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S. No. of Question Paper : 5574

Unique Paper Code : 2352013603

Name of the Paper : Complex Analysis

Name of the Course : B.Sc. (Hons.) Mathematics (NEP-UGCF 2022)

Semester : VI

Duration : 3 Hours

Maximum Marks : 90

(Write your Roll No. on the top immediately on receipt of this question paper.)

All questions are compulsory and carry equal marks.

Each question contains three parts (a), (b) and (c), attempt only two parts.

1. (a) (i) Sketch the set $S = \left\{ z \in \mathbf{C} : |z| \leq 1, -\frac{\pi}{2} \leq \arg z \leq \frac{\pi}{2} \right\}$. 3

(ii) Under the map $f(z) = z^2$, draw the image of the set : 4.5

$$S = \left\{ z \in \mathbf{C} : |z| = 1, -\frac{\pi}{4} \leq \arg z \leq \frac{\pi}{4} \right\}.$$

(b) Under the map $f(z) = e^z$ draw the image of the rectangular region

$$R = \left\{ z = x + iy \in \mathbf{C} : 1 \leq x \leq 2, -\frac{\pi}{4} \leq y \leq \frac{\pi}{2} \right\}. 7.5$$

P.T.O.

(c) Does $\lim_{z \rightarrow 0} \left(\frac{z}{\bar{z}} \right)^2$ exist ? Justify your answer. 7.5

2. (a) Show that the function $f(z) = \bar{z}$ is nowhere differentiable on \mathbf{C} . 7.5

(b) (i) Giving proper justification, write two differences between the functions : 3.5

$$f(x) = e^x, x \in \mathbf{R} \text{ and } f(z) = e^z, z \in \mathbf{C}$$

(ii) Solve the equation $e^z = -2$. 4

(c) Find out the values of $\sin^{-1} 2$. 7.5

3. (a) (i) Let a function f be integrable along a contour C . Prove that : 3.5

$$\int_{-C} f(z) dz = - \int_C f(z) dz.$$

(ii) Define a simple arc. Give an example to explain that a same set of points may represent different arcs. 4

(b) Let C be any simple closed contour, described in the positive sense in the z plane, then compute the value of the integral :

$$\int_C \frac{s^3 + 2s}{(s - z_0)^3} ds,$$

(i) when z_0 is inside C , and

(ii) when z_0 is outside C . 7.5

(c) (i) Let $w(t)$ be a piecewise continuous complex valued function defined on an interval $a \leq t \leq b$, then show that : 3.5

$$\left| \int_a^b w(t) dt \right| \leq \int_a^b |w(t)| dt.$$

(ii) Let C be the arc of the circle $|z| = 2$ such that $0 \leq \arg z \leq \frac{\pi}{2}$.

Show that :

4

$$\left| \int_C \frac{z+4}{z^3-1} dz \right| \leq \frac{6\pi}{7}.$$

4. (a) State and prove Cauchy integral formula. 7.5

(b) Let f be an entire function such that $|f(z)| \leq A|z|$ for all z , where A is a fixed positive number. Show that $f(z) = bz$, where b is a complex constant. 7.5

(c) (i) Evaluate the following integral along the line segment, say L , from 0 to $\pi + 2i$: 3.5

$$\int_L \cos(z/2) dz.$$

(ii) Evaluate the following integral :

$$\int_C \frac{z dz}{(9-z^2)(z+i)},$$

where C is the positively oriented circle $|z| = 2$.

4

5. (a) State Laurent's series theorem. Give two series expansions in powers of z for the function $f(z) = \frac{z+1}{z-1}$ specifying the regions in which those expansions are valid. 7.5

(b) If a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges when $z = z_1$ ($z_1 \neq z_0$), then prove that it is absolutely convergent at each point z in the open disk $|z-z_0| < R_1$ where $R_1 = |z_1 - z_0|$. 7.5

P.T.O.

- (c) Define residue. Evaluate the residue of the function $f(z) = \frac{e^z - 1}{z^4}$ and use it to find the integral $\int_C \frac{e^z - 1}{z^4} dz$, where C is the positively oriented unit circle $|z| = 1$. 7.5

6. (a) Define an isolated singular point. For the below given functions, find the isolated singular point and determine whether it is a removable singular point, an essential singular point, or a pole : 7.5

(i) $\frac{\sin z}{z}$ and

(ii) $\frac{e^{2z}}{(z-1)^2}$.

- (b) Let z_0 be an isolated singular point of a function f . Then show that the following two statements are equivalent :

(i) z_0 is a pole of order m ($m = 1, 2, \dots$) of f ;

(ii) $f(z)$ can be written in the form $f(z) = \frac{\varphi(z)}{(z-z_0)^m}$, where $\varphi(z)$ is analytic

and non-zero at z_0 .

Further, establish that

$$\operatorname{Res}_{z=z_0} f(z) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!} \text{ when } m = 1, 2, 3, \dots$$

where $\varphi^{(0)} \equiv \varphi$.

7.5

- (c) Using residues, evaluate the following integral :

7.5

$$\int_0^{2\pi} \frac{d\theta}{5 + 4 \sin \theta} d\theta.$$