1. **(a) Show that the function f, defined by $f(x) = (1+3x)^{1/x}$ when $x \neq 0$, $f(0) = e^3$, is continuous for x = 0.**

   - For the function $f(x)$ to be continuous at $x = 0$, we need to show that $\lim_{x\to 0} f(x) = f(0)$.
   - We are given $f(0) = e^3$.
   - Now, let's evaluate $\lim_{x\to 0}(1+3x)^{1/x}$. This is of the form 1^∞.
   - Let $L = \lim_{x\to 0}(1+3x)^{1/x}$.
   - Take the natural logarithm of both sides: $\ln L = \lim_{x\to 0}\frac{1}{x}\ln(1+3x)$.
   - This is of the form $\frac{0}{0}$, so we can apply L'Hôpital's Rule.
   - $\ln L = \lim_{x\to 0}\frac{\frac{d}{dx}(\ln(1+3x))}{\frac{d}{dx}(x)} = \lim_{x\to 0}\frac{\frac{3}{1+3x}}{1} = \frac{3}{1+0} = 3.$
   - So, $\ln L = 3$, which implies $L = e^3$.
   - Since $\lim_{x\to 0} f(x) = e^3$ and $f(0) = e^3$, we have $\lim_{x\to 0} f(x) = f(0)$.
   - Therefore, the function $f(x)$ is continuous at $x = 0$.

   (b) If $y = x\log\frac{x-1}{x+1}$, prove that $\frac{d^n y}{dx^n} = (-1)^n(n-2)!\left[\frac{(x-n)}{(x-1)^n} - \frac{(x+n)}{(x+1)^n}\right]$.

   - Given $y = x\log\left(\frac{x-1}{x+1}\right) = x[\log(x-1) - \log(x+1)]$.
   - First derivative: $\frac{dy}{dx} = 1 \cdot [\log(x-1) - \log(x+1)] + x\left[\frac{1}{x-1} - \frac{1}{x+1}\right]$ $\frac{dy}{dx} = \log\left(\frac{x-1}{x+1}\right) + x\left[\frac{(x+1)-(x-1)}{(x-1)(x+1)}\right]$ $\frac{dy}{dx} = \log\left(\frac{x-1}{x+1}\right) + x\,\frac{2}{x^2-1}$ $\frac{dy}{dx} = \log\left(\frac{x-1}{x+1}\right) + \frac{2x}{x^2-1}$.

- Second derivative: $\frac{d^2y}{dx^2} = \frac{1}{x-1} - \frac{1}{x+1} + \frac{2(x^2-1)-2x(2x)}{(x^2-1)^2}$ $\frac{d^2y}{dx^2} = \frac{(x+1)-(x-1)}{(x-1)(x+1)} + \frac{2x^2-2-4x^2}{(x^2-1)^2}$ $\frac{d^2y}{dx^2} = \frac{2}{x^2-1} + \frac{-2x^2-2}{(x^2-1)^2}$ $\frac{d^2y}{dx^2} = \frac{2(x^2-1)-2x^2-2}{(x^2-1)^2} = \frac{2x^2-2-2x^2-2}{(x^2-1)^2} = \frac{-4}{(x^2-1)^2} = -4(x^2-1)^{-2}$.

- We use the general formula for the n-th derivative of $\log(ax+b)$ and $1/(ax+b)$. For $f(x) = \log(ax+b)$, $f^{(n)}(x) = (-1)^{n-1}(n-1)!\frac{a^n}{(ax+b)^n}$. For $f(x) = \frac{1}{ax+b}$, $f^{(n)}(x) = (-1)^n n!\frac{a^n}{(ax+b)^{n+1}}$.

- Let $y = x[\log(x-1) - \log(x+1)]$.

- Using Leibniz's theorem for the n-th derivative of a product $(uv)^{(n)} = \sum_{k=0}^{n}\binom{n}{k} u^{(n-k)}v^{(k)}$.

- Let $u = x$ and $v = \log(x-1) - \log(x+1)$.

- $u' = 1, u^{(k)} = 0$ for $k \geq 2$.

- $v^{(n)} = \frac{d^n}{dx^n}[\log(x-1)] - \frac{d^n}{dx^n}[\log(x+1)] = (-1)^{n-1}(n-1)!\left[\frac{1}{(x-1)^n} - \frac{1}{(x+1)^n}\right]$.

- $v^{(n-1)} = (-1)^{n-2}(n-2)!\left[\frac{1}{(x-1)^{n-1}} - \frac{1}{(x+1)^{n-1}}\right]$.

- $\frac{d^ny}{dx^n} = \binom{n}{0}u^{(0)}v^{(n)} + \binom{n}{1}u^{(1)}v^{(n-1)}$

- $\frac{d^ny}{dx^n} = x \cdot (-1)^{n-1}(n-1)!\left[\frac{1}{(x-1)^n} - \frac{1}{(x+1)^n}\right] + n \cdot 1 \cdot (-1)^{n-2}(n-2)!\left[\frac{1}{(x-1)^{n-1}} - \frac{1}{(x+1)^{n-1}}\right]$

- $\frac{d^ny}{dx^n} = (-1)^{n-1}(n-1)!\left[\frac{x}{(x-1)^n} - \frac{x}{(x+1)^n}\right] + n(-1)^{n-2}(n-2)!\left[\frac{1}{(x-1)^{n-1}} - \frac{1}{(x+1)^{n-1}}\right]$

- Recall $(-1)^{n-2} = (-1)^n$ and $(n-1)! = (n-1)(n-2)!$.

- $\frac{d^ny}{dx^n} = (-1)^{n-1}(n-1)!\left[\frac{x}{(x-1)^n} - \frac{x}{(x+1)^n}\right] + (-1)^n(n-1)(n-2)!\left[\frac{x-1}{(x-1)^n} - \frac{x+1}{(x+1)^n}\right]$
- $\frac{d^ny}{dx^n} = (-1)^{n-1}(n-1)!\left[\frac{x}{(x-1)^n} - \frac{x}{(x+1)^n}\right] + (-1)^n(n-1)!\left[\frac{x-1}{(x-1)^n} - \frac{x+1}{(x+1)^n}\right]$
- $\frac{d^ny}{dx^n} = (-1)^n(n-1)!\left[-\frac{x}{(x-1)^n} + \frac{x}{(x+1)^n} + \frac{x-1}{(x-1)^n} - \frac{x+1}{(x+1)^n}\right]$
- $\frac{d^ny}{dx^n} = (-1)^n(n-1)!\left[\frac{x-1-x}{(x-1)^n} + \frac{x-(x+1)}{(x+1)^n}\right]$
- $\frac{d^ny}{dx^n} = (-1)^n(n-1)!\left[\frac{-1}{(x-1)^n} + \frac{-1}{(x+1)^n}\right]$
- This result does not match the desired proof. Let's re-evaluate using a different approach or verify the question's target expression.
- Let's check the given expression for n=2: $(-1)^2(2-2)!\left[\frac{(x-2)}{(x-1)^2} - \frac{(x+2)}{(x+1)^2}\right] = 0!\left[\frac{x-2}{(x-1)^2} - \frac{x+2}{(x+1)^2}\right] = \frac{(x-2)(x+1)^2-(x+2)(x-1)^2}{(x-1)^2(x+1)^2} = \frac{(x-2)(x^2+2x+1)-(x+2)(x^2-2x+1)}{(x^2-1)^2} = \frac{x^3+2x^2+x-2x^2-4x-2-(x^3-2x^2+x+2x^2-4x+2)}{(x^2-1)^2} = \frac{x^3-3x-2-(x^3-3x+2)}{(x^2-1)^2} = \frac{-4}{(x^2-1)^2}$.
- Our calculated $y''(x) = \frac{-4}{(x^2-1)^2}$ matches the form for $n = 2$. This indicates the proposed formula might be correct.
- Let's use the result for $\frac{d}{dx}\left(\frac{2x}{x^2-1}\right) = \frac{d}{dx}\left(\frac{1}{x-1} + \frac{1}{x+1}\right)$.
- $\frac{d^n}{dx^n}\left(\frac{1}{x-1}\right) = (-1)^n n!\,(x-1)^{-(n+1)}$.
- $\frac{d^n}{dx^n}\left(\frac{1}{x+1}\right) = (-1)^n n!\,(x+1)^{-(n+1)}$.

- From $y' = \log\left(\frac{x-1}{x+1}\right) + \frac{2x}{x^2-1}$.
- $\frac{d^ny}{dx^n} = \frac{d^{n-1}}{dx^{n-1}}[\log(x-1) - \log(x+1)] + \frac{d^{n-1}}{dx^{n-1}}\left[\frac{1}{x-1} + \frac{1}{x+1}\right]$ for $n \geq 2$.
- $\frac{d^ny}{dx^n} = (-1)^{n-2}(n-2)!\left[\frac{1}{(x-1)^{n-1}} - \frac{1}{(x+1)^{n-1}}\right] + (-1)^{n-1}(n-1)!\left[\frac{1}{(x-1)^n} + \frac{1}{(x+1)^n}\right]$.
- This still doesn't directly simplify to the given expression. There seems to be a discrepancy in the provided formula for the nth derivative. Let's assume the question expects the result to be proven, and recheck for small n.
- The formula in the question is likely derived by considering the terms in a slightly different way.
- Consider $y = x[\log(x-1) - \log(x+1)]$.
- $\frac{y}{x} = \log(x-1) - \log(x+1)$.
- $\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{x-1} - \frac{1}{x+1}$.
- $\frac{d^n}{dx^n}\left(\frac{y}{x}\right) = \frac{d^{n-1}}{dx^{n-1}}\left[\frac{1}{x-1} - \frac{1}{x+1}\right]$.
- $\frac{d^n}{dx^n}\left(\frac{y}{x}\right) = (-1)^{n-1}(n-1)!\left[\frac{1}{(x-1)^n} - \frac{1}{(x+1)^n}\right]$.
- Using Leibniz rule for $y = x \cdot (\frac{y}{x})$.
- $\frac{d^ny}{dx^n} = \binom{n}{0}x\frac{d^n}{dx^n}\left(\frac{y}{x}\right) + \binom{n}{1}1\frac{d^{n-1}}{dx^{n-1}}\left(\frac{y}{x}\right)$.
- $\frac{d^ny}{dx^n} = x(-1)^{n-1}(n-1)!\left[\frac{1}{(x-1)^n} - \frac{1}{(x+1)^n}\right] + n(-1)^{n-2}(n-2)!\left[\frac{1}{(x-1)^{n-1}} - \frac{1}{(x+1)^{n-1}}\right]$.

- $\frac{d^n y}{dx^n} = (-1)^{n-1}(n-1)!\left[\frac{x}{(x-1)^n} - \frac{x}{(x+1)^n}\right] + n(-1)^n(n-2)!\left[\frac{x-1}{(x-1)^n} - \frac{x+1}{(x+1)^n}\right]$.
- $\frac{d^n y}{dx^n} = (-1)^{n-1}(n-1)!\left[\frac{x}{(x-1)^n} - \frac{x}{(x+1)^n}\right] + (-1)^n n(n-2)!\left[\frac{x-1}{(x-1)^n} - \frac{x+1}{(x+1)^n}\right]$.
- $\frac{d^n y}{dx^n} = (-1)^n(n-2)!\left[-(n-1)\left(\frac{x}{(x-1)^n} - \frac{x}{(x+1)^n}\right) + n\left(\frac{x-1}{(x-1)^n} - \frac{x+1}{(x+1)^n}\right)\right]$.
- $\frac{d^n y}{dx^n} = (-1)^n(n-2)!\left[\frac{-(n-1)x+n(x-1)}{(x-1)^n} - \frac{-(n-1)x+n(x+1)}{(x+1)^n}\right]$.
- $\frac{d^n y}{dx^n} = (-1)^n(n-2)!\left[\frac{-nx+x+nx-n}{(x-1)^n} - \frac{-nx+x+nx+n}{(x+1)^n}\right]$.
- $\frac{d^n y}{dx^n} = (-1)^n(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$.
- This matches the required proof.

2. **(a) Determine** $\lim_{x\to 0}\left(\frac{1}{x^2} - \cot^2 x\right)$**.**
   - We have $\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{\cos^2 x}{\sin^2 x}\right)$.
   - This is of the form $\infty - \infty$.
   - Combine the terms: $\lim_{x\to 0}\frac{\sin^2 x - x^2\cos^2 x}{x^2\sin^2 x}$.
   - This is of the form $\frac{0}{0}$. We can use L'Hôpital's Rule or Taylor series expansion.
   - Using Taylor series: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ $\sin^2 x = \left(x - \frac{x^3}{6} + \cdots\right)^2 = x^2 - \frac{x^4}{3} + \frac{x^6}{36} + \frac{x^6}{15} - \cdots = x^2 - \frac{x^4}{3} + O(x^6)$. $\cos x = 1 - \frac{x^2}{2!} +$

$\frac{x^4}{4!} - \cdots \cos^2 x = \left(1 - \frac{x^2}{2} + \cdots\right)^2 = 1 - x^2 + \frac{x^4}{4} + \frac{x^4}{12} - \cdots = 1 - x^2 + \frac{x^4}{3} + O(x^6)$.

- Numerator: $\sin^2 x - x^2 \cos^2 x = (x^2 - \frac{x^4}{3}) - x^2(1 - x^2 + \frac{x^4}{3}) = x^2 - \frac{x^4}{3} - x^2 + x^4 - \frac{x^6}{3} = x^4 - \frac{x^4}{3} = \frac{2x^4}{3} + O(x^6)$.
- Denominator: $x^2 \sin^2 x = x^2(x^2 - \frac{x^4}{3} + \cdots) = x^4 - \frac{x^6}{3} + \cdots = x^4 + O(x^6)$.
- The limit becomes $\lim_{x \to 0} \frac{\frac{2x^4}{3} + O(x^6)}{x^4 + O(x^6)} = \lim_{x \to 0} \frac{\frac{2}{3} + O(x^2)}{1 + O(x^2)} = \frac{2}{3}$.

(b) If A, B, C are the angles of a triangle such that $\sin^2 A + \sin^2 B + \sin^2 C =$ **Constant, prove that** $\frac{dA}{dB} = \frac{\tan A - \tan B}{\tan C - \tan A}$**.**

- Given $A + B + C = \pi$ (since A, B, C are angles of a triangle).
- So, $C = \pi - (A + B)$.
- $\sin C = \sin(\pi - (A + B)) = \sin(A + B)$.
- $\sin^2 A + \sin^2 B + \sin^2(A + B) = k$ (Constant).
- Differentiate with respect to B: $2\sin A \cos A \frac{dA}{dB} + 2\sin B \cos B + 2\sin(A + B)\cos(A + B)\left(\frac{dA}{dB} + 1\right) = 0$.
- $\sin 2A \frac{dA}{dB} + \sin 2B + \sin 2(A + B)\left(\frac{dA}{dB} + 1\right) = 0$.
- $\frac{dA}{dB}(\sin 2A + \sin 2(A + B)) = -(\sin 2B + \sin 2(A + B))$.
- $\frac{dA}{dB}(\sin 2A - \sin 2C) = -(\sin 2B - \sin 2C)$ (since $A + B = \pi - C$, $2(A + B) = 2\pi - 2C$, so $\sin 2(A + B) = -\sin 2C$).
- $\frac{dA}{dB}(\sin 2A - \sin 2C) = \sin 2C - \sin 2B$.

- $\frac{dA}{dB} = \frac{\sin 2C - \sin 2B}{\sin 2A - \sin 2C}$.
- Using $\sin X - \sin Y = 2\cos\left(\frac{X+Y}{2}\right)\sin\left(\frac{X-Y}{2}\right)$: Numerator: $2\cos(C + B)\sin(C - B)$. Denominator: $2\cos(A + C)\sin(A - C)$.
- $\frac{dA}{dB} = \frac{\cos(C+B)\sin(C-B)}{\cos(A+C)\sin(A-C)}$.
- Since $A + B + C = \pi$: $C + B = \pi - A \Longrightarrow \cos(C + B) = -\cos A$. $A + C = \pi - B \Longrightarrow \cos(A + C) = -\cos B$.
- $\frac{dA}{dB} = \frac{-\cos A\sin(C-B)}{-\cos B\sin(A-C)} = \frac{\cos A\sin(C-B)}{\cos B\sin(A-C)}$.
- Now, let's look at the RHS of the expression to prove: $\frac{\tan A - \tan B}{\tan C - \tan A}$.
- $\frac{\frac{\sin A}{\cos A} - \frac{\sin B}{\cos B}}{\frac{\sin C}{\cos C} - \frac{\sin A}{\cos A}} = \frac{\sin A\cos B - \cos A\sin B}{\cos A\cos B} \cdot \frac{\cos C\cos A}{\sin C\cos A - \cos C\sin A}$
- $= \frac{\sin(A-B)}{\cos A\cos B} \cdot \frac{\cos C\cos A}{\sin(C-A)}$
- $= \frac{\sin(A-B)\cos C}{\cos B\sin(C-A)}$.
- We need to show $\frac{\cos A\sin(C-B)}{\cos B\sin(A-C)} = \frac{\sin(A-B)\cos C}{\cos B\sin(C-A)}$.
- This means $\cos A\sin(C - B)\sin(C - A) = \sin(A - B)\cos C\sin(A - C)$.
- Since $\sin(A - C) = -\sin(C - A)$, this becomes $\cos A\sin(C - B) = -\sin(A - B)\cos C$.
- $\cos A(\sin C\cos B - \cos C\sin B) = -(\sin A\cos B - \cos A\sin B)\cos C$.
- $\cos A\sin C\cos B - \cos A\cos C\sin B = -\sin A\cos B\cos C + \cos A\sin B\cos C$.
- $\cos A\sin C\cos B + \sin A\cos B\cos C = 2\cos A\cos C\sin B$.
- Divide by $\cos A\cos B\cos C$: $\tan C + \tan A = 2\tan B$.

- This is only true if A, B, C are in arithmetic progression, which is not generally given.
- Let's recheck the differentiation steps carefully.
- $\sin 2A \frac{dA}{dB} + \sin 2B + \sin 2(A+B)\left(\frac{dA}{dB} + 1\right) = 0.$
- $\frac{dA}{dB}(\sin 2A + \sin 2(A+B)) = -(\sin 2B + \sin 2(A+B)).$
- Since $A + B + C = \pi$, $A + B = \pi - C$, so $\sin 2(A+B) = \sin(2\pi - 2C) = -\sin 2C$.
- $\frac{dA}{dB}(\sin 2A - \sin 2C) = -(\sin 2B - \sin 2C).$
- $\frac{dA}{dB} = \frac{\sin 2C - \sin 2B}{\sin 2A - \sin 2C}.$
- Using the identity $\sin X - \sin Y = 2\cos\left(\frac{X+Y}{2}\right)\sin\left(\frac{X-Y}{2}\right)$:
- Numerator: $2\cos(C+B)\sin(C-B)$.
- Denominator: $2\cos(A+C)\sin(A-C)$.
- $\frac{dA}{dB} = \frac{\cos(C+B)\sin(C-B)}{\cos(A+C)\sin(A-C)}.$
- We know $C + B = \pi - A$, so $\cos(C+B) = -\cos A$.
- We know $A + C = \pi - B$, so $\cos(A+C) = -\cos B$.
- So, $\frac{dA}{dB} = \frac{-\cos A \sin(C-B)}{-\cos B \sin(A-C)} = \frac{\cos A \sin(C-B)}{\cos B \sin(A-C)}.$
- Now, let's expand the RHS:
- $\frac{\tan A - \tan B}{\tan C - \tan A} = \frac{\frac{\sin A}{\cos A} - \frac{\sin B}{\cos B}}{\frac{\sin C}{\cos C} - \frac{\sin A}{\cos A}} = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B} \cdot \frac{\cos C \cos A}{\sin C \cos A - \cos C \sin A}$
- $= \frac{\sin(A-B)}{\cos A \cos B} \cdot \frac{\cos C \cos A}{\sin(C-A)} = \frac{\sin(A-B)\cos C}{\cos B \sin(C-A)}.$
- We need to prove $\frac{\cos A \sin(C-B)}{\cos B \sin(A-C)} = \frac{\sin(A-B)\cos C}{\cos B \sin(C-A)}.$

- Since $\sin(A-C) = -\sin(C-A)$, the equality becomes:
- $\frac{\cos A \sin(C-B)}{\cos B \sin(A-C)} = \frac{\sin(A-B)\cos C}{-\cos B \sin(A-C)}$.
- This implies $\cos A \sin(C-B) = -\sin(A-B)\cos C$.
- $\cos A(\sin C \cos B - \cos C \sin B) = -(\sin A \cos B - \cos A \sin B)\cos C$.
- $\cos A \sin C \cos B - \cos A \cos C \sin B = -\sin A \cos B \cos C + \cos A \sin B \cos C$.
- $\cos A \sin C \cos B + \sin A \cos B \cos C = 2\cos A \sin B \cos C$.
- Divide by $\cos A \cos B \cos C$ (assuming none are zero, which is generally true for angles of a triangle):
- $\frac{\sin C}{\cos C} + \frac{\sin A}{\cos A} = 2\frac{\sin B}{\cos B}$.
- $\tan C + \tan A = 2\tan B$.
- This is a condition for an arithmetic progression of tangents, not a general identity for any triangle with $\sin^2 A + \sin^2 B + \sin^2 C =$ Constant.
- There might be a typo in the question's target expression or the conditions. However, the derivative step is correct. Assuming the target expression is correct, let's verify if there is an algebraic manipulation missing or a standard identity for this condition.
- A known result for $\sin^2 A + \sin^2 B + \sin^2 C =$ constant is that the triangle must be right-angled. If the constant is 2, then it's a right-angled triangle.
- If $A + B + C = \pi$ and $\sin^2 A + \sin^2 B + \sin^2 C = K$, then $2\sin A \cos A dA + 2\sin B \cos B dB + 2\sin C \cos C dC = 0$.
- $\sin 2A dA + \sin 2B dB + \sin 2C dC = 0$.
- Since $A + B + C = \pi$, $dA + dB + dC = 0$, so $dC = -(dA + dB)$.

- $\sin 2A dA + \sin 2B dB - \sin 2C(dA + dB) = 0$.
- $(\sin 2A - \sin 2C)dA + (\sin 2B - \sin 2C)dB = 0$.
- $\frac{dA}{dB} = -\frac{\sin 2B - \sin 2C}{\sin 2A - \sin 2C} = \frac{\sin 2C - \sin 2B}{\sin 2A - \sin 2C}$. This part is correct.
- The simplification to the tangent form seems to rely on the specific identity $\tan A + \tan C = 2\tan B$. This is not generally true. It is possible that the problem expects a specific relation for the angles. Without that, the proof might not hold.
- Let's check alternative forms of $\frac{\sin 2C - \sin 2B}{\sin 2A - \sin 2C}$.
- We found $\frac{dA}{dB} = \frac{\cos A \sin(C-B)}{\cos B \sin(A-C)}$.
- The RHS is $\frac{\tan A - \tan B}{\tan C - \tan A} = \frac{\sin(A-B)\cos C}{\cos B \sin(C-A)}$.
- For them to be equal: $\frac{\cos A \sin(C-B)}{\sin(A-C)} = \frac{\sin(A-B)\cos C}{\sin(C-A)}$.
- $\cos A \sin(C - B)\sin(C - A) = \sin(A - B)\cos C \sin(A - C)$.
- Since $\sin(A - C) = -\sin(C - A)$, we have:
- $\cos A \sin(C - B) = -\sin(A - B)\cos C$.
- $\cos A(\sin C \cos B - \cos C \sin B) = -(\sin A \cos B - \cos A \sin B)\cos C$.
- $\cos A \sin C \cos B - \cos A \cos C \sin B = -\sin A \cos B \cos C + \cos A \sin B \cos C$.
- $\cos A \sin C \cos B + \sin A \cos B \cos C = 2\cos A \cos C \sin B$.
- Divide by $\cos A \cos B \cos C$:
- $\tan C + \tan A = 2\tan B$.
- This identity means that $\tan A, \tan B, \tan C$ are in arithmetic progression. This condition is not implied by $\sin^2 A + \sin^2 B +$

$\sin^2 C =$ Constant. Therefore, the statement to prove is likely incorrect or requires additional conditions not specified.

3. **(a) Evaluate** $\iint xy(x+y), dx, dy$ **over the area between** $y = x^2$ **and** $y = x$**.**

   - First, find the points of intersection of $y = x^2$ and $y = x$.
   - $x^2 = x \Longrightarrow x^2 - x = 0 \Longrightarrow x(x-1) = 0$.
   - So, $x = 0$ and $x = 1$. The corresponding y-values are $y = 0$ and $y = 1$.
   - The region of integration is bounded by $y = x^2$ from below and $y = x$ from above, for x from 0 to 1.
   - The integral is $\int_0^1 \int_{x^2}^{x} (x^2y + xy^2), dy, dx$.
   - Integrate with respect to y: $\int_{x^2}^{x} (x^2y + xy^2), dy = \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{y=x^2}^{y=x} = \left(\frac{x^2(x)^2}{2} + \frac{x(x)^3}{3}\right) - \left(\frac{x^2(x^2)^2}{2} + \frac{x(x^2)^3}{3}\right) = \left(\frac{x^4}{2} + \frac{x^4}{3}\right) - \left(\frac{x^6}{2} + \frac{x^7}{3}\right) = \frac{3x^4+2x^4}{6} - \left(\frac{3x^6+2x^7}{6}\right) = \frac{5x^4}{6} - \frac{3x^6+2x^7}{6}$.
   - Now, integrate with respect to x: $\int_0^1 \left(\frac{5x^4}{6} - \frac{3x^6}{6} - \frac{2x^7}{6}\right), dx = \frac{1}{6}\int_0^1 (5x^4 - 3x^6 - 2x^7), dx = \frac{1}{6}\left[5\frac{x^5}{5} - 3\frac{x^7}{7} - 2\frac{x^8}{8}\right]_0^1 = \frac{1}{6}\left[x^5 - \frac{3}{7}x^7 - \frac{1}{4}x^8\right]_0^1 = \frac{1}{6}\left[1^5 - \frac{3}{7}(1)^7 - \frac{1}{4}(1)^8\right] - 0 = \frac{1}{6}\left(1 - \frac{3}{7} - \frac{1}{4}\right) = \frac{1}{6}\left(\frac{28-12-7}{28}\right) = \frac{1}{6}\left(\frac{9}{28}\right) = \frac{9}{168} = \frac{3}{56}$.

   (b) Solve $x\cos x \frac{dy}{dx} + y(\sin x + \cos x) = 1$**.**

   - This is a first-order linear differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$.
   - Divide by $x\cos x$: $\frac{dy}{dx} + \frac{\sin x + \cos x}{x\cos x} y = \frac{1}{x\cos x}$.

- So, $P(x) = \frac{\sin x + \cos x}{x\cos x} = \frac{\sin x}{x\cos x} + \frac{\cos x}{x\cos x} = \frac{\tan x}{x} + \frac{1}{x}$.
- The integrating factor (IF) is $e^{\int P(x),dx}$.
- $\int P(x), dx = \int \left(\frac{\tan x}{x} + \frac{1}{x}\right), dx$. This integral is not straightforward in terms of elementary functions for $\frac{\tan x}{x}$.
- Let's rewrite $P(x)$ as $\frac{\sin x + \cos x}{x\cos x} = \frac{1}{x} + \frac{\sin x}{\cos x} = \frac{1}{x} + \tan x$.
- $\text{IF} = e^{\int (\frac{1}{x} + \tan x),dx} = e^{\ln|x| + \ln|\sec x|} = e^{\ln|x\sec x|} = x\sec x$.
- Multiply the entire equation by the integrating factor: $x\sec x \frac{dy}{dx} + yx\sec x\left(\frac{1}{x} + \tan x\right) = x\sec x \cdot \frac{1}{x\cos x}$. $x\sec x \frac{dy}{dx} + y(\sec x + x\sec x\tan x) = \sec x \cdot \frac{1}{\cos x} = \sec^2 x$.
- The LHS is the derivative of $(y \cdot IF)$: $\frac{d}{dx}(y \cdot x\sec x) = \sec^2 x$.
- Integrate both sides with respect to x: $y \cdot x\sec x = \int \sec^2 x, dx$. $y \cdot x\sec x = \tan x + C$.
- Solve for y: $y = \frac{\tan x + C}{x\sec x} = \frac{\sin x/\cos x + C}{x/\cos x}$. $y = \frac{\sin x + C\cos x}{x}$.

4. **(a) Solve the partial differential equation $p(q^2 + 1) + (b - z)q = 0$.**

- This is a first-order non-linear partial differential equation. It is of Clairaut's form $z = px + qy + f(p, q)$ or can be solved using Charpit's method.
- Rearrange the equation: $p(q^2 + 1) = (z - b)q$.
- $p\frac{q^2+1}{q} = z - b$.
- $p(q + \frac{1}{q}) = z - b$.
- This equation can be separated: $\frac{p}{z-b} = \frac{q}{q^2+1}$.

- Let this common ratio be equal to a constant a.
- So, $\frac{p}{z-b} = a \Rightarrow p = a(z-b)$.
- And $\frac{q}{q^2+1} = a \Rightarrow q = a(q^2+1)$.
- We have $dz = pdx + qdy$.
- Substitute p and q: $dz = a(z-b)dx + a(q^2+1)dy$.
- From $q = a(q^2+1)$, we have $aq^2 - q + a = 0$.
- Solving for q using the quadratic formula: $q = \frac{-(-1)\pm\sqrt{(-1)^2-4(a)(a)}}{2a} = \frac{1\pm\sqrt{1-4a^2}}{2a}$.
- For q to be real, $1 - 4a^2 \geq 0 \Rightarrow 4a^2 \leq 1 \Rightarrow a^2 \leq \frac{1}{4} \Rightarrow -\frac{1}{2} \leq a \leq \frac{1}{2}$.
- Now substitute $p = a(z-b)$ into $dz = pdx + qdy$: $dz = a(z-b)dx + qdy$. $\frac{dz}{z-b} = adx + \frac{q}{z-b}dy$.
- We have $\frac{q}{z-b} = \frac{q}{p/a} = \frac{aq}{p}$. This approach seems to complicate things.
- Let's go back to $\frac{p}{z-b} = \frac{q}{q^2+1} = a$.
- We need q in terms of a. $q = \frac{1\pm\sqrt{1-4a^2}}{2a}$.
- Now, $dz = pdx + qdy$.
- $dz = a(z-b)dx + \frac{1\pm\sqrt{1-4a^2}}{2a}dy$.
- $\frac{dz}{z-b} = adx + \frac{1\pm\sqrt{1-4a^2}}{2a(z-b)}dy$. This substitution is incorrect as q is also a function of x, y, z.

- The method of separation of variables (for a non-linear PDE) is applicable if we can write $F(x,p) = G(y,q)$. This is not the case here.
- Let's try to find if it fits Charpit's auxiliary equations.
- $F = p(q^2+1) + (b-z)q = 0$.
- $\frac{dp}{F_x+pF_z} = \frac{dq}{F_y+qF_z} = \frac{dz}{-pF_p-qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$.
- $F_x = 0$, $F_y = 0$, $F_z = -q$, $F_p = q^2+1$, $F_q = 2pq + (b-z)$.
- $\frac{dp}{p(-q)} = \frac{dq}{q(-q)}$. This gives $\frac{dp}{-pq} = \frac{dq}{-q^2}$.
- $\frac{dp}{p} = \frac{dq}{q} \Longrightarrow \ln p = \ln q + \ln c \Longrightarrow p = cq$.
- Substitute $p = cq$ into the original equation: $cq(q^2+1) + (b-z)q = 0$. Since q is not identically zero (otherwise $p = 0$, and $0 = 0$), we can divide by q: $c(q^2+1) + (b-z) = 0$. $cq^2 + c + b - z = 0$. $cq^2 = z - b - c$. $q^2 = \frac{z-b-c}{c}$. $q = \sqrt{\frac{z-b-c}{c}}$.
- Then $p = c\sqrt{\frac{z-b-c}{c}} = \sqrt{c(z-b-c)}$.
- Now use $dz = pdx + qdy$.
- $dz = \sqrt{c(z-b-c)}dx + \sqrt{\frac{z-b-c}{c}}dy$.
- Divide by $\sqrt{z-b-c}$: $\frac{dz}{\sqrt{z-b-c}} = \sqrt{c}dx + \frac{1}{\sqrt{c}}dy$.
- Integrate both sides: $\int \frac{dz}{\sqrt{z-b-c}} = \int \sqrt{c}dx + \int \frac{1}{\sqrt{c}}dy$. $2\sqrt{z-b-c} = \sqrt{c}x + \frac{1}{\sqrt{c}}y + k$, where k is an integration constant.
- Squaring both sides (general solution): $4(z-b-c) = \left(\sqrt{c}x + \frac{1}{\sqrt{c}}y + k\right)^2$. This is the complete integral.

(b) Show that $\int_0^{\pi/2} \sqrt{\tan\theta}\,, d\theta = \frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{4}\right) = 2\int_0^{\infty} \frac{x^2}{1+x^4}\,, dx$.

- Let $I = \int_0^{\pi/2} \sqrt{\tan\theta}\,, d\theta$.
- Substitute $\tan\theta = x^2$. Then $\theta = \arctan(x^2)$.
- $d\theta = \frac{2x}{1+x^4} dx$.
- When $\theta = 0, x = 0$. When $\theta = \pi/2, x \to \infty$.
- $I = \int_0^{\infty} x \frac{2x}{1+x^4} dx = \int_0^{\infty} \frac{2x^2}{1+x^4} dx = 2\int_0^{\infty} \frac{x^2}{1+x^4} dx$. This proves the second equality.
- Now, we need to show $I = \frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{4}\right)$.
- The Beta function is defined as $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta, d\theta$.
- We have $\int_0^{\pi/2} \sqrt{\tan\theta}\,, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta, d\theta$.
- Comparing with the Beta function definition: $2m - 1 = 1/2 \Longrightarrow 2m = 3/2 \Longrightarrow m = 3/4$. $2n - 1 = -1/2 \Longrightarrow 2n = 1/2 \Longrightarrow n = 1/4$.
- So, $\int_0^{\pi/2} \sqrt{\tan\theta}\,, d\theta = \frac{1}{2} \cdot 2\int_0^{\pi/2} \sin^{2(3/4)-1}\theta \cos^{2(1/4)-1}\theta, d\theta = \frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{4}\right)$.
- This proves the first equality.

5. **(a) Change the order of integration in the integral $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{ax}} f\,(x,y), dy, dx$.**

   - The region of integration is defined by: $0 \le x \le a\ \sqrt{ax - x^2} \le y \le \sqrt{ax}$
   - Let's analyze the curves: $y = \sqrt{ax - x^2} \Longrightarrow y^2 = ax - x^2 \Longrightarrow x^2 - ax + y^2 = 0$. Completing the square for x: $(x - a/2)^2 - a^2/4 +$

$y^2 = 0 \Longrightarrow (x - a/2)^2 + y^2 = (a/2)^2$. This is a circle centered at $(a/2, 0)$ with radius $a/2$. Since $y \geq 0$, it's the upper semi-circle. $y = \sqrt{ax} \Longrightarrow y^2 = ax$. This is a parabola opening to the right, symmetric about the x-axis. Since $y \geq 0$, it's the upper half of the parabola.

- Intersection points of $y = x^2$ and $y = x$: The prompt had $y = x^2$ and $y = x$ in 3a, this is a different problem.
- Let's find the intersection of the two curves for this problem. $\sqrt{ax - x^2} = \sqrt{ax}$ $ax - x^2 = ax$ $-x^2 = 0 \Longrightarrow x = 0$. So they intersect at (0,0).
- Consider the limits for x: $0 \leq x \leq a$.
- For $y = \sqrt{ax}$, if $x = a$, $y = \sqrt{a^2} = a$.
- For $y = \sqrt{ax - x^2}$, if $x = a$, $y = \sqrt{a^2 - a^2} = 0$.
- The region is above the semi-circle $(x - a/2)^2 + y^2 = (a/2)^2$ and below the parabola $y^2 = ax$.
- When changing the order of integration to $dxdy$, we need to express x in terms of y.
- From $y^2 = ax$, we have $x = y^2/a$. This is the lower bound for x on the right side of the region.
- From $(x - a/2)^2 + y^2 = (a/2)^2$, we have $(x - a/2)^2 = (a/2)^2 - y^2$. $x - a/2 = \pm\sqrt{(a/2)^2 - y^2}$. $x = a/2 \pm \sqrt{(a/2)^2 - y^2}$.
- The region lies from $x = 0$ to $x = a$. The upper semi-circle goes from $x = 0$ to $x = a$.
- The parabola $y^2 = ax$ also starts at (0,0). For $x = a$, $y^2 = a^2 \Longrightarrow y = a$.
- The maximum y value in the region is a (from $y = \sqrt{ax}$ at $x = a$).

- The overall region is complex. It's best to split it into two or three simpler regions.
- The maximum value of y from $(x - a/2)^2 + y^2 = (a/2)^2$ is $y = a/2$ (when $x = a/2$).
- The maximum value of y from $y^2 = ax$ for $x \in [0, a]$ is $y = a$ (when $x = a$).
- Let's sketch the region: The parabola $y^2 = ax$ starts at origin and goes up. The circle $(x - a/2)^2 + y^2 = (a/2)^2$ also starts at origin and goes up to $y = a/2$ at $x = a/2$, then down to $x = a$. The region is bounded below by the circle and above by the parabola.
- Region 1: $0 \le y \le a/2$. For a given y, x varies from $x = a/2 + \sqrt{(a/2)^2 - y^2}$ (right side of circle) to $x = y^2/a$ (from parabola). This is incorrect.
- The region is defined as: x from 0 to a. For each x, y goes from the circle to the parabola.
- Let's find the intersection of the circle and parabola: $y^2 = ax$. Substitute into $(x - a/2)^2 + y^2 = (a/2)^2$: $(x - a/2)^2 + ax = (a/2)^2$ $x^2 - ax + a^2/4 + ax = a^2/4$ $x^2 = 0 \Longrightarrow x = 0$. So the only intersection point is $(0,0)$. This confirms the initial analysis.
- The region is from the semicircle to the parabola.
- For change of order, we need to split the region.
- The full region is bounded by $y = \sqrt{ax - x^2}$, $y = \sqrt{ax}$, $x = 0$, $x = a$.
- Maximum y value is a (at $x = a$ for $y = \sqrt{ax}$).
- The region can be divided based on the y-values.
- Case 1: $0 \le y \le a/2$. For this range of y, x comes from two different parts of the circle: $x_1 = a/2 - \sqrt{(a/2)^2 - y^2}$ (left side of circle) and $x_2 = a/2 + \sqrt{(a/2)^2 - y^2}$ (right side of circle). The parabola is $x =$

y^2/a. So, for $0 \le y \le a/2$: We have x from y^2/a to $a/2 - \sqrt{(a/2)^2 - y^2}$ and from $a/2 + \sqrt{(a/2)^2 - y^2}$ to some upper limit. This is becoming too complicated.

- Let's redefine the curves: Lower curve: $x^2 - ax + y^2 = 0 \Rightarrow x = \frac{a \pm \sqrt{a^2 - 4y^2}}{2}$. Upper curve: $x = y^2/a$.
- Maximum y in the region: $y_{max} = a$ (from $y = \sqrt{ax}$ at $x = a$).
- For the circle, max y is $a/2$.
- So, we need to split the region based on y: from $y = 0$ to $y = a/2$ and from $y = a/2$ to $y = a$.
- Region 1: $0 \le y \le a/2$. For a given y, x varies from the right half of the circle to the left half of the circle. This is not correct from the description. The region is bounded below by the circle and above by the parabola. Let $x_L(y) = y^2/a$ (from parabola). Let $x_R(y) = a/2 + \sqrt{(a/2)^2 - y^2}$ (right side of circle). Let $x_C(y) = a/2 - \sqrt{(a/2)^2 - y^2}$ (left side of circle).
- The original integral is $\int_0^a d\,x \int_{\text{circle}}^{\text{parabola}} d\,y$.
- This implies for each x, the integration is from the circle to the parabola.
- This means the region is above the semi-circle and below the parabola.
- Intersection points of $y = \sqrt{ax}$ and $y = \sqrt{ax - x^2}$ is only at (0,0).
- The region looks like this: It starts at $(0,0)$. The upper boundary is $y = \sqrt{ax}$. The lower boundary is $y = \sqrt{ax - x^2}$. For $x = 0$, both are 0. For $x = a$, $y = \sqrt{a^2} = a$ and $y = \sqrt{a^2 - a^2} = 0$. This means at $x = a$, the region spans from $y = 0$ to $y = a$.

- The circle boundary goes from $(0,0)$ to $(a, 0)$ passing through $(a/2, a/2)$.
- The parabola boundary goes from $(0,0)$ to (a, a).
- The region looks like a crescent shape between the upper part of the circle and the upper part of the parabola.
- To change the order of integration, we need to split the region.
- Region A: $0 \le y \le a/2$. For a fixed y, x varies from $x = y^2/a$ (parabola) to $x = a/2 - \sqrt{(a/2)^2 - y^2}$ (left circle). This is if the region is bounded by $y = x^2$ and $y = x$ in 3a, this problem is $y = \sqrt{ax - x^2}$ and $y = \sqrt{ax}$. Let's be clear about the boundaries. $y_1(x) = \sqrt{ax - x^2}$ and $y_2(x) = \sqrt{ax}$. The integral is $\int_0^a \left(\int_{y_1(x)}^{y_2(x)} f(x, y) dy \right) dx$. This means the region is above $y_1(x)$ and below $y_2(x)$. The intersection of $y_1(x)$ and $y_2(x)$ occurs only at $x = 0$. At $x = a$, $y_1(a) = 0$ and $y_2(a) = a$. The region is from $x = 0$ to $x = a$. The maximum y value for the circle is $a/2$ (at $x = a/2$). The maximum y value for the parabola $y = \sqrt{ax}$ in $x \in [0, a]$ is a (at $x = a$).
- This means we have two parts of the region:
  i. $0 \le y \le a/2$. Here x goes from the parabola $x = y^2/a$ to the left arm of the circle $x = a/2 - \sqrt{(a/2)^2 - y^2}$. AND x goes from the right arm of the circle $x = a/2 + \sqrt{(a/2)^2 - y^2}$ to the vertical line $x = a$.

  ii. $a/2 < y \le a$. Here x goes from the parabola $x = y^2/a$ to the vertical line $x = a$. (The circle part is not active here).
- This is tricky. Let's re-evaluate the region. The description "over the area between $y = x^2$ and $y = x$" in 3a is much simpler. This one is "between $\sqrt{ax - x^2}$ and $\sqrt{ax}$".
- The phrasing means the region is bounded by these two curves.

- The region is for $0 \le x \le a$.
- Region 1 (R_1): From $y = 0$ to $y = a/2$. In this range, the circle has two parts: $x = a/2 \pm \sqrt{(a/2)^2 - y^2}$. The parabola is $x = y^2/a$. The lower curve is $y_1(x) = \sqrt{ax - x^2}$ (circle), and the upper curve is $y_2(x) = \sqrt{ax}$ (parabola). The region defined is for $x \in [0, a]$, for each x, y is between the circle and the parabola. So, y starts at 0 and goes up to a. When $0 \le y \le a/2$: x varies from y^2/a to $a/2 - \sqrt{(a/2)^2 - y^2}$. Also x from $a/2 + \sqrt{(a/2)^2 - y^2}$ to a. When $a/2 < y \le a$: x varies from y^2/a to a.
- So the integral splits into three parts.
- $\int_0^{a/2} \int_{y^2/a}^{a/2-\sqrt{(a/2)^2-y^2}} f\,(x,y), dx, dy + \int_0^{a/2} \int_{a/2+\sqrt{(a/2)^2-y^2}}^{a} f\,(x,y), dx, dy + \int_{a/2}^{a} \int_{y^2/a}^{a} f\,(x,y), dx, dy$.

(b) Find $y_n(0)$ when $y = \log(x + \sqrt{1 + x^2})$.

- Given $y = \log(x + \sqrt{1 + x^2})$.
- First derivative: $\frac{dy}{dx} = \frac{1}{x+\sqrt{1+x^2}} \cdot \left(1 + \frac{2x}{2\sqrt{1+x^2}}\right) \frac{dy}{dx} = \frac{1}{x+\sqrt{1+x^2}} \cdot \left(1 + \frac{x}{\sqrt{1+x^2}}\right) \frac{dy}{dx} = \frac{1}{x+\sqrt{1+x^2}} \cdot \left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right) \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$.
- So, $\sqrt{1 + x^2}\,\frac{dy}{dx} = 1$.
- Square both sides: $(1 + x^2)\left(\frac{dy}{dx}\right)^2 = 1$.
- Differentiate using Leibniz rule for products: $2x\left(\frac{dy}{dx}\right)^2 + (1 + x^2) \cdot 2\frac{dy}{dx}\frac{d^2y}{dx^2} = 0$.
- Divide by $2\frac{dy}{dx}$ (assuming $\frac{dy}{dx} \neq 0$): $x\frac{dy}{dx} + (1 + x^2)\frac{d^2y}{dx^2} = 0$.
- This is the differential equation for y.

- Let's find $y_n(0)$ using Maclaurin's series for $y(x) = \sum_{n=0}^{\infty} \frac{y_n(0)}{n!} x^n$.
- From $y = \log(x + \sqrt{1 + x^2})$: $y(0) = \log(0 + \sqrt{1 + 0}) = \log(1) = 0$. So $y_0(0) = 0$.
- From $y_1 = \frac{1}{\sqrt{1+x^2}} = (1 + x^2)^{-1/2}$: $y_1(0) = (1 + 0)^{-1/2} = 1$.
- From $(1 + x^2)y_2 + xy_1 = 0$: At $x = 0$: $(1 + 0)y_2(0) + 0 \cdot y_1(0) = 0 \Longrightarrow y_2(0) = 0$.
- Differentiate $(1 + x^2)y_2 + xy_1 = 0$ using Leibniz theorem $n - 2$ times (for y_n): $(1 + x^2)y_{n-2+2} + (n - 2)(2x)y_{n-2+1} + \frac{(n-2)(n-3)}{2}(2)y_{n-2} + xy_{n-2+1} + (n - 2)(1)y_{n-2} = 0$. $(1 + x^2)y_n + 2(n - 2)xy_{n-1} + (n - 2)(n - 3)y_{n-2} + xy_{n-1} + (n - 2)y_{n-2} = 0$. $(1 + x^2)y_n + (2n - 4 + 1)xy_{n-1} + ((n - 2)(n - 3) + (n - 2))y_{n-2} = 0$. $(1 + x^2)y_n + (2n - 3)xy_{n-1} + (n - 2)(n - 3 + 1)y_{n-2} = 0$. $(1 + x^2)y_n + (2n - 3)xy_{n-1} + (n - 2)^2y_{n-2} = 0$.
- This is Legendre's differential equation.
- Now, substitute $x = 0$: $(1 + 0)y_n(0) + (2n - 3)(0)y_{n-1}(0) + (n - 2)^2y_{n-2}(0) = 0$. $y_n(0) + (n - 2)^2y_{n-2}(0) = 0$. $y_n(0) = -(n - 2)^2y_{n-2}(0)$.
- This is a recurrence relation.
- $y_0(0) = 0$.
- $y_1(0) = 1$.
- For $n = 2$: $y_2(0) = -(2 - 2)^2y_0(0) = 0$. (Matches previous calculation).
- For $n = 3$: $y_3(0) = -(3 - 2)^2y_1(0) = -(1)^2 \cdot 1 = -1$.
- For $n = 4$: $y_4(0) = -(4 - 2)^2y_2(0) = -(2)^2 \cdot 0 = 0$.
- For $n = 5$: $y_5(0) = -(5 - 2)^2y_3(0) = -(3)^2 \cdot (-1) = 9$.

- So, $y_n(0) = 0$ if n is even and $n \geq 2$.

- If n is odd, let $n = 2k + 1$. $y_{2k+1}(0) = -(2k-1)^2 y_{2k-1}(0)$. $y_1(0) = 1$. $y_3(0) = -1^2 y_1(0) = -1$. $y_5(0) = -3^2 y_3(0) = -9(-1) = 9 = (3 \cdot 1)^2$. No. $(3)^2 = 9$. $y_5(0) = -(5-2)^2 y_3(0) = -3^2(-1) = 9$. $y_7(0) = -(7-2)^2 y_5(0) = -5^2(9) = -25 \cdot 9 = -225$.

- The pattern for odd n: $y_n(0) = (-1)^{(n-1)/2} \cdot 1^2 \cdot 3^2 \cdot \ldots \cdot (n-2)^2$. $y_n(0) = (-1)^{(n-1)/2}[(n-2)!!]^2$ where $n!!$ is the double factorial. Or more precisely: $y_n(0) = (-1)^{(n-1)/2} \frac{[(n-2)!!]^2}{(n-2)!(n-4)!\ldots}$ No. $y_n(0) = (-1)^{(n-1)/2} \prod_{k=1}^{(n-1)/2} (2k-1)^2$. For $n = 1$, $y_1(0) = (-1)^0 \prod_{k=1}^{0} = 1$. (empty product is 1). For $n = 3$, $y_3(0) = (-1)^1 \prod_{k=1}^{1} (2k-1)^2 = -1^2 = -1$. For $n = 5$, $y_5(0) = (-1)^2 \prod_{k=1}^{2} (2k-1)^2 = 1^2 \cdot 3^2 = 9$. For $n = 7$, $y_7(0) = (-1)^3 \prod_{k=1}^{3} (2k-1)^2 = -(1^2 \cdot 3^2 \cdot 5^2) = -225$.

- So, $y_n(0) = 0$ for even $n \geq 2$.

- For odd n: $y_n(0) = (-1)^{(n-1)/2}((n-2)!!)^2$.

6. **(a) Assuming the validity of differentiation under integral sign, show that $\int_0^\infty e^{-x^2} \cos ax, dx = \frac{1}{2}\sqrt{\pi} e^{-a^2/4}$.**

- Let $I(a) = \int_0^\infty e^{-x^2} \cos ax, dx$.

- Differentiate with respect to a: $\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}(e^{-x^2} \cos ax), dx = \int_0^\infty e^{-x^2} (-x \sin ax), dx$.

- Integrate by parts: $\int u dv = uv - \int v du$. Let $u = \sin ax$, $dv = -x e^{-x^2} dx$. Then $du = a \cos ax dx$, $v = \frac{1}{2} e^{-x^2}$.

- $\frac{dI}{da} = \left[\frac{1}{2} e^{-x^2} \sin ax\right]_0^\infty - \int_0^\infty \frac{1}{2} e^{-x^2} (a \cos ax), dx$.

- The first term: $\lim_{x\to\infty} \frac{1}{2} e^{-x^2} \sin ax - \frac{1}{2} e^0 \sin(0) = 0 - 0 = 0$.

- So, $\frac{dI}{da} = -\frac{a}{2}\int_0^\infty e^{-x^2} \cos ax, dx = -\frac{a}{2}I(a)$.
- This is a first-order linear differential equation: $\frac{dI}{I} = -\frac{a}{2}da$.
- Integrate both sides: $\ln I = -\frac{a^2}{4} + C'$.
- $I(a) = e^{-\frac{a^2}{4}+C'} = Ke^{-a^2/4}$.
- To find K, we use a known value of $I(a)$.
- When $a = 0$, $I(0) = \int_0^\infty e^{-x^2} \cos(0), dx = \int_0^\infty e^{-x^2}, dx$.
- This is a Gaussian integral. We know $\int_0^\infty e^{-x^2}, dx = \frac{\sqrt{\pi}}{2}$.
- So, $I(0) = Ke^0 = K = \frac{\sqrt{\pi}}{2}$.
- Therefore, $I(a) = \frac{1}{2}\sqrt{\pi}e^{-a^2/4}$.

(b) Solve the partial differential equation, $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos x \cos 2y$.

- This is a non-homogeneous linear partial differential equation with constant coefficients.
- The symbolic form is $(D_x^2 - D_x D_y)z = \cos x \cos 2y$.
- $(D_x(D_x - D_y))z = \cos x \cos 2y$.
- Auxiliary equation for the complementary function (CF): $m(m - 1) = 0 \Rightarrow m = 0,1$.
- So, $CF = f_1(y + 0x) + f_2(y + 1x) = f_1(y) + f_2(y + x)$.
- For the particular integral (PI): $PI = \frac{1}{D_x^2 - D_x D_y}(\cos x \cos 2y)$. We can write $\cos x \cos 2y = \mathrm{Re}(e^{ix})\mathrm{Re}(e^{i2y})$. Or use trigonometric identities: $\cos x \cos 2y = \frac{1}{2}[\cos(x + 2y) + \cos(x - 2y)]$.

- $PI = \frac{1}{2}\text{Re}\left[\frac{1}{D_x^2 - D_x D_y} e^{i(x+2y)} + \frac{1}{D_x^2 - D_x D_y} e^{i(x-2y)}\right]$.
- For e^{ax+by}, substitute $D_x = a$ and $D_y = b$.
- For $e^{i(x+2y)}$, $a = i, b = 2i$. Denominator: $i^2 - i(2i) = -1 - 2i^2 = -1 + 2 = 1$. So, $\frac{1}{1} e^{i(x+2y)} = e^{i(x+2y)}$.
- For $e^{i(x-2y)}$, $a = i, b = -2i$. Denominator: $i^2 - i(-2i) = -1 + 2i^2 = -1 - 2 = -3$. So, $\frac{1}{-3} e^{i(x-2y)}$.
- $PI = \frac{1}{2}\text{Re}\left[e^{i(x+2y)} - \frac{1}{3} e^{i(x-2y)}\right]$.
- $e^{i(x+2y)} = \cos(x + 2y) + i\sin(x + 2y)$.
- $e^{i(x-2y)} = \cos(x - 2y) + i\sin(x - 2y)$.
- $PI = \frac{1}{2}\left[\cos(x + 2y) - \frac{1}{3}\cos(x - 2y)\right]$.
- The general solution is $z = CF + PI$.
- $z = f_1(y) + f_2(y + x) + \frac{1}{2}\left[\cos(x + 2y) - \frac{1}{3}\cos(x - 2y)\right]$.

7. **(a) Solve $(D^3 - 2D + 4)y = x^4 + 3x^2 - 5x + 2$.**
   - This is a non-homogeneous linear ordinary differential equation with constant coefficients.
   - Auxiliary equation for the complementary function (CF): $m^3 - 2m + 4 = 0$.
   - By inspection, if $m = -2$, $(-2)^3 - 2(-2) + 4 = -8 + 4 + 4 = 0$. So $m = -2$ is a root.
   - Divide $(m^3 - 2m + 4)$ by $(m + 2)$: $(m + 2)(m^2 - 2m + 2) = 0$.
   - For $m^2 - 2m + 2 = 0$, use quadratic formula: $m = \frac{2\pm\sqrt{(-2)^2 - 4(1)(2)}}{2} = \frac{2\pm\sqrt{4-8}}{2} = \frac{2\pm\sqrt{-4}}{2} = \frac{2\pm 2i}{2} = 1 \pm i$.

- So the roots are $m = -2, 1 + i, 1 - i$.
- $CF = C_1 e^{-2x} + e^{x}(C_2 \cos x + C_3 \sin x)$.
- For the particular integral (PI): $PI = \frac{1}{D^3 - 2D + 4}(x^4 + 3x^2 - 5x + 2)$.
- Since the RHS is a polynomial, we use the method of undetermined coefficients or expand the operator in ascending powers of D.
- $PI = \frac{1}{4 - 2D + D^3}(x^4 + 3x^2 - 5x + 2)$
- $= \frac{1}{4(1 - \frac{2D - D^3}{4})}(x^4 + 3x^2 - 5x + 2)$
- $= \frac{1}{4}\left(1 - \frac{2D - D^3}{4}\right)^{-1}(x^4 + 3x^2 - 5x + 2)$
- Using $(1 - u)^{-1} = 1 + u + u^2 + u^3 + \cdots$ where $u = \frac{2D - D^3}{4}$.
- We need derivatives up to the fourth order of x^4. So, we need terms in the expansion up to D^4.
- $u = \frac{1}{2}D - \frac{1}{4}D^3$.
- $u^2 = (\frac{1}{2}D - \frac{1}{4}D^3)^2 = \frac{1}{4}D^2 - 2 \cdot \frac{1}{2}D \cdot \frac{1}{4}D^3 + \frac{1}{16}D^6 = \frac{1}{4}D^2 - \frac{1}{4}D^4 + O(D^6)$.
- $u^3 = (\frac{1}{2}D)^3 = \frac{1}{8}D^3 + O(D^5)$.
- $u^4 = (\frac{1}{2}D)^4 = \frac{1}{16}D^4 + O(D^5)$.
- So, $\frac{1}{4}\left(1 + (\frac{1}{2}D - \frac{1}{4}D^3) + (\frac{1}{4}D^2 - \frac{1}{4}D^4) + \frac{1}{8}D^3 + \frac{1}{16}D^4 + \cdots\right)(x^4 + 3x^2 - 5x + 2)$.
- Combining terms by powers of D: $\frac{1}{4}\left(1 + \frac{1}{2}D + \frac{1}{4}D^2 + (-\frac{1}{4} + \frac{1}{8})D^3 + (-\frac{1}{4} + \frac{1}{16})D^4 + \cdots\right)$. $= \frac{1}{4}\left(1 + \frac{1}{2}D + \frac{1}{4}D^2 - \frac{1}{8}D^3 - \frac{3}{16}D^4 + \cdots\right)(x^4 + 3x^2 - 5x + 2)$.

- Now apply this to the polynomial: $y_p = \frac{1}{4}[(x^4 + 3x^2 - 5x + 2)$ $+\frac{1}{2}D(x^4 + 3x^2 - 5x + 2) = \frac{1}{2}(4x^3 + 6x - 5) + \frac{1}{4}D^2(x^4 + 3x^2 - 5x + 2) = \frac{1}{4}(12x^2 + 6) - \frac{1}{8}D^3(x^4 + 3x^2 - 5x + 2) = -\frac{1}{8}(24x)$ $-\frac{3}{16}D^4(x^4 + 3x^2 - 5x + 2) = -\frac{3}{16}(24)$].
- $y_p = \frac{1}{4}[x^4 + 3x^2 - 5x + 2 + 2x^3 + 3x - \frac{5}{2} + 3x^2 + \frac{3}{2} - 3x - \frac{9}{2}$].
- Combine terms by powers of x: x^4: $\frac{1}{4}x^4$. x^3: $\frac{2}{4}x^3 = \frac{1}{2}x^3$. x^2: $\frac{1}{4}(3x^2 + 3x^2) = \frac{6x^2}{4} = \frac{3}{2}x^2$. x: $\frac{1}{4}(-5x + 3x - 3x) = \frac{-5x}{4}$. Constant: $\frac{1}{4}(2 - \frac{5}{2} + \frac{3}{2} - \frac{9}{2}) = \frac{1}{4}(2 - \frac{11}{2}) = \frac{1}{4}(\frac{4-11}{2}) = \frac{1}{4}(-\frac{7}{2}) = -\frac{7}{8}$.
- $PI = \frac{1}{4}x^4 + \frac{1}{2}x^3 + \frac{3}{2}x^2 - \frac{5}{4}x - \frac{7}{8}$.
- The general solution is $y = CF + PI$.
- $y = C_1 e^{-2x} + e^x(C_2\cos x + C_3\sin x) + \frac{1}{4}x^4 + \frac{1}{2}x^3 + \frac{3}{2}x^2 - \frac{5}{4}x - \frac{7}{8}$.

(b) Show that $\int \cos^3 x, dx = \frac{3}{4}\sin x + \frac{1}{12}\sin 3x$.

- We can use the power reduction formula or trigonometric identity.
- Recall $\cos 3x = 4\cos^3 x - 3\cos x$.
- So, $4\cos^3 x = \cos 3x + 3\cos x$.
- $\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$.
- Integrate both sides: $\int \cos^3 x, dx = \int \frac{1}{4}(\cos 3x + 3\cos x), dx. = \frac{1}{4}\left(\int \cos 3x, dx + \int 3\cos x, dx\right). = \frac{1}{4}\left(\frac{\sin 3x}{3} + 3\sin x\right) + C. = \frac{1}{12}\sin 3x + \frac{3}{4}\sin x + C.$
- This matches the required proof (ignoring the constant of integration, as usually done for indefinite integrals in such proofs).

8. **(a) Discuss the derivability of the function:** $f(x) = \begin{cases} x, & x < 1 \\ 2 - x, & 1 \le x \le 2 \\ -2 + 3x - x^2, & x > 2 \end{cases}$ **at x = 1 and 2.**

- For a function to be derivable at a point, it must first be continuous at that point.
- **At x = 1:**
  - **Continuity at x=1:** $\lim_{x\to1^-} f(x) = \lim_{x\to1^-} x = 1$. $\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(2-x) = 2-1 = 1$. $f(1) = 2-1 = 1$. Since $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = f(1) = 1$, the function is continuous at $x = 1$.
  - **Derivability at x=1:** Left-hand derivative (LHD) at $x = 1$: $f'(1^-) = \lim_{h\to0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to0^-} \frac{(1+h)-1}{h} = \lim_{h\to0^-} \frac{h}{h} = 1$. Alternatively, $f'(x) = 1$ for $x < 1$. So $f'(1^-) = 1$. Right-hand derivative (RHD) at $x = 1$: $f'(1^+) = \lim_{h\to0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to0^+} \frac{(2-(1+h))-1}{h} = \lim_{h\to0^+} \frac{2-1-h-1}{h} = \lim_{h\to0^+} \frac{-h}{h} = -1$. Alternatively, $f'(x) = -1$ for $1 < x < 2$. So $f'(1^+) = -1$. Since $f'(1^-) \ne f'(1^+)$ $(1 \ne -1)$, the function is not derivable at $x = 1$.
- **At x = 2:**
  - **Continuity at x=2:** $\lim_{x\to2^-} f(x) = \lim_{x\to2^-}(2-x) = 2-2 = 0$. $\lim_{x\to2^+} f(x) = \lim_{x\to2^+}(-2+3x-x^2) = -2+3(2)-(2)^2 = -2+6-4 = 0$. $f(2) = 2-2 = 0$. Since $\lim_{x\to2^-} f(x) = \lim_{x\to2^+} f(x) = f(2) = 0$, the function is continuous at $x = 2$.
  - **Derivability at x=2:** Left-hand derivative (LHD) at $x = 2$: $f'(2^-) = \lim_{h\to0^-} \frac{f(2+h)-f(2)}{h} = \lim_{h\to0^-} \frac{(2-(2+h))-0}{h} = \lim_{h\to0^-} \frac{-h}{h} = -1$. Alternatively, $f'(x) = -1$ for $1 < x < 2$.

So $f'(2^-) = -1$. Right-hand derivative (RHD) at $x = 2$: $f'(2^+) = \lim_{h\to0^+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to0^+} \frac{(-2+3(2+h)-(2+h)^2)-0}{h}$ $= \lim_{h\to0^+} \frac{-2+6+3h-(4+4h+h^2)}{h} = \lim_{h\to0^+} \frac{4+3h-4-4h-h^2}{h} =$ $\lim_{h\to0^+} \frac{-h-h^2}{h} = \lim_{h\to0^+}(-1-h) = -1$. Alternatively, $f'(x) = 3-2x$ for $x > 2$. So $f'(2^+) = 3-2(2) = 3-4 = -1$. Since $f'(2^-) = f'(2^+) = -1$, the function is derivable at $x = 2$.

(b) By the elimination of the constants h and k, find the differential equation for which $(x-h)^2 + (y-k)^2 = a^2$, is a solution.

- Given equation: $(x-h)^2 + (y-k)^2 = a^2$. (1 constant a, but h, k are constants to be eliminated).
- Differentiate with respect to x: $2(x-h) + 2(y-k)\frac{dy}{dx} = 0$. $(x-h) + (y-k)y' = 0$. (2)
- Differentiate (2) with respect to x: $1 + (y-k)y'' + (y')^2 = 0$. (3)
- From (2), $x-h = -(y-k)y'$. Substitute this into (1): $(-(y-k)y')^2 + (y-k)^2 = a^2$. $(y-k)^2(y')^2 + (y-k)^2 = a^2$. $(y-k)^2((y')^2+1) = a^2$. (4)
- From (3), $y-k = -\frac{1+(y')^2}{y''}$.
- Substitute this into (4): $\left(-\frac{1+(y')^2}{y''}\right)^2((y')^2+1) = a^2$. $\frac{(1+(y')^2)^2}{(y'')^2}(1+(y')^2) = a^2$. $\frac{(1+(y')^2)^3}{(y'')^2} = a^2$. $(1+(y')^2)^3 = a^2(y'')^2$.
- This is the differential equation for which the given family of circles is a solution.
- This is a third-order differential equation for a general circle (with h, k, a as constants). However, since a is also a constant, we only need to eliminate h and k. This is a second-order differential equation.

- Let's check the number of arbitrary constants. The given equation $(x-h)^2+(y-k)^2=a^2$ represents a family of circles. There are three independent arbitrary constants to be eliminated: h,k,a. So the differential equation should be of order 3.
- Let's re-do the differentiation process assuming a is also an arbitrary constant.
- $(x-h)^2+(y-k)^2=a^2$ (1)
- $2(x-h)+2(y-k)y'=0 \Longrightarrow (x-h)+(y-k)y'=0$ (2)
- $1+(y-k)y''+(y')^2=0$ (3)
- Differentiate (3) again with respect to x: $0+(y-k)y'''+y'y''+2y'y''=0$. $(y-k)y'''+3y'y''=0$. (4)
- From (4), $(y-k)y'''=-3y'y''$.
- If $y'''\neq 0$, then $y-k=-\frac{3y'y''}{y'''}$.
- Substitute $y-k$ into (3): $1+\left(-\frac{3y'y''}{y'''}\right)y''+(y')^2=0$. $1-\frac{3y'(y'')^2}{y'''}+(y')^2=0$. Multiply by y''': $y'''-3y'(y'')^2+(y')^2y'''=0$. $y'''(1+(y')^2)=3y'(y'')^2$.
- This is a third-order differential equation, as expected for three arbitrary constants.