Question 1

(i) Show that the time average Poynting vector for time varying fields is given by: ⟨S⟩ = ½ Re (Ē × H̄)*

- The instantaneous Poynting vector is given by $\vec{S} = \vec{E} \times \vec{H}$.
- For time-varying fields, we can represent the electric and magnetic fields as complex phasors: $\vec{E}(t) = \mathrm{Re}(\vec{E}e^{j\omega t})$ $\vec{H}(t) = \mathrm{Re}(\vec{H}e^{j\omega t})$
- Here, $\vec{E}$ and $\vec{H}$ are the complex amplitudes.
- The product $\vec{E}(t) \times \vec{H}(t)$ involves terms like $\mathrm{Re}(A)\mathrm{Re}(B)$.
- Using the identity $\mathrm{Re}(A)\mathrm{Re}(B) = \frac{1}{2}\mathrm{Re}(AB^* + AB)$, where A and B are complex numbers.
- In our case, $A = \vec{E}e^{j\omega t}$ and $B = \vec{H}e^{j\omega t}$.
- So, $\vec{S}(t) = \frac{1}{2}\mathrm{Re}[(\vec{E}e^{j\omega t}) \times (\vec{H}e^{j\omega t})^* + (\vec{E}e^{j\omega t}) \times (\vec{H}e^{j\omega t})]$.
- $\vec{S}(t) = \frac{1}{2}\mathrm{Re}[\vec{E} \times \vec{H}^* + \vec{E} \times \vec{H}e^{j2\omega t}]$.
- To find the time average, we integrate over one period $T = 2\pi/\omega$: $\langle\vec{S}\rangle = \frac{1}{T}\int_0^T \vec{S}\,(t)dt$.
- The integral of the term $\vec{E} \times \vec{H}e^{j2\omega t}$ over a period is zero.
- Therefore, the time average Poynting vector is $\langle\vec{S}\rangle = \frac{1}{2}\mathrm{Re}(\vec{E} \times \vec{H}^*)$.

(ii) The electric field intensity for an electromagnetic wave is expressed as $E = 120\sin(10^{14}t)\hat{\imath}$ V/m. Calculate the magnitude of displacement current density and the conduction current density. Given: $\sigma = 8.0$ S/m and $\epsilon = 1$.

- Given electric field intensity: $\vec{E} = 120\sin(10^{14}t)\hat{\imath}$ V/m.
- From this, we can identify the amplitude $E_0 = 120$ V/m and angular frequency $\omega = 10^{14}$ rad/s.

- Given conductivity $\sigma = 8.0$ S/m.
- Given relative permittivity $\epsilon_r = 1$.
- The permittivity of the medium is $\epsilon = \epsilon_r \epsilon_0 = 1 \times 8.85 \times 10^{-12} = 8.85 \times 10^{-12}$ F/m.
- Conduction current density ($\vec{J}_c$):
  - $\vec{J}_c = \sigma \vec{E}$
  - $J_c = \sigma E = 8.0 \times 120 \sin(10^{14} t)$
  - $J_c = 960 \sin(10^{14} t)$ A/m^2.
  - The magnitude of conduction current density is $|J_c| = 960$ A/m^2.
- Displacement current density ($\vec{J}_d$):
  - $\vec{J}_d = \frac{\partial \vec{D}}{\partial t} = \epsilon \frac{\partial \vec{E}}{\partial t}$
  - $\vec{J}_d = \epsilon \frac{\partial}{\partial t}(120 \sin(10^{14} t)\hat{\imath})$
  - $\vec{J}_d = \epsilon(120 \times 10^{14} \cos(10^{14} t)\hat{\imath})$
  - $\vec{J}_d = (8.85 \times 10^{-12}) \times (120 \times 10^{14}) \cos(10^{14} t)\hat{\imath}$
  - $\vec{J}_d = 1.062 \times 10^5 \cos(10^{14} t)\hat{\imath}$ A/m^2.
  - The magnitude of displacement current density is $|J_d| = 1.062 \times 10^5$ A/m^2.

(iii) A light beam is incident from denser medium (n_1=2.0) on a rarer medium (n_2=1). Plot the reflection coefficients for the parallel and perpendicular components as a function of the angle of incidence.

- This question asks for a plot, which I am unable to provide directly in text format. However, I can describe the behavior of the reflection coefficients.
- The reflection coefficients are given by Fresnel's equations.

- For perpendicular polarization (s-polarization): $R_\perp = \left(\frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t}\right)^2$
- For parallel polarization (p-polarization): $R_\parallel = \left(\frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t}\right)^2$
- Where θ_i is the angle of incidence and θ_t is the angle of transmission, related by Snell's Law: $n_1 \sin\theta_i = n_2 \sin\theta_t$.
- Since $n_1 > n_2$, there will be a critical angle for total internal reflection (θ_c).
- $\sin\theta_c = n_2/n_1 = 1/2 = 0.5$, so $\theta_c = 30°$.
- For angles of incidence less than the critical angle ($\theta_i < \theta_c = 30°$):
  - Both $R_\perp$ and $R_\parallel$ will be non-zero and less than 1.
  - As θ_i increases, $R_\perp$ generally increases monotonically.
  - $R_\parallel$ will initially decrease, reach a minimum (Brewster's angle, where $R_\parallel = 0$), and then increase. However, in this case, $n_1 > n_2$, so there is no Brewster's angle for reflection from denser to rarer medium if the reflected light is required to be completely polarized (i.e. if $R_\parallel = 0$ it occurs for transmission. Brewster angle is $\theta_B = \arctan(n_2/n_1) = \arctan(1/2) \approx 26.56°$).
- For angles of incidence greater than or equal to the critical angle ($\theta_i \geq 30°$):
  - Total internal reflection occurs.
  - Both $R_\perp$ and $R_\parallel$ will be equal to 1, meaning all incident light is reflected.
- A plot would show $R_\perp$ increasing from a small value (at normal incidence) to 1 at 30° and staying at 1. $R_\parallel$ would also increase from a small value, but may have a dip before 30° (which corresponds to Brewster's angle for external reflection or specific internal reflection conditions) and then rise to 1 at 30° and stay at 1.

(iv) What will be the minimum thickness of a calcite plate that would convert a plane polarized light of wavelength 8000 Å into circularly polarized light? (Given: n_o=1.5533 and n_e=1.5443).

- To convert plane polarized light into circularly polarized light, a quarter-wave plate is needed.
- A quarter-wave plate introduces a phase difference of $\pi/2$ (or 90°) between the ordinary and extraordinary rays.
- The thickness t of a quarter-wave plate is given by: $t = \frac{\lambda}{4|n_o - n_e|}$
- Given:
  - Wavelength $\lambda = 8000$ Å $= 8000 \times 10^{-10}$ m $= 8 \times 10^{-7}$ m.
  - Ordinary refractive index $n_o = 1.5533$.
  - Extraordinary refractive index $n_e = 1.5443$.
- Calculate the difference in refractive indices: $|n_o - n_e| = |1.5533 - 1.5443| = 0.0090$.
- Calculate the minimum thickness t: $t = \frac{8\times10^{-7}}{4\times0.0090}$ $t = \frac{8\times10^{-7}}{0.036}$ $t = 2.222 \times 10^{-5}$ m $t = 0.02222$ mm or 22.22 μm.

(v) Show that if the electric field of the incident wave lies in the plane of incidence, the electric fields of the reflected and transmitted waves will also lie in the plane of incidence.

- The plane of incidence is defined by the normal to the surface and the propagation vector of the incident wave.
- When the electric field of the incident wave lies in the plane of incidence, it is referred to as p-polarization (or parallel polarization).
- According to the boundary conditions for electromagnetic fields at an interface:

a. The tangential component of $\vec{E}$ is continuous across the interface: $E_{1t} = E_{2t}$.

b. The tangential component of $\vec{H}$ is continuous across the interface: $H_{1t} = H_{2t}$.

c. The normal component of $\vec{D}$ is continuous across the interface: $D_{1n} = D_{2n}$.

d. The normal component of $\vec{B}$ is continuous across the interface: $B_{1n} = B_{2n}$.

- Consider the electric field vectors: $\vec{E}_{inc}$, $\vec{E}_{refl}$, $\vec{E}_{trans}$.
- For p-polarization, the incident electric field $\vec{E}_{inc}$ lies in the plane of incidence. Since $\vec{k}_{inc}$ and the surface normal define the plane, $\vec{E}_{inc}$ is perpendicular to $\vec{k}_{inc}$ and has a component parallel to the interface and a component normal to the interface, both within the plane of incidence.
- Maxwell's equations, particularly Faraday's law ($\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$) and Ampere-Maxwell's law ($\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$), and the wave equation derived from them, show that electromagnetic waves are transverse. This means $\vec{E}$ is always perpendicular to the direction of propagation ($\vec{k}$).
- Also, $\vec{E}$, $\vec{H}$, and $\vec{k}$ form a right-handed triplet (i.e., $\vec{E} \times \vec{H}$ is in the direction of $\vec{k}$).
- For p-polarization, $\vec{H}$ is perpendicular to the plane of incidence.
- Due to the continuity of tangential $\vec{E}$ and $\vec{H}$ components at the boundary, and the fact that the directions of reflected and transmitted waves ($\vec{k}_{refl}$ and $\vec{k}_{trans}$) are also in the plane of incidence (as per Snell's Law for $\vec{k}$ vectors), the electric fields of the reflected and transmitted waves must also adjust.

- If the incident $\vec{E}$ is in the plane of incidence, its $\vec{H}$ is perpendicular to it. For the reflected and transmitted waves, their $\vec{H}$ fields must also be perpendicular to the plane of incidence to satisfy the boundary conditions (especially the continuity of tangential $\vec{H}$).
- Since $\vec{E}$ is always perpendicular to $\vec{H}$ and perpendicular to the direction of propagation, and knowing that $\vec{H}$ (for p-polarization) is perpendicular to the plane of incidence, it logically follows that the $\vec{E}$ fields of the reflected and transmitted waves must lie within the plane of incidence. If they didn't, their $\vec{H}$ components would not be perpendicular to the plane of incidence, violating the consistency required by the incident wave's $\vec{H}$ field.

(vi) The refractive indices of quartz for right handed and left-handed circularly polarized light of wavelength 7000 Å are 1.62207 and 1.63201 respectively. Calculate the rotation of plane of polarization of light produced by a plate of thickness 0.9 mm.

- This phenomenon is called optical activity or circular birefringence.
- The rotation of the plane of polarization (θ) is given by the formula: $\theta = \frac{\pi t}{\lambda}|n_L - n_R|$
- Given:
  - Thickness $t = 0.9$ mm $= 0.9 \times 10^{-3}$ m.
  - Wavelength $\lambda = 7000$ Å $= 7000 \times 10^{-10}$ m $= 7 \times 10^{-7}$ m.
  - Refractive index for left-handed circularly polarized light $n_L = 1.63201$.
  - Refractive index for right-handed circularly polarized light $n_R = 1.62207$.
- Calculate the difference in refractive indices: $|n_L - n_R| = |1.63201 - 1.62207| = 0.00994$.

- Calculate the rotation θ: $\theta = \frac{\pi\times(0.9\times10^{-3})}{7\times10^{-7}} \times 0.00994$ $\theta = \frac{0.9\times3.14159\times0.00994}{7\times10^{-4}}$ $\theta = \frac{0.02809}{7\times10^{-4}}$ $\theta = 40.128$ radians.
- To convert to degrees: $\theta_{degrees} = \theta_{radians} \times \frac{180}{\pi}$ $\theta_{degrees} = 40.128 \times \frac{180}{3.14159}$ $\theta_{degrees} = 2298.9$ degrees.

(vii) Show that a beam of plane polarized light may be regarded as composed of two equal and opposite circularly polarized light.

- A linearly polarized (plane polarized) light wave oscillating along a specific direction can be decomposed into two counter-rotating circularly polarized components.
- Consider a plane polarized wave with its electric field oscillating along the x-axis: $\vec{E}(z,t) = E_0\cos(kz-\omega t)\hat{\imath}$
- We can represent this as the sum of two circularly polarized waves:
  e. A right-handed circularly polarized (RHCP) wave: $\vec{E}_{RHCP} = \frac{E_0}{2}[\cos(kz-\omega t)\hat{\imath} + \sin(kz-\omega t)\hat{\jmath}]$
  f. A left-handed circularly polarized (LHCP) wave: $\vec{E}_{LHCP} = \frac{E_0}{2}[\cos(kz-\omega t)\hat{\imath} - \sin(kz-\omega t)\hat{\jmath}]$
- Adding these two components: $\vec{E}_{RHCP} + \vec{E}_{LHCP} = \frac{E_0}{2}[\cos(kz-\omega t)\hat{\imath} + \sin(kz-\omega t)\hat{\jmath} + \cos(kz-\omega t)\hat{\imath} - \sin(kz-\omega t)\hat{\jmath}] = \frac{E_0}{2}[2\cos(kz-\omega t)\hat{\imath}] = E_0\cos(kz-\omega t)\hat{\imath}$
- This sum is indeed the original plane polarized wave.
- Both circularly polarized components have the same amplitude $(E_0/2)$ and the same frequency and wavelength. They rotate in opposite directions in the xy-plane as the wave propagates along the z-axis. The superposition of these two counter-rotating fields results in a net electric field that oscillates only

along a single direction (the x-axis in this case), thus forming a plane polarized wave.

Question 2:

(a) State and establish Poynting's theorem for electromagnetic fields. Compare it with the equation of continuity and give an interpretation of the Poynting vector.

- **Poynting's Theorem Statement:** Poynting's theorem states that the rate of energy flow out of a given volume is equal to the rate of decrease in the electromagnetic energy stored within that volume, minus the rate of work done by the electromagnetic field on the charges within that volume.
- **Establishment of Poynting's Theorem:**
  - We start with Maxwell's equations in differential form:
    i. $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's Law)
    ii. $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ (Ampere-Maxwell Law)
  - Dot product equation (2) with $\vec{E}$: $\vec{E} \cdot (\nabla \times \vec{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}$ (Equation A)
  - Dot product equation (1) with $\vec{H}$: $\vec{H} \cdot (\nabla \times \vec{E}) = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t}$ (Equation B)
  - Subtract Equation B from Equation A: $\vec{E} \cdot (\nabla \times \vec{H}) - \vec{H} \cdot (\nabla \times \vec{E}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}$
  - Use the vector identity: $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$.
  - Applying this identity to the left side: $\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H})$.
  - So, $-\nabla \cdot (\vec{E} \times \vec{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}$.
  - For linear, isotropic, and non-dispersive media, $\vec{D} = \epsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$.

- $\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = \epsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{1}{2} \epsilon \frac{\partial}{\partial t} (\vec{E} \cdot \vec{E}) = \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon E^2 \right)$
- $\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = \mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} = \frac{1}{2} \mu \frac{\partial}{\partial t} (\vec{H} \cdot \vec{H}) = \frac{\partial}{\partial t} \left(\frac{1}{2} \mu H^2 \right)$

- Substitute these back into the equation: $-\nabla \cdot (\vec{E} \times \vec{H}) = \vec{E} \cdot \vec{J} + \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right)$
- Define the Poynting vector $\vec{S} = \vec{E} \times \vec{H}$.
- Define the electromagnetic energy density $u = \frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2$.
- The equation becomes: $-\nabla \cdot \vec{S} = \vec{E} \cdot \vec{J} + \frac{\partial u}{\partial t}$.
- Integrate over a volume V bounded by a surface A: $\int_V (- \nabla \cdot \vec{S}) dV = \int_V \vec{E} \cdot \vec{J} dV + \int_V \frac{\partial u}{\partial t} dV$
- Using the divergence theorem for the left side: $-\oint_A \vec{S} \cdot d\vec{a} = \int_V \vec{E} \cdot \vec{J} dV + \frac{\partial}{\partial t} \int_V u\, dV$
- This is Poynting's theorem.

- **Comparison with the Equation of Continuity:**
  - The equation of continuity for charge conservation is: $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$.
  - This equation states that the rate of decrease of charge density within a volume is equal to the net current flowing out of that volume. It describes the conservation of electric charge.
  - Poynting's theorem: $\nabla \cdot \vec{S} = -\frac{\partial u}{\partial t} - \vec{E} \cdot \vec{J}$.
  - This equation describes the conservation of energy.
  - **Similarities:** Both are continuity equations for conserved quantities (charge for the equation of continuity, and energy for Poynting's theorem). Both relate a divergence of a flux density (current density $\vec{J}$

and Poynting vector $\vec{S}$) to the rate of change of a density (charge density ρ and energy density u) plus a source/sink term ($\vec{E} \cdot \vec{J}$ representing power dissipated or delivered to charges).

- **Differences:**
  - The conserved quantity is different (charge vs. energy).
  - Poynting's theorem has an additional term $\vec{E} \cdot \vec{J}$, which represents the rate at which the electromagnetic field does work on the charges (or dissipated as heat in a resistive medium, σE^2). This term is absent in the simple charge continuity equation. It signifies the conversion of electromagnetic energy into other forms (e.g., heat, mechanical energy).

- **Interpretation of the Poynting Vector ($\vec{S}$):**
  - The Poynting vector $\vec{S} = \vec{E} \times \vec{H}$ represents the instantaneous rate of electromagnetic energy flow per unit area at a given point, in a direction perpendicular to both $\vec{E}$ and $\vec{H}$.
  - Its direction indicates the direction of energy propagation.
  - Its magnitude represents the power flux density (W/m^2).
  - In simple terms, it tells us where and how much electromagnetic energy is flowing at any instant. For a propagating electromagnetic wave, $\vec{S}$ is in the direction of wave propagation.

(b) What are electromagnetic potentials? Discuss their non-uniqueness and hence explain the significance of gauge transformation.

- **Electromagnetic Potentials:**
  - Electromagnetic potentials are auxiliary fields introduced to simplify the solution of Maxwell's equations. They consist of two components:

    iii. **Scalar Potential (V):** Related to the electric field. It is a scalar field, similar to electrostatic potential.

iv. **Vector Potential ($\vec{A}$):** Related to the magnetic field. It is a vector field.

- Their relationship to the electric and magnetic fields is given by: $\vec{B} = \nabla \times \vec{A}$ $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$
- These definitions automatically satisfy two of Maxwell's equations (the homogeneous ones, which contain no sources):
  - $\nabla \cdot \vec{B} = 0$ (since the divergence of a curl is always zero: $\nabla \cdot (\nabla \times \vec{A}) = 0$)
  - $\nabla \times \vec{E} = \nabla \times (-\nabla V - \frac{\partial \vec{A}}{\partial t}) = -\nabla \times (\nabla V) - \frac{\partial}{\partial t}(\nabla \times \vec{A}) = 0 - \frac{\partial \vec{B}}{\partial t} = -\frac{\partial \vec{B}}{\partial t}$ (since the curl of a gradient is always zero: $\nabla \times (\nabla V) = 0$).

- **Non-Uniqueness of Electromagnetic Potentials:**
  - The choice of scalar and vector potentials is not unique for a given set of $\vec{E}$ and $\vec{B}$ fields.
  - Consider a new scalar potential V' and a new vector potential $\vec{A}'$ related to the original potentials by: $V' = V - \frac{\partial \Lambda}{\partial t}$ $\vec{A}' = \vec{A} + \nabla \Lambda$ where Λ is an arbitrary scalar function of position and time.
  - Let's check if these new potentials yield the same $\vec{E}$ and $\vec{B}$ fields:
    - For $\vec{B}'$: $\vec{B}' = \nabla \times \vec{A}' = \nabla \times (\vec{A} + \nabla \Lambda) = \nabla \times \vec{A} + \nabla \times (\nabla \Lambda)$ Since $\nabla \times (\nabla \Lambda) = 0$, we have $\vec{B}' = \nabla \times \vec{A} = \vec{B}$. The magnetic field remains unchanged.
    - For $\vec{E}'$: $\vec{E}' = -\nabla V' - \frac{\partial \vec{A}'}{\partial t} = -\nabla(V - \frac{\partial \Lambda}{\partial t}) - \frac{\partial}{\partial t}(\vec{A} + \nabla \Lambda)$ $\vec{E}' = -\nabla V + \nabla\left(\frac{\partial \Lambda}{\partial t}\right) - \frac{\partial \vec{A}}{\partial t} - \frac{\partial}{\partial t}(\nabla \Lambda)$ Since spatial and temporal

derivatives commute, $\nabla\left(\frac{\partial\Lambda}{\partial t}\right) = \frac{\partial}{\partial t}(\nabla\Lambda)$. So, $\vec{E}' = -\nabla V - \frac{\partial \vec{A}}{\partial t} = \vec{E}$. The electric field also remains unchanged.

- This demonstrates that adding a gradient of an arbitrary scalar function to $\vec{A}$ and a corresponding time derivative of that function to V does not change the physical fields $\vec{E}$ and $\vec{B}$. This property is known as the non-uniqueness of the potentials.

- **Significance of Gauge Transformation:**
  - The transformation from $(\vec{V}, \vec{A})$ to $(\vec{V}', \vec{A}')$ is called a **gauge transformation**.
  - Its significance lies in the fact that:

v. **Freedom of Choice:** It provides a degree of freedom in choosing the potentials without altering the physical electromagnetic fields. This freedom can be exploited to simplify calculations.

vi. **Gauge Conditions:** We can impose additional conditions on the potentials, called "gauge conditions," to uniquely determine them or simplify Maxwell's equations expressed in terms of potentials. Common gauges include:

- **Lorentz Gauge:** $\nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t} = 0$. This gauge leads to symmetric wave equations for V and A, and it is Lorentz invariant, making it suitable for relativistic electromagnetism.

- **Coulomb Gauge (or Radiation Gauge):** $\nabla \cdot \vec{A} = 0$. This gauge is particularly useful in situations where sources are static or for separating instantaneous Coulombic interactions from retarded radiation effects.

vii. **Physical Invariance:** The fact that $\vec{E}$ and $\vec{B}$ fields are invariant under gauge transformations highlights that only the fields themselves are physically measurable, not the potentials in isolation. The potentials are mathematical constructs that aid in solving the equations.

viii. **Quantum Electrodynamics (QED):** Gauge invariance is a fundamental principle in quantum field theories, including QED. It ensures the consistency and renormalizability of the theory, indicating that the fundamental laws of physics are independent of the specific choice of gauge.

Question 3:

A starting from Maxwell's equations in an isotropic, homogeneous dielectric material, show that electromagnetic waves are transverse in nature. Calculate the time average of momentum density stored in these fields. (12 marks)

- **Maxwell's Equations in Isotropic, Homogeneous Dielectric Material:** For a source-free, isotropic, homogeneous dielectric medium (where $\vec{J} = 0$ and $\rho = 0$), Maxwell's equations are:

  g. $\nabla \cdot \vec{E} = 0$ (Gauss's Law for Electric Fields)

  h. $\nabla \cdot \vec{B} = 0$ (Gauss's Law for Magnetic Fields)

  i. $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's Law)

  j. $\nabla \times \vec{B} = \mu\epsilon\frac{\partial \vec{E}}{\partial t}$ (Ampere-Maxwell Law, with $\vec{D} = \epsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$)

- **Showing Transverse Nature of Electromagnetic Waves:**

  - Take the curl of equation (3): $\nabla \times (\nabla \times \vec{E}) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$
  - Using the vector identity $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A}$: $\nabla(\nabla \cdot \vec{E}) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$
  - From equation (1), $\nabla \cdot \vec{E} = 0$.

- Substitute equation (4) into the right side: $0 - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu\epsilon \frac{\partial \vec{E}}{\partial t}\right)$
  $\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$
- This is the wave equation for the electric field. A similar derivation can be done for the magnetic field: $\nabla^2 \vec{B} = \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}$.
- The wave equation suggests that electromagnetic waves propagate with a speed $v = \frac{1}{\sqrt{\mu\epsilon}}$.
- **To show transverse nature:**
  - Consider a plane wave propagating in the z-direction. For such a wave, the field depends on $(z - vt)$.
  - From equation (1), $\nabla \cdot \vec{E} = 0$. For a wave propagating in z-direction, $\vec{E} = E_x(z,t)\hat{\imath} + E_y(z,t)\hat{\jmath} + E_z(z,t)\hat{k}$. $\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$. Since the wave is a plane wave propagating in z-direction, $\frac{\partial}{\partial x} = 0$ and $\frac{\partial}{\partial y} = 0$. Therefore, $\frac{\partial E_z}{\partial z} = 0$. This implies that E_z is constant in space for a plane wave. As there are no sources, E_z must be zero for a freely propagating wave (a non-zero constant E_z would imply a static electric field, not part of a propagating wave). Thus, $E_z = 0$.
  - This means the electric field $\vec{E}$ has no component in the direction of propagation (z-direction). Hence, $\vec{E}$ is perpendicular to the direction of propagation.
  - Similarly, for the magnetic field, from equation (2), $\nabla \cdot \vec{B} = 0$. Following the same logic, $B_z = 0$.
  - Thus, the magnetic field $\vec{B}$ also has no component in the direction of propagation and is therefore perpendicular to the direction of propagation.

- Furthermore, we can show that $\vec{E}$ and $\vec{B}$ are perpendicular to each other. From Faraday's law ($\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$), if $\vec{E}$ is in the xy-plane and propagating in z, then $\vec{E}$ changes in space to produce a changing $\vec{B}$ in the xy-plane. For instance, if $\vec{E} = E_x \hat{\imath}$, then $\nabla \times \vec{E} = -\frac{\partial E_x}{\partial z} \hat{\jmath}$. This implies $\frac{\partial B_y}{\partial t} = \frac{\partial E_x}{\partial z}$, showing B_y is related to E_x. In general, $\vec{E}$ and $\vec{B}$ are mutually perpendicular and perpendicular to the direction of propagation. This confirms that electromagnetic waves are transverse in nature.

- **Time Average of Momentum Density Stored in These Fields:**
  - The momentum density of an electromagnetic field is given by: $\vec{g} = \frac{1}{c^2}\vec{S} = \frac{1}{c^2}(\vec{E} \times \vec{H})$ where $c = \frac{1}{\sqrt{\mu\epsilon}}$ is the speed of light in the medium.
  - So, $\vec{g} = \epsilon\mu(\vec{E} \times \vec{H})$.
  - For time-harmonic fields, $\vec{E} = \mathrm{Re}(\vec{E}_0 e^{j(\vec{k}\cdot\vec{r} - \omega t)})$ and $\vec{H} = \mathrm{Re}(\vec{H}_0 e^{j(\vec{k}\cdot\vec{r} - \omega t)})$.
  - The instantaneous Poynting vector is $\vec{S}(t) = \vec{E}(t) \times \vec{H}(t)$.
  - The time average Poynting vector is $\langle \vec{S} \rangle = \frac{1}{2}\mathrm{Re}(\vec{E}_0 \times \vec{H}_0^*)$.
  - Therefore, the time average momentum density is: $\langle \vec{g} \rangle = \frac{1}{c^2}\langle \vec{S} \rangle = \frac{1}{2c^2}\mathrm{Re}(\vec{E}_0 \times \vec{H}_0^*)$.
  - For a plane wave in a dielectric medium, $\vec{H}_0 = \frac{1}{\eta}\hat{k} \times \vec{E}_0$, where $\eta = \sqrt{\frac{\mu}{\epsilon}}$ is the intrinsic impedance of the medium, and $\hat{k}$ is the direction of propagation.
  - So, $\vec{H}_0^* = \frac{1}{\eta}\hat{k} \times \vec{E}_0^*$.

- $\langle\vec{S}\rangle = \frac{1}{2}\mathrm{Re}\left(\vec{E}_0 \times \left(\frac{1}{\eta}\hat{k} \times \vec{E}_0^*\right)\right)$.
- Using vector identity $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$: $\vec{E}_0 \times (\hat{k} \times \vec{E}_0^*) = \hat{k}(\vec{E}_0 \cdot \vec{E}_0^*) - \vec{E}_0^*(\vec{E}_0 \cdot \hat{k})$.
- Since $\vec{E}_0$ is perpendicular to $\hat{k}$ (transverse wave), $\vec{E}_0 \cdot \hat{k} = 0$.
- So, $\langle\vec{S}\rangle = \frac{1}{2\eta}\mathrm{Re}(\hat{k}|\vec{E}_0|^2) = \frac{|\vec{E}_0|^2}{2\eta}\hat{k}$.
- Therefore, the time average momentum density is: $\langle\vec{g}\rangle = \frac{1}{c^2}\frac{|\vec{E}_0|^2}{2\eta}\hat{k}$.
- We know that the time average energy density $\langle u\rangle = \frac{1}{4}(\epsilon|\vec{E}_0|^2 + \mu|\vec{H}_0|^2) = \frac{1}{2}\epsilon|\vec{E}_0|^2$ (since $\epsilon|\vec{E}_0|^2 = \mu|\vec{H}_0|^2$ for plane waves in lossless media).
- Also, $\eta = \sqrt{\mu/\epsilon}$. So, $\frac{1}{\eta} = \sqrt{\epsilon/\mu}$.
- $\langle\vec{g}\rangle = \frac{1}{\mu\epsilon}\frac{|\vec{E}_0|^2}{2\sqrt{\mu/\epsilon}}\hat{k} = \frac{\sqrt{\epsilon}}{2\mu^{3/2}}|\vec{E}_0|^2\hat{k}$.
- Alternatively, using the relation between energy density and Poynting vector, $\langle S\rangle = v\langle u\rangle$: $\langle\vec{g}\rangle = \frac{\langle\vec{S}\rangle}{c^2} = \frac{v\langle u\rangle}{c^2} = \frac{\frac{1}{\sqrt{\mu\epsilon}}\langle u\rangle}{\frac{1}{\mu\epsilon}} = \sqrt{\mu\epsilon}\langle u\rangle\hat{k}$.
- Substituting $\langle u\rangle = \frac{1}{2}\epsilon|\vec{E}_0|^2$: $\langle\vec{g}\rangle = \sqrt{\mu\epsilon}\left(\frac{1}{2}\epsilon|\vec{E}_0|^2\right)\hat{k} = \frac{1}{2}\epsilon^{3/2}\mu^{1/2}|\vec{E}_0|^2\hat{k}$.
- This can also be expressed in terms of the time-average energy density $\langle u\rangle$: $\langle\vec{g}\rangle = \frac{\langle u\rangle}{v}\hat{k}$ or $\langle\vec{g}\rangle = \frac{\langle u\rangle}{c}\hat{k}$ (where c is the speed of light in the medium).

Question 4:

(a) Calculate the intrinsic impedance and wave velocity for a conducting medium with $\sigma = 60$ MS/m and $\mu = 1$, at frequency of 120 MHz.

- Given:
  - Conductivity $\sigma = 60$ MS/m $= 60 \times 10^6$ S/m.
  - Relative permeability $\mu_r = 1$, so permeability $\mu = \mu_r \mu_0 = 1 \times 4\pi \times 10^{-7}$ H/m.
  - Frequency $f = 120$ MHz $= 120 \times 10^6$ Hz.
  - Angular frequency $\omega = 2\pi f = 2\pi \times 120 \times 10^6 = 2.4\pi \times 10^8$ rad/s.
  - Assume $\epsilon = \epsilon_0 = 8.85 \times 10^{-12}$ F/m for a good conductor unless specified otherwise.
- **Intrinsic Impedance (η):**
  - For a conducting medium, the intrinsic impedance is given by: $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$
  - First, compare the magnitudes of σ and $\omega\epsilon$: $\omega\epsilon = (2.4\pi \times 10^8) \times (8.85 \times 10^{-12}) \approx 6.67 \times 10^{-3}$ S/m. $\sigma = 60 \times 10^6$ S/m.
  - Since $\sigma \gg \omega\epsilon$, it's a good conductor approximation. In this case, $\sigma + j\omega\epsilon \approx \sigma$.
  - So, $\eta \approx \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}} e^{j\pi/4}$
  - $\eta \approx \sqrt{\frac{(2.4\pi \times 10^8) \times (4\pi \times 10^{-7})}{60 \times 10^6}}$
  - $\eta \approx \sqrt{\frac{9.6\pi^2 \times 10^1}{60 \times 10^6}} = \sqrt{\frac{94.75}{60 \times 10^6}} = \sqrt{1.579 \times 10^{-6}}$
  - $\eta \approx 1.256 \times 10^{-3}(1 + j) \approx 0.001256\sqrt{2}e^{j\pi/4}$ Ohms.

- Magnitude: $|\eta| = \sqrt{\frac{\omega\mu}{\sigma}} = 0.001256$ Ohms.
- Angle: 45° or $\pi/4$ radians.
- So, $\eta \approx 1.256 \times 10^{-3} e^{j\pi/4}$ Ohms.

- **Wave Velocity (v):**
    - For a good conductor, the phase velocity is given by: $v = \sqrt{\frac{2\omega}{\mu\sigma}}$
    - $v = \sqrt{\frac{2\times(2.4\pi\times10^8)}{(4\pi\times10^{-7})\times(60\times10^6)}}$
    - $v = \sqrt{\frac{4.8\pi\times10^8}{240\pi\times10^{-1}}}$
    - $v = \sqrt{\frac{4.8\times10^8}{24}} = \sqrt{0.2 \times 10^8} = \sqrt{2 \times 10^7}$
    - $v = \sqrt{20 \times 10^6} = 4.472 \times 10^3$ m/s.

(b) Derive Fresnel's relation for reflection and transmission of plane electromagnetic waves at an interface between two dielectric media when an electric vector of the incident wave is perpendicular to the plane of incidence.

- **Setup:**
    - Consider a plane electromagnetic wave incident from medium 1 (with parameters ϵ_1, μ_1) onto medium 2 (with parameters ϵ_2, μ_2).
    - The interface is the xy-plane (z=0).
    - The plane of incidence is the xz-plane.
    - The electric vector of the incident wave is perpendicular to the plane of incidence. This is known as **s-polarization** or **TE (Transverse Electric) mode**.

- For s-polarization, the electric fields of the incident, reflected, and transmitted waves are all parallel to the y-axis (perpendicular to the plane of incidence).
- $\vec{E}_i = E_{i0} e^{-j\vec{k}_i \cdot \vec{r}} \hat{y}$
- $\vec{E}_r = E_{r0} e^{-j\vec{k}_r \cdot \vec{r}} \hat{y}$
- $\vec{E}_t = E_{t0} e^{-j\vec{k}_t \cdot \vec{r}} \hat{y}$
- The wave vectors are: $\vec{k}_i = k_1(\sin\theta_i \hat{x} + \cos\theta_i \hat{z})$ $\vec{k}_r = k_1(\sin\theta_r \hat{x} - \cos\theta_r \hat{z})$ (z-component is negative for reflection) $\vec{k}_t = k_2(\sin\theta_t \hat{x} + \cos\theta_t \hat{z})$
- At the boundary (z=0), the phases must match for all x and y: $\vec{k}_i \cdot \vec{r} = \vec{k}_r \cdot \vec{r} = \vec{k}_t \cdot \vec{r}$ at z=0. This leads to the laws of reflection ($\theta_i = \theta_r$) and Snell's law ($k_1 \sin\theta_i = k_2 \sin\theta_t \Rightarrow n_1 \sin\theta_i = n_2 \sin\theta_t$).

- **Magnetic Fields (s-polarization):**
  - Since $\vec{H} = \frac{1}{\mu\omega}(\vec{k} \times \vec{E})$, for s-polarization ($\vec{E}$ along y-axis):
  - $\vec{H}_i = \frac{1}{\omega\mu_1}(\vec{k}_i \times \vec{E}_i) = \frac{E_{i0}}{\eta_1}(-\cos\theta_i \hat{x} + \sin\theta_i \hat{z}) e^{-j\vec{k}_i \cdot \vec{r}}$
  - $\vec{H}_r = \frac{1}{\omega\mu_1}(\vec{k}_r \times \vec{E}_r) = \frac{E_{r0}}{\eta_1}(\cos\theta_r \hat{x} + \sin\theta_r \hat{z}) e^{-j\vec{k}_r \cdot \vec{r}}$
  - $\vec{H}_t = \frac{1}{\omega\mu_2}(\vec{k}_t \times \vec{E}_t) = \frac{E_{t0}}{\eta_2}(-\cos\theta_t \hat{x} + \sin\theta_t \hat{z}) e^{-j\vec{k}_t \cdot \vec{r}}$
  - Here $\eta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance.
- **Boundary Conditions at z=0:**

  k. **Continuity of tangential $\vec{E}$:** $(E_i + E_r)_{tangential} = (E_t)_{tangential}$

  - Since all $\vec{E}$ fields are along $\hat{y}$ (which is tangential to the interface): $E_{i0} + E_{r0} = E_{t0}$ (Equation 1)

1. **Continuity of tangential $\vec{H}$:** $(H_i + H_r)_{tangential} = (H_t)_{tangential}$

   - The tangential components of $\vec{H}$ are along the x-axis.
   - $\frac{E_{i0}}{\eta_1}(-\cos\theta_i) + \frac{E_{r0}}{\eta_1}(\cos\theta_r) = \frac{E_{t0}}{\eta_2}(-\cos\theta_t)$
   - Since $\theta_i = \theta_r$, we replace $\cos\theta_r$ with $\cos\theta_i$: $-\frac{E_{i0}}{\eta_1}\cos\theta_i + \frac{E_{r0}}{\eta_1}\cos\theta_i = -\frac{E_{t0}}{\eta_2}\cos\theta_t$ $\frac{\cos\theta_i}{\eta_1}(E_{r0} - E_{i0}) = -\frac{E_{t0}}{\eta_2}\cos\theta_t$ (Equation 2)

- **Solving for Reflection and Transmission Coefficients:**
  - From Equation 1, $E_{t0} = E_{i0} + E_{r0}$. Substitute this into Equation 2: $\frac{\cos\theta_i}{\eta_1}(E_{r0} - E_{i0}) = -\frac{(E_{i0}+E_{r0})}{\eta_2}\cos\theta_t$ Multiply by $\eta_1\eta_2$: $\eta_2\cos\theta_i(E_{r0} - E_{i0}) = -\eta_1\cos\theta_t(E_{i0} + E_{r0})$ $\eta_2\cos\theta_i E_{r0} - \eta_2\cos\theta_i E_{i0} = -\eta_1\cos\theta_t E_{i0} - \eta_1\cos\theta_t E_{r0}$ Group terms with E_{r0} and E_{i0}: $E_{r0}(\eta_2\cos\theta_i + \eta_1\cos\theta_t) = E_{i0}(\eta_2\cos\theta_i - \eta_1\cos\theta_t)$
  - **Reflection Coefficient ($r_\perp$):** $r_\perp = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$ Since $\eta = \sqrt{\mu/\epsilon}$, and for non-magnetic media $\mu_1 = \mu_2 = \mu_0$: $r_\perp = \frac{\sqrt{\epsilon_1}\cos\theta_i - \sqrt{\epsilon_2}\cos\theta_t}{\sqrt{\epsilon_1}\cos\theta_i + \sqrt{\epsilon_2}\cos\theta_t}$ Using refractive indices $n = c/\sqrt{\mu\epsilon} = \sqrt{\epsilon/\epsilon_0}$ for non-magnetic media ($c = 1/\sqrt{\mu_0\epsilon_0}$): $r_\perp = \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t}$
  - **Transmission Coefficient ($t_\perp$):** Use $E_{t0} = E_{i0} + E_{r0} = E_{i0}(1 + r_\perp)$: $t_\perp = \frac{E_{t0}}{E_{i0}} = 1 + r_\perp = 1 + \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$ $t_\perp = \frac{(\eta_2\cos\theta_i + \eta_1\cos\theta_t) + (\eta_2\cos\theta_i - \eta_1\cos\theta_t)}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$ $t_\perp = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$ In terms of refractive indices: $t_\perp = \frac{2n_1\cos\theta_i}{n_1\cos\theta_i + n_2\cos\theta_t}$

Question 5:

(a) A uniform plane wave is incident on planar boundary separating regions 1 and 2, with $\epsilon_1 = \epsilon_0$ and $\mu_1 = \mu_0$, and $\epsilon_2 = 0$ and $\mu_2 = 1$. Find the ratio of ϵ_2/ϵ_1, if 20

% of the incident wave energy is (a) reflected and (b) transmitted. (Assume normal incidence).

- The problem statement contains a potential typo with "$\epsilon_2 = 0$". Permittivity cannot be zero. It's likely intended to mean $\epsilon_2 = \epsilon_0$ (similar to ϵ_1) or some other value for ϵ_2. Let's assume there's a typo and treat it as a general interface where ϵ_2 is unknown.
- Given:
  - $\epsilon_1 = \epsilon_0$, $\mu_1 = \mu_0$. So medium 1 is free space.
  - $\mu_2 = \mu_0$ (assuming $\mu_2 = 1$ means $\mu_r = 1$, so $\mu_2 = \mu_0$).
  - Normal incidence ($\theta_i = \theta_t = 0$).
- **Reflection Coefficient (r) and Transmission Coefficient (t) for Normal Incidence:**
  - The reflection coefficient for electric field amplitude at normal incidence is: $r = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$
  - The transmission coefficient for electric field amplitude at normal incidence is: $t = \frac{2\eta_2}{\eta_2 + \eta_1}$
  - Where $\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} = \sqrt{\frac{\mu_0}{\epsilon_0}} = \eta_0$ (intrinsic impedance of free space, approx 377 Ohms).
  - $\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}} = \sqrt{\frac{\mu_0}{\epsilon_2}}$.
- **Reflected Power (R) and Transmitted Power (T):**
  - The reflection coefficient of power (reflectivity) is $R = |r|^2$.
  - The transmission coefficient of power (transmissivity) is $T = \frac{\eta_1}{\eta_2}|t|^2$.
  - Also, $R + T = 1$ (conservation of energy).

- **Case (a): 20% of incident wave energy is reflected (R = 0.20)**
  - $R = |r|^2 = \left|\frac{\eta_2-\eta_1}{\eta_2+\eta_1}\right|^2 = 0.20$
  - $\frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \pm\sqrt{0.20} = \pm 0.4472$
  - Consider the positive root first: $\eta_2 - \eta_1 = 0.4472(\eta_2 + \eta_1)\ \eta_2 - \eta_1 = 0.4472\eta_2 + 0.4472\eta_1\ 0.5528\eta_2 = 1.4472\eta_1\ \eta_2 = \frac{1.4472}{0.5528}\eta_1 \approx 2.618\eta_1$
  - Consider the negative root: $\eta_2 - \eta_1 = -0.4472(\eta_2 + \eta_1)\ \eta_2 - \eta_1 = -0.4472\eta_2 - 0.4472\eta_1\ 1.4472\eta_2 = 0.5528\eta_1\ \eta_2 = \frac{0.5528}{1.4472}\eta_1 \approx 0.382\eta_1$
  - Now, relate η_2 to ϵ_2: $\eta_2 = \sqrt{\frac{\mu_0}{\epsilon_2}}$ and $\eta_1 = \sqrt{\frac{\mu_0}{\epsilon_0}}$. So, $\eta_2/\eta_1 = \sqrt{\epsilon_0/\epsilon_2} = \sqrt{1/(\epsilon_2/\epsilon_0)}$. Therefore, $\epsilon_2/\epsilon_0 = (\eta_1/\eta_2)^2$.
  - For $\eta_2 \approx 2.618\eta_1$: $\epsilon_2/\epsilon_0 = (1/2.618)^2 \approx 0.146$. So, $\epsilon_2 = 0.146\epsilon_0$.
  - For $\eta_2 \approx 0.382\eta_1$: $\epsilon_2/\epsilon_0 = (1/0.382)^2 \approx 6.84$. So, $\epsilon_2 = 6.84\epsilon_0$.
  - The ratio of ϵ_2/ϵ_1 is ϵ_2/ϵ_0. So the possible ratios are **0.146** or **6.84**.
- **Case (b): 20% of incident wave energy is transmitted (T = 0.20)**
  - If $T = 0.20$, then $R = 1 - T = 1 - 0.20 = 0.80$.
  - Using the reflection coefficient formula again: $R = |r|^2 = \left|\frac{\eta_2-\eta_1}{\eta_2+\eta_1}\right|^2 = 0.80$
  - $\frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \pm\sqrt{0.80} = \pm 0.8944$
  - Consider the positive root: $\eta_2 - \eta_1 = 0.8944(\eta_2 + \eta_1)\ \eta_2 - \eta_1 = 0.8944\eta_2 + 0.8944\eta_1\ 0.1056\eta_2 = 1.8944\eta_1\ \eta_2 = \frac{1.8944}{0.1056}\eta_1 \approx 17.939\eta_1$

- Consider the negative root: $\eta_2 - \eta_1 = -0.8944(\eta_2 + \eta_1)$ $\eta_2 - \eta_1 = -0.8944\eta_2 - 0.8944\eta_1$ $1.8944\eta_2 = 0.1056\eta_1$ $\eta_2 = \frac{0.1056}{1.8944}\eta_1 \approx 0.0557\eta_1$
- For $\eta_2 \approx 17.939\eta_1$: $\epsilon_2/\epsilon_0 = (1/17.939)^2 \approx 0.0031$. So, $\epsilon_2 = 0.0031\epsilon_0$.
- For $\eta_2 \approx 0.0557\eta_1$: $\epsilon_2/\epsilon_0 = (1/0.0557)^2 \approx 322.8$. So, $\epsilon_2 = 322.8\epsilon_0$.
- The possible ratios are **0.0031** or **322.8**.

- **Self-Correction for $\epsilon_2 = 0$ in the problem:** If $\epsilon_2 = 0$ was truly intended, it implies an unusual medium. If $\epsilon_2 = 0$, then $\eta_2 = \sqrt{\mu_0/0}$, which would be infinite. This would mean $r = (\infty - \eta_1)/(\infty + \eta_1) = 1$, leading to $R = 1$ (total reflection) and $T = 0$. This contradicts the given 20% reflection/transmission. Therefore, the interpretation of ϵ_2 being some unknown value is more appropriate for the given conditions of 20% reflection or transmission. The problem statement for $\epsilon_2 = 0$ seems to be a conceptual error.

(b) Distinguish between positive and negative crystals in terms of double refraction. How are these crystals used to make quarter wave plates? Explain how the quarter wave plate is used in producing elliptically and circularly polarized light.

- **Distinction between Positive and Negative Crystals in terms of Double Refraction:**
  - Double refraction (or birefringence) is the property of certain crystals to split an unpolarized light ray into two rays upon entering the crystal. These two rays, the ordinary ray (o-ray) and the extraordinary ray (e-ray), are polarized perpendicular to each other and travel at different speeds within the crystal, leading to different refractive indices.

- The optical axis is a specific direction in the crystal where the two rays travel at the same speed (and thus have the same refractive index).
- **Positive Crystals:**
    - In positive crystals, the refractive index for the extraordinary ray (n_e) is greater than the refractive index for the ordinary ray (n_o) ($n_e > n_o$).
    - This means the ordinary ray travels faster than the extraordinary ray ($v_o > v_e$).
    - The extraordinary ray is "slow" and the ordinary ray is "fast".
    - Examples: Quartz, Ice, Tourmaline.
- **Negative Crystals:**
    - In negative crystals, the refractive index for the extraordinary ray (n_e) is less than the refractive index for the ordinary ray (n_o) ($n_e < n_o$).
    - This means the extraordinary ray travels faster than the ordinary ray ($v_e > v_o$).
    - The extraordinary ray is "fast" and the ordinary ray is "slow".
    - Examples: Calcite, Ruby.

- **How these Crystals are used to make Quarter Wave Plates:**
    - A quarter-wave plate (QWP) is an optical device that introduces a phase difference of $\pi/2$ (or 90°) between the ordinary and extraordinary components of a polarized light beam.
    - The crystal (either positive or negative) is cut such that its optical axis is parallel to the surface of the plate.
    - When linearly polarized light is incident normally on a quarter-wave plate, it is resolved into two orthogonal components: one parallel to

the optical axis (extraordinary ray) and one perpendicular to the optical axis (ordinary ray).

- Due to the difference in refractive indices (n_e and n_o), these two components travel at different speeds through the crystal.
- The thickness (t) of the QWP is chosen such that one ray lags behind the other by a quarter of a wavelength ($\lambda/4$) in terms of path difference, which corresponds to a phase difference of $\pi/2$.
- The required thickness is given by: $t = \frac{\lambda}{4|n_o - n_e|}$.
- For a positive crystal ($n_e > n_o$), the e-ray is slowed down. For a negative crystal ($n_o > n_e$), the o-ray is slowed down. The choice of positive or negative crystal doesn't fundamentally change the function of a QWP, as long as the correct thickness is used to achieve the desired phase difference.

- **How the Quarter Wave Plate is used in Producing Elliptically and Circularly Polarized Light:**
  - **Producing Circularly Polarized Light:**
    - Take a linearly polarized light beam. Orient its plane of polarization such that it makes an angle of 45° with the optical axis (or fast/slow axis) of the quarter-wave plate.
    - When this linearly polarized light enters the QWP, its electric field is resolved into two equal amplitude components: one along the fast axis and one along the slow axis.
    - As these two components travel through the QWP, a phase difference of $\pi/2$ is introduced between them.
    - When two equal amplitude waves with a phase difference of $\pi/2$ are superimposed, they result in circularly polarized light.
    - If the fast axis is at 45° to the incident polarization and the fast component lags by 90° relative to the slow component (or vice

versa, depending on the crystal), the output light will be circularly polarized (either right-handed or left-handed, depending on the relative orientation).

- **Producing Elliptically Polarized Light:**
  - If the linearly polarized light is incident on the quarter-wave plate such that its plane of polarization is *not* at 45° (i.e., at an arbitrary angle $\theta \neq 0^\circ, 45^\circ, 90^\circ$) with respect to the optical axis of the QWP.
  - In this case, the amplitudes of the two components (parallel and perpendicular to the optical axis) will not be equal.
  - The QWP still introduces a phase difference of $\pi/2$ between these two components.
  - The superposition of two orthogonal waves with unequal amplitudes and a phase difference of $\pi/2$ results in elliptically polarized light. The shape and orientation of the ellipse depend on the relative amplitudes of the components and the sign of the phase difference.
  - If the incident linearly polarized light is aligned with either the fast or slow axis (0° or 90° relative to the optical axis), no phase difference is introduced (only one component exists), and the output remains linearly polarized.

Question 6:

(a) Show that in an electrically anisotropic dielectric medium, the permittivity tensor is symmetric in nature.

- In an electrically anisotropic dielectric medium, the electric displacement field $\vec{D}$ is not necessarily parallel to the electric field $\vec{E}$. Their relationship is described by a permittivity tensor $[\epsilon]$: $D_i = \sum_j \epsilon_{ij} E_j$ In matrix form: $\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$

- To show that the permittivity tensor is symmetric (i.e., $\epsilon_{ij} = \epsilon_{ji}$), we consider the energy stored in the electric field.
- The incremental work done by the electric field when the displacement field changes by $d\vec{D}$ is given by: $dW = \vec{E} \cdot d\vec{D}$
- The total energy density (u_e) stored in the electric field when the field is established from zero to its final values is: $u_e = \int_0^D \vec{E} \cdot d\vec{D}$
- This energy density must be a unique function of the final state (i.e., independent of the path taken to reach that state). This implies that dW must be an exact differential.
- For $dW = E_x dD_x + E_y dD_y + E_z dD_z$ to be an exact differential, it must satisfy the integrability conditions (similar to partial derivatives in thermodynamics, if $Pdx + Qdy$ is exact, then $\partial P/\partial y = \partial Q/\partial x$): $\frac{\partial E_x}{\partial D_y} = \frac{\partial E_y}{\partial D_x}$ $\frac{\partial E_x}{\partial D_z} = \frac{\partial E_z}{\partial D_x}$ $\frac{\partial E_y}{\partial D_z} = \frac{\partial E_z}{\partial D_y}$
- We can express $\vec{E}$ in terms of $\vec{D}$ using the inverse of the permittivity tensor, which is the impermeability tensor $[\chi]$ (or reciprocal permittivity): $E_i = \sum_j \chi_{ij} D_j$ where $[\chi] = [\epsilon]^{-1}$.
- Then, the integrability conditions become: $\frac{\partial}{\partial D_y}(\sum_k \chi_{xk} D_k) = \frac{\partial}{\partial D_x}(\sum_k \chi_{yk} D_k)$ This simplifies to $\chi_{xy} = \chi_{yx}$. Similarly, we get $\chi_{xz} = \chi_{zx}$ and $\chi_{yz} = \chi_{zy}$.
- Therefore, the impermeability tensor $[\chi]$ is symmetric.
- Since the permittivity tensor $[\epsilon]$ is the inverse of a symmetric tensor, it must also be symmetric. Mathematically, if $[\chi]$ is symmetric, then $[\chi]^T = [\chi]$.

Since $[\epsilon][\chi] = I$ (identity matrix), then $[\chi]^T[\epsilon]^T = I$. Also, $[\chi][\epsilon]^T = I$. So $[\chi]^T[\epsilon]^T = [\chi][\epsilon]$. Since $[\chi]^T = [\chi]$, we have $[\chi][\epsilon]^T = [\chi][\epsilon]$. Multiplying by $[\chi]^{-1}$ (which is $[\epsilon]$) from the left: $[\epsilon]^T = [\epsilon]$.

- This proves that the permittivity tensor $[\epsilon]$ is symmetric. This symmetry is a consequence of the conservation of energy and the fact that the stored electric energy is a state function.

(b) Derive wave equation for E of electromagnetic wave in a symmetric planar dielectric wave guide with refractive index profile as: $n = n_1$ for $-d/2 < x < d/2$ $n = n_2$ for $|x| > d/2$ $(n_2 < n_1)$ where d is the width of the guide. Using the boundary conditions, obtain the eigenvalue equation for symmetric TE modes.

- **Wave Equation in a Dielectric Medium:**
  - Starting from Maxwell's equations in a source-free dielectric medium: $\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$ $\nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t}$
  - Take the curl of the first equation: $\nabla \times (\nabla \times \vec{E}) = -\mu \frac{\partial}{\partial t}(\nabla \times \vec{H})$
  - Substitute $\nabla \times \vec{H}$: $\nabla \times (\nabla \times \vec{E}) = -\mu \frac{\partial}{\partial t}\left(\epsilon \frac{\partial \vec{E}}{\partial t}\right)$
  - Using vector identity $\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$:
  - Since $\nabla \cdot \vec{D} = 0$ for source-free medium, and $\vec{D} = \epsilon \vec{E}$, then $\nabla \cdot (\epsilon \vec{E}) = 0$. If ϵ is constant (homogeneous regions), then $\nabla \cdot \vec{E} = 0$.
  - So, $-\nabla^2 \vec{E} = -\mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$ $\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$
  - This is the wave equation for $\vec{E}$ in a homogeneous dielectric medium. We can write $\mu\epsilon = n^2/c_0^2$, where n is the refractive index and c_0 is the speed of light in vacuum. $\nabla^2 \vec{E} - \frac{n^2}{c_0^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$

- **Symmetric Planar Dielectric Waveguide (TE Modes):**

- Consider propagation along the z-axis: $\vec{E}(x,y,z,t) = \vec{E}(x,y)e^{j(\omega t-\beta z)}$. For planar waveguide, assume no y-variation, so $\vec{E}(x,z,t) = \vec{E}(x)e^{j(\omega t-\beta z)}$.
- For TE (Transverse Electric) modes, the electric field is entirely transverse to the direction of propagation (z-axis). In a planar waveguide, this means $\vec{E} = E_y(x)\hat{y}$.
- The wave equation for E_y becomes: $\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} - \frac{n^2}{c_0^2}\frac{\partial^2 E_y}{\partial t^2} = 0$
- Substituting $E_y(x,z,t) = E_y(x)e^{j(\omega t-\beta z)}$: $\frac{d^2 E_y}{dx^2} + (-\beta^2)E_y - \frac{n^2}{c_0^2}(-\omega^2)E_y = 0$ $\frac{d^2 E_y}{dx^2} + (\frac{n^2\omega^2}{c_0^2} - \beta^2)E_y = 0$
- Let $k_0 = \omega/c_0$ be the wave number in vacuum. The term $\frac{n^2\omega^2}{c_0^2} = n^2k_0^2$. $\frac{d^2 E_y}{dx^2} + (n^2k_0^2 - \beta^2)E_y = 0$

- **Solutions in Different Regions:**
  - **Region 1 (Core):** $-d/2 < x < d/2$**,** $n = n_1$ Let $k_x^2 = n_1^2k_0^2 - \beta^2$. For guided modes, $n_1^2k_0^2 > \beta^2$. $\frac{d^2 E_y}{dx^2} + k_x^2E_y = 0$ General solution: $E_y(x) = A\cos(k_x x) + B\sin(k_x x)$. For symmetric modes, $E_y(x)$ must be an even function, so $B = 0$. $E_y(x) = A\cos(k_x x)$
  - **Region 2 (Cladding):** $|x| > d/2$**,** $n = n_2$ Let $\gamma^2 = \beta^2 - n_2^2k_0^2$. For guided modes, $\beta^2 > n_2^2k_0^2$. $\frac{d^2 E_y}{dx^2} - \gamma^2E_y = 0$ General solution: $E_y(x) = Ce^{-\gamma x} + De^{\gamma x}$. For bounded modes (fields decaying away from the core), $D = 0$ for $x > d/2$ and $C = 0$ for $x < -d/2$. $E_y(x) = Ce^{-\gamma x}$ for $x > d/2$ $E_y(x) = Ce^{\gamma x}$ for $x < -d/2$ (for symmetric modes, magnitude is same, only sign difference for x)

- **Boundary Conditions at** $x = d/2$**:**

m. **Continuity of** E_y**:** $A\cos(k_x d/2) = Ce^{-\gamma d/2}$ (Equation 1)

n. **Continuity of H_z (tangential component of $\vec{H}$):** From Maxwell's equations, $\nabla \times \vec{E} = -j\omega\mu\vec{H}$. For TE modes, E_y is the only electric field component. $(\nabla \times \vec{E})_z = \frac{\partial E_y}{\partial x} = -j\omega\mu H_z$. So $H_z = -\frac{1}{j\omega\mu}\frac{\partial E_y}{\partial x}$. The tangential component H_z must be continuous at the boundary. $\frac{\partial E_y}{\partial x}|_{x=d/2^-} = \frac{\partial E_y}{\partial x}|_{x=d/2^+}$ (assuming μ is constant across the boundary) Derivative of E_y in Region 1: $-Ak_x\sin(k_x x)$ Derivative of E_y in Region 2: $-C\gamma e^{-\gamma x}$ At $x = d/2$: $-Ak_x\sin(k_x d/2) = -C\gamma e^{-\gamma d/2}$ $Ak_x\sin(k_x d/2) = C\gamma e^{-\gamma d/2}$ (Equation 2)

- **Eigenvalue Equation for Symmetric TE Modes:** Divide Equation 2 by Equation 1: $\frac{Ak_x\sin(k_x d/2)}{A\cos(k_x d/2)} = \frac{C\gamma e^{-\gamma d/2}}{Ce^{-\gamma d/2}}$ $k_x\tan(k_x d/2) = \gamma$

  This is the eigenvalue equation for symmetric TE modes in a planar dielectric waveguide.

  - Recall $k_x = \sqrt{n_1^2 k_0^2 - \beta^2}$ and $\gamma = \sqrt{\beta^2 - n_2^2 k_0^2}$.
  - The equation $k_x\tan(k_x d/2) = \gamma$ is a transcendental equation that needs to be solved graphically or numerically to find the allowed values of β (propagation constant) for different modes.

(c) Find the state of polarization of electromagnetic wave having electric field vector: $E = 2\cos(\omega t - kz)\hat{\imath} - 2\cos(\omega t - kz - \pi/2)\hat{\jmath}$

- Given electric field vector: $\vec{E} = 2\cos(\omega t - kz)\hat{\imath} - 2\cos(\omega t - kz - \pi/2)\hat{\jmath}$
- Let $\phi = \omega t - kz$.
- The x-component is $E_x = 2\cos(\phi)$.
- The y-component is $E_y = -2\cos(\phi - \pi/2)$.
- We know that $\cos(\theta - \pi/2) = \sin(\theta)$.
- So, $E_y = -2\sin(\phi)$.
- We have: $E_x = 2\cos(\phi)$ $E_y = -2\sin(\phi)$

- From these, we can see: $E_x^2 = 4\cos^2(\phi)$ $E_y^2 = 4\sin^2(\phi)$
- Adding them: $E_x^2 + E_y^2 = 4(\cos^2(\phi) + \sin^2(\phi)) = 4$.
- This equation $E_x^2 + E_y^2 = (2)^2$ represents a circle in the $E_x E_y$ plane.
- The amplitudes of the x and y components are equal ($A_x = A_y = 2$).
- The phase difference between the two components is $\delta =$ phase of $E_y -$ phase of $E_x = (-\pi/2) - 0 = -\pi/2$. (Or, considering $\cos(\phi)$ vs $\sin(\phi)$ is a $\pi/2$ phase shift).
- Since the amplitudes are equal and the phase difference is $\pm\pi/2$, the wave is circularly polarized.
- To determine the handedness (right-handed or left-handed):
  - At $z = 0$, $t = 0$: $E_x = 2\cos(0) = 2$, $E_y = -2\sin(0) = 0$. $\vec{E} = 2\hat{\imath}$.
  - At $z = 0$, $\omega t = \pi/2$: $E_x = 2\cos(\pi/2) = 0$, $E_y = -2\sin(\pi/2) = -2$. $\vec{E} = -2\hat{\jmath}$.
  - As time increases, the electric field vector rotates from $+x$ towards $-y$.
  - For an observer looking in the direction of propagation (z-axis), if the electric field vector rotates counter-clockwise, it's right-handed (RHCP). If it rotates clockwise, it's left-handed (LHCP).
  - In this case, from $+x$ to $-y$ is a clockwise rotation.
- Therefore, the state of polarization is **Left-Handed Circularly Polarized (LHCP)**.