SL No of QP : 5534

Unique Paper Code : 2352012401

Name of the Paper : Sequences and Series of Functions

Name of the Course : B.Sc. (H) Mathematics

Semester : IV

Duration : 3 hours

Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on the receipt of this question paper.
2. **All** questions carry equal marks.
3. Attempt any **two** parts from each question.

1. (a) Define uniform convergence for sequence of functions (f_n) defined on $\mathbb{R}$. Show that a sequence (f_n) of bounded functions on $A \subseteq \mathbb{R}$ converges uniformly on A to f if and only if $||f_n - f||_A \to 0$.

   (b) Show that the sequence $f_n(x) = \frac{nx}{1+n^2x^2}$ converges uniformly in the interval $[a, \infty)$ where $a > 0$ but it does not converge uniformly on the interval $[0, \infty)$.

   (c) Let $(f_n), (g_n)$ be sequences of bounded functions on A that converge uniformly on A to f, g, respectively. Show that $(f_n g_n)$ converges uniformly on A to fg.

2. (a) Let $f_n(x): [0,1] \to \mathbb{R}$ be defined for $n \geq 2$ by

$$f_n(x) := \begin{cases} n^2 x & for\ 0 \leq x \leq \frac{1}{n} \\ -n^2\left(x - \frac{2}{n}\right) & for\ \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & for\ \frac{2}{n} \leq x \leq 1 \end{cases}$$

   Evaluate pointwise limit f of f_n and show that (f_n) does not converge uniformly to f on the interval $[0,1]$.

   (b) Let (f_n) be a sequence of integrable functions on $[a, b]$ and suppose that (f_n) converges uniformly to f on $[a, b]$. Show that f is integrable on $[a, b]$ and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

   (c) Show that the sequence $f_n(x) = n^2x^2e^{-nx}$ converges uniformly on $[a, \infty)$ where $a > 0$ but does not converge uniformly on the interval $[0, \infty)$.

3. (a) State and prove Weierstrass M-test for uniform convergence of series of functions. Hence, check for uniform convergence of series $\sum_{n=1}^{\infty}\frac{1}{x^2+n^2}$, $x \in \mathbb{R}$.

(b) Discuss the pointwise convergence of the series $\sum_{n=1}^{\infty}\frac{x^n}{x^n+1}$, $x \geq 0$. Also, show that the given series is not uniformly convergent on $[0,1)$ but is uniformly convergent on $[0,\frac{1}{2})$.

(c) Prove that if $\sum f_n$ converges uniformly to f on a domain $D \subseteq \mathbb{R}$ and each f_n is continuous on $D \subseteq \mathbb{R}$ to $\mathbb{R}$ then f is continuous on D.

4. (a) Let f_n be a real valued function on $[a,b]$ that has derivative f_n' on $[a,b]$ for each $n \in \mathbb{N}$. Suppose that the series $\sum f_n$ converges for at least one point of $[a,b]$ and the series of derivatives $\sum f_n'$ converges uniformly on $[a,b]$. Show that there exists a real valued function f on $[a,b]$ such that $\sum f_n$ converges uniformly to f on $[a,b]$. Also, that f has a derivative on $[a,b]$ and $f' = \sum f_n'$.

(b) State and prove the Cauchy criterion for the uniform convergence of series of functions $\sum f_n$. Use it to prove the non uniform convergence of $\sum_{n=1}^{\infty}\frac{1}{n^2x^2}$ for $|x| < 1$.

(c) Prove that the series $\sum_{n=1}^{\infty}\frac{1}{x^n+1}$ is convergent for $x > 1$ and divergent for $0 \leq x \leq 1$. Also, show that the series converges uniformly on $[a,\infty)$, $a > 1$.

5. (a) (i) Find the exact interval of convergence for the power series $\sum_{n=1}^{\infty} a_n x^n$ where $a_n = \frac{3^n}{n4^n}$.

(ii) Let $f(x) = |x|$ for $x \in \mathbb{R}$. Is there a power series $\sum_{n=0}^{\infty} a_n x^n$ such that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for all x ? Justify.

(b) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. If $0 < R_1 < R$, then show that the power series converges uniformly on $[-R_1, R_1]$ to a continuous function.

(c) Show that $\arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$ for $|x| < 1$. Apply Abel's theorem to show that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

6. (a) Define logarithm function L on $(0,\infty)$. Show that the function L satisfies the following:

(i) $L'(x) = \frac{1}{x}$ for $x > 0$.

(ii) $L(xy) = L(x) + L(y)$ for $x > 0, y > 0$.

(b) Show that $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$. Evaluate $\sum_{n=1}^{\infty}\frac{n}{3^n}$.

(c) Show that there does not exist a sequence of polynomials converging uniformly on $\mathbb{R}$ to $f(x) = e^x$.