[This question paper contains 4 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 5649 **J**

Unique Paper Code : 2223010021

Name of the Paper : Advanced Mathematical Physics – II

Name of the Course : **B. Sc. (H) Physics (NEP UGCF)**

Semester : VI

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt five questions in all.
3. All questions carry equal marks.
4. Question number 1 is compulsory.

1. Attempt ***any six*** questions (6 × 3 = 18)

(a) If A_i is a vector, show that $F_{ij} = \frac{\partial A_i}{\partial x_j}$ is a Cartesian tensor of rank two.

(b) If A_i and B_i are two Cartesian tensors, prove that $A_i B_i$ is a scalar.

(c) Using Cartesian tensors, show that the curl of gradient of a scalar is zero.

(d) Show that $\frac{\partial A_p}{\partial x^q}$ is not a tensor under a general coordinate transformation, even though A_p is a covariant tensor of rank one.

(e) Let $A(i, j, k)$ be a set of N^3 functions of coordinates in N-dimensions. If $A_{ijk}B^{jk} = C_i$ where B^{jk} is an arbitrary contravariant tensor of rank-2 and C_i is a covariant tensor of rank-1. What can you conclude about A_{ijk} ?

(f) What is polarizability tensor?

(g) Prove that $\left\{ {s \atop p\ q} \right\} = \left\{ {s \atop q\ p} \right\}$ where $\left\{ {s \atop p\ q} \right\}$ is the Christoffel Symbol of the second kind.

2. (a) Show that δ_{ij} is an isotropic Cartesian tensor of rank two and ε_{ijk} is an isotropic Cartesian tensor of rank three. (8)

(b) Let X_{ijkl} be a Cartesian tensor of rank four such that $X_{ijkl} = X_{jikl}$ and $X_{ijkl} = -X_{ijlk}$ i.e. it is symmetric with respect to the first two indices and anti-symmetric with respect to the last two indices. How many independent components are there in X_{ijkl} in three dimensions? (6)

(c) Show that any rank two Cartesian tensor can be expressed as a sum of symmetric and antisymmetric tensors. (4)

3. (a) A rigid body consists of three point masses of 1 kg, 2 kg and 1 kg, connected by massless rods. The coordinates of the three masses are (1,1,0) , (2, -1,2) and (0, -1, -1) in meters, respectively. Determine the inertia tensor of the system. If the body is rotating with an angular velocity $\boldsymbol{\omega} = 3\hat{\imath} - 2\hat{\jmath} + 4\hat{\boldsymbol{k}}$, what is the angular momentum of the body? (12)

(b) Stress tensor σ_{ij} and strain tensor e_{kl} are related as $\sigma_{ij} = C_{ijkl}e_{kl}$ where, elastic tensor C_{ijkl} is symmetric in i, j and k, l and its most general isotropic form is given by $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\,\delta_{il}\delta_{jk}$ where λ, μ and ν are constants. Prove that $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ (6)

4. (a) If $ds^2 = 5(dx^1)^2 + 4(dx^2)^2 - 3(dx^3)^2 + 4\,dx^1dx^2 - 6\,dx^2dx^3$, find the following matrices:

(i) $[g_{ij}]$

(ii) $[g^{ij}]$

(iii) the product of $[g_{ij}]$ and $[g^{ij}]$ (4, 4, 2)

(b) If A_i are the component of a covariant vector, show that $\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$ are components of a skew-symmetric covariant tensor of rank 2 (8)

5. (a) Write the Lorentz transformation for coordinates (ct, x, y, z) in an inertial frame $\boldsymbol{S}$ to coordinates (ct', x', y', z') in another frame $\boldsymbol{S'}$ moving with velocity v along the x axis. Write these equations in matrix form. Show that the invariance of spacetime interval in the two frames leads to the condition

$$\eta_{\mu\nu}\Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta} = \eta_{\alpha\beta}$$

where $\Lambda^{\mu}_{\ \alpha}$ and $\Lambda^{\nu}_{\ \beta}$ are Lorentz transformation matrices and $\eta_{\alpha\beta}$ *and* $\eta_{\mu\nu}$ are Minkowski metric tensors. (10)

(b) Calculate the values of the following Christoffel symbols of the first kind for cylindrical coordinates $(x^1 = \rho,\ x^2 = \phi,\ x^3 = z)$ for which the metric is given by $ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$

(*i*) [12,2]

(*ii*) [22,1] (8)

6. (a) Prove the following vector identities using Cartesian tensors:

(i) $\vec{\mathbf{A}} \times (\vec{\mathbf{B}} \times \vec{\mathbf{C}}) = (\vec{\mathbf{A}} \cdot \vec{\mathbf{C}})\vec{\mathbf{B}} - (\vec{\mathbf{A}} \cdot \vec{\mathbf{B}})\vec{\mathbf{C}}$

(ii) $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2\vec{A}$ (5,5)

(b) Check if the matrix T given by $T = \begin{bmatrix} -x_1x_2 & -x_2^2 \\ -x_1^2 & -x_1x_2 \end{bmatrix}$ is a tensor of rank two. (8)