Question 1: (a) Let (X, d) be a metric space. Define the function d': X × X → R by d'(x,y) = |x-y| / (1+|x-y|). Show that d' is a metric on X. Besides, d'(x,y) < 1 for all x, y ∈ X.

- To show that d' is a metric, we need to verify the following four properties:
  - **Non-negativity:** $d'(x,y) = \frac{|x-y|}{1+|x-y|}$. Since $|x-y| \geq 0$, it follows that $d'(x,y) \geq 0$.
  - **Identity of indiscernibles:**
    - If $d'(x,y) = 0$, then $\frac{|x-y|}{1+|x-y|} = 0$. This implies $|x-y| = 0$, which means $x = y$.
    - If $x = y$, then $|x-y| = 0$, so $d'(x,y) = \frac{0}{1+0} = 0$.
  - **Symmetry:** $d'(x,y) = \frac{|x-y|}{1+|x-y|} = \frac{|y-x|}{1+|y-x|} = d'(y,x)$.
  - **Triangle inequality:** Let $a, b, c \in X$. We need to show $d'(a,c) \leq d'(a,b) + d'(b,c)$. Let $u = |a-b|$ and $v = |b-c|$. By the triangle inequality for the standard metric $|\cdot|$, we have $|a-c| \leq |a-b| + |b-c| = u + v$. Consider the function $f(t) = \frac{t}{1+t} = 1 - \frac{1}{1+t}$. This function is increasing for $t \geq 0$ because $f'(t) = \frac{(1+t)(1)-t(1)}{(1+t)^2} = \frac{1}{(1+t)^2} > 0$. Since $|a-c| \leq u+v$, and f is increasing, we have $f(|a-c|) \leq f(u+v)$. So, $d'(a,c) = \frac{|a-c|}{1+|a-c|} \leq \frac{u+v}{1+u+v}$. We know that $\frac{u+v}{1+u+v} = \frac{u}{1+u+v} + \frac{v}{1+u+v}$. Since $1 + u + v \geq 1 + u$, we have $\frac{u}{1+u+v} \leq \frac{u}{1+u}$. Since $1 + u + v \geq 1 + v$, we have $\frac{v}{1+u+v} \leq \frac{v}{1+v}$. Therefore, $d'(a,c) \leq \frac{u}{1+u} + \frac{v}{1+v} = d'(a,b) + d'(b,c)$.
- Besides, $d'(x,y) < 1$ for all $x, y \in X$. For any $x, y \in X$, $|x-y| \geq 0$. So, $1 + |x-y| > |x-y|$ (since $1 > 0$). Dividing both sides by $1 + |x-y|$

(which is positive), we get $\frac{|x-y|}{1+|x-y|} < 1$. Thus, $d'(x, y) < 1$ for all $x, y \in X$.

(b) Let X = C[a,b] be the space of all continuous functions on [a,b]. Define $d(x, y) = \int_a^b | f(x) - g(x)|dx$. Then check whether this metric imply pointwise Convergence or not.

- No, this metric does not imply pointwise convergence.
- Consider a sequence of functions $f_n \in C[0,1]$ defined as follows: Let $[a, b] = [0,1]$. For $n \geq 2$, let $f_n(x)$ be a "tent" function. $f_n(x) = 0$ for $x \in [0, \frac{1}{2} - \frac{1}{2n}] \cup [\frac{1}{2} + \frac{1}{2n}, 1]$. $f_n(x)$ rises linearly from 0 to 1 over $[\frac{1}{2} - \frac{1}{2n}, \frac{1}{2}]$ and falls linearly from 1 to 0 over $[\frac{1}{2}, \frac{1}{2} + \frac{1}{2n}]$. The peak of the tent is at $x = \frac{1}{2}$, where $f_n(\frac{1}{2}) = 1$.
- The integral $\int_0^1 | f_n(x) - 0|dx$ represents the area of the triangle. The base of the triangle is $(\frac{1}{2} + \frac{1}{2n}) - (\frac{1}{2} - \frac{1}{2n}) = \frac{1}{n}$. The height of the triangle is 1. So, $d(f_n, 0) = \int_0^1 | f_n(x)|dx = \frac{1}{2} \times$ base $\times$ height $= \frac{1}{2} \times \frac{1}{n} \times 1 = \frac{1}{2n}$.
- As $n \to \infty$, $d(f_n, 0) \to 0$. This means the sequence f_n converges to the zero function in the L^1 metric.
- However, the sequence f_n does not converge pointwise to the zero function. For example, at $x = \frac{1}{2}$, $f_n(\frac{1}{2}) = 1$ for all n. So, $\lim_{n\to\infty} f_n(\frac{1}{2}) = 1 \neq 0$.
- Therefore, convergence in the L^1 metric does not imply pointwise convergence.

(c) Define Cauchy Sequence and Complete metric space. Let X be any non-empty set and d be defined by d(x,y) = {0, x=y; 1, x≠y}. Then show that (X, d) is a Complete metric space.

- **Cauchy Sequence:** A sequence $\{x_n\}$ in a metric space (X, d) is called a Cauchy sequence if for every $\epsilon > 0$, there exists a positive integer N such that for all $m, n > N$, we have $d(x_m, x_n) < \epsilon$.
- **Complete Metric Space:** A metric space (X, d) is said to be complete if every Cauchy sequence in (X, d) converges to a point in X.
- **Show that (X, d) is a Complete metric space:**
  - Let $\{x_n\}$ be a Cauchy sequence in (X, d).
  - By definition of a Cauchy sequence, for $\epsilon = 1/2 > 0$, there exists an integer N such that for all $m, n > N$, $d(x_m, x_n) < 1/2$.
  - From the definition of the metric $d(x, y)$:
    - If $x_m \neq x_n$, then $d(x_m, x_n) = 1$.
    - If $x_m = x_n$, then $d(x_m, x_n) = 0$.
  - Since $d(x_m, x_n) < 1/2$, it must be that $d(x_m, x_n) = 0$.
  - Therefore, for all $m, n > N$, $x_m = x_n$.
  - This means that the sequence $\{x_n\}$ is eventually constant. Let $x = x_{N+1}$.
  - For any $\epsilon > 0$, choose $N_0 = N$. Then for all $n > N_0$, $x_n = x$.
  - So, $d(x_n, x) = d(x, x) = 0 < \epsilon$.
  - This shows that the sequence $\{x_n\}$ converges to $x \in X$.
  - Since every Cauchy sequence in (X, d) converges to a point in X, the metric space (X, d) is complete.

Question 2: (a) Let (X, d) be a metric space. Then show that: (i) Ø and X are open sets in (X, d);

- **Ø is open:** By definition, a set G is open if for every $x \in G$, there exists an $\epsilon > 0$ such that $S(x, \epsilon) \subseteq G$. Since the empty set Ø contains no points, the condition is vacuously true. Thus, Ø is open.
- **X is open:** For any point $x \in X$, we can choose any $\epsilon > 0$ (for example, $\epsilon = 1$). The open ball $S(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ will always be a subset of X. Thus, X is open.

(ii) the union of an arbitrary family of open sets is open;

- Let $\{G_\alpha\}_{\alpha \in I}$ be an arbitrary family of open sets in (X, d), where I is an index set. Let $G = \bigcup_{\alpha \in I} G_\alpha$.
- We need to show that G is an open set.
- Let $x \in G$. By the definition of union, there must exist at least one index $\alpha_0 \in I$ such that $x \in G_{\alpha_0}$.
- Since G_{α_0} is an open set, by definition, there exists an $\epsilon > 0$ such that the open ball $S(x, \epsilon) \subseteq G_{\alpha_0}$.
- Since $G_{\alpha_0} \subseteq G$, it follows that $S(x, \epsilon) \subseteq G$.
- Therefore, for every $x \in G$, there exists an $\epsilon > 0$ such that $S(x, \epsilon) \subseteq G$. This proves that G is an open set.

(iii) the intersection of any finite family of open sets is open.

- Let $G_1, G_2, \ldots, G_n$ be a finite family of open sets in (X, d). Let $G = \bigcap_{i=1}^{n} G_i$.
- We need to show that G is an open set.
- If $G = \emptyset$, then G is open by part (i).
- Assume $G \neq \emptyset$. Let $x \in G$.
- By the definition of intersection, $x \in G_i$ for all $i = 1, 2, \ldots, n$.

- Since each G_i is an open set, for each $x \in G_i$, there exists an $\epsilon_i > 0$ such that $S(x, \epsilon_i) \subseteq G_i$.
- Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$. Since there are a finite number of ϵ_i's and each $\epsilon_i > 0$, ϵ will be positive ($\epsilon > 0$).
- Now, for this ϵ, we have $S(x, \epsilon) \subseteq S(x, \epsilon_i)$ for all $i = 1,2, \dots, n$.
- Since $S(x, \epsilon_i) \subseteq G_i$, it follows that $S(x, \epsilon) \subseteq G_i$ for all $i = 1,2, \dots, n$.
- Therefore, $S(x, \epsilon) \subseteq \bigcap_{i=1}^{n} G_i = G$.
- Thus, for every $x \in G$, there exists an $\epsilon > 0$ such that $S(x, \epsilon) \subseteq G$. This proves that G is an open set.

(b) Let A be a subset of a metric space (X, d). Then prove that: (i) A° is the largest open subset of A.

- **Definition of A°:** The interior of A, denoted by A°, is the set of all interior points of A. A point $x \in A$ is an interior point of A if there exists an $\epsilon > 0$ such that $S(x, \epsilon) \subseteq A$.
- **A° is open:** Let $x \in A^\circ$. By definition, there exists an $\epsilon_0 > 0$ such that $S(x, \epsilon_0) \subseteq A$. We need to show that $S(x, \epsilon_0)$ is contained in A°. Let $y \in S(x, \epsilon_0)$. Then $d(x, y) < \epsilon_0$. Let $\delta = \epsilon_0 - d(x, y) > 0$. For any $z \in S(y, \delta)$, we have $d(y, z) < \delta$. By the triangle inequality, $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \delta = d(x, y) + \epsilon_0 - d(x, y) = \epsilon_0$. So $S(y, \delta) \subseteq S(x, \epsilon_0)$. Since $S(x, \epsilon_0) \subseteq A$, we have $S(y, \delta) \subseteq A$. This means y is an interior point of A, so $y \in A^\circ$. Thus, $S(x, \epsilon_0) \subseteq A^\circ$. This shows that for every $x \in A^\circ$, there is an open ball around x entirely contained in A°. Hence, A° is open.
- **A° is a subset of A:** By definition, if $x \in A^\circ$, then there exists an open ball $S(x, \epsilon)$ such that $S(x, \epsilon) \subseteq A$. Since $x \in S(x, \epsilon)$, it directly implies $x \in A$. Thus, $A^\circ \subseteq A$.
- **A° is the largest open subset of A:** Let G be any open set such that $G \subseteq A$. We need to show that $G \subseteq A^\circ$. Let $x \in G$. Since G is open,

there exists an $\epsilon > 0$ such that $S(x, \epsilon) \subseteq G$. Since $G \subseteq A$, it follows that $S(x, \epsilon) \subseteq A$. By definition, this means x is an interior point of A, i.e., $x \in A°$. Since this holds for every $x \in G$, we have $G \subseteq A°$. Combining these points, $A°$ is an open subset of A, and it contains every other open subset of A. Therefore, $A°$ is the largest open subset of A.

(ii) A is open if and only if A = A°.

- **If A is open, then A = A°:**
    - We already know that $A° \subseteq A$.
    - If A is open, then by part (b)(i) (A° is the largest open subset of A), A itself must be contained in A° (because A is an open subset of A). So $A \subseteq A°$.
    - Combining $A° \subseteq A$ and $A \subseteq A°$, we get $A = A°$.
- **If A = A°, then A is open:**
    - Since $A°$ is always an open set (as shown in (b)(i)), and we are given $A = A°$, it directly follows that A is an open set.

(c) (i) Let (X, d) be a metric space and F ⊆ X. Then show that a point x_0 is a limit point of F if and only if it is possible to select from the set F a sequence of distinct x_1, x_2, ... xn, ... such that lim_(n→∞) d(xn, x_0) = 0.

- **Definition of a limit point:** A point $x_0 \in X$ is a limit point of F if every open ball $S(x_0, \epsilon)$ contains at least one point of F other than x_0. That is, $S(x_0, \epsilon) \cap (F \setminus \{x_0\}) \neq \emptyset$ for all $\epsilon > 0$.
- **(⇒) Assume x_0 is a limit point of F. Show there exists a sequence of distinct points in F converging to x_0.**
    - Since x_0 is a limit point of F, for each $n \in \mathbb{N}$, the open ball $S(x_0, 1/n)$ contains a point $x_n \in F$ such that $x_n \neq x_0$.
    - We need to ensure the sequence $\{x_n\}$ has distinct terms.
    - Choose $x_1 \in S(x_0, 1) \cap (F \setminus \{x_0\})$.

- Suppose $x_1, x_2, \dots, x_k$ have been chosen such that $x_i \in S(x_0, 1/i) \cap (F \backslash \{x_0\})$ and all x_i are distinct and $x_i \neq x_0$.
- Consider the set $S_k = \{x_0\}$ if x_0 is not equal to any of $x_1, \dots, x_k$. If x_0 is equal to one of them, then it's $\{x_0\}$ still. Let $r_k = \min(\{d(x_0, x_i): i = 1, \dots, k \text{ and } x_i \neq x_0\} \cup \{1/(k+1)\})$. Since all x_i are distinct from x_0, $d(x_0, x_i) > 0$. So $r_k > 0$.
- The ball $S(x_0, r_k)$ must contain a point $x_{k+1} \in F$ such that $x_{k+1} \neq x_0$.
- Also, x_{k+1} cannot be any of $x_1, \dots, x_k$ because $d(x_0, x_{k+1}) < r_k \leq d(x_0, x_i)$ for $i = 1, \dots, k$.
- Thus, we can construct a sequence of distinct points $\{x_n\}$ in $F \backslash \{x_0\}$ such that $x_n \in S(x_0, 1/n)$.
- This implies $d(x_n, x_0) < 1/n$. As $n \to \infty$, $1/n \to 0$, so $d(x_n, x_0) \to 0$. Therefore, $\lim_{n\to\infty} d(x_n, x_0) = 0$.

- **(⇐) Assume there exists a sequence of distinct $x_n \in F$ such that $\lim_{n\to\infty} d(x_n, x_0) = 0$. Show x_0 is a limit point of F.**
  - Since $\lim_{n\to\infty} d(x_n, x_0) = 0$, for any $\epsilon > 0$, there exists an integer N such that for all $n > N$, $d(x_n, x_0) < \epsilon$.
  - This means that $x_n \in S(x_0, \epsilon)$ for all $n > N$.
  - Since the terms x_n are distinct and $x_n \in F$, and since $d(x_n, x_0) < \epsilon$ for $n > N$, there are infinitely many points of the sequence in $S(x_0, \epsilon)$.
  - In particular, there exists at least one point $x_n \in S(x_0, \epsilon)$ such that $x_n \in F$ and $x_n \neq x_0$ (since the terms are distinct and converge to x_0, for sufficiently large n, x_n cannot be x_0 unless the sequence is eventually constant at x_0 and $x_0 \in F$. Even then, if x_0 is a limit point, there must be other points).

- If x_0 is one of the x_n terms, say $x_k = x_0$, then for $n > k$, $x_n \neq x_k = x_0$ (because the terms are distinct). So $S(x_0, \epsilon)$ contains points from $F \setminus \{x_0\}$.
- Thus, $S(x_0, \epsilon) \cap (F \setminus \{x_0\}) \neq \emptyset$ for every $\epsilon > 0$.
- Therefore, x_0 is a limit point of F.

(ii) Let A ⊆ and F = {f ∈ C: f(t) = 0, ∀t ∈ A}. Show that F is a closed subset of C equipped with the uniform metric.

- Assuming "C" refers to $C[a,b]$, the space of continuous functions on $[a,b]$, and the uniform metric is $d_\infty(f,g) = \sup_{t \in [a,b]} |f(t) - g(t)|$.
- We need to show that $F = \{f \in C[a,b]: f(t) = 0, \forall t \in A\}$ is a closed subset of $C[a,b]$.
- A set is closed if it contains all its limit points. Alternatively, a set is closed if its complement is open. A common way to prove a set is closed is to show that if a sequence in the set converges, its limit is also in the set.
- Let $\{f_n\}$ be a sequence in F such that $f_n \to f$ in the uniform metric, for some $f \in C[a,b]$. We need to show that $f \in F$, i.e., $f(t) = 0$ for all $t \in A$.
- Since $f_n \in F$ for all n, we have $f_n(t) = 0$ for all $t \in A$ and for all n.
- Since $f_n \to f$ in the uniform metric, we have $\lim_{n\to\infty} d_\infty(f_n, f) = 0$.
- This means $\lim_{n\to\infty} \sup_{t \in [a,b]} |f_n(t) - f(t)| = 0$.
- By the property of uniform convergence, for any $t_0 \in [a,b]$, if $f_n \to f$ uniformly, then $f_n(t_0) \to f(t_0)$ pointwise.
- In particular, for any $t_0 \in A$, we have $f_n(t_0) = 0$ for all n.
- Therefore, $\lim_{n\to\infty} f_n(t_0) = 0$.
- Since $f_n(t_0) \to f(t_0)$ pointwise, we must have $f(t_0) = 0$.

- This holds for all $t_0 \in A$.
- Thus, $f(t) = 0$ for all $t \in A$, which means $f \in F$.
- Therefore, F is a closed subset of $C[a, b]$ equipped with the uniform metric.

Question 3: (a) Let (X, d) be a metric space and F ⊆ X. Then show that the following statements are equivalent: (i) x ∈ $\overline{F}$; (ii) S(x, ε) ∩ F ≠ Ø for every open ball S(x, ε) centred at x; (iii) There exists an infinite sequence {x_n} of points (not necessarily distinct) of F such that lim_(n→∞) x_n = x.

- **Definitions:**
  - $\overline{F}$ (closure of F) is the smallest closed set containing F. Equivalently, $\overline{F}$ = F \cup F', where F' is the set of limit points of F.
  - A point x is in $\overline{F}$ if and only if every open ball centered at x intersects F. This is exactly statement (ii). So (i) and (ii) are equivalent by definition.
- **Equivalence of (i) and (ii):**
  - (⇒) Assume x \in $\overline{F}$. If $x \in F$, then for any $S(x, \epsilon)$, $x \in S(x, \epsilon) \cap F$, so $S(x, \epsilon) \cap F \neq \emptyset$. If $x \in F'$ (limit point of F), then by definition of limit point, every $S(x, \epsilon)$ contains a point of $F \backslash \{x\}$. So $S(x, \epsilon) \cap F \neq \emptyset$. Thus, (i) implies (ii).
  - (⇐) Assume $S(x, \epsilon) \cap F \neq \emptyset$ for every $S(x, \epsilon)$. This is the definition of a point in the closure of F. Thus, x \in $\overline{F}$. So (ii) implies (i).
  - Therefore, (i) ⇔ (ii).
- **Equivalence of (ii) and (iii):**
  - **(⇒) Assume (ii) is true. Show (iii) is true.**

- Since $S(x, \epsilon) \cap F \neq \emptyset$ for every $\epsilon > 0$, for each $n \in \mathbb{N}$, consider $\epsilon = 1/n$.
- Then $S(x, 1/n) \cap F \neq \emptyset$.
- This means we can choose a point $x_n \in S(x, 1/n) \cap F$.
- By definition of $S(x, 1/n)$, we have $d(x_n, x) < 1/n$.
- As $n \to \infty$, $1/n \to 0$, so $d(x_n, x) \to 0$.
- Thus, $\lim_{n\to\infty} x_n = x$. The sequence $\{x_n\}$ consists of points from F. These points are not necessarily distinct. So (ii) implies (iii).

  - **(⇐) Assume (iii) is true. Show (ii) is true.**
    - Assume there exists a sequence $\{x_n\}$ of points of F such that $\lim_{n\to\infty} x_n = x$.
    - This means for any $\epsilon > 0$, there exists an integer N such that for all $n > N$, $d(x_n, x) < \epsilon$.
    - So, for all $n > N$, $x_n \in S(x, \epsilon)$.
    - Since $x_n \in F$ for all n, this implies that $S(x, \epsilon)$ contains at least one point from F (specifically, $x_{N+1} \in S(x, \epsilon) \cap F$).
    - Therefore, $S(x, \epsilon) \cap F \neq \emptyset$ for every $\epsilon > 0$. So (iii) implies (ii).

- Since (i) ⇔ (ii) and (ii) ⇔ (iii), all three statements are equivalent.

(b) State and prove Cantor's intersection theorem.

- **Statement of Cantor's Intersection Theorem:** Let (X, d) be a complete metric space. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of non-empty closed subsets of X such that:

  a. $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ (nested sequence, i.e., $F_{n+1} \subseteq F_n$ for all n).

b. $\lim_{n\to\infty}\text{diam}(F_n) = 0$, where $\text{diam}(F_n) = \sup\{d(x,y): x, y \in F_n\}$ is the diameter of F_n. Then, the intersection $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

- **Proof:**
  - **Existence of a point:**
    - Since each F_n is non-empty, for each n, we can choose an arbitrary point $x_n \in F_n$.
    - Consider the sequence $\{x_n\}$. We will show that it is a Cauchy sequence.
    - Let $\epsilon > 0$. Since $\lim_{n\to\infty}\text{diam}(F_n) = 0$, there exists an integer N such that for all $n > N$, $\text{diam}(F_n) < \epsilon$.
    - Now, for any $m, k > N$, without loss of generality, assume $m \geq k$.
    - Since the sequence is nested ($F_k \supseteq F_{k+1} \supseteq \cdots$), if $m, k > N$, then $x_m \in F_m \subseteq F_N$ and $x_k \in F_k \subseteq F_N$. More specifically, if $m, n > N$, then both x_m and x_n belong to F_N (since F_N contains all subsequent F_k).
    - Thus, $d(x_m, x_n) \leq \text{diam}(F_N)$.
    - Since N is chosen such that $\text{diam}(F_N) < \epsilon$, we have $d(x_m, x_n) < \epsilon$ for all $m, n > N$.
    - Therefore, $\{x_n\}$ is a Cauchy sequence in X.
    - Since (X, d) is a complete metric space, every Cauchy sequence converges to a point in X. So, there exists a point $x_0 \in X$ such that $\lim_{n\to\infty} x_n = x_0$.
  - **The point x_0 is in the intersection:**
    - We need to show that $x_0 \in F_k$ for every k.

    - Consider an arbitrary F_k. For all $n \geq k$, we have $x_n \in F_n \subseteq F_k$.
    - So, the sequence $\{x_n\}_{n=k}^{\infty}$ is a sequence of points in F_k.
    - Since F_k is a closed set and $x_n \to x_0$, the limit point x_0 must belong to F_k.
    - Since this holds for every k, $x_0 \in \bigcap_{n=1}^{\infty} F_n$. Thus, the intersection is non-empty.
  - **Uniqueness of the point:**
    - Assume there are two points $x_0, y_0 \in \bigcap_{n=1}^{\infty} F_n$.
    - Then $x_0 \in F_n$ and $y_0 \in F_n$ for all n.
    - Therefore, $d(x_0, y_0) \leq \text{diam}(F_n)$ for all n.
    - Since $\lim_{n\to\infty} \text{diam}(F_n) = 0$, we have $d(x_0, y_0) \leq 0$.
    - Since distances are non-negative, $d(x_0, y_0) = 0$, which implies $x_0 = y_0$.
    - Thus, the intersection contains exactly one point.

(c) Show that the metric spaces (X, d) and (X, ρ) where $\rho(x, y) = d(x, y)/(1 + d(x, y))$ are equivalent.

- Two metric spaces (X, d) and (X, ρ) are equivalent if they induce the same topology. This means that a set $G \subseteq X$ is open in (X, d) if and only if it is open in (X, ρ).
- Equivalently, it means that for every point $x \in X$ and every $\epsilon > 0$, there exists a $\delta_1 > 0$ such that $S_\rho(x, \delta_1) \subseteq S_d(x, \epsilon)$, and there exists a $\delta_2 > 0$ such that $S_d(x, \delta_2) \subseteq S_\rho(x, \epsilon)$.
  - $S_d(x, \epsilon) = \{y \in X: d(x, y) < \epsilon\}$
  - $S_\rho(x, \delta_1) = \{y \in X: \rho(x, y) < \delta_1\}$

- **Part 1: Show that for any $S_d(x,\epsilon)$, there exists $S_\rho(x,\delta_1)$ such that $S_\rho(x,\delta_1) \subseteq S_d(x,\epsilon)$.**
  - Let $x \in X$ and $\epsilon > 0$. We want to find $\delta_1 > 0$ such that if $\rho(x,y) < \delta_1$, then $d(x,y) < \epsilon$.
  - Let $\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$.
  - Consider the function $f(t) = \frac{t}{1+t}$. We know $f'(t) = \frac{1}{(1+t)^2} > 0$, so $f(t)$ is an increasing function for $t \geq 0$.
  - Also, $\rho(x,y) < \delta_1 \Leftrightarrow \frac{d(x,y)}{1+d(x,y)} < \delta_1$.
  - Since $f(t)$ is increasing, its inverse $f^{-1}(s) = \frac{s}{1-s}$ (for $0 \leq s < 1$) is also increasing.
  - So, $d(x,y) < \frac{\delta_1}{1-\delta_1}$ (if $\delta_1 < 1$).
  - We want $d(x,y) < \epsilon$.
  - We can choose δ_1 such that $\frac{\delta_1}{1-\delta_1} = \epsilon$.
  - Solving for δ_1: $\delta_1 = \epsilon(1-\delta_1) \Rightarrow \delta_1 = \epsilon - \epsilon\delta_1 \Rightarrow \delta_1(1+\epsilon) = \epsilon \Rightarrow \delta_1 = \frac{\epsilon}{1+\epsilon}$.
  - Since $\epsilon > 0$, we have $0 < \delta_1 < 1$.
  - So, if we choose $\delta_1 = \frac{\epsilon}{1+\epsilon}$, then $\rho(x,y) < \delta_1$ implies $d(x,y) < \epsilon$.
  - Thus, $S_\rho(x,\frac{\epsilon}{1+\epsilon}) \subseteq S_d(x,\epsilon)$.
- **Part 2: Show that for any $S_\rho(x,\epsilon)$, there exists $S_d(x,\delta_2)$ such that $S_d(x,\delta_2) \subseteq S_\rho(x,\epsilon)$.**
  - Let $x \in X$ and $\epsilon > 0$. We want to find $\delta_2 > 0$ such that if $d(x,y) < \delta_2$, then $\rho(x,y) < \epsilon$.

- We know that $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$.
- Since the function $f(t) = \frac{t}{1+t}$ is increasing, if we choose $\delta_2 = \epsilon$ (assuming $\epsilon < 1$, as $\rho(x, y)$ is always less than 1), then $d(x, y) < \delta_2$ implies $d(x, y) < \epsilon$.
- Since $d(x, y) < \epsilon$, and $f(t)$ is increasing, we have $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)} < \frac{\epsilon}{1+\epsilon}$.
- Since we want $\rho(x, y) < \epsilon$, we can choose $\delta_2 = \epsilon$ (if $\epsilon < 1$). If $\epsilon \geq 1$, we can choose any positive δ_2, say $\delta_2 = 1$, because $\rho(x, y)$ will always be less than 1.
- Let's be more precise. We need $\frac{d(x,y)}{1+d(x,y)} < \epsilon$.
- Choose $\delta_2 = \epsilon$. If $d(x, y) < \epsilon$, then $1 + d(x, y) > 1$.
- Then $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)} < d(x, y) < \epsilon$.
- Thus, if we choose $\delta_2 = \epsilon$, then $S_d(x, \epsilon) \subseteq S_\rho(x, \epsilon)$.

- Since both conditions are met, the metric spaces (X, d) and (X, ρ) are equivalent.

Question 4: (a) Prove that a mapping f: X → Y is continuous on X if and only if $f^{-1}(G)$ is open in X for all open subsets G of Y.

- **Definition of Continuity (using ϵ-δ):** A mapping $f: (X, d_X) \to (Y, d_Y)$ is continuous at a point $x \in X$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x' \in X$, if $d_X(x, x') < \delta$, then $d_Y(f(x), f(x')) < \epsilon$. The mapping f is continuous on X if it is continuous at every point in X.
- **Proof (⇒): Assume f is continuous on X. Show $f^{-1}(G)$ is open in X for all open subsets G of Y.**
  - Let G be an arbitrary open subset of Y. We want to show that $f^{-1}(G)$ is open in X.

- If $f^{-1}(G) = \emptyset$, then it is open.
- Assume $f^{-1}(G) \neq \emptyset$. Let $x \in f^{-1}(G)$.
- By definition of inverse image, $f(x) \in G$.
- Since G is an open set in Y, there exists an $\epsilon > 0$ such that the open ball $S_Y(f(x), \epsilon) \subseteq G$.
- Since f is continuous at x, for this ϵ, there exists a $\delta > 0$ such that if $x' \in X$ and $d_X(x, x') < \delta$, then $d_Y(f(x), f(x')) < \epsilon$.
- This means that $f(S_X(x, \delta)) \subseteq S_Y(f(x), \epsilon)$.
- Since $S_Y(f(x), \epsilon) \subseteq G$, it follows that $f(S_X(x, \delta)) \subseteq G$.
- Taking the inverse image of both sides, $S_X(x, \delta) \subseteq f^{-1}(G)$.
- Thus, for every $x \in f^{-1}(G)$, we have found an open ball $S_X(x, \delta)$ centered at x which is entirely contained in $f^{-1}(G)$.
- Therefore, $f^{-1}(G)$ is open in X.

- **Proof (⇐): Assume $f^{-1}(G)$ is open in X for all open subsets G of Y. Show f is continuous on X.**
  - We want to show that f is continuous at an arbitrary point $x \in X$.
  - Let $\epsilon > 0$ be given.
  - Consider the open ball $S_Y(f(x), \epsilon)$ centered at $f(x)$ in Y. This is an open set in Y.
  - By hypothesis, its inverse image $f^{-1}(S_Y(f(x), \epsilon))$ is an open set in X.
  - Since $f(x) \in S_Y(f(x), \epsilon)$, it means $x \in f^{-1}(S_Y(f(x), \epsilon))$.

- Since $f^{-1}(S_Y(f(x),\epsilon))$ is an open set and contains x, by definition of an open set, there exists a $\delta > 0$ such that the open ball $S_X(x,\delta) \subseteq f^{-1}(S_Y(f(x),\epsilon))$.
- This means that for any $x' \in S_X(x,\delta)$, we have $x' \in f^{-1}(S_Y(f(x),\epsilon))$, which implies $f(x') \in S_Y(f(x),\epsilon)$.
- By definition of $S_Y(f(x),\epsilon)$, this means $d_Y(f(x),f(x')) < \epsilon$.
- Thus, for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $d_X(x,x') < \delta$, then $d_Y(f(x),f(x')) < \epsilon$.
- Therefore, f is continuous at x. Since x was arbitrary, f is continuous on X.

(b) Let T: X→X be a contraction mapping of the complete metric space (X, d). Then show that T has a unique fixed point.

- **Definitions:**
  - **Contraction Mapping:** A mapping $T: X \to X$ is a contraction mapping if there exists a constant $k \in [0,1)$ (i.e., $0 \leq k < 1$) such that for all $x, y \in X$, $d(T(x),T(y)) \leq kd(x,y)$. The constant k is called the contraction constant.
  - **Fixed Point:** A point $x \in X$ is a fixed point of T if $T(x) = x$.
- **Proof (Banach Fixed Point Theorem):**
  - **Existence:**
    - Let x_0 be an arbitrary point in X.
    - Construct a sequence $\{x_n\}$ by iterating T: $x_{n+1} = T(x_n)$ for $n \geq 0$.
    - Consider the distance between consecutive terms: $d(x_{n+1},x_n) = d(T(x_n),T(x_{n-1})) \leq kd(x_n,x_{n-1})$.
    - By repeating this, we get: $d(x_{n+1},x_n) \leq kd(x_n,x_{n-1}) \leq k^2 d(x_{n-1},x_{n-2}) \leq \cdots \leq k^n d(x_1,x_0)$.

- Now, we show that $\{x_n\}$ is a Cauchy sequence. For $m > n$: $d(x_m, x_n) = d(x_m, x_{m-1} + x_{m-1} + \cdots + x_n)$ By the triangle inequality: $d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n)$ $d(x_m, x_n) \leq k^{m-1} d(x_1, x_0) + k^{m-2} d(x_1, x_0) + \cdots + k^n d(x_1, x_0)$ $d(x_m, x_n) \leq d(x_1, x_0)(k^n + k^{n+1} + \cdots + k^{m-1})$ $d(x_m, x_n) \leq d(x_1, x_0) k^n (1 + k + \cdots + k^{m-1-n})$ $d(x_m, x_n) \leq d(x_1, x_0) k^n \frac{1-k^{m-n}}{1-k}$ Since $0 \leq k < 1$, we have $\frac{1-k^{m-n}}{1-k} < \frac{1}{1-k}$. So, $d(x_m, x_n) \leq \frac{k^n}{1-k} d(x_1, x_0)$.
- Since $0 \leq k < 1$, as $n \to \infty$, $k^n \to 0$.
- Therefore, for any $\epsilon > 0$, we can choose N large enough such that $\frac{k^n}{1-k} d(x_1, x_0) < \epsilon$ for all $n > N$.
- This means $d(x_m, x_n) < \epsilon$ for all $m, n > N$. Hence, $\{x_n\}$ is a Cauchy sequence.
- Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $x^* \in X$. Let $x^* = \lim_{n\to\infty} x_n$.
- Now we show x^* is a fixed point. Since T is a contraction mapping, it is continuous. (If $d(x, y) < \delta$, choose $\delta = \epsilon/k$, then $d(T(x), T(y)) \leq k d(x, y) < k(\epsilon/k) = \epsilon$). Since $x_{n+1} = T(x_n)$ and T is continuous, taking the limit as $n \to \infty$: $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n)$ $x^* = T(\lim_{n\to\infty} x_n)$ $x^* = T(x^*)$. Thus, x^* is a fixed point of T.

- **Uniqueness:**
  - Suppose x^* and y^* are two fixed points of T.
  - Then $T(x^*) = x^*$ and $T(y^*) = y^*$.
  - Consider the distance $d(x^*, y^*)$.

- Since T is a contraction mapping: $d(x^*, y^*) = d(T(x^*), T(y^*)) \leq kd(x^*, y^*)$.
- So, $d(x^*, y^*) \leq kd(x^*, y^*)$.
- Rearranging, $(1-k)d(x^*, y^*) \leq 0$.
- Since $0 \leq k < 1$, we have $1-k > 0$.
- For $(1-k)d(x^*, y^*) \leq 0$ to be true, and $1-k > 0$, it must be that $d(x^*, y^*) \leq 0$.
- Since distance is non-negative, $d(x^*, y^*) = 0$.
- By the property of a metric, $d(x^*, y^*) = 0$ implies $x^* = y^*$.
- Therefore, the fixed point is unique.

Question 5: (a) Let (X, d) be a metric space. Then show that the following statements are equivalent: (i) (X, d) is disconnected; (ii) there exists a continuous mapping of (X, d) onto the discrete two element space (X_0, d_0).

- **Definitions:**
  - **Disconnected Metric Space:** A metric space (X, d) is disconnected if it can be written as the union of two non-empty disjoint open sets. That is, $X = A \cup B$ where $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$, and A and B are both open in X.
  - **Discrete Two Element Space (X_0, d_0):** Let $X_0 = \{0,1\}$. The discrete metric d_0 is defined as $d_0(x, y) = 0$ if $x = y$ and $d_0(x, y) = 1$ if $x \neq y$. In a discrete space, every subset is open (and closed).
- **Proof (⇒): Assume (X, d) is disconnected. Show there exists a continuous mapping from X onto X_0.**
  - Since (X, d) is disconnected, there exist non-empty open sets $A, B \subseteq X$ such that $X = A \cup B$ and $A \cap B = \emptyset$.

- Define a mapping $f: X \to X_0$ as follows: $f(x) = 0$ if $x \in A$ $f(x) = 1$ if $x \in B$
- Since A and B are non-empty, f maps X onto $\{0,1\}$.
- To show f is continuous, we need to show that the inverse image of every open set in X_0 is open in X.
- The open sets in X_0 are $\emptyset, \{0\}, \{1\}, \{0,1\}$.
  - $f^{-1}(\emptyset) = \emptyset$, which is open in X.
  - $f^{-1}(\{0\}) = A$, which is given as open in X.
  - $f^{-1}(\{1\}) = B$, which is given as open in X.
  - $f^{-1}(\{0,1\}) = A \cup B = X$, which is open in X.
- Since the inverse image of every open set in X_0 is open in X, f is a continuous mapping.

- **Proof (⇐): Assume there exists a continuous mapping $f: X \to X_0$ onto X_0. Show (X, d) is disconnected.**
  - Let $f: X \to X_0$ be a continuous mapping onto $X_0 = \{0,1\}$.
  - Since f is onto, there must exist $x_0 \in X$ such that $f(x_0) = 0$ and $x_1 \in X$ such that $f(x_1) = 1$.
  - Consider the sets $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$.
  - Since $\{0\}$ and $\{1\}$ are open sets in the discrete space X_0, and f is continuous, their inverse images A and B must be open sets in X.
  - Since f is onto, A and B are non-empty (because 0 and 1 are in the image).
  - Also, $A \cup B = f^{-1}(\{0\}) \cup f^{-1}(\{1\}) = f^{-1}(\{0\} \cup \{1\}) = f^{-1}(X_0) = X$.

- And $A \cap B = f^{-1}(\{0\}) \cap f^{-1}(\{1\}) = f^{-1}(\{0\} \cap \{1\}) = f^{-1}(\emptyset) = \emptyset$.
- Thus, X is expressed as the union of two non-empty, disjoint open sets A and B.
- Therefore, (X, d) is disconnected.

(b) Let I = [0,1] and let f: I → I be continuous. Then show that there exists a point c ∈ I such that f(c) = c. Discuss the result if I = [-1,1]. Discuss the result if I = [-1,1] and I = [-1, ∞).

- **Part 1: Let I = [0,1] and let f: I → I be continuous. Then show that there exists a point c ∈ I such that f(c) = c.**
  - This is a direct application of the **Intermediate Value Theorem** (or Brouwer Fixed Point Theorem for 1D).
  - Consider the function $g(x) = f(x) - x$.
  - Since f is continuous on $[0,1]$ and x is continuous, $g(x)$ is continuous on $[0,1]$.
  - Evaluate $g(x)$ at the endpoints:
    - $g(0) = f(0) - 0 = f(0)$. Since $f: [0,1] \to [0,1]$, we have $f(0) \in [0,1]$, so $f(0) \geq 0$. Thus, $g(0) \geq 0$.
    - $g(1) = f(1) - 1$. Since $f: [0,1] \to [0,1]$, we have $f(1) \in [0,1]$, so $f(1) \leq 1$. Thus, $g(1) \leq 0$.
  - Case 1: If $g(0) = 0$, then $f(0) = 0$, so $c = 0$ is a fixed point.
  - Case 2: If $g(1) = 0$, then $f(1) = 1$, so $c = 1$ is a fixed point.
  - Case 3: If $g(0) > 0$ and $g(1) < 0$. Since g is continuous on $[0,1]$ and $g(0)$ and $g(1)$ have opposite signs, by the Intermediate Value Theorem, there exists a point $c \in (0,1)$ such that $g(c) = 0$.
  - If $g(c) = 0$, then $f(c) - c = 0$, which means $f(c) = c$.

- Therefore, in all cases, there exists a point $c \in [0,1]$ such that $f(c) = c$.

- **Part 2: Discuss the result if I = [-1,1].**
  - The result holds for $I = [-1,1]$ as well. The interval $[-1,1]$ is a closed and bounded interval, hence compact and connected.
  - If $f: [-1,1] \to [-1,1]$ is continuous, consider $g(x) = f(x) - x$.
  - $g(-1) = f(-1) - (-1) = f(-1) + 1$. Since $f(-1) \in [-1,1]$, $f(-1) \geq -1$, so $f(-1) + 1 \geq 0$. Thus $g(-1) \geq 0$.
  - $g(1) = f(1) - 1$. Since $f(1) \in [-1,1]$, $f(1) \leq 1$, so $f(1) - 1 \leq 0$. Thus $g(1) \leq 0$.
  - By the Intermediate Value Theorem, there exists $c \in [-1,1]$ such that $g(c) = 0$, which means $f(c) = c$.
  - The result holds: a continuous function from $[-1,1]$ to $[-1,1]$ has a fixed point.

- **Part 3: Discuss the result if I = [-1,1] and I = [-1, ∞).** (The phrasing "if I = [-1,1] and I = [-1, ∞)" suggests two separate discussions).
  - **For I = [-1,1]:** (Already discussed above) The result holds. Continuous functions $f: [-1,1] \to [-1,1]$ always have a fixed point. This is because $[-1,1]$ is a compact and convex set, and Brouwer's Fixed Point Theorem applies to continuous mappings from a convex compact subset of $\mathbb{R}^n$ to itself. For $n = 1$, this is a closed interval.
  - **For I = [-1, ∞):**
    - The result (existence of a fixed point) **does not necessarily hold** if the domain/codomain is not compact (i.e., not closed and bounded).
    - Consider $I = [-1, \infty)$. Let $f: [-1, \infty) \to [-1, \infty)$ be defined by $f(x) = x + 1$.

- $f(x)$ is continuous.
- We are looking for c such that $f(c) = c$, i.e., $c + 1 = c$.
- This equation simplifies to $1 = 0$, which has no solution.
- Thus, the function $f(x) = x + 1$ has no fixed point in $[-1, \infty)$.
- The reason the fixed point theorem fails here is that $[-1, \infty)$ is not a compact set (it's not bounded). The fixed point property relies on the domain being compact and convex.

(c) (i) If C is a connected subset of a disconnected metric space X = A ∪ B, where A, B are nonempty and $\overline{A}$ ∩ B = Ø = A ∩ $\overline{B}$, then show that either C ⊆ A or C ⊆ B.

- **Definition of Disconnected Space:** A metric space X is disconnected if there exist non-empty sets $A, B \subseteq X$ such that $X = A \cup B$, $A \cap B = \emptyset$, and both A and B are open (equivalently, both A and B are closed, as $A = X \setminus B$ and $B = X \setminus A$).
- The condition given, $\overline{A}$ \cap B = \emptyset = A \cap $\overline{B}$, means that A and B form a separation of X.
    - This is equivalent to X being disconnected, where A and B are closed and disjoint, and $A \cup B = X$.
    - If A and B are open and disjoint, then $A \cap \overline{B} = \emptyset$ (if $y \in B$, $S(y, \epsilon) \subseteq B$, so no point in A can be a limit point of B; similar for $\overline{A} \cap B = \emptyset$).
- **Proof:**
    - Assume C is a connected subset of $X = A \cup B$, where A, B are non-empty, $A \cap B = \emptyset$, and A and B are separated (i.e., $\overline{A}$ \cap B = \emptyset and A \cap $\overline{B}$ = \emptyset). This implies that A and B are open and closed in X.

- Consider the sets $C_A = C \cap A$ and $C_B = C \cap B$.
- Since A and B are open in X, their intersections with C are open in the subspace topology of C. That is, C_A and C_B are open in C.
- We also have $C = (C \cap A) \cup (C \cap B) = C_A \cup C_B$.
- And $C_A \cap C_B = (C \cap A) \cap (C \cap B) = C \cap (A \cap B) = C \cap \emptyset = \emptyset$.
- Since C is connected, it cannot be expressed as the union of two non-empty, disjoint open sets (in its own subspace topology).
- Therefore, either $C_A = \emptyset$ or $C_B = \emptyset$.
- If $C_A = \emptyset$, then $C \cap A = \emptyset$. Since $C \subseteq A \cup B$, this implies $C \subseteq B$.
- If $C_B = \emptyset$, then $C \cap B = \emptyset$. Since $C \subseteq A \cup B$, this implies $C \subseteq A$.
- Thus, either $C \subseteq A$ or $C \subseteq B$.

(ii) If Y is a connected set in a metric space (X, d) then show that any set Z such that Y ⊆ Z ⊆ $\overline{\text{Y}}$ connected.

- **Proof:**
  - Assume Y is a connected set in (X, d). Let Z be a set such that Y \subseteq Z \subseteq $\overline{\text{Y}}$.
  - We want to show that Z is connected. We will use the definition of connected: A set S is connected if it cannot be written as the union of two non-empty disjoint open sets (in the subspace topology of S). Equivalently, if $f: S \to \{0,1\}$ is continuous, then f must be constant.
  - Let $f: Z \to \{0,1\}$ be a continuous function. We want to show that f is constant on Z.
  - Since $Y \subseteq Z$, the restriction of f to Y, denoted as $f|_Y$, is a continuous function from Y to $\{0,1\}$.

- Since Y is connected, $f|_Y$ must be constant on Y. Let $f(y) = c$ for all $y \in Y$, where $c \in \{0,1\}$.
- Now, we need to show that $f(z) = c$ for all $z \in Z$.
- Let $z \in Z$. Since Z \subseteq Ȳ, z is in the closure of Y.
- By the property of closure (from Question 3(a)), if z \in Ȳ, then for every $\epsilon > 0$, the open ball $S_X(z, \epsilon)$ intersects Y. That is, $S_X(z, \epsilon) \cap Y \neq \emptyset$.
- Since f is continuous on Z, for every $\epsilon' > 0$ (in the metric of $\{0,1\}$), there exists $\delta > 0$ such that if $z' \in Z$ and $d_X(z, z') < \delta$, then $d_0(f(z), f(z')) < \epsilon'$.
- Let's take $\epsilon' = 1/2$. Then there exists $\delta > 0$ such that if $z' \in Z$ and $d_X(z, z') < \delta$, then $d_0(f(z), f(z')) < 1/2$.
- In the discrete metric d_0, $d_0(a, b) < 1/2$ implies $a = b$. So, if $d_X(z, z') < \delta$, then $f(z') = f(z)$.
- Since z \in Ȳ, there exists a point $y \in S_X(z, \delta) \cap Y$.
- Since $y \in Y$, we know $f(y) = c$.
- Since $y \in S_X(z, \delta)$ and $y \in Z$ (because $Y \subseteq Z$), we have $d_X(z, y) < \delta$.
- By the continuity of f at z, since $d_X(z, y) < \delta$, it must be that $f(z) = f(y)$.
- Since $f(y) = c$, we have $f(z) = c$.
- Since z was an arbitrary point in Z, f is constant on Z.
- Therefore, Z is connected.

Question 6: (a) Let (X, d) be a metric space. Then show that the following statements are equivalent: (i) every infinite set in (X, d) has at least one limit point in X; (ii) every infinite sequence in (X, d) contains a convergent subsequence.

- **Definitions:**
  - **(i) This property is equivalent to X being sequentially compact.** (Though often in general topology, it's part of the definition of compact).
  - **(ii) This property is called sequential compactness.**
- **Proof (i) ⇔ (ii): This is the Bolzano-Weierstrass property.**
- **Proof (⇒): Assume (i) is true. Show (ii) is true.**
  - Let $\{x_n\}$ be an arbitrary infinite sequence in (X, d).
  - Consider the set $A = \{x_n : n \in \mathbb{N}\}$ (the set of distinct values in the sequence).
  - **Case 1:** A **is a finite set.**
    - If A is finite, then at least one element $x_k \in A$ must appear infinitely many times in the sequence $\{x_n\}$.
    - Let x^* be such an element. We can form a subsequence $\{x_{n_j}\}$ such that $x_{n_j} = x^*$ for all j.
    - This subsequence clearly converges to x^* (as $d(x_{n_j}, x^*) = 0$ for all j).
    - So, $\{x_n\}$ has a convergent subsequence.
  - **Case 2:** A **is an infinite set.**
    - By assumption (i), every infinite set in X has at least one limit point in X.
    - So, A has a limit point, say $x^* \in X$.
    - Since x^* is a limit point of A, by Question 2(c)(i) (or 3(a)(iii)), there exists a sequence of distinct points from A that converges to x^*.

- Let this sequence be $\{y_k\}$, where $y_k \in A$ and $y_k \to x^*$.
- Since each $y_k \in A = \{x_n : n \in \mathbb{N}\}$, each y_k is some x_{n_k} for some n_k.
- We can choose the indices n_k to be strictly increasing. For example, choose x_{n_1} such that $d(x_{n_1}, x^*) < 1$. Then choose x_{n_2} with $n_2 > n_1$ such that $d(x_{n_2}, x^*) < 1/2$, and so on. (If x^* is a limit point of the sequence values, we can always find such distinct points and increasing indices).
- Thus, we obtain a subsequence $\{x_{n_k}\}$ that converges to x^*.
- So, in both cases, $\{x_n\}$ contains a convergent subsequence.

- **Proof (⇐): Assume (ii) is true. Show (i) is true.**
  - Let A be an arbitrary infinite set in (X, d).
  - We need to show that A has a limit point in X.
  - Since A is infinite, we can choose an infinite sequence of distinct points $\{x_n\}$ from A.
  - By assumption (ii), this sequence $\{x_n\}$ contains a convergent subsequence, say $\{x_{n_k}\}$, that converges to some point $x^* \in X$.
  - Since $x_{n_k} \in A$ for all k, and the terms x_{n_k} are distinct (because they are chosen from a sequence of distinct points), x^* is a limit point of the set A (as seen in Question 2(c)(i) or 3(a)(iii)).
  - Thus, every infinite set in X has at least one limit point in X.

(b) If f is a one-to-one continuous mapping of a compact metric space (X, d_X) onto a metric space (Y, d_Y), then show that f^{-1} is continuous on Y and, hence, f is a homeomorphism of (X, d_X) onto (Y, d_Y).

- **Definitions:**
  - **Homeomorphism:** A mapping $f: X \to Y$ is a homeomorphism if it is bijective (one-to-one and onto), continuous, and its inverse $f^{-1}: Y \to X$ is also continuous.
  - **Compact Metric Space:** A metric space is compact if every open cover has a finite subcover. In metric spaces, compactness is equivalent to sequential compactness (every sequence has a convergent subsequence) and also to completeness and total boundedness.
- **Given:**
  - $f: (X, d_X) \to (Y, d_Y)$ is one-to-one (injective) and onto (surjective), so it is bijective.
  - f is continuous on X.
  - (X, d_X) is a compact metric space.
- **Goal:** Show $f^{-1}: Y \to X$ is continuous. (This will imply f is a homeomorphism).
- **Proof:**
  - To show f^{-1} is continuous, we need to show that for every open set $U \subseteq X$, its inverse image under f^{-1} is open in Y.
  - The inverse image of U under f^{-1} is $(f^{-1})^{-1}(U) = f(U)$.
  - So, we need to show that for every open set $U \subseteq X$, the set $f(U)$ is open in Y.
  - Alternatively, it is often easier to show that f^{-1} maps closed sets to closed sets. That is, for every closed set $F \subseteq X$, $f(F)$ is closed in Y.
  - Let F be any closed subset of X.

- Since X is a compact metric space, and F is a closed subset of a compact space, F itself is compact.
    - (Proof that closed subset of compact space is compact: Let $\{G_\alpha\}$ be an open cover of F. Then $\{G_\alpha\} \cup \{X \setminus F\}$ is an open cover of X. Since X is compact, there is a finite subcover $\{G_{\alpha_1}, \dots, G_{\alpha_n}, X \setminus F\}$. Then $\{G_{\alpha_1}, \dots, G_{\alpha_n}\}$ is a finite subcover of F.)
- Now, we use a key property of continuous functions on compact spaces: **A continuous image of a compact set is compact.**
- Since F is compact and f is continuous, the image $f(F)$ must be compact in Y.
- In any metric space, **every compact set is closed.** (Proof: Let K be a compact set. Let $y \notin K$. For each $k \in K$, there are disjoint open balls U_k around k and V_k around y. $\{U_k\}_{k \in K}$ is an open cover of K. Take a finite subcover. The union of the corresponding U_k's covers K. The intersection of the corresponding V_k's is an open neighborhood of y disjoint from K. So K is closed).
- Therefore, $f(F)$ is a closed set in Y.
- We have shown that for every closed set F in X, its image $f(F)$ is a closed set in Y. This is precisely the condition for f^{-1} to be continuous.
- Since f is bijective, continuous, and f^{-1} is continuous, f is a homeomorphism.

(c) Let A be a compact subset of a metric space (X, d). Show that for any B ⊆ X, there is a point p ∈ A such that d(p, B) = d(A, B).

- **Definitions:**
    - Distance from a point to a set: $d(x, B) = \inf\{d(x, b) : b \in B\}$.

- Distance between two sets: $d(A,B) = \inf\{d(a,b): a \in A, b \in B\}$.

- **Proof:**
  - Let A be a compact subset of (X,d), and $B \subseteq X$.
  - We know that $d(A,B) = \inf_{a\in A}\{d(a,B)\}$.
  - Consider the function $g: A \to \mathbb{R}$ defined by $g(x) = d(x,B)$.
  - We need to show that this function $g(x)$ is continuous on A.
  - For any $x_1, x_2 \in X$, and any $b \in B$, by the triangle inequality, $d(x_1,b) \le d(x_1,x_2) + d(x_2,b)$.
  - Taking the infimum over $b \in B$ on the right side, we get $d(x_1,B) \le d(x_1,x_2) + d(x_2,B)$.
  - Rearranging, $d(x_1,B) - d(x_2,B) \le d(x_1,x_2)$.
  - Similarly, $d(x_2,B) - d(x_1,B) \le d(x_1,x_2)$.
  - Combining these, $|d(x_1,B) - d(x_2,B)| \le d(x_1,x_2)$. This shows that $g(x)$ is continuous (it is even Lipschitz continuous with constant 1).
  - Since A is a compact set and $g: A \to \mathbb{R}$ is a continuous function, by the **Extreme Value Theorem** (a continuous real-valued function on a compact set attains its minimum and maximum values).
  - Therefore, $g(x)$ attains its minimum value on A.
  - This means there exists a point $p \in A$ such that $g(p) = \inf_{x\in A} g(x)$.
  - In other words, there exists a point $p \in A$ such that $d(p,B) = \inf_{a\in A}\{d(a,B)\}$.
  - By definition, $\inf_{a\in A}\{d(a,B)\} = d(A,B)$.
  - Hence, there is a point $p \in A$ such that $d(p,B) = d(A,B)$.