1. (a) If a · b = 0, then either a = 0 or b = 0.

This statement is known as the **Zero Product Property** for real numbers.

- **Proof:**
    - Assume $a \cdot b = 0$.
    - **Case 1:** $a \neq 0$**.**
        - Since $a \neq 0$, $1/a$ exists as a real number.
        - Multiply both sides of $a \cdot b = 0$ by $1/a$: $(1/a) \cdot (a \cdot b) = (1/a) \cdot 0$
        - Using associativity of multiplication: $((1/a) \cdot a) \cdot b = 0$
        - Since $(1/a) \cdot a = 1$: $1 \cdot b = 0$
        - Therefore, $b = 0$.
    - **Case 2:** $b \neq 0$**.**
        - The proof is analogous to Case 1. Since $b \neq 0$, $1/b$ exists.
        - Multiply both sides of $a \cdot b = 0$ by $1/b$: $(a \cdot b) \cdot (1/b) = 0 \cdot (1/b)$
        - Using associativity of multiplication: $a \cdot (b \cdot (1/b)) = 0$
        - Since $b \cdot (1/b) = 1$: $a \cdot 1 = 0$
        - Therefore, $a = 0$.
    - From both cases, we conclude that if $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

1. (b) State the order properties of ℝ. Using it prove that if a, b, c are real numbers such that a > b, then a + c > b + c.

- **Order Properties of ℝ (Axioms of Order):**

  - **Trichotomy Property:** For any two real numbers a and b, exactly one of the following is true: $a < b$, $a = b$, or $a > b$.
  - **Transitivity Property:** If $a < b$ and $b < c$, then $a < c$.
  - **Addition Property:** If $a < b$, then $a + c < b + c$ for any real number c.
  - **Multiplication Property:** If $a < b$ and $c > 0$, then $ac < bc$. If $a < b$ and $c < 0$, then $ac > bc$.
- **Proof that if $a > b$, then $a + c > b + c$:**
  - Given $a > b$.
  - By definition of $>$, this means $b < a$.
  - According to the **Addition Property** of order, if $b < a$, then $b + c < a + c$ for any real number c.
  - By definition of $<$, $b + c < a + c$ is equivalent to $a + c > b + c$.
  - Therefore, if $a > b$, then $a + c > b + c$.

1. (c) Find all values of x satisfying |x - 2| ≤ x + 1.

- We need to solve the inequality $|x - 2| \leq x + 1$.
- **Case 1: $x - 2 \geq 0$, i.e., $x \geq 2$.**
  - In this case, $|x - 2| = x - 2$.
  - The inequality becomes: $x - 2 \leq x + 1$
  - Subtract x from both sides: $-2 \leq 1$.
  - This statement is always true.
  - So, for $x \geq 2$, all values of x satisfy the inequality.
- **Case 2: $x - 2 < 0$, i.e., $x < 2$.**
  - In this case, $|x - 2| = -(x - 2) = 2 - x$.

- The inequality becomes: $2 - x \le x + 1$
- Add x to both sides: $2 \le 2x + 1$
- Subtract 1 from both sides: $1 \le 2x$
- Divide by 2: $x \ge 1/2$.
- Combining this with the condition $x < 2$, we get $1/2 \le x < 2$.

- **Combining both cases:**
  - From Case 1, we have $x \in [2, \infty)$.
  - From Case 2, we have $x \in [1/2, 2)$.
  - The union of these two intervals is $[1/2, \infty)$.
- Also, for $|x - 2| \le x + 1$ to be defined, we must have $x + 1 \ge 0$, which means $x \ge -1$. Since our solution $x \ge 1/2$ already satisfies $x \ge -1$, this condition is met.
- Therefore, the values of x satisfying the inequality are $x \ge 1/2$.

1. (d) Write the definition of Supremum and Infimum of a set. Give an example of a set having supremum and infimum, where the set: (i) contains its supremum and infimum (ii) does not contain its supremum and infimum

- **Definition of Supremum (Least Upper Bound):**
  - Let S be a non-empty subset of $\mathbb{R}$. A real number M is called the **supremum** (or least upper bound) of S, denoted as $\sup S$, if:
    1. M is an upper bound of S (i.e., for all $x \in S$, $x \le M$).
    2. M is the least among all upper bounds of S (i.e., if M' is any other upper bound of S, then $M \le M'$).
- **Definition of Infimum (Greatest Lower Bound):**

- Let S be a non-empty subset of $\mathbb{R}$. A real number m is called the **infimum** (or greatest lower bound) of S, denoted as $\inf S$, if:
  - iii. m is a lower bound of S (i.e., for all $x \in S$, $x \geq m$).
  - iv. m is the greatest among all lower bounds of S (i.e., if m' is any other lower bound of S, then $m' \leq m$).

- **Example of a set having supremum and infimum:**
  - **(i) Set contains its supremum and infimum:**
    - Let $A = [0,5] = \{x \in \mathbb{R} \mid 0 \leq x \leq 5\}$.
    - Supremum of A, $\sup A = 5$. Since $5 \in A$, the set contains its supremum.
    - Infimum of A, $\inf A = 0$. Since $0 \in A$, the set contains its infimum.
  - **(ii) Set does not contain its supremum and infimum:**
    - Let $B = (0,5) = \{x \in \mathbb{R} \mid 0 < x < 5\}$.
    - Supremum of B, $\sup B = 5$. However, $5 \notin B$.
    - Infimum of B, $\inf B = 0$. However, $0 \notin B$.

2. (a) State and prove Archimedean property.

- **Archimedean Property (of Real Numbers):**
  - For any two positive real numbers a and b, there exists a positive integer n such that $na > b$.
  - **Equivalent forms:**
    - For any real number x, there exists an integer n such that $n > x$.
    - For any $\epsilon > 0$, there exists a positive integer n such that $1/n < \epsilon$.

- **Proof of Archimedean Property (using Completeness Axiom):**
  - We will prove the equivalent form: For any $x \in \mathbb{R}$, there exists an integer n such that $n > x$.
  - Assume, for contradiction, that the property does not hold.
  - This means there exists some real number x such that for all integers n, $n \leq x$.
  - Consider the set of natural numbers $\mathbb{N} = \{1,2,3,\ldots\}$.
  - If $n \leq x$ for all $n \in \mathbb{N}$, then x is an upper bound for the set $\mathbb{N}$.
  - Since $\mathbb{N}$ is a non-empty set and is bounded above (by x), by the **Completeness Axiom (or Supremum Property) of Real Numbers**, $\mathbb{N}$ must have a supremum in $\mathbb{R}$.
  - Let $s = \sup\mathbb{N}$.
  - Since s is the supremum, $s - 1$ is not an upper bound for $\mathbb{N}$ (because s is the *least* upper bound).
  - Therefore, there must exist some natural number $m \in \mathbb{N}$ such that $m > s - 1$.
  - Adding 1 to both sides of the inequality, we get $m + 1 > s$.
  - However, $m + 1$ is also a natural number, i.e., $m + 1 \in \mathbb{N}$.
  - This contradicts the fact that s is an upper bound for $\mathbb{N}$ (since s must be greater than or equal to all elements in $\mathbb{N}$).
  - This contradiction arises from our initial assumption that the Archimedean Property does not hold.
  - Therefore, the Archimedean Property must be true.

2. (b) Let S be a non-empty subset of $\mathbb{R}$ and a > 0, then show that sup(aS) = a sup S.

- **Given:**

- S is a non-empty subset of $\mathbb{R}$.
- $a > 0$ is a real number.
- $aS = \{ax \mid x \in S\}$.

- **To prove:** $\sup(aS) = a\sup S$.
- **Proof:**
  - Since S is a non-empty subset of $\mathbb{R}$ and we are talking about its supremum, we assume S is bounded above. Let $M = \sup S$.
  - By definition of supremum, for all $x \in S$, we have $x \leq M$.
  - Since $a > 0$, we can multiply the inequality by a without changing its direction: $ax \leq aM$ for all $x \in S$.
  - This implies that aM is an upper bound for the set aS.
  - Now, we need to show that aM is the *least* upper bound for aS.
  - Let M' be any upper bound for aS.
  - Then, for all $y \in aS$, we have $y \leq M'$.
  - Since any $y \in aS$ can be written as ax for some $x \in S$, we have $ax \leq M'$ for all $x \in S$.
  - Since $a > 0$, we can divide the inequality by a without changing its direction: $x \leq M'/a$ for all $x \in S$.
  - This means M'/a is an upper bound for S.
  - Since $M = \sup S$ is the least upper bound for S, we must have $M \leq M'/a$.
  - Multiplying by a (which is positive), we get $aM \leq M'$.
  - This shows that aM is indeed the least upper bound for aS.
- Therefore, $\sup(aS) = a\sup S$.

2. (c) Let (x_n) be a sequence in $\mathbb{R}$ and let $x \in \mathbb{R}$. If (a_n) is a sequence of positive real numbers with $\lim(a_n)=0$ and for some constant $K > 0$ and some $m \in \mathbb{N}$ we have $|x_n - x| \leq Ka_n$ for all $n \geq m$, then prove that $\lim(x_n) = x$.

- **Given:**
  - (x_n) is a sequence in $\mathbb{R}$.
  - $x \in \mathbb{R}$.
  - (a_n) is a sequence of positive real numbers (i.e., $a_n > 0$ for all n).
  - $\lim_{n\to\infty} a_n = 0$.
  - There exists a constant $K > 0$ and an integer $m \in \mathbb{N}$ such that $|x_n - x| \leq Ka_n$ for all $n \geq m$.
- **To prove:** $\lim_{n\to\infty} x_n = x$.
- **Proof:**
  - Let $\epsilon > 0$ be given.
  - Since $\lim_{n\to\infty} a_n = 0$, by the definition of a limit, for the given $\epsilon/K > 0$, there exists an integer $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have: $|a_n - 0| < \epsilon/K$ $|a_n| < \epsilon/K$
  - Since a_n is a sequence of positive real numbers, $|a_n| = a_n$.
  - So, $a_n < \epsilon/K$ for all $n \geq N_1$.
  - Let $N = \max(m, N_1)$.
  - Now, for any $n \geq N$, we know that $n \geq m$ and $n \geq N_1$.
  - From the given condition, for $n \geq m$, we have $|x_n - x| \leq Ka_n$.
  - Since $n \geq N_1$, we also have $a_n < \epsilon/K$.

- Substituting this into the inequality: $|x_n - x| \leq K(\epsilon/K)$ $|x_n - x| \leq \epsilon$
- Thus, for every $\epsilon > 0$, there exists an integer N (namely $N = \max(m, N_1)$) such that for all $n \geq N$, $|x_n - x| \leq \epsilon$.
- This is precisely the definition of the limit of a sequence.

- Therefore, $\lim_{n\to\infty} x_n = x$.

2. (d) Using the definition of limit, show that lim (4n+5)/(3n+4) = 4/3.

- **To prove:** $\lim_{n\to\infty} \frac{4n+5}{3n+4} = \frac{4}{3}$ using the $\epsilon - N$ definition of a limit.
- **Definition of Limit:** For every $\epsilon > 0$, there exists a natural number N such that for all $n \geq N$, we have $\left|\frac{4n+5}{3n+4} - \frac{4}{3}\right| < \epsilon$.
- **Proof:**
  - Consider the expression $\left|\frac{4n+5}{3n+4} - \frac{4}{3}\right|$.
  - Combine the fractions: $\left|\frac{3(4n+5)-4(3n+4)}{3(3n+4)}\right| = \left|\frac{12n+15-12n-16}{3(3n+4)}\right| = \left|\frac{-1}{3(3n+4)}\right| = \frac{1}{3(3n+4)}$ (since $3(3n + 4)$ is positive for $n \geq 1$).
  - We want to find N such that $\frac{1}{3(3n+4)} < \epsilon$ for all $n \geq N$.
  - Rearrange the inequality: $1 < 3\epsilon(3n + 4)$ $\frac{1}{3\epsilon} < 3n + 4$ $\frac{1}{3\epsilon} - 4 < 3n$ $\frac{1}{3}\left(\frac{1}{3\epsilon} - 4\right) < n$ $n > \frac{1}{9\epsilon} - \frac{4}{3}$
  - Let N be a natural number such that $N > \frac{1}{9\epsilon} - \frac{4}{3}$. (By the Archimedean property, such an N always exists).
  - Then for all $n \geq N$, we have: $n > \frac{1}{9\epsilon} - \frac{4}{3}$ $3n > \frac{1}{3\epsilon} - 4$ $3n + 4 > \frac{1}{3\epsilon}$ $\frac{1}{3n+4} < 3\epsilon$ $\frac{1}{3(3n+4)} < \epsilon$

  - Thus, for every $\epsilon > 0$, there exists an N such that for all $n \geq N$, $\left|\frac{4n+5}{3n+4} - \frac{4}{3}\right| < \epsilon$.
- Therefore, $\lim_{n\to\infty} \frac{4n+5}{3n+4} = \frac{4}{3}$.

3. (a) Let (x_n) and (y_n) be sequences of real number such that lim(x_n)=x and lim(y_n)=y, then show that lim(x_n+y_n) = x+y.

- **Given:**
  - (x_n) and (y_n) are sequences of real numbers.
  - $\lim_{n\to\infty} x_n = x$.
  - $\lim_{n\to\infty} y_n = y$.
- **To prove:** $\lim_{n\to\infty}(x_n + y_n) = x + y$.
- **Proof:**
  - Let $\epsilon > 0$ be given.
  - Since $\lim_{n\to\infty} x_n = x$, by definition, for $\epsilon/2 > 0$, there exists an integer $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$: $|x_n - x| < \epsilon/2$.
  - Since $\lim_{n\to\infty} y_n = y$, by definition, for $\epsilon/2 > 0$, there exists an integer $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$: $|y_n - y| < \epsilon/2$.
  - Let $N = \max(N_1, N_2)$.
  - Then, for all $n \geq N$, both conditions hold: $|x_n - x| < \epsilon/2$ and $|y_n - y| < \epsilon/2$.
  - Consider the expression $|(x_n + y_n) - (x + y)|$: $|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)|$.
  - By the Triangle Inequality, $|(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y|$.
  - Substituting the inequalities for $n \geq N$: $|(x_n + y_n) - (x + y)| < \epsilon/2 + \epsilon/2$ $|(x_n + y_n) - (x + y)| < \epsilon$.

- Thus, for every $\epsilon > 0$, there exists an integer N such that for all $n \geq N$, $|(x_n + y_n) - (x + y)| < \epsilon$.

- Therefore, $\lim_{n\to\infty}(x_n + y_n) = x + y$.

3. (b) Let (x_n) be a sequence of positive real numbers such that L = lim (x_{n+1})/x_n exists. Show that if L < 1, then (x_n) converges and lim (x_n) = 0.

- **Given:**
  - (x_n) is a sequence of positive real numbers ($x_n > 0$ for all n).
  - $L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ exists.
  - $L < 1$.
- **To prove:** (x_n) converges and $\lim_{n\to\infty} x_n = 0$.
- **Proof:**
  - Since $L < 1$, we can choose a real number r such that $L < r < 1$. For example, $r = (L + 1)/2$.
  - Since $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$, by the definition of a limit, for $\epsilon = r - L > 0$, there exists a natural number N such that for all $n \geq N$: $\left|\frac{x_{n+1}}{x_n} - L\right| < r - L$.
  - This implies $-(r - L) < \frac{x_{n+1}}{x_n} - L < r - L$.
  - Adding L to all parts: $L - (r - L) < \frac{x_{n+1}}{x_n} < L + (r - L)$.
  - $2L - r < \frac{x_{n+1}}{x_n} < r$.
  - Since we are interested in the upper bound, we have $\frac{x_{n+1}}{x_n} < r$ for all $n \geq N$.
  - Since $x_n > 0$, we can write $x_{n+1} < r x_n$ for all $n \geq N$.

- Let's write out the terms starting from $n = N$: $x_{N+1} < rx_N$ $x_{N+2} < rx_{N+1} < r(rx_N) = r^2x_N$ $x_{N+3} < rx_{N+2} < r(r^2x_N) = r^3x_N$ ... In general, for $k \geq 1$: $x_{N+k} < r^k x_N$.
- Let $n = N + k$. Then $k = n - N$.
- So, $x_n < r^{n-N}x_N = (r^{-N}x_N)r^n$ for all $n > N$.
- Let $C = r^{-N}x_N$. Since $r > 0$ and $x_N > 0$, C is a positive constant.
- Thus, we have $0 < x_n < Cr^n$ for all $n > N$.
- We know that $0 < r < 1$. Therefore, $\lim_{n\to\infty} r^n = 0$.
- By the Squeeze Theorem (or a direct result of limit properties), since $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} Cr^n = C \cdot 0 = 0$, it follows that $\lim_{n\to\infty} x_n = 0$.
- Since the limit exists and is a finite value (0), the sequence (x_n) converges.

3. (c) State Squeeze theorem and show that if $z_n = (2^n + 3^n)^{1/n}$ then lim $z_n = 3$.

- **Squeeze Theorem (or Sandwich Theorem):**
  - Let (x_n), (y_n), and (z_n) be sequences of real numbers.
  - If there exists a natural number N such that $x_n \leq y_n \leq z_n$ for all $n \geq N$,
  - And if $\lim_{n\to\infty} x_n = L$ and $\lim_{n\to\infty} z_n = L$,
  - Then $\lim_{n\to\infty} y_n = L$.
- **Show that if $z_n = (2^n + 3^n)^{1/n}$ then $\lim z_n = 3$.**
- **Proof:**
  - We have $z_n = (2^n + 3^n)^{1/n}$.

- We know that $3^n < 2^n + 3^n$.
- Also, $2^n + 3^n < 3^n + 3^n = 2 \cdot 3^n$.
- So, we have the inequality: $3^n < 2^n + 3^n < 2 \cdot 3^n$.
- Now, take the n-th root of each part (since all terms are positive, the inequality direction is preserved): $(3^n)^{1/n} < (2^n + 3^n)^{1/n} < (2 \cdot 3^n)^{1/n}$ $3 < (2^n + 3^n)^{1/n} < 2^{1/n} \cdot (3^n)^{1/n}$ $3 < z_n < 3 \cdot 2^{1/n}$
- Let $x_n = 3$ and $y_n = 3 \cdot 2^{1/n}$.
- We know that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} 3 = 3$.
- For $y_n = 3 \cdot 2^{1/n}$, we know that $\lim_{n\to\infty} 2^{1/n} = 2^0 = 1$ (since $\lim_{n\to\infty} 1/n = 0$ and the exponential function $f(x) = a^x$ is continuous).
- So, $\lim_{n\to\infty} y_n = \lim_{n\to\infty}(3 \cdot 2^{1/n}) = 3 \cdot \lim_{n\to\infty} 2^{1/n} = 3 \cdot 1 = 3$.
- Since $3 < z_n < 3 \cdot 2^{1/n}$ for all $n \geq 1$, and $\lim_{n\to\infty} 3 = 3$ and $\lim_{n\to\infty}(3 \cdot 2^{1/n}) = 3$,
- By the Squeeze Theorem, $\lim_{n\to\infty} z_n = 3$.

3. (d) Let X = (x_n) be a sequence of real numbers defined by x_1 = 1 and x_{n+1} = √(2+x_n) for n ∈ ℝ. Show that the sequence (x_n) is convergent and find its limit.

- **Given:** The sequence (x_n) is defined by $x_1 = 1$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \in \mathbb{N}$.
- **To show convergence, we need to show that the sequence is monotone and bounded.**
- **Step 1: Check Monotonicity (Is it increasing or decreasing?)**
  - $x_1 = 1$.

- $x_2 = \sqrt{2 + x_1} = \sqrt{2 + 1} = \sqrt{3} \approx 1.732$.
- Since $x_2 > x_1$, the sequence appears to be increasing. Let's prove by induction that $x_{n+1} > x_n$.
- **Base Case:** $n = 1$, $x_2 = \sqrt{3} > 1 = x_1$. The base case holds.
- **Inductive Hypothesis:** Assume $x_k > x_{k-1}$ for some $k \geq 2$.
- **Inductive Step:** We want to show $x_{k+1} > x_k$.
  - $x_{k+1} = \sqrt{2 + x_k}$
  - $x_k = \sqrt{2 + x_{k-1}}$
  - Since $x_k > x_{k-1}$ (by inductive hypothesis),
  - $2 + x_k > 2 + x_{k-1}$
  - $\sqrt{2 + x_k} > \sqrt{2 + x_{k-1}}$ (since the square root function is strictly increasing for non-negative values).
  - $x_{k+1} > x_k$.
- Thus, by induction, the sequence (x_n) is strictly increasing.

- **Step 2: Check Boundedness**
  - Since $x_1 = 1$, and the sequence is increasing, it is bounded below by 1.
  - Let's hypothesize an upper bound. If the sequence converges to a limit L, then $L = \sqrt{2 + L}$.
    - $L^2 = 2 + L$
    - $L^2 - L - 2 = 0$
    - $(L - 2)(L + 1) = 0$

- So, $L = 2$ or $L = -1$. Since $x_n = \sqrt{2 + x_{n-1}}$ must be positive (as $x_1 = 1$ and square roots are non-negative), the limit must be non-negative. Thus, $L = 2$.

- Let's prove by induction that $x_n < 2$ for all n.
- **Base Case:** $x_1 = 1 < 2$. The base case holds.
- **Inductive Hypothesis:** Assume $x_k < 2$ for some $k \geq 1$.
- **Inductive Step:** We want to show $x_{k+1} < 2$.
  - Since $x_k < 2$,
  - $2 + x_k < 2 + 2 = 4$.
  - $\sqrt{2 + x_k} < \sqrt{4}$
  - $x_{k+1} < 2$.
- Thus, by induction, the sequence (x_n) is bounded above by 2.

- **Step 3: Conclusion on Convergence**
  - Since the sequence (x_n) is monotone increasing and bounded above, by the **Monotone Convergence Theorem**, the sequence is convergent.
- **Step 4: Find the Limit**
  - Let $\lim_{n\to\infty} x_n = L$.
  - Since $x_{n+1} = \sqrt{2 + x_n}$, taking the limit of both sides: $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2 + x_n}$
  - Since the square root function is continuous, we can pass the limit inside: $L = \sqrt{2 + \lim_{n\to\infty} x_n}$ $L = \sqrt{2 + L}$
  - Squaring both sides: $L^2 = 2 + L$ $L^2 - L - 2 = 0$ $(L - 2)(L + 1) = 0$

  - This gives two possible values for L: $L = 2$ or $L = -1$.
  - Since $x_n > 0$ for all n (as $x_1 = 1$ and subsequent terms are square roots of positive numbers), the limit L must be non-negative.
- Therefore, the limit of the sequence is $L = 2$.

4. (a) Prove that if a sequence (x_n) is a monotone decreasing and bounded below sequence of real numbers, then it is convergent.

- **Given:**
  - (x_n) is a sequence of real numbers.
  - (x_n) is monotone decreasing, meaning $x_{n+1} \leq x_n$ for all n.
  - (x_n) is bounded below, meaning there exists a real number m such that $x_n \geq m$ for all n.
- **To prove:** (x_n) is convergent (i.e., $\lim_{n\to\infty} x_n$ exists and is a finite real number).
- **Proof:**
  - Consider the set $S = \{x_n \mid n \in \mathbb{N}\}$ which contains all terms of the sequence.
  - Since the sequence (x_n) is bounded below, the set S is bounded below.
  - Since $x_1 \in S$, S is a non-empty set.
  - By the **Completeness Axiom (or Supremum Property) of Real Numbers**, every non-empty set of real numbers that is bounded below has a greatest lower bound (infimum) in $\mathbb{R}$.
  - Let $L = \inf S$. We will show that $\lim_{n\to\infty} x_n = L$.
  - Let $\epsilon > 0$ be given.

- Since $L = \inf S$, L is a lower bound for S. This means $x_n \geq L$ for all $n \in \mathbb{N}$.
- Also, since L is the *greatest* lower bound, $L + \epsilon$ is no longer a lower bound for S.
- Therefore, there must exist some term x_N in the sequence such that $x_N < L + \epsilon$.
- Since the sequence (x_n) is monotone decreasing, for all $n \geq N$, we have $x_n \leq x_N$.
- Combining these inequalities: $L \leq x_n \leq x_N < L + \epsilon$ for all $n \geq N$.
- From $L \leq x_n$ and $x_n < L + \epsilon$, we have: $L \leq x_n < L + \epsilon$
- This can be written as: $0 \leq x_n - L < \epsilon$ $|x_n - L| < \epsilon$ (since $x_n - L \geq 0$).
- Thus, for every $\epsilon > 0$, there exists an integer N such that for all $n \geq N$, $|x_n - L| < \epsilon$.

- This is the definition of the limit of a sequence. Therefore, the sequence (x_n) is convergent, and $\lim_{n\to\infty} x_n = L = \inf\{x_n\}$.

4. (b) State Bolzano Weierstrass Theorem for Sequences. Show that the sequence ((-1)^n) is divergent.

- **Bolzano-Weierstrass Theorem for Sequences:**
  - Every bounded sequence of real numbers has a convergent subsequence.
- **Show that the sequence $((-1)^n)$ is divergent.**
- **Proof by contradiction using the definition of convergence:**
  - The sequence is $x_n = (-1)^n$, which is $x_1 = -1, x_2 = 1, x_3 = -1, x_4 = 1, ...$

- Assume, for contradiction, that the sequence $((-1)^n)$ converges to some limit L.
- By the definition of convergence, for every $\epsilon > 0$, there exists an integer N such that for all $n \geq N$, $|(-1)^n - L| < \epsilon$.
- Let's choose $\epsilon = 1/2$.
- Then there must exist an N such that for all $n \geq N$: $|(-1)^n - L| < 1/2$.
- Consider two cases for $n \geq N$:
  - If n is even, then $(-1)^n = 1$. So, $|1 - L| < 1/2$. This implies $-1/2 < 1 - L < 1/2$. Subtracting 1: $-3/2 < -L < -1/2$. Multiplying by -1 (and reversing inequalities): $1/2 < L < 3/2$.
  - If n is odd, then $(-1)^n = -1$. So, $|-1 - L| < 1/2$. This implies $-1/2 < -1 - L < 1/2$. Adding 1: $1/2 < -L < 3/2$. Multiplying by -1 (and reversing inequalities): $-3/2 < L < -1/2$.
- We have found that L must satisfy both $(1/2 < L < 3/2)$ and $(-3/2 < L < -1/2)$.
- These two intervals are disjoint. There is no real number L that can be in both intervals simultaneously.
- This is a contradiction.

- Therefore, our initial assumption that the sequence converges must be false. Hence, the sequence $((-1)^n)$ is divergent.
- **Alternatively, using the Bolzano-Weierstrass Theorem:**
  - The sequence $((-1)^n)$ is bounded (e.g., $-1 \leq (-1)^n \leq 1$ for all n).

- By Bolzano-Weierstrass, it must have a convergent subsequence.
- Consider the subsequence of even terms: $x_{2k} = (-1)^{2k} = 1$ for all $k \in \mathbb{N}$. This subsequence converges to 1.
- Consider the subsequence of odd terms: $x_{2k-1} = (-1)^{2k-1} = -1$ for all $k \in \mathbb{N}$. This subsequence converges to -1.
- A fundamental property of convergent sequences is that if a sequence converges, then every subsequence must converge to the *same* limit.
- Since we found two subsequences that converge to different limits (1 and -1), the original sequence $((-1)^n)$ cannot converge.

- Therefore, the sequence $((-1)^n)$ is divergent.

4. (c) Find limit inferior and limit superior of the following sequences: (i) sin(nπ/4) (ii) (3 + (-1)^n)

- **Definitions:**
  - **Limit Superior ($\limsup_{n\to\infty} x_n$):** The largest accumulation point of the sequence. It can also be defined as $\inf_{k\in\mathbb{N}}(\sup_{n\geq k} x_n)$.
  - **Limit Inferior ($\liminf_{n\to\infty} x_n$):** The smallest accumulation point of the sequence. It can also be defined as $\sup_{k\in\mathbb{N}}(\inf_{n\geq k} x_n)$.
- **(i)** $x_n = \sin(n\pi/4)$
  - Let's list the values of $\sin(n\pi/4)$ for different n:
    - $n = 1: \sin(\pi/4) = 1/\sqrt{2}$
    - $n = 2: \sin(2\pi/4) = \sin(\pi/2) = 1$
    - $n = 3: \sin(3\pi/4) = 1/\sqrt{2}$
    - $n = 4: \sin(4\pi/4) = \sin(\pi) = 0$

- $n = 5: \sin(5\pi/4) = -1/\sqrt{2}$
- $n = 6: \sin(6\pi/4) = \sin(3\pi/2) = -1$
- $n = 7: \sin(7\pi/4) = -1/\sqrt{2}$
- $n = 8: \sin(8\pi/4) = \sin(2\pi) = 0$
- The values repeat in a cycle of 8. The set of all values is $\{0, 1/\sqrt{2}, 1, -1/\sqrt{2}, -1\}$.

- The accumulation points of this sequence are the values it repeatedly takes: $\{-1, -1/\sqrt{2}, 0, 1/\sqrt{2}, 1\}$.
- **Limit Superior:** The largest accumulation point is 1.
  - $\limsup_{n\to\infty} \sin(n\pi/4) = 1$.
- **Limit Inferior:** The smallest accumulation point is -1.
  - $\liminf_{n\to\infty} \sin(n\pi/4) = -1$.

- **(ii)** $y_n = (3 + (-1)^n)$
  - Let's list the values of y_n for different n:
    - If n is even, $(-1)^n = 1$. So $y_n = 3 + 1 = 4$.
    - If n is odd, $(-1)^n = -1$. So $y_n = 3 - 1 = 2$.
  - The sequence is $2,4,2,4,2,4, \ldots$.
  - The accumulation points of this sequence are the values it repeatedly takes: $\{2,4\}$.
  - **Limit Superior:** The largest accumulation point is 4.
    - $\limsup_{n\to\infty} (3 + (-1)^n) = 4$.
  - **Limit Inferior:** The smallest accumulation point is 2.
    - $\liminf_{n\to\infty} (3 + (-1)^n) = 2$.

4. (d) Show that every Cauchy sequence of real numbers is bounded. Is the converse true? Justify your answer.

- **Show that every Cauchy sequence of real numbers is bounded.**
- **Proof:**
  - Let (x_n) be a Cauchy sequence.
  - By the definition of a Cauchy sequence, for every $\epsilon > 0$, there exists a natural number N such that for all $m, n \geq N$, $|x_n - x_m| < \epsilon$.
  - Let's choose $\epsilon = 1$.
  - Then there exists an integer N such that for all $n \geq N$, $|x_n - x_N| < 1$.
  - This implies $-1 < x_n - x_N < 1$.
  - Adding x_N to all parts: $x_N - 1 < x_n < x_N + 1$ for all $n \geq N$.
  - This shows that all terms of the sequence from x_N onwards are bounded between $x_N - 1$ and $x_N + 1$.
  - Now, consider the set of all terms of the sequence: $\{x_1, x_2, \ldots, x_{N-1}, x_N, x_{N+1}, \ldots\}$.
  - Let $M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |x_N - 1|, |x_N + 1|\}$.
  - Then, for all $n \in \mathbb{N}$, we have $|x_n| \leq M$.
  - Specifically, for $n < N$, $|x_n|$ is bounded by $\max\{|x_1|, \ldots, |x_{N-1}|\}$.
  - For $n \geq N$, we have $x_N - 1 < x_n < x_N + 1$. This implies $|x_n| < \max(|x_N - 1|, |x_N + 1|)$.
  - Thus, the sequence (x_n) is bounded.
- **Is the converse true? Justify your answer.**
- **No, the converse is not true.**

- The converse would state: "Every bounded sequence of real numbers is a Cauchy sequence." This is false.

- **Justification (Counterexample):**
  - Consider the sequence $x_n = (-1)^n$.
  - This sequence is bounded, as $-1 \leq x_n \leq 1$ for all n.
  - However, we have already shown in part 4(b) that this sequence is divergent.
  - We also know that every convergent sequence is a Cauchy sequence (and conversely, in $\mathbb{R}$, every Cauchy sequence is convergent).
  - Since $x_n = (-1)^n$ is divergent, it cannot be a Cauchy sequence.
  - To demonstrate it's not Cauchy directly:
    - Take $\epsilon = 1$.
    - For any N, we can find $m, n \geq N$ such that x_n and x_m have different signs.
    - For instance, let n be an even integer $\geq N$ (so $x_n = 1$) and m be an odd integer $\geq N$ (so $x_m = -1$).
    - Then $|x_n - x_m| = |1 - (-1)| = |2| = 2$.
    - Since $2 \nless 1$ (our chosen ϵ), the condition for a Cauchy sequence is not met.
  - Thus, $x_n = (-1)^n$ is a bounded sequence that is not Cauchy.
- Therefore, the converse is false.

5. (a) State and prove Cauchy Criterion for convergence of a series Σ $\mathbf{a_n}$.

- **Cauchy Criterion for Convergence of a Series Σa_n:**

- The infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if for every $\epsilon > 0$, there exists a natural number N such that for all $m > n \geq N$, $|a_{n+1} + a_{n+2} + \cdots + a_m| < \epsilon$.

- **Proof:**
  - Let $S_n = a_1 + a_2 + \cdots + a_n$ be the n-th partial sum of the series $\sum a_n$.
  - By definition, the series $\sum a_n$ converges if and only if the sequence of its partial sums (S_n) converges.
  - We know that a sequence of real numbers converges if and only if it is a Cauchy sequence (this is the Cauchy Convergence Criterion for sequences in $\mathbb{R}$).
  - Therefore, the series $\sum a_n$ converges if and only if the sequence of partial sums (S_n) is a Cauchy sequence.
  - By the definition of a Cauchy sequence, (S_n) is Cauchy if for every $\epsilon > 0$, there exists a natural number N such that for all $m > n \geq N$, we have $|S_m - S_n| < \epsilon$.
  - Now, let's look at the expression $S_m - S_n$: $S_m - S_n = (a_1 + \cdots + a_n + a_{n+1} + \cdots + a_m) - (a_1 + \cdots + a_n)$ $S_m - S_n = a_{n+1} + a_{n+2} + \cdots + a_m$.
  - Substituting this into the Cauchy condition for sequences: For every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m > n \geq N$, $|a_{n+1} + a_{n+2} + \cdots + a_m| < \epsilon$.
- This proves the Cauchy Criterion for the convergence of a series.

5. (b) Test the convergence of the following series: (i) Σ n/e^n (ii) Σ ln n/n^2

- **(i)** $\sum_{n=1}^{\infty} \frac{n}{e^n}$
  - We can use the **Ratio Test** for convergence.

- Let $a_n = \frac{n}{e^n}$. Then $a_{n+1} = \frac{n+1}{e^{n+1}}$.
- Consider the limit $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$: $L = \lim_{n\to\infty} \left|\frac{(n+1)/e^{n+1}}{n/e^n}\right|$ $L = \lim_{n\to\infty} \frac{n+1}{e^{n+1}} \cdot \frac{e^n}{n}$ $L = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{e^n}{e^{n+1}}$ $L = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \cdot \frac{1}{e}$ $L = (1+0) \cdot \frac{1}{e} = \frac{1}{e}$.
- Since $e \approx 2.718$, $L = 1/e \approx 1/2.718 < 1$.
- By the Ratio Test, since $L < 1$, the series $\sum_{n=1}^{\infty} \frac{n}{e^n}$ **converges**.

- **(ii)** $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$
  - We can use the **Comparison Test** or **Limit Comparison Test**.
  - For $n \geq 1$, we know that $\ln n < n$. (Actually, for $n \geq 1$, $\ln n < n^{\alpha}$ for any $\alpha > 0$).
  - Specifically, for sufficiently large n, $\ln n < n^{1/2} = \sqrt{n}$. (A stronger bound is useful here).
  - Let's compare with the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ (since $p = 3/2 > 1$).
  - Consider the terms $a_n = \frac{\ln n}{n^2}$ and $b_n = \frac{1}{n^{3/2}}$.
  - We need to evaluate $\lim_{n\to\infty} \frac{a_n}{b_n}$: $\lim_{n\to\infty} \frac{\ln n/n^2}{1/n^{3/2}} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot n^{3/2}$ $= \lim_{n\to\infty} \frac{\ln n}{n^{1/2}}$.
  - This limit is of the form ∞/∞, so we can use L'Hopital's Rule (treating n as a continuous variable x): $\lim_{x\to\infty} \frac{\ln x}{x^{1/2}} =$ $\lim_{x\to\infty} \frac{1/x}{(1/2)x^{-1/2}} = \lim_{x\to\infty} \frac{1}{x} \cdot \frac{2}{x^{-1/2}} = \lim_{x\to\infty} \frac{2}{x^{1/2}} = 0$.
  - Since the limit is 0, and $\sum b_n = \sum \frac{1}{n^{3/2}}$ is a convergent p-series ($p = 3/2 > 1$), by the **Limit Comparison Test** (specifically, if

$\lim_{n\to\infty} a_n/b_n = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges), the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ **converges**.

5. (c) Prove that Σ 1/(n(ln n)p), p > 0 is convergent for p > 1 and divergent for p ≤ 1.

- **Given:** The series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ for $p > 0$. (The sum starts from $n = 2$ because $\ln 1 = 0$).
- **Proof using the Integral Test:**
  - Let $f(x) = \frac{1}{x(\ln x)^p}$.
  - For $x \geq 2$, $f(x)$ is positive, continuous, and decreasing (since x and $\ln x$ are increasing, and $p > 0$).
  - Therefore, the Integral Test can be applied. The series converges if and only if the improper integral $\int_2^{\infty} \frac{1}{x(\ln x)^p} dx$ converges.
  - Let $u = \ln x$. Then $du = \frac{1}{x} dx$.
  - When $x = 2$, $u = \ln 2$.
  - When $x \to \infty$, $u \to \infty$.
  - The integral becomes: $\int_{\ln 2}^{\infty} \frac{1}{u^p} du$.
  - **Case 1:** $p = 1$
    - $\int_{\ln 2}^{\infty} \frac{1}{u} du = [\ln|u|]_{\ln 2}^{\infty} = \lim_{b\to\infty}(\ln b - \ln(\ln 2))$.
    - This limit goes to ∞. So, the integral **diverges** for $p = 1$.
  - **Case 2:** $p \neq 1$
    - $\int_{\ln 2}^{\infty} u^{-p}\, du = \left[\frac{u^{-p+1}}{-p+1}\right]_{\ln 2}^{\infty} = \left[\frac{1}{(1-p)u^{p-1}}\right]_{\ln 2}^{\infty}$.

- If $p > 1$, then $p - 1 > 0$. As $u \to \infty$, $u^{p-1} \to \infty$, so $\frac{1}{(1-p)u^{p-1}} \to 0$.
    - The integral evaluates to $0 - \frac{1}{(1-p)(\ln 2)^{p-1}} = \frac{1}{(p-1)(\ln 2)^{p-1}}$.
    - This is a finite value, so the integral **converges** for $p > 1$.
- If $0 < p < 1$, then $p - 1 < 0$. Let $-(p-1) = q > 0$. So $u^{p-1} = u^{-q} = 1/u^q$.
    - The integral becomes $\left[\frac{u^{1-p}}{1-p}\right]_{\ln 2}^{\infty}$.
    - As $u \to \infty$, $u^{1-p} \to \infty$ (since $1 - p > 0$).
    - So the integral **diverges** for $0 < p < 1$.

- **Conclusion:**
    - Based on the Integral Test, the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is:
        - **Convergent for $p > 1$.**
        - **Divergent for $p \leq 1$.**

5. (d) Show that if the series Σ u_n converges, then lim u_n = 0. Is the converse true? Justify your answer.

- **Show that if the series $\sum u_n$ converges, then $\lim u_n = 0$.**
- **Proof:**
    - Let the series $\sum_{n=1}^{\infty} u_n$ converge to a sum S.
    - This means the sequence of partial sums $S_k = u_1 + u_2 + \cdots + u_k$ converges to S, i.e., $\lim_{k\to\infty} S_k = S$.

- We can write the n-th term u_n in terms of partial sums: $u_n = S_n - S_{n-1}$ for $n \geq 2$.
- Consider the limit of u_n as $n \to \infty$: $\lim_{n\to\infty} u_n = \lim_{n\to\infty}(S_n - S_{n-1})$.
- Since $\lim_{n\to\infty} S_n = S$, it also means $\lim_{n\to\infty} S_{n-1} = S$ (as (S_{n-1}) is just a shifted version of the convergent sequence (S_n)).
- Using the property of limits of sequences (limit of a difference is the difference of limits): $\lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$.

- Therefore, if the series $\sum u_n$ converges, then $\lim_{n\to\infty} u_n = 0$. This is often called the **n-th Term Test for Divergence**.
- **Is the converse true? Justify your answer.**
- **No, the converse is not true.**
  - The converse would state: "If $\lim u_n = 0$, then the series $\sum u_n$ converges." This is false.
- **Justification (Counterexample):**
  - Consider the **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$.
  - First, let's check the limit of the n-th term: $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$. So the condition $\lim u_n = 0$ is satisfied.
  - However, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is a well-known **divergent** series (it's a p-series with $p = 1 \leq 1$).
  - We can prove its divergence by grouping terms (or using the integral test as in 5(c) with $p = 1$): $1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots 1 + \frac{1}{2} + \left(> \frac{1}{4} + \frac{1}{4}\right) + \left(> \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots 1 +$

$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$ The sum grows indefinitely, so the series diverges.

- Therefore, the converse is false. The condition $\lim u_n = 0$ is a necessary condition for convergence, but not a sufficient one.

6. (a) State the Alternating Series test. Show that the alternating series Σ $(-1)^{n+1}/n^2$ is convergent.

- **Alternating Series Test (Leibniz Test):**
    - Consider an alternating series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ or $\sum_{n=1}^{\infty}(-1)^n b_n$, where $b_n > 0$.
    - The series converges if the following two conditions are met:
        v. The sequence (b_n) is decreasing (i.e., $b_{n+1} \le b_n$ for all n).
        vi. $\lim_{n\to\infty} b_n = 0$.
- **Show that the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ is convergent.**
- **Proof:**
    - The given series is $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$.
    - Here, $b_n = \frac{1}{n^2}$.
    - We need to check the two conditions of the Alternating Series Test:
    - **Condition 1: Is (b_n) decreasing?**
        - We need to check if $b_{n+1} \le b_n$, i.e., $\frac{1}{(n+1)^2} \le \frac{1}{n^2}$.
        - This is true because $(n+1)^2 > n^2$ for all $n \ge 1$, and since both are positive, taking reciprocals reverses the inequality.

- So, the sequence (b_n) is indeed decreasing.

- **Condition 2: Does** $\lim_{n\to\infty} b_n = 0$**?**
  - $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^2}$.
  - As $n \to \infty$, $n^2 \to \infty$, so $\frac{1}{n^2} \to 0$.
  - So, $\lim_{n\to\infty} b_n = 0$.
- Since both conditions of the Alternating Series Test are satisfied, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ **converges**.

6. (b) Test the convergence of the series 1/e + 4/e² + 27/e³ + 256/e⁴ + 3125/e⁵ + ...

- The terms of the series appear to be of the form n^n/e^n.
- Let the series be $\sum_{n=1}^{\infty} a_n$, where $a_n = \frac{n^n}{e^n} = \left(\frac{n}{e}\right)^n$.
- We can use the **Root Test** for convergence.
- Consider the limit $L = \lim_{n\to\infty} |a_n|^{1/n}$: $L = \lim_{n\to\infty} \left|\left(\frac{n}{e}\right)^n\right|^{1/n}$ $L = \lim_{n\to\infty} \frac{n}{e}$ $L = \frac{1}{e} \lim_{n\to\infty} n$.
- As $n \to \infty$, $\lim_{n\to\infty} n = \infty$.
- So, $L = \infty$.
- By the Root Test, since $L = \infty > 1$, the series $\sum_{n=1}^{\infty} \frac{n^n}{e^n}$ **diverges**.

6. (c) Define a conditionally convergent series and an absolutely convergent series. Test the series Σ (-1)ⁿ sin n/n³/² for absolute or conditional convergence.

- **Definition of Absolutely Convergent Series:**

- A series $\sum a_n$ is said to be **absolutely convergent** if the series of the absolute values of its terms, $\sum|a_n|$, converges.
- If a series is absolutely convergent, then it is also convergent.

- **Definition of Conditionally Convergent Series:**
  - A series $\sum a_n$ is said to be **conditionally convergent** if the series itself converges, but the series of the absolute values of its terms, $\sum|a_n|$, diverges.
- **Test the series $\sum_{n=1}^{\infty} \frac{(-1)^n \sin n}{n^{3/2}}$ for absolute or conditional convergence.**
- **Step 1: Test for Absolute Convergence.**
  - Consider the series of absolute values: $\sum_{n=1}^{\infty} \left|\frac{(-1)^n \sin n}{n^{3/2}}\right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^{3/2}}$.
  - We know that $0 \leq |\sin n| \leq 1$ for all n.
  - So, we have the inequality: $\frac{|\sin n|}{n^{3/2}} \leq \frac{1}{n^{3/2}}$.
  - Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. This is a **p-series** with $p = 3/2$.
  - Since $p = 3/2 > 1$, the p-series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges.
  - By the **Direct Comparison Test**, since $0 \leq \frac{|\sin n|}{n^{3/2}} \leq \frac{1}{n^{3/2}}$ and $\sum \frac{1}{n^{3/2}}$ converges, the series $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^{3/2}}$ also converges.
  - Since the series of absolute values converges, the original series $\sum_{n=1}^{\infty} \frac{(-1)^n \sin n}{n^{3/2}}$ is **absolutely convergent**.
- **Step 2: Conclusion.**

- Because the series is absolutely convergent, it is also convergent. We don't need to test for conditional convergence separately.

- Therefore, the series $\sum_{n=1}^{\infty} \frac{(-1)^n \sin n}{n^{3/2}}$ is **absolutely convergent**.

6. (d) State D'Alembert's Ratio test for a series. Find if the series, 1/2 + (1·2)/(3·5) + (1·2·3)/(3·5·7) + (1·2·3·4)/(3·5·7·9) + ... is convergent.

- **D'Alembert's Ratio Test:**
  - Let $\sum a_n$ be a series with positive terms (or consider $|a_n|$ for general terms).
  - Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.
  - The test concludes:

    vii. If $L < 1$, the series converges absolutely.

    viii. If $L > 1$ (or $L = \infty$), the series diverges.

    ix. If $L = 1$, the test is inconclusive (the series may converge or diverge).

- **Find if the series,** $1/2 + (1 \cdot 2)/(3 \cdot 5) + (1 \cdot 2 \cdot 3)/(3 \cdot 5 \cdot 7) + (1 \cdot 2 \cdot 3 \cdot 4)/(3 \cdot 5 \cdot 7 \cdot 9) + \ldots$ **is convergent.**

- **Step 1: Write the general term a_n.**
  - The numerator is the product of integers from 1 to n, which is $n!$.
  - The denominator is the product of odd integers: $3 \cdot 5 \cdot 7 \cdot \ldots$. Let's find the n-th term in this sequence.
    - For $n = 1$, denominator is $2 \cdot 1 + 1 = 3$. Oh, no, it's just 2. Wait, the first term is $1/2$.
    - For $n = 1$: Numerator is $1! = 1$. Denominator is 2.

- For $n = 2$: Numerator is $2! = 1 \cdot 2$. Denominator is $3 \cdot 5$.
- For $n = 3$: Numerator is $3! = 1 \cdot 2 \cdot 3$. Denominator is $3 \cdot 5 \cdot 7$.
- For $n = 4$: Numerator is $4! = 1 \cdot 2 \cdot 3 \cdot 4$. Denominator is $3 \cdot 5 \cdot 7 \cdot 9$.

  - The denominator for a_n is the product of the first n terms of the arithmetic progression $3,5,7,\dots$. The general term of this progression is $3 + (k-1)2 = 2k + 1$.
  - So, the denominator for a_n is $3 \cdot 5 \cdot 7 \cdot \dots \cdot (2n+1)$.
  - Therefore, the general term $a_n = \frac{n!}{3 \cdot 5 \cdot 7 \cdot \dots \cdot (2n+1)}$.

- **Step 2: Find a_{n+1}.**
  - $a_{n+1} = \frac{(n+1)!}{3 \cdot 5 \cdot 7 \cdot \dots \cdot (2n+1) \cdot (2(n+1)+1)}$
  - $a_{n+1} = \frac{(n+1)!}{3 \cdot 5 \cdot 7 \cdot \dots \cdot (2n+1) \cdot (2n+3)}$.

- **Step 3: Calculate the ratio a_{n+1}/a_n.**
  - $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{3 \cdot 5 \cdot \dots \cdot (2n+1) \cdot (2n+3)} \cdot \frac{3 \cdot 5 \cdot \dots \cdot (2n+1)}{n!}$
  - $\frac{a_{n+1}}{a_n} = \frac{(n+1) \cdot n!}{n!} \cdot \frac{1}{2n+3}$
  - $\frac{a_{n+1}}{a_n} = \frac{n+1}{2n+3}$.

- **Step 4: Find the limit $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.**
  - $L = \lim_{n\to\infty} \frac{n+1}{2n+3}$
  - Divide numerator and denominator by n: $L = \lim_{n\to\infty} \frac{1+1/n}{2+3/n}$ $L = \frac{1+0}{2+0} = \frac{1}{2}$.

- **Step 5: Apply the Ratio Test conclusion.**
  - Since $L = 1/2 < 1$, by D'Alembert's Ratio Test, the series **converges**.