Question 1:

(a) The displacement y of a particle executing periodic motion is given by, y = 4 $\cos^2$ (½ t) sin(1000 t). Show that this expression may be a result of the superposition of three independent harmonic motions.

- The given equation for displacement is $y = 4\cos^2(\frac{1}{2}t)\sin(1000t)$.
- We use the trigonometric identity $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$.
- Substituting $\theta = \frac{1}{2}t$, we get $\cos^2(\frac{1}{2}t) = \frac{1+\cos(2\times\frac{1}{2}t)}{2} = \frac{1+\cos(t)}{2}$.
- Substitute this back into the expression for y:
  - $y = 4\left(\frac{1+\cos(t)}{2}\right)\sin(1000t)$
  - $y = 2(1 + \cos(t))\sin(1000t)$
  - $y = 2\sin(1000t) + 2\cos(t)\sin(1000t)$.
- Now, we use the product-to-sum trigonometric identity: $\cos A\sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$.
- Here, $A = t$ and $B = 1000t$.
  - $2\cos(t)\sin(1000t) = 2 \times \frac{1}{2}[\sin(t + 1000t) - \sin(t - 1000t)]$
  - $= \sin(1001t) - \sin(-999t)$
  - $= \sin(1001t) + \sin(999t)$ (since $\sin(-\theta) = -\sin\theta$).
- Substitute this back into the expression for y:
  - $y = 2\sin(1000t) + \sin(1001t) + \sin(999t)$.
- This expression is a sum of three terms, each of the form $A'\sin(\omega t)$, which represents a simple harmonic motion.

- The first term is $y_1 = 2\sin(1000t)$ with amplitude 2 and angular frequency 1000 rad/s.
- The second term is $y_2 = \sin(1001t)$ with amplitude 1 and angular frequency 1001 rad/s.
- The third term is $y_3 = \sin(999t)$ with amplitude 1 and angular frequency 999 rad/s.

- Thus, the given expression for displacement can be shown to be a result of the superposition of three independent harmonic motions.

(b) Explain the conditions for obtaining a straight line as a Lissajous figure on an oscilloscope.

- Lissajous figures are graphical representations of the superposition of two simple harmonic motions that are perpendicular to each other. Let the two motions be described by:
  - $x = A_x \sin(\omega_x t + \phi_x)$
  - $y = A_y \sin(\omega_y t + \phi_y)$

- For a Lissajous figure to be a straight line, two primary conditions must be met:
  - **Condition 1: Equal Frequencies:** The angular frequencies of the two perpendicular simple harmonic motions must be equal.
    - That is, $\omega_x = \omega_y = \omega$.
    - This ensures that both oscillations complete their cycles in the same amount of time.
  - **Condition 2: Specific Phase Difference:** The phase difference between the two motions, $\Delta\phi = \phi_y - \phi_x$, must be a multiple of π.
    - **Case 1: Phase difference $\Delta\phi = 0$ (or $2n\pi$, where n is an integer).**

- In this case, $x = A_x \sin(\omega t)$ and $y = A_y \sin(\omega t)$.
- Dividing the two equations, we get $\frac{y}{x} = \frac{A_y}{A_x}$, which implies $y = \left(\frac{A_y}{A_x}\right) x$.
- This is the equation of a straight line passing through the origin with a positive slope. The line extends from $(-A_x, -A_y)$ to (A_x, A_y).

▪ **Case 2: Phase difference $\Delta\phi = \pm\pi$ (or $(2n+1)\pi$, where n is an integer).**

- In this case, $x = A_x \sin(\omega t)$ and $y = A_y \sin(\omega t \pm \pi) = -A_y \sin(\omega t)$.
- Dividing the two equations, we get $\frac{y}{x} = -\frac{A_y}{A_x}$, which implies $y = -\left(\frac{A_y}{A_x}\right) x$.
- This is the equation of a straight line passing through the origin with a negative slope. The line extends from $(-A_x, A_y)$ to $(A_x, -A_y)$.

- Therefore, a straight line Lissajous figure is obtained when the two perpendicular simple harmonic motions have the same frequency and their phase difference is 0 or π.

(c) A particle of mass 3 moves along the axis attracted toward origin by a force whose magnitude is numerically equal to 12x. The particle is also subjected to a damping force whose magnitude is numerically equal to 12 times the instantaneous speed. If it is initially at rest at x = 10, find the position and the velocity of the particle at any time.

- Given:
  - Mass of the particle, $m = 3$.

- Restoring force (attracted toward origin), $F_k = -12x$. From Hooke's Law $F_k = -kx$, so spring constant $k = 12$.

- Damping force, $F_d = -12\frac{dx}{dt}$. From the damping force equation $F_d = -b\frac{dx}{dt}$, so damping coefficient $b = 12$.

- Initial conditions: At $t = 0$, position $x(0) = 10$, and velocity $v(0) = \frac{dx}{dt}(0) = 0$ (initially at rest).

- The general differential equation for a damped harmonic oscillator is:
  - $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0.$
- Substitute the given values into the equation:
  - $3\frac{d^2x}{dt^2} + 12\frac{dx}{dt} + 12x = 0.$
- Divide the entire equation by $m = 3$:
  - $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0.$
- To solve this second-order linear homogeneous differential equation, we form the characteristic equation:
  - $r^2 + 4r + 4 = 0.$
- Factor the quadratic equation:
  - $(r + 2)^2 = 0.$
- This gives a repeated real root: $r = -2$. This indicates a critically damped system.
- The general solution for a critically damped system is of the form:
  - $x(t) = (C_1 + C_2t)e^{rt} = (C_1 + C_2t)e^{-2t}.$
- Now, apply the initial conditions to find the constants C_1 and C_2.

- **At $t = 0$, $x = 10$:**
  - $10 = (C_1 + C_2 \times 0)e^{-2\times 0}$
  - $10 = C_1 \times 1$
  - $C_1 = 10.$

- So, the position equation becomes $x(t) = (10 + C_2 t)e^{-2t}$.
- Next, find the velocity $v(t)$ by differentiating $x(t)$ with respect to time:
  - $v(t) = \frac{dx}{dt} = C_2 e^{-2t} + (10 + C_2 t)(-2)e^{-2t}$.
  - $v(t) = e^{-2t}[C_2 - 2(10 + C_2 t)]$.
  - $v(t) = e^{-2t}[C_2 - 20 - 2C_2 t]$.
- **At $t = 0$, $v = 0$:**
  - $0 = e^{-2\times 0}[C_2 - 20 - 2C_2 \times 0]$
  - $0 = 1 \times [C_2 - 20]$
  - $C_2 = 20.$
- Substitute the values of C_1 and C_2 back into the equations for position and velocity.
- **Position of the particle at any time:** $x(t) = (10 + 20t)e^{-2t}$.
- **Velocity of the particle at any time:** $v(t) = e^{-2t}[20 - 20 - 2(20)t] = e^{-2t}[-40t] = -40te^{-2t}$.

(d) Explain qualitatively the normal coordinates and normal modes of a coupled oscillatory system.

- **Coupled Oscillatory System:** A system consisting of two or more individual oscillating components (e.g., masses on springs, pendulums) that are interconnected in such a way that the motion of one component influences the motion of the others. This interconnection allows for the exchange of energy between the components. The oscillations are no longer independent, leading to complex motion.

- **Normal Modes:**
    - Normal modes are specific, characteristic patterns of oscillation in a coupled system where all parts of the system oscillate *synchronously* (at the same frequency) and with a *fixed phase relationship* between them.
    - When a coupled system vibrates in a normal mode, every part of the system moves with the same characteristic frequency, known as a **normal frequency** or **eigenfrequency**.
    - If a system is set into motion in one of its normal modes, it will continue to oscillate in that specific pattern without transferring energy to other parts of the system (in the absence of external forces or damping).
    - A system with N degrees of freedom (independent ways it can move) will typically have N distinct normal modes, each with its own unique normal frequency.
    - For example, in two identical coupled pendulums, there are two normal modes:
        - **Symmetric Mode:** Both pendulums swing in the same direction (in phase) with equal amplitudes.
        - **Antisymmetric Mode:** Both pendulums swing in opposite directions (180 degrees out of phase) with equal amplitudes.
    - Any complex motion of a coupled system can be expressed as a superposition (linear combination) of its normal modes.
- **Normal Coordinates:**
    - Normal coordinates are a special set of generalized coordinates that simplify the description of motion in coupled oscillatory systems.

- When the system's motion is expressed in terms of its normal coordinates, the equations of motion for each normal coordinate become **decoupled**. This means that each normal coordinate oscillates independently, behaving like a simple harmonic oscillator with its corresponding normal frequency.
- By transforming from the original physical coordinates (e.g., x_1, x_2) to normal coordinates (e.g., q_1, q_2), the complex problem of coupled oscillators is transformed into a set of simpler, independent simple harmonic oscillator problems.
- This transformation simplifies the analysis of the system's dynamics, allowing for a clearer understanding of its fundamental vibrational patterns and their associated frequencies.

(e) Distinguish between stationary and progressive waves.

- **Stationary Waves (Standing Waves):**
  - **Formation:** Formed by the superposition of two identical progressive waves traveling in opposite directions in the same medium.
  - **Energy Transfer:** No net transfer of energy from one point to another. Energy is trapped and oscillates between kinetic and potential forms within segments of the wave.
  - **Amplitude Variation:** The amplitude of oscillation varies along the length of the wave.
    - **Nodes:** Points of zero displacement (minimum amplitude) that remain fixed in position.
    - **Antinodes:** Points of maximum displacement (maximum amplitude) that also remain fixed in position.
  - **Phase Relationship:** All particles between two consecutive nodes oscillate in phase. Particles in adjacent loops (separated by a node) are 180° out of phase.

- **Wave Form Movement:** The wave pattern itself does not travel; it appears stationary, hence the name "standing wave."
- **Example:** Vibrations on a string fixed at both ends (e.g., guitar string), sound waves in closed organ pipes.

- **Progressive Waves (Traveling Waves):**
  - **Formation:** Generated by a continuous disturbance that propagates through a medium or space.
  - **Energy Transfer:** Continuously transfers energy from the source to distant points in the direction of propagation.
  - **Amplitude Variation:** All particles in the medium oscillate with the same amplitude (assuming no damping or dispersion).
  - **Phase Relationship:** The phase of oscillation changes continuously as the wave propagates. Different particles are at different stages of their oscillation cycle at any given instant.
  - **Wave Form Movement:** The wave pattern (crest, trough, compression, rarefaction) visibly moves through the medium.
  - **Example:** Sound waves in air, light waves, ripples on the surface of water, seismic waves.

(f) Find the quality factor Q for the damped oscillations of an object with a frequency of 1 Hz, and whose amplitude of vibration gets halved in 5 s.

- Given:
  - Frequency of oscillation, $f = 1$ Hz.
  - Time for amplitude to halve, $t = 5$ s.
- The amplitude of a damped oscillation is given by $A(t) = A_0 e^{-\gamma t}$, where A_0 is the initial amplitude and γ is the damping coefficient (or decay constant).
- We are given that when $t = 5$ s, $A(5) = A_0/2$.

- $\frac{A_0}{2} = A_0 e^{-\gamma(5)}$.
- $\frac{1}{2} = e^{-5\gamma}$.

- Take the natural logarithm of both sides:
  - $\ln\left(\frac{1}{2}\right) = -5\gamma$.
  - $-\ln(2) = -5\gamma$.
  - $\gamma = \frac{\ln(2)}{5}$.
  - Using $\ln(2) \approx 0.693$: $\gamma = \frac{0.693}{5} = 0.1386\ \text{s}^{-1}$.
- The quality factor Q for a damped oscillator is defined as $Q = \frac{\omega_0}{2\gamma}$, where ω_0 is the angular frequency of the undamped oscillator (or the damped frequency if damping is small, which is typical for Q-factor calculations).
- First, calculate the angular frequency ω_0 from the given frequency f:
  - $\omega_0 = 2\pi f = 2\pi(1\ \text{Hz}) = 2\pi$ rad/s.
- Now, substitute the values of ω_0 and γ into the Q-factor formula:
  - $Q = \frac{2\pi}{2\times\frac{\ln(2)}{5}}$.
  - $Q = \frac{\pi}{\frac{\ln(2)}{5}} = \frac{5\pi}{\ln(2)}$.
- Using $\pi \approx 3.14159$ and $\ln(2) \approx 0.693147$:
  - $Q = \frac{5\times3.14159}{0.693147} = \frac{15.70795}{0.693147} \approx 22.66$.
- The quality factor Q for the damped oscillations is approximately 22.66.

Question 2:

(a) A uniform spring of spring constant K and a finite mass 'ms' is loaded with a mass 'M'. If 'ms' is not negligible compared to 'M', show that the period of oscillations of mass spring system is, T = 2π√(M + ms/3)/K.

- Consider a uniform spring of mass m_s and spring constant K. A point mass M is attached to its free end.
- When the system oscillates, different parts of the spring move with different velocities.
- Let the spring have a total length L. Consider a small segment of the spring of length dx at a distance x from the fixed end.
- The mass of this small segment is $dm = (\frac{m_s}{L})dx$.
- If the free end of the spring (where mass M is attached) has a displacement $y(t)$, then the displacement of the element dx at position x is proportional to its distance from the fixed end, i.e., $\frac{x}{L}y(t)$.
- The velocity of this element is $v_x = \frac{d}{dt}\left(\frac{x}{L}y(t)\right) = \frac{x}{L}\frac{dy}{dt}$.
- The kinetic energy of this small segment $d(KE_{spring})$ is:
    - $d(KE_{spring}) = \frac{1}{2}dmv_x^2 = \frac{1}{2}\left(\frac{m_s}{L}dx\right)\left(\frac{x}{L}\frac{dy}{dt}\right)^2 = \frac{1}{2}\frac{m_s}{L^3}\left(\frac{dy}{dt}\right)^2 x^2dx.$
- To find the total kinetic energy of the spring (KE_{spring}), we integrate this expression from $x = 0$ to $x = L$:
    - $KE_{spring} = \int_0^L \frac{1}{2}\frac{m_s}{L^3}\left(\frac{dy}{dt}\right)^2 x^2dx$
    - $KE_{spring} = \frac{1}{2}\frac{m_s}{L^3}\left(\frac{dy}{dt}\right)^2 \int_0^L x^2\,dx$
    - $KE_{spring} = \frac{1}{2}\frac{m_s}{L^3}\left(\frac{dy}{dt}\right)^2 \left[\frac{x^3}{3}\right]_0^L$
    - $KE_{spring} = \frac{1}{2}\frac{m_s}{L^3}\left(\frac{dy}{dt}\right)^2 \left(\frac{L^3}{3}\right)$

- $KE_{spring} = \frac{1}{2}\left(\frac{m_s}{3}\right)\left(\frac{dy}{dt}\right)^2$.

- The total kinetic energy of the system is the sum of the kinetic energy of the attached mass M and the spring:
  - $KE_{total} = KE_M + KE_{spring} = \frac{1}{2}M\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}\left(\frac{m_s}{3}\right)\left(\frac{dy}{dt}\right)^2$.
  - $KE_{total} = \frac{1}{2}\left(M + \frac{m_s}{3}\right)\left(\frac{dy}{dt}\right)^2$.
- From this expression, we can identify the effective mass of the system as $M_{eff} = M + \frac{m_s}{3}$.
- The potential energy stored in the spring when displaced by y is $PE = \frac{1}{2}Ky^2$.
- For a simple harmonic oscillator, the angular frequency ω is given by $\omega = \sqrt{\frac{K}{M_{eff}}}$.
  - $\omega = \sqrt{\frac{K}{M+m_s/3}}$.
- The period of oscillation T is related to the angular frequency by $T = \frac{2\pi}{\omega}$.
  - $T = \frac{2\pi}{\sqrt{\frac{K}{M+m_s/3}}} = 2\pi\sqrt{\frac{M+m_s/3}{K}}$.
- Thus, the period of oscillations of the mass-spring system, when the spring's mass is not negligible, is $T = 2\pi\sqrt{(M + m_s/3)/K}$.

(b) Two vibrations at right angles to each other are described by the equations x = 10 cos(5πt), y = 10 cos(10πt-π/4). Draw graphically the Lissajous figure of the resulting motion.

- The equations of the two perpendicular vibrations are:

- $x(t) = 10\cos(5\pi t)$
- $y(t) = 10\cos(10\pi t - \pi/4)$

- From the equations:
  - Amplitude in x-direction, $A_x = 10$. Angular frequency in x-direction, $\omega_x = 5\pi$ rad/s.
  - Amplitude in y-direction, $A_y = 10$. Angular frequency in y-direction, $\omega_y = 10\pi$ rad/s.
- **Frequency Ratio:** The ratio of the y-frequency to the x-frequency is $\frac{\omega_y}{\omega_x} = \frac{10\pi}{5\pi} = 2$. This means $\omega_y = 2\omega_x$.
- **Phase Difference:**
  - The phase of x can be considered 0 (if we consider cosine as the reference).
  - The phase of y is $-\pi/4$.
- To graphically draw the Lissajous figure, we can plot points $(x(t), y(t))$ for various values of t. It is helpful to consider the relationship between x and y.
- Let $\theta = 5\pi t$. Then $x = 10\cos\theta$.
- And $y = 10\cos(2\theta - \pi/4)$.
- We can use the identity $\cos(2\theta) = 2\cos^2\theta - 1$.
- We can also use $\cos(A - B) = \cos A\cos B + \sin A\sin B$.
  - $y = 10[\cos(2\theta)\cos(\pi/4) + \sin(2\theta)\sin(\pi/4)]$
  - $y = 10[\cos(2\theta)(\frac{\sqrt{2}}{2}) + \sin(2\theta)(\frac{\sqrt{2}}{2})]$
  - $y = 5\sqrt{2}[\cos(2\theta) + \sin(2\theta)]$.

- This is still in terms of θ. Substituting $\cos\theta = x/10$, and knowing $\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\sqrt{1-(x/10)^2} = \pm\frac{\sqrt{100-x^2}}{10}$.
- The shape for a frequency ratio of $2:1$ is generally a figure-eight (lemniscate) shape. The phase difference will determine its orientation and specific form.
- **Plotting Points:**
  - $t = 0$: $x = 10\cos(0) = 10$. $y = 10\cos(-\pi/4) = 10(\frac{\sqrt{2}}{2}) \approx 7.07$. Point: $(10, 7.07)$.
  - $5\pi t = \pi/4 \Longrightarrow t = 1/20$: $x = 10\cos(\pi/4) = 7.07$. $y = 10\cos(2(\pi/4) - \pi/4) = 10\cos(\pi/4) = 7.07$. Point: $(7.07, 7.07)$.
  - $5\pi t = \pi/2 \Longrightarrow t = 1/10$: $x = 10\cos(\pi/2) = 0$. $y = 10\cos(2(\pi/2) - \pi/4) = 10\cos(\pi - \pi/4) = 10\cos(3\pi/4) = -7.07$. Point: $(0, -7.07)$.
  - $5\pi t = 3\pi/4 \Longrightarrow t = 3/20$: $x = 10\cos(3\pi/4) = -7.07$. $y = 10\cos(2(3\pi/4) - \pi/4) = 10\cos(3\pi/2 - \pi/4) = 10\cos(5\pi/4) = -7.07$. Point: $(-7.07, -7.07)$.
  - $5\pi t = \pi \Longrightarrow t = 1/5$: $x = 10\cos(\pi) = -10$. $y = 10\cos(2\pi - \pi/4) = 10\cos(-\pi/4) = 7.07$. Point: $(-10, 7.07)$.
  - $5\pi t = 5\pi/4 \Longrightarrow t = 1/4$: $x = 10\cos(5\pi/4) = -7.07$. $y = 10\cos(2(5\pi/4) - \pi/4) = 10\cos(5\pi/2 - \pi/4) = 10\cos(9\pi/4) = 7.07$. Point: $(-7.07, 7.07)$.
  - $5\pi t = 3\pi/2 \Longrightarrow t = 3/10$: $x = 10\cos(3\pi/2) = 0$. $y = 10\cos(2(3\pi/2) - \pi/4) = 10\cos(3\pi - \pi/4) = 10\cos(11\pi/4) = -7.07$. Point: $(0, -7.07)$.
  - $5\pi t = 7\pi/4 \Longrightarrow t = 7/20$: $x = 10\cos(7\pi/4) = 7.07$. $y = 10\cos(2(7\pi/4) - \pi/4) = 10\cos(7\pi/2 - \pi/4) = 10\cos(13\pi/4) = -7.07$. Point: $(7.07, -7.07)$.

- $5\pi t = 2\pi \Longrightarrow t = 2/5$: $x = 10\cos(2\pi) = 10$. $y = 10\cos(4\pi - \pi/4) = 10\cos(-\pi/4) = 7.07$. Point: $(10, 7.07)$. The cycle repeats.

- The Lissajous figure will trace a tilted figure-eight shape within the square boundaries of $x = \pm 10$ and $y = \pm 10$. It will have two points where it touches the x-axis and one point where it touches the y-axis, but due to the phase shift, it does not exactly touch the central axes at the origin. It forms a closed loop that is somewhat skewed. *(A graphical drawing cannot be produced in text format, but the above description and key points allow for manual plotting.)*

Question 3:

(a) Two collinear simple harmonic motions of nearly equal frequencies and different amplitude are superimposed on each other. Find out resultant equation and explain the formation of beats.

- Let the two collinear simple harmonic motions be represented by:
  - $y_1(t) = A_1 \sin(\omega_1 t)$
  - $y_2(t) = A_2 \sin(\omega_2 t)$
  - Here, $A_1 \neq A_2$ (different amplitudes) and $\omega_1 \approx \omega_2$ (nearly equal frequencies).
- The resultant displacement $y(t)$ is the superposition of these two motions:
  - $y(t) = y_1(t) + y_2(t) = A_1 \sin(\omega_1 t) + A_2 \sin(\omega_2 t)$.
- To simplify this, let $\omega_1 = \omega - \Delta\omega$ and $\omega_2 = \omega + \Delta\omega$, where $\omega = \frac{\omega_1 + \omega_2}{2}$ is the average angular frequency, and $2\Delta\omega = |\omega_2 - \omega_1|$ is the small difference in angular frequencies.
- The resultant equation can be complex to express in a single amplitude and frequency term directly. However, we can qualitatively understand the behavior and the formation of beats. For a more explicit expression, let's consider the case where we can factor out common terms by assuming one

amplitude is larger than the other, or by using a method that shows the varying amplitude.

- Let $y(t) = A_1 \sin(\omega_1 t) + A_2 \sin(\omega_2 t)$.
- If we were to simplify, we would look for a resultant amplitude and phase. The key is that the resultant amplitude will not be constant due to the slight frequency difference.
- Consider a simpler case for understanding: $y(t) = A_1 \sin(\omega t - \phi_1) + A_2 \sin(\omega t - \phi_2)$.
- For beats, we are dealing with $\omega_1 \neq \omega_2$.
- The resultant wave is characterized by an amplitude that varies periodically. The maximum amplitude is $(A_1 + A_2)$ when the waves constructively interfere, and the minimum amplitude is $|A_1 - A_2|$ when they destructively interfere.
- The overall frequency of the resultant wave is the average frequency, $\omega_{avg} = \frac{\omega_1 + \omega_2}{2}$.
- The frequency of the amplitude variation (the beat frequency) is $f_{beat} = |f_1 - f_2| = \frac{|\omega_1 - \omega_2|}{2\pi}$.

- **Explanation of the Formation of Beats:**
  - Beats are the periodic variations in the intensity (amplitude) of a sound (or other wave) due to the superposition of two waves of slightly different frequencies.
  - When two waves with nearly equal frequencies, say f_1 and f_2, propagate in the same direction and superimpose:
    - **Constructive Interference:** At certain moments, the crests of both waves (or troughs of both waves) coincide. This leads to maximum displacement and therefore maximum resultant amplitude. If these are sound waves, this results in a loud sound. The combined amplitude is $(A_1 + A_2)$.

- **Destructive Interference:** Due to the slight difference in frequencies, after some time, one wave will gain or lose phase relative to the other. At other moments, the crest of one wave coincides with the trough of the other wave. This leads to minimum resultant displacement and thus minimum resultant amplitude. If these are sound waves, this results in a soft sound (or silence if amplitudes are equal). The combined amplitude is $|A_1 - A_2|$.

  - This continuous alternation between constructive and destructive interference, occurring periodically, causes the amplitude of the resultant wave to wax and wane. These periodic fluctuations in amplitude are what we perceive as "beats."
  - The rate at which these amplitude fluctuations occur is the beat frequency, which is equal to the absolute difference between the two individual frequencies: $f_{beat} = |f_1 - f_2|$. This means that f_{beat} is the number of times per second that the sound goes from loud to soft and back to loud.

(b) An alternating emf of peak-to-peak value 40 volts is applied across the series combination of an inductor of inductance 100 mH, capacitor of capacitance 1pF and resistance 100 Ω. Determine resonance frequency, quality factor and bandwidth.

- Given values for the series RLC circuit:
  - Peak-to-peak voltage, $V_{pp} = 40$ V. (This information is not directly needed for resonance frequency, quality factor, or bandwidth, but useful for understanding the circuit operation).
  - Inductance, $L = 100$ mH $= 100 \times 10^{-3}$ H = 0.1 H.
  - Capacitance, $C = 1$ pF $= 1 \times 10^{-12}$ F.
  - Resistance, $R = 100\ \Omega$.
- **1. Resonance Frequency (ω_0 and f_0):**

- For a series RLC circuit, resonance occurs when the inductive reactance equals the capacitive reactance ($X_L = X_C$), leading to minimum impedance. The resonance angular frequency ω_0 is given by:
  - $\omega_0 = \frac{1}{\sqrt{LC}}$.
- Substitute the given values of L and C:
  - $\omega_0 = \frac{1}{\sqrt{(0.1\text{ H})\times(1\times10^{-12}\text{ F})}} = \frac{1}{\sqrt{10^{-13}\text{ s}^2}}$.
  - $\omega_0 = \frac{1}{\sqrt{10\times10^{-14}}} = \frac{1}{10^{-7}\sqrt{10}} = \frac{10^7}{\sqrt{10}}$ rad/s.
  - $\omega_0 \approx \frac{10^7}{3.162277} \approx 3.162\times10^6$ rad/s.
- To find the resonance frequency in Hertz (f_0), use the relation $\omega_0 = 2\pi f_0$:
  - $f_0 = \frac{\omega_0}{2\pi} = \frac{3.162\times10^6\text{ rad/s}}{2\pi} \approx 503292\text{ Hz} \approx 503.3\text{ kHz}$.

- **2. Quality Factor (Q-factor):**
  - The quality factor for a series RLC circuit at resonance is a measure of the circuit's selectivity. It can be calculated using the formula:
    - $Q = \frac{\omega_0 L}{R}$.
  - Substitute the calculated ω_0 and given L and R:
    - $Q = \frac{(3.162277\times10^6\text{ rad/s})\times(0.1\text{ H})}{100\ \Omega}$.
    - $Q = \frac{3.162277\times10^5}{100} = 3162.277$.
  - Alternatively, $Q = \frac{1}{R}\sqrt{\frac{L}{C}}$:

- $Q = \frac{1}{100\,\Omega}\sqrt{\frac{0.1\text{ H}}{1\times10^{-12}\text{ F}}} = \frac{1}{100}\sqrt{10^{11}} = \frac{1}{100}(10^5\sqrt{10}) = 1000\sqrt{10} \approx 1000 \times 3.162277 \approx 3162.277.$

  - The quality factor is approximately 3162.3.

- **3. Bandwidth (Δf):**
  - The bandwidth of a series RLC circuit is the range of frequencies over which the power delivered to the circuit is at least half of the maximum power (at resonance). It is given by:
    - $\Delta f = \frac{f_0}{Q}$.
  - Substitute the values of f_0 and Q:
    - $\Delta f = \frac{503292\text{ Hz}}{3162.277} \approx 159.15$ Hz.
  - Alternatively, in terms of angular frequency, $\Delta\omega = \frac{R}{L}$:
    - $\Delta\omega = \frac{100\,\Omega}{0.1\text{ H}} = 1000$ rad/s.
    - Then $\Delta f = \frac{\Delta\omega}{2\pi} = \frac{1000}{2\pi} \approx 159.15$ Hz.
  - The bandwidth is approximately 159.2 Hz.

Question 4:

(a) Show that for a forced and damped harmonic oscillator in steady state, the average power is equal to the average power dissipated by system.

- The equation of motion for a forced and damped harmonic oscillator is given by:
  - $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0\cos(\omega t)$

- Where m is mass, b is the damping coefficient, k is the spring constant, F_0 is the amplitude of the driving force, and ω is the angular frequency of the driving force.

- In the steady state, the oscillator vibrates with the same frequency as the driving force, but with a certain amplitude A and a phase lag ϕ:
  - Position: $x(t) = A\cos(\omega t - \phi)$.
  - Velocity: $v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t - \phi)$.
- **Average Power Input by the Driving Force ($\langle P_{in} \rangle$):**
  - The instantaneous power supplied by the driving force is $P_{in}(t) = F(t) \cdot v(t)$.
  - $P_{in}(t) = [F_0\cos(\omega t)] \cdot [-A\omega\sin(\omega t - \phi)]$.
  - We use the trigonometric identity $\cos(\omega t) = \cos((\omega t - \phi) + \phi) = \cos(\omega t - \phi)\cos\phi - \sin(\omega t - \phi)\sin\phi$.
  - $P_{in}(t) = -F_0A\omega[\cos(\omega t - \phi)\cos\phi - \sin(\omega t - \phi)\sin\phi]\sin(\omega t - \phi)$.
  - $P_{in}(t) = -F_0A\omega[\cos(\omega t - \phi)\sin(\omega t - \phi)\cos\phi - \sin^2(\omega t - \phi)\sin\phi]$.
  - To find the average power over a complete cycle ($T = 2\pi/\omega$), we use the average values:
    - $\langle\cos(\theta)\sin(\theta)\rangle = 0$ over a full cycle.
    - $\langle\sin^2(\theta)\rangle = \frac{1}{2}$ over a full cycle.
  - So, $\langle P_{in} \rangle = -F_0A\omega[0 \cdot \cos\phi - \frac{1}{2} \cdot \sin\phi]$.
  - $\langle P_{in} \rangle = \frac{1}{2}F_0A\omega\sin\phi$.

- From the impedance analysis of a forced oscillator, the amplitude A and phase ϕ are related to the force and system parameters. Specifically, the component of the driving force that is in phase with the velocity is $F_0 \sin\phi$. This force component is balanced by the damping force. The damping force is $F_d = -bv = -b(-A\omega \sin(\omega t - \phi)) = bA\omega \sin(\omega t - \phi)$.
- At steady state, $F_0 \sin\phi = bA\omega$.
- Substitute this into the average power input equation:
  - $\langle P_{in} \rangle = \frac{1}{2}(bA\omega)A\omega = \frac{1}{2}bA^2\omega^2$.

- **Average Power Dissipated by Damping System ($\langle P_{diss} \rangle$):**
  - The damping force is $F_d(t) = -bv(t)$.
  - The instantaneous power dissipated by the damping force is $P_{diss}(t) = -F_d(t) \cdot v(t)$ (negative sign because the force opposes motion, so power dissipated is positive).
  - $P_{diss}(t) = -[-bv(t)]v(t) = b[v(t)]^2$.
  - Substitute $v(t) = -A\omega \sin(\omega t - \phi)$:
    - $P_{diss}(t) = b[-A\omega \sin(\omega t - \phi)]^2 = bA^2\omega^2 \sin^2(\omega t - \phi)$.
  - To find the average power dissipated over a complete cycle:
    - $\langle P_{diss} \rangle = \frac{1}{T}\int_0^T b\, A^2\omega^2 \sin^2(\omega t - \phi)dt$.
    - $\langle P_{diss} \rangle = bA^2\omega^2 \langle \sin^2(\omega t - \phi) \rangle$.
    - Since $\langle \sin^2(\theta) \rangle = \frac{1}{2}$ over a full cycle:
    - $\langle P_{diss} \rangle = \frac{1}{2}bA^2\omega^2$.

- **Conclusion:**

- From the calculations, we find that $\langle P_{in} \rangle = \frac{1}{2} b A^2 \omega^2$ and $\langle P_{diss} \rangle = \frac{1}{2} b A^2 \omega^2$.
- Therefore, in steady state, the average power supplied by the driving force to the forced and damped harmonic oscillator is exactly equal to the average power dissipated by the damping force. This is consistent with the principle of energy conservation in a steady-state system, where there is no net change in the stored energy over a cycle.

(b) Two identical masses of mass 'M' are connected with three identical springs of same spring constant 'K' and placed on a smooth surface as shown in Fig. 1. Find out the normal mode frequencies and corresponding configuration/shapes. Figure 1 Description: The image (Fig. 1) displays a linear arrangement of two masses, each denoted as 'm', connected by three springs, each denoted as 'k'. The springs are positioned between the masses and at the outer ends, forming a system k-m-k-m-k.

- Let the two identical masses be M_1 and M_2.
- Let x_1 be the displacement of M_1 from its equilibrium position.
- Let x_2 be the displacement of M_2 from its equilibrium position.
- The springs are arranged in a linear fashion: Wall - K - M_1 - K - M_2 - K - Wall. (The description implies 'k-m-k-m-k', where the 'k's at the ends are connected to fixed walls).
- **Equations of Motion:**
  - **For Mass M_1:**
    - The leftmost spring exerts a force $-Kx_1$.
    - The middle spring exerts a force $-K(x_1 - x_2)$ (it is stretched by $x_1 - x_2$ if $x_1 > x_2$, or compressed if $x_1 < x_2$).
    - Newton's second law for M_1: $M \frac{d^2 x_1}{dt^2} = -Kx_1 - K(x_1 - x_2)$
    - $M \frac{d^2 x_1}{dt^2} = -2Kx_1 + Kx_2$. (Equation 1)

- **For Mass M_2:**
  - The middle spring exerts a force $-K(x_2 - x_1)$ (it is stretched by $x_2 - x_1$ if $x_2 > x_1$, or compressed if $x_2 < x_1$).
  - The rightmost spring exerts a force $-Kx_2$.
  - Newton's second law for M_2: $M\frac{d^2x_2}{dt^2} = -Kx_2 - K(x_2 - x_1)$
  - $M\frac{d^2x_2}{dt^2} = Kx_1 - 2Kx_2$. (Equation 2)

- **Finding Normal Mode Frequencies and Configurations:**
  - Assume harmonic solutions for the displacements in the form:
    - $x_1(t) = A_1\cos(\omega t)$
    - $x_2(t) = A_2\cos(\omega t)$
  - Substitute these into the equations of motion (the $\cos(\omega t)$ term will cancel out):
    - $-M\omega^2 A_1 = -2KA_1 + KA_2$
    - $-M\omega^2 A_2 = KA_1 - 2KA_2$
  - Rearrange these into a system of homogeneous linear equations for A_1 and A_2:
    - $(2K - M\omega^2)A_1 - KA_2 = 0$
    - $-KA_1 + (2K - M\omega^2)A_2 = 0$
  - For non-trivial solutions (i.e., A_1 and A_2 are not both zero), the determinant of the coefficient matrix must be zero:
    - $\begin{vmatrix} (2K - M\omega^2) & -K \\ -K & (2K - M\omega^2) \end{vmatrix} = 0.$
    - $(2K - M\omega^2)(2K - M\omega^2) - (-K)(-K) = 0.$

- $(2K - M\omega^2)^2 - K^2 = 0.$
- Let $\lambda = (2K - M\omega^2)$. Then $\lambda^2 - K^2 = 0$.
- $\lambda^2 = K^2 \Longrightarrow \lambda = \pm K.$

- **Case 1:** $\lambda = K$
  - $2K - M\omega^2 = K.$
  - $M\omega^2 = K.$
  - $\omega_1^2 = \frac{K}{M}.$
  - **Normal Mode Frequency 1:** $\omega_1 = \sqrt{\frac{K}{M}}.$
  - Substitute $\lambda = K$ back into the first equation:
    - $(K)A_1 - KA_2 = 0 \Longrightarrow K(A_1 - A_2) = 0 \Longrightarrow A_1 = A_2.$
  - **Configuration/Shape 1 (Symmetric Mode):** Both masses oscillate in phase with equal amplitudes. They move together, to the right then to the left, symmetrically. The middle spring's length effectively doesn't change from its equilibrium value because $x_1 - x_2 = 0$.

- **Case 2:** $\lambda = -K$
  - $2K - M\omega^2 = -K.$
  - $M\omega^2 = 3K.$
  - $\omega_2^2 = \frac{3K}{M}.$
  - **Normal Mode Frequency 2:** $\omega_2 = \sqrt{\frac{3K}{M}}.$
  - Substitute $\lambda = -K$ back into the first equation:

- $(-K)A_1 - KA_2 = 0 \Rightarrow -K(A_1 + A_2) = 0 \Rightarrow A_1 = -A_2$.

- **Configuration/Shape 2 (Antisymmetric Mode):** Both masses oscillate out of phase by 180° with equal amplitudes. When M_1 moves to the right, M_2 moves to the left by the same amount, and vice versa. The center point of the middle spring remains stationary, acting as a node.

Question 5:

(a) Find the expression for the normal mode of vibration and displacement for a string fixed at both the ends having tension T and mass per unit length μ.

- **Wave Equation:** The transverse displacement $y(x,t)$ of a vibrating string is governed by the one-dimensional wave equation:
  - $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$
  - Where v is the speed of the wave on the string. For a string with tension T and mass per unit length μ, the wave speed is $v = \sqrt{T/\mu}$.
  - So, the wave equation becomes: $\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$.
- **Boundary Conditions:** Since the string is fixed at both ends, the displacement at the ends must always be zero. Let the length of the string be L.
  - $y(0,t) = 0$ for all t.
  - $y(L,t) = 0$ for all t.
- **Method of Separation of Variables:**
  - Assume a solution of the form $y(x,t) = X(x)T(t)$, where $X(x)$ is a function of position only and $T(t)$ is a function of time only.
  - Substitute this into the wave equation:

    - $T(t)\frac{d^2X}{dx^2} = \frac{\mu}{T}X(x)\frac{d^2T}{dt^2}$.
  - Divide both sides by $X(x)T(t)$:
    - $\frac{1}{X(x)}\frac{d^2X}{dx^2} = \frac{\mu}{T}\frac{1}{T(t)}\frac{d^2T}{dt^2}$.
  - Since the left side depends only on x and the right side depends only on t, both sides must be equal to a constant. For oscillatory solutions, this constant must be negative, so let's call it $-\omega^2/v^2$ (or $-\lambda^2$).
    - $\frac{1}{X(x)}\frac{d^2X}{dx^2} = -\frac{\omega^2}{v^2} \Longrightarrow \frac{d^2X}{dx^2} + \left(\frac{\omega}{v}\right)^2 X = 0$.
    - $\frac{\mu}{T}\frac{1}{T(t)}\frac{d^2T}{dt^2} = -\frac{\omega^2}{v^2} \Longrightarrow \frac{d^2T}{dt^2} + \omega^2 T = 0$.
- **Solutions for $X(x)$ and $T(t)$:**
  - The general solution for $X(x)$ is $X(x) = C_1\cos\left(\frac{\omega}{v}x\right) + C_2\sin\left(\frac{\omega}{v}x\right)$.
  - The general solution for $T(t)$ is $T(t) = C_3\cos(\omega t) + C_4\sin(\omega t)$, which can be written as $T(t) = A_{time}\sin(\omega t + \phi)$.
- **Applying Boundary Conditions to $X(x)$:**
  - At $x = 0$, $y(0,t) = X(0)T(t) = 0$. Since $T(t)$ is not always zero, $X(0)$ must be zero.
    - $X(0) = C_1\cos(0) + C_2\sin(0) = C_1 = 0$.
    - So, $X(x) = C_2\sin\left(\frac{\omega}{v}x\right)$.
  - At $x = L$, $y(L,t) = X(L)T(t) = 0$. So, $X(L)$ must be zero.
    - $X(L) = C_2\sin\left(\frac{\omega}{v}L\right) = 0$.
    - Since C_2 cannot be zero (otherwise $X(x)$ would be zero for all x, meaning no vibration), we must have $\sin\left(\frac{\omega}{v}L\right) = 0$.

    - This implies $\frac{\omega}{v}L = n\pi$, where n is an integer ($n = 1,2,3,\ldots$). (n=0 would mean $\omega = 0$, no oscillation).

- **Normal Mode Frequencies:**
  - From $\frac{\omega_n}{v}L = n\pi$, we get $\omega_n = \frac{n\pi v}{L}$.
  - Substitute $v = \sqrt{T/\mu}$:
    - $\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}$. These are the angular frequencies of the normal modes.
  - The corresponding frequencies in Hertz are $f_n = \frac{\omega_n}{2\pi} = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$.
    - For $n = 1$, $f_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$ (fundamental frequency or first harmonic).
    - For $n = 2$, $f_2 = \frac{2}{2L}\sqrt{\frac{T}{\mu}} = 2f_1$ (first overtone or second harmonic), and so on.
- **Expression for Displacement (Normal Modes):**
  - Combining $X(x)$ and $T(t)$, and letting $A_n = C_2 A_{time}$, the displacement for the n-th normal mode is:
    - $y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\sin(\omega_n t + \phi_n)$.
  - The term $\sin\left(\frac{n\pi x}{L}\right)$ describes the spatial shape (mode shape or eigenfunction) of the n-th mode, while $\sin(\omega_n t + \phi_n)$ describes its harmonic time evolution.
  - The most general displacement of the vibrating string is a superposition of all its normal modes:

- $y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \sin(\omega_n t + \phi_n)$.
- The coefficients A_n and phases ϕ_n are determined by the initial conditions (initial shape and initial velocity) of the string.

(b) A string of length $l = 0.5$ m and mass per unit length 0.01 kg/m has a fundamental frequency of 250 Hz. What is the tension in the string?

- Given:
  - Length of the string, $L = 0.5$ m.
  - Mass per unit length, $\mu = 0.01$ kg/m.
  - Fundamental frequency (for $n = 1$), $f_1 = 250$ Hz.
- The formula for the fundamental frequency of a string fixed at both ends is:
  - $f_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$.
- We need to find the tension T. Let's rearrange the formula to solve for T:
  - Multiply both sides by $2L$: $2Lf_1 = \sqrt{\frac{T}{\mu}}$.
  - Square both sides: $(2Lf_1)^2 = \frac{T}{\mu}$.
  - Multiply by μ: $T = \mu(2Lf_1)^2$.
- Now, substitute the given values into the formula:
  - $T = (0.01$ kg/m$) \times (2 \times 0.5$ m $\times 250$ Hz$)^2$.
  - $T = (0.01$ kg/m$) \times (1$ m $\times 250$ s$^{-1})^2$.
  - $T = (0.01) \times (250)^2$ N.
  - $T = (0.01) \times (62500)$ N.
  - $T = 625$ N.

- The tension in the string is 625 N.