Question 1:

(a) Determine the linear dependence (or independence) of set of the functions: -1, $\sin^2 x$, $\cos^2 x$

- To determine linear dependence or independence, we need to check if there exist constants c_1, c_2, c_3, not all zero, such that $c_1(-1) + c_2 \sin^2 x + c_3 \cos^2 x = 0$.
- We know the trigonometric identity $\sin^2 x + \cos^2 x = 1$.
- Rearranging this identity, we get $1 - \sin^2 x - \cos^2 x = 0$, or $-1 + \sin^2 x + \cos^2 x = 0$.
- Comparing this to $c_1(-1) + c_2 \sin^2 x + c_3 \cos^2 x = 0$, we can see that if we choose $c_1 = 1$, $c_2 = 1$, and $c_3 = 1$, the equation holds true.
- Since we found non-zero constants c_1, c_2, c_3 that satisfy the linear combination equaling zero, the set of functions $\{-1, \sin^2 x, \cos^2 x\}$ is linearly dependent.

(b) Solve the differential equation: $[y(1 + 1/x) + \cos y]dx + [x + \log x - x \sin y]dy = 0$

- This is a first-order differential equation of the form $M(x, y)dx + N(x, y)dy = 0$.
- Let $M(x, y) = y(1 + 1/x) + \cos y = y + y/x + \cos y$.
- Let $N(x, y) = x + \log x - x \sin y$.
- Check for exactness:

- $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y + y/x + \cos y) = 1 + 1/x - \sin y$.
- $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x + \log x - x\sin y) = 1 + 1/x - \sin y$.

- Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the differential equation is exact.
- Therefore, there exists a function $f(x, y)$ such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$.
- Integrate M with respect to x (treating y as a constant):
  - $f(x, y) = \int (y + y/x + \cos y)dx = yx + y\log x + x\cos y + h(y)$.
- Now, differentiate $f(x, y)$ with respect to y and set it equal to N:
  - $\frac{\partial f}{\partial y} = x + \log x - x\sin y + h'(y)$.
  - We know $\frac{\partial f}{\partial y} = N = x + \log x - x\sin y$.
  - Comparing these, we get $h'(y) = 0$.
- Integrate $h'(y)$ to find $h(y)$:
  - $h(y) = \int 0dy = C_0$, where C_0 is an arbitrary constant.
- Substitute $h(y)$ back into the expression for $f(x, y)$:
  - $f(x, y) = yx + y\log x + x\cos y + C_0$.
- The general solution to the exact differential equation is $f(x, y) = C$, where C is a constant.
- So, the solution is $yx + y\log x + x\cos y = C$.

(c) Using index notation, verify that: $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{c} \times \vec{a})$

- The scalar triple product $\vec{a} \cdot (\vec{b} \times \vec{c})$ can be written in index notation as $\epsilon_{ijk} a_i b_j c_k$.
- Consider $\vec{c} \cdot (\vec{a} \times \vec{b})$:
    - In index notation, this is $\epsilon_{ijk} c_i a_j b_k$.
    - We can rearrange the indices using the property of the Levi-Civita symbol: $\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki}$.
    - So, $\epsilon_{ijk} c_i a_j b_k = \epsilon_{kij} c_i a_j b_k$. Now, if we relabel the dummy indices $(i \to k, j \to i, k \to j)$, we get $\epsilon_{ijk} a_i b_j c_k$.
    - Alternatively, we can cyclically permute the vectors: $\vec{c} \cdot (\vec{a} \times \vec{b}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$.
    - Using the property of the scalar triple product, a cyclic permutation of the vectors does not change its value: $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$.
    - Thus, $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b})$.
- Consider $\vec{b} \cdot (\vec{c} \times \vec{a})$:
    - In index notation, this is $\epsilon_{ijk} b_i c_j a_k$.
    - Again, using the cyclic property of the Levi-Civita symbol, $\epsilon_{ijk} b_i c_j a_k = \epsilon_{jki} b_i c_j a_k$. If we relabel the dummy indices $(i \to j, j \to k, k \to i)$, we get $\epsilon_{ijk} a_i b_j c_k$.
    - Alternatively, $\vec{b} \cdot (\vec{c} \times \vec{a}) = (\vec{c} \times \vec{a}) \cdot \vec{b}$.

- By cyclic permutation, $(\vec{c} \times \vec{a}) \cdot \vec{b} = \vec{c} \cdot (\vec{a} \times \vec{b})$.
- Since we already showed $\vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{a} \cdot (\vec{b} \times \vec{c})$, it follows that $\vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{a} \cdot (\vec{b} \times \vec{c})$.

- Therefore, $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{c} \times \vec{a})$ is verified.

(d) A certain student population, consisting of 70% from the government schools, selects 15 representatives to attend an international student meet. Find the mean representation of the students from government schools in the sample and calculate its standard deviation.

- This problem can be modeled using a binomial distribution.
- Let n be the total number of representatives selected, $n = 15$.
- Let p be the proportion of students from government schools in the population, $p = 70\% = 0.70$.
- Let X be the number of representatives from government schools in the sample.
- Mean of a binomial distribution: $\mu = np$.
  - $\mu = 15 \times 0.70 = 10.5$.
- Standard deviation of a binomial distribution: $\sigma = \sqrt{np(1-p)}$.
  - $\sigma = \sqrt{15 \times 0.70 \times (1 - 0.70)}$
  - $\sigma = \sqrt{15 \times 0.70 \times 0.30}$
  - $\sigma = \sqrt{15 \times 0.21}$

  - $\sigma = \sqrt{3.15}$
  - $\sigma \approx 1.7748$.

- The mean representation of students from government schools in the sample is 10.5.
- The standard deviation of the representation is approximately 1.7748.

(e) Evaluate $\iint_S \vec{r} \cdot \vec{n} dS$, where S is a closed surface.

- This integral can be evaluated using the Divergence Theorem (Gauss's Theorem).
- The Divergence Theorem states that for a vector field $\vec{F}$ and a closed surface S enclosing a volume V, $\iint_S \vec{F} \cdot \vec{n} dS = \iiint_V (\nabla \cdot \vec{F}) dV$.
- In this case, $\vec{F} = \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.
- Calculate the divergence of $\vec{r}$:
  - $\nabla \cdot \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$.
- Substitute this into the Divergence Theorem:
  - $\iint_S \vec{r} \cdot \vec{n} dS = \iiint_V 3\, dV$.
- Since the integral of a constant over a volume is the constant multiplied by the volume:
  - $\iiint_V 3\, dV = 3 \iiint_V d\,V = 3V$.

- Therefore, $\iint_S \vec{r} \cdot \vec{n} dS = 3V$, where V is the volume enclosed by the closed surface S.

(f) Find the unit vector normal to the surface: $x^2 + y^2 + z^2 = 4$ at the point $(1, \sqrt{2}, -1)$.

- The given surface is a sphere centered at the origin with radius 2.
- To find the normal vector to a surface given by $f(x, y, z) = C$, we can use the gradient ∇f.
- Let $f(x, y, z) = x^2 + y^2 + z^2 - 4$.
- Calculate the gradient of f:
    - $\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$
    - $\nabla f = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$.
- Evaluate the gradient at the given point $(1, \sqrt{2}, -1)$:
    - $\nabla f|_{(1,\sqrt{2},-1)} = 2(1)\hat{\imath} + 2(\sqrt{2})\hat{\jmath} + 2(-1)\hat{k} = 2\hat{\imath} + 2\sqrt{2}\hat{\jmath} - 2\hat{k}$.
- This vector is a normal vector to the surface at the given point. To find the unit normal vector $\hat{n}$, we divide the normal vector by its magnitude.
- Magnitude of the normal vector:
    - $|\nabla f| = \sqrt{(2)^2 + (2\sqrt{2})^2 + (-2)^2}$
    - $|\nabla f| = \sqrt{4 + 8 + 4}$

  - $|\nabla f| = \sqrt{16} = 4$.

- The unit vector normal to the surface at the point $(1, \sqrt{2}, -1)$ is:
  - $\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2\hat{\imath} + 2\sqrt{2}\hat{\jmath} - 2\hat{k}}{4}$
  - $\hat{n} = \frac{1}{2}\hat{\imath} + \frac{\sqrt{2}}{2}\hat{\jmath} - \frac{1}{2}\hat{k}$.

Question 2:

(a) Solve: $dy/dx - y\tan x = -y^2 \sec x$.

- This is a Bernoulli differential equation of the form $dy/dx + P(x)y = Q(x)y^n$.
- Here, $P(x) = -\tan x$, $Q(x) = -\sec x$, and $n = 2$.
- Divide by y^2: $y^{-2}dy/dx - y^{-1}\tan x = -\sec x$.
- Let $v = y^{1-n} = y^{1-2} = y^{-1}$.
- Then $dv/dx = -y^{-2}dy/dx$.
- So, $-dv/dx - v\tan x = -\sec x$.
- Multiply by -1: $dv/dx + v\tan x = \sec x$.
- This is now a first-order linear differential equation of the form $dv/dx + P(x)v = Q(x)$.
- The integrating factor is $IF = e^{\int P(x)dx} = e^{\int \tan x dx} = e^{\ln|\sec x|} = \sec x$.
- Multiply the equation by the integrating factor:

- $\sec x \frac{dv}{dx} + v \sec x \tan x = \sec^2 x.$

- The left side is the derivative of $(v \sec x)$ with respect to x:
  - $\frac{d}{dx}(v \sec x) = \sec^2 x.$
- Integrate both sides with respect to x:
  - $\int \frac{d}{dx}(v \sec x) dx = \int \sec^2 x dx.$
  - $v \sec x = \tan x + C.$
- Substitute back $v = y^{-1}$:
  - $y^{-1} \sec x = \tan x + C.$
- Solve for y:
  - $\frac{\sec x}{y} = \tan x + C.$
  - $y = \frac{\sec x}{\tan x + C}.$
- We can also write $\sec x = 1/\cos x$ and $\tan x = \sin x / \cos x$:
  - $y = \frac{1/\cos x}{(\sin x/\cos x) + C} = \frac{1/\cos x}{(\sin x + C\cos x)/\cos x} = \frac{1}{\sin x + C\cos x}.$

(b) Solve the differential equation $d^2x/dt^2 + g/ldx/dt + g/lx = g/L$ where g, l and L are constants subject to the conditions $x = a, dx/dt = 0$ at $t = 0$.

- This is a second-order linear non-homogeneous differential equation with constant coefficients. Let's assume the question meant $d^2x/dt^2 + g/ldx/dt + g/lx = g/L$ and not $d^2y/dx^2 + g/ldy/dx + g/lx = g/L$ as the variable changes. We will proceed with x as the dependent variable and t as the independent variable.

- The homogeneous equation is $d^2x/dt^2 + (g/l)dx/dt + (g/l)x = 0$.
- The characteristic equation is $m^2 + (g/l)m + (g/l) = 0$.
- The roots are $m = \frac{-(g/l)\pm\sqrt{(g/l)^2-4(g/l)}}{2}$.
- Let $\omega_0^2 = g/l$ and $2\zeta\omega_0 = g/l$. So $m = \frac{-2\zeta\omega_0\pm\sqrt{(2\zeta\omega_0)^2-4\omega_0^2}}{2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2-1}$.
- The form of the homogeneous solution $x_h(t)$ depends on the discriminant $(g/l)^2 - 4(g/l)$:
  - Case 1: $(g/l)^2 - 4(g/l) > 0$ (Overdamped): m_1, m_2 are distinct real roots. $x_h(t) = C_1e^{m_1t} + C_2e^{m_2t}$.
  - Case 2: $(g/l)^2 - 4(g/l) = 0$ (Critically damped): $m_1 = m_2 = -g/(2l)$. $x_h(t) = (C_1 + C_2t)e^{-gt/(2l)}$.
  - Case 3: $(g/l)^2 - 4(g/l) < 0$ (Underdamped): $m = \alpha \pm i\beta$ where $\alpha = -g/(2l)$ and $\beta = \frac{\sqrt{4(g/l)-(g/l)^2}}{2}$. $x_h(t) = e^{\alpha t}(C_1\cos(\beta t) + C_2\sin(\beta t))$.
- For the particular solution $x_p(t)$, since the right-hand side is a constant (g/L), we assume $x_p(t) = A$.
- Then $dx_p/dt = 0$ and $d^2x_p/dt^2 = 0$.

- Substitute into the differential equation: $0 + (g/l)(0) + (g/l)A = g/L$.
- $(g/l)A = g/L$.
- $A = (g/L) \times (l/g) = l/L$.
- So, $x_p(t) = l/L$.
- The general solution is $x(t) = x_h(t) + x_p(t) = x_h(t) + l/L$.

  Let's assume the critically damped case for simplicity unless specific values of g and l are given, as the question does not provide them. This is often a common scenario in physics problems of this type. If $(g/l)^2 = 4(g/l)$, then $g/l = 4$. In this specific case, $m = -2$. $x_h(t) = (C_1 + C_2 t)e^{-2t}$. If we don't assume a specific case, the general solution requires keeping m_1, m_2 symbolic.

- Let's keep the general form for $x_h(t)$ based on the roots m_1, m_2.
  - $x(t) = C_1 e^{m_1 t} + C_2 e^{m_2 t} + l/L$. (Using the distinct real roots case as an example, but the method is general).
  - $dx/dt = m_1 C_1 e^{m_1 t} + m_2 C_2 e^{m_2 t}$.
- Apply the initial conditions:
  - At $t = 0, x = a$: $a = C_1 e^0 + C_2 e^0 + l/L \Longrightarrow a = C_1 + C_2 + l/L$.
  - $C_1 + C_2 = a - l/L$. (Equation 1)
  - At $t = 0, dx/dt = 0$: $0 = m_1 C_1 e^0 + m_2 C_2 e^0 \Longrightarrow 0 = m_1 C_1 + m_2 C_2$. (Equation 2)

- From Equation 2, $C_2 = -\frac{m_1}{m_2}C_1$.
- Substitute into Equation 1: $C_1 - \frac{m_1}{m_2}C_1 = a - l/L$.
- $C_1(1 - m_1/m_2) = a - l/L$.
- $C_1\left(\frac{m_2 - m_1}{m_2}\right) = a - l/L$.
- $C_1 = \left(a - \frac{l}{L}\right)\frac{m_2}{m_2 - m_1}$.
- $C_2 = -\frac{m_1}{m_2}C_1 = -\frac{m_1}{m_2}\left(a - \frac{l}{L}\right)\frac{m_2}{m_2 - m_1} = -\left(a - \frac{l}{L}\right)\frac{m_1}{m_2 - m_1}$.
- Substitute C_1 and C_2 back into the general solution for the specific case of the roots. The specific form of the solution will depend on the relationship between g/l and 4.

(c) Evaluate $\nabla \cdot [\vec{r}\nabla(1/r^3)]$ where $r = \sqrt{x^2 + y^2 + z^2}$.

- First, let's find $\nabla(1/r^3)$.
  - We know $\nabla(f(r)) = f'(r)\hat{r} = f'(r)\frac{\vec{r}}{r}$.
  - Here $f(r) = r^{-3}$, so $f'(r) = -3r^{-4}$.
  - $\nabla(1/r^3) = -3r^{-4}\frac{\vec{r}}{r} = -3\frac{\vec{r}}{r^5}$.
- Now we need to evaluate $\nabla \cdot [\vec{r}(-3\frac{\vec{r}}{r^5})] = \nabla \cdot (-3\frac{\vec{r}\cdot\vec{r}}{r^5})$.
- We know $\vec{r} \cdot \vec{r} = |\vec{r}|^2 = r^2$.
- So the expression becomes $\nabla \cdot (-3\frac{r^2}{r^5}) = \nabla \cdot (-3\frac{1}{r^3})$.

- Let $G(r) = -3/r^3$. We need to find $\nabla \cdot (G(r)\vec{r})$.
- Using the vector identity $\nabla \cdot (\phi\vec{A}) = (\nabla\phi) \cdot \vec{A} + \phi(\nabla \cdot \vec{A})$, where $\phi = -3/r^3$ and $\vec{A} = \vec{r}$.
- We already found $\nabla(-3/r^3) = -3\nabla(r^{-3}) = -3(-3r^{-4}\frac{\vec{r}}{r}) = 9\frac{\vec{r}}{r^5}$.
- And we know $\nabla \cdot \vec{r} = 3$.
- So, $\nabla \cdot (-3/r^3\vec{r}) = \left(9\frac{\vec{r}}{r^5}\right) \cdot \vec{r} + \left(-\frac{3}{r^3}\right)(3)$.
- $= 9\frac{\vec{r}\cdot\vec{r}}{r^5} - \frac{9}{r^3}$.
- $= 9\frac{r^2}{r^5} - \frac{9}{r^3}$.
- $= 9\frac{1}{r^3} - \frac{9}{r^3}$.
- $= 0$.
- Therefore, $\nabla \cdot [\vec{r}\nabla(1/r^3)] = 0$.

Question 3:

(a) Solve by the method of Undetermined Coefficients: $d^2y/dx^2 + 2dy/dx + 2y = e^x + x$.

- This is a second-order linear non-homogeneous differential equation.
- First, find the complementary solution y_c by solving the homogeneous equation: $d^2y/dx^2 + 2dy/dx + 2y = 0$.
- The characteristic equation is $m^2 + 2m + 2 = 0$.

- Using the quadratic formula, $m = \frac{-2\pm\sqrt{2^2-4(1)(2)}}{2} = \frac{-2\pm\sqrt{4-8}}{2} = \frac{-2\pm\sqrt{-4}}{2} = \frac{-2\pm 2i}{2} = -1 \pm i$.
- The roots are complex conjugates, so the complementary solution is $y_c(x) = e^{-x}(C_1\cos x + C_2\sin x)$.
- Next, find the particular solution y_p using the method of Undetermined Coefficients. The right-hand side is $g(x) = e^x + x$. We can treat this as two parts: $g_1(x) = e^x$ and $g_2(x) = x$.
- For $g_1(x) = e^x$: Assume a particular solution of the form $y_{p1}(x) = Ae^x$.
    - $y'_{p1} = Ae^x$.
    - $y''_{p1} = Ae^x$.
    - Substitute into the differential equation: $Ae^x + 2Ae^x + 2Ae^x = e^x$.
    - $5Ae^x = e^x$.
    - $5A = 1 \Longrightarrow A = 1/5$.
    - So, $y_{p1}(x) = \frac{1}{5}e^x$.
- For $g_2(x) = x$: Assume a particular solution of the form $y_{p2}(x) = Bx + D$.
    - $y'_{p2} = B$.
    - $y''_{p2} = 0$.
    - Substitute into the differential equation: $0 + 2(B) + 2(Bx + D) = x$.

- $2B + 2Bx + 2D = x$.
- $2Bx + (2B + 2D) = x$.
- Comparing coefficients:
  - Coefficient of x: $2B = 1 \Longrightarrow B = 1/2$.
  - Constant term: $2B + 2D = 0$. Substitute $B = 1/2$: $2(1/2) + 2D = 0 \Longrightarrow 1 + 2D = 0 \Longrightarrow 2D = -1 \Longrightarrow D = -1/2$.
- So, $y_{p2}(x) = \frac{1}{2}x - \frac{1}{2}$.

- The total particular solution is $y_p(x) = y_{p1}(x) + y_{p2}(x) = \frac{1}{5}e^x + \frac{1}{2}x - \frac{1}{2}$.
- The general solution is $y(x) = y_c(x) + y_p(x)$.
- $y(x) = e^{-x}(C_1 \cos x + C_2 \sin x) + \frac{1}{5}e^x + \frac{1}{2}x - \frac{1}{2}$.

(b) Verify $\nabla \times \vec{B} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$ where the vector $\vec{B}$ is defined as $\vec{B} = \nabla \times \vec{A}$.

- We need to verify the vector identity $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$.
- Let $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$.
- The curl of a curl in index notation is given by:
  - $[\nabla \times (\nabla \times \vec{A})]_i = \epsilon_{ijk}\, \partial_j (\nabla \times \vec{A})_k$
  - $= \epsilon_{ijk}\, \partial_j (\epsilon_{klm}\, \partial_l A_m)$
  - $= \epsilon_{ijk}\epsilon_{klm}\, \partial_j\, \partial_l A_m$.
- Using the epsilon-delta identity $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$:

  - $= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\partial_j\,\partial_l A_m$.
  - $= \delta_{il}\delta_{jm}\,\partial_j\,\partial_l A_m - \delta_{im}\delta_{jl}\,\partial_j\,\partial_l A_m$.

- Applying the Kronecker deltas:
  - The first term: $\delta_{il}\delta_{jm}\,\partial_j\,\partial_l A_m$.
    - Sum over l: $\delta_{jm}\,\partial_j\,\partial_i A_m$.
    - Sum over m: $\partial_j\,\partial_i A_j$. (since m becomes j)
    - This can be written as $\partial_i(\partial_j A_j) = \frac{\partial}{\partial x_i}(\nabla \cdot \vec{A}) = [\nabla(\nabla \cdot \vec{A})]_i$.
- The second term: $-\delta_{im}\delta_{jl}\,\partial_j\,\partial_l A_m$.
  - Sum over m: $-\delta_{jl}\,\partial_j\,\partial_l A_i$. (since m becomes i)
  - Sum over l: $-\,\partial_j\,\partial_j A_i$. (since l becomes j)
  - This is $-\nabla^2 A_i = -[\nabla^2 \vec{A}]_i$.
- Combining the two terms, we get:
  - $[\nabla \times (\nabla \times \vec{A})]_i = [\nabla(\nabla \cdot \vec{A})]_i - [\nabla^2 \vec{A}]_i$.
- Since this holds for each component, the vector identity is verified:
  - $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$.
- Given $\vec{B} = \nabla \times \vec{A}$, substituting $\vec{B}$ into the identity gives $\nabla \times \vec{B} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$. This verifies the statement.

Question 4:

(a) Solve the given differential equation using Variation of Parameters:
$d^2y/dx^2 + y = x - \cot x$.

- First, find the complementary solution y_c by solving the homogeneous equation: $d^2y/dx^2 + y = 0$.
- The characteristic equation is $m^2 + 1 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i$.
- So, $y_c(x) = C_1\cos x + C_2\sin x$.
- From y_c, we identify $y_1(x) = \cos x$ and $y_2(x) = \sin x$.
- Calculate the Wronskian $W(y_1, y_2)$:
    - $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - (-\sin^2 x) = \cos^2 x + \sin^2 x = 1$.
- The non-homogeneous term is $f(x) = x - \cot x$.
- The particular solution $y_p(x)$ is given by $y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W}dx + y_2(x)\int \frac{y_1(x)f(x)}{W}dx$.
- Calculate the first integral:
    - $\int \frac{\sin x(x-\cot x)}{1}dx = \int (x\sin x - \sin x\frac{\cos x}{\sin x})dx = \int (x\sin x - \cos x)dx$.
    - $\int x\sin x dx$: Use integration by parts, $\int u dv = uv - \int v du$. Let $u = x, dv = \sin x dx \Rightarrow du = dx, v = -\cos x$.
        - $\int x\sin x dx = -x\cos x - \int (-\cos x)dx = -x\cos x + \sin x$.
    - So, $\int (x\sin x - \cos x)dx = (-x\cos x + \sin x) - \sin x = -x\cos x$.

- Calculate the second integral:
  - $\int \frac{\cos x(x-\cot x)}{1} dx = \int (x\cos x - \cos x \frac{\cos x}{\sin x}) dx = \int (x\cos x - \frac{\cos^2 x}{\sin x}) dx.$
  - $\int x\cos x dx$: Use integration by parts. Let $u = x, dv = \cos x dx \Rightarrow du = dx, v = \sin x.$
    - $\int x\cos x dx = x\sin x - \int \sin x dx = x\sin x + \cos x.$
  - $\int \frac{\cos^2 x}{\sin x} dx = \int \frac{1-\sin^2 x}{\sin x} dx = \int (\frac{1}{\sin x} - \sin x) dx = \int (\csc x - \sin x) dx.$
    - $\int \csc x dx = \ln|\csc x - \cot x|.$
    - $\int \sin x dx = -\cos x.$
    - So, $\int \frac{\cos^2 x}{\sin x} dx = \ln|\csc x - \cot x| + \cos x.$
  - Therefore, $\int (x\cos x - \frac{\cos^2 x}{\sin x}) dx = (x\sin x + \cos x) - (\ln|\csc x - \cot x| + \cos x) = x\sin x - \ln|\csc x - \cot x|.$
- Now, substitute these back into the formula for $y_p(x)$:
  - $y_p(x) = -\cos x(-x\cos x) + \sin x(x\sin x - \ln|\csc x - \cot x|).$
  - $y_p(x) = x\cos^2 x + x\sin^2 x - \sin x \ln|\csc x - \cot x|.$
  - $y_p(x) = x(\cos^2 x + \sin^2 x) - \sin x \ln|\csc x - \cot x|.$
  - $y_p(x) = x - \sin x \ln|\csc x - \cot x|.$

- The general solution is $y(x) = y_c(x) + y_p(x)$.
- $y(x) = C_1 \cos x + C_2 \sin x + x - \sin x \ln|\csc x - \cot x|$.

(b) Verify that a scalar product of vectors $\vec{A}$ and $\vec{B}$ is invariant under rotation.

- Let $\vec{A}$ and $\vec{B}$ be two vectors. Their scalar product is $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$. In index notation, $\vec{A} \cdot \vec{B} = A_i B_i$.
- Consider a rotation of the coordinate system. Let the new coordinates be denoted by primes. The components of a vector transform according to $A'_i = R_{ij} A_j$ and $B'_i = R_{ij} B_j$, where R_{ij} is the rotation matrix.
- The scalar product in the new coordinate system is $\vec{A}' \cdot \vec{B}' = A'_i B'_i$.
- Substitute the transformation rules:
  - $A'_i B'_i = (R_{ij} A_j)(R_{ik} B_k)$. (Note: We use k for the second sum to avoid repeating j)
  - $= R_{ij} R_{ik} A_j B_k$.
- We know that for an orthogonal rotation matrix, $R_{ij} R_{ik} = \delta_{jk}$ (where δ_{jk} is the Kronecker delta). This is because $R^T R = I$.
- So, $A'_i B'_i = \delta_{jk} A_j B_k$.
- Applying the Kronecker delta, the sum over k means that only when $k = j$ is the term non-zero.
  - $\delta_{jk} A_j B_k = A_j B_j$.
- Therefore, $\vec{A}' \cdot \vec{B}' = A_j B_j = \vec{A} \cdot \vec{B}$.

- This shows that the scalar product of two vectors remains unchanged under a rotation of the coordinate system, meaning it is invariant under rotation.

(c) Using Green's theorem, show that the area enclosed by the curve C is $1/2 \oint_C (xdy - ydx)$.

- Green's Theorem states that for a simply connected region R with a positively oriented boundary curve C, if $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives in R, then:
  - $\oint_C (Pdx + Qdy) = \iint_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA$.
- We know that the area A of a region R can be calculated by the double integral $A = \iint_R d\,A$.
- We want to find P and Q such that $(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) = 1$.
- Consider the integrand given: $1/2(xdy - ydx)$.
- Let's rewrite this in the form $Pdx + Qdy$:
  - $1/2(-ydx + xdy)$.
  - So, $P(x, y) = -y/2$ and $Q(x, y) = x/2$.
- Now, calculate the partial derivatives:
  - $\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x/2) = 1/2$.
  - $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(-y/2) = -1/2$.
- Apply Green's Theorem:

- $\oint_C (Pdx + Qdy) = \iint_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA.$
- $\oint_C (-y/2dx + x/2dy) = \iint_R (1/2 - (-1/2)) dA.$
- $\oint_C \frac{1}{2}(xdy - ydx) = \iint_R (1/2 + 1/2) dA.$
- $\frac{1}{2}\oint_C (xdy - ydx) = \iint_R 1\, dA.$

- Since $\iint_R 1\, dA$ represents the area enclosed by the curve C, we have:
  - Area $= \frac{1}{2}\oint_C (xdy - ydx).$
- This verifies the given formula for the area enclosed by the curve C using Green's theorem.

Question 5:

(a) Obtain the expression for the mean and variance of the Poisson distribution.

- A Poisson distribution describes the probability of a given number of events occurring in a fixed interval of time or space if these events occur with a known constant mean rate and independently of the time since the last event.
- The probability mass function (PMF) of a Poisson distribution is given by:
  - $P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$, for $k = 0,1,2, \ldots$, where $\lambda > 0$ is the average number of events in the interval.
- **Mean of the Poisson Distribution ($E[X]$):**
  - The mean is defined as $E[X] = \sum_{k=0}^{\infty} k\, P(X = k)$.

- $E[X] = \sum_{k=0}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!}$.
- For $k = 0$, the term is 0, so we can start the sum from $k = 1$:
  - $E[X] = \sum_{k=1}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!}$.
- Since $k! = k \times (k-1)!$:
  - $E[X] = \sum_{k=1}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k(k-1)!} = \sum_{k=1}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k-1)!}$.
- Factor out $\lambda e^{-\lambda}$:
  - $E[X] = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$.
- Let $j = k - 1$. When $k = 1$, $j = 0$.
  - $E[X] = \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}$.
- We know that the Taylor series expansion for e^x is $\sum_{j=0}^{\infty} \frac{x^j}{j!} = e^x$.
- So, $\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = e^{\lambda}$.
- Therefore, $E[X] = \lambda e^{-\lambda} e^{\lambda} = \lambda e^0 = \lambda$.
- The mean of the Poisson distribution is λ.

- **Variance of the Poisson Distribution ($Var(X)$):**
  - The variance is defined as $Var(X) = E[X^2] - (E[X])^2$.
  - We already found $E[X] = \lambda$. So we need to find $E[X^2]$.

- It's often easier to calculate $E[X(X-1)]$ first:
  - $E[X(X-1)] = \sum_{k=0}^{\infty} k\,(k-1)\frac{\lambda^k e^{-\lambda}}{k!}$.
  - For $k = 0$ and $k = 1$, the term is 0, so we can start the sum from $k = 2$:
    - $E[X(X-1)] = \sum_{k=2}^{\infty} k\,(k-1)\frac{\lambda^k e^{-\lambda}}{k(k-1)(k-2)!}$.
    - $E[X(X-1)] = \sum_{k=2}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k-2)!}$.
  - Factor out $\lambda^2 e^{-\lambda}$:
    - $E[X(X-1)] = \lambda^2 e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!}$.
  - Let $j = k-2$. When $k = 2$, $j = 0$.
    - $E[X(X-1)] = \lambda^2 e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}$.
  - Again, $\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = e^{\lambda}$.
  - So, $E[X(X-1)] = \lambda^2 e^{-\lambda} e^{\lambda} = \lambda^2$.
- Now, we know that $E[X(X-1)] = E[X^2 - X] = E[X^2] - E[X]$.
- Therefore, $E[X^2] - E[X] = \lambda^2$.
- $E[X^2] = \lambda^2 + E[X]$.
- Substitute $E[X] = \lambda$: $E[X^2] = \lambda^2 + \lambda$.
- Finally, calculate the variance:

- $Var(X) = E[X^2] - (E[X])^2 = (\lambda^2 + \lambda) - (\lambda)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$

  - The variance of the Poisson distribution is λ.

(b) Verify Stokes' theorem for the vector field $\vec{F} = y^2\hat{\imath} - (x+z)\hat{\jmath} + yz\hat{k}$ over the unit square bounded by $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$.

- Stokes' Theorem states that for a vector field $\vec{F}$ and an open surface S bounded by a closed curve C, $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$.
- The problem describes a unit square bounded by $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$. This defines a unit cube. The "unit square" implies we should choose one face of the cube. Let's assume the question refers to the face in the xy-plane where $z = 0$, and $0 \le x \le 1, 0 \le y \le 1$.
- **Part 1: Calculate the surface integral $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$.**
  - Calculate the curl of $\vec{F} = y^2\hat{\imath} - (x+z)\hat{\jmath} + yz\hat{k}$:
    - $\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & -(x+z) & yz \end{vmatrix}$
    - $= \hat{\imath}\left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(-(x+z))\right)$
    - $-\hat{\jmath}\left(\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial z}(y^2)\right)$
    - $+\hat{k}\left(\frac{\partial}{\partial x}(-(x+z)) - \frac{\partial}{\partial y}(y^2)\right)$
    - $= \hat{\imath}(z - (-1)) - \hat{\jmath}(0 - 0) + \hat{k}(-1 - 2y)$

- $= (z+1)\hat{\imath} + 0\hat{\jmath} + (-1-2y)\hat{k} = (z+1)\hat{\imath} - (1+2y)\hat{k}$.

- For the surface S, we take the face in the xy-plane, so $z = 0$.
- The outward normal vector for this surface (pointing in the positive z direction for a positive orientation of C) is $d\vec{S} = \hat{k}dxdy$.
- On this surface, $\nabla \times \vec{F}$ becomes $(0+1)\hat{\imath} - (1+2y)\hat{k} = \hat{\imath} - (1+2y)\hat{k}$.
- $(\nabla \times \vec{F}) \cdot d\vec{S} = (\hat{\imath} - (1+2y)\hat{k}) \cdot (\hat{k}dxdy) = -(1+2y)dxdy$.
- $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_0^1 \int_0^1 -(1+2y)dxdy$.
- $= \int_0^1 [-(1+2y)x]_0^1 dy = \int_0^1 -(1+2y)dy$.
- $= [-y-y^2]_0^1 = -(1) - (1)^2 - (0-0) = -1-1 = -2$.
- So, $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = -2$.

- **Part 2: Calculate the line integral $\oint_C \vec{F} \cdot d\vec{r}$.**
  - The boundary curve C for the unit square in the xy-plane (z=0) consists of four line segments:

i. C_1: From $(0,0,0)$ to $(1,0,0)$ (y=0, z=0, dx). $\vec{r} = x\hat{\imath}$, $d\vec{r} = dx\hat{\imath}$.

- $\vec{F} = 0^2\hat{\imath} - (x+0)\hat{\jmath} + 0\hat{k} = -x\hat{\jmath}$.
- $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 (-x\hat{\jmath}) \cdot (dx\hat{\imath}) = \int_0^1 0\, dx = 0$.

ii. C_2: From $(1,0,0)$ to $(1,1,0)$ (x=1, z=0, dy). $\vec{r} = \hat{\imath} + y\hat{\jmath}$, $d\vec{r} = dy\hat{\jmath}$.

- $\vec{F} = y^2\hat{\imath} - (1+0)\hat{\jmath} + y(0)\hat{k} = y^2\hat{\imath} - \hat{\jmath}$.
- $\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 (y^2\hat{\imath} - \hat{\jmath}) \cdot (dy\hat{\jmath}) = \int_0^1 -1dy = [-y]_0^1 = -1$.

iii. C_3: From (1,1,0) to (0,1,0) (y=1, z=0, dx). $\vec{r} = x\hat{\imath} + \hat{\jmath}$, $d\vec{r} = dx\hat{\imath}$.

- $\vec{F} = 1^2\hat{\imath} - (x+0)\hat{\jmath} + 1(0)\hat{k} = \hat{\imath} - x\hat{\jmath}$.
- $\int_{C_3} \vec{F} \cdot d\vec{r} = \int_1^0 (\hat{\imath} - x\hat{\jmath}) \cdot (dx\hat{\imath}) = \int_1^0 1\, dx = [x]_1^0 = 0 - 1 = -1$.

iv. C_4: From (0,1,0) to (0,0,0) (x=0, z=0, dy). $\vec{r} = y\hat{\jmath}$, $d\vec{r} = dy\hat{\jmath}$.

- $\vec{F} = y^2\hat{\imath} - (0+0)\hat{\jmath} + y(0)\hat{k} = y^2\hat{\imath}$.
- $\int_{C_4} \vec{F} \cdot d\vec{r} = \int_1^0 (y^2\hat{\imath}) \cdot (dy\hat{\jmath}) = \int_1^0 0\, dy = 0$.

○ $\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 0 + (-1) + (-1) + 0 = -2$.

- Since $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = -2$ and $\oint_C \vec{F} \cdot d\vec{r} = -2$, Stokes' Theorem is verified for this case.

Question 6:

(a) Show that the vector field $\vec{F} = 2x(y^2 + z^3)\hat{\imath} - 2x^2y\hat{\jmath} + 3x^2z^2\hat{k}$ is conservative. Find the corresponding scalar potential and compute the work done in moving the particle from $(-1,2,1)$ to $(2,3,4)$.

- **Show that the vector field is conservative:**

- A vector field $\vec{F}$ is conservative if its curl is zero, i.e., $\nabla \times \vec{F} = \vec{0}$.
- Let $F_x = 2x(y^2 + z^3)$, $F_y = -2x^2y$, $F_z = 3x^2z^2$.
- Calculate $\nabla \times \vec{F}$:
  - $(\nabla \times \vec{F})_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = \frac{\partial}{\partial y}(3x^2z^2) - \frac{\partial}{\partial z}(-2x^2y) = 0 - 0 = 0$.
  - $(\nabla \times \vec{F})_y = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = \frac{\partial}{\partial z}(2x(y^2 + z^3)) - \frac{\partial}{\partial x}(3x^2z^2) = (2x)(3z^2) - (6xz^2) = 6xz^2 - 6xz^2 = 0$.
  - $(\nabla \times \vec{F})_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial}{\partial x}(-2x^2y) - \frac{\partial}{\partial y}(2x(y^2 + z^3)) = -4xy - (2x)(2y) = -4xy - 4xy = -8xy$.
- Since $(\nabla \times \vec{F})_z = -8xy$ is not identically zero, the vector field is **not** conservative.
- *(Self-correction based on calculation: The given vector field is not conservative. If the question intended it to be conservative, there might be a typo in the field definition. However, as per the definition, it's not conservative. If it's not conservative, we cannot find a scalar potential in the usual sense, and work done would be path-dependent.)*

Let's assume there was a typo in the vector field, and it was meant to be conservative for the rest of the problem to make sense. Let's try to adjust one term to make it conservative for demonstration purposes, or state that it is not conservative.

Given the instruction to "Find the corresponding scalar potential and compute the work done", this strongly implies it should be conservative. Let's re-examine my partial derivatives, it's possible for $\vec{F}$ to be conservative, that the $2x(y^2+z^3)$ term is problematic.

Let's check the partials again: $F_x = 2x(y^2+z^3) = 2xy^2 + 2xz^3$ $F_y = -2x^2y$ $F_z = 3x^2z^2$

$\frac{\partial F_z}{\partial y} = 0$ $\frac{\partial F_y}{\partial z} = 0$ So, $(\nabla \times \vec{F})_x = 0$. This is correct.

$\frac{\partial F_x}{\partial z} = 6xz^2$ $\frac{\partial F_z}{\partial x} = 6xz^2$ So, $(\nabla \times \vec{F})_y = 6xz^2 - 6xz^2 = 0$. This is correct.

$\frac{\partial F_y}{\partial x} = -4xy$ $\frac{\partial F_x}{\partial y} = 4xy$ So, $(\nabla \times \vec{F})_z = -4xy - 4xy = -8xy$. This is indeed non-zero.

Conclusion: The given vector field $\vec{F} = 2x(y^2+z^3)\hat{\imath} - 2x^2y\hat{\jmath} + 3x^2z^2\hat{k}$ is NOT conservative.

If a question asks to "Show that it is conservative" and it's not, the correct answer is to show the curl is not zero. If it then asks to "Find the corresponding scalar potential" and "compute the work done", these steps cannot be performed in the manner expected for a conservative field. For a non-conservative field, the work done depends on the path.

Assuming the question implicitly implies it should be conservative and there's a typo, let's assume $F_y = 2x^2y$ (positive) to make it conservative for the sake of demonstrating the next parts. If $\vec{F} = 2x(y^2+z^3)\hat{\imath} + 2x^2y\hat{\jmath} + 3x^2z^2\hat{k}$ (Changed F_y sign):

- $(\nabla \times \vec{F})_z = \frac{\partial}{\partial x}(2x^2 y) - \frac{\partial}{\partial y}(2x(y^2 + z^3)) = 4xy - 4xy = 0.$
- In this hypothetical case, it would be conservative.

However, sticking to the provided question, the field is NOT conservative. Therefore, a scalar potential does not exist in the conventional sense, and the work done would be path-dependent.

If I must answer, I will state that it's not conservative and explain why the subsequent steps are not straightforward.

- **Is F conservative?**: No, because $\nabla \times \vec{F} \neq \vec{0}$. Specifically, $(\nabla \times \vec{F})_z = -8xy$.
- **Find the corresponding scalar potential**: Since the field is not conservative, a scalar potential ϕ such that $\vec{F} = \nabla\phi$ does not exist. If it were conservative, we would integrate the components.
  - If $\vec{F} = \nabla\phi$, then $\frac{\partial\phi}{\partial x} = F_x$, $\frac{\partial\phi}{\partial y} = F_y$, $\frac{\partial\phi}{\partial z} = F_z$.
  - $\phi = \int F_x dx = \int 2x(y^2 + z^3)dx = x^2(y^2 + z^3) + g(y, z)$.
  - $\frac{\partial\phi}{\partial y} = 2x^2 y + \frac{\partial g}{\partial y}$. We need this to be equal to $F_y = -2x^2 y$.
  - So, $2x^2 y + \frac{\partial g}{\partial y} = -2x^2 y \Longrightarrow \frac{\partial g}{\partial y} = -4x^2 y$. This term depends on x, which means $g(y, z)$ cannot exist as a function of only y and z, confirming the non-conservativeness.
- **Compute the work done**: For a non-conservative field, the work done depends on the path. The problem does not specify a path. If it

were conservative, the work done would be $\phi(P_2) - \phi(P_1)$. Since it's not conservative, we cannot compute work done with just start and end points without a path.

Given the phrasing of the question, it's highly probable there's a typo in the vector field definition and it was intended to be conservative. Let's proceed as if F_y was $2x^2y$ instead of $-2x^2y$ to fulfill the rest of the question.

Assuming the intended field was $\vec{F} = 2x(y^2 + z^3)\hat{\imath} + 2x^2y\hat{\jmath} + 3x^2z^2\hat{k}$ (hypothetical, made conservative for demonstration):

- **Show that the vector field is conservative (with modified F_y):**
  - $(\nabla \times \vec{F})_x = \frac{\partial}{\partial y}(3x^2z^2) - \frac{\partial}{\partial z}(2x^2y) = 0 - 0 = 0.$
  - $(\nabla \times \vec{F})_y = \frac{\partial}{\partial z}(2x(y^2 + z^3)) - \frac{\partial}{\partial x}(3x^2z^2) = 6xz^2 - 6xz^2 = 0.$
  - $(\nabla \times \vec{F})_z = \frac{\partial}{\partial x}(2x^2y) - \frac{\partial}{\partial y}(2x(y^2 + z^3)) = 4xy - 4xy = 0.$
  - So, $\nabla \times \vec{F} = \vec{0}$, hence the (modified) vector field is conservative.
- **Find the corresponding scalar potential $\phi(x, y, z)$ such that $\vec{F} = \nabla\phi$:**

  v. $\frac{\partial \phi}{\partial x} = 2x(y^2 + z^3) = 2xy^2 + 2xz^3.$

    - Integrate with respect to x: $\phi(x, y, z) = x^2y^2 + x^2z^3 + g(y, z).$

vi. $\frac{\partial \phi}{\partial y} = 2x^2y + \frac{\partial g}{\partial y}$. This must equal $F_y = 2x^2y$ (from the modified field).

- $2x^2y + \frac{\partial g}{\partial y} = 2x^2y \Longrightarrow \frac{\partial g}{\partial y} = 0$.
- So, $g(y,z)$ is actually a function of z only, let's call it $h(z)$.
- $\phi(x,y,z) = x^2y^2 + x^2z^3 + h(z)$.

vii. $\frac{\partial \phi}{\partial z} = 3x^2z^2 + h'(z)$. This must equal $F_z = 3x^2z^2$.

- $3x^2z^2 + h'(z) = 3x^2z^2 \Longrightarrow h'(z) = 0$.
- So, $h(z)$ is a constant, let $h(z) = C$. We can choose $C = 0$.

▪ The scalar potential is $\phi(x,y,z) = x^2y^2 + x^2z^3$.

- **Compute the work done in moving the particle from** $(-1,2,1)$ **to** $(2,3,4)$**:**
  - For a conservative field, work done $W = \phi(P_2) - \phi(P_1)$.
  - $P_1 = (-1,2,1)$, $P_2 = (2,3,4)$.
  - $\phi(P_2) = (2)^2(3)^2 + (2)^2(4)^3 = 4 \times 9 + 4 \times 64 = 36 + 256 = 292$.
  - $\phi(P_1) = (-1)^2(2)^2 + (-1)^2(1)^3 = 1 \times 4 + 1 \times 1 = 4 + 1 = 5$.
  - Work done $W = 292 - 5 = 287$.

Given the ambiguity, the best approach is to state that the given field is not conservative based on the calculation, and therefore the latter parts of the question cannot be answered as expected. If force to proceed, state the assumption made.

Final Answer for (a) based on the *given* F:

- **Show that the vector field is conservative:**
  - No, the vector field $\vec{F} = 2x(y^2 + z^3)\hat{\imath} - 2x^2y\hat{\jmath} + 3x^2z^2\hat{k}$ is **not conservative**.
  - We compute the curl:
    - $\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x(y^2 + z^3) & -2x^2y & 3x^2z^2 \end{vmatrix}$
    - $= \hat{\imath}\left(\frac{\partial}{\partial y}(3x^2z^2) - \frac{\partial}{\partial z}(-2x^2y)\right) - \hat{\jmath}\left(\frac{\partial}{\partial x}(3x^2z^2) - \frac{\partial}{\partial z}(2x(y^2 + z^3))\right) + \hat{k}\left(\frac{\partial}{\partial x}(-2x^2y) - \frac{\partial}{\partial y}(2x(y^2 + z^3))\right)$
    - $= \hat{\imath}(0 - 0) - \hat{\jmath}(6xz^2 - 6xz^2) + \hat{k}(-4xy - 4xy)$
    - $= 0\hat{\imath} - 0\hat{\jmath} - 8xy\hat{k} = -8xy\hat{k}$.
  - Since $\nabla \times \vec{F} \neq \vec{0}$, the vector field is not conservative.
- **Find the corresponding scalar potential:**

- Since the vector field is not conservative, a scalar potential ϕ such that $\vec{F} = \nabla\phi$ does not exist.

- **Compute the work done in moving the particle from** $(-1,2,1)$ **to** $(2,3,4)$**:**
  - For a non-conservative vector field, the work done is path-dependent. As no path is specified, the work done cannot be computed using only the start and end points.

(b) Find Taylor expansion of $f(x) = \ln(1+x)$ near $x = 0$ and approximate $f(x) = -0.1$ by taking first four terms of the series.

- **Taylor expansion of** $f(x) = \ln(1+x)$ **near** $x = 0$ **(Maclaurin series):**
  - The Taylor series expansion of a function $f(x)$ around $x = a$ is given by:
    - $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$
  - For Maclaurin series, $a = 0$.
  - $f(x) = \ln(1+x)$
    - $f(0) = \ln(1) = 0$.
  - $f'(x) = \frac{1}{1+x}$
    - $f'(0) = 1$.
  - $f''(x) = -\frac{1}{(1+x)^2}$

- $f''(0) = -1.$
- $f'''(x) = \frac{2}{(1+x)^3}$
  - $f'''(0) = 2.$
- $f^{(4)}(x) = -\frac{6}{(1+x)^4}$
  - $f^{(4)}(0) = -6.$
- Substitute these values into the Maclaurin series formula:
  - $f(x) = 0 + 1(x-0) + \frac{-1}{2!}(x-0)^2 + \frac{2}{3!}(x-0)^3 + \frac{-6}{4!}(x-0)^4 + \cdots$
  - $f(x) = x - \frac{x^2}{2} + \frac{2x^3}{6} - \frac{6x^4}{24} + \cdots$
  - $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

- **Approximate $f(-0.1)$ by taking the first four terms of the series:**
  - The first four terms are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$.
  - Substitute $x = -0.1$:
    - $f(-0.1) \approx (-0.1) - \frac{(-0.1)^2}{2} + \frac{(-0.1)^3}{3} - \frac{(-0.1)^4}{4}.$
    - $f(-0.1) \approx -0.1 - \frac{0.01}{2} + \frac{-0.001}{3} - \frac{0.0001}{4}.$
    - $f(-0.1) \approx -0.1 - 0.005 - 0.0003333\ldots - 0.000025.$
    - $f(-0.1) \approx -0.10535833\ldots$

- The actual value of $\ln(1 - 0.1) = \ln(0.9) \approx -0.1053605$. The approximation is quite good.

(c) Solve: $x^2 d^2y/dx^2 + 4xdy/dx + 2y = e^x$.

- This is a non-homogeneous Cauchy-Euler (or Euler-Cauchy) differential equation.
- First, solve the homogeneous equation: $x^2 d^2y/dx^2 + 4xdy/dx + 2y = 0$.
- Assume a solution of the form $y = x^m$.
- Then $dy/dx = mx^{m-1}$ and $d^2y/dx^2 = m(m-1)x^{m-2}$.
- Substitute into the homogeneous equation:
  - $x^2[m(m-1)x^{m-2}] + 4x[mx^{m-1}] + 2x^m = 0$.
  - $m(m-1)x^m + 4mx^m + 2x^m = 0$.
  - $x^m[m(m-1) + 4m + 2] = 0$.
- Since $x^m \neq 0$, we solve the auxiliary equation:
  - $m^2 - m + 4m + 2 = 0$.
  - $m^2 + 3m + 2 = 0$.
  - $(m+1)(m+2) = 0$.
- The roots are $m_1 = -1$ and $m_2 = -2$.
- The complementary solution is $y_c(x) = C_1x^{-1} + C_2x^{-2} = \frac{C_1}{x} + \frac{C_2}{x^2}$.

- Next, find the particular solution $y_p(x)$ using the method of Variation of Parameters.
- The equation must be in standard form: $d^2y/dx^2 + P(x)dy/dx + Q(x)y = f(x)$.
- Divide the original equation by x^2:
  - $d^2y/dx^2 + \frac{4}{x}dy/dx + \frac{2}{x^2}y = \frac{e^x}{x^2}$.
- Here, $y_1 = x^{-1}$ and $y_2 = x^{-2}$.
- $y_{1'} = -x^{-2}$ and $y_{2'} = -2x^{-3}$.
- Calculate the Wronskian $W(y_1, y_2)$:
  - $W(y_1, y_2) = \begin{vmatrix} x^{-1} & x^{-2} \\ -x^{-2} & -2x^{-3} \end{vmatrix} = (x^{-1})(-2x^{-3}) - (x^{-2})(-x^{-2})$
  - $= -2x^{-4} + x^{-4} = -x^{-4}$.
- The non-homogeneous term in standard form is $f(x) = e^x/x^2$.
- The particular solution $y_p(x)$ is given by $y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W}dx + y_2(x)\int \frac{y_1(x)f(x)}{W}dx$.
- Calculate the first integral:
  - $\int \frac{x^{-2}(e^x/x^2)}{-x^{-4}}dx = \int \frac{e^x/x^4}{-x^{-4}}dx = \int -e^x dx = -e^x$.
- Calculate the second integral:
  - $\int \frac{x^{-1}(e^x/x^2)}{-x^{-4}}dx = \int \frac{e^x/x^3}{-x^{-4}}dx = \int -xe^x dx$.

- Integrate by parts: $\int udv = uv - \int vdu$. Let $u = -x, dv = e^x dx \Longrightarrow du = -dx, v = e^x$.
  - $\int -xe^x dx = -xe^x - \int (-e^x)dx = -xe^x + e^x$.

- Substitute these back into the formula for $y_p(x)$:
  - $y_p(x) = -(x^{-1})(-e^x) + (x^{-2})(-xe^x + e^x)$.
  - $y_p(x) = x^{-1}e^x - x^{-1}e^x + x^{-2}e^x$.
  - $y_p(x) = \frac{e^x}{x^2}$.
- The general solution is $y(x) = y_c(x) + y_p(x)$.
- $y(x) = \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{e^x}{x^2}$.