(9)

[This question paper contains 8 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 1064 **I**

Unique Paper Code : 2352013501

Name of the Paper : Metric Spaces

Name of the Course : **B.Sc. (Hons.) Mathematics (DSC-13)**

Semester : V

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll. No. on the top immediately on receipt of this question paper.

2. Attempt all question by selecting two parts from each question.

3. Part of the questions to be attempted together.

4. All questions carry equal marks.

5. Use of Calculator not allowed.

1. (a) Let (X, d) be a metric space. Define the function $d': X \times X \to \boldsymbol{R}$ by $d' = \frac{|x-y|}{1+|x-y|}$ Show that d' is a metric on X. Besides, $d'(x, y) < 1$ for all $x, y \in$ X. (7.5)

(b) Let $X = C[a,b]$ be the space of all continuous functions on $[a,b]$. Define $d(x,y) = \int_a^b |f(x) - g(x)|\, dx$, then check whether this metric imply pointwise Convergence or not. (7.5)

(c) Define Cauchy Sequence and Complete metric space. Let X be any non-empty set and d be defined by

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

then show that (X, d) is a Complete metric space. (7.5)

2. (a) Let (X, d) be a metric space. Then show that

(i) $\emptyset$ and X are open sets in (X, d);

(ii) the union of an arbitrary family of open sets is open;

(iii) the intersection of any finite family of open sets is open. (7.5)

(b) Let A be a subset of a metric space (X, d). Then prove that

(i) A° is the largest open subset of A. (3.5)

(ii) A is open if and only if $A = A^\circ$. (4)

(c) (i) Let (X, d) be a metric space and $F \subseteq X$. Then show that a point x_o is a limit point of F if and only if it is possible to select from the set F a sequence of distinct

$x_1, x_2, \ldots, x_n, \ldots$ such that $\lim_{n\to\infty} d(x_n, x_o) = 0$.

(4.5)

(ii) Let $A \subseteq [0, 1]$ and $F = \{f \in C[0, 1]: f(t) = 0, \forall t \in A\}$. Show that F is a closed subset of C[0, 1] equipped with the uniform metric. (3)

3. (a) Let (X, d) be a metric space and $F \subseteq X$. Then show that the following statements are equivalent:

(i) $x \in \bar{F}$;

(ii) $S(x, \varepsilon) \cap F \neq \emptyset$ for every open ball $S(x, \varepsilon)$ centred at x;

(iii) There exists an infinite sequence $\{x_n\}$ of points (not necessarily distinct) of F such that

$$\lim_{n\to\infty} x_n = x.$$

(7.5)

(b) State and prove Cantor's intersection theorem. (7.5)

(c) (i) If Y is a nonempty subset of a metric space (X, d), and (Y, d_y) is complete, then show that Y is a closed in X. (3.5)

(ii) Let (X, d) be a complete metric space and Y a closed subset of X. Then show that (Y, d_y) is a complete space. (4)

4. (a) Prove that a mapping $f: X \to Y$ is continuous on X if and only if $f^{-1}(G)$ is open in X for all open subsets G of Y. (7.5)

(b) Let $T: X \to X$ be a contraction mapping of the complete metric space (X, d). Then show that T has a unique fixed point. (7.5)

(c) Show that the metric spaces (X, d) and (X, ρ)

where $\rho(x,y)=\dfrac{d(x,y)}{1+d(x,y)}$ are equivalent. (7.5)

5. (a) Let (X, d) be a metric space. Then show that the following statements are equivalent:

(i) (X, d) is disconnected;

(ii) there exists a continuous mapping of (X, d) onto the discrete two element space (X_o, d_o). (7.5)

(b) Let $I=[1,1]$and let $f: I \to I$ be continuous. Then show that there exists a point $c \in I$ such that $f(c) = c$. Discuss the result if $I = [-1,1]$. Discuss the result if $I = [-1,1)$ and $I = [-1, \infty)$. (4+3.5)

(c) (i) If C is a connected subset of a disconnected metric space $X = A \cup B$, where A, B are nonempty and $\bar{A} \cap B = \emptyset = A \cap \bar{B}$, then show that either $C \subseteq A$ or $C \subseteq B$.

(ii) If Y is a connected set in a metric space (X, d) then show that any set Z such that is $Y \subseteq Z \subseteq \bar{Y}$ connected. (4+3.5)

6. (a) Let (X, d) be a metric space. Then show that the following statements are equivalent:

(i) every infinite set in (X, d) has at least one limit point in X;

(ii) every infinite sequence in (X, d) contains a convergent subsequence. (7.5)

(b) If f is a one-to-one continuous mapping of a compact metric space (X, d_x) onto a metric space (Y, d_y), then show that f^{-1} is continuous on Y and, hence, f is a homeomorphism of (X, d_x) onto (Y, d_y). (7.5)

(c) Let A be a compact subset of a metric space (X, d). Show that for any $B \subseteq X$, there is a point $p \in A$ such that $d(p, B) = d(A, B)$.. (7.5)