Question 1: (a) Find the upper and lower Darboux integrals for f(x) = x^2 on the interval [0, b] and show that $\int_0^b x^2 = b^3/3$.

- To find the upper and lower Darboux integrals for $f(x) = x^2$ on $[0, b]$, we consider a partition $P_n = \{x_0, x_1, \ldots, x_n\}$ of $[0, b]$, where $x_i = \frac{ib}{n}$ for $i = 0, 1, \ldots, n$.
- For each subinterval $[x_{i-1}, x_i]$, the function $f(x) = x^2$ is increasing.
- Therefore, the infimum m_i of $f(x)$ on $[x_{i-1}, x_i]$ is $f(x_{i-1}) = (\frac{(i-1)b}{n})^2$.
- The supremum M_i of $f(x)$ on $[x_{i-1}, x_i]$ is $f(x_i) = (\frac{ib}{n})^2$.
- The length of each subinterval $\Delta x_i = x_i - x_{i-1} = \frac{b}{n}$.
- The lower Darboux sum is $L(f, P_n) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} (\frac{(i-1)b}{n})^2 \frac{b}{n} = \frac{b^3}{n^3} \sum_{i=1}^{n} (i-1)^2 = \frac{b^3}{n^3} \sum_{k=0}^{n-1} k^2$.
- Using the formula $\sum_{k=0}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$, we have $\sum_{k=0}^{n-1} k^2 = \frac{(n-1)n(2n-1)}{6}$.
- So, $L(f, P_n) = \frac{b^3}{n^3} \frac{(n-1)n(2n-1)}{6} = \frac{b^3}{6} \frac{(n-1)(2n-1)}{n^2} = \frac{b^3}{6} (1 - \frac{1}{n})(2 - \frac{1}{n})$.
- The lower Darboux integral is $\underline{\int_0^b} x^2 dx = \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} \frac{b^3}{6} (1 - \frac{1}{n})(2 - \frac{1}{n}) = \frac{b^3}{6} (1)(2) = \frac{b^3}{3}$.

- The upper Darboux sum is $U(f, P_n) = \sum_{i=1}^{n} M_i \, \Delta x_i = \sum_{i=1}^{n} (\frac{ib}{n})^2 \frac{b}{n} = \frac{b^3}{n^3} \sum_{i=1}^{n} i^2$.
- Using the formula $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
- So, $U(f, P_n) = \frac{b^3}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{b^3}{6} \frac{(n+1)(2n+1)}{n^2} = \frac{b^3}{6} (1 + \frac{1}{n})(2 + \frac{1}{n})$.
- The upper Darboux integral is $\overline{\int_0^b} x^2 dx = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \frac{b^3}{6} (1 + \frac{1}{n})(2 + \frac{1}{n}) = \frac{b^3}{6} (1)(2) = \frac{b^3}{3}$.
- Since the upper and lower Darboux integrals are equal, the function is integrable, and $\int_0^b x^2 \, dx = \frac{b^3}{3}$.

(b) Let f be a bounded function on [a, b]. If P and Q are partitions of [a, b] and P⊆Q, then prove that L(f, P) ≤ L(f, Q) ≤ U(f, Q) ≤ U(f, P).

- Let P be a partition of $[a, b]$, and Q be a refinement of P, meaning P ⊆ Q. This implies that Q contains all the points of P, plus some additional points.
- Consider a single subinterval $[x_{i-1}, x_i]$ from the partition P. Let $m_i = \inf_{x\in[x_{i-1},x_i]} f(x)$ and $M_i = \sup_{x\in[x_{i-1},x_i]} f(x)$.
- When we refine the partition P to Q by adding a point c in (x_{i-1}, x_i), the interval $[x_{i-1}, x_i]$ is split into two subintervals: $[x_{i-1}, c]$ and $[c, x_i]$.
- Let $m_{i'}$ be the infimum on $[x_{i-1}, c]$ and $m_{i''}$ be the infimum on $[c, x_i]$. We know that $m_i \leq m_{i'}$ and $m_i \leq m_{i''}$.

- The contribution to the lower sum from the interval $[x_{i-1}, x_i]$ in P is $m_i(x_i - x_{i-1})$.
- The contribution to the lower sum from the corresponding subintervals in Q is $m_{i'}(c - x_{i-1}) + m_{i''}(x_i - c)$.
- Since $m_i \leq m_{i'}$ and $m_i \leq m_{i''}$, we have $m_i(c - x_{i-1}) \leq m_{i'}(c - x_{i-1})$ and $m_i(x_i - c) \leq m_{i''}(x_i - c)$.
- Summing these, $m_i(c - x_{i-1}) + m_i(x_i - c) = m_i(x_i - x_{i-1}) \leq m_{i'}(c - x_{i-1}) + m_{i''}(x_i - c)$.
- This shows that the lower sum either increases or stays the same when a partition is refined. Thus, $L(f, P) \leq L(f, Q)$.
- Similarly, for the upper sum, let $M_{i'}$ be the supremum on $[x_{i-1}, c]$ and $M_{i''}$ be the supremum on $[c, x_i]$. We know that $M_i \geq M_{i'}$ and $M_i \geq M_{i''}$.
- The contribution to the upper sum from the interval $[x_{i-1}, x_i]$ in P is $M_i(x_i - x_{i-1})$.
- The contribution to the upper sum from the corresponding subintervals in Q is $M_{i'}(c - x_{i-1}) + M_{i''}(x_i - c)$.
- Since $M_i \geq M_{i'}$ and $M_i \geq M_{i''}$, we have $M_i(c - x_{i-1}) \geq M_{i'}(c - x_{i-1})$ and $M_i(x_i - c) \geq M_{i''}(x_i - c)$.
- Summing these, $M_i(c - x_{i-1}) + M_i(x_i - c) = M_i(x_i - x_{i-1}) \geq M_{i'}(c - x_{i-1}) + M_{i''}(x_i - c)$.

- This shows that the upper sum either decreases or stays the same when a partition is refined. Thus, $U(f,Q) \leq U(f,P)$.
- Finally, for any partition Q, it is always true that $L(f,Q) \leq U(f,Q)$ because for each subinterval, $m_i \leq M_i$.
- Combining these inequalities, we get $L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$.

(c) Let f: [a, b] → R be a bounded function on [a, b]. Prove that if f is integrable on [a, b], then for each ε > 0, there exists a partition P of [a, b] such that U(f, P) – L(f, P) < ε.

- By definition, a bounded function f on $[a,b]$ is Darboux integrable if its lower Darboux integral equals its upper Darboux integral, i.e., $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$.
- Let $I = \int_a^b f\,(x)dx$.
- By the definition of the lower Darboux integral, for any $\varepsilon > 0$, there exists a partition P_1 such that $I - \varepsilon/2 < L(f,P_1) \leq I$.
- By the definition of the upper Darboux integral, for any $\varepsilon > 0$, there exists a partition P_2 such that $I \leq U(f,P_2) < I + \varepsilon/2$.
- Let P be a common refinement of P_1 and P_2, i.e., $P = P_1 \cup P_2$.
- From part (b), we know that if $P_1 \subseteq P$, then $L(f,P_1) \leq L(f,P)$. So, $I - \varepsilon/2 < L(f,P)$.

- Also from part (b), if $P_2 \subseteq P$, then $U(f,P) \leq U(f,P_2)$. So, $U(f,P) < I + \varepsilon/2$.
- Combining these inequalities, we have: $U(f,P) - L(f,P) < (I + \varepsilon/2) - (I - \varepsilon/2)$ $U(f,P) - L(f,P) < I + \varepsilon/2 - I + \varepsilon/2$ $U(f,P) - L(f,P) < \varepsilon$.
- Thus, if f is integrable on $[a,b]$, then for each $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that $U(f,P) – L(f,P) < \varepsilon$.

(d) Let $f(x) = 2x + 1$ over the interval. Let $P = \{0,1/2,1,3/2,2\}$ be a partition of. Compute U(f, P), L(f, P) and U(f, P) – L(f, P).

- The interval is not explicitly stated, but based on the partition $P = \{0,1/2,1,3/2,2\}$, the interval is $[0,2]$.
- The function is $f(x) = 2x + 1$. This is an increasing function.
- The subintervals are:
  - $[0,1/2]$
  - $[1/2,1]$
  - $[1,3/2]$
  - $[3/2,2]$
- For an increasing function on an interval $[x_{i-1}, x_i]$:
  - $m_i = f(x_{i-1})$ (infimum)
  - $M_i = f(x_i)$ (supremum)

- $\Delta x_i = x_i - x_{i-1}$

- Calculations for each subinterval:
  - Interval 1: $[0,1/2]$
    - $m_1 = f(0) = 2(0) + 1 = 1$
    - $M_1 = f(1/2) = 2(1/2) + 1 = 2$
    - $\Delta x_1 = 1/2 - 0 = 1/2$
  - Interval 2: $[1/2,1]$
    - $m_2 = f(1/2) = 2(1/2) + 1 = 2$
    - $M_2 = f(1) = 2(1) + 1 = 3$
    - $\Delta x_2 = 1 - 1/2 = 1/2$
  - Interval 3: $[1,3/2]$
    - $m_3 = f(1) = 2(1) + 1 = 3$
    - $M_3 = f(3/2) = 2(3/2) + 1 = 4$
    - $\Delta x_3 = 3/2 - 1 = 1/2$
  - Interval 4: $[3/2,2]$
    - $m_4 = f(3/2) = 2(3/2) + 1 = 4$
    - $M_4 = f(2) = 2(2) + 1 = 5$
    - $\Delta x_4 = 2 - 3/2 = 1/2$

- Lower Darboux Sum $L(f,P)$: $L(f,P) = m_1\Delta x_1 + m_2\Delta x_2 + m_3\Delta x_3 + m_4\Delta x_4$ $L(f,P) = (1)(1/2) + (2)(1/2) + (3)(1/2) + (4)(1/2)$ $L(f,P) = 1/2 + 1 + 3/2 + 2 = 5$
- Upper Darboux Sum $U(f,P)$: $U(f,P) = M_1\Delta x_1 + M_2\Delta x_2 + M_3\Delta x_3 + M_4\Delta x_4$ $U(f,P) = (2)(1/2) + (3)(1/2) + (4)(1/2) + (5)(1/2)$ $U(f,P) = 1 + 3/2 + 2 + 5/2 = 7$
- Difference $U(f,P) – L(f,P)$: $U(f,P) – L(f,P) = 7 - 5 = 2$.

Question 2: (a) Let f be an integrable function on [a, b]. Show that –f is integrable on [a, b] and $\int_a^b (–f) = –\int_a^b f$.

- Given that f is an integrable function on $[a,b]$, by the definition of integrability, for any $\varepsilon > 0$, there exists a partition P such that $U(f,P) - L(f,P) < \varepsilon$.
- Let $g(x) = -f(x)$.
- For any subinterval $[x_{i-1}, x_i]$ of a partition P, let m_i and M_i be the infimum and supremum of f on this interval, respectively.
- The infimum of $g(x) = -f(x)$ on $[x_{i-1}, x_i]$ is $-M_i$. (Because if $M_i = \sup f(x)$, then $-M_i = \inf(-f(x))$.)
- The supremum of $g(x) = -f(x)$ on $[x_{i-1}, x_i]$ is $-m_i$. (Because if $m_i = \inf f(x)$, then $-m_i = \sup(-f(x))$.)
- Now, let's look at the Darboux sums for $g(x)$:
    - $L(g,P) = \sum_{i=1}^{n}(-M_i)\Delta x_i = -\sum_{i=1}^{n} M_i \Delta x_i = -U(f,P)$.

- $U(g,P) = \sum_{i=1}^{n}(-m_i)\Delta x_i = -\sum_{i=1}^{n} m_i \Delta x_i = -L(f,P)$.

- Now consider the difference $U(g,P) - L(g,P)$:
  - $U(g,P) - L(g,P) = (-L(f,P)) - (-U(f,P)) = U(f,P) - L(f,P)$.
- Since f is integrable, for any $\varepsilon > 0$, there exists a partition P such that $U(f,P) - L(f,P) < \varepsilon$.
- Therefore, $U(g,P) - L(g,P) < \varepsilon$, which implies that $g(x) = -f(x)$ is integrable on $[a,b]$.
- Now, let's show that $\int_a^b(-f) = -\int_a^b f$.
- We know that $\int_a^b(-f) = \underline{\int_a^b}(-f) = \lim_{||P||\to 0} L(-f,P) = \lim_{||P||\to 0}(-U(f,P))$.
- Since f is integrable, $\lim_{||P||\to 0} U(f,P) = \int_a^b f$.
- Therefore, $\int_a^b(-f) = -\int_a^b f$.

(b) Let f: → R be defined as f(x) = {1, if x is rational; –1, if x is irrational}. Calculate the upper and lower Darboux Integrals for f on the interval. Is f integrable on?

- The interval is not explicitly stated but is implicitly $[a,b]$ as usually understood for Darboux integrals. Let's assume the interval is $[a,b]$.
- Let P be any partition of $[a,b]$, $P = \{x_0, x_1, \ldots, x_n\}$.
- Consider any subinterval $[x_{i-1}, x_i]$.

- Since every non-empty interval of real numbers contains both rational and irrational numbers:
  - The supremum of $f(x)$ on $[x_{i-1}, x_i]$ is $M_i = \sup_{x\in[x_{i-1},x_i]} f(x) = 1$ (because there's always a rational number in the interval).
  - The infimum of $f(x)$ on $[x_{i-1}, x_i]$ is $m_i = \inf_{x\in[x_{i-1},x_i]} f(x) = -1$ (because there's always an irrational number in the interval).
- Now, let's calculate the Darboux sums:
  - Lower Darboux Sum $L(f,P) = \sum_{i=1}^{n} m_i \, \Delta x_i = \sum_{i=1}^{n}(-1)(x_i - x_{i-1})$.
  - $L(f,P) = -(x_1 - x_0) - (x_2 - x_1) - \ldots - (x_n - x_{n-1})$.
  - This is a telescoping sum: $L(f,P) = -(x_n - x_0) = -(b-a)$.
  - Upper Darboux Sum $U(f,P) = \sum_{i=1}^{n} M_i \, \Delta x_i = \sum_{i=1}^{n}(1)(x_i - x_{i-1})$.
  - $U(f,P) = (x_1 - x_0) + (x_2 - x_1) + \ldots + (x_n - x_{n-1})$.
  - This is a telescoping sum: $U(f,P) = (x_n - x_0) = (b-a)$.
- The lower Darboux integral is $\underline{\int_a^b} f(x)dx = \sup_P L(f,P) = \sup_P(-(b-a)) = -(b-a)$.
- The upper Darboux integral is $\overline{\int_a^b} f(x)dx = \inf_P U(f,P) = \inf_P(b-a) = (b-a)$.
- Is f integrable on?

- For f to be integrable, the lower Darboux integral must be equal to the upper Darboux integral.
- Here, $\underline{\int_a^b} f(x)dx = -(b-a)$ and $\overline{\int_a^b} f(x)dx = (b-a)$.
- Since $a \neq b$, $-(b-a) \neq (b-a)$. For example, if $a = 0, b = 1$, then the lower integral is -1 and the upper integral is 1.
- Therefore, the function $f(x)$ is not integrable on the given interval.

(c) Let f: [a, b] → R be a bounded function. Show that if f is integrable (Darboux) on [a, b], then it is Riemann integrable on [a, b].

- A function f is Darboux integrable on $[a, b]$ if for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$. Also, the common value of the upper and lower Darboux integrals is the Darboux integral.
- A function f is Riemann integrable on $[a, b]$ if there exists a number I such that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every partition P with norm $||P|| < \delta$ and any choice of sample points $c_i \in [x_{i-1}, x_i]$, we have $|R(f, P) - I| < \varepsilon$, where $R(f, P) = \sum_{i=1}^{n} f(c_i)\Delta x_i$ is the Riemann sum.
- Let f be Darboux integrable on $[a, b]$ with integral $I = \int_a^b f(x)dx$.
- By the definition of Darboux integrability, for every $\varepsilon > 0$, there exists a partition P_0 such that $U(f, P_0) - L(f, P_0) < \varepsilon$.
- For any partition P, and any choice of sample points $c_i \in [x_{i-1}, x_i]$:

- We know that $m_i \le f(c_i) \le M_i$ for each subinterval.
- Multiplying by Δx_i and summing over all subintervals: $L(f,P) = \sum_{i=1}^{n} m_i \, \Delta x_i \le \sum_{i=1}^{n} f\,(c_i)\Delta x_i = R(f,P) \le \sum_{i=1}^{n} M_i \, \Delta x_i = U(f,P)$.

* Also, by the property of Darboux integrals, we know that for any partition P:
  - $L(f,P) \le \underline{\int_a^b} f(x)dx = I = \overline{\int_a^b} f(x)dx \le U(f,P)$.
* Combining these, we have: $L(f,P) \le R(f,P) \le U(f,P)$ and $L(f,P) \le I \le U(f,P)$.
* This implies that both $R(f,P)$ and I lie within the interval $[L(f,P), U(f,P)]$.
* Therefore, the distance between $R(f,P)$ and I must be less than or equal to the length of this interval: $|R(f,P) - I| \le U(f,P) - L(f,P)$.
* Since f is Darboux integrable, for any $\varepsilon > 0$, there exists a partition P such that $U(f,P) - L(f,P) < \varepsilon$.
* This implies that for any such partition P and any choice of sample points c_i, we have $|R(f,P) - I| < \varepsilon$.
* This is precisely the definition of Riemann integrability. Therefore, if f is Darboux integrable, it is Riemann integrable.

(d) For a bounded function f on [a, b], define the Riemann Sum associated with a partition P. Hence, give Riemann's definition of integrability.

- **Riemann Sum:** Let f be a bounded function on the interval $[a, b]$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ such that $a = x_0 < x_1 < \ldots < x_n = b$. Let $\Delta x_i = x_i - x_{i-1}$ be the length of the i-th subinterval. For each subinterval $[x_{i-1}, x_i]$, choose an arbitrary sample point $c_i \in [x_{i-1}, x_i]$. The Riemann sum for f corresponding to the partition P and the chosen sample points c_i is defined as: $R(f, P) = \sum_{i=1}^{n} f(c_i)\Delta x_i$.

- **Riemann's Definition of Integrability:** A bounded function f on $[a, b]$ is said to be Riemann integrable if there exists a unique real number I such that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every partition P of $[a, b]$ with norm $||P|| = \max_i \Delta x_i < \delta$, and for any choice of sample points $c_i \in [x_{i-1}, x_i]$, we have: $|R(f, P) - I| < \varepsilon$. The number I is called the Riemann integral of f over $[a, b]$, denoted by $\int_a^b f(x)dx$.

Question 3: (a) Prove that every bounded piecewise monotonic function f on [a, b] is integrable.

- A function f is piecewise monotonic on $[a, b]$ if the interval $[a, b]$ can be divided into a finite number of subintervals such that f is monotonic on each subinterval.

- A function is monotonic on an interval if it is either increasing or decreasing on that interval.

- We know that every monotonic function on a closed and bounded interval is integrable.

- Let f be a bounded piecewise monotonic function on $[a, b]$.

- This means there exists a partition of $[a, b]$, say $P_0 = \{x_0, x_1, \dots, x_n\}$ such that on each subinterval $[x_{j-1}, x_j]$, f is monotonic.
- Since f is monotonic on each $[x_{j-1}, x_j]$, f is integrable on each $[x_{j-1}, x_j]$.
- This implies that for each subinterval $[x_{j-1}, x_j]$ and for any $\varepsilon_j > 0$, there exists a partition P_j of $[x_{j-1}, x_j]$ such that $U(f|_{[x_{j-1},x_j]}, P_j) - L(f|_{[x_{j-1},x_j]}, P_j) < \varepsilon_j$.
- Let $P = P_0 \cup P_1 \cup \dots \cup P_n$ be a partition of $[a, b]$ formed by combining all the partition points.
- Then $U(f, P) - L(f, P) = \sum_{j=1}^{n}(U(f|_{[x_{j-1},x_j]}, P_j) - L(f|_{[x_{j-1},x_j]}, P_j))$.
- We can choose $\varepsilon_j = \varepsilon/n$ for each subinterval.
- Then $U(f, P) - L(f, P) < \sum_{j=1}^{n} \varepsilon/n = n \cdot (\varepsilon/n) = \varepsilon$.
- Since for any $\varepsilon > 0$, we can find such a partition P, f is integrable on $[a, b]$.

(b) Show that if a function f is integrable on [a, b], then |f| is integrable on [a, b] and $|\int_a^b f| \leq \int_a^b |f|$.

- **Part 1: Show that |f| is integrable.**
    - Given that f is integrable on $[a, b]$, it means f is bounded on $[a, b]$. If f is bounded, then there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. This also means that $|f|$ is bounded.

- For any subinterval $[x_{i-1}, x_i]$ of a partition P, let m_i and M_i be the infimum and supremum of f on this interval, and $m_{i'}$ and $M_{i'}$ be the infimum and supremum of $|f|$ on this interval.
- We know that for any $x, y \in [x_{i-1}, x_i]$, we have $||f(x)| - |f(y)|| \leq |f(x) - f(y)|$.
- This implies that $M_{i'} - m_{i'} \leq M_i - m_i$. (The oscillation of $|f|$ is less than or equal to the oscillation of f).
- Now, consider the difference between the upper and lower Darboux sums for $|f|$: $U(|f|, P) - L(|f|, P) = \sum_{i=1}^{n}(M_{i'} - m_{i'})\Delta x_i$.
- We have $U(|f|, P) - L(|f|, P) \leq \sum_{i=1}^{n}(M_i - m_i)\Delta x_i = U(f, P) - L(f, P)$.
- Since f is integrable, for any $\varepsilon > 0$, there exists a partition P such that $U(f, P) - L(f, P) < \varepsilon$.
- Therefore, $U(|f|, P) - L(|f|, P) < \varepsilon$, which implies that $|f|$ is integrable on $[a, b]$.

- **Part 2: Show that $|\int_a^b f| \leq \int_a^b |f|$.**
  - We know that for any real number x, $-|x| \leq x \leq |x|$.
  - Therefore, for any $x \in [a, b]$, we have $-|f(x)| \leq f(x) \leq |f(x)|$.
  - Since integration preserves inequalities: $\int_a^b(- |f(x)|)dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx$.

- From part (a) of Question 2, we know that $\int_a^b (-|f(x)|)dx = -\int_a^b |f(x)|dx$.
- So, $-\int_a^b |f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx$.
- This inequality is equivalent to saying that $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$.

(c) If f is a continuous, non-negative function on [a, b] and if $\int_a^b f = 0$, then prove that f is identically 0 on [a, b]. Give an example of a discontinuous non-zero function f on for which $\int_0^1 f = 0$.

- **Part 1: Proof for continuous, non-negative function.**
  - Assume, for the sake of contradiction, that f is not identically 0 on $[a, b]$.
  - Since f is non-negative, this means there exists at least one point $c \in [a, b]$ such that $f(c) > 0$.
  - Since f is continuous at c and $f(c) > 0$, by the definition of continuity, for $\varepsilon = f(c)/2 > 0$, there exists a $\delta > 0$ such that for all $x \in [c - \delta, c + \delta] \cap [a, b]$, we have $|f(x) - f(c)| < f(c)/2$.
  - This implies $f(c) - f(c)/2 < f(x) < f(c) + f(c)/2$, or $f(c)/2 < f(x) < 3f(c)/2$.
  - So, there exists an interval $[c_1, c_2] \subseteq [a, b]$ (where $[c_1, c_2]$ is $[c - \delta, c + \delta] \cap [a, b]$) such that for all $x \in [c_1, c_2]$, $f(x) \geq f(c)/2 > 0$. Let $k = f(c)/2$.

- Now, consider the integral of f over $[a,b]$: $\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \int_{c_2}^{b} f(x)dx$.
- Since $f(x) \geq 0$ on $[a,b]$, $\int_a^{c_1} f(x)dx \geq 0$ and $\int_{c_2}^{b} f(x)dx \geq 0$.
- On the interval $[c_1, c_2]$, we have $f(x) \geq k > 0$.
- Therefore, $\int_{c_1}^{c_2} f(x)dx \geq \int_{c_1}^{c_2} k\,dx = k(c_2 - c_1)$.
- Since $k > 0$ and $c_2 - c_1 > 0$, it follows that $k(c_2 - c_1) > 0$.
- Thus, $\int_a^b f(x)dx \geq k(c_2 - c_1) > 0$.
- This contradicts our assumption that $\int_a^b f(x)dx = 0$.
- Therefore, our initial assumption must be false, meaning $f(x)$ must be identically 0 on $[a,b]$.

- **Part 2: Example of a discontinuous non-zero function f on for which $\int_0^1$ f = 0.**
    - The interval is implicitly $[0,1]$.
    - Let $f: [0,1] \to \mathbb{R}$ be defined as: $f(x) = \begin{cases} 1 & \text{if } x = 1/2 \\ 0 & \text{if } x \neq 1/2 \end{cases}$
    - This function is discontinuous at $x = 1/2$. It is not identically zero on $[0,1]$ (because $f(1/2) = 1$).
    - However, when calculating the integral, the value of the function at a single point does not affect the value of the definite integral. The set of discontinuities is a set of measure zero.

- More formally, for any partition P of $[0,1]$, if $1/2$ is not a partition point, it falls into one subinterval. The contribution of this subinterval to the integral will approach zero as the norm of the partition approaches zero. If $1/2$ is a partition point, it's an endpoint of two intervals.
- The integral $\int_0^1 f\,(x)dx$ can be evaluated. The function is 0 everywhere except at a single point.
- The lower Darboux sum will always be 0 (since the infimum in any interval containing 0 will be 0, and in intervals not containing 1/2, it is 0).
- The upper Darboux sum will also approach 0. For any interval $[x_{i-1}, x_i]$ containing $1/2$, the supremum is 1, so the contribution is $1 \cdot (x_i - x_{i-1})$. For other intervals, the supremum is 0. As the norm of the partition goes to 0, the length of the interval containing $1/2$ goes to 0, so the upper sum also goes to 0.
- Therefore, $\int_0^1 f\,(x)dx = 0$.

(d) State and prove Fundamental Theorem of Calculus I.

- **Statement of Fundamental Theorem of Calculus I (FTC I):** Let f be a continuous function on the closed interval $[a, b]$. Let $F(x) = \int_a^x f\,(t)dt$ for $x \in [a, b]$. Then F is differentiable on (a, b) and $F'(x) = f(x)$ for all $x \in (a, b)$. If f is continuous at a, F is right-differentiable at a and $F'(a^+) = f(a)$. If f is continuous at b, F is left-differentiable at b and $F'(b^-) = f(b)$.

- **Proof:**
  - Let $x \in (a, b)$. We want to show that $F'(x) = f(x)$, which means we need to evaluate the limit: $F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$.
  - By the definition of $F(x)$: $F(x+h) - F(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt$.
  - So, we need to evaluate $\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)dt$.
  - Since f is continuous on $[a, b]$, it is continuous on any subinterval, including $[x, x+h]$ (or $[x+h, x]$ if $h < 0$).
  - By the Extreme Value Theorem, on this closed interval, f attains its minimum value m_h and maximum value M_h.
  - So, $m_h \le f(t) \le M_h$ for all t between x and $x + h$.
  - Integrating this inequality over the interval from x to $x + h$: If $h > 0$: $\int_x^{x+h} m_h \, dt \le \int_x^{x+h} f(t)dt \le \int_x^{x+h} M_h \, dt$ $m_h h \le \int_x^{x+h} f(t)dt \le M_h h$ Dividing by h (since $h > 0$): $m_h \le \frac{1}{h} \int_x^{x+h} f(t)dt \le M_h$. If $h < 0$: The integral is from x to $x + h$, which means $x + h < x$. $\int_x^{x+h} m_h \, dt \ge \int_x^{x+h} f(t)dt \ge \int_x^{x+h} M_h \, dt$ (reversing limits changes sign, or multiplying by negative h flips inequality) $m_h h \ge \int_x^{x+h} f(t)dt \ge M_h h$ Dividing by h (since $h < 0$, we also flip the inequalities): $m_h \le \frac{1}{h} \int_x^{x+h} f(t)dt \le M_h$.
  - In both cases ($h > 0$ or $h < 0$), we have $m_h \le \frac{F(x+h) - F(x)}{h} \le M_h$.

- As $h \to 0$, the interval $[x, x+h]$ shrinks to the point x.
- Since f is continuous at x, as $h \to 0$, $m_h \to f(x)$ and $M_h \to f(x)$.
- By the Squeeze Theorem, $\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x)$.
- Therefore, $F'(x) = f(x)$.
- The statements about right-differentiability at a and left-differentiability at b follow similarly by considering one-sided limits.

Question 4: (a) If u and v are continuous functions on [a, b] that are differentiable on (a, b), and u' and v' are integrable, prove that $\int_a^b$ uv' + $\int_a^b$ u'v = u(b)v(b) – u(a)v(a). Hence evaluate $\int_0$^(π/2) x cos x.

- **Proof of Integration by Parts Formula:**
  - Consider the product function $P(x) = u(x)v(x)$.
  - Since u and v are differentiable on (a, b) and continuous on $[a, b]$, their product $P(x)$ is also differentiable on (a, b) and continuous on $[a, b]$.
  - By the product rule for differentiation, $P'(x) = u'(x)v(x) + u(x)v'(x)$.
  - Given that u' and v' are integrable, and u and v are continuous (and thus bounded), it implies that $u'v$ and uv' are also integrable (products of integrable/bounded functions are integrable).

- Now, apply the Fundamental Theorem of Calculus II, which states that if P' is integrable on $[a, b]$, then $\int_a^b P'(x)dx = P(b) - P(a)$.
- So, $\int_a^b (u'(x)v(x) + u(x)v'(x))dx = u(b)v(b) - u(a)v(a)$.
- By the linearity of integration: $\int_a^b u'(x)v(x)dx + \int_a^b u(x)v'(x)dx = u(b)v(b) - u(a)v(a)$.
- This is the integration by parts formula: $\int_a^b u\,v'dx = [uv]_a^b - \int_a^b u'\,vdx$. (Rearranged form).
- The given form is $\int_a^b u\,v' + \int_a^b u'\,v = u(b)v(b) - u(a)v(a)$.

- **Evaluate ∫₀^(π/2) x cos x:**
  - We use the integration by parts formula. Let $u(x) = x$ and $dv = \cos x dx$.
  - Then $du = dx$ and $v = \int \cos x dx = \sin x$.
  - Applying the formula $\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$: $\int_0^{\pi/2} x\cos x dx = [x\sin x]_0^{\pi/2} - \int_0^{\pi/2} \sin x dx$.
  - Evaluate the first term: $[x\sin x]_0^{\pi/2} = (\frac{\pi}{2}\sin(\frac{\pi}{2})) - (0 \cdot \sin(0)) = (\frac{\pi}{2} \cdot 1) - 0 = \frac{\pi}{2}$.
  - Evaluate the second term: $\int_0^{\pi/2} \sin x dx = [-\cos x]_0^{\pi/2} = (-\cos(\frac{\pi}{2})) - (-\cos(0)) = (-0) - (-1) = 1$.

- Substitute these values back: $\int_0^{\pi/2} x\cos x dx = \frac{\pi}{2} - 1$.

(b) Use the Fundamental Theorem of Calculus to calculate $\lim_{x\to 0} (1/x)\int_0^x e^{t^2} dt$.

- This limit has the form $\lim_{x\to 0} \frac{\int_0^x e^{t^2} dt}{x}$. This is an indeterminate form of type $0/0$ because $\int_0^0 e^{t^2}\, dt = 0$.
- We can use L'Hôpital's Rule.
- Let $F(x) = \int_0^x e^{t^2}\, dt$. Then by the Fundamental Theorem of Calculus I, $F'(x) = e^{x^2}$.
- The derivative of the denominator x is 1.
- Applying L'Hôpital's Rule: $\lim_{x\to 0} \frac{\int_0^x e^{t^2} dt}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}(\int_0^x e^{t^2} dt)}{\frac{d}{dx}(x)}. =$

  $\lim_{x\to 0} \frac{e^{x^2}}{1}. = e^{0^2} = e^0 = 1.$
- Alternatively, this limit is precisely the definition of the derivative of the function $F(x) = \int_0^x e^{t^2}\, dt$ at $x = 0$, i.e., $F'(0)$.
- By FTC I, $F'(x) = e^{x^2}$. So, $F'(0) = e^{0^2} = 1$.

(c) Let f be an integrable function on [a, b]. For x in [a, b], let $F(x) = \int_a^x f(t)dt$. Then show that F is uniformly continuous on [a, b]. For F(x) = {0, t<0; t, 0≤t≤1; 4, t>1} (i) Determine the function $F(x) = \int_0^x f(t)dt$. (ii) Where is F continuous?

- **Part 1: Show F is uniformly continuous.**

- Given that f is an integrable function on $[a, b]$, it implies that f is bounded on $[a, b]$.
- So, there exists a constant $M > 0$ such that $|f(t)| \leq M$ for all $t \in [a, b]$.
- Let $F(x) = \int_a^x f(t)dt$.
- Consider any $x_1, x_2 \in [a, b]$ with $x_1 < x_2$.
- $|F(x_2) - F(x_1)| = |\int_a^{x_2} f(t)dt - \int_a^{x_1} f(t)dt| = |\int_{x_1}^{x_2} f(t)dt|$.
- Using the property that $|\int_c^d g(t)dt| \leq \int_c^d |g(t)|dt$: $|\int_{x_1}^{x_2} f(t)dt| \leq \int_{x_1}^{x_2} |f(t)|dt$.
- Since $|f(t)| \leq M$: $\int_{x_1}^{x_2} |f(t)|dt \leq \int_{x_1}^{x_2} M\, dt = M(x_2 - x_1)$.
- So, $|F(x_2) - F(x_1)| \leq M|x_2 - x_1|$. (We can use $|x_2 - x_1|$ to cover both $x_1 < x_2$ and $x_2 < x_1$).
- This shows that F is Lipschitz continuous on $[a, b]$ with Lipschitz constant M.
- Since every Lipschitz continuous function on a closed interval is uniformly continuous, F is uniformly continuous on $[a, b]$.

- **Part 2: Given** $f(t) = \{0, t < 0; t, 0 \leq t \leq 1; 4, t > 1\}$
  - The question seems to have a typo for $F(x) = \{0, t < 0; t, 0 \leq t \leq 1; 4, t > 1\}$. This looks like a definition of a function, let's call it $g(t)$, which is not $f(t)$ from the previous context. Let's

assume the question meant "Let $f(t)$ be defined as:" and then proceeds to define a piecewise function.

- So, let $f(t)$ be defined as: $f(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \le t \le 1 \\ 4 & \text{if } t > 1 \end{cases}$
- **(i) Determine the function $F(x) = \int_0^x f(t)dt$.**
  - We need to consider different cases for x.
  - Case 1: $x < 0$ $F(x) = \int_0^x f(t)dt = 0$ (since $f(t) = 0$ for $t < 0$ and the upper limit is less than the lower limit, $\int_0^x = -\int_x^0$). More rigorously, $\int_0^x 0\,dt = 0$.
  - Case 2: $0 \le x \le 1$ $F(x) = \int_0^x f(t)dt = \int_0^x t\,dt = [\frac{t^2}{2}]_0^x = \frac{x^2}{2} - 0 = \frac{x^2}{2}$.
  - Case 3: $x > 1$ $F(x) = \int_0^x f(t)dt = \int_0^1 f(t)dt + \int_1^x f(t)dt$. $F(x) = \int_0^1 t\,dt + \int_1^x 4\,dt$. $F(x) = [\frac{t^2}{2}]_0^1 + [4t]_1^x$. $F(x) = (\frac{1^2}{2} - 0) + (4x - 4(1)) = \frac{1}{2} + 4x - 4 = 4x - \frac{7}{2}$.
  - So, the function $F(x)$ is: $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2/2 & \text{if } 0 \le x \le 1 \\ 4x - 7/2 & \text{if } x > 1 \end{cases}$
- **(ii) Where is F continuous?**
  - Each piece of $F(x)$ (0, $x^2/2$, $4x - 7/2$) is a polynomial, and thus continuous within its defined interval. We need

to check continuity at the transition points $x = 0$ and $x = 1$.

- At $x = 0$:
  - $\lim_{x \to 0^-} F(x) = \lim_{x \to 0^-} 0 = 0$.
  - $\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} x^2/2 = 0^2/2 = 0$.
  - $F(0) = 0^2/2 = 0$.
  - Since the left limit, right limit, and function value are all equal at $x = 0$, F is continuous at $x = 0$.
- At $x = 1$:
  - $\lim_{x \to 1^-} F(x) = \lim_{x \to 1^-} x^2/2 = 1^2/2 = 1/2$.
  - $\lim_{x \to 1^+} F(x) = \lim_{x \to 1^+} (4x - 7/2) = 4(1) - 7/2 = 8/2 - 7/2 = 1/2$.
  - $F(1) = 1^2/2 = 1/2$.
  - Since the left limit, right limit, and function value are all equal at $x = 1$, F is continuous at $x = 1$.
- Therefore, $F(x)$ is continuous for all $x \in \mathbb{R}$.

(d) For t ∈, define F(t) = {0, t<1/2; 1, t≥1/2} and let f(x) = x^2, x ∈. Show that f is F-integrable and that $\int_0^1$ f dF = f(1/2).

- This question refers to the Riemann-Stieltjes integral, denoted by $\int_a^b f\, dF$.

- The interval for integration is $[0,1]$.
- The integrator function is $F(t) = \begin{cases} 0 & \text{if } t < 1/2 \\ 1 & \text{if } t \geq 1/2 \end{cases}$. This is a step function with a jump at $t = 1/2$.
- The integrand function is $f(x) = x^2$. This is a continuous function.
- **Showing f is F-integrable:**
  - A common theorem states that if f is continuous on $[a, b]$ and F is a function of bounded variation on $[a, b]$, then f is Riemann-Stieltjes integrable with respect to F.
  - In our case, $f(x) = x^2$ is continuous on $[0,1]$.
  - The function $F(t)$ is a step function with a single jump. Such functions are of bounded variation. The total variation is $|F(1/2) - F(1/2^-)| = |1 - 0| = 1$.
  - Therefore, f is F-integrable on $[0,1]$.
- **Calculating $\int_0^1$ f dF:**
  - For a function F that is a step function with a single jump at $c \in (a, b)$, and f is continuous at c, the Riemann-Stieltjes integral $\int_a^b f\, dF$ simplifies to: $\int_a^b f\,(x) dF(x) = f(c)[F(c^+) - F(c^-)]$.
  - In our case, the jump occurs at $c = 1/2$.
  - $F(1/2^+) = 1$ (since $F(t) = 1$ for $t \geq 1/2$).
  - $F(1/2^-) = 0$ (since $F(t) = 0$ for $t < 1/2$).

- The jump size is $[F(1/2^+) - F(1/2^-)] = 1 - 0 = 1$.
- The integrand function is $f(x) = x^2$.
- We need to evaluate f at the jump point $c = 1/2$: $f(1/2) = (1/2)^2 = 1/4$.
- Therefore, $\int_0^1 f\,dF = f(1/2) \cdot (1) = f(1/2)$.
- So, $\int_0^1 x^2\,dF(x) = (1/2)^2 = 1/4$.

Question 5: (a) Find the volume of the solid generated when the region enclosed by the curves x = √y and x = y/4 is revolved about the x – axis.

- First, find the points of intersection of the two curves: $x = \sqrt{y} \Longrightarrow x^2 = y$ $x = y/4 \Longrightarrow y = 4x$
- Substitute $y = 4x$ into $x^2 = y$: $x^2 = 4x$ $x^2 - 4x = 0$ $x(x-4) = 0$ So, $x = 0$ or $x = 4$.
- If $x = 0$, $y = 0$. Point is $(0,0)$.
- If $x = 4$, $y = 4^2 = 16$ (from $y = x^2$) or $y = 4(4) = 16$ (from $y = 4x$). Point is $(4,16)$.
- The region is enclosed by $y = x^2$ and $y = 4x$.
- We are revolving about the x-axis. We will use the Washer Method.
- The outer radius $R(x)$ is the upper curve, and the inner radius $r(x)$ is the lower curve.

- On the interval $[0,4]$, $4x \geq x^2$. To check, pick $x = 1$, $4(1) \geq 1^2 \Longrightarrow 4 \geq 1$. So $y = 4x$ is the outer curve, and $y = x^2$ is the inner curve.
- $R(x) = 4x$
- $r(x) = x^2$
- The volume V is given by the integral: $V = \int_a^b \pi\,([R(x)]^2 - [r(x)]^2)dx$. $V = \int_0^4 \pi\,((4x)^2 - (x^2)^2)dx$. $V = \pi \int_0^4 (\,16x^2 - x^4)dx$.
- Integrate term by term: $V = \pi[\frac{16x^3}{3} - \frac{x^5}{5}]_0^4$. $V = \pi[(\frac{16(4)^3}{3} - \frac{4^5}{5}) - (0 - 0)]$. $V = \pi[\frac{16\cdot 64}{3} - \frac{1024}{5}]$. $V = \pi[\frac{1024}{3} - \frac{1024}{5}]$. $V = 1024\pi[\frac{1}{3} - \frac{1}{5}]$. $V = 1024\pi[\frac{5-3}{15}]$. $V = 1024\pi\,\frac{2}{15} = \frac{2048\pi}{15}$.

(b) Use cylindrical shells to find the volume of the solid generated when the region under y = x² is revolved about the line y = –1.

- The region is under $y = x^2$. This typically means from $y = 0$ to $y = x^2$. Let's assume the interval for x is from 0 to some value, say b, for a meaningful region. If not specified, we usually mean the region bounded by $y = x^2$ and $y = 0$ (the x-axis). Let's assume the region is from $x = 0$ to $x = 2$ for a specific example, or more generally an interval $[0, a]$. Let's assume we are integrating from $x = 0$ to some X.
- However, revolving about $y = -1$ with cylindrical shells usually implies integrating with respect to y. This means we need to express x in terms of y.
- From $y = x^2$, we have $x = \sqrt{y}$ (assuming $x \geq 0$).

- The region is bounded by $y = x^2$, $x = 0$, and some upper limit for y. Let's assume the region is under $y = x^2$ from $x = 0$ to $x = 2$. So y goes from 0 to $2^2 = 4$.
- The axis of revolution is $y = -1$.
- For cylindrical shells when revolving about a horizontal line $y = k$:
  - Shell height is $x_{right} - x_{left}$ in terms of y. So $h(y) = \sqrt{y} - 0 = \sqrt{y}$.
  - Shell radius is the distance from the axis of revolution $y = -1$ to the strip at height y. So $r(y) = y - (-1) = y + 1$.
- The volume V is given by $V = \int_c^d 2\,\pi \cdot \text{radius} \cdot \text{height}dy$.
- The limits of integration for y are from 0 to 4 (since x goes from 0 to 2, $y = x^2$ goes from $y = 0$ to $y = 4$). $V = \int_0^4 2\,\pi(y + 1)\sqrt{y}dy$. $V = 2\pi \int_0^4 (\,y^{3/2} + y^{1/2})dy$.
- Integrate term by term: $V = 2\pi[\frac{y^{5/2}}{5/2} + \frac{y^{3/2}}{3/2}]_0^4$. $V = 2\pi[\frac{2}{5}y^{5/2} + \frac{2}{3}y^{3/2}]_0^4$. $V = 2\pi[(\frac{2}{5}(4)^{5/2} + \frac{2}{3}(4)^{3/2}) - (0)]$. $V = 2\pi[(\frac{2}{5}(32) + \frac{2}{3}(8))]$. $V = 2\pi[\frac{64}{5} + \frac{16}{3}]$. $V = 2\pi[\frac{64\cdot 3+16\cdot 5}{15}]$. $V = 2\pi[\frac{192+80}{15}]$. $V = 2\pi\,\frac{272}{15} = \frac{544\pi}{15}$.

(c) Find the exact arc length of the curve x = (1/3)(y^2 + 2)$^{3/2}$ from y = 0 to y = 1.

- The arc length formula for a curve given by $x = g(y)$ from $y = c$ to $y = d$ is: $L = \int_c^d \sqrt{1 + (\frac{dx}{dy})^2}\, dy$.
- Given $x = \frac{1}{3}(y^2 + 2)^{3/2}$.
- Find $\frac{dx}{dy}$: $\frac{dx}{dy} = \frac{1}{3} \cdot \frac{3}{2}(y^2 + 2)^{1/2} \cdot (2y)$. $\frac{dx}{dy} = y(y^2 + 2)^{1/2}$.
- Now, calculate $(\frac{dx}{dy})^2$: $(\frac{dx}{dy})^2 = [y(y^2 + 2)^{1/2}]^2 = y^2(y^2 + 2) = y^4 + 2y^2$.
- Substitute into the arc length formula: $L = \int_0^1 \sqrt{1 + (y^4 + 2y^2)}\, dy$. $L = \int_0^1 \sqrt{y^4 + 2y^2 + 1}\, dy$. $L = \int_0^1 \sqrt{(y^2 + 1)^2}\, dy$. $L = \int_0^1 (y^2 + 1) dy$ (since $y^2 + 1$ is always positive).
- Integrate: $L = [\frac{y^3}{3} + y]_0^1$. $L = (\frac{1^3}{3} + 1) - (\frac{0^3}{3} + 0)$. $L = \frac{1}{3} + 1 = \frac{4}{3}$.

(d) Find the area of the surface that is generated by revolving the portion of the curve y = x² between x = 1 and x = 2 about the y – axis.

- The surface area formula for revolving about the y-axis for a curve $y = f(x)$ from $x = a$ to $x = b$ is: $S = \int_a^b 2\pi x \sqrt{1 + (\frac{dy}{dx})^2} dx$.
- Given $y = x^2$.
- Find $\frac{dy}{dx}$: $\frac{dy}{dx} = 2x$.
- Calculate $(\frac{dy}{dx})^2$: $(\frac{dy}{dx})^2 = (2x)^2 = 4x^2$.

- Substitute into the surface area formula: $S = \int_1^2 2\,\pi x\sqrt{1+4x^2}dx$.
- To evaluate this integral, use a u-substitution. Let $u = 1 + 4x^2$. Then $du = 8xdx$, so $xdx = \frac{1}{8}du$.
- Change the limits of integration: When $x = 1$, $u = 1 + 4(1)^2 = 5$. When $x = 2$, $u = 1 + 4(2)^2 = 1 + 16 = 17$.
- Substitute into the integral: $S = 2\pi \int_5^{17} \sqrt{u} \cdot \frac{1}{8} du$. $S = \frac{2\pi}{8} \int_5^{17} u^{1/2}\, du$. $S = \frac{\pi}{4}[\frac{u^{3/2}}{3/2}]_5^{17}$. $S = \frac{\pi}{4} \cdot \frac{2}{3}[u^{3/2}]_5^{17}$. $S = \frac{\pi}{6}[17^{3/2} - 5^{3/2}]$. $S = \frac{\pi}{6}[17\sqrt{17} - 5\sqrt{5}]$.

Question 6: (a) Discuss the convergence or divergence of the following improper integrals: (i) $\int_0^1 (1/\sqrt{x})dx$; (ii) ∫_(-∞)^(+∞) e^(-x²)dx.

- **(i) $\int_0^1 (1/\sqrt{x})dx$**
  - This is an improper integral of Type I because the integrand $1/\sqrt{x}$ has an infinite discontinuity at $x = 0$ within the interval $[0,1]$.
  - We evaluate it as a limit: $\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{a\to 0^+} \int_a^1 x^{-1/2}\, dx$.
  - Integrate $x^{-1/2}$: $\int x^{-1/2} dx = \frac{x^{1/2}}{1/2} = 2\sqrt{x}$.
  - Now apply the limits: $\lim_{a\to 0^+}[2\sqrt{x}]_a^1. = \lim_{a\to 0^+}(2\sqrt{1} - 2\sqrt{a}). = \lim_{a\to 0^+}(2 - 2\sqrt{a}). = 2 - 2(0) = 2$.
  - Since the limit exists and is a finite number (2), the improper integral converges.

- **(ii) ∫_(-∞)^(+∞) e^(-x²)dx**
  - This is an improper integral of Type II because both limits of integration are infinite.
  - We split the integral into two (or three) parts. Let's split it at 0: $\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \int_{-\infty}^{0} e^{-x^2}\,dx + \int_{0}^{+\infty} e^{-x^2}\,dx$.
  - Consider $\int_{0}^{+\infty} e^{-x^2}\,dx = \lim_{b\to+\infty} \int_{0}^{b} e^{-x^2}\,dx$.
  - The integral $\int e^{-x^2} dx$ is not expressible in terms of elementary functions. This is the Gaussian integral.
  - However, it is known that $\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$.
  - Since the value is finite, the integral converges.
  - (More formally, one can show convergence using comparison tests. For $x \geq 1$, $e^{-x^2} \leq e^{-x}$. We know $\int_{1}^{\infty} e^{-x}\,dx = [-e^{-x}]_{1}^{\infty} = 0 - (-e^{-1}) = e^{-1}$, which converges. Since e^{-x^2} is positive, and bounded by an integrable function, $\int_{1}^{\infty} e^{-x^2}\,dx$ converges. Similarly for $\int_{-\infty}^{-1} e^{-x^2}\,dx$. The integral over $[-1,1]$ is a definite integral of a continuous function, so it exists. Thus, the entire integral converges.)

(b) Find the value of r for which the integral $\int_1$^(+∞) x⁻ʳ dx exists or converges, and determine the value of the integral.

- This is an improper integral of Type II.

- We evaluate it as a limit: $\int_1^{+\infty} x^{-r}\, dx = \lim_{b\to+\infty} \int_1^b x^{-r}\, dx$.

- **Case 1:** $r = 1$**.** $\lim_{b\to+\infty} \int_1^b x^{-1}\, dx = \lim_{b\to+\infty} \int_1^b \frac{1}{x} dx. =$ $\lim_{b\to+\infty}[\ln|x|]_1^b. = \lim_{b\to+\infty}(\ln b - \ln 1). = \lim_{b\to+\infty}\ln b = +\infty$. So, for $r = 1$, the integral diverges.

- **Case 2:** $r \neq 1$**.** $\lim_{b\to+\infty} \int_1^b x^{-r}\, dx = \lim_{b\to+\infty}[\frac{x^{-r+1}}{-r+1}]_1^b. =$ $\lim_{b\to+\infty}(\frac{b^{1-r}}{1-r} - \frac{1^{1-r}}{1-r}). = \frac{1}{1-r}\lim_{b\to+\infty}(b^{1-r} - 1)$.
  - For the limit to exist, b^{1-r} must go to 0 as $b \to +\infty$. This happens if and only if the exponent $(1 - r)$ is negative.
  - $1 - r < 0 \Longrightarrow 1 < r$.
  - If $r > 1$, then $1 - r$ is negative, so $\lim_{b\to+\infty} b^{1-r} = 0$.
  - In this case, the value of the integral is $\frac{1}{1-r}(0 - 1) = \frac{-1}{1-r} = \frac{1}{r-1}$.

- **Conclusion:** The integral $\int_1^{+\infty} x^{-r}\, dx$ converges if and only if $r > 1$. When it converges, its value is $\frac{1}{r-1}$.

(c) Show that the improper integral $\int_1$^(+∞) (sin x / x^2)dx converges absolutely.

- For an integral to converge absolutely, $\int_1^{+\infty} |\frac{\sin x}{x^2}| dx$ must converge.

- We use the Comparison Test for improper integrals.

- We know that $|\sin x| \leq 1$ for all x.

- Therefore, $|\frac{\sin x}{x^2}| = \frac{|\sin x|}{x^2} \leq \frac{1}{x^2}$ for all $x \geq 1$.
- Now, consider the integral of the dominating function: $\int_1^{+\infty} \frac{1}{x^2} dx$.
- This is a p-integral of the form $\int_1^{+\infty} \frac{1}{x^p} dx$ with $p = 2$.
- From part (b), we know that such an integral converges if $p > 1$. Since $2 > 1$, $\int_1^{+\infty} \frac{1}{x^2} dx$ converges.
  - Let's verify: $\lim_{b \to +\infty} \int_1^b x^{-2}\, dx = \lim_{b \to +\infty} [-x^{-1}]_1^b = \lim_{b \to +\infty} (-\frac{1}{b} - (-\frac{1}{1})) = \lim_{b \to +\infty} (1 - \frac{1}{b}) = 1 - 0 = 1$.
- Since $0 \leq |\frac{\sin x}{x^2}| \leq \frac{1}{x^2}$ and $\int_1^{+\infty} \frac{1}{x^2} dx$ converges, by the Comparison Test, $\int_1^{+\infty} |\frac{\sin x}{x^2}| dx$ also converges.
- Therefore, the improper integral $\int_1^{+\infty} \frac{\sin x}{x^2} dx$ converges absolutely.

(d) Define the Gamma function Γ(m). Prove that Γ(m) converges if m > 0.

- **Definition of the Gamma Function Γ(m):** The Gamma function, $\Gamma(m)$, is defined by the improper integral: $\Gamma(m) = \int_0^{+\infty} x^{m-1}\, e^{-x} dx$. This definition is valid for complex numbers m with a positive real part (Re$(m) > 0$). For real values of m, it is defined for $m > 0$.
- **Proof of Convergence for m > 0:** We need to show that the integral converges for $m > 0$. This is an improper integral of Type I and Type II (if $m - 1 < 0$). We need to split the integral into two parts to

address both potential discontinuities: $\Gamma(m) = \int_0^{+\infty} x^{m-1} e^{-x} dx = \int_0^1 x^{m-1} e^{-x} dx + \int_1^{+\infty} x^{m-1} e^{-x} dx$.

- **Part 1: Convergence of $\int_0^1 x^{m-1} e^{-x} dx$ (Improper at $x = 0$ if $m - 1 < 0$, i.e., $m < 1$).**
    - For $x \in (0,1]$, e^{-x} is bounded between e^{-1} and $e^0 = 1$. So, $e^{-x} \leq 1$.
    - Thus, $0 \leq x^{m-1} e^{-x} \leq x^{m-1}$ for $x \in (0,1]$.
    - Consider the integral $\int_0^1 x^{m-1} dx$. This is a p-integral of the form $\int_0^1 \frac{1}{x^p} dx$ where $p = 1 - m$.
    - This integral converges if $p < 1$.
    - So, $1 - m < 1 \Longrightarrow m > 0$.
    - If $m > 0$, then $\int_0^1 x^{m-1} dx = [\frac{x^m}{m}]_0^1 = \frac{1^m}{m} - \lim_{a \to 0^+} \frac{a^m}{m} = \frac{1}{m} - 0 = \frac{1}{m}$, which is finite.
    - Since $\int_0^1 x^{m-1} dx$ converges for $m > 0$, by the Comparison Test, $\int_0^1 x^{m-1} e^{-x} dx$ converges for $m > 0$.
- **Part 2: Convergence of $\int_1^{+\infty} x^{m-1} e^{-x} dx$ (Improper at $x = +\infty$).**
    - For large x, the exponential term e^{-x} dominates any polynomial term x^{m-1}.

- We can use the Limit Comparison Test. Consider a comparison function $g(x) = \frac{1}{x^2}$. (We know $\int_1^{+\infty} \frac{1}{x^2} dx$ converges).
- Calculate the limit: $\lim_{x\to+\infty} \frac{x^{m-1}e^{-x}}{1/x^2} = \lim_{x\to+\infty} \frac{x^{m+1}}{e^x}$.
- By repeated application of L'Hôpital's Rule (if $m + 1 \geq 0$) or by the fact that exponential functions grow faster than any polynomial, this limit is 0 for any finite m.
- Since the limit is 0 (a finite non-negative number), and $\int_1^{+\infty} \frac{1}{x^2} dx$ converges, by the Limit Comparison Test, $\int_1^{+\infty} x^{m-1} e^{-x} dx$ converges.

- **Conclusion:** Since both parts of the integral converge for $m > 0$, the Gamma function $\Gamma(m) = \int_0^{+\infty} x^{m-1} e^{-x} dx$ converges if $m > 0$.