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[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1102 **I**

Unique Paper Code : 2352013502

Name of the Paper : Ring Theory

Name of the Course : **B.Sc. (H) Mathematics**

Semester : V – DSC-14

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. **All** questions are compulsory and are of **15** marks each.

3. Attempt any **six** parts from Question **1**. Each part is of **2.5** marks.

4. Attempt any **two** parts from each of the Questions **2** to **6**. Each part is of **7.5** marks.

5. Use of calculator is not allowed.

1. (i) Find all the units of $\mathbb{Z}_7[x]$.

(ii) Check whether $\mathbb{Q} \oplus \mathbb{Q}$ is an integral domain or not.

(iii) Give an example of a subring S of a ring R which is not an ideal of R.

(iv) Prove that a ring homomorphism carries an idempotent to an idempotent.

(v) Let ϕ be a ring homomorphism from a ring R to a ring S. If R has unity 1, $S \neq \{0\}$ and ϕ is onto then prove that $\phi(1)$ is the unity of S.

(vi) Let $f(x) = 2x^5 + 14x^2 - 21x + 7$. Is f(x) an irreducible polynomial over $\mathbb{Q}$? Justify your answer.

(vii) Let D be an intergral domain. Suppose that $p, q \in D$ and $q \neq 0$. Show that if p is not a unit, then $\langle pq \rangle$ is a proper subset of $\langle q \rangle$.

(viii) Explain why $3x^2 + 6$ is reducible over $\mathbb{Z}$.

2. (a) Prove that intersection of two subrings in a ring R is a subring of R. Is the union of two subrings necessarily a subring of R? Justify your answer.

(b) Find all the units, zero divisors and idempotent elements in $\mathbb{Z}_3 \oplus \mathbb{Z}_6$.

(c) Prove that $\mathbb{Z}_n$, the ring of integers modulo n, is a field if and only if n is a prime.

3. (a) Let R be a commutative ring with unity and let U(R) denote the set of units of R. Prove that U(R) is a group under multiplication. Also, find $U(\mathbb{Z}[i])$.

(b) Define the characteristic of a ring. Prove that the characteristic of an integral domain is either 0 or prime.

(c) Prove that in a commutative ring R with unity, an ideal A is a maximal ideal if and only if $\frac{R}{A}$ is a field.

4. (a) Prove that the ideal $\langle x \rangle$ is a prime ideal in $\mathbb{Z}[x]$ but not a maximal ideal in $\mathbb{Z}[x]$.

(b) Let ϕ be a ring homomorphism from a ring R onto a ring S. Prove that $\frac{R}{\text{Ker}\,\phi} \approx S$.

(c) Determine all ring homomorphisms from $\mathbb{Z}_4 \rightarrow \mathbb{Z}_{10}$.

5. (a) Let F be a field and let $I = \{a_0 + a_1x + \ldots + a_nx^n : a_0, a_1, \ldots, a_n \in F \text{ and } a_0 + a_1 + \ldots + a_n = 0\}$. Show that I is an ideal of F[x] and find a generator for I.

(b) Let $f(x) = 5x^4 + 3x^3 + 1$ and $g(x) = 3x^2 + 2x + 1 \in \mathbb{Z}_7[x]$. Determine the quotient and remainder obtained when f(x) is divided by g(x).

(c) Prove that the product of two primitive polynomials is a primitive polynomial.

6. (a) Show that $p(x) = x^3 + x + 1$ is an irreducible polynomial over $\mathbb{Z}_2$.

Let $M = \langle x^3 + x + 1 \rangle$ be an ideal of $\mathbb{Z}_2[x]$.

Show that $F = \dfrac{\mathbb{Z}_2[x]}{M}$ is a field of order 8. Exhibit all the 8 elements of F. Find the product of $x^2 + x + 1 + M$ and $x^2 + 1 + M$ and express it as a member of F.

(b) In a principal ideal domain, prove that the element is irreducible if and only if it is prime.

(c) Show that integral domain $\mathbb{Z}[t]$ is Euclidean Domain. Is $\mathbb{Z}[i]$ a Unique Factorization Domain? Justify.