(ii) Let T be a linear operator on a finite dimensional complex inner product space V. Show that T is unitary if and only if T is normal and $|\lambda| = 1$ for every eigen value λ of T. (2+5.5)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 5554 **J**

Unique Paper Code : 2352013602

Name of the Paper : Advanced Linear Algebra

Name of the Course : **Bachelor of Science (Honours Course) Mathematics**

Semester : VI

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **TWO** parts from each Question.

1. (a) Let T be a linear operator on R^2 defined as

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a+b \\ a-3b \end{pmatrix}.$$

For the ordered basis $\beta = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ and

$\beta' = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$, of R^2, find the change of coordinate matrix Q that changes β'-coordinates into β-coordinates. Also, verify that

$[T]_{\beta'} = Q^{-1}[T]_{\beta}Q.$ (2.5+5)

(b) Let $V = P_1(R)$ and for $p(x) \in V$, let $f_1, f_2 \in V^*$ be defined as

$$f_1(p(x)) = \int_0^1 p(t)dt$$

(c) Find the best fit linear function for the data $\{(-3,9), (-2,6), (0,2), (1,1)\}$ using the least squares approximation. Also, compute the error E.

(5+2.5)

6. (a) Let T be a normal operator defined on a finite dimensional real inner product space V whose characteristic polynomial splits. Prove that V has an orthonormal basis of eigen vectors of T. Hence prove that T is self-adjoint.

(5+2.5)

(b) For the following matrix A, find an orthogonal matrix P and a diagonal matrix D such that $P^tAP = D$.

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$ (7.5)

(c) (i) State the Spectral theorem.

(c) Let W = span ({(i, 0, 1)}) in C^3. Find the orthonormal bases for W and $W^{\perp}$. (7.5)

5. (a) Let V = P(R) with the inner product defined as

$$\langle f(x), g(x)\rangle = \int_0^1 f(t)\, g(t)dt,\ \forall\, f(x),\ g(x) \in V.$$

Find the orthogonal projection of the vector $h(x) = 4 + 3x - 2x^2$ on the subspace $W = P_1(R)$. (7.5)

(b) (i) Let V be a finite dimensional inner product space and β be an orthonormal basis for V. If T is a linear operator on V, show that

$$[T^*]_\beta = ([T]_\beta)^*.$$

(ii) For the inner product space $V = C^2$ and linear operator

$$T(z_1, z_2) = (2z_1 + iz_2, (1-i)z_1),$$

evaluate T^* at $z = (3 - i,\ 1 + 2i)$. (4.5+3)

and $f_2(p(x)) = \int_0^2 p(t)dt$.

Prove that $\{f_1, f_2\}$ is a basis for V^* and find a basis of V for which it is the dual basis. (3+4.5)

(c) Let V be finite dimensional vector space. Define the annihilator S^0 of a subset S of V and prove that S^0 is a subspace of V^*. If W_1 and W_2 are subspaces of V, prove that

$$(W_1 + W_2)^0 = W_1^0 \cap W_2^0. \quad (3.5+4)$$

2. (a) Let $A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix} \in M_{2\times 2}(C)$. Determine all eigen values of A and for each eigen value λ of A, find the set of eigen vectors corresponding to λ. Also, find a basis for C^2 consisting of eigenvectors of A. (2.5+5)

(b) Let T be a linear operator on $P_2(R)$ defined as

$$T(f(x)) = f(0) + f(1)(x + x^2).$$

Test the linear operator T for diagonalizability. If T is diagonalizable, then find a basis β for V such that $[T]_\beta$ is a diagonal matrix. (2.5+5)

(c) Let T be a diagonalizable linear operator on a finite dimensional vector space V and $\lambda_1, \lambda_2, \ldots \ldots \lambda_k$ be the distinct eigen values of T. Prove that

$V = E_{\lambda_1} \oplus E_{\lambda_2} \ldots\ldots \oplus E_{\lambda_k}$, where E_{λ_i} is the eigen space of λ_i, for all i. (7.5)

3. (a) Let T be a linear operator on the vector space $V = R^4$ defined as

$$T(a, b, c, d) = (a + b,\ b - c,\ a + c,\ a + d)$$

Find an ordered basis of the T-cyclic subspace W of V generated by $z = e_1$. Also, find the characteristic polynomial of T_W. (3+4.5)

(b) Let T be a linear operator defined on a finite dimensional vector space V. Prove that the characteristic polynomial and the minimal polynomial of T have the same zeros. (7.5)

(c) Let T be a linear operator on $V = M_{n\times n}(R)$ defined as $T(A) = A^t$. Find the minimal polynomial of T. Hence show that T is diagonalizable. (7.5)

4. (a) Let V be an inner product space, prove that the following inequality holds

$$|\langle x, y\rangle| \le \|x\|\,\|y\|, \text{ for all } x, y \in V.$$

Also, verify that the inequality holds for $x = (1, 2i, 1 + i)$, $y = (5 + i, 1, 2)$ in C^3. (5+2.5)

(b) Let V be an inner product space and let S be an orthogonal subset of V consisting of nonzero vectors. Prove that S is linearly independent. (7.5)