Question 1: (a) Suppose $N_0 = N \cup \{0\} = \{0,1,2,.....\}$. Define $m \leq n$ in N_0, if and only if there exists $k \in N_0$, such that $n = km$. Prove that $(N_0, \leq)$ is a partially ordered set. Is $(N_0, \leq)$ a chain, an antichain or none? Justify your answer.

- To prove that $(N_0, \leq)$ is a partially ordered set, we need to show that the relation $\leq$ is reflexive, antisymmetric, and transitive.
  - Reflexive: For any $m \in N_0$, we need to show that $m \leq m$. This means there exists $k \in N_0$ such that $m = km$. If $m \neq 0$, then $k = 1$, which is in N_0. If $m = 0$, then $0 = k \cdot 0$ is true for any $k \in N_0$. So, $0 \leq 0$. Thus, $m \leq m$ for all $m \in N_0$.
  - Antisymmetric: For any $m, n \in N_0$, if $m \leq n$ and $n \leq m$, we need to show that $m = n$.
    - If $m \leq n$, then $n = k_1 m$ for some $k_1 \in N_0$.
    - If $n \leq m$, then $m = k_2 n$ for some $k_2 \in N_0$.
    - Substituting the second equation into the first, we get $n = k_1(k_2 n)$.
    - If $n \neq 0$, then $1 = k_1 k_2$. Since $k_1, k_2 \in N_0$, the only possibility is $k_1 = 1$ and $k_2 = 1$. This implies $n = m$.
    - If $n = 0$, then from $n = k_1 m$, we have $0 = k_1 m$. If $m \neq 0$, then $k_1 = 0$. If $m = 0$, then $n = m = 0$.
    - From $m = k_2 n$, we have $m = k_2 \cdot 0 = 0$. So if $n = 0$, then $m = 0$.
    - Therefore, in all cases, if $m \leq n$ and $n \leq m$, then $m = n$.
  - Transitive: For any $l, m, n \in N_0$, if $l \leq m$ and $m \leq n$, we need to show that $l \leq n$.
    - If $l \leq m$, then $m = k_1 l$ for some $k_1 \in N_0$.
    - If $m \leq n$, then $n = k_2 m$ for some $k_2 \in N_0$.

- Substituting the first equation into the second, we get $n = k_2(k_1 l) = (k_2 k_1)l$.
- Since $k_1, k_2 \in N_0$, their product $k_2 k_1$ is also in N_0. Let $k = k_2 k_1$. Then $n = kl$, so $l \leq n$.

- Therefore, $(N_0, \leq)$ is a partially ordered set.
- Is $(N_0, \leq)$ a chain, an antichain or none? Justify your answer.
  - It is neither a chain nor an antichain.
  - Justification:
    - It is not a chain because not every pair of elements is comparable. For example, consider 2 and 3. $2 \nleq 3$ since there is no $k \in N_0$ such that $3 = 2k$. Also, $3 \nleq 2$ since there is no $k \in N_0$ such that $2 = 3k$. Therefore, 2 and 3 are incomparable.
    - It is not an antichain because there exist comparable elements. For example, $2 \leq 4$ because $4 = 2 \cdot 2$, and $2 \in N_0$. Therefore, not all distinct pairs are incomparable.

(b) Define when we say that the two sets have the same cardinality. Prove that the sets (0,1) and (a, ∞) have the same cardinality.

- Definition of same cardinality: Two sets A and B are said to have the same cardinality if there exists a bijection (one-to-one and onto function) from set A to set B.
- Prove that the sets (0,1) and (a, ∞) have the same cardinality.
  - We need to find a bijection $f: (0,1) \rightarrow (a, \infty)$.
  - Consider the function $f(x) = a + \frac{1}{x} - 1$.
    - Let's check if this function works. As x approaches 0 from the right, $\frac{1}{x}$ approaches ∞, so $f(x)$ approaches ∞. As x

approaches 1 from the left, $\frac{1}{x}$ approaches 1, so $f(x)$ approaches $a+1-1=a$. This suggests a different transformation might be needed for a simple bijection.

- Let's try a different approach. We can construct a function that maps the interval $(0,1)$ to (a,∞).
- Consider the function $f(x)=a+\frac{1-x}{x}$.
  - As $x\to 0^{+}$, $\frac{1-x}{x}\to\frac{1}{0^{+}}\to\infty$, so $f(x)\to a+\infty=\infty$.
  - As $x\to 1^{-}$, $\frac{1-x}{x}\to\frac{0}{1}\to 0$, so $f(x)\to a+0=a$.
  - So, this function maps $(0,1)$ to (a,∞). Now, we need to show it's a bijection.
  - Injectivity (One-to-one): Assume $f(x_1)=f(x_2)$. $a+\frac{1-x_1}{x_1}=a+\frac{1-x_2}{x_2}\,\frac{1-x_1}{x_1}=\frac{1-x_2}{x_2}\,\frac{1}{x_1}-1=\frac{1}{x_2}-1\,\frac{1}{x_1}=\frac{1}{x_2}\,x_1=x_2$. Thus, f is injective.
  - Surjectivity (Onto): Let $y\in(a,\infty)$. We need to find $x\in(0,1)$ such that $f(x)=y$. $y=a+\frac{1-x}{x}\,y-a=\frac{1-x}{x}\,(y-a)x=1-x\,(y-a)x+x=1\,x(y-a+1)=1\,x=\frac{1}{y-a+1}$. Since $y\in(a,\infty)$, we have $y>a$, so $y-a>0$. Therefore, $y-a+1>1$. This implies $0<\frac{1}{y-a+1}<1$. So $x\in(0,1)$. Thus, f is surjective.
- Since f is both injective and surjective, it is a bijection. Therefore, the sets $(0,1)$ and (a,∞) have the same cardinality.

(c) Draw the Hasse diagrams for the following ordered sets: (i) (P(X), ⊆), with X = {1,2,3}. Here P(X) is the power set of X.

- P(X) = { ∅, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3} }

- The Hasse diagram will have 8 nodes.
- Level 0: ∅
- Level 1: {1}, {2}, {3} (each is a subset of {1,2,3})
- Level 2: {1,2}, {1,3}, {2,3}
- Level 3: {1,2,3}
- Lines will connect sets where one is a direct subset of another (i.e., difference in size is 1).
    - ∅ is connected to {1}, {2}, {3}.
    - {1} is connected to {1,2}, {1,3}.
    - {2} is connected to {1,2}, {2,3}.
    - {3} is connected to {1,3}, {2,3}.
    - {1,2} is connected to {1,2,3}.
    - {1,3} is connected to {1,2,3}.
    - {2,3} is connected to {1,2,3}.

(ii) Dual of $\oplus M_3$, where $M_3 = 1 \oplus \bar{n} \oplus 1$. (Assuming n is a single element from context, otherwise it would be a chain of n elements. Given M_3 is a specific lattice, n likely represents a particular element for the structure of M_3.)

- The notation M_3 is commonly associated with the "diamond lattice". If n refers to a chain, the expression $1 \oplus \bar{n} \oplus 1$ implies a construction. Let's interpret M_3 as the standard diamond lattice. The standard M_3 is a modular lattice with 5 elements. Let the elements be $0, a, b, c, 1$.
    - $0 < a, 0 < b, 0 < c$
    - $a < 1, b < 1, c < 1$
    - No other comparability relations between a, b, c.

- Hasse diagram of M_3:
  - Top element: 1
  - Middle elements: a, b, c (incomparable to each other)
  - Bottom element: 0
  - Lines: 0 to a, 0 to b, 0 to c; a to 1, b to 1, c to 1.
- Dual of M_3: The dual lattice has the same elements but the order relations are reversed.
  - New bottom element: 1
  - New middle elements: a, b, c (still incomparable to each other)
  - New top element: 0
  - Lines: 1 to a, 1 to b, 1 to c; a to 0, b to 0, c to 0.

(iii) 2 x 3

- This notation usually refers to the direct product of two chains. So, it's $C_2 \times C_3$.
- Let $C_2 = \{0,1\}$ with $0 < 1$.
- Let $C_3 = \{0,1,2\}$ with $0 < 1 < 2$.
- The elements of 2×3 are pairs (x, y) where $x \in C_2$ and $y \in C_3$.
  - Elements: (0,0), (0,1), (0,2), (1,0), (1,1), (1,2)
- The order relation is $(x_1, y_1) \leq (x_2, y_2)$ if $x_1 \leq x_2$ in C_2 AND $y_1 \leq y_2$ in C_3.
- Hasse diagram for 2×3:
  - (0,0) is the bottom element.
  - Elements directly above (0,0): (0,1) and (1,0).

- Elements directly above (0,1): (0,2) and (1,1).
- Elements directly above (1,0): (1,1).
- Elements directly above (0,2): None in this lattice unless there are elements with a 0 in the first component and something larger than 2 in the second, which there are not.
- Elements directly above (1,1): (1,2).
- (1,2) is the top element.

- Connections:
  - (0,0) ≤ (0,1)
  - (0,0) ≤ (1,0)
  - (0,1) ≤ (0,2)
  - (0,1) ≤ (1,1)
  - (1,0) ≤ (1,1)
  - (0,2) ≤ (1,2)
  - (1,1) ≤ (1,2)

Question 2: (a) Define maximal element of an ordered set. Give an example of an ordered set which has exactly one maximal element but does not have a greatest (or maximum) element. Give one example of an ordered set with exactly 3 maximal elements.

- Definition of maximal element: In an ordered set $(P, \leq)$, an element $m \in P$ is called a maximal element if there is no element $x \in P$, $x \neq m$, such that $m \leq x$. (i.e., no element is strictly greater than m).
- Example of an ordered set with exactly one maximal element but no greatest element:
  - Let $P = \{(0, n) \mid n \in N\} \cup \{(1,0)\}$.

- Define the order $(x_1, y_1) \leq (x_2, y_2)$ if $x_1 = x_2$ and $y_1 \leq y_2$ (standard order on natural numbers), OR if $x_1 = 0$ and $x_2 = 1$.
- Alternatively, consider the set $P = \{(n, 0) \mid n \in \mathbb{N}\} \cup \{(0,0)\}$. Define $(a, b) \leq (c, d)$ if $a = c$ and $b = d$, or if $a = 0$ and $b = 0$. This is not a partial order.
- Let's use a standard example for this: Consider the set $S = \{a, b, c, d, e\}$ with the following Hasse diagram description:
  - $a < b$
  - $a < c$
  - $a < d$
  - $e < b$
  - $e < c$
  - $e < d$
  - No other comparabilities, and b, c, d are mutually incomparable.
  - In this set, b, c, d are maximal elements.
- Let's refine the example to have exactly one maximal element but no greatest.
- Consider the set $A = \{a_n \mid n \in \mathbb{N}\} \cup \{b\}$ with the order relation: $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$, and $a_n \leq b$ for all $n \in \mathbb{N}$.
  - This creates a chain $a_1 < a_2 < a_3 < \cdots$ and an element b which is an upper bound for all a_n.
  - In this set, b is the greatest element. So this is not the example.
- Consider the set $P = \mathbb{Z}^- \cup \{0\}$ with the usual order $\leq$. So $\ldots < -3 < -2 < -1 < 0$.

- The maximal element is 0.
- There is no greatest element, because for any $x \in P$, $x - 1$ is also in P and $x - 1 < x$. This example is wrong. 0 is the greatest element here.

- Let $P = \{a_i \mid i \in \mathbb{N}\} \cup \{b_i \mid i \in \mathbb{N}\}$ with the order $a_i < a_{i+1}$ and $b_i < b_{i+1}$ for all i. Also, a_i is incomparable to b_j for all i, j.
  - This set has no maximal elements.
- A good example: Let $S = \{x \in \mathbb{R} \mid 0 < x < 1\} \cup \{2\}$.
  - Define the order as the usual order on $\mathbb{R}$ within $(0,1)$, and $x < 2$ for all $x \in (0,1)$.
  - The only maximal element is 2.
  - There is no greatest element in the set $(0,1)$ itself.
  - Wait, 2 is the greatest element here.
- Let $P = \{(n, 0) \mid n \in \mathbb{N}\} \cup \{(0,1)\}$.
  - Order relation: $(x_1, y_1) \leq (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ in the usual sense.
  - This does not lead to exactly one maximal element and no greatest.
- Let's try a set with no top: $P = \{(x, y) \in \mathbb{R}^2 \mid x + y < 1\}$. Order $(x_1, y_1) \leq (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$.
  - There are no maximal elements.
- Consider the set $S = \{a_1, a_2, a_3, \ldots\} \cup \{b\}$ where $a_i < b$ for all i, but a_i are incomparable to each other.
  - b is the greatest element here.

- Correct Example: Let $P = \{a_n \mid n \in \mathbb{N}\} \cup \{b\}$ with the ordering $a_n < b$ for all n, and a_i are incomparable with a_j for $i \neq j$.
  - In this set, b is the greatest element, and thus the unique maximal element. This is not the example.
- Consider the set $P = \mathbb{N} \cup \{x\}$, where x is an element not in $\mathbb{N}$.
  - Order: For $m, n \in \mathbb{N}$, $m \leq n$ is the usual order.
  - For $n \in \mathbb{N}$, $n \leq x$ is not true. $x \leq n$ is not true. So x is incomparable with all $n \in \mathbb{N}$.
  - Maximal elements: x is a maximal element. Also, $\mathbb{N}$ has no maximal element under its usual ordering.
  - This doesn't work.
- Let's use the standard "fork" example: Consider $S = \{a, b, c, d\}$ with $a < b$, $a < c$, and $d < c$. Assume b and d are incomparable, and a is the minimum.
  - Maximal elements are b and c. So not one.
- The standard example is an infinite ascending chain with an isolated top element or an antichain with a top element.
- Consider $P = \{x_1, x_2, x_3, \dots\} \cup \{y\}$ with the ordering $x_1 < x_2 < x_3 < \cdots$ (an infinite chain), and y is incomparable to all x_i.
  - Maximal elements: y is a maximal element. There are no other maximal elements since the chain x_i continues infinitely.
  - Is y the greatest element? No, because y is not comparable to all x_i. For example, $y \ngeq x_1$.
  - This example works! $P = \mathbb{N} \cup \{y\}$ with standard order on $\mathbb{N}$ and y is incomparable to all elements in $\mathbb{N}$. y is the only maximal element. There is no greatest element because

y is not greater than any element in $\mathbb{N}$, and there is no element in $\mathbb{N}$ that is greatest.

- Example of an ordered set with exactly 3 maximal elements:
  - Consider the set $S = \{a, b, c, d, e, f\}$ with the following relations:
    - $a < d$
    - $a < e$
    - $a < f$
    - $b < d$
    - $c < e$
    - All other relations are only those implied by transitivity or reflexivity.
    - Assume d, e, f are incomparable to each other, and a, b, c are incomparable to each other.
    - Maximal elements: d, e, f.
    - This example works.

(b) (i) State duality principle in ordered sets.

- Duality Principle in Ordered Sets: If a statement is true for all ordered sets, then its dual statement is also true for all ordered sets. The dual statement is obtained by replacing every instance of '≤' with '≥', 'join (∨)' with 'meet (∧)', 'bottom element' with 'top element', 'maximal element' with 'minimal element', and 'greatest element' with 'least element', and vice versa.

(ii) Define an order preserving map between two ordered sets and prove that the composite map of two order preserving maps is order preserving.

- Definition of an order preserving map: Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two ordered sets. A map $f: P \to Q$ is said to be order preserving if for all $a, b \in P$, if $a \leq_P b$, then $f(a) \leq_Q f(b)$.
- Prove that the composite map of two order preserving maps is order preserving:
  - Let $(P, \leq_P)$, $(Q, \leq_Q)$, and $(R, \leq_R)$ be three ordered sets.
  - Let $f: P \to Q$ be an order preserving map.
  - Let $g: Q \to R$ be an order preserving map.
  - We need to prove that the composite map $g \circ f: P \to R$ is order preserving.
  - Let $a, b \in P$ such that $a \leq_P b$.
  - Since f is order preserving and $a \leq_P b$, we have $f(a) \leq_Q f(b)$.
  - Now, let $f(a) = x$ and $f(b) = y$. So $x, y \in Q$ and $x \leq_Q y$.
  - Since g is order preserving and $x \leq_Q y$, we have $g(x) \leq_R g(y)$.
  - Substituting back $x = f(a)$ and $y = f(b)$, we get $g(f(a)) \leq_R g(f(b))$.
  - This means $(g \circ f)(a) \leq_R (g \circ f)(b)$.
  - Therefore, the composite map $g \circ f$ is order preserving.

(c) Define bottom and top element in an ordered set. Give one example of an ordered set in which bottom and top both exist and one example in which none of them exist.

- Definition of bottom element: In an ordered set $(P, \leq)$, an element $b \in P$ is called a bottom element (or least element or minimum element) if for all $x \in P$, $b \leq x$.

- Definition of top element: In an ordered set $(P, \leq)$, an element $t \in P$ is called a top element (or greatest element or maximum element) if for all $x \in P$, $x \leq t$.
- Example of an ordered set in which bottom and top both exist:
  - Consider the set $S = \{1,2,3,4,5\}$ with the usual order $\leq$.
  - The bottom element is 1 (because $1 \leq x$ for all $x \in S$).
  - The top element is 5 (because $x \leq 5$ for all $x \in S$).
- Example of an ordered set in which none of them exist:
  - Consider the set of integers $\mathbb{Z}$ with the usual order $\leq$.
  - There is no bottom element because for any integer x, $x - 1$ is also an integer and $x - 1 < x$. So, no element is less than or equal to all other elements.
  - There is no top element because for any integer x, $x + 1$ is also an integer and $x < x + 1$. So, no element is greater than or equal to all other elements.

Question 3: (a) (i) Prove that in a lattice L, for any a, b, c, d ∈L, a ≤ c, b ≤ d implies that avb≤ cvd.

- Proof:
  - Let L be a lattice. Let $a, b, c, d \in L$.
  - We are given that $a \leq c$ and $b \leq d$.
  - By definition of join, $a \leq a \vee b$ and $b \leq a \vee b$.
  - Since $a \leq c$ and $a \leq a \vee b$, we have $a \vee b$ as an upper bound for a.
  - Since c is an upper bound for a and d is an upper bound for b, then $c \vee d$ is an upper bound for c and d.

- We know that $a \leq c$. Also $c \leq c \vee d$. By transitivity, $a \leq c \vee d$.
- Similarly, we know that $b \leq d$. Also $d \leq c \vee d$. By transitivity, $b \leq c \vee d$.
- Since $c \vee d$ is an upper bound for both a and b, and $a \vee b$ is the least upper bound of a and b, it follows that $a \vee b \leq c \vee d$.

(ii) Prove that a lattice L is a chain if and only if every nonempty subset of L is a sublattice of L.

- Proof:
  - (⇒) Assume L is a chain. This means for any $x, y \in L$, either $x \leq y$ or $y \leq x$.
    - Let S be a nonempty subset of L.
    - We need to show that for any $x, y \in S$, $x \vee y \in S$ and $x \wedge y \in S$.
    - Since L is a chain, either $x \leq y$ or $y \leq x$.
    - If $x \leq y$, then $x \vee y = y$ and $x \wedge y = x$. Since $x, y \in S$, it follows that $x \wedge y \in S$ and $x \vee y \in S$.
    - If $y \leq x$, then $x \vee y = x$ and $x \wedge y = y$. Since $x, y \in S$, it follows that $x \wedge y \in S$ and $x \vee y \in S$.
    - Thus, every nonempty subset of L is a sublattice of L.
  - (⇐) Assume every nonempty subset of L is a sublattice of L.
    - We need to show that L is a chain. This means we need to show that for any $x, y \in L$, either $x \leq y$ or $y \leq x$.
    - Consider any two elements $x, y \in L$. Let $S = \{x, y\}$.
    - Since S is a nonempty subset of L, by our assumption, S must be a sublattice of L.

- This implies that $x \vee y \in S$ and $x \wedge y \in S$.
- If $x \vee y \in \{x, y\}$, then either $x \vee y = x$ or $x \vee y = y$.
  - If $x \vee y = x$, then by definition of join, $y \leq x$.
  - If $x \vee y = y$, then by definition of join, $x \leq y$.
- In either case, x and y are comparable.
- Since this holds for any arbitrary $x, y \in L$, it means that L is a chain.
- Therefore, a lattice L is a chain if and only if every nonempty subset of L is a sublattice of L.

(b) Prove that in any lattice L, the following holds $((x \wedge y) \vee (x \wedge z)) \wedge ((x \vee y) \vee (y \wedge z)) = x \wedge y$, for all x, y, z ∈ L.

- This identity is known as a specific property or a simplification. Let's try to prove it using lattice properties.
- We know that in any lattice:
  - $x \wedge y \leq x$ and $x \wedge y \leq y$.
  - $x \wedge z \leq x$ and $x \wedge z \leq z$.
  - $x \vee y \geq x$ and $x \vee y \geq y$.
  - $y \wedge z \leq y$ and $y \wedge z \leq z$.
- Let LHS = $((x \wedge y) \vee (x \wedge z)) \wedge ((x \vee y) \vee (y \wedge z))$.
- Consider the first term: $(x \wedge y) \vee (x \wedge z)$.
  - Since $x \wedge y \leq x$ and $x \wedge z \leq x$, we have $(x \wedge y) \vee (x \wedge z) \leq x$. (Distributive inequality)
  - Also, by absorption, if we assume distributivity, but we can't assume distributivity in a general lattice.

- By idempotence and associativity of ∧ and ∨.
- We know $x \wedge y \leq x$ and $x \wedge z \leq x$. Thus $(x \wedge y) \vee (x \wedge z) \leq x$.
- Also, $x \wedge y \leq y$.

- Consider the second term: $(x \vee y) \vee (y \wedge z)$.
  - We know $y \leq x \vee y$.
  - We know $y \wedge z \leq y$.
  - So $(x \vee y) \vee (y \wedge z) = x \vee y$ since $y \wedge z \leq y \leq x \vee y$. So $(y \wedge z) \vee (x \vee y) = x \vee y$.
  - This is incorrect. This is only true if $y \wedge z$ is less than or equal to $x \vee y$.
    - Since $y \wedge z \leq y$, and $y \leq x \vee y$, by transitivity, $y \wedge z \leq x \vee y$.
    - Therefore, $(x \vee y) \vee (y \wedge z) = x \vee y$. This step is correct.
- Now substitute this back into the LHS:
  - LHS = $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y)$.
- We need to show that $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y) = x \wedge y$.
- We know $x \wedge y \leq x$ and $x \wedge y \leq y$.
- This means $x \wedge y \leq x \vee y$.
- Also, $x \wedge y \leq x \wedge y$.
- Since $x \wedge y \leq x \wedge y$ and $x \wedge y \leq x \vee y$, we need to check the other component.
- This identity does not hold in a general lattice. It is a specific property. Let's recheck the statement.

- This identity is related to modularity or distributivity properties. In a general lattice, it might not hold.
- Let's check the identity in M_3 (the diamond lattice), which is modular but not distributive.
- Elements: $0, a, b, c, 1$. (0 is bottom, 1 is top, a, b, c are incomparable but above 0 and below 1).
- Let $x = a, y = b, z = c$.
- $x \wedge y = a \wedge b = 0$.
- $x \wedge z = a \wedge c = 0$.
- $y \wedge z = b \wedge c = 0$.
- $x \vee y = a \vee b = 1$.
- LHS = $((0) \vee (0)) \wedge ((1) \vee (0)) = 0 \wedge 1 = 0$.
- RHS = $x \wedge y = a \wedge b = 0$.
- It holds for this case in M_3.
- Let $x = 1, y = a, z = b$.
- $x \wedge y = 1 \wedge a = a$.
- $x \wedge z = 1 \wedge b = b$.
- $y \wedge z = a \wedge b = 0$.
- $x \vee y = 1 \vee a = 1$.
- LHS = $((a) \vee (b)) \wedge ((1) \vee (0)) = 1 \wedge 1 = 1$.
- RHS = $x \wedge y = 1 \wedge a = a$.
- Here LHS = 1 and RHS = a. So $1 = a$ which is false.
- Therefore, the identity $((x \wedge y) \vee (x \wedge z)) \wedge ((x \vee y) \vee (y \wedge z)) = x \wedge y$ is NOT true in every lattice. There must be a typo in the

question, or an implicit assumption that the lattice is distributive, or the question expects a proof of specific instances or specific conditions.

- Recheck the identity using a different approach.
- We have: $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y)$.
- By absorption laws ($A \vee (A \wedge B) = A$) and ($A \wedge (A \vee B) = A$).
- We know $x \wedge y \leq x$.
- We know $x \wedge z \leq x$.
- So $(x \wedge y) \vee (x \wedge z) \leq x$.
- Let $P = (x \wedge y) \vee (x \wedge z)$. Then $P \leq x$.
- The expression is $P \wedge (x \vee y)$.
- Since $P \leq x$, and $x \leq x \vee y$, we have $P \leq x \vee y$.
- So $P \wedge (x \vee y) = P = (x \wedge y) \vee (x \wedge z)$.
- Therefore, the identity simplifies to $(x \wedge y) \vee (x \wedge z) = x \wedge y$.
- This equality $(x \wedge y) \vee (x \wedge z) = x \wedge y$ is true if and only if $x \wedge z \leq x \wedge y$. This means $x \wedge z$ must be less than or equal to $x \wedge y$. This is not generally true for all x, y, z in a lattice.
- Given the problem structure, it's highly likely this is an identity that *should* hold, possibly a known one in lattice theory that reduces to something else or relies on an implicit assumption (e.g., modularity, or even distributivity for a simpler path).
- Let's assume the question implicitly refers to a distributive lattice, as some identities are only provable in such cases.
- If L is a distributive lattice, then $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$.
- So, $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y)$

- Let $A = x \vee y$.
- The expression is $((x \wedge y) \vee (x \wedge z)) \wedge A$.
- This is not in the form for direct application of distributivity.
- Let's re-evaluate the simplification of the second term: $(x \vee y) \vee (y \wedge z)$.
- We correctly established that $y \wedge z \leq y$. Since $y \leq x \vee y$, by transitivity $y \wedge z \leq x \vee y$.
- Therefore $(x \vee y) \vee (y \wedge z) = x \vee y$. This step is definitely correct for any lattice.
- So the identity simplifies to $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y) = x \wedge y$.
- Let's use the property $A \wedge (B \vee C) \leq (A \wedge B) \vee (A \wedge C)$ (meet distributes over join, one inequality).
- Let's also use $(A \vee B) \wedge C \leq (A \wedge C) \vee (B \wedge C)$.
- Let $U = (x \wedge y) \vee (x \wedge z)$ and $V = x \vee y$.
- We need to prove $U \wedge V = x \wedge y$.
- We know $x \wedge y \leq U$ and $x \wedge y \leq V$. So $x \wedge y \leq U \wedge V$.
- Now we need to prove $U \wedge V \leq x \wedge y$.
- $U \wedge V = ((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y)$.
- Using modular inequality: $A \wedge (B \vee C) \leq (A \wedge B) \vee C$ if $C \leq A$.
- Let $A = x \vee y$, $B = x \wedge y$, $C = x \wedge z$.
- $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y)$.
- We know $x \wedge y \leq x \vee y$.
- We know $x \wedge z \leq x$. And $x \leq x \vee y$. So $x \wedge z \leq x \vee y$.

- This is the property: $(A \vee B) \wedge C = (A \wedge C) \vee B$ if $B \leq C$. Not directly applicable.
- This identity $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y) = x \wedge y$ is known as a property of modular lattices, specifically. It is not true in a general lattice, but it is true in distributive lattices (which are a stronger form of modular lattices).
- Let's prove it assuming it is a distributive lattice.
- $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y)$
- $= (x \wedge (y \vee z)) \wedge (x \vee y)$ (by distributivity: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$)
- This is incorrect. The distributive law states $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, not $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$. The first term is of the form $(A \vee B) \wedge C$.
- Using dual distributivity: $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$.
- So, $((x \wedge y) \vee (x \wedge z))$ is not necessarily $x \wedge (y \vee z)$.
- Let's retry:
- LHS = $((x \wedge y) \vee (x \wedge z)) \wedge ((x \vee y) \vee (y \wedge z))$.
- First, we simplified $(x \vee y) \vee (y \wedge z) = x \vee y$ because $y \wedge z \leq y \leq x \vee y$. This is correct.
- So LHS = $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y)$.
- Now we need to prove $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y) = x \wedge y$.
- We know that $x \wedge y \leq (x \wedge y) \vee (x \wedge z)$ and $x \wedge y \leq x \vee y$.
- So, $x \wedge y \leq ((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y)$. This proves one side of the equality.
- For the other side, we need to show $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y) \leq x \wedge y$.

- We know that $(x \wedge y) \vee (x \wedge z) \leq x$.
- So $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y) \leq x \wedge (x \vee y) = x$. (Since $A \wedge (A \vee B) = A$)
- Also, $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y)$
- $\leq ((x \wedge y) \vee (x \wedge z)) \wedge (y \vee y)$ (This is not valid step)
- This identity is not generally true in a lattice. It might be specific to distributive lattices, or there's a typo.
- If it is a distributive lattice:
  - LHS = $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y)$
  - $= (x \wedge (y \vee z)) \wedge (x \vee y)$ (Distributivity: $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ is used here in reverse direction)
  - $= x \wedge (y \vee z) \wedge (x \vee y)$. This doesn't seem to simplify to $x \wedge y$.
- Let's consider the inequality using isotony.
- $(x \wedge y) \vee (x \wedge z) \leq x$.
- $x \vee y$.
- Consider $x \wedge y$.
- We need to prove $A \wedge B = x \wedge y$ where $A = (x \wedge y) \vee (x \wedge z)$ and $B = x \vee y$.
- Since $x \wedge y \leq A$ and $x \wedge y \leq B$, we have $x \wedge y \leq A \wedge B$.
- Now prove $A \wedge B \leq x \wedge y$.
- $A = (x \wedge y) \vee (x \wedge z)$.
- $A \wedge B = ((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y)$.
- By modularity (if L is modular): If $x \wedge y \leq x \vee y$, then $x \wedge y \vee (x \wedge z) \wedge (x \vee y) = (x \wedge y) \vee ((x \wedge z) \wedge (x \vee y))$.

- This identity is called the "distributive law for join and meet" or a simplification. It is a well-known identity called the **absorptive law for joins and meets**, typically $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y) = x \wedge y$.
- This is true in a distributive lattice.
- If a lattice is distributive, then for any $a, b, c \in L$:
  - $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
  - $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- Let's assume the lattice is distributive.
- LHS = $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y)$
- Let $A = x$.
- LHS = $(x \wedge (y \vee z)) \wedge (x \vee y)$ (using $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for distributive lattice)
- This step is incorrect. $((x \wedge y) \vee (x \wedge z))$ is not necessarily $x \wedge (y \vee z)$.
- Example: $X = \{1,2,3\}$, $P(X)$. $x = \{1,2\}$, $y = \{1,3\}$, $z = \{2,3\}$.
  - $x \wedge y = \{1\}$.
  - $x \wedge z = \{2\}$.
  - $(x \wedge y) \vee (x \wedge z) = \{1\} \vee \{2\} = \{1,2\} = x$.
  - $x \wedge (y \vee z) = \{1,2\} \wedge (\{1,3\} \vee \{2,3\}) = \{1,2\} \wedge \{1,2,3\} = \{1,2\} = x$.
  - So $((x \wedge y) \vee (x \wedge z)) = x \wedge (y \vee z)$ holds in distributive lattices. This is the first distributive law.
- So, if L is distributive:
  - LHS = $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y)$

    - $= (x \wedge (y \vee z)) \wedge (x \vee y)$ (using $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ in reverse)
    - $= x \wedge (y \vee z) \wedge (x \vee y)$ (associativity of $\wedge$)
    - $= x \wedge (y \vee (z \wedge x)) \wedge y$ (not useful)
- Let's try to prove that $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y) = x \wedge y$ without assuming distributivity, but using the other established part of the question.
- From part (a) (i), we know $a \leq c, b \leq d \Longrightarrow a \vee b \leq c \vee d$.
- We know $x \wedge y \leq x$ and $x \wedge y \leq y$.
- $x \wedge z \leq x$.
- So $(x \wedge y) \vee (x \wedge z) \leq x$.
- And $x \wedge y \leq y$.
- Therefore, $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y)$
- Let $P = (x \wedge y) \vee (x \wedge z)$. We know $P \leq x$.
- So the expression is $P \wedge (x \vee y)$.
- Since $P \leq x$, and $x \leq x \vee y$, we have $P \leq x \vee y$.
- This implies $P \wedge (x \vee y) = P$.
- So, the identity reduces to $(x \wedge y) \vee (x \wedge z) = x \wedge y$.
- This is only true if $x \wedge z \leq x \wedge y$. This is not generally true.
- Conclusion for part (b): The given identity $((x \wedge y) \vee (x \wedge z)) \wedge ((x \vee y) \vee (y \wedge z)) = x \wedge y$ is NOT true in a general lattice. It is true in a distributive lattice. If the question implicitly asks to prove this in a distributive lattice, then the steps would involve applying distributive laws. If it's a general lattice, a counterexample exists (as shown with M_3). Given it's a "prove

that in any lattice L", there must be an error in my understanding or in the question's premise.

- However, let's assume the property is true and there is a subtle identity.
- We have $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y)$.
- Let $A = x \wedge y$.
- Let $B = x \wedge z$.
- Let $C = x \vee y$.
- We need to show $(A \vee B) \wedge C = A$.
- This requires $A \vee B \leq C$ and $A \leq C$. We know $A \leq C$ (since $x \wedge y \leq x \leq x \vee y$ and $x \wedge y \leq y \leq x \vee y$).
- We also need $A \vee B \leq C$.
- $(x \wedge y) \vee (x \wedge z) \leq x$. And $x \leq x \vee y$.
- So $(x \wedge y) \vee (x \wedge z) \leq x \vee y$.
- Therefore, $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y) = (x \wedge y) \vee (x \wedge z)$.
- So the identity simplifies to $(x \wedge y) \vee (x \wedge z) = x \wedge y$.
- This means $x \wedge z \leq x \wedge y$. This is not true in general.
- Perhaps the problem intended to use a more complex structure or property.
- Given it says "Prove that in any lattice L", this implies it holds universally. The previous simplification of $((x \vee y) \vee (y \wedge z)) = x \vee y$ is correct for any lattice.
- So, we are left with proving $((x \wedge y) \vee (x \wedge z)) \wedge (x \vee y) = x \wedge y$.
- This is the same as $(P \vee Q) \wedge R = P$ where $P = x \wedge y$, $Q = x \wedge z$, $R = x \vee y$.

- We know $P \leq R$ and $Q \leq R$. This implies $P \vee Q \leq R$.
- So $(P \vee Q) \wedge R = P \vee Q$.
- Therefore, the identity is equivalent to $(x \wedge y) \vee (x \wedge z) = x \wedge y$.
- This means $x \wedge z \leq x \wedge y$. This is not true in any lattice. Example: Let $L = D_6 = \{1,2,3,6\}$ (divisors of 6), ordered by divisibility.
  - Let $x = 6, y = 2, z = 3$.
  - $x \wedge y =$ gcd$(6,2) = 2$.
  - $x \wedge z =$ gcd$(6,3) = 3$.
  - $(x \wedge y) \vee (x \wedge z) = 2 \vee 3 =$ lcm$(2,3) = 6$.
  - We need $6 = 2$, which is false.
- Therefore, the identity as stated in question 3(b) is not true in any lattice. There must be a mistake in the problem statement.

(c) Let L and K be lattices and f: L → K be a map. Show that the following are equivalent: (i) f is order preserving. (ii) (∀a, b ∈ L), f(a ∧ b) ≤ f(a) ∧ f(b).

- Proof of equivalence: We need to show that (i) ⇒ (ii) and (ii) ⇒ (i).
- (i) ⇒ (ii): Assume f is order preserving.
  - Let $a, b \in L$.
  - By definition of meet, $a \wedge b \leq a$ and $a \wedge b \leq b$.
  - Since f is order preserving:
    - From $a \wedge b \leq a$, we have $f(a \wedge b) \leq f(a)$.
    - From $a \wedge b \leq b$, we have $f(a \wedge b) \leq f(b)$.
  - Since $f(a \wedge b)$ is less than or equal to both $f(a)$ and $f(b)$, it must be less than or equal to their greatest lower bound (meet).

- Therefore, $f(a \wedge b) \leq f(a) \wedge f(b)$.

- (ii) ⇒ (i): Assume that for all $a, b \in L$, $f(a \wedge b) \leq f(a) \wedge f(b)$.
  - We need to show that f is order preserving. That is, if $a \leq b$, then $f(a) \leq f(b)$.
  - Let $a, b \in L$ such that $a \leq b$.
  - If $a \leq b$, then $a \wedge b = a$.
  - Using the given condition (ii) with a and b:
    - $f(a \wedge b) \leq f(a) \wedge f(b)$.
    - Since $a \wedge b = a$, we substitute this: $f(a) \leq f(a) \wedge f(b)$.
  - By definition of meet, $f(a) \wedge f(b) \leq f(b)$.
  - From $f(a) \leq f(a) \wedge f(b)$ and $f(a) \wedge f(b) \leq f(b)$, by transitivity of $\leq$, we get $f(a) \leq f(b)$.
  - Therefore, f is order preserving.
- Thus, (i) and (ii) are equivalent.

Question 4: (a) Let L and M be two ordered sets and f be an order isomorphism from L onto M. Prove that if L is a lattice, then M is also a lattice, and f is a lattice isomorphism.

- Proof:
  - Given $f: L \to M$ is an order isomorphism from an ordered set L onto an ordered set M.
  - This means f is a bijection, and for all $x, y \in L$, $x \leq_L y \Leftrightarrow f(x) \leq_M f(y)$.
  - We are given that L is a lattice. This means for any $a, b \in L$, $a \vee b$ and $a \wedge b$ exist in L.

- We need to prove that M is a lattice, and f is a lattice isomorphism.

- Part 1: Prove M is a lattice.
  - To prove M is a lattice, we need to show that for any $u, v \in M$, their join $u \vee_M v$ and their meet $u \wedge_M v$ exist in M.
  - Since f is onto, for any $u, v \in M$, there exist unique elements $a, b \in L$ such that $f(a) = u$ and $f(b) = v$. (Unique because f is one-to-one).
  - Since L is a lattice, the join $a \vee_L b$ and the meet $a \wedge_L b$ exist in L.
  - Consider $f(a \vee_L b)$. We claim this is $u \vee_M v$.
    - Since $a \leq_L a \vee_L b$, and f is order preserving, $f(a) \leq_M f(a \vee_L b)$, i.e., $u \leq_M f(a \vee_L b)$.
    - Since $b \leq_L a \vee_L b$, and f is order preserving, $f(b) \leq_M f(a \vee_L b)$, i.e., $v \leq_M f(a \vee_L b)$.
    - Thus, $f(a \vee_L b)$ is an upper bound for u and v in M.
    - Now, let $w \in M$ be any upper bound for u and v. So $u \leq_M w$ and $v \leq_M w$.
    - Since f is onto, there exists $c \in L$ such that $f(c) = w$.
    - Since $u \leq_M w$, i.e., $f(a) \leq_M f(c)$, and f is an order isomorphism, $a \leq_L c$.
    - Since $v \leq_M w$, i.e., $f(b) \leq_M f(c)$, and f is an order isomorphism, $b \leq_L c$.
    - Since $a \leq_L c$ and $b \leq_L c$, and $a \vee_L b$ is the least upper bound in L, it follows that $a \vee_L b \leq_L c$.
    - Since f is order preserving, $f(a \vee_L b) \leq_M f(c) = w$.

   - Thus, $f(a \vee_L b)$ is the least upper bound (join) of u and v in M. So $u \vee_M v = f(a \vee_L b)$.
 - Similarly, consider $f(a \wedge_L b)$. We claim this is $u \wedge_M v$.
   - Since $a \wedge_L b \leq_L a$, and f is order preserving, $f(a \wedge_L b) \leq_M f(a)$, i.e., $f(a \wedge_L b) \leq_M u$.
   - Since $a \wedge_L b \leq_L b$, and f is order preserving, $f(a \wedge_L b) \leq_M f(b)$, i.e., $f(a \wedge_L b) \leq_M v$.
   - Thus, $f(a \wedge_L b)$ is a lower bound for u and v in M.
   - Now, let $w' \in M$ be any lower bound for u and v. So $w' \leq_M u$ and $w' \leq_M v$.
   - Since f is onto, there exists $c' \in L$ such that $f(c') = w'$.
   - Since $f(c') \leq_M f(a)$, and f is an order isomorphism, $c' \leq_L a$.
   - Since $f(c') \leq_M f(b)$, and f is an order isomorphism, $c' \leq_L b$.
   - Since $c' \leq_L a$ and $c' \leq_L b$, and $a \wedge_L b$ is the greatest lower bound in L, it follows that $c' \leq_L a \wedge_L b$.
   - Since f is order preserving, $f(c') \leq_M f(a \wedge_L b)$.
   - Thus, $f(a \wedge_L b)$ is the greatest lower bound (meet) of u and v in M. So $u \wedge_M v = f(a \wedge_L b)$.
 - Since joins and meets exist for any pair of elements in M, M is a lattice.
- Part 2: Prove f is a lattice isomorphism.
  - A lattice isomorphism is an order isomorphism that preserves join and meet operations.
  - We have already shown that f is an order isomorphism.

- We just showed that for any $u, v \in M$, if $f(a) = u$ and $f(b) = v$:
  - $u \vee_M v = f(a \vee_L b)$, which is $f(f^{-1}(u) \vee_L f^{-1}(v))$. This shows f preserves joins.
  - $u \wedge_M v = f(a \wedge_L b)$, which is $f(f^{-1}(u) \wedge_L f^{-1}(v))$. This shows f preserves meets.
- Alternatively, for any $a, b \in L$:
  - $f(a \vee_L b) = f(a) \vee_M f(b)$
  - $f(a \wedge_L b) = f(a) \wedge_M f(b)$
- This is precisely what it means for f to be a lattice homomorphism. Since it's also an order isomorphism (and thus a bijection), it's a lattice isomorphism.

(b) Prove that the direct product L x K of two distributive lattices L and K is also a distributive lattice.

- Proof:
  - Let L and K be two distributive lattices.
  - The direct product $L \times K$ is the set of ordered pairs (a, b) where $a \in L$ and $b \in K$.
  - The order relation is defined as $(a_1, b_1) \leq (a_2, b_2)$ if $a_1 \leq_L a_2$ and $b_1 \leq_K b_2$.
  - The join and meet operations in $L \times K$ are defined component-wise:
    - $(a_1, b_1) \vee (a_2, b_2) = (a_1 \vee_L a_2, b_1 \vee_K b_2)$
    - $(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge_L a_2, b_1 \wedge_K b_2)$
  - To prove that $L \times K$ is a distributive lattice, we need to show that for any three elements $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in L \times K$, the distributive laws hold:

  - $((x_1, y_1) \wedge (x_2, y_2)) \vee ((x_1, y_1) \wedge (x_3, y_3)) = (x_1, y_1) \wedge ((x_2, y_2) \vee (x_3, y_3))$
  - $((x_1, y_1) \vee (x_2, y_2)) \wedge ((x_1, y_1) \vee (x_3, y_3)) = (x_1, y_1) \vee ((x_2, y_2) \wedge (x_3, y_3))$

- Let's prove the first distributive law:
  - LHS = $((x_1 \wedge_L x_2, y_1 \wedge_K y_2)) \vee ((x_1 \wedge_L x_3, y_1 \wedge_K y_3))$
  - $= ((x_1 \wedge_L x_2) \vee_L (x_1 \wedge_L x_3), (y_1 \wedge_K y_2) \vee_K (y_1 \wedge_K y_3))$
  - Since L is distributive, $(x_1 \wedge_L x_2) \vee_L (x_1 \wedge_L x_3) = x_1 \wedge_L (x_2 \vee_L x_3)$.
  - Since K is distributive, $(y_1 \wedge_K y_2) \vee_K (y_1 \wedge_K y_3) = y_1 \wedge_K (y_2 \vee_K y_3)$.
  - So, LHS = $(x_1 \wedge_L (x_2 \vee_L x_3), y_1 \wedge_K (y_2 \vee_K y_3))$
  - RHS = $(x_1, y_1) \wedge ((x_2 \vee_L x_3, y_2 \vee_K y_3))$
  - $= (x_1 \wedge_L (x_2 \vee_L x_3), y_1 \wedge_K (y_2 \vee_K y_3))$
  - Since LHS = RHS, the first distributive law holds for $L \times K$.

- Now, let's prove the second distributive law (dual distributivity):
  - LHS = $((x_1, y_1) \vee (x_2, y_2)) \wedge ((x_1, y_1) \vee (x_3, y_3))$
  - $= ((x_1 \vee_L x_2, y_1 \vee_K y_2)) \wedge ((x_1 \vee_L x_3, y_1 \vee_K y_3))$
  - $= ((x_1 \vee_L x_2) \wedge_L (x_1 \vee_L x_3), (y_1 \vee_K y_2) \wedge_K (y_1 \vee_K y_3))$
  - Since L is distributive, $(x_1 \vee_L x_2) \wedge_L (x_1 \vee_L x_3) = x_1 \vee_L (x_2 \wedge_L x_3)$.
  - Since K is distributive, $(y_1 \vee_K y_2) \wedge_K (y_1 \vee_K y_3) = y_1 \vee_K (y_2 \wedge_K y_3)$.
  - So, LHS = $(x_1 \vee_L (x_2 \wedge_L x_3), y_1 \vee_K (y_2 \wedge_K y_3))$

      - RHS = $(x_1, y_1) \vee ((x_2 \wedge_L x_3, y_2 \wedge_K y_3))$
      - $= (x_1 \vee_L (x_2 \wedge_L x_3), y_1 \vee_K (y_2 \wedge_K y_3))$
      - Since LHS = RHS, the second distributive law holds for $L \times K$.
   - Since both distributive laws hold, the direct product $L \times K$ is also a distributive lattice.

(c) (i) Prove that every sublattice of a modular lattice L is modular.

- Proof:
   - Let L be a modular lattice. This means that for all $x, y, z \in L$, if $x \leq z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$.
   - Let S be a sublattice of L.
   - This means that $S \subseteq L$, and for any $a, b \in S$, $a \vee_S b = a \vee_L b$ and $a \wedge_S b = a \wedge_L b$. (The operations in S are the same as in L restricted to elements of S).
   - We need to prove that S is modular. This means that for all $x, y, z \in S$, if $x \leq_S z$, then $x \vee_S (y \wedge_S z) = (x \vee_S y) \wedge_S z$.
   - Let $x, y, z \in S$ such that $x \leq_S z$.
   - Since S is a sublattice of L, the order relation in S is inherited from L. So $x \leq_L z$.
   - Since x, y, z are also elements of L, and L is modular, and $x \leq_L z$, the modular law holds in L:
      - $x \vee_L (y \wedge_L z) = (x \vee_L y) \wedge_L z$.
   - Since S is a sublattice, the join and meet operations in S are the same as in L for elements of S.
      - $x \vee_S (y \wedge_S z) = x \vee_L (y \wedge_L z)$
      - $(x \vee_S y) \wedge_S z = (x \vee_L y) \wedge_L z$

- Therefore, $x \vee_S (y \wedge_S z) = (x \vee_S y) \wedge_S z$.
- Thus, every sublattice of a modular lattice L is modular.