(d) Suppose f is differentiable on $\mathbb{R}$, $1 \le f'(x) \le 2$ for $x \in \mathbb{R}$, and $f(0) = 0$. Prove $x \le f(x) \le 2x$ for all $x \ge 0$. (5)

5. (a) Let f be differentiable function on an open interval (a, b). Then show that f is strictly increasing on (a, b) if $f'(x) > 0$. (5)

(b) If $y = \sin^{-1} x$, prove that (5)

$$(1 - x^2)y_{n+2} - (2n+1)x\, y_{n+1} - n^2 y_n = 0.$$

(c) If $y = \left[x + \sqrt{1+x^2}\right]^m$, find $y_n(0)$. (5)

(d) State Taylor's theorem. Find Taylor series expansion of cos x. (5)

6. (a) Find all values of k and l such that

$$\lim_{x \to 0} \frac{k + \cos lx}{x^2} = -4.$$ (5)

(b) Determine the position and nature of the double points on the curve (5)

$$y^2 = (x-2)^2(x-1).$$

(c) Sketch a graph of the rational function showing the horizontal, vertical and oblique asymptote (if any) of $y = x^2 - \frac{1}{x}$. (5)

(d) Sketch the curve in polar coordinates of $r = \cos 2\theta$. (5)

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 4121 **H**

Unique Paper Code : 2352011202

Name of the Paper : CALCULUS – DSC 5

Name of the Course : **B.Sc. (H) Mathematics UGCF-2022**

Semester : II

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt all questions by selecting **three** parts from each question.
3. **All** questions carry equal marks.
4. Use of Calculator is not allowed.

1. (a) If $f: A \to \mathbb{R}$ and if c is a cluster point of A then prove that f can have only one limit at c. (5)

(b) Use $\in - \delta$ definition of limit to establish the following limit : (5)

$$\lim_{x \to -1} \frac{x+5}{2x+3} = 4.$$

(c) Determine whether the following limit exists in $\mathbb{R}$

$$\lim_{x \to 0} sgn \sin\left(\frac{1}{x}\right) \quad (5)$$

(d) Let $A \subseteq \mathbb{R}$, let $f, g, h: A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be cluster point of A. If $f(x) \leq g(x) \leq h(x)$ for all $x \in A, x \neq c$ and if $\lim_{x \to c} f = L = \lim_{x \to c} h$, then show that $\lim_{x \to c} g = L$. (5)

2. (a) State and prove sequential criterion for continuity. (5)

(b) Let $f: \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that f is discontinuous at every x in $\mathbb{R}$. (5)

(c) Let f and g be continuous real-valued function on (a, b) such that $f(r) = g(r)$ for each rational number r in (a, b) then prove that $f(x) = g(x)$ for all $x \in (a, b)$. (5)

(d) State Intermediate Value Theorem. Show that $x = \cos x$ for some x in $\left(0, \frac{\pi}{2}\right)$. (5)

3. (a) Prove that every continuous function defined on a closed interval is bounded therein. (5)

(b) If f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then prove that $(f(s_n))$ is a Cauchy sequence. (5)

(c) Show that the function $f(x) = \frac{1}{x^2}$ is not uniformly continuous on (0,1) but it is uniformly continuous on $[a, \infty)$ where $a > 0$. (5)

(d) Let $f(x) = x^2 \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable on $\mathbb{R}$ and also show that f′ is not continuous at $x \neq 0$. (5)

4. (a) Let f be defined on an open interval containing x_0. If f assumes its maximum or minimum at x_0 and if f is differentiable at x_0 then show that $f'(x_0) = 0$. (5)

(b) State Intermediate Value Theorem for derivatives. Suppose f is differentiable on E and $f(0) = 0$, $f(1) = 1$, $f(2) = 1$.

(i) Show that $f'(x) = \frac{1}{3}$ for some $x \in (0,2)$.

(ii) Show that $f'(x) = \frac{2}{5}$ for some $x \in (0,2)$. (5)

(c) Show that $ex \leq e^x$ for all $x \in \mathbb{R}$. (5)