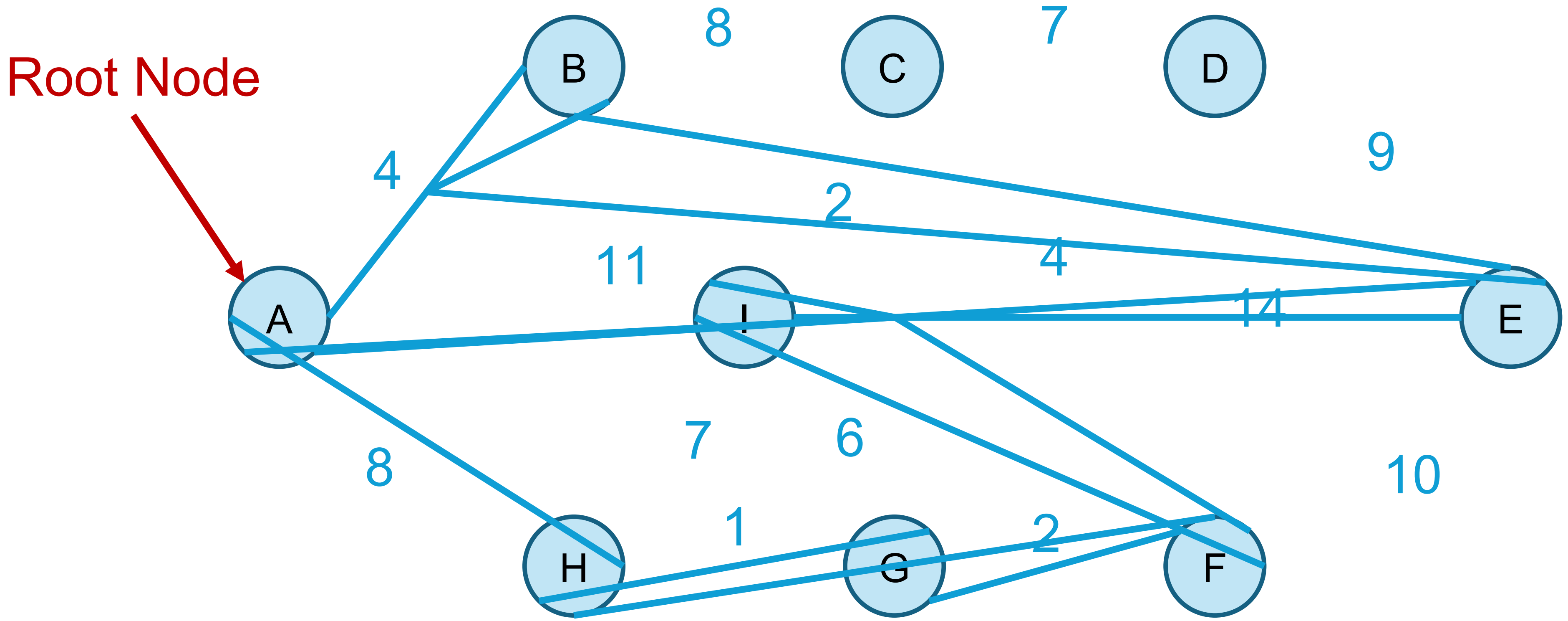
Minimum Spanning Tree

- The strategy:
  - Make a **series of greedy choices**, adding edges to the tree.
  - Show that each edge we add is **safe to add**:
    - we do not rule out the possibility of success
  - **Keep going** until we have an MST.

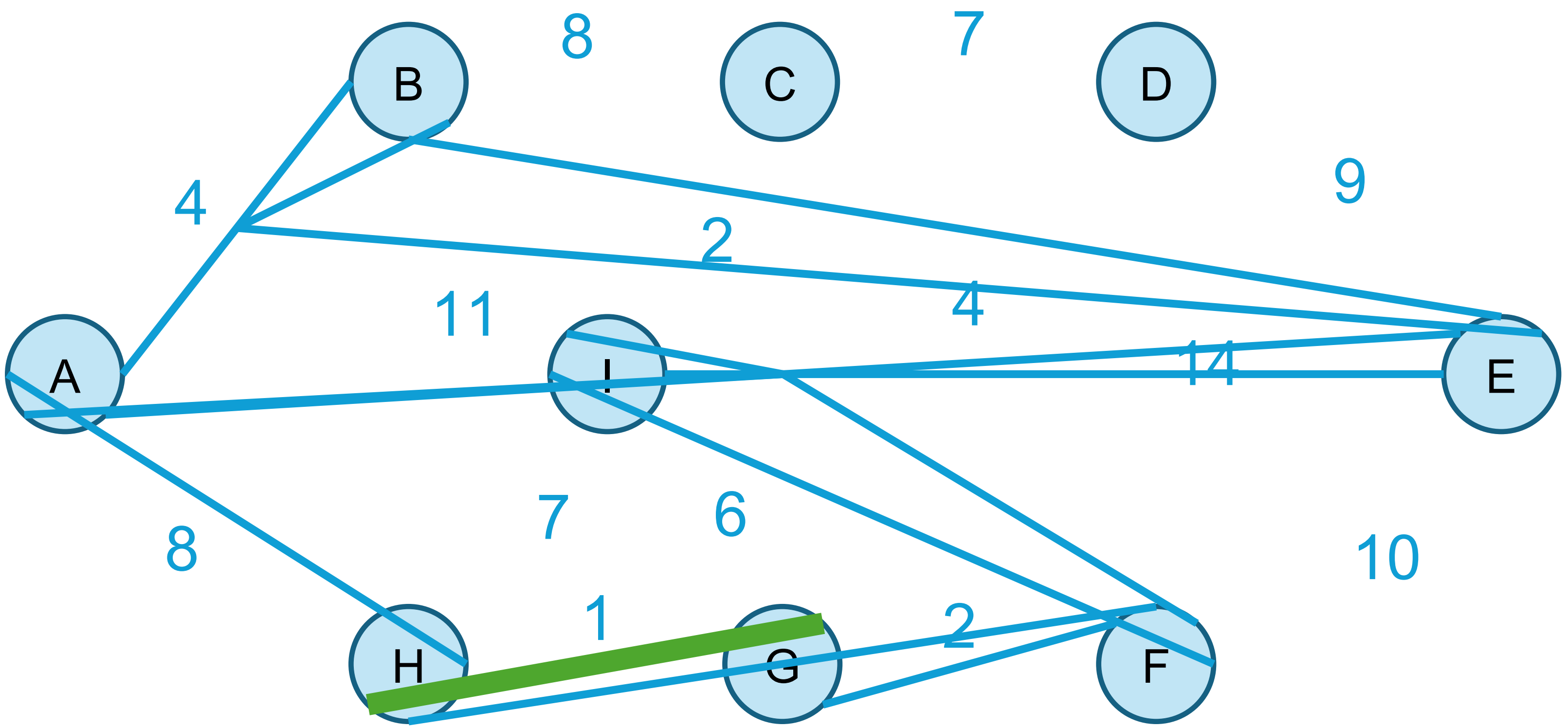
Minimum Spanning Tree

- Greedy strategy 1 (Prim's Algorithm):
  - Start from an empty tree (a cut)
  - At each step, grow the tree (a cut) with a node that can be connected with minimum cost (i.e., grow the tree by adding the node on the other end of the light edge)
  - Terminate when all nodes are included in the tree

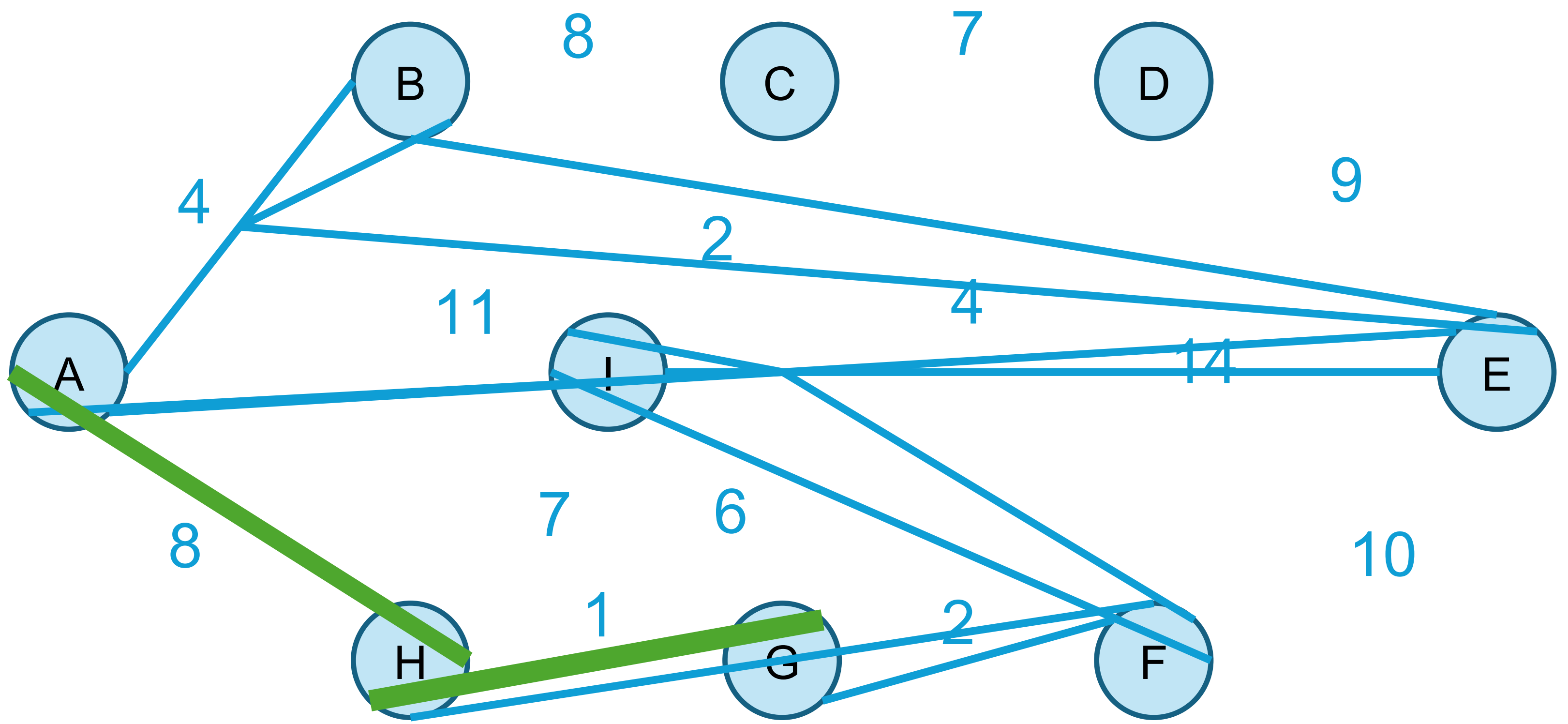
Prim’s Algorithm



Prim’s Algorithm



Prim's Algorithm



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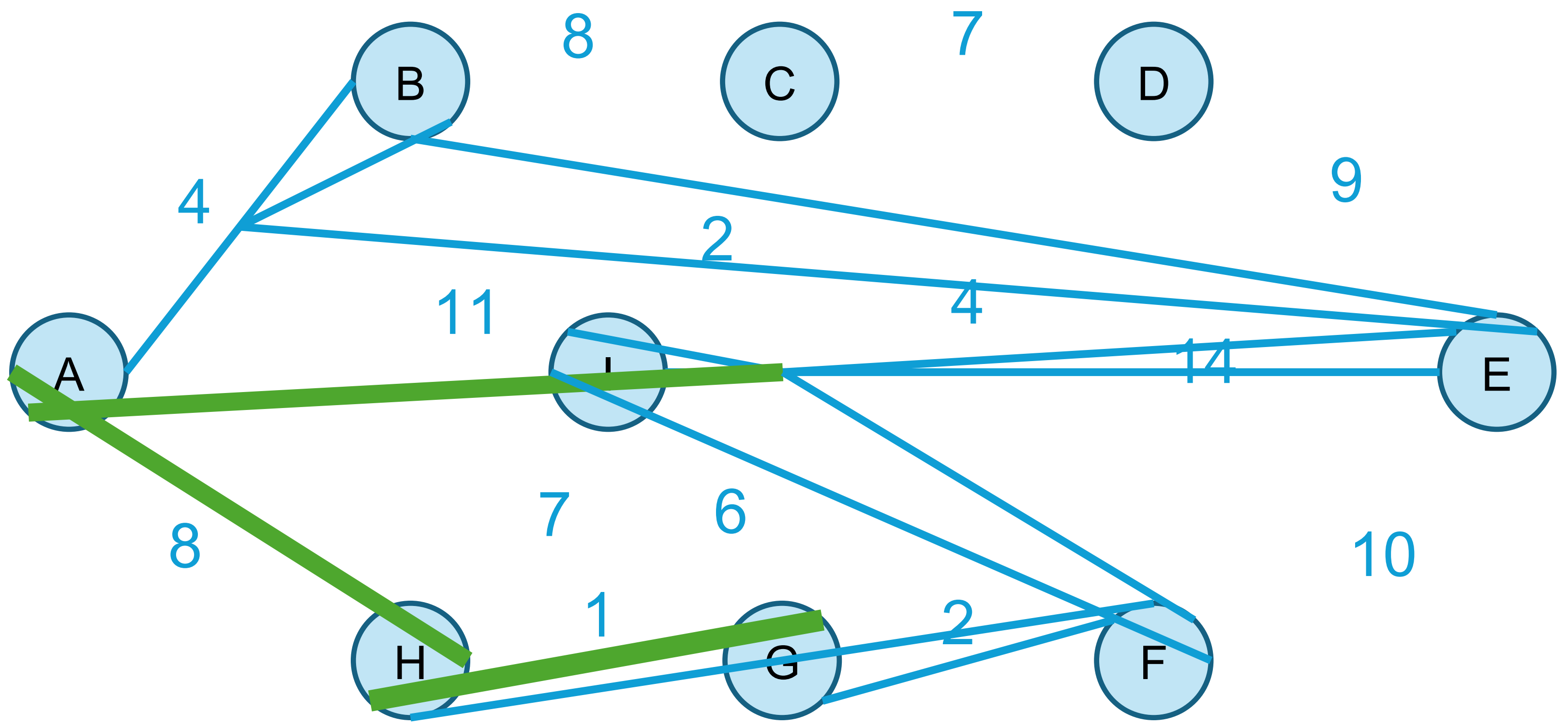

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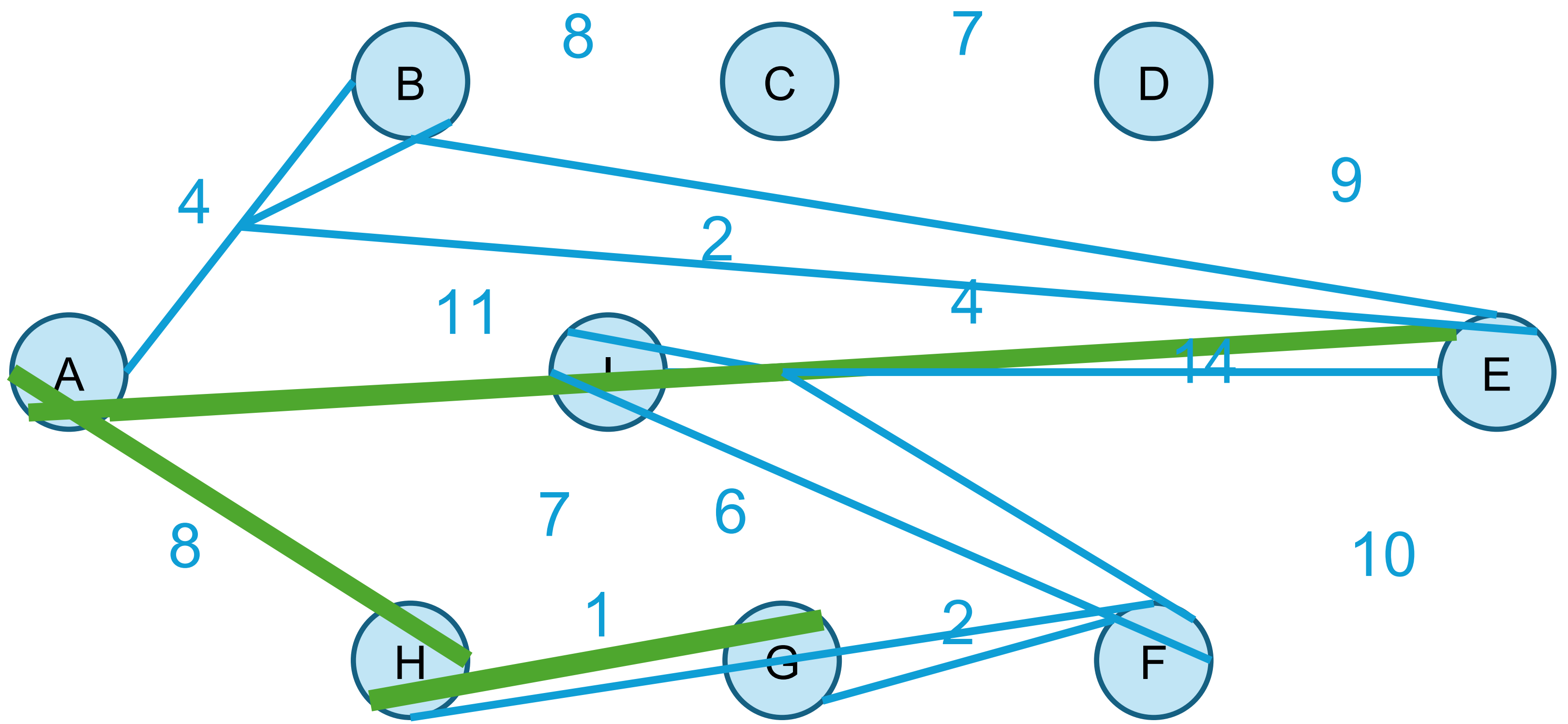


Prim's Algorithm

```
MST-PRIM(G, w, r)
for each vertex u ∈ G.V
    u.key = ∞
    u.π = NIL
r.key = 0
Q = ∅
for each vertex u ∈ G.V
    INSERT(Q, u)
while Q ≠ ∅
    u = EXTRACT-MIN(Q)          // add u to the tree
    for each vertex v in G.Adj[u]   // update keys of u's non-tree neighbors
        if v ∈ Q and w(u, v) < v.key
            v.π = u
            v.key = w(u, v)
            DECREASE-KEY(Q, v, w(u, v))
```

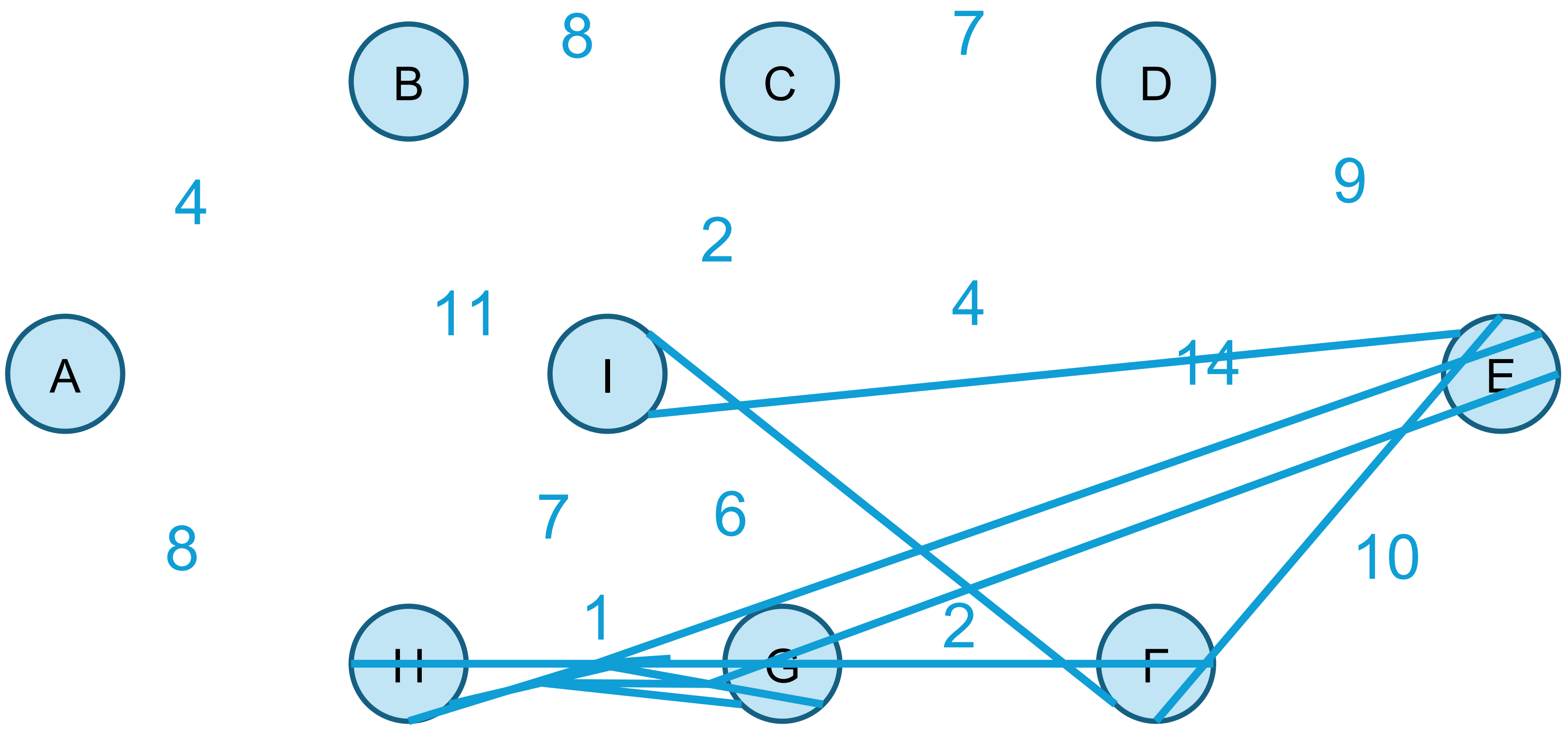
Prim’s Algorithm

- Proof of correctness?
  - Later

Minimum Spanning Tree

- Greedy Strategy 2 (Kruskal’s Algorithm):
  - Start with each node as a separate tree
  - Consider the edges in ascending order of their weights
  - Include the minimum weight edge between two disjoint trees to connect them into a single tree
    - Discard the edge if it creates a cycle
  - Terminate when all the nodes are included

Kruskal's Algorithm



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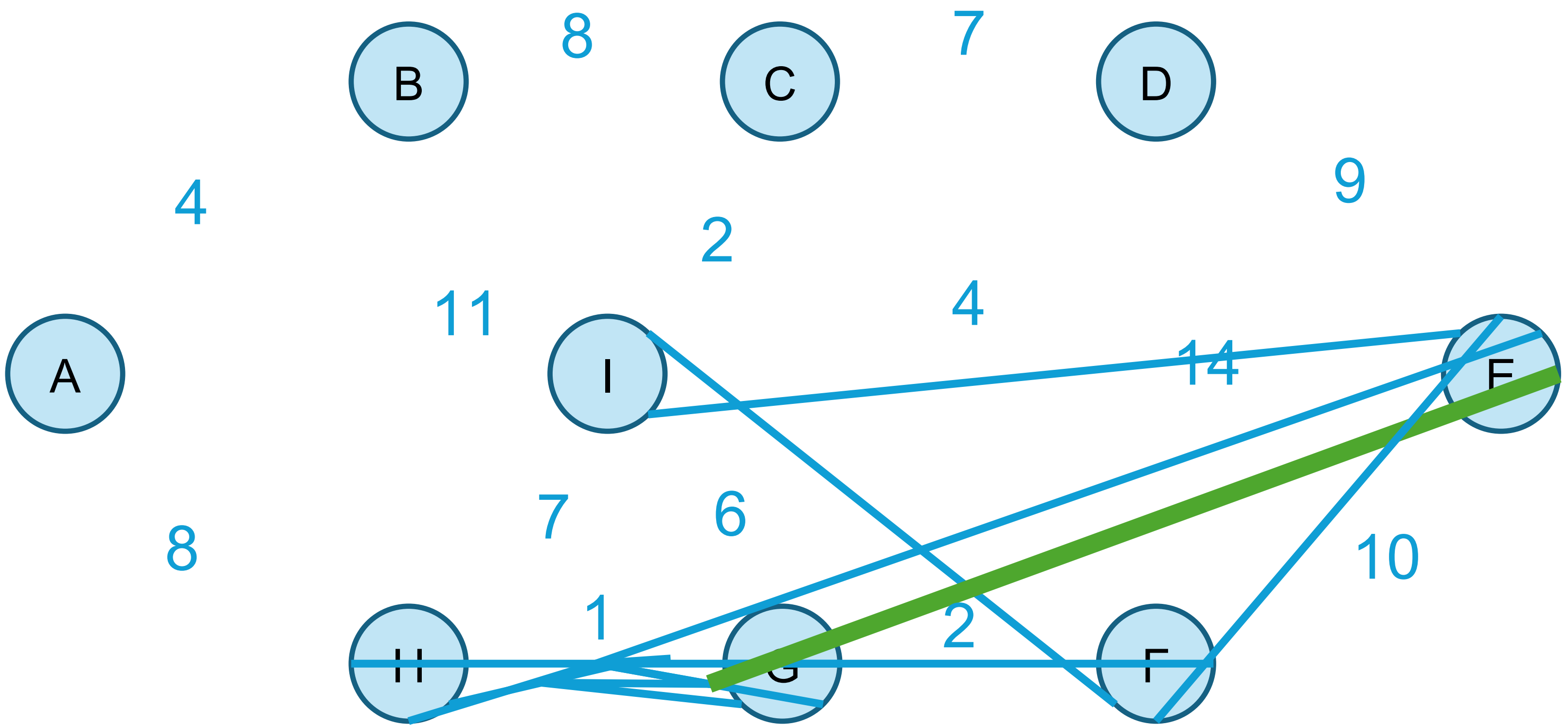

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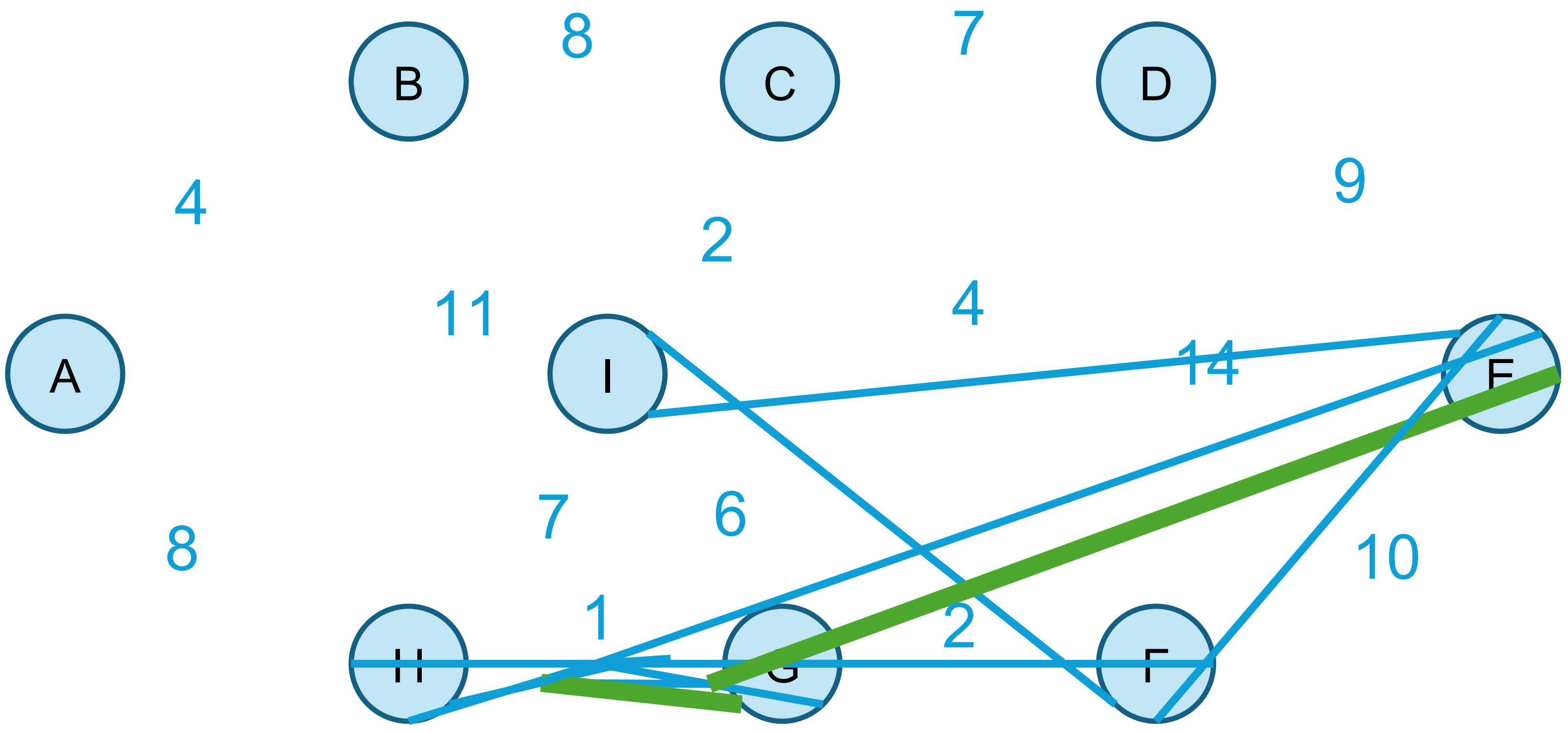

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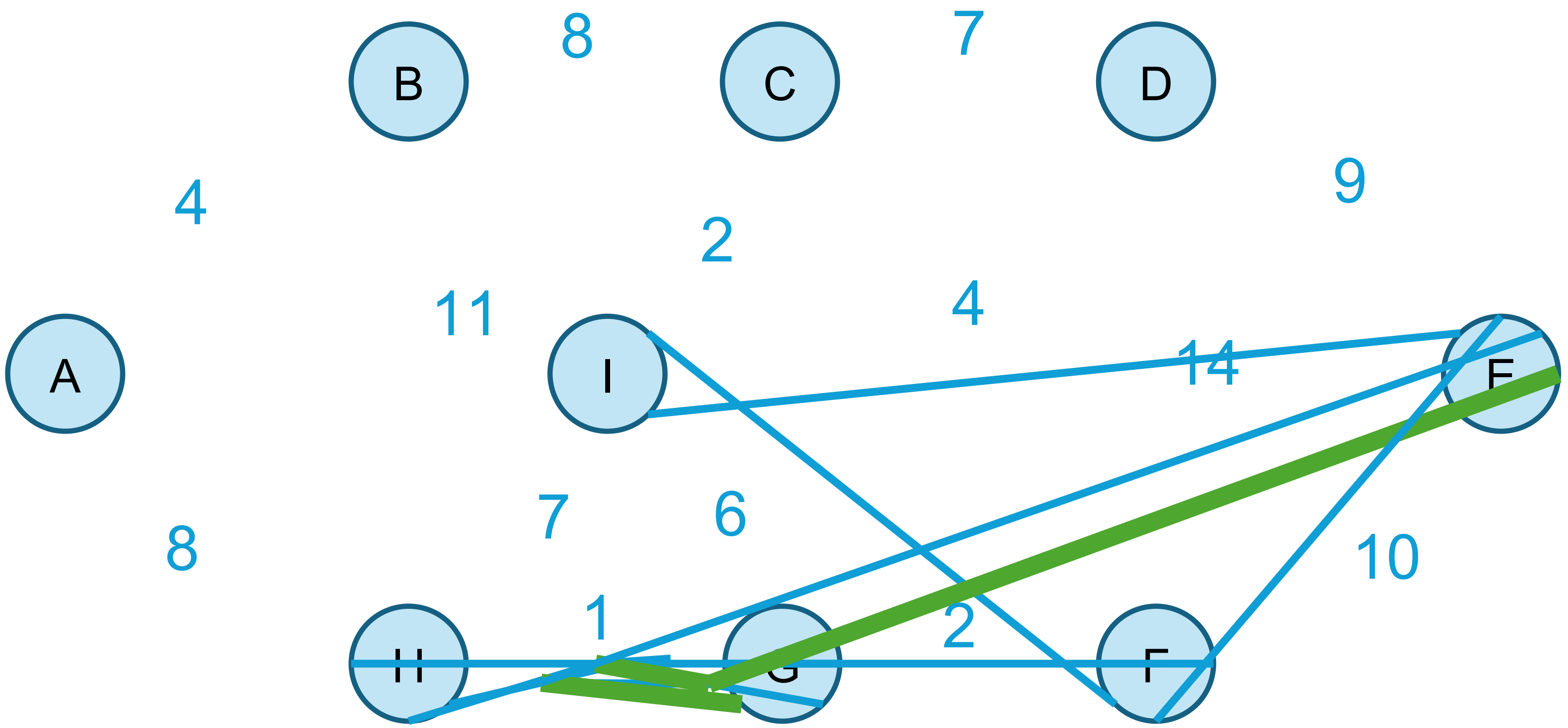

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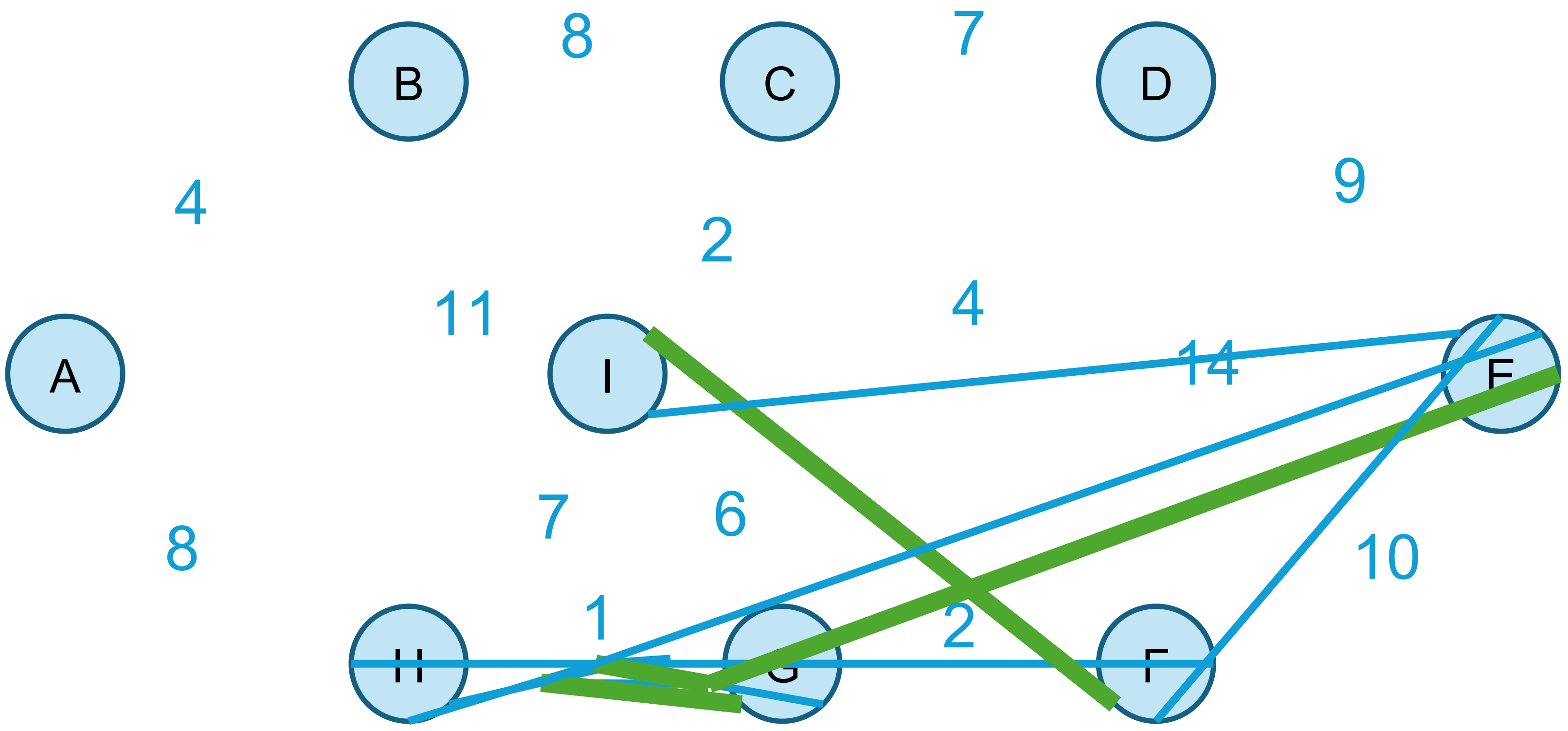

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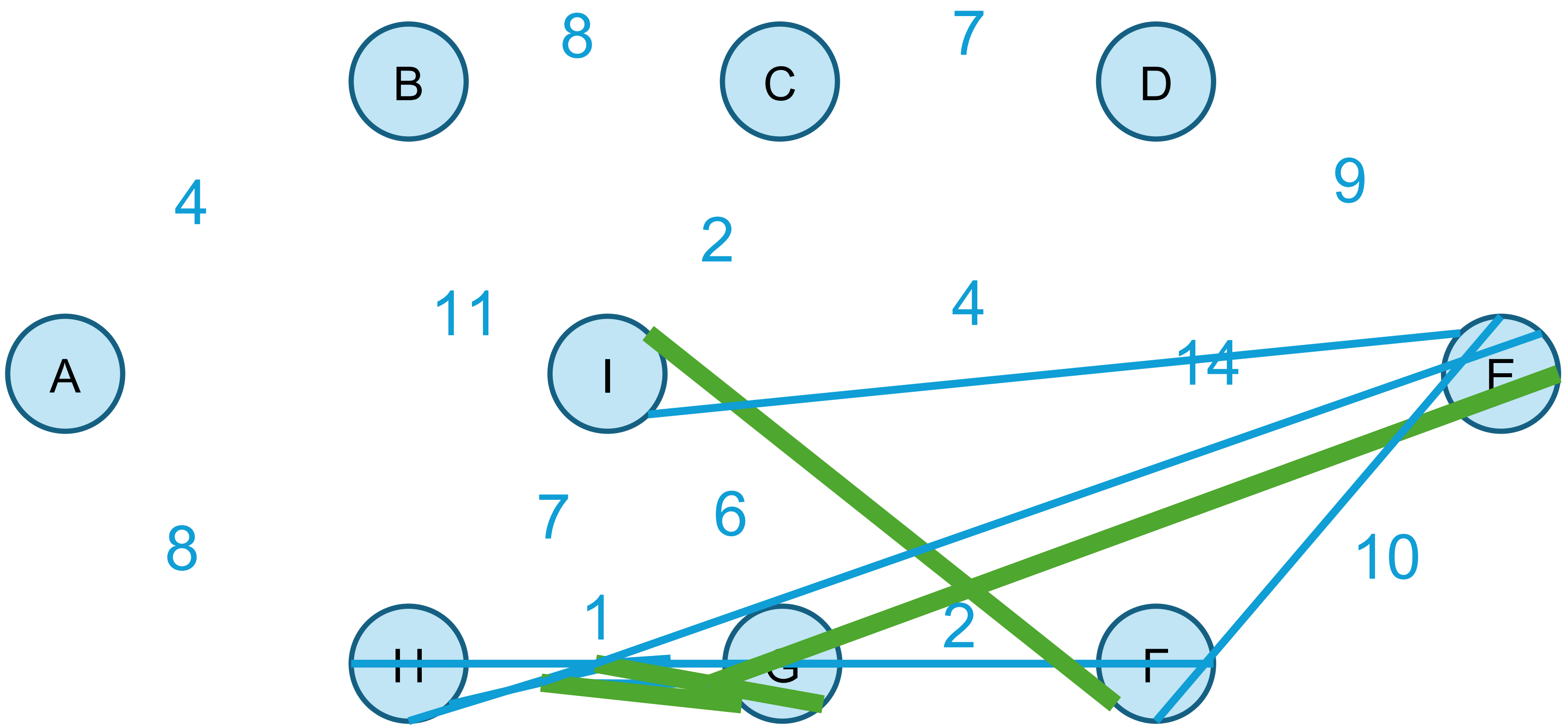

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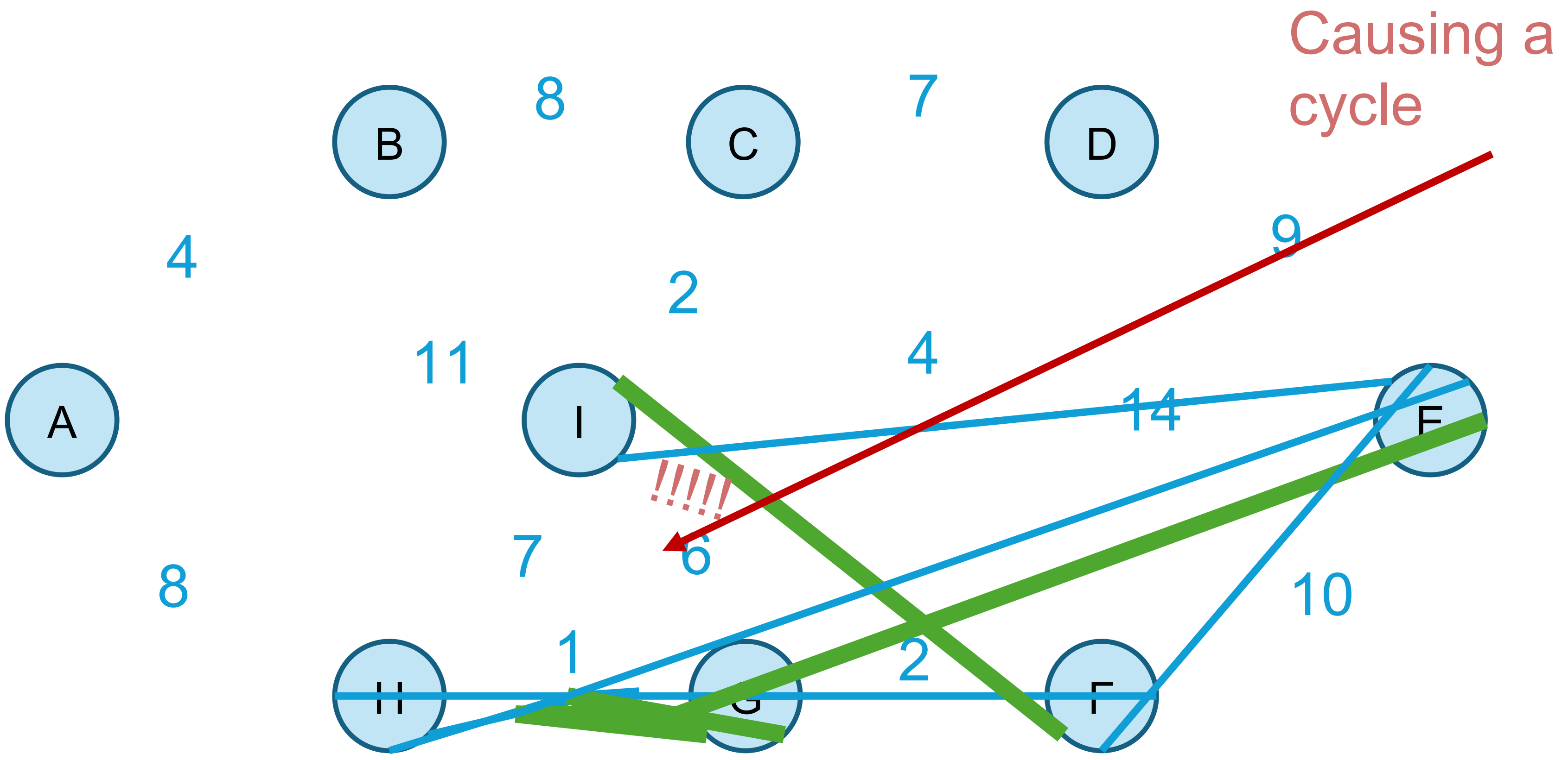

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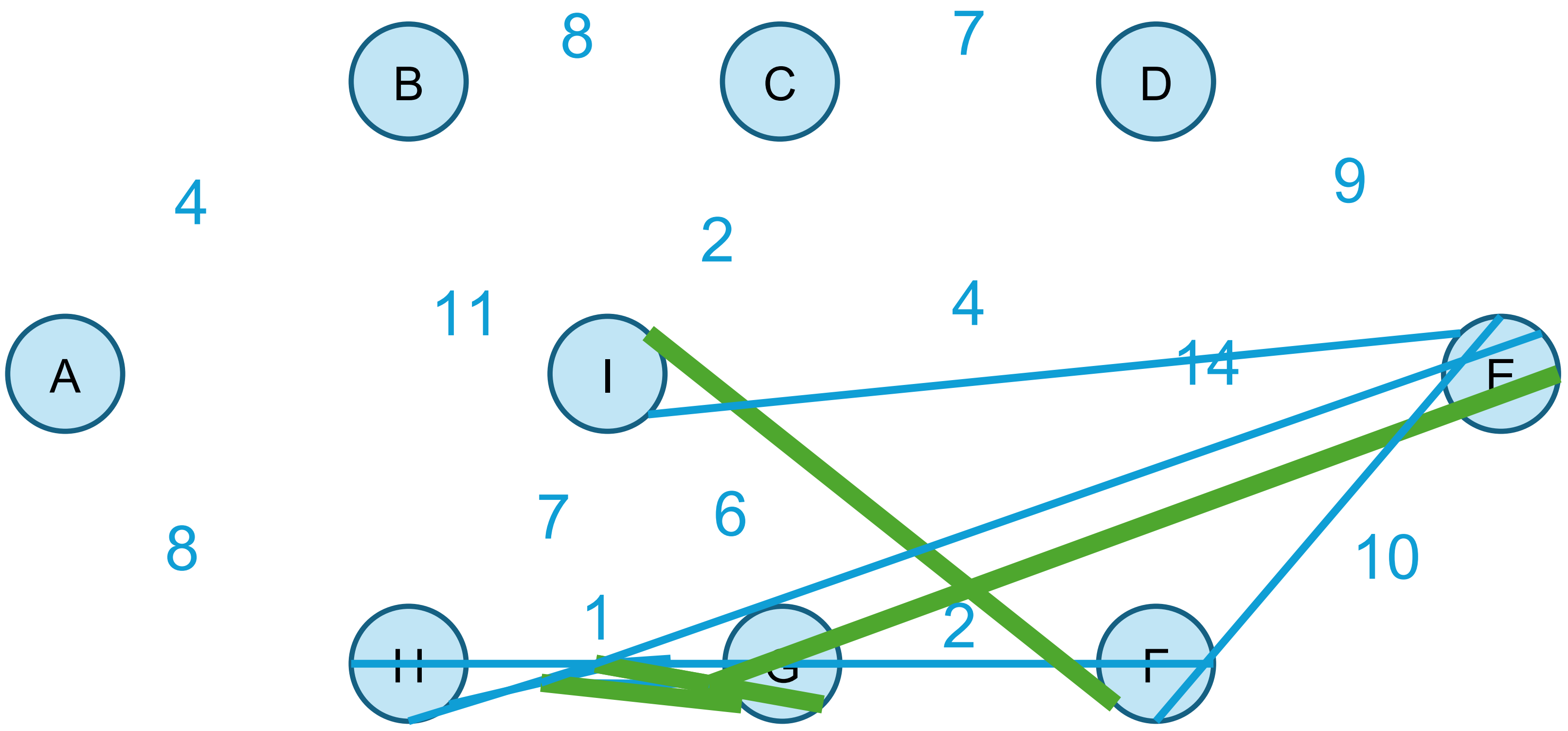

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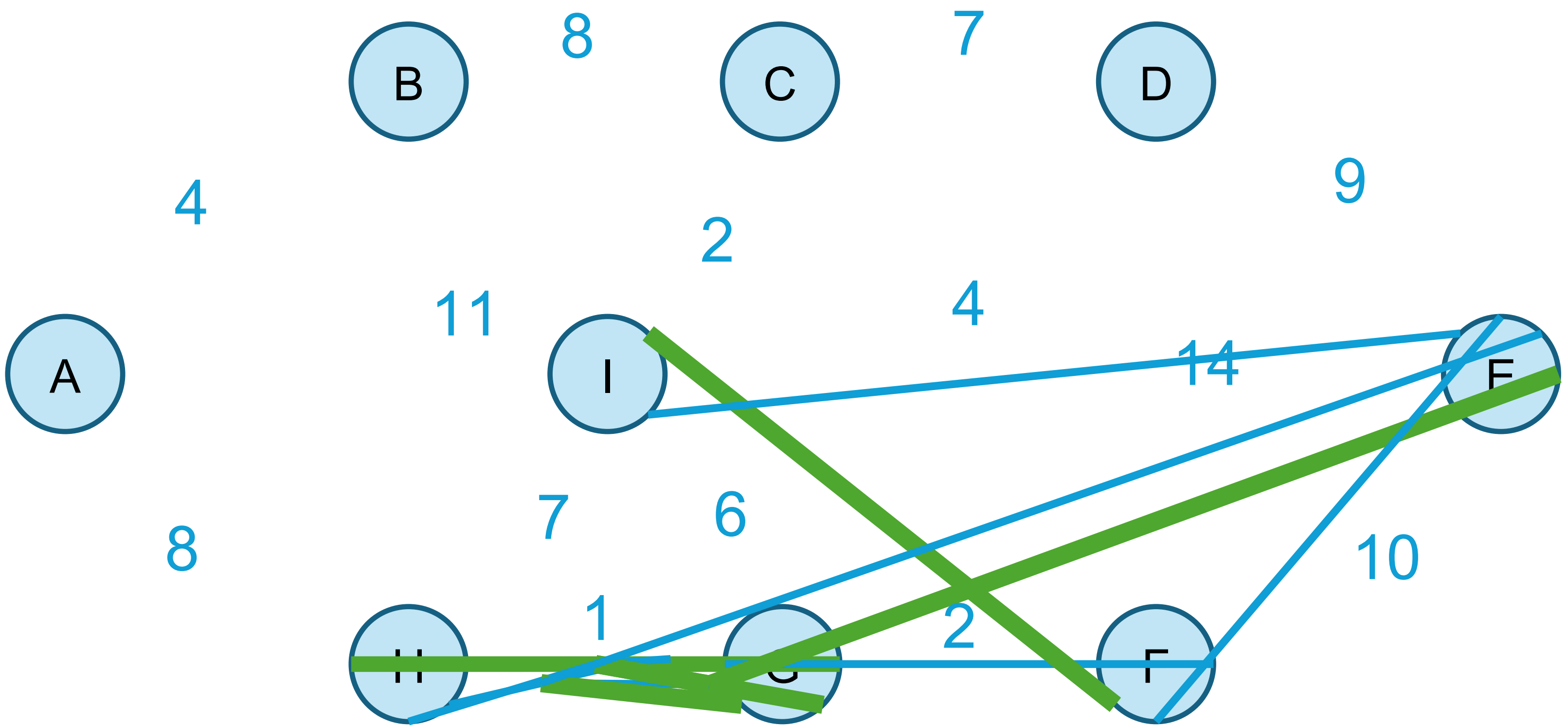

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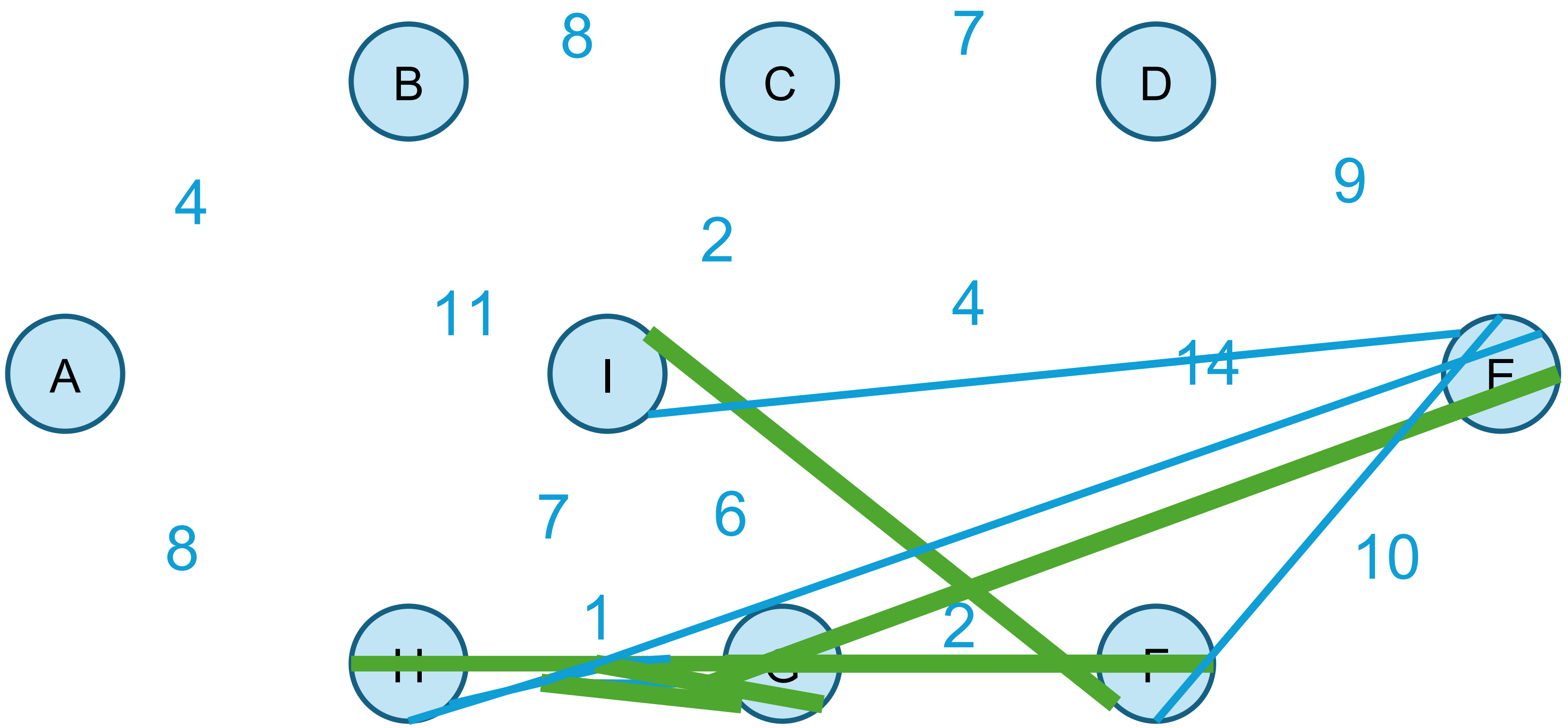

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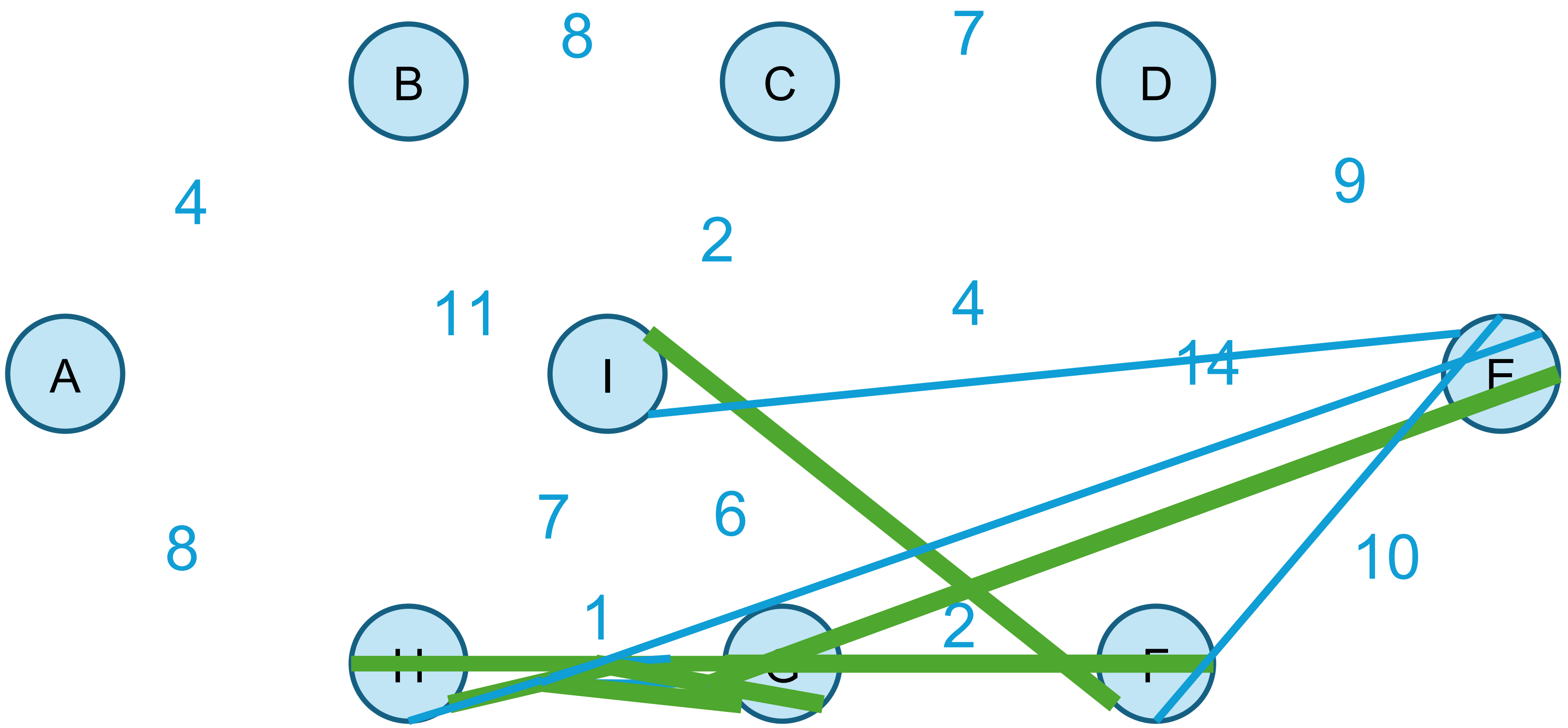

Kruskal's Algorithm

```
MST-KRUSKAL(G, w)
A = ∅
for each vertex v ∈ G.V
    MAKE-SET(v)
create a single list of the edges in G.E
sort the list of edges into monotonically increasing order by weight w
for each edge (u, v) taken from the sorted list in order
    if FIND-SET(u) ≠ FIND-SET(v)
        A = A ∪ {(u, v)}
        UNION(u, v)
return A
```

Kruskal’s Algorithm

Proof of Correctness?

- Later

Cut Property

- Assume that all edge costs are distinct.
- Let S be any subset of nodes that is neither empty nor equal to all of V, and let edge e =(v, w) be the minimum cost edge with one end in S and the other in V −S.

- Then every minimum spanning tree contains the edge e

Correctness

- Kruskal’s Algorithm
  - Apply cut property

- Prim’s Algorithm
  - Apply cut property

Reference

- Greedy Algorithms
  - CLRS 4th ed. Sections 15.1, 15.2
- Dijkstra’s Algorithm
  - CLRS 4th ed. Sections 22 (intro), 22.3
- Minimum Spanning Tree
  - CLRS 4th ed. Sections 21.1, 21.2
  - KT Section 4.5 (Correctness)