This question paper contains **7** printed pages]

Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **5737**

Unique Paper Code : **2353012005**

Name of the Paper : **Mathematical Modeling**

Type of the Paper : **DSE**

Name of the Course : **B.Sc. (H) Mathematics**

Semester : **IV**

Duration : **3 Hours** Maximum Marks : **90**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Each question contains three parts.

Attempt *all* questions by selecting *two* parts from each question.

Parts of questions to be attempted together.

All questions carry equal marks.

Use of non-programmable scientific calculators is allowed.

1. (*a*) Define Mathematical Modeling. Support you live on planet of radius R. Place a rope around the planet on its equators. If the length of the rope increased by 10 meters while keeping it in the plane of the equator, how far above the surface of the planet is the lengthened rope ?

P.T.O.

(*b*) Define physical and mathematical equations. Convert the following differential equation of decay of an atomic nucleus as in radioactivity, into a dimensionless equation

$$\frac{dc}{dt} = -\lambda c, \; c(0) = c_0 > 0,$$

where $c(t)$ has the physical units of atoms per unit volume.

(*c*) Find the dimensions of the perameters a, D_1, ι_1 and ι_2 from the following differential equation by assuming it to be dynamically consistent equation :

$$\frac{\partial u}{\partial t} + au\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2} + \lambda_1\sqrt{u} - \lambda_2 u^2.$$

2. (*a*) Using mathemaical modeling, construct an SIR model. Mention the assumptions and symbols used. Give the properties satisfied by the transition functions $T_1(S, I)$ and $T_2(I)$.

(*b*) Consider an epidemic model, where the infected individuals eventually recover and the dynamics of the population are described by the differential equations :

$$\frac{dS}{dt} = -\beta\sqrt{S}\sqrt{I}, \; \frac{dI}{dt} = \beta\sqrt{S}\sqrt{I} - \gamma\sqrt{I}.$$

Using the parameters values $b = 0.02$ and $g = 0.4$ and assuming initially there is only one infected individual but there are only 500 susceptible individuals within a population :

(*i*) Calculate the basic reproduction number r_0 using the given parameters.

(*ii*) How many individuals remain susceptible and never get infected throughout the epidemic ?

(*iii*) What is the maximum number of individuals infected an any point in time during the epidemic ?

(*c*) Consider the dieting model $\frac{dM}{dt} = \lambda - \beta M^{\frac{3}{4}}$, where l, b are constant parameters. Define time scale T^* and show that $T^* = \left(\frac{\lambda}{\beta^4}\right)^{\frac{1}{3}}$. Find the critical point (M^*) for the given dieting model. Is it globally stable ? Draw the graph of $M(t)$ over a period of time if $M(t) > M^*$ and if $M(t) < M^*$.

3. (*a*) Find the equilibrium solution of $x¢¢ + (x^2 - 1)x¢ + x = 0$. Discuss the nature and stability of the critical point.

(*b*) Explain and analyze the Predator-Prey system by identifying and discussing all isolated critical points.

(*c*) Find all critical points of the following systems of differential equations and investigate their type and stability :

(*i*) $\frac{dx}{dt} = y^2 - 1, \frac{dy}{dt} = x^3 - y$

(*ii*) $\frac{dx}{dt} = x - y, \frac{dy}{dt} = x + 3y - 4.$

4\. (*a*) Consider the following system of differential equations,

$$\frac{dx}{dt} = 60x - 3x^2 - 4xy$$

$$\frac{dy}{dt} = 42y - 3y^2 - 2xy.$$

Find the eigenvalues and the corresponding eigenvectors of the coefficient matrix of linear system and hence, determine the type and stability of all non-zero critical points of the system.

(*b*) Find the critical point (x^*, y^*) for the system of differential equations :

$$\frac{dx}{dt} = 4x + 6y - 52$$

$$\frac{dy}{dt} = 4x + 3y - 28.$$

Using appropriate transformation, rewrite the system to the following form :

$$\frac{du}{dt} = au + bv$$

$$\frac{dv}{dt} = cu + dv.$$

Analyze the nature and stability of critical points of the original system and the transformed system.

(c) Consider a damped non-linear spring-mass system. Let m denote the mass of an object attached to a spring and let $x(t)$ be the displacement of the mass at any time t from its equilibrium position. Assume that the mass on the spring is connected to a dashpot, exerting a force of resistance proportional to velocity $y = x¢ = \frac{dx}{dt}$ of the mass. Let the force exerted by the spring on the mass be given by $F(x) = m\frac{dy}{dt} = mx¢¢$. The equation of motion of the mass is given by $mx¢¢ = -cx¢ - kx + \mathrm{b}x^3$, where $c, m, k, \mathrm{b} > 0$. If $m = 1$, $c = 2$, $k = 5$ and $\mathrm{b} = \frac{5}{4}$, then write the corresponding system of first order differential equations and discuss the stability of all critical points of the system.

5. (a) Using Monte Carlo Simulation, write an algorithm to approximate the area under the curve $f(x) = \sqrt{x}$ over the interval $\frac{1}{2} \le x \le \frac{3}{2}$.

(b) Define random numbers. Explain Middle-Square Method for generating random numbers. Give the drawback if it has any. Use it to generate 15 random numbers using 3043, as the seed number.

(c) Using Monte Carlo Simulation, write an algorithm to calculate that part of the volume of the sphere $x^2 + y^2 + z^2 \le 1$ that lies in first octant, $x > 0$, $y > 0$, $z > 0$.

6. (a) Use Simplex method to solve the following problem :

$$\text{Max. } Z = 4x + 3y$$

$$\text{s. to } \quad -x + y \le 6,$$
$$2x + y \le 20,$$
$$x + y \le 12,$$
$$x, y \ge 0.$$

(b) Consider the following Linear Programming Problem :

$$\text{Max. } Z = 6x + 5y$$

$$\text{s. to } \quad x + y \le 6,$$
$$2x + y \le 9,$$
$$x, y \ge 0.$$

Using algebraic methods find the possible number of points of intersection in xy-plane. Are all the points feasible ? If not, then how many are feasible and how many are infeasible. List the feasible extreme points along with the value of the objective function.

(c) Solve the following Linear Programming problem graphically :

Max. $Z = 124x + 60y$

s. to $10x + 3y \leq 16,$

$4x + 5y \leq 14,$

$x, y \geq 0.$

Perform sensitivity analysis to find the range of values for the coefficient of y in the objective function for which the current extreme point remains optimal.