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[This question paper contains 8 printed pages.]

Your Roll No.............

Sr. No. of Question Paper : 1085 I

Unique Paper Code : 2222012301

Name of the Paper : Mathematical Physics – III

Name of the Course : B.Sc. (H) Physics

Semester : III

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt five questions in all.

3. All questions carry equal marks.

4. Question number 1 is compulsory.

5. The principal branch of argument of complex number is to be taken as $0 \leq \theta \leq 2\pi$.

5. Use the following definition for the Fourier transform of f(x) :

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

6. Use the following definition for the Fourier sine transform of f(x) :

$$\mathcal{F}_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin(kx)\, dx$$

7. Use the following definition for the Fourier cosine transform of f(x) :

$$\mathcal{F}_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos(kx)\, dx$$

8. Use the following definition for the convolution of two functions f(x) and g(x) :

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$$

9. Some useful formulae are given at the end.

1. Attempt any **six** questions : (6×3=18)

(a) Obtain all the roots of the equation :

$$z^3 - i = 0;\ i = \sqrt{-1}$$

(b) Show that $tanh^{-1}(z) = \frac{1}{2} ln \frac{1+z}{1-z}$

(c) In the finite z-plane, determine and classify the singularities of the function:

$$f(z) = tan^{-1}(z^2 + 4z + 5)$$

(d) Solve $\frac{1}{2\pi i} \oint_C \frac{e^z\, dz}{z-1}$; C is $|z - 1| = 2$.

(e) Find the residue at $z = 0$ for $f(z) = \frac{\cosh(z)}{z^3}$.

(f) If $\mathcal{F}^{-1}[F(k)] = f(x)$, show that

$$\mathcal{F}^{-1}[F(k-a)] = e^{iax} f(x); a > 0.$$

(g) General solution of 1-d wave equation is given as:

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi t}{L}\right)$$

If $0 \leq x \leq L$ and $y(x, 0) = x$, determine c_n.

2. (a) If $z = 4\, e^{i\pi/3}$, evaluate $|e^{iz}|$. (4)

(b) Given $\tan(x + iy) = u + iv$, show that

$$u = \frac{\sin(2x)}{\cos(2x) + \cosh(2y)}$$

and $$v = \frac{\sinh(2y)}{\cos(2x) + \cosh(2y)}$$ (6)

(c) Prove that the function $u(x, y) = 2x(1 - y)$ is harmonic and hence find $v(x, y)$ such that $f(z) = u + iv$ is analytic. Also, express $f(z)$ in terms of z, where $z = x + iy$. (8)

3. (a) State and verify Cauchy's theorem for the function:

$$f(z) = 3z + 2i$$

and C is a triangle with vertices $1 + i$, $-1 \pm i$. (8)

(b) Solve the integral

$$\oint_C \frac{z^2\, dz}{(z^2 + 9)(z^2 + 4)^2}; \quad C \text{ is } |z| = 1$$ (5)

(c) Expand

$$f(z) = \frac{z}{(z + 1)(z - 2)}$$

in a Laurent series valid for the annular domain $0 < |z - 2| < 3$. (5)

4. Using residue theorem and suitable contour, solve any two real integrals : (2×9=18)

(a) $\int_0^{\infty} \frac{x^2}{x^4+1} dx$

(b) $\int_0^{2\pi} \frac{d\theta}{\cos\theta + 2\sin\theta + 3}$

(c) $\int_0^{\infty} \frac{x \sin 2x}{x^2+9} dx$

5. (a) Find $\mathcal{F}^{-1}\left(\frac{1}{k^2 - 4k + 29}\right)$. (8)

(b) Show that $\mathcal{F}_c\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\sqrt{k}}$. (5)

(c) Obtain the function $q(x)$, if $\mathcal{F}_s[q(x)] = e^{-2k}$. (5)

6. (a) Using the method of separation of variables, find the solution of the following partial differential equation :

$$4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$$

such that $u(0, y) = 3e^{-y}$. (4)

(b) Solve one-dimensional heat equation:

$$\frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2} \qquad (0 \le x \le L).$$

such that $u(0, t) = 0, u(L, t) = 0$

and $u(x, 0) = x(L - x)$ (14)

Some useful formulae:

1. $\int_0^\infty x^n \exp(-ax^m)\, dx = \frac{1}{m\, a^{(n+1)/m}} \Gamma\left(\frac{n+1}{m}\right)$;

$n > -1;\ a, m > 0$

2. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

3. $\mathcal{F}^{-1}\left(\frac{1}{k^2 + a^2}\right) = \frac{\sqrt{2\pi}}{2a} e^{-a|x|};\ a > 0$

4. $\mathcal{F}^{-1}[a\, g(k) + b\, h(k)] = a\, \mathcal{F}^{-1}[g(k)] + b\, \mathcal{F}^{-1}[h(k)]$
(*a* and *b* are constants)