This question paper contains 7 printed pages]

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S. No. of Question Paper : **5736**

Unique Paper Code : **2353012004**

Name of the Paper : **Biomathematics**

Name of the Course : **B.Sc. (H) Mathematics**

Semester : **IV**

Duration : **3 Hours** Maximum Marks : **90**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any *two* parts from each question.

Each part of questions is of **7.5** marks.

1. (*a*) Define the concept of an equilibrium state for $\frac{dP}{dt} = f(P)$, and refer to Figure 1 for the graph of $f(P)$. Identify the equilibrium points and categorize them as either stable or unstable.

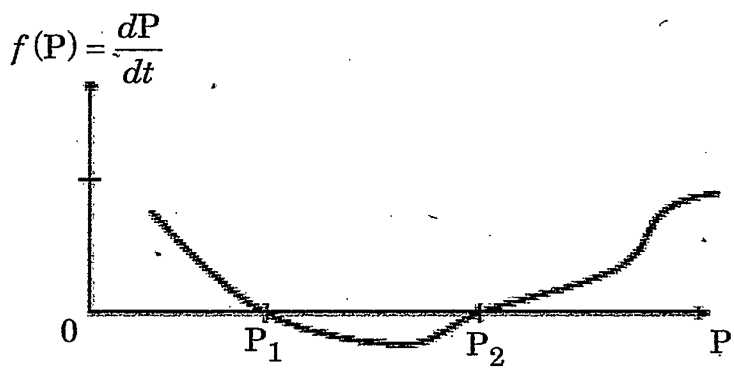

Figure 1. Plot of dP/dt *versus* P.

P.T.O.

(b) Suppose a virus enters the blood stream and develops at a rate proportional to its concentration. An antidote to the virus is administered at a time h and decays according to the law

$$\frac{da(t)}{dt} = -\frac{a(t)}{\tau}.$$

Write a simple model of Virus antidote needed to eradicate the viral infection and show initial dose a_0 at time h should satisfy

$$a_0 > \frac{C_0}{p}\left(k + \frac{1}{\tau}\right)e^{kh},$$

where p is rate at which the antidote kills virus cell and $k > 0$ is rate of proportionality for growth of virus. C_0 is initial concentration of virus.

(c) From the data given below showing the initial phase of yeast culture :

Time (hrs)	Biomass
0	9.6
1	18.3
2	29
3	47.2
4	71.1
5	119.1
6	174.6
7	257.3

(*i*) Find the best value of

$k = (P_{n+1} - P_n)/P_n$ and $r = \ln(P_{n+1}) - \ln(P_n)$,

where P_n represents the population aftet n hours, for the discrete model and continuous model for growth respectively.

(*ii*) Use determined values of k and r to find predicted values from the continuous model. Do you expect the continuous model to remain accurate in predicting the long term growth behavior of the culture ?

2. (*a*) State Law of mass action. Use the law to derive the mathematical model governed by the intermediaries X and Y in the trimolecular reaction :

$$A \rightarrow X$$

$$B + X \rightarrow Y + D$$

$$2X + Y \rightarrow 3X$$

$$X \rightarrow E$$

where A, B, D and E are initial and final products and all rate constants are equal to 1.

(*b*) Compute Volterra-Lotka model of predator-prey interaction and find conditions for a steady state to exist.

(c) Find equilibrium points and discuss their nature for discrete Verhulst model :

$$P_{n+1} - P_n = a\left(1 - \frac{P_n}{K}\right)P_n.$$

Also, discuss what happens if initial population size, P(0), lies between 0 and K.

3. (a) Assuming that the population with size N is divided into two non-intersecting groups — the group of those who have the disease and can infect others (I) and the group of those who do not have the disease and can be infected (S). Write the SIS model for infectious diseases along with properly defining all the parameters used. Also mention the assumptions undertaken to derive the model. Mention the relation that $\bar{d}$, average length of the infection, exhibits with the per capita recovery rate β in the SIS model. Also, explain how the value of β will affect the length of the disease.

(b) Show that in the SIR model,

$$\frac{dS}{dR} = -\frac{\alpha}{\beta}S.$$

Further deduce that :

$$S(t) = S(0)e^{-\frac{\alpha}{\beta}R(t)} \text{ for } R(0) = 0.$$

Also show that in SIR model, in the long run, a fraction of the population will never get infected.

(c) Consider the following system :

$$\frac{dx}{dt} = -0.00001\, xy + 0.2$$

$$\frac{dy}{dt} = 0.00001\, xy - 0.02\, y$$

Determine the equilibrium points; and check whether they are asymptotically stable ?

4\. (a) Sketch the trajectories in the phase plane of the system :

$$\frac{dx}{dt} = 6x + 12y$$

$$\frac{dy}{dt} = 3x + y.$$

(b) State the Poincaré-Bendixson Theorem and use it to show that the system

$$\frac{dx}{dt} = x - y - x^3,$$

$$\frac{dy}{dt} = x + y - y^3,$$

has a stable periodic orbit.

(c) Prove that :

$$\frac{d^2x}{dt} - \in \left(1 - x^4\right)\frac{dx}{dt} + x = 0$$

has a stable limit cycle.

5. (a) What is bifurcation ? Show that the non-linear system with $\mu \geq 0$,

$$\frac{dx}{dt} = y,$$

$$\frac{dy}{dt} = \mu \sin x - x,$$

has one equilibrium point for $0 \leq \mu < 1$ and three for $\mu > 1$. Also discuss the nature of these equilibrium points.

(b) What is Hopf bifurcation ? Show that Hopf bifurcation holds for the following system :

$$\frac{dx}{dt} = -y + x(\mu - x^2 - y^2),$$

$$\frac{dy}{dt} = x + y(\mu - x^2 - y^2).$$

(c) On a Poincare plane successive points are related by

$$x_{n+1} = \frac{1}{2} y_n,$$

$$y_{n+1} = -x_n + \frac{1}{2}\mu x_n - y_n^3.$$

Show that there is a bifurcation at $\mu = 3$. Show that the limit cycle in $0 < \mu < 3$ is stable and becomes of saddle type when μ exceeds 3.

6. (a) For the iteration scheme

$$x_{n+1} = \mu x_n(1 - x_n),\ n \geq 1,$$

$$x_0 = \lim_{x \to \infty} x_n,$$

Show that there are bifurcations at $\mu = 1$ and $\mu = 3$.

(*b*) Explain Jukes-Cantor model (JC model). Estimate the proportion of the sites that will have a base **A** in the ancestral sequence and a base **T** in the descendent sequence after one time step.

(*c*) Derive the formula for the Jukes-Cantor distance (d_{JC}) given that all diagonal entries of Jukes-Cantor matrix M^t are $\frac{1}{4}+\frac{3}{4}\left(1-\frac{4}{3}\alpha\right)^t$, where α is the mutation rate. Compute the Jukes-Cantor distance $d_{JC}(S_0, S_1)$ to 6 decimal digits, from the following 400 base table :

$S_1 \backslash S_0$	A	G	C	T
A	92	15	2	2
G	13	84	4	4
C	0	1	77	16
T	4	2	14	70