1. (a) Find the dot product, cross product and angle between the vector $\vec{a} = \hat{\imath} + 5\widehat{4\jmath} - 2\hat{k}$ and $\vec{b} = 5\hat{\imath} - \hat{\jmath} + 3\hat{k}$

   - The given vectors are $\vec{a} = \hat{\imath} + 4j - 2\hat{k}$ and $\vec{b} = 5\hat{\imath} - \hat{\jmath} + 3\hat{k}$.
   - Dot product: $\vec{a} \cdot \vec{b} = (1)(5) + (4)(-1) + (-2)(3) = 5 - 4 - 6 = -5$
   - Cross product: $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 4 & -2 \\ 5 & -1 & 3 \end{vmatrix} = \hat{\imath}((4)(3) - (-2)(-1)) - \hat{\jmath}((1)(3) - (-2)(5)) + \hat{k}((1)(-1) - (4)(5)) = \hat{\imath}(12 - 2) - \hat{\jmath}(3 + 10) + \hat{k}(-1 - 20) = 10\hat{\imath} - 13\hat{\jmath} - 21\hat{k}$
   - Angle between the vectors: $cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$ $|\vec{a}| = \sqrt{1^2 + 4^2 + (-2)^2} = \sqrt{1 + 16 + 4} = \sqrt{21}$ $|\vec{b}| = \sqrt{5^2 + (-1)^2 + 3^2} = \sqrt{25 + 1 + 9} = \sqrt{35}$ $cos\theta = \frac{-5}{\sqrt{21}\sqrt{35}} = \frac{-5}{\sqrt{735}}$ $\theta = arccos(\frac{-5}{\sqrt{735}})$

2. (b) Is $Q(x) = 6x_1^2 + 3x_2^2 + x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3$ is positive definite ?

   - The quadratic form is $Q(x) = 6x_1^2 + 3x_2^2 + x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3$.
   - The symmetric matrix A associated with this quadratic form is: $A = \begin{bmatrix} 6 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
   - To check if it's positive definite, we evaluate the leading principal minors.
   - $D_1 = |6| = 6 > 0$
   - $D_2 = \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} = (6)(3) - (2)(2) = 18 - 4 = 14 > 0$

- $D_3 = \begin{vmatrix} 6 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 6(3-1) - 2(2-1) + 1(2-3) = 6(2) - 2(1) + 1(-1) = 12 - 2 - 1 = 9 > 0$
- Since all leading principal minors are positive, the quadratic form $Q(x)$ is positive definite.

3. (c) Define bases of vector space. Check whether $A =$„ is a bases of vector space R^3 or not?
   - **Definition of Bases of Vector Space:** A set of vectors $B = \{v_1, v_2, \ldots, v_n\}$ in a vector space V is called a basis for V if it satisfies two conditions:
     - The set B is linearly independent.
     - The set B spans V (i.e., every vector in V can be written as a linear combination of the vectors in B).
   - **Check if $A = \{[1,0,0], [0,1,0], [0,0,1]\}$ is a basis of R^3:** (The question seems to have a typo, assuming the set is the standard basis vectors for R^3: $A = \{[1,0,0], [0,1,0], [0,0,1]\}$)
   - **Linear Independence:** Consider the equation $c_1[1,0,0] + c_2[0,1,0] + c_3[0,0,1] = [0,0,0]$. This simplifies to $[c_1, c_2, c_3] = [0,0,0]$. Therefore, $c_1 = 0, c_2 = 0, c_3 = 0$. Since the only solution is the trivial solution, the vectors are linearly independent.
   - **Spanning R^3:** Let $[x, y, z]$ be an arbitrary vector in R^3. We need to find scalars c_1, c_2, c_3 such that: $c_1[1,0,0] + c_2[0,1,0] + c_3[0,0,1] = [x, y, z]$ This gives $[c_1, c_2, c_3] = [x, y, z]$. So, $c_1 = x, c_2 = y, c_3 = z$. Since any vector in R^3 can be expressed as a linear combination of the vectors in A, the set A spans R^3.
   - Since the set A is both linearly independent and spans R^3, it is a basis for the vector space R^3.
4. (d) Find Rank of the following matrix using reduced row echelon form

$[3 \quad 0 \quad 2 \quad 2]\ [-6 \quad 42 \quad 24 \quad 54]$

- The given matrix is: $M = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \end{bmatrix}$
- Apply row operations to get it into row echelon form.
- $R_2 \rightarrow R_2 + 2R_1$: $M = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \end{bmatrix}$
- $R_1 \rightarrow \frac{1}{3}R_1$: $M = \begin{bmatrix} 1 & 0 & 2/3 & 2/3 \\ 0 & 42 & 28 & 58 \end{bmatrix}$
- $R_2 \rightarrow \frac{1}{42}R_2$: $M = \begin{bmatrix} 1 & 0 & 2/3 & 2/3 \\ 0 & 1 & 28/42 & 58/42 \end{bmatrix}$ $M = \begin{bmatrix} 1 & 0 & 2/3 & 2/3 \\ 0 & 1 & 2/3 & 29/21 \end{bmatrix}$
- This is the reduced row echelon form. The number of non-zero rows is 2.
- Therefore, the rank of the matrix is 2.

5. (e) Determine whether $f: R^3 \rightarrow R^3$ such that $f([x_1, x_2, x_3]) = [x_2, x_3, x_1]$ is linear transformation or not.
   - To check if f is a linear transformation, we need to verify two conditions:
     i. Additivity: $f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v})$ for any vectors $\vec{u}, \vec{v} \in R^3$.
     ii. Homogeneity (Scalar Multiplication): $f(c\vec{u}) = cf(\vec{u})$ for any scalar c and vector $\vec{u} \in R^3$.
   - Let $\vec{u} = [x_1, x_2, x_3]$ and $\vec{v} = [y_1, y_2, y_3]$.
   - **Condition 1: Additivity** $\vec{u} + \vec{v} = [x_1 + y_1, x_2 + y_2, x_3 + y_3]$ $f(\vec{u} + \vec{v}) = f([x_1 + y_1, x_2 + y_2, x_3 + y_3]) = [x_2 + y_2, x_3 + y_3, x_1 + y_1]$ $f(\vec{u}) = [x_2, x_3, x_1]$ $f(\vec{v}) = [y_2, y_3, y_1]$ $f(\vec{u}) + f(\vec{v}) = [x_2 + y_2, x_3 + y_3, x_1 + y_1]$ Since $f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v})$, the additivity condition is satisfied.

- **Condition 2: Homogeneity (Scalar Multiplication)** $c\vec{u} = [cx_1, cx_2, cx_3]$ $f(c\vec{u}) = f([cx_1, cx_2, cx_3]) = [cx_2, cx_3, cx_1]$ $cf(\vec{u}) = c[x_2, x_3, x_1] = [cx_2, cx_3, cx_1]$ Since $f(c\vec{u}) = cf(\vec{u})$, the homogeneity condition is satisfied.
- Since both conditions are satisfied, $f([x_1, x_2, x_3]) = [x_2, x_3, x_1]$ is a linear transformation.

6. (f) Find the directional derivative of $F(x, y, z) = 4x^2 + y^2 + 3z^2$ at P(3,2,4) in the direction $5\hat{\imath} + 6\hat{k}$.
   - The function is $F(x, y, z) = 4x^2 + y^2 + 3z^2$.
   - The point is $P(3,2,4)$.
   - The direction vector is $\vec{v} = 5\hat{\imath} + 6\hat{k}$.
   - First, find the gradient of F: $\nabla F = \frac{\partial F}{\partial x}\hat{\imath} + \frac{\partial F}{\partial y}\hat{\jmath} + \frac{\partial F}{\partial z}\hat{k}$ $\frac{\partial F}{\partial x} = 8x$ $\frac{\partial F}{\partial y} = 2y$ $\frac{\partial F}{\partial z} = 6z$ $\nabla F = 8x\hat{\imath} + 2y\hat{\jmath} + 6z\hat{k}$
   - Evaluate the gradient at point $P(3,2,4)$: $\nabla F(3,2,4) = 8(3)\hat{\imath} + 2(2)\hat{\jmath} + 6(4)\hat{k} = 24\hat{\imath} + 4\hat{\jmath} + 24\hat{k}$
   - Find the unit vector in the direction of $\vec{v}$: $|\vec{v}| = \sqrt{5^2 + 0^2 + 6^2} = \sqrt{25 + 36} = \sqrt{61}$ $\hat{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{5\hat{\imath}+6\hat{k}}{\sqrt{61}}$
   - The directional derivative is $D_{\hat{u}}F = \nabla F \cdot \hat{u}$: $D_{\hat{u}}F = (24\hat{\imath} + 4\hat{\jmath} + 24\hat{k}) \cdot \left(\frac{5\hat{\imath}+6\hat{k}}{\sqrt{61}}\right) = \frac{(24)(5)+(4)(0)+(24)(6)}{\sqrt{61}} = \frac{120+0+144}{\sqrt{61}} = \frac{264}{\sqrt{61}}$

7. (a) The set R^2 defined with the addition operation $[x, y] \oplus [w, z] = [x + w - 2, y + z + 3]$ and scalar multiplication $a \odot [x, y] = [ax - 2a + 2, ay + 3a - 3]$. Show that R^2 is a vector space over addition and scalar multiplication.
   - To show R^2 is a vector space, we need to verify the 10 axioms: Let $\vec{u} = [x_1, y_1]$, $\vec{v} = [x_2, y_2]$, $\vec{w} = [x_3, y_3]$ be vectors in R^2, and a, b be scalars.

- **Axioms for Addition:**
  i. **Closure under Addition:** $\vec{u} \oplus \vec{v} = [x_1 + x_2 - 2, y_1 + y_2 + 3]$ which is in R^2. (Closed)

  ii. **Commutativity of Addition:** $\vec{u} \oplus \vec{v} = [x_1 + x_2 - 2, y_1 + y_2 + 3]$ $\vec{v} \oplus \vec{u} = [x_2 + x_1 - 2, y_2 + y_1 + 3]$ Since $x_1 + x_2 = x_2 + x_1$ and $y_1 + y_2 = y_2 + y_1$, $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$. (Commutative)

  iii. **Associativity of Addition:** $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = ([x_1 + x_2 - 2, y_1 + y_2 + 3]) \oplus [x_3, y_3] = [(x_1 + x_2 - 2) + x_3 - 2, (y_1 + y_2 + 3) + y_3 + 3] = [x_1 + x_2 + x_3 - 4, y_1 + y_2 + y_3 + 6]$ $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = [x_1, y_1] \oplus ([x_2 + x_3 - 2, y_2 + y_3 + 3]) = [x_1 + (x_2 + x_3 - 2) - 2, y_1 + (y_2 + y_3 + 3) + 3] = [x_1 + x_2 + x_3 - 4, y_1 + y_2 + y_3 + 6]$ So, $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$. (Associative)

  iv. **Existence of Zero Vector:** Let $\vec{0} = [0_x, 0_y]$ be the zero vector. $\vec{u} \oplus \vec{0} = [x_1 + 0_x - 2, y_1 + 0_y + 3] = [x_1, y_1]$ So, $x_1 + 0_x - 2 = x_1 \Rightarrow 0_x = 2$ And $y_1 + 0_y + 3 = y_1 \Rightarrow 0_y = -3$ Thus, the zero vector is $[2, -3]$. Check: $[x_1, y_1] \oplus [2, -3] = [x_1 + 2 - 2, y_1 - 3 + 3] = [x_1, y_1]$. (Exists)

  v. **Existence of Additive Inverse:** Let $-\vec{u} = [-x_{1'}, -y_{1'}]$ be the additive inverse of $\vec{u} = [x_1, y_1]$. $\vec{u} \oplus (-\vec{u}) = [x_1 + x_{1'} - 2, y_1 + y_{1'} + 3] = [2, -3]$ (the zero vector) $x_1 + x_{1'} - 2 = 2 \Rightarrow x_{1'} = 4 - x_1$ $y_1 + y_{1'} + 3 = -3 \Rightarrow y_{1'} = -6 - y_1$ So, $-\vec{u} = [4 - x_1, -6 - y_1]$. (Exists)

- **Axioms for Scalar Multiplication:** 6. **Closure under Scalar Multiplication:** $a \odot \vec{u} = [ax_1 - 2a + 2, ay_1 + 3a - 3]$ which is in R^2. (Closed) 7. **Distributivity over Vector Addition:** $a \odot (\vec{u} \oplus \vec{v}) = a \odot ([x_1 + x_2 - 2, y_1 + y_2 + 3]) = [a(x_1 + x_2 - 2) - 2a + 2, a(y_1 + y_2 + 3) + 3a - 3] = [ax_1 + ax_2 - 2a - 2a + 2, ay_1 + ay_2 + 3a + 3a - 3] = [ax_1 + ax_2 - 4a + 2, ay_1 + ay_2 + 6a - 3]$

$a \odot \vec{u} \oplus a \odot \vec{v} = [ax_1 - 2a + 2, ay_1 + 3a - 3] \oplus [ax_2 - 2a + 2, ay_2 + 3a - 3] = [(ax_1 - 2a + 2) + (ax_2 - 2a + 2) - 2, (ay_1 + 3a - 3) + (ay_2 + 3a - 3) + 3] = [ax_1 + ax_2 - 4a + 4 - 2, ay_1 + ay_2 + 6a - 6 + 3] = [ax_1 + ax_2 - 4a + 2, ay_1 + ay_2 + 6a - 3]$ So, $a \odot (\vec{u} \oplus \vec{v}) = a \odot \vec{u} \oplus a \odot \vec{v}$. (Distributive over vector addition) 8. **Distributivity over Scalar Addition:** $(a + b) \odot \vec{u} = [(a + b)x_1 - 2(a + b) + 2, (a + b)y_1 + 3(a + b) - 3] = [ax_1 + bx_1 - 2a - 2b + 2, ay_1 + by_1 + 3a + 3b - 3]$ $a \odot \vec{u} \oplus b \odot \vec{u} = [ax_1 - 2a + 2, ay_1 + 3a - 3] \oplus [bx_1 - 2b + 2, by_1 + 3b - 3] = [(ax_1 - 2a + 2) + (bx_1 - 2b + 2) - 2, (ay_1 + 3a - 3) + (by_1 + 3b - 3) + 3] = [ax_1 + bx_1 - 2a - 2b + 4 - 2, ay_1 + by_1 + 3a + 3b - 6 + 3] = [ax_1 + bx_1 - 2a - 2b + 2, ay_1 + by_1 + 3a + 3b - 3]$ So, $(a + b) \odot \vec{u} = a \odot \vec{u} \oplus b \odot \vec{u}$. (Distributive over scalar addition) 9. **Associativity of Scalar Multiplication:** $(ab) \odot \vec{u} = [(ab)x_1 - 2(ab) + 2, (ab)y_1 + 3(ab) - 3]$ $a \odot (b \odot \vec{u}) = a \odot [bx_1 - 2b + 2, by_1 + 3b - 3] = [a(bx_1 - 2b + 2) - 2a + 2, a(by_1 + 3b - 3) + 3a - 3] = [abx_1 - 2ab + 2a - 2a + 2, aby_1 + 3ab - 3a + 3a - 3] = [abx_1 - 2ab + 2, aby_1 + 3ab - 3]$ So, $(ab) \odot \vec{u} = a \odot (b \odot \vec{u})$. (Associative) 10. **Identity Element for Scalar Multiplication:** $1 \odot \vec{u} = [1x_1 - 2(1) + 2, 1y_1 + 3(1) - 3] = [x_1 - 2 + 2, y_1 + 3 - 3] = [x_1, y_1] = \vec{u}$ So, $1 \odot \vec{u} = \vec{u}$. (Identity exists)

- Since all 10 axioms are satisfied, R^2 with the given operations is a vector space.

8. (b) Define inner product space. Consider a real vector space R^2, which is defined as $< x, y >= x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2$. Show that it is inner product space.

   - **Definition of Inner Product Space:** An inner product space is a vector space V over the field of real numbers or complex numbers, equipped with an inner product, which is a function that associates a scalar with each pair of vectors in V, denoted by $< u, v >$, satisfying the following axioms for all vectors $u, v, w \in V$ and scalar c:

i. **Conjugate Symmetry (Symmetry for Real Vector Spaces):** $< u, v >= \overline{< v, u >}$. For real vector spaces, this simplifies to $< u, v >=< v, u >$.

ii. **Linearity in the First Argument (Additivity):** $< u + v, w >=< u, w > +< v, w >$.

iii. **Linearity in the First Argument (Homogeneity):** $< cu, v >= c < u, v >$.

iv. **Positive-Definiteness:** $< u, u >\geq 0$, and $< u, u >= 0$ if and only if $u = 0$.

- **Show that** $< x, y >= x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2$ **is an inner product space for** R^2**:** Let $x = [x_1, x_2]$, $y = [y_1, y_2]$, $z = [z_1, z_2]$ be vectors in R^2, and c be a scalar.

v. **Symmetry:** $< x, y >= x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2 < y, x >= y_1x_1 - y_1x_2 - y_2x_1 + 2y_2x_2$ Since multiplication of real numbers is commutative, $x_1y_1 = y_1x_1$, $x_1y_2 = y_2x_1$, $x_2y_1 = y_1x_2$, $x_2y_2 = y_2x_2$. Therefore, $< x, y >=< y, x >$. (Symmetric)

vi. **Linearity in the First Argument (Additivity):** Let $x = [x_1, x_2]$ and $y = [y_1, y_2]$, $z = [z_1, z_2]$. $< x + y, z >=< [x_1 + y_1, x_2 + y_2], [z_1, z_2] > = (x_1 + y_1)z_1 - (x_1 + y_1)z_2 - (x_2 + y_2)z_1 + 2(x_2 + y_2)z_2 = x_1z_1 + y_1z_1 - x_1z_2 - y_1z_2 - x_2z_1 - y_2z_1 + 2x_2z_2 + 2y_2z_2 < x, z > +< y, z >= (x_1z_1 - x_1z_2 - x_2z_1 + 2x_2z_2) + (y_1z_1 - y_1z_2 - y_2z_1 + 2y_2z_2) = x_1z_1 - x_1z_2 - x_2z_1 + 2x_2z_2 + y_1z_1 - y_1z_2 - y_2z_1 + 2y_2z_2$ Thus, $< x + y, z >=< x, z > +< y, z >$. (Additive)

vii. **Linearity in the First Argument (Homogeneity):** $< cx, y >=< [cx_1, cx_2], [y_1, y_2] > = (cx_1)y_1 - (cx_1)y_2 -$

$(cx_2)y_1 + 2(cx_2)y_2 = c(x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2) = c < x, y >$ Thus, $< cx, y >= c < x, y >$. (Homogeneous)

viii. **Positive-Definiteness:** $< x, x >= x_1^2 - x_1x_2 - x_2x_1 + 2x_2^2 = x_1^2 - 2x_1x_2 + 2x_2^2$ We can rewrite this by completing the square: $= (x_1^2 - 2x_1x_2 + x_2^2) + x_2^2 = (x_1 - x_2)^2 + x_2^2$ Since squares of real numbers are non-negative, $(x_1 - x_2)^2 \geq 0$ and $x_2^2 \geq 0$. Therefore, $< x, x >= (x_1 - x_2)^2 + x_2^2 \geq 0$. Now, we need to check when $< x, x >= 0$. $(x_1 - x_2)^2 + x_2^2 = 0$ This implies $(x_1 - x_2)^2 = 0$ AND $x_2^2 = 0$. From $x_2^2 = 0$, we get $x_2 = 0$. Substitute $x_2 = 0$ into $(x_1 - x_2)^2 = 0$, we get $(x_1 - 0)^2 = 0 \Longrightarrow x_1^2 = 0 \Longrightarrow x_1 = 0$. So, $< x, x >= 0$ if and only if $x_1 = 0$ and $x_2 = 0$, which means $x = [0,0]$ (the zero vector). (Positive-definite)

- Since all four axioms are satisfied, the given function defines an inner product on R^2, and thus R^2 with this inner product is an inner product space.

9. (a) Solve using the Gauss Jordan Method

$2x_1 + x_2 + 3x_3 = 16$ $3x_1 + 2x_2 + x_4 = 16$ $2x_1 + 12x_2 - 5x_4 = 5$

- The system of equations is: $2x_1 + x_2 + 3x_3 + 0x_4 = 16$ $3x_1 + 2x_2 + 0x_3 + x_4 = 16$ $2x_1 + 12x_2 + 0x_3 - 5x_4 = 5$
- Write the augmented matrix: $\begin{bmatrix} 2 & 1 & 3 & 0 & | & 16 \\ 3 & 2 & 0 & 1 & | & 16 \\ 2 & 12 & 0 & -5 & | & 5 \end{bmatrix}$
- $R_1 \rightarrow \frac{1}{2}R_1$: $\begin{bmatrix} 1 & 1/2 & 3/2 & 0 & | & 8 \\ 3 & 2 & 0 & 1 & | & 16 \\ 2 & 12 & 0 & -5 & | & 5 \end{bmatrix}$

- $R_2 \rightarrow R_2 - 3R_1$ and $R_3 \rightarrow R_3 - 2R_1$:
$$\begin{bmatrix} 1 & 1/2 & 3/2 & 0 & | & 8 \\ 0 & 1/2 & -9/2 & 1 & | & -8 \\ 0 & 11 & -3 & -5 & | & -11 \end{bmatrix}$$

- $R_2 \rightarrow 2R_2$: $\begin{bmatrix} 1 & 1/2 & 3/2 & 0 & | & 8 \\ 0 & 1 & -9 & 2 & | & -16 \\ 0 & 11 & -3 & -5 & | & -11 \end{bmatrix}$

- $R_1 \rightarrow R_1 - \frac{1}{2}R_2$ and $R_3 \rightarrow R_3 - 11R_2$:
$$\begin{bmatrix} 1 & 0 & 3/2 - (1/2)(-9) & 0 - (1/2)(2) & | & 8 - (1/2)(-16) \\ 0 & 1 & -9 & 2 & | & -16 \\ 0 & 0 & -3 - 11(-9) & -5 - 11(2) & | & -11 - 11(-16) \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 3/2 + 9/2 & -1 & | & 8 + 8 \\ 0 & 1 & -9 & 2 & | & -16 \\ 0 & 0 & -3 + 99 & -5 - 22 & | & -11 + 176 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 12/2 & -1 & | & 16 \\ 0 & 1 & -9 & 2 & | & -16 \\ 0 & 0 & 96 & -27 & | & 165 \end{bmatrix}\begin{bmatrix} 1 & 0 & 6 & -1 & | & 16 \\ 0 & 1 & -9 & 2 & | & -16 \\ 0 & 0 & 96 & -27 & | & 165 \end{bmatrix}$$

- $R_3 \rightarrow \frac{1}{96}R_3$: $\begin{bmatrix} 1 & 0 & 6 & -1 & | & 16 \\ 0 & 1 & -9 & 2 & | & -16 \\ 0 & 0 & 1 & -27/96 & | & 165/96 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 6 & -1 & | & 16 \\ 0 & 1 & -9 & 2 & | & -16 \\ 0 & 0 & 1 & -9/32 & | & 55/32 \end{bmatrix}$ (simplified $27/96$ to $9/32$ and $165/96$ to $55/32$ by dividing by 3)

- $R_1 \rightarrow R_1 - 6R_3$ and $R_2 \rightarrow R_2 + 9R_3$:
$$\begin{bmatrix} 1 & 0 & 0 & -1 - 6(-9/32) & | & 16 - 6(55/32) \\ 0 & 1 & 0 & 2 + 9(-9/32) & | & -16 + 9(55/32) \\ 0 & 0 & 1 & -9/32 & | & 55/32 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & -1 + 54/32 & | & 16 - 330/32 \\ 0 & 1 & 0 & 2 - 81/32 & | & -16 + 495/32 \\ 0 & 0 & 1 & -9/32 & | & 55/32 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & (-32+54)/32 & | & (512-330)/32 \\ 0 & 1 & 0 & (64-81)/32 & | & (-512+495)/32 \\ 0 & 0 & 1 & -9/32 & | & 55/32 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 22/32 & | & 182/32 \\ 0 & 1 & 0 & -17/32 & | & -17/32 \\ 0 & 0 & 1 & -9/32 & | & 55/32 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 11/16 & | & 91/16 \\ 0 & 1 & 0 & -17/32 & | & -17/32 \\ 0 & 0 & 1 & -9/32 & | & 55/32 \end{bmatrix}$$

- This is the reduced row echelon form.
- The system has 4 variables and 3 equations, so there will be one free variable. Let $x_4 = t$. $x_1 + (11/16)x_4 = 91/16 \Rightarrow x_1 = 91/16 - (11/16)t$ $x_2 - (17/32)x_4 = -17/32 \Rightarrow x_2 = -17/32 + (17/32)t$ $x_3 - (9/32)x_4 = 55/32 \Rightarrow x_3 = 55/32 + (9/32)t$
- The solution is: $x_1 = \frac{91-11t}{16}$ $x_2 = \frac{-17+17t}{32}$ $x_3 = \frac{55+9t}{32}$ $x_4 = t$ where t is any real number.

10. (b) Find the bases of row space and null space of the following matrix

$[1 \quad 0 \quad 1]$ $[2 \quad 1 \quad 1]$ $[-1 \quad 1 \quad -2]$

- The given matrix is: $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$
- **Row Space:** First, reduce the matrix to row echelon form: $R_2 \to R_2 - 2R_1$: $R_3 \to R_3 + R_1$: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ $R_3 \to R_3 - R_2$: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ The non-zero rows in the row echelon form are $[1,0,1]$ and $[0,1,-1]$. These vectors form a basis for the row space. Basis for Row Space: $\{[1,0,1], [0,1,-1]\}$

- **Null Space:** To find the null space, we solve $Ax = 0$:
  $\begin{bmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ From the reduced row echelon form, we have: $x_1 + x_3 = 0 \Longrightarrow x_1 = -x_3$ $x_2 - x_3 = 0 \Longrightarrow x_2 = x_3$ Let $x_3 = t$ (free variable). Then $x_1 = -t$ and $x_2 = t$. The solution vector x can be written as: $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-t\\t\\t\end{bmatrix} = t\begin{bmatrix}-1\\1\\1\end{bmatrix}$ The vector $[-1,1,1]$ forms a basis for the null space. Basis for Null Space: $\{[-1,1,1]\}$

11. (a) Solve the following set of equations using Gauss Elimination method.

$5x - 5y - 15z = 40$ $4x - 2y - 6z = 19$ $3x - 6y - 17z = 41$

- Write the augmented matrix: $\begin{bmatrix}5 & -5 & -15 & | & 40\\4 & -2 & -6 & | & 19\\3 & -6 & -17 & | & 41\end{bmatrix}$

- $R_1 \to \frac{1}{5}R_1$: $\begin{bmatrix}1 & -1 & -3 & | & 8\\4 & -2 & -6 & | & 19\\3 & -6 & -17 & | & 41\end{bmatrix}$

- $R_2 \to R_2 - 4R_1$ and $R_3 \to R_3 - 3R_1$:
  $\begin{bmatrix}1 & -1 & -3 & | & 8\\0 & -2-4(-1) & -6-4(-3) & | & 19-4(8)\\0 & -6-3(-1) & -17-3(-3) & | & 41-3(8)\end{bmatrix}$
  $\begin{bmatrix}1 & -1 & -3 & | & 8\\0 & 2 & 6 & | & -13\\0 & -3 & -8 & | & 17\end{bmatrix}$

- $R_2 \to \frac{1}{2}R_2$: $\begin{bmatrix}1 & -1 & -3 & | & 8\\0 & 1 & 3 & | & -13/2\\0 & -3 & -8 & | & 17\end{bmatrix}$

- $R_3 \to R_3 + 3R_2$: $\begin{bmatrix} 1 & -1 & -3 & | & 8 \\ 0 & 1 & 3 & | & -13/2 \\ 0 & 0 & -8+3(3) & | & 17+3(-13/2) \end{bmatrix}$

  $\begin{bmatrix} 1 & -1 & -3 & | & 8 \\ 0 & 1 & 3 & | & -13/2 \\ 0 & 0 & -8+9 & | & 17-39/2 \end{bmatrix}$

  $\begin{bmatrix} 1 & -1 & -3 & | & 8 \\ 0 & 1 & 3 & | & -13/2 \\ 0 & 0 & 1 & | & (34-39)/2 \end{bmatrix} \begin{bmatrix} 1 & -1 & -3 & | & 8 \\ 0 & 1 & 3 & | & -13/2 \\ 0 & 0 & 1 & | & -5/2 \end{bmatrix}$

- This is the row echelon form. Now, perform back substitution. From the third row: $z = -5/2$ From the second row: $y + 3z = -13/2$ $y + 3(-5/2) = -13/2$ $y - 15/2 = -13/2$ $y = -13/2 + 15/2 = 2/2 = 1$ From the first row: $x - y - 3z = 8$ $x - 1 - 3(-5/2) = 8$ $x - 1 + 15/2 = 8$ $x + (-2 + 15)/2 = 8$ $x + 13/2 = 8$ $x = 8 - 13/2 = (16 - 13)/2 = 3/2$

- The solution is $x = 3/2$, $y = 1$, $z = -5/2$.

12. (b) Find value(s) of λ for which following system of equations is consistent.

$2x + 3y = 4$ $x + y + z = 4$ $x + 2y - z = \lambda$

- Write the augmented matrix: $\begin{bmatrix} 2 & 3 & 0 & | & 4 \\ 1 & 1 & 1 & | & 4 \\ 1 & 2 & -1 & | & \lambda \end{bmatrix}$

- Swap R_1 and R_2 to get a leading 1: $\begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 2 & 3 & 0 & | & 4 \\ 1 & 2 & -1 & | & \lambda \end{bmatrix}$

- $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$:

  $\begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 0 & 3-2(1) & 0-2(1) & | & 4-2(4) \\ 0 & 2-1(1) & -1-1(1) & | & \lambda-1(4) \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 0 & 1 & -2 & | & -4 \\ 0 & 1 & -2 & | & \lambda-4 \end{bmatrix}$

- $R_3 \to R_3 - R_2$: $\begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 0 & 1 & -2 & | & -4 \\ 0 & 0 & 0 & | & (\lambda - 4) - (-4) \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 0 & 1 & -2 & | & -4 \\ 0 & 0 & 0 & | & \lambda \end{bmatrix}$

- For the system to be consistent, the last row must represent a true statement. This means the last element in the augmented part must be 0.
- Therefore, $\lambda = 0$.
- The system is consistent only when $\lambda = 0$.

13. (a) Diagonalize the following matrix

$[-4 \quad 8 \quad -12] \; [6 \quad -6 \quad 12] \; [6 \quad -8 \quad 14]$

- Let the matrix be $A = \begin{bmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{bmatrix}$.
- **Step 1: Find the eigenvalues.** Characteristic equation: $det(A - \lambda I) = 0 \begin{vmatrix} -4-\lambda & 8 & -12 \\ 6 & -6-\lambda & 12 \\ 6 & -8 & 14-\lambda \end{vmatrix} = 0$ Let's try to simplify the determinant calculation. $R_2 \to R_2 + R_3$: $\begin{vmatrix} -4-\lambda & 8 & -12 \\ 12 & -14-\lambda & 26 \\ 6 & -8 & 14-\lambda \end{vmatrix} = 0$ (This does not simplify well)

  Let's calculate the determinant directly: $(-4-\lambda)[(-6-\lambda)(14-\lambda) - (12)(-8)] - 8[6(14-\lambda) - 12(6)] - 12[6(-8) - (-6-\lambda)(6)] = 0$ $(-4-\lambda)[-84 + 6\lambda - 14\lambda + \lambda^2 + 96] - 8[84 - 6\lambda - 72] - 12[-48 - 36 - 6\lambda] = 0$ $(-4-\lambda)[\lambda^2 - 8\lambda + 12] - 8[12 - 6\lambda] - 12[-84 - 6\lambda] = 0$ $(-4-\lambda)(\lambda-2)(\lambda-6) - 96 + 48\lambda + 1008 + 72\lambda = 0$ $-(\lambda+4)(\lambda-2)(\lambda-6) + 120\lambda + 912 = 0$ $-(\lambda+4)(\lambda^2 - 8\lambda + 12) + 120\lambda + 912 = 0$ $-(\lambda^3 - 8\lambda^2 + 12\lambda + 4\lambda^2 - 32\lambda + 48) + 120\lambda + 912 = 0$ $-(\lambda^3 - 4\lambda^2 - 20\lambda + 48) +$

$120\lambda + 912 = 0$ $-\lambda^3 + 4\lambda^2 + 20\lambda - 48 + 120\lambda + 912 = 0$ $-\lambda^3 + 4\lambda^2 + 140\lambda + 864 = 0$ $\lambda^3 - 4\lambda^2 - 140\lambda - 864 = 0$

Let's check for integer roots that divide 864. This is quite tedious. A quicker way might be to look for simple eigenvalues. Sum of rows: Row 1: $-4 + 8 - 12 = -8$ Row 2: $6 - 6 + 12 = 12$ Row 3: $6 - 8 + 14 = 12$ This doesn't immediately suggest an eigenvalue.

Let's reconsider the determinant calculation from a simpler step. $\begin{vmatrix} -4-\lambda & 8 & -12 \\ 6 & -6-\lambda & 12 \\ 6 & -8 & 14-\lambda \end{vmatrix}$ Try $C_2 \rightarrow C_2 + C_1$, $C_3 \rightarrow C_3 + C_1$ (doesn't always help) Consider if $\lambda = 2$ is an eigenvalue. $A - 2I = \begin{bmatrix} -6 & 8 & -12 \\ 6 & -8 & 12 \\ 6 & -8 & 12 \end{bmatrix}$ Rows are linearly dependent (Row 2 = -Row 1, Row 3 = -Row 1). So $\lambda = 2$ is an eigenvalue. For $\lambda = 2$: $\begin{bmatrix} -6 & 8 & -12 \\ 6 & -8 & 12 \\ 6 & -8 & 12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $-6x_1 + 8x_2 - 12x_3 = 0$ Divide by -2: $3x_1 - 4x_2 + 6x_3 = 0$ Let $x_2 = s$ and $x_3 = t$. $3x_1 = 4s - 6t \Rightarrow x_1 = \frac{4}{3}s - 2t$ Eigenvector: $x = \begin{bmatrix} (4/3)s - 2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 4/3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ Let's choose integer vectors: For $s = 3, t = 0 \Rightarrow v_1 = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$ For $s = 0, t = 1 \Rightarrow v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ So, $\lambda = 2$ has geometric multiplicity 2.

Since the sum of columns gives a multiple of a vector, let's try $C_1 + C_2 + C_3$: $-4 + 8 - 12 = -8$ $6 - 6 + 12 = 12$ $6 - 8 + 14 = 12$ This doesn't seem to simplify finding other eigenvalues.

Let's re-calculate the characteristic polynomial carefully. $\lambda^3 - (tr(A))\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - det(A) = 0$ $tr(A) = -4 - 6 + 14 = 4$ $M_{11} = (-6)(14) - (12)(-8) = -84 + 96 = 12$ $M_{22} =$

$(-4)(14)-(-12)(6)=-56+72=16$ $M_{33}=(-4)(-6)-(8)(6)=24-48=-24$ Sum of principal minors $=12+16-24=4$ $det(A)=-4(-6\cdot 14-12\cdot(-8))-8(6\cdot 14-12\cdot 6)-12(6\cdot(-8)-(-6)\cdot 6)=-4(-84+96)-8(84-72)-12(-48+36)=-4(12)-8(12)-12(-12)=-48-96+144=0$ Since $det(A)=0$, $\lambda=0$ is an eigenvalue.

So the characteristic equation is $\lambda^3-4\lambda^2+4\lambda-0=0$ $\lambda(\lambda^2-4\lambda+4)=0$ $\lambda(\lambda-2)^2=0$ The eigenvalues are $\lambda_1=0$, $\lambda_2=2$ (with multiplicity 2). This matches our finding that $\lambda=2$ is an eigenvalue with geometric multiplicity 2.

- **Step 2: Find the eigenvectors for each eigenvalue.**
  - **For $\lambda=0$:** $(A-0I)x=0 \implies Ax=0$ $\begin{bmatrix}-4 & 8 & -12\\ 6 & -6 & 12\\ 6 & -8 & 14\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ $R_1\to -\frac{1}{4}R_1$: $\begin{bmatrix}1 & -2 & 3\\ 6 & -6 & 12\\ 6 & -8 & 14\end{bmatrix}$ $R_2\to R_2-6R_1$, $R_3\to R_3-6R_1$: $\begin{bmatrix}1 & -2 & 3\\ 0 & 6 & -6\\ 0 & 4 & -4\end{bmatrix}$ $R_2\to\frac{1}{6}R_2$: $\begin{bmatrix}1 & -2 & 3\\ 0 & 1 & -1\\ 0 & 4 & -4\end{bmatrix}$ $R_3\to R_3-4R_2$: $\begin{bmatrix}1 & -2 & 3\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$ $R_1\to R_1+2R_2$: $\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$ From this, $x_1+x_3=0\implies x_1=-x_3$ $x_2-x_3=0\implies x_2=x_3$ Let $x_3=t$. Then $x_1=-t, x_2=t$.
    Eigenvector for $\lambda=0$: $v_3=\begin{bmatrix}-1\\1\\1\end{bmatrix}$ (taking $t=1$)
  - **For $\lambda=2$:** (already found eigenvectors above) $v_1=\begin{bmatrix}4\\3\\0\end{bmatrix}$ and $v_2=\begin{bmatrix}-2\\0\\1\end{bmatrix}$

- **Step 3: Form the diagonalizing matrix P and the diagonal matrix D.** The matrix P is formed by the eigenvectors as columns: $P = \begin{bmatrix} 4 & -2 & -1 \\ 3 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ The diagonal matrix D has the eigenvalues on its diagonal, in the same order as their corresponding eigenvectors in P: $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- The matrix A is diagonalizable because the algebraic multiplicity of each eigenvalue equals its geometric multiplicity (multiplicity of 2 for $\lambda = 2$, and multiplicity of 1 for $\lambda = 0$).
- The diagonalization is $A = PDP^{-1}$.

14. (b) Define Cayley-Hamilton theorem and verify it for the following matrix

$[7 \quad 1 \quad -1]\ [-11 \quad -3 \quad 2]\ [18 \quad 2 \quad -4]$

- **Cayley-Hamilton Theorem Definition:** The Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation. If A is an $n \times n$ matrix and its characteristic polynomial is $p(\lambda) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$, then $p(A) = c_nA^n + c_{n-1}A^{n-1} + \cdots + c_1A + c_0I = 0$, where I is the $n \times n$ identity matrix and 0 is the $n \times n$ zero matrix.
- **Verification for the given matrix:** Let $A = \begin{bmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{bmatrix}$.

  Step 1: Find the characteristic polynomial $p(\lambda) = det(A - \lambda I)$**.** $A - \lambda I = \begin{bmatrix} 7-\lambda & 1 & -1 \\ -11 & -3-\lambda & 2 \\ 18 & 2 & -4-\lambda \end{bmatrix}$ $p(\lambda) = (7-\lambda)[(-3-\lambda)(-4-\lambda) - (2)(2)] - 1[(-11)(-4-\lambda) - (2)(18)] + (-1)[(-11)(2) - (-3-\lambda)(18)] = (7-\lambda)[(12 + 3\lambda + 4\lambda + \lambda^2) - 4] - 1[44 + 11\lambda - 36] - 1[-22 - 54 - 18\lambda] = (7-\lambda)[\lambda^2 + 7\lambda +$

$8] - [8 + 11\lambda] - 1[-76 - 18\lambda] = (7\lambda^2 + 49\lambda + 56 - \lambda^3 - 7\lambda^2 - 8\lambda) - 8 - 11\lambda + 76 + 18\lambda = -\lambda^3 + (7\lambda^2 - 7\lambda^2) + (49\lambda - 8\lambda - 11\lambda + 18\lambda) + (56 - 8 + 76) = -\lambda^3 + (49 - 8 - 11 + 18)\lambda + (56 - 8 + 76) = -\lambda^3 + 48\lambda + 124$ So, the characteristic polynomial is $p(\lambda) = -\lambda^3 + 48\lambda + 124$.

Step 2: Verify $p(A) = -A^3 + 48A + 124I = 0$. We need to calculate A^2 and A^3. $A^2 = A \cdot A =$

$$\begin{bmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{bmatrix}\begin{bmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{bmatrix} A^2 =$$

$$\begin{bmatrix} (49 - 11 - 18) & (7 - 3 - 2) & (-7 + 2 + 4) \\ (-77 + 33 + 36) & (-11 + 9 + 4) & (11 - 6 - 8) \\ (126 - 22 - 72) & (18 - 6 - 8) & (-18 + 4 + 16) \end{bmatrix} =$$

$$\begin{bmatrix} 20 & 2 & -1 \\ -8 & 2 & -3 \\ 32 & 4 & 2 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 20 & 2 & -1 \\ -8 & 2 & -3 \\ 32 & 4 & 2 \end{bmatrix}\begin{bmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{bmatrix} A^3 =$$

$$\begin{bmatrix} (140 - 22 - 18) & (20 - 6 - 2) & (-20 + 4 + 4) \\ (-56 - 22 - 54) & (-8 - 6 - 6) & (8 + 4 + 12) \\ (224 - 44 + 36) & (32 - 12 + 4) & (-32 + 8 - 8) \end{bmatrix} =$$

$$\begin{bmatrix} 100 & 12 & -12 \\ -132 & -20 & 24 \\ 216 & 24 & -32 \end{bmatrix}$$

Now substitute into $p(A) = -A^3 + 48A + 124I$: $-A^3 =$

$$\begin{bmatrix} -100 & -12 & 12 \\ 132 & 20 & -24 \\ -216 & -24 & 32 \end{bmatrix} 48A = 48\begin{bmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{bmatrix} =$$

$$\begin{bmatrix} 336 & 48 & -48 \\ -528 & -144 & 96 \\ 864 & 96 & -192 \end{bmatrix} 124I = 124\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 124 & 0 & 0 \\ 0 & 124 & 0 \\ 0 & 0 & 124 \end{bmatrix}$$

$-A^3 + 48A + 124I =$
$\begin{bmatrix} -100+336+124 & -12+48+0 & 12-48+0 \\ 132-528+0 & 20-144+124 & -24+96+0 \\ -216+864+0 & -24+96+0 & 32-192+124 \end{bmatrix} =$
$\begin{bmatrix} 360 & 36 & -36 \\ -396 & 0 & 72 \\ 648 & 72 & -36 \end{bmatrix}$

There is a calculation error somewhere, as this matrix is not the zero matrix. Let me recheck the determinant or A^2 or A^3.

Recheck $p(\lambda)$ calculation: $p(\lambda) = det(A - \lambda I) = (7-\lambda)[(-3-\lambda)(-4-\lambda)-4] - 1[-11(-4-\lambda)-36] - 1[-22-18(-3-\lambda)] = (7-\lambda)[\lambda^2+7\lambda+12-4] - [44+11\lambda-36] - [-22+54+18\lambda] = (7-\lambda)[\lambda^2+7\lambda+8] - [11\lambda+8] - [18\lambda+32] = 7\lambda^2+49\lambda+56-\lambda^3-7\lambda^2-8\lambda-11\lambda-8-18\lambda-32 = -\lambda^3+(49-8-11-18)\lambda+(56-8-32) = -\lambda^3+12\lambda+16$ So, $p(\lambda) = -\lambda^3+12\lambda+16$. Now we need to verify $-A^3+12A+16I=0$.

$12A = 12\begin{bmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 84 & 12 & -12 \\ -132 & -36 & 24 \\ 216 & 24 & -48 \end{bmatrix}$ $16I =$
$\begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix}$

$-A^3 + 12A + 16I = \begin{bmatrix} -100 & -12 & 12 \\ 132 & 20 & -24 \\ -216 & -24 & 32 \end{bmatrix} +$
$\begin{bmatrix} 84 & 12 & -12 \\ -132 & -36 & 24 \\ 216 & 24 & -48 \end{bmatrix} + \begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix} =$
$\begin{bmatrix} -100+84+16 & -12+12+0 & 12-12+0 \\ 132-132+0 & 20-36+16 & -24+24+0 \\ -216+216+0 & -24+24+0 & 32-48+16 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
The verification holds.

15. (a) Apply Gram Schmidt orthonormalization process to obtain an orthonormal bases for given bases of R^3: $[1,0,-1], [-1,4,-1]$,

(The problem states "given bases of R^3: $[1,0,-1], [-1,4,-1]$,", which appears to be missing a third vector. Assuming it meant to ask for orthonormalization of the *given vectors* as part of a basis, and implying they are linearly independent. For a basis of R^3, we need 3 linearly independent vectors. Let's assume the question meant to give three vectors or asks to orthonormalize the given two vectors within their span. If it implies a full basis of R^3, a third vector is needed.)

Let's assume the provided vectors are $v_1 = [1,0,-1]$ and $v_2 = [-1,4,-1]$. We will find an orthonormal basis for the subspace spanned by these two vectors.

- **Step 1: Normalize the first vector to get u_1.** $u_1 = \frac{v_1}{||v_1||}$ $||v_1|| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{1+0+1} = \sqrt{2}$ $u_1 = \frac{1}{\sqrt{2}}[1,0,-1] = [\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}]$
- **Step 2: Find $v_{2'}$ orthogonal to u_1.** $v_{2'} = v_2 - proj_{u_1} v_2 = v_2 - <v_2, u_1> u_1$ $<v_2, u_1> = (-1)(\frac{1}{\sqrt{2}}) + (4)(0) + (-1)(-\frac{1}{\sqrt{2}}) = -\frac{1}{\sqrt{2}} + 0 + \frac{1}{\sqrt{2}} = 0$ Since $<v_2, u_1> = 0$, v_2 is already orthogonal to u_1. This means $v_{2'} = v_2 = [-1,4,-1]$. (This suggests that the original vectors v_1 and v_2 are already orthogonal. Let's check: $v_1 \cdot v_2 = (1)(-1) + (0)(4) + (-1)(-1) = -1 + 0 + 1 = 0$. Yes, they are orthogonal.)
- **Step 3: Normalize $v_{2'}$ to get u_2.** $u_2 = \frac{v_{2'}}{||v_{2'}||}$ $||v_{2'}|| = ||[-1,4,-1]|| = \sqrt{(-1)^2 + 4^2 + (-1)^2} = \sqrt{1+16+1} = \sqrt{18} = 3\sqrt{2}$ $u_2 = \frac{1}{3\sqrt{2}}[-1,4,-1] = [-\frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}]$ Or $u_2 = [-\frac{\sqrt{2}}{6}, \frac{4\sqrt{2}}{6}, -\frac{\sqrt{2}}{6}] = [-\frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}, -\frac{\sqrt{2}}{6}]$
- The orthonormal basis for the subspace spanned by the given two vectors is: $\{[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}], [-\frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, -\frac{1}{3\sqrt{2}}]\}$

If the intention was to find an orthonormal basis for R^3, we would need a third linearly independent vector, say v_3, that is not in the span of v_1 and v_2. Then we would apply the Gram-Schmidt process for v_3 after obtaining u_1 and u_2.

Assuming the question meant to provide only two basis vectors for a 2-dimensional subspace: The orthonormal basis is $\{u_1, u_2\}$ as calculated above.

16. (b) Find inverse of the following matrix using row echelon form.

$[2 \quad -6 \quad 5]\ [-4 \quad 12 \quad -9]\ [2 \quad -9 \quad 8]$

- Let the matrix be $A = \begin{bmatrix} 2 & -6 & 5 \\ -4 & 12 & -9 \\ 2 & -9 & 8 \end{bmatrix}$.
- Augment the matrix with the identity matrix: $[A|I]$
$$\begin{bmatrix} 2 & -6 & 5 & | & 1 & 0 & 0 \\ -4 & 12 & -9 & | & 0 & 1 & 0 \\ 2 & -9 & 8 & | & 0 & 0 & 1 \end{bmatrix}$$
- $R_1 \to \frac{1}{2}R_1$: $\begin{bmatrix} 1 & -3 & 5/2 & | & 1/2 & 0 & 0 \\ -4 & 12 & -9 & | & 0 & 1 & 0 \\ 2 & -9 & 8 & | & 0 & 0 & 1 \end{bmatrix}$
- $R_2 \to R_2 + 4R_1$ and $R_3 \to R_3 - 2R_1$:
$$\begin{bmatrix} 1 & -3 & 5/2 & | & 1/2 & 0 & 0 \\ 0 & 0 & -9+4(5/2) & | & 0+4(1/2) & 1 & 0 \\ 0 & -9-2(-3) & 8-2(5/2) & | & 0-2(1/2) & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -3 & 5/2 & | & 1/2 & 0 & 0 \\ 0 & 0 & -9+10 & | & 2 & 1 & 0 \\ 0 & -3 & 8-5 & | & -1 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -3 & 5/2 & | & 1/2 & 0 & 0 \\ 0 & 0 & 1 & | & 2 & 1 & 0 \\ 0 & -3 & 3 & | & -1 & 0 & 1 \end{bmatrix}$$
- Swap R_2 and R_3: $\begin{bmatrix} 1 & -3 & 5/2 & | & 1/2 & 0 & 0 \\ 0 & -3 & 3 & | & -1 & 0 & 1 \\ 0 & 0 & 1 & | & 2 & 1 & 0 \end{bmatrix}$

- $R_2 \to -\frac{1}{3}R_2$: $\begin{bmatrix} 1 & -3 & 5/2 & | & 1/2 & 0 & 0 \\ 0 & 1 & -1 & | & 1/3 & 0 & -1/3 \\ 0 & 0 & 1 & | & 2 & 1 & 0 \end{bmatrix}$

- $R_1 \to R_1 - \frac{5}{2}R_3$ and $R_2 \to R_2 + R_3$:

$$\begin{bmatrix} 1 & -3 & 0 & | & 1/2 - 5/2(2) & 0 - 5/2(1) & 0 \\ 0 & 1 & 0 & | & 1/3 + 2 & 0 + 1 & -1/3 \\ 0 & 0 & 1 & | & 2 & 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -3 & 0 & | & 1/2 - 5 & -5/2 & 0 \\ 0 & 1 & 0 & | & 7/3 & 1 & -1/3 \\ 0 & 0 & 1 & | & 2 & 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -3 & 0 & | & -9/2 & -5/2 & 0 \\ 0 & 1 & 0 & | & 7/3 & 1 & -1/3 \\ 0 & 0 & 1 & | & 2 & 1 & 0 \end{bmatrix}$$

- $R_1 \to R_1 + 3R_2$:

$$\begin{bmatrix} 1 & 0 & 0 & | & -9/2 + 3(7/3) & -5/2 + 3(1) & 0 + 3(-1/3) \\ 0 & 1 & 0 & | & 7/3 & 1 & -1/3 \\ 0 & 0 & 1 & | & 2 & 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & | & -9/2 + 7 & -5/2 + 3 & -1 \\ 0 & 1 & 0 & | & 7/3 & 1 & -1/3 \\ 0 & 0 & 1 & | & 2 & 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & | & (-9 + 14)/2 & (-5 + 6)/2 & -1 \\ 0 & 1 & 0 & | & 7/3 & 1 & -1/3 \\ 0 & 0 & 1 & | & 2 & 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & | & 5/2 & 1/2 & -1 \\ 0 & 1 & 0 & | & 7/3 & 1 & -1/3 \\ 0 & 0 & 1 & | & 2 & 1 & 0 \end{bmatrix}$$

- The inverse matrix is: $A^{-1} = \begin{bmatrix} 5/2 & 1/2 & -1 \\ 7/3 & 1 & -1/3 \\ 2 & 1 & 0 \end{bmatrix}$

17. (a) Calculate grad(div(curl $\vec{F}$)) of the following vector field $\vec{F} = x^3y^3z\hat{\imath} + x^2y^3z^4\hat{\jmath} + xyz\hat{k}$

- The vector field is $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$, where $P = x^3y^3z$ $Q = x^2y^3z^4$ $R = xyz$

- **Step 1: Calculate curl $\vec{F}$** $curl\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3y^3z & x^2y^3z^4 & xyz \end{vmatrix} = \hat{\imath}\left(\frac{\partial}{\partial y}(xyz) - \frac{\partial}{\partial z}(x^2y^3z^4)\right) - \hat{\jmath}\left(\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial z}(x^3y^3z)\right) + \hat{k}\left(\frac{\partial}{\partial x}(x^2y^3z^4) - \frac{\partial}{\partial y}(x^3y^3z)\right) = \hat{\imath}(xz - 4x^2y^3z^3) - \hat{\jmath}(yz - x^3y^3) + \hat{k}(2xy^3z^4 - 3x^3y^2z)$ $curl\vec{F} = (xz - 4x^2y^3z^3)\hat{\imath} + (x^3y^3 - yz)\hat{\jmath} + (2xy^3z^4 - 3x^3y^2z)\hat{k}$
- **Step 2: Calculate div(curl $\vec{F}$)** Recall the vector identity: $div(curl\vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = 0$. Let's verify this by calculating it directly. Let $curl\vec{F} = M\hat{\imath} + N\hat{\jmath} + O\hat{k}$, where $M = xz - 4x^2y^3z^3$ $N = x^3y^3 - yz$ $O = 2xy^3z^4 - 3x^3y^2z$ $div(curl\vec{F}) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial O}{\partial z}$ $\frac{\partial M}{\partial x} = z - 8xy^3z^3$ $\frac{\partial N}{\partial y} = 3x^3y^2 - z$ $\frac{\partial O}{\partial z} = 8xy^3z^3 - 3x^3y^2$ $div(curl\vec{F}) = (z - 8xy^3z^3) + (3x^3y^2 - z) + (8xy^3z^3 - 3x^3y^2) = z - 8xy^3z^3 + 3x^3y^2 - z + 8xy^3z^3 - 3x^3y^2 = 0$ So, $div(curl\vec{F}) = 0$.
- **Step 3: Calculate grad(div(curl $\vec{F}$))** Since $div(curl\vec{F}) = 0$, which is a scalar function. $grad(0) = \nabla(0) = \frac{\partial}{\partial x}(0)\hat{\imath} + \frac{\partial}{\partial y}(0)\hat{\jmath} + \frac{\partial}{\partial z}(0)\hat{k} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} = \vec{0}$
- Therefore, grad(div(curl $\vec{F}$)) = $\vec{0}$.

18. (b) A weather model uses a Markov chain to predict daily weather based on the states Sunny (S), Rainy (R) and Cloudy (C) with transition matrix

$[0.7 \quad 0.2 \quad 0.1]$ $[0.3 \quad 0.4 \quad 0.3]$ $[0.2 \quad 0.3 \quad 0.5]$ S R C

(i) If today is Sunny, what is the probability that it will be Cloudy tomorrow? (ii) If today is Rainy, what is the probability that it will be Sunny after two days? (iii) If the initial state vector is:

$[1]$ $[0]$ $[0]$ What is the state probability vector after 2 days?

- The transition matrix P is given as: $P = \begin{bmatrix} P_{SS} & P_{SR} & P_{SC} \\ P_{RS} & P_{RR} & P_{RC} \\ P_{CS} & P_{CR} & P_{CC} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$ (Rows represent 'from' state, columns represent 'to' state)
- (i) If today is Sunny, what is the probability that it will be Cloudy tomorrow?
  - This is the probability of transitioning from Sunny (S) to Cloudy (C) in one step.
  - This corresponds to the element P_{SC} in the transition matrix.
  - $P_{SC} = 0.1$
  - The probability that it will be Cloudy tomorrow if today is Sunny is 0.1.
- (ii) If today is Rainy, what is the probability that it will be Sunny after two days?
  - We need to find the 2-step transition matrix P^2.
  - $P^2 = P \cdot P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$
  - Calculate the element for going from Rainy (R) to Sunny (S) in two days, which is P_{RS}^2 (second row, first column).
  - $P_{RS}^2 = (0.3)(0.7) + (0.4)(0.3) + (0.3)(0.2) = 0.21 + 0.12 + 0.06 = 0.39$
  - The probability that it will be Sunny after two days if today is Rainy is 0.39.

- (iii) If the initial state vector is: [1] [0] [0] What is the state probability vector after 2 days?
  - The initial state vector $s_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ means today is Sunny (100% Sunny, 0% Rainy, 0% Cloudy).
  - The state probability vector after 2 days is $s_2 = P^2 s_0$.
  - We already calculated P^2 in part (ii). $P^2 =$

    $$\begin{bmatrix} (0.7)(0.7) + (0.2)(0.3) + (0.1)(0.2) & (0.7)(0.2) + (0.2)(0.4) + (0.1)(0. \\ (0.3)(0.7) + (0.4)(0.3) + (0.3)(0.2) & (0.3)(0.2) + (0.4)(0.4) + (0.3)(0. \\ (0.2)(0.7) + (0.3)(0.3) + (0.5)(0.2) & (0.2)(0.2) + (0.3)(0.4) + (0.5)(0. \end{bmatrix}$$

    $P^2 =$

    $$\begin{bmatrix} 0.49 + 0.06 + 0.02 & 0.14 + 0.08 + 0.03 & 0.07 + 0.06 + 0.05 \\ 0.21 + 0.12 + 0.06 & 0.06 + 0.16 + 0.09 & 0.03 + 0.12 + 0.15 \\ 0.14 + 0.09 + 0.10 & 0.04 + 0.12 + 0.15 & 0.02 + 0.09 + 0.25 \end{bmatrix}$$

    $$P^2 = \begin{bmatrix} 0.57 & 0.25 & 0.18 \\ 0.39 & 0.31 & 0.30 \\ 0.33 & 0.31 & 0.36 \end{bmatrix}$$

  - Now, calculate $s_2 = P^2 s_0$: $s_2 = \begin{bmatrix} 0.57 & 0.25 & 0.18 \\ 0.39 & 0.31 & 0.30 \\ 0.33 & 0.31 & 0.36 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} =$

    $$\begin{bmatrix} (0.57)(1) + (0.25)(0) + (0.18)(0) \\ (0.39)(1) + (0.31)(0) + (0.30)(0) \\ (0.33)(1) + (0.31)(0) + (0.36)(0) \end{bmatrix} = \begin{bmatrix} 0.57 \\ 0.39 \\ 0.33 \end{bmatrix}$$

  - The state probability vector after 2 days is $\begin{bmatrix} 0.57 \\ 0.39 \\ 0.33 \end{bmatrix}$, meaning there is a 57% chance of Sunny, 39% chance of Rainy, and 33% chance of Cloudy after 2 days.