Lecture-2

Matrices

4.21 PROPERTIES OF MATRIX ADDITION

Only matrices of the same order can be added or subtracted.

(*i*) **Commutative Law.** $A + B = B + A$.

(*ii*) **Associative law.** $A + (B + C) = (A + B) + C$.

4.22 SUBTRACTION OF MATRICES

The difference of two matrices is a matrix, each element of which is obtained by subtracting the elements of the second matrix from the corresponding element of the first.

$$A - B = [a_{ij} - b_{ij}]$$

Thus $\begin{bmatrix} 8 & 6 & 4 \\ 1 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 5 & 1 \\ 7 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 8-3 & 6-5 & 4-1 \\ 1-7 & 2-6 & 0-2 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 3 \\ -6 & -4 & -2 \end{bmatrix}$ **Ans.**

4.23 SCALAR MULTIPLE OF A MATRIX

If a matrix is multiplied by a scalar quantity k, then each element is multiplied by k, *i.e.*

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix}$$

$$3A = 3\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 3\times 2 & 3\times 3 & 3\times 4 \\ 3\times 4 & 3\times 5 & 3\times 6 \\ 3\times 6 & 3\times 7 & 3\times 9 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 12 \\ 12 & 15 & 18 \\ 18 & 21 & 27 \end{bmatrix}$$

4.24 MULTIPLICATION

The product of two matrices A and B is only possible if the number of columns in A is equal to the number of rows in B.

Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be an $n \times p$ matrix. Then the product AB of these matrices is an $m \times p$ matrix $C = [c_{ij}]$ where

$$c_{ij} = a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + a_{i3}\, b_{3j} + \ldots. + a_{in}\, b_{nj}$$

4.25 $(AB)' = B'A'$

If A and B are two matrices conformal for product AB, then show that $(AB)' = B'A'$, where dash represents transpose of a matrix.

Solution. Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{ij})$ be $n \times p$ matrix.

Since AB is $m \times p$ matrix, $(AB)'$ is a $p \times m$ matrix.

Further B' *is* $p \times n$ matrix and A' an $n \times m$ matrix and therefore $B'\, A'$ is a $p \times m$ matrix.

Then $(AB)'$ and $B'\, A'$ are matrices of the same order.

Now the (j, i)th element of $(AB)' = (i, j)$th element of $(AB) = \sum_{k=1}^{n} a_{ik} b_{kj}$ ...(1)

Also the jth row of B' is $b_{1j}, b_{2j} \ldots. b_{nj}$, and ith column of A' is $a_{i1}, a_{i2}, a_{i3} \ldots. a_{in}$.

$\therefore$ (j, i)th element of $B'A' = \sum_{k=1}^{n} b_{kj} a_{ik}$ ...(2)

From (1) and (2), we have (j, i)th element of $(AB)' = (j, i)$ th element of $B'A'$.

As the matrices $(AB)'$ and $B'A'$ are of the same order and their corresponding elements are equal, we have $(AB)' = B'A'$. **Proved.**

4.26 PROPERTIES OF MATRIX MULTIPLICATION

1. Multiplication of matrices is not commutative. $AB \neq BA$
2. Matrix multiplication is associative, if conformability is assured. $A(BC) = (AB)C$
3. Matrix multiplication is distributive with respect to addition. $A(B + C) = AB + AC$
4. Multiplication of matrix A by unit matrix. $AI = IA = A$
5. Multiplicative inverse of a matrix exists if $|A| \neq 0$. $A \cdot A^{-1} = A^{-1} \cdot A = I$
6. If A is a square then $A \times A = A^2$, $A \times A \times A = A^3$.
7. $A^0 = I$
8. $I^n = I$, where n is positive integer.

Example 5. *If* $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

from the products AB and BA, and show that $AB \neq BA$.

Solution. Here,

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1-0+3 & 0-2+6 & 2-4+0 \\ 2+0-1 & 0+3-2 & 4+6-0 \\ -3+0+2 & 0+1+4 & -6+2+0 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 1 & 1 & 10 \\ -1 & 5 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1+0-6 & -2+0+2 & 3-0+4 \\ 0+2-6 & 0+3+2 & 0-1+4 \\ 1+4+0 & -2+6+0 & 3-2+0 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 7 \\ -4 & 5 & 3 \\ 5 & 4 & 1 \end{bmatrix}$$

$AB \neq BA$ **Proved.**

Example 12. *Verify that*

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \textit{ is orthogonal.}$$

Solution. $$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \quad \therefore \quad A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$AA' = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, A is an orthogonal matrix. **Verified.**

Example 13. *Determine the values of* α, β, γ *when*

$$\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \textit{ is orthogonal.}$$

Solution. Let $$A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$$

On transposing A, we have

$$A' = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$$

If A is orthogonal, then $AA' = I$

$$\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4\beta^2 + \gamma^2 & 2\beta^2 - \gamma^2 & -2\beta^2 + \gamma^2 \\ 2\beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 \\ -2\beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we have

$$\left.\begin{aligned} 4\beta^2 + \gamma^2 &= 1 \\ 2\beta^2 - \gamma^2 &= 0 \end{aligned}\right\} \Rightarrow \beta = \pm\frac{1}{\sqrt{6}}, \gamma = \pm\frac{1}{\sqrt{3}}$$

But $\alpha^2 + \beta^2 + \gamma^2 = 1$ as $\beta = \pm\frac{1}{\sqrt{6}}, \gamma = \pm\frac{1}{\sqrt{3}}, \alpha = \pm\frac{1}{\sqrt{2}}$

Example 14. *Prove that*

$$(AB)^n = A^n \cdot B^n, \ \textit{if } A \cdot B = B \cdot A$$

Solution.

$$(AB)^1 = AB = (A) \cdot (B)$$

$$(AB)^2 = (AB) \cdot (AB) = (ABA) \cdot B = \{ A\,(AB) \} \cdot B$$

$$= (A^2B) \cdot B = A^2\,(B \cdot B) = A^2 \cdot B^2$$

Suppose that

$$(AB)^n = A^n \cdot B^n$$

$$(AB)^{n+1} = (AB)^n \cdot (AB) = (A^n \cdot B^n) \cdot (AB) = A^n \cdot (B^nA) \cdot B$$

$$= A^n \cdot (B^{n-1} \cdot BA) \cdot B = A^n \cdot (B^{n-1} \cdot AB) \cdot B$$

$$= A^n \cdot (B^{n-2} \cdot B \cdot AB) \cdot B = A^n \cdot (B^{n-2} \cdot AB \cdot B) \cdot B$$

$$= A^n \cdot (B^{n-2} \cdot AB^2) \cdot B, \text{ continuing the process } n \text{ times.}$$

$$= A^n \cdot (A \cdot B^n) \cdot B = A^n \cdot (A \cdot B^{n+1}) = A^{n+1} \cdot B^{n+1}$$

Hence, taking the above to be true for $n = n$, we have shown that it is true for $n = n + 1$ and also it was true for $n = 1, 2, ...$ so it is universally true. **Proved.**

4.30 INVERSE OF A MATRIX

If A and B are two square matrices of the same order, such that

$$AB = BA = I \qquad (I = \text{unit matrix})$$

then B is called the inverse of A *i.e.* $B = A^{-1}$ and A is the inverse of B.

Condition for a square matrix A to possess an inverse is that matrix A is non-singular, i.e., $|A| \neq 0$

If A is a square matrix and B be its inverse, then $AB = I$

Taking determinant of both sides, we get $|AB| = |I|$ or $|A||B| = I$

From this relation it is clear that $|A| \neq 0$

i.e. the matrix A is non-singular.

To find the inverse matrix with the help of adjoint matrix

We know that $\quad A \cdot (Adj.\, A) = |A|\, I$

$$\Rightarrow \quad A \cdot \frac{1}{|A|}(A\,dj.\, A) = I \qquad \text{[Provided } |A| \neq 0] \qquad \ldots(1)$$

and $\quad A \cdot A^{-1} = I \qquad \ldots(2)$

From (1) and (2), we have

$$\therefore \quad \boxed{\boldsymbol{A^{-1} = \frac{1}{|A|}(Adj.\, A)}}$$

Example 16. *If* $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, *find* A^{-1}.

Solution. $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$$|A| = 3(-3+4) + 3(2-0) + 4(-2-0) = 3 + 6 - 8 = 1$$

The co-factors of elements of various rows of $|A|$ are

$$\begin{bmatrix} (-3+4) & (-2-0) & (-2-0) \\ (3-4) & (3-0) & (3-0) \\ (-12+12) & (-12+8) & (-9+6) \end{bmatrix}$$

Therefore, the matrix formed by the co-factors of $|A|$ is

$$\begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}, \quad Adj.\ A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{1}{|A|} Adj.\ A = \frac{1}{1}\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$ **Ans.**

Example 19. *If A and B are non-singular matrices of the same order then,*

$$(AB)^{-1} = B^{-1} \cdot A^{-1}$$

Hence prove that $(A^{-1})^m = (A^m)^{-1}$ *for any positive integer m.*

Solution. We know that,

$$(AB) \cdot (B^{-1} A^{-1}) = [(AB) B^{-1}] \cdot A^{-1} = [A (BB^{-1}] \cdot A^{-1}$$

$$= [AI] A^{-1} = A \cdot A^{-1} = I$$

Also, $$B^{-1} A^{-1} \cdot (AB) = B^{-1}[A^{-1} \cdot (AB)] = B^{-1} [(A^{-1} A) \cdot B]$$

$$= B^{-1} [I . B] = \mathrm{B}^{-1} \cdot B = I$$

By definition of the inverse of a matrix, $B^{-1} A^{-1}$ is inverse of AB.

$\Rightarrow$ $$B^{-1} A^{-1} = (AB)^{-1}$$ **Proved.**

$$(A^m)^{-1} = [A \cdot A^{m-1}]^{-1} = (A^{m-1})^{-1} A^{-1}$$

$$= (A \cdot A^{m-2})^{-1} \cdot A^{-1} = [(A^{m-2})^{-1} \cdot A^{-1}] \cdot A^{-1} = (A^{m-2})^{-1} (A^{-1})^2$$

$$= (A \cdot A^{m-3})^{-1} \cdot (A^{-1})^2 = [(A^{m-3})^{-1} \cdot A^{-1}] (A^{-1})^2 = (A^{m-3})^{-1} (A^{-1})^3$$

$$= A^{-1} (A^{-1})^{m-1} = (A^{-1})^m$$ **Proved.**

Row Echelon and Reduced Row Echelon Form

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a ***leading 1***.
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else in that column.

A matrix that has the first three properties is said to be in ***row echelon form***. (Thus, a matrix in reduced row echelon form is of necessity in row echelon form, but not conversely.)

EXAMPLE 1 Row Echelon and Reduced Row Echelon Form

The following matrices are in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The following matrices are in row echelon form but not reduced row echelon form.

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$