A New Approach to Screw Theory using Geometric Algebra

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Abstract

Since it was first developed by Sir Robert S. Ball at the end of the XIXth century, the Theory of Screws has known a considerable variety of reformulations, each of them underlining a different interpretation of screws: as geometrical, mechanical or algebraic objects. Beginning with an overview of the main existing formalisms of Screw Theory, this article determines what prerequisites must be satisfied to represent screws and conciliate their geometric and algebraic aspects in a clear, elegant and pedagogical way. The mathematical framework of Geometric Algebra was precisely designed to reveal the geometrical significance of the algebraic objects of physics. A new formalism for Screw Theory is hence introduced, based on the geometric algebra $\mathbb{G}^{3,0}$ and intended to generalize the concept of a screw (and therefore the extent of Screw Theory) and conciliate geometrical insight with algebraic efficiency. Moreover, this approach is coordinate-free and origin-independent, which makes it the correct point of view on affine geometry. A simple and straightforward description of barycentration and of finite motions appears as a natural feature of this new formulation.

Keywords: screw theory, geometric algebra, screw algebra, rigid body mechanics, finite motions, barycentration

Introduction

Robert S. Ball introduced screws in rigid body mechanics as a way to unify the descriptions of motions and forces [1]. His formulation of the Theory, however, was almost purely geometrical and made calculations rather difficult. Since then, various mathematical formalisms have been proposed to complete Ball's work and address this problem. Screw Theory, might we say, is a theory in much the same sense as Integration Theory: there are multiple ways to define the integral, each of them having its specificities, certain being more general than others, but all give the same results for the most common cases. And as

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a good integration theory must satisfy certain conditions (geometric interpretation as an area, connection to primitivation, Stokes' theorem…), a good screw theory must meet specific requirements, that we will determine by means of an overview of the main existing formulations. Specifically, we will identify three compulsory criteria, and a variety of additional requirements that are not satisfied by all formalisms but remain quite useful. Even if many other definitions and approaches to Screw Theory have been proposed throughout the years, we chose to reduce our survey to those which seemed the most common, the most efficient and the most useful to the development of a new formalism.

In the wake of Ball's work (detailed in section 1.1), mathematicians and physicists investigated the use of Plücker coordinates [2] (see section 1.2) and of hypercomplex numbers such as Clifford biquaternions [3] or dual vectors [4] (see section 1.3). From the second half of the XXth century and with the development of automation and robotism, researchers as F. M. Dimentberg [5] and K. H. Hunt [6] reinvigorated Screw Theory by promoting screws as the main mathematical description of mechanism kinematics. By the same time, Screw Theory underwent a completely different evolution in France [7, 8], where it was standardized (and is still widely used) for the purpose of teaching rigid body mechanics. This resulted in what we will accordingly call the French formalism, presented in section 1.4.

The development of Geometric Algebra by David Hestenes in the late XXth century introduced a new and efficient framework for geometry and its applications in physics [9]. Geometric algebras are axiomatically defined real Clifford algebras, based on a euclidian or pseudo-euclidian vector space. As screws are fundamentally geometric objects, and moreover *affine* objects, it seemed quite natural to formulate Screw Theory in Conformal Geometric Algebra (CGA) $\mathbb{G}^{4,1}$ [10]. Geometric Algebra will be properly introduced in section 1.5.

In section 2, we will use the three criteria and the additional requirements to elaborate a new and comprehensive formalism, which we believe to be the correct synthesis of the others, and the correct point of view on affine geometry.

Section 3 is dedicated to the formulation of rigid body mechanics in this new framework. We will particularly introduce in 3.2 a simple and efficient description of finite motions that perfectly fits into our formalism.

1. A quick overview of Screw Theory

The introduction of Dimentberg's famous book [5] provides much information on the early history of Screw Theory. A concise chronology of the development of rigid body dynamics in general can also be found in [11].

1.1. Ball's original formulation

Ball defines a screw α as an oriented axis in space, together with a real quantity p_α called the pitch of the screw [1]. A rigid body is said to receive a twist of amplitude α' about α if it undergoes both a rotation by an angle α' around the axis of α and a translation of amplitude $p_\alpha \alpha'$ along the same axis. Similarly, a rigid body is submitted to a wrench of intensity α'' on α if it is acted upon simultaneously by a force of intensity α'' and a couple $p_\alpha \alpha''$.

The *virtual coefficient* of two screws α and β is the quantity:

$$\varpi_{\alpha\beta} \stackrel{\text{def}}{=} \frac{1}{2}[(p_\alpha + p_\beta)\cos\theta - d\sin\theta] \tag{1}$$

where θ is the angle and d the orthogonal distance between the axes of α and β.
If a rigid body is acted upon by a wrench on β while it receives an infinitesimal twist about α, then the (infinitesimal) quantity of work done is $\alpha'\beta''\varpi_{\alpha\beta}$.
If $\varpi_{\alpha\beta} = 0$, then α and β are said to be *reciprocal*.

It can be demonstrated that we can find six screws $(\omega_i)_{i=1,\ldots,6}$ that are reciprocal to one another; their pitches are noted $(p_i)_{i=1,\ldots,6}$. Let α be a screw. Ball proves that there exist six real quantities $(\alpha_i)_{i=1,\ldots,6}$ such that:

$$p_\alpha = \sum_{i=1}^{6} p_i \alpha_i^2 \tag{2}$$

These quantities are termed the *coordinates* of α, and the (ω_i) are referred to as the *screws of reference*. The coordinates of a twist or a wrench about α are respectively defined by multiplying (2) by α' or α''.

From there, the kinematics and dynamics of a rigid body are discussed, either by means of purely geometrical arguments, or thanks to the screw coordinates and pitches. However, no operation on the screws themselves can be defined. Additionally, screw coordinates are abstract quantities that do not have physical significance. These issues make the calculations both cumbersome and difficult to interpret. Nevertheless, Ball's formulation uncovers the most important geometrical properties of screws, and defines the interesting virtual coefficient, interpreted as a work. It gives rise to the important principles:

- First criterion. *A screw must be able to represent an axis in space, with given amplitude and pitch.*
- Second criterion. *A simple operation on screws must yield their virtual coefficient.*

Ball doesn't include the amplitude in the concept of screw, because he distinguishes between what is purely geometrical (the line and the pitch) and what is physical (the amplitude). Such a distinction rather complicates the Theory.

All modern formalisms attach the amplitude to the definition of the screw.

1.2. Matrix formalism

The matrix formalism originated in the work of Julius Plücker [2] and reached its current form with the development of vector algebra, from which it arises quite naturally. It allows for a systematic representation of screws and of linear transformations on screws. A more detailed approach can be found in [12].

Plücker coordinates are a way to represent a straight line Δ in a coordinate frame $\mathcal{R}$. Let $\mathbf{S}$ be a vector collinear to Δ, $\mathbf{r}$ a vector pointing from the origin of $\mathcal{R}$ to a point on Δ. The Plücker coordinates of Δ are the six scalar quantities $(\mathbf{S}, \mathbf{r} \times \mathbf{S})$ where the symbol $\times$ denotes the vector cross product, and $\mathbf{r} \times \mathbf{S}$ is termed the *moment* of the line. Note that these coordinates are *homogeneous*, i.e. if λ be any non zero scalar, then $(\lambda\mathbf{S}, \lambda\mathbf{r} \times \mathbf{S})$ represents the same line Δ. We will take advantage of this freedom to define a line vector.

A *line vector* is a Plücker coordinates representation of a *given* vector $\mathbf{S}$ attached to an axis passing through point $\mathbf{r}$. It is thus written $\begin{bmatrix} \mathbf{S} \\ \mathbf{r} \times \mathbf{S} \end{bmatrix}_{\mathcal{R}}$.

A *couple vector* is a representation of a free vector $\mathbf{C}$ in a six-component column vector: $\begin{bmatrix} \mathbf{0} \\ \mathbf{C} \end{bmatrix}_{\mathcal{R}}$.

A screw \$ is defined as the sum of a couple vector and a line vector.

$$\$ = \begin{bmatrix} \mathbf{S} \\ \mathbf{r} \times \mathbf{S} + \mathbf{C} \end{bmatrix}_{\mathcal{R}} = \begin{bmatrix} \mathbf{S} \\ \mathbf{M} \end{bmatrix}_{\mathcal{R}} \tag{3}$$

Conversely, given a column vector $\mathsf{S} = \begin{bmatrix} \mathbf{S} \\ \mathbf{M} \end{bmatrix}_{\mathcal{R}}$, and defining $h \stackrel{\text{def}}{=} \frac{\mathbf{S} \cdot \mathbf{M}}{\mathbf{S}^2}$, we can decompose:

$$\mathsf{S} = \begin{bmatrix} \mathbf{S} \\ \mathbf{M} - h\mathbf{S} \end{bmatrix}_{\mathcal{R}} + \begin{bmatrix} \mathbf{0} \\ h\mathbf{S} \end{bmatrix}_{\mathcal{R}} \tag{4}$$

with $\mathbf{M} - h\mathbf{S} = \frac{\mathbf{S} \times (\mathbf{M} - h\mathbf{S})}{\mathbf{S}^2} \times \mathbf{S}$, thus the first term is a line vector, the second is a couple vector, and S is a screw. It can be verified that this is its unique decomposition into a line vector and a couple vector that is collinear to the axis of the line vector. We call this the *canonical decomposition* of S. $\mathbf{S}$ is called the *direction* or *principal vector*, $\mathbf{M}$ the *moment* and $h\mathbf{S}$ the *pure moment* of S.

Therefore:

1. The set of all screws, with the addition of screws and the product by a scalar, is a six-dimensional vector space.

2. The axis of a screw is the axis of the line vector in the canonical decomposition.
3. The amplitude of the screw is $\|\mathbf{S}\|$; the amplitude of the translation (for the twist) or of the force couple (for the wrench) is $\|h\mathbf{S}\|$.
4. h is the pitch of the screw.

The aforementionned first criterion is thus satisfied, *and* we have introduced the useful algebraic structure of the screw space.

We define on two screws $\$_1$ and $\$_2$ the *reciprocal product* $\$_1 \cdot \$_2$ and the *cross product* $\$_1 \times \$_2$:

$$\$_1 \cdot \$_2 = \$_1^\top \begin{bmatrix} \bar{\bar{0}}_3 & \bar{\bar{1}}_3 \\ \bar{\bar{1}}_3 & \bar{\bar{0}}_3 \end{bmatrix} \$_2 = \mathbf{S}_1 \cdot \mathbf{S}_2^0 + \mathbf{S}_1^0 \cdot \mathbf{S}_2 \tag{5}$$

$$\$_1 \times \$_2 = \begin{bmatrix} \mathbf{S}_1 \times \mathbf{S}_2 \\ \mathbf{S}_1 \times \mathbf{S}_2^0 + \mathbf{S}_1^0 \times \mathbf{S}_2 \end{bmatrix}_{\mathcal{R}} \tag{6}$$

where $\bar{\bar{1}}_3$ is the 3×3 identity matrix, and $\bar{\bar{0}}_3$ the 3×3 zero matrix.
Note that the reciprocal product is a symmetric non degenerate bilinear form (of signature $(3, 3)$), and the cross product is a Lie bracket.

A wrench is a screw representing a force and a torque. An instantaneous twist is a screw representing a rotational and a translational velocity. If $\$_1$ is an instantaneous twist, and $\$_2$ a wrench applied on a rigid body, then $\$_1 \cdot \$_2$ is the power delivered by the wrench to the body as it moves. The second criterion of the previous section is thereby also satisfied. A *finite twist* describes a finite motion: a rotation of angle θ around the axis (of which $\mathbf{s}$ a unit vector), combined with a translation of amplitude t along $\mathbf{s}$, is represented by:

$$\$_f = \begin{bmatrix} 2 \tan \frac{\theta}{2} \mathbf{s} \\ t\mathbf{s} + 2\mathbf{r} \times \tan \frac{\theta}{2} \mathbf{s} \end{bmatrix}_{\mathcal{R}} \tag{7}$$

where $\mathbf{r}$ denotes the position of a point on the axis.
Two finite twists $\$_f^1$ and $\$_f^2$ can be composed through the *triangle product* (whose form is given by Eqn. (13) in [13]), which produces the finite motion resulting from the successive application of $\$_f^1$ and $\$_f^2$. Endowed with the triangle product, the finite twists thus form the special euclidian group SE(3), a Lie group associated to the Lie algebra of instantaneous twists with the screw cross product [13].
Hence, the study of the matrix formalism allows us to discuss the algebraic structure of the set of screws:

Third criterion. *The space of screws must be a six-dimensional Lie algebra, endowed with a symmetric, non degenerate bilinear form of signature* $(3, 3)$.

This formalism is quite simple to construct and its properties are directly derived from those of $\mathbb{R}^6$. Matrix computations are also simple to implement algorithmically. However, it remains coordinate-dependent, origin-dependent, and, as we will see in the next section, has low computational efficiency in comparison with other formalisms [14]. Moreover, its representation of finite motion composition is difficult to use, as the triangle product takes a most complex form.

1.3. Study-Kotelnikov formalism and Dimentberg calculus

Alexandr P. Kotelnikov and Eduard Study developed a powerful formalism based on a type of hypercomplex numbers called dual numbers [4]. A *dual number* is a number of the form $a+\varepsilon b$ where $a, b \in \mathbb{R}$ and ε is an imaginary unit such that $\varepsilon^2 = 0$. Dual quantities are denoted by a hat. Addition and multiplication of dual numbers are defined from real addition and multiplication. A complete presentation of this formalism is given by Dimentberg [5].

A screw $\widehat{S}$ is a dual vector $\mathbf{S} + \varepsilon\mathbf{M}$ where $\mathbf{S}$ is the direction vector and $\mathbf{M}$ the moment of $\widehat{S}$. It can be easily verified that the addition and multiplication by a real number follow the same rules as in matrix formalism. Let us calculate the dot product of two dual vectors:

$$\begin{aligned}\widehat{S}_1 \cdot \widehat{S}_2 &= (\mathbf{S}_1 + \varepsilon\mathbf{S}_1^0) \cdot (\mathbf{S}_2 + \varepsilon\mathbf{S}_2^0) \\ &= \mathbf{S}_1 \cdot \mathbf{S}_2 + \varepsilon(\mathbf{S}_1 \cdot \mathbf{S}_2^0 + \mathbf{S}_1^0 \cdot \mathbf{S}_2) + \varepsilon^2\mathbf{S}_1^0 \cdot \mathbf{S}_2^0 \\ &= \mathbf{S}_1 \cdot \mathbf{S}_2 + \varepsilon(\mathbf{S}_1 \cdot \mathbf{S}_2^0 + \mathbf{S}_1^0 \cdot \mathbf{S}_2)\end{aligned} \tag{8}$$

We see by comparing this expression with (5) that the reciprocal product of two screws is the dual part of the dot product of the dual vectors. Let us examine the cross product:

$$\widehat{S}_1 \times \widehat{S}_2 = (\mathbf{S}_1 + \varepsilon\mathbf{S}_1^0) \times (\mathbf{S}_2 + \varepsilon\mathbf{S}_2^0) = \mathbf{S}_1 \times \mathbf{S}_2 + \varepsilon(\mathbf{S}_1 \times \mathbf{S}_2^0 + \mathbf{S}_1^0 \times \mathbf{S}_2) \tag{9}$$

The dual vector cross product is thus exactly the screw cross product. The space of dual vectors is a Lie algebra isomorphic to the Lie algebra of screws. The three critera previously established are thus satisfied.

The Study-Kotelnikov formalism treats finite motions in a way that is much more efficient than the finite screw of the matrix formalism. It uses dual quaternions, i.e. quaternions on the ring of dual numbers. A dual quaternion can be most simply seen as the sum of a dual scalar and a dual vector:

$$\widehat{q} = \widehat{a} + \widehat{S} = a + \varepsilon b + \mathbf{S} + \varepsilon\mathbf{M} \tag{10}$$

Let $\widehat{a}_1, \widehat{a}_2$ be two dual scalars, $\widehat{S}_1, \widehat{S}_2$ two screws. The product of two dual quaternions can be written:

$$(\widehat{a}_1 + \widehat{S}_1) \times (\widehat{a}_2 + \widehat{S}_2) = \widehat{a}_1\widehat{a}_2 + \widehat{a}_1\widehat{S}_2 + \widehat{a}_2\widehat{S}_1 + \widehat{S}_1 \times \widehat{S}_2 \tag{11}$$

A rotation of angle θ around an axis of unit vector $\mathbf{s}$ together with a translation of amplitude t along the same axis is represented by:

$$\widehat{q} = \cos\frac{\theta}{2} - \varepsilon\frac{t}{2}\sin\frac{\theta}{2} + \frac{1}{2}\cos\frac{\theta}{2}\widehat{S} \tag{12}$$

$$\text{with } \widehat{S} = 2\tan\frac{\theta}{2}\mathbf{s} + \varepsilon\left(2\tan\frac{\theta}{2}\mathbf{r} \times \mathbf{s} + t\mathbf{s}\right) \tag{13}$$

where $\mathbf{r}$ indicates the position of an arbitrary point on the axis.

If a rigid body undergoes a motion $\widehat{q}_a$ and then a motion $\widehat{q}_b$, then it undergoes an equivalent motion $\widehat{q}_a \times \widehat{q}_b$.

Some authors [15] choose to represent finite motions with proper orthogonal dual tensors.

Dual quaternions allow for much easier calculations of finite motions than rotation matrices, and their computational efficiency is considerably better [14]. The definition of the composition operation, which is simply the dual quaternion multiplication, is much more natural than the triangle product of finite screws in matrix formalism. We'll be seeking to obtain a comparable efficiency in our new formalism, though without introducing the somewhat artificial imaginary unit ε.

1.4. French formalism

Although it seems that it did not originate in France, but in the works of Richard von Mises [16, 17] and perhaps Gilbert H. Lovell III [18], this formalism was standardized in French universities and *classes préparatoires*, where screws ("*torseurs*") are widely employed to teach rigid body kinematics and dynamics. This approach differs drastically from the previous ones in its presentation, as it was adopted for its mathematical elegance and the explicitness of its notations. A thorough and most clear introduction to this formalism can be found in [19]. Any french book on rigid body mechanics can also be consulted.

A vector field $\mathbf{M}$, defined on the the n-dimensional affine euclidian space $\mathbb{E}^n$, is said to be *equiprojective* if:

$$\forall A, B \in \mathbb{E}^n,\ \mathbf{M}(A) \cdot \overrightarrow{AB} = \mathbf{M}(B) \cdot \overrightarrow{AB} \tag{14}$$

A screw is an equiprojective field on $\mathbb{E}^3$. It can be proved that a vector field on $\mathbb{E}^3$ is equiprojective if, and only if, there exists a vector $\mathbf{S}$ such that:

$$\forall A, B \in \mathbb{E}^3,\ \mathbf{M}(A) = \mathbf{M}(B) + \overrightarrow{AB} \times \mathbf{S} \tag{15}$$

This formula is called the *Varignon relation* of the screw, $\mathbf{S}$ is unique and is referred to as the *resultant* of the screw. A screw is thus completely characterized by its resultant and its value at a given point, which is termed its *moment* at this point. We characterize a screw $\left\{S\right\}$ with resultant $\mathbf{S}$ and moment $\mathbf{M}(A)$ at a point A by writing:

$$\left\{S\right\} = \left\{\begin{matrix} \mathbf{S} \\ \mathbf{M}(A) \end{matrix}\right\}_A \tag{16}$$

The *central axis* of $\left\{S\right\}$ is the (unique if $\mathbf{S} \neq \mathbf{0}$) axis where the moments are collinear to the resultant. The Varignon relation shows that all moments are equal on the central axis, and that a point I thereon can be computed from any point A with the formula:

$$I = A + \frac{\mathbf{S} \times \mathbf{M}(A)}{\mathbf{S}^2} \tag{17}$$

The sum $\left\{S_3\right\}$ of two screws $\left\{S_1\right\}$ and $\left\{S_2\right\}$ is defined to be the pointwise addition of the associate vector fields $\mathbf{M}_1$ and $\mathbf{M}_2$. The Varignon relations of $\left\{S_1\right\}$ and $\left\{S_2\right\}$ yield:

$$\begin{aligned} \mathbf{M}_3(A) &= \mathbf{M}_1(A) + \mathbf{M}_2(A) \\ &= \mathbf{M}_1(B) + \overrightarrow{AB} \times \mathbf{S}_1 + \mathbf{M}_2(B) + \overrightarrow{AB} \times \mathbf{S}_2 \\ &= \mathbf{M}_3(B) + \overrightarrow{AB} \times (\mathbf{S}_1 + \mathbf{S}_2) \end{aligned} \tag{18}$$

where we recognise the Varignon relation for the screw $\left\{S_3\right\}$. Thus, its resultant is $\mathbf{S}_1 + \mathbf{S}_2$.

Multiplication of a screw by a scalar is defined to be the pointwise multiplication of the associate vector field by this scalar. Analogously, $\lambda\mathbf{S}$ is the resultant of $\lambda\left\{S\right\}$.

The *comoment* of two screws $\left\{S_1\right\}$ and $\left\{S_2\right\}$ is the quantity:

$$\left\{S_1\right\} \otimes \left\{S_2\right\} = \mathbf{S}_1 \cdot \mathbf{M}_2(A) + \mathbf{S}_2 \cdot \mathbf{M}_1(A) \tag{19}$$

It can be verified that this definition is invariant by change of point (i.e. by taking B instead of A). The *scalar invariant* of a screw $\left\{S\right\}$ is the quantity $\frac{1}{2}\left\{S\right\}\otimes\left\{S\right\} = \mathbf{S}\cdot\mathbf{M}(A)$, and the pitch is the scalar invariant divided by $\mathbf{S}^2$.

The *product* of two screws $\left\{S_1\right\}$ and $\left\{S_2\right\}$ is defined as:

$$\left\{S_1\right\} \times \left\{S_2\right\} = \left\{\begin{matrix} \mathbf{S}_1 \times \mathbf{S}_2 \\ \mathbf{S}_1 \times \mathbf{M}_2(A) + \mathbf{S}_2 \times \mathbf{M}_1(A) \end{matrix}\right\}_A \tag{20}$$

This definition is also invariant by change of point. The product of two screws is indeed a Lie bracket on the screw space.

A screw can be a function of time. Such a function is a differentiable curve in the screw space. The derivative of a screw is its *eulerian* derivative, i.e. it is obtained by deriving pointwise the associate vector field.

The *kinematic screw* $\left\{\mathscr{V}_{\mathcal{S}/\mathcal{R}}\right\}$ is the velocity field of a rigid body $\mathcal{S}$ with respect to the frame $\mathcal{R}$. Its resultant is the angular velocity of $\mathcal{S}$ and its axis is the instantaneous axis of rotation. The *wrench* $\left\{\mathcal{T}_{\Sigma_1\to\Sigma_2}\right\}$ is the torque field applied by Σ_1 upon Σ_2. Its resultant is the total force applied by Σ_1 on Σ_2 and its axis is the axis of application of the force. The *kinetic screw* $\left\{\mathscr{C}_{\Sigma/\mathcal{R}}\right\}$ is the angular momemtum field of a mechanical system Σ with respect to the frame $\mathcal{R}$. Its resultant is the linear momentum of Σ.

The French formalism allows us to rewrite various equations of kinematics and dynamics in a much compact and elegant way. The linear and angular velocity composition law between three frames $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$:

$$\left\{\mathscr{V}_{\mathcal{R}_1/\mathcal{R}_3}\right\} = \left\{\mathscr{V}_{\mathcal{R}_1/\mathcal{R}_2}\right\} + \left\{\mathscr{V}_{\mathcal{R}_2/\mathcal{R}_3}\right\} \tag{21}$$

The power $P_{\Sigma\to\mathcal{S}/\mathcal{R}}$ developed by a mechanical system Σ acting on a rigid body $\mathcal{S}$, with respect to the frame $\mathcal{R}$, and its kinetic energy $T_{\mathcal{S}/\mathcal{R}}$:

$$P_{\Sigma\to\mathcal{S}/\mathcal{R}} = \left\{\mathcal{T}_{\Sigma\to\mathcal{S}}\right\} \otimes \left\{\mathscr{V}_{\mathcal{S}/\mathcal{R}}\right\} \tag{22}$$

$$T_{\mathcal{S}/\mathcal{R}} = \frac{1}{2}\left\{\mathscr{C}_{\mathcal{S}/\mathcal{R}}\right\} \otimes \left\{\mathscr{V}_{\mathcal{S}/\mathcal{R}}\right\} \tag{23}$$

The Euler-Newton Equations for a mechanical system Σ in a galilean frame $\mathcal{R}$ (where $\overline{\Sigma}$ denotes the exterior of the system):

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\left\{\mathscr{C}_{\Sigma/\mathcal{R}}\right\}\right|_{\mathcal{R}} = \left\{\mathcal{T}_{\overline{\Sigma}\to\Sigma}\right\} \tag{24}$$

Newton's Third Law of Motion for any system Σ_1 upon which a system Σ_2 is acting:

$$\left\{\mathcal{T}_{\Sigma_1\to\Sigma_2}\right\} = -\left\{\mathcal{T}_{\Sigma_2\to\Sigma_1}\right\} \tag{25}$$

The main advantage of the French formalism is its frame-independance, which allows for a most clear presentation of rigid body mechanics, and the fact that most results of Screw Theory are easy to derive algebraically. Once a vector field is proven to be a screw thanks to Eqn.(14), interesting formulae as the Varignon relation or Eqn.(17) can be used. The main disadvantage, on the other hand, is the lack of computational efficiency: all computations must be done by choosing a particular reference frame, i.e. by recovering the matrix formalism. Furthermore, the representation of finite motion remains quite odd. As far as we are aware, the French formalism is not compatible with dual quaternions.

1.5. Screws in Geometric Algebra

1.5.1. A presentation of Geometric Algebra

A clear and concise introduction to Geometric Algebra (GA) can be found in [20]. Chris Doran's PhD thesis [21] provides more information about the construction of geometric algebras and their applications in physics. A geometric algebra $\mathbb{G}^{p,q}$ is an associative algebra based on the pseudo-euclidian space $\mathbb{R}^{p,q}$ (with (p,q) the signature of the space). The elements of $\mathbb{R}^{p,q}$ are called vectors. The exterior product $\mathbf{a} \wedge \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ is called a 2-*blade* and represents an oriented parallelogram whose sides are $\mathbf{a}$ and $\mathbf{b}$. The norm of $\mathbf{a} \wedge \mathbf{b}$ is the area of the parallelogram. Because the parallelogramm is oriented, $\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$. Just as a vector is an oriented line segment, a 2-blade is an oriented "plane segment", a 3-blade $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ is an oriented finite volume, and a k-blade is an oriented finite k-dimensional volume. The integer k is termed the *grade* of the k-blade. Because $\mathbb{R}^{p,q}$ is a $(p+q)$-dimensional space, there cannot be any finite volume of dimension $k > p+q$, so k-blades only exist for $0 \leq k \leq p+q$, the 0-blades being the scalars, and the 1-blades the vectors. The geometric algebra $\mathbb{G}^{p,q}$ is composed by all k-blades and their linear combinations. An element of $\mathbb{G}^{p,q}$ is called a *multivector*, and a multivector with given grade k (a sum of k-blades) is called a k-vector. A $(p+q)$-vector is called a *pseudoscalar*. Any multivector M can be decomposed into a sum of elements of each grade:

$$M = \sum_{k=0}^{p+q} \langle M \rangle_k \tag{26}$$

where $\langle M \rangle_k$ is the grade k component of M.

For any given k, the set of k-vectors is subspace of $\mathbb{G}^{p,q}$ of dimension $\binom{p+q}{k}$. This shows that the space of k-vectors is isomorphic to the space of $(p+q-k)$-vectors.

The geometric product (denoted by the absence of operation symbol: ab) is axiomatically defined on all multivectors [9] as an associative, distributive, non commutative multiplication. The inner product (27) and the exterior (or outer) product (28) of a k-blade A and a l-blade B are defined from the geometric product:

$$A \cdot B = \langle AB \rangle_{|k-l|} \tag{27}$$

$$A \wedge B = \langle AB \rangle_{k+l} \tag{28}$$

The geometric product of a vector $\mathbf{a}$ with a blade B is the sum of their inner product and of their exterior product:

$$\mathbf{a}B = \mathbf{a} \cdot B + \mathbf{a} \wedge B \tag{29}$$

The definitions (27) and (28) are extended to any multivectors by bilinearity. Note that the outer product is always associative.

The following unary operations on a multivector M are also useful:

$$\text{Inversion (if it exists)} \;\; M^{-1} = \frac{M}{M^2} \tag{30}$$

$$\text{Norm} \;\; |M| = \sqrt{\sum_k |\langle M\rangle_k \cdot \langle M\rangle_k|} \tag{31}$$

$$\text{Grade involution} \;\; \widehat{M} = \sum_k (-1)^k \langle M\rangle_k \tag{32}$$

$$\text{Reversion} \;\; M^\dagger = \sum_k (-1)^{\frac{k(k-1)}{2}} \langle M\rangle_k \tag{33}$$

$$\text{Dual} \;\; M^* = MI^{-1} \tag{34}$$

Let $n \in \mathbb{N}^*$, and $(a_i)_{i=,\dots,n}$ be a set of vectors. The reversion has the notable property:

$$(a_1 a_2 \dots a_n)^\dagger = a_n^\dagger \dots a_2^\dagger a_1^\dagger \tag{35}$$

The quantity I in the definition of the dual is the unit pseudoscalar $I = e_1 e_2 \dots e_{p+q}$, where the vectors e_i form an orthonormal basis of $\mathbb{R}^{p,q}$.

The *Conformal Geometric Algebra* (CGA) is the geometric algebra $\mathbb{G}^{4,1}$. Details on the construction of CGA are given in [22] and a quick but more physical introduction in [23]. We take $(e_1, e_2, e_3, e_+, e_-)$ as a basis for $\mathbb{R}^{4,1}$ such that:

$$e_1^2 = e_2^2 = e_3^2 = e_+^2 = +1 \quad ; \quad e_-^2 = -1 \tag{36}$$

From there, we build the *point at infinity* n_∞ and the *origin* n_o:

$$n_\infty = e_- + e_+ \quad ; \quad n_o = \frac{1}{2}(e_- - e_+) \tag{37}$$

These are null vectors, i.e. $n_\infty^2 = n_o^2 = 0$. A point $\mathbf{x}$ in $\mathbb{R}^3$ is mapped to a CGA vector $F(\mathbf{x})$ by the formula:

$$F(\mathbf{x}) = \frac{1}{2}\mathbf{x}^2 n_\infty + \mathbf{x} + n_o \tag{38}$$

Other definition choices, up to a sign or to a scaling factor, are possible for Eqn.(37) and Eqn.(38) (see [23] for example). Some authors choose to operate a *conformal split* to relate $\mathbb{R}^3$ to $\mathbb{G}^{4,1}$, while our choice is called an *additive split* (this is discussed in [10]).

1.5.2. The Cambridge formalism of Screw Theory

There are several ways to represent screws in GA. Two of the earliest attempts in this direction were made by David Hestenes [9, 10] who successively identified screws with sums of a vector and a bivector in $\mathbb{G}^{3,0}$, and with special bivectors in $\mathbb{G}^{4,1}$ with a conformal split. The approach we present in this paper is founded upon CGA and has been developed by researchers of Cambridge University [23, 24].

In CGA, a straight line L passing through $\mathbf{p}$ and $\mathbf{q}$ is represented by the trivector $F(\mathbf{p}) \wedge F(\mathbf{q}) \wedge n_\infty$.

$$\begin{aligned} L &= \left(\tfrac{1}{2}\mathbf{p}^2 n_\infty + \mathbf{p} + n_o\right) \wedge \left(\tfrac{1}{2}\mathbf{q}^2 n_\infty + \mathbf{q} + n_o\right) \wedge n_\infty \\ &= (\mathbf{p} - \mathbf{q}) \wedge n_o \wedge n_\infty + \mathbf{p} \wedge \mathbf{q} \wedge n_\infty \\ &= \mathbf{S}E + \overset{\hookrightarrow}{M} n_\infty \end{aligned} \tag{39}$$

with $\mathbf{S} = \mathbf{q} - \mathbf{p}$, $\overset{\hookrightarrow}{M} = \mathbf{p} \wedge \mathbf{q} = \mathbf{p} \wedge \mathbf{S}$ and $E = n_\infty \wedge n_o$ the unit pseudoscalar of the Minkowski plane[2]. Conversely, let L' be a 3-vector that can be factored by n_∞. L' can be uniquely decomposed into a term factored by E and one that is not, which gives us a relation of the same form than in Eqn.(39), where $\mathbf{S}$ and $\overset{\hookrightarrow}{M}$ are taken to be, respectively, a vector and a bivector of $\mathbb{G}^3$, the geometric algebra of our physical space. Defining $\mathbf{r} \overset{\text{def}}{=} \mathbf{S}^{-1} \cdot \overset{\hookrightarrow}{M}$, we obtain via the fundamental identity (29):

$$\begin{aligned} \overset{\hookrightarrow}{M} &= \mathbf{S}^{-1}(\mathbf{S} \cdot \overset{\hookrightarrow}{M}) + \mathbf{S}^{-1}(\mathbf{S} \wedge \overset{\hookrightarrow}{M}) \\ &= \mathbf{S} \wedge (\mathbf{S}^{-1} \cdot \overset{\hookrightarrow}{M}) + \mathbf{S}^{-1}(\mathbf{S} \wedge \overset{\hookrightarrow}{M}) \\ &= \mathbf{S} \wedge \mathbf{r} + \mathbf{S}^{-1}(\mathbf{S} \wedge \overset{\hookrightarrow}{M}) \end{aligned} \tag{40}$$

where the first term is indeed the moment of a line and the second term is a couple. It follows that the trivectors factorizable by n_∞ could be interpreted as screws. However, the operations on the screws cannot be directly recovered by this means; this issue is addressed by taking the dual of a line. We define a screw by multiplying (39) by the unit pseudoscalar $I_5 = I_3 E$ (with I_3 the unit pseudoscalar of the euclidian 3D space):

$$S = \mathbf{S} I_3 - \overset{\hookrightarrow}{M} I_3 n_\infty \tag{41}$$

Both the resultant and the moment can be extracted by algebraic operations:

[2] In this paper, the right hook superscript denotes a bivector.

$$\mathbf{S} = -((Sn_\infty) \wedge n_o)I_5 \tag{42}$$

$$\overset{\hookrightarrow}{M} = ((S - ((Sn_\infty) \wedge n_o)E) \wedge n_o)I_5 \tag{43}$$

Let us calculate the geometric product of two screws S_1 and S_2:

$$\begin{aligned} S_1 S_2 &= \mathbf{S}_1 I_3 \mathbf{S}_2 I_3 - \mathbf{S}_1 I_3 \overset{\hookrightarrow}{M}_2 I_3 n_\infty - \overset{\hookrightarrow}{M}_1 I_3 n_\infty \mathbf{S}_2 I_3 + \overset{\hookrightarrow}{M}_1 I_3 n_\infty \overset{\hookrightarrow}{M}_2 I_3 n_\infty \\ &= -\mathbf{S}_1 \mathbf{S}_2 + (\mathbf{S}_1 \overset{\hookrightarrow}{M}_2 + \overset{\hookrightarrow}{M}_1 \mathbf{S}_2) n_\infty \end{aligned} \tag{44}$$

Separating the parts of different grades, we recognize three operations:

$$S_1 \cdot S_2 = -\mathbf{S}_1 \cdot \mathbf{S}_2 \tag{45}$$

$$[S_1, S_2] = \mathbf{S}_2 \wedge \mathbf{S}_1 - (\mathbf{S}_2 \cdot \overset{\hookrightarrow}{M}_1 - \mathbf{S}_1 \cdot \overset{\hookrightarrow}{M}_2) n_\infty \tag{46}$$

$$S_1 \wedge S_2 = (\mathbf{S}_1 \wedge \overset{\hookrightarrow}{M}_2 + \overset{\hookrightarrow}{M}_1 \wedge \mathbf{S}_2) n_\infty \tag{47}$$

Equation (45) is an invariant of the two screws. Equation (46) is the Lie Bracket, and Eqn.(47) is the reciprocal product, which doesn't produce a scalar, but a 3D pseudoscalar multiplied by n_∞.

As in the Study-Kotelnikov formalism, finite motions are not represented by screws, but by spinors (i.e. elements of the even subalgebra of $\mathbb{G}^{4,1}$), and we compose motions by multipliying the corresponding spinors. A brief and general presentation of spinors in GA is given in [20]. [23] provides further information on the applications of CGA to rigid body mechanics.

Screws appear as natural objects in CGA. Geometric Algebra allows for efficient computations, and usually provides much geometric insight. However, the introduction of two supplementary dimensions to the physical space $\mathbb{R}^3$ and of null vectors complicates the Theory by adding objects that have no straightforward physical meaning. For the same reason, calculations become a bit unwieldy when done manually; the different "components" of a screw (resultant, line moment, pure moment) cannot be directly read and must be algebraically extracted. By comparison, manual calculations are much more practical and readable in the Study-Kotelnikov and French formalisms. The Cambridge formalism is thus a powerful tool, particularly for computing, but it suffers from the heaviness of CGA, which we believe should be avoided in an ideal formulation of Screw Theory.

2. Introducing a new formalism

Our survey of Screw Theory being now complete, we have at our disposal the three key criteria (determined in section 1.1 and 1.2) and the additional requirements that an ideal theory of screws should be:

- coordinate-free, origin-independent;
- computationally efficient;
- straightforward in the physical *and* geometrical meaning of its objects;
- able to easily represent and compose finite motions.

Each of these features is useful and possessed by at least one of these formalisms, but not by all formalisms.

We now face the challenge of elaborating a formalism verifying all of these conditions while introducing the fewest mathematical concepts possible, and namely without introducing any concept that doesn't have a direct physical interpretation. This is what we try to achieve in the subsequent section.

The following notations are used throughout this section. Let $n \geq 2$. We consider the n-dimensional affine euclidian space $\mathbb{E}^n$ and the geometric algebra $\mathbb{G}^{n,0}$ written $\mathbb{G}^n$ for brevity.

2.1. The screw algebra

Definition. A **first moment field** M is a multivector field on $\mathbb{E}^n$ such that there exists a multivector S satisfying:

$$\forall A, B \in \mathbb{E}^n,\ M(A) = M(B) + \overrightarrow{AB} \wedge S \tag{48}$$

Equation (48) is called a *Varignon relation* for M, and S is a *resultant* of M.

Theorem. The resultant of a first moment field is unique up to a pseudoscalar.

Proof. Indeed, let S_1, S_2 be two resultants of the same first moment field M. Let A, B be two points of $\mathbb{E}^n$. Then:

$$M(A) = M(B) + \overrightarrow{AB} \wedge S_1 \quad ; \quad M(A) = M(B) + \overrightarrow{AB} \wedge S_2 \tag{49}$$

Subtracting these equations yields:

$$0 = \overrightarrow{AB} \wedge (S_1 - S_2) \tag{50}$$

This being true for *any* two points A and B, we conclude that $\sum_{k \neq n} \langle S_1 - S_2 \rangle_k = 0$. The pseudoscalar part of $S_1 - S_2$ may however be non zero because the exterior product of a pseudoscalar by a vector is zero anyway. □

Definition. A **(generalized) screw** $\left|\mathscr{S}\right\}$ is a first moment field M with fixed resultant S. Let $A, B \in \mathbb{E}^n$. We write:

$$\left|\mathscr{S}\right\} = \left|\begin{matrix} S \\ M(A) \end{matrix}\right\}_A = \left|\begin{matrix} S \\ M(A) + \overrightarrow{BA} \wedge S \end{matrix}\right\}_B \tag{51}$$

$M(A)$ is called the *moment* of $\left|\mathscr{S}\right\}$ at the point A. The resultant and the moment at A are called the *reduction elements* of $\left|\mathscr{S}\right\}$ at A.

Equation (51) provides a unique characterization of $\left|\mathscr{S}\right\}$: if $(S_1, M_1) \neq (S_2, M_2)$, then:

$$\left|\begin{matrix} S_1 \\ M_1 \end{matrix}\right\}_A \neq \left|\begin{matrix} S_2 \\ M_2 \end{matrix}\right\}_A$$

Definitions. Let $k \in \mathbb{N}$.

1. A **k-screw** is a screw whose moments are k-vectors and whose resultant is a $(k-1)$-vector. In particular, the resultant of a 0-screw is zero. The moments of a $(n+1)$-screw are zero.

2. A **couple** is a screw with zero resultant. Couples have uniform moment fields. A k-couple is a couple which is also a k-screw.

3. A **blade screw** is a screw whose reduction elements at any point are blades.

Definition. Let S_1, M_1, S_2, M_2 be four multivectors. We define the **addition** and the **wedge product** of screws as:

$$\left|\begin{matrix} S_1 \\ M_1 \end{matrix}\right\}_A + \left|\begin{matrix} S_2 \\ M_2 \end{matrix}\right\}_A = \left|\begin{matrix} S_1 + S_2 \\ M_1 + M_2 \end{matrix}\right\}_A \tag{52}$$

$$\left|\begin{matrix} S_1 \\ M_1 \end{matrix}\right\}_A \wedge \left|\begin{matrix} S_2 \\ M_2 \end{matrix}\right\}_A = \left|\begin{matrix} S_1 \wedge M_2 + \widehat{M_1} \wedge S_2 \\ M_1 \wedge M_2 \end{matrix}\right\}_A \tag{53}$$

To prove that these definitions are invariant by change of point, we compute the difference between $M(A)$ and $M(B)$ using the Varignon relations of the operand screws $\left|\mathscr{S}_1\right\}$ and $\left|\mathscr{S}_2\right\}$ to prove the Varignon relation of the result screw $\left|\mathscr{S}\right\}$:

Invariance of the addition.

$$\begin{aligned} M\Big|_B^A &= M_1(A) + M_2(A) - M_1(B) - M_2(B) \\ &= \overrightarrow{AB} \wedge (S_1 + S_2) \quad \square \end{aligned}$$

Invariance of the wedge product.

$$
\begin{aligned}
M\Big|_B^A &= M_1(A) \wedge M_2(A) - M_1(B) \wedge M_2(B) \\
&= M_1(A) \wedge M_2(A) - \Big(M_1(A) - \overrightarrow{AB} \wedge S_1\Big) \wedge \Big(M_2(A) - \overrightarrow{AB} \wedge S_2\Big) \\
&= \overrightarrow{AB} \wedge S_1 \wedge M_2(A) + M_1(A) \wedge \overrightarrow{AB} \wedge S_2 \\
&= \overrightarrow{AB} \wedge (S_1 \wedge M_2(A) + \widehat{M}_1 \wedge S_2) \quad \square
\end{aligned}
$$

The 0-screws are all the screws of the form $\left|\begin{matrix}0\\\lambda\end{matrix}\right\}_A$, with $\lambda \in \mathbb{R}$. Therefore, they are exactly the uniform scalar-valued fields on $\mathbb{E}^n$, allowing us to identify them with the corresponding scalars:

$$
\left|\begin{matrix}0\\\lambda\end{matrix}\right\} \stackrel{\text{def}}{=} \lambda \tag{54}
$$

(Due to the uniformity of the field, we omit the subscript indicating the point of reduction.) The wedge product of a 0-screw $\left|\begin{matrix}0\\\lambda\end{matrix}\right\}$ with an arbitrary screw $\left|\mathscr{S}\right\}$ defines the multiplication of $\left|\mathscr{S}\right\}$ by λ. The set of all screws on $\mathbb{E}^n$, noted $\mathbb{S}^n$, endowed with the addition and the multiplication by a scalar, is a linear space. For a given k, the set of k-screws $\mathbb{S}^n_k$ is a linear subspace of $\mathbb{S}^n$.

The zero element of $\mathbb{S}^n$ is $0 = \left|\begin{matrix}0\\0\end{matrix}\right\}$ (and this notation is consistent with the identification of Eqn.(54)).

Theorem. $\mathbb{S}^n$ and $\mathbb{S}^n_k$ are linear spaces of dimension 2^{n+1} and $\binom{n+1}{k}$ respectively.

Proof. Let $A \in \mathcal{E}^n$. The linear map

$$
\Phi_A \quad : \quad \begin{matrix} \mathbb{G}^n \times \mathbb{G}^n & \longrightarrow & \mathbb{S}^n \\ (S, M) & \longmapsto & \left|\begin{matrix}S\\M\end{matrix}\right\}_A \end{matrix}
$$

is a linear isomorphism. Hence:

$$
\dim(\mathbb{S}^n) = 2\dim(\mathbb{G}^n) = 2^{n+1}
$$

Noting $\mathbb{G}^n_k$ the subalgebra of k-vectors of $\mathbb{G}^n$, $\Phi_A(\mathbb{G}^n_{k-1} \times \mathbb{G}^n_k) = \mathbb{S}^n_k$. It follows that:

$$
\dim(\mathbb{S}^n_k) = \dim(\mathbb{G}^n_{k-1}) + \dim(\mathbb{G}^n_k) = \binom{n+1}{k} \quad \square
$$

Theorem.

$$\mathbb{S}^n = \bigoplus_{k=0}^{n+1} \mathbb{S}_k^n \tag{55}$$

Proof. We decompose a screw into a sum of k-screws by decomposing the moment at a point A and the resultant into a sum of k-vectors:

$$\begin{aligned} \left| \begin{matrix} S \\ M \end{matrix} \right\}_A &= \left| \begin{matrix} \sum_{k=1}^{n+1} \langle S \rangle_{k-1} \\ \sum_{k=0}^{n} \langle M \rangle_k \end{matrix} \right\}_A \\ &= \left| \begin{matrix} 0 \\ \langle M \rangle_0 \end{matrix} \right\}_A + \sum_{k=1}^{n} \left| \begin{matrix} \langle S \rangle_{k-1} \\ \langle M \rangle_k \end{matrix} \right\}_A + \left| \begin{matrix} \langle S \rangle_n \\ 0 \end{matrix} \right\}_A \end{aligned} \tag{56}$$

For all $k \in [\![0\,; n+1]\!]$, let $\mathcal{B}_k$ be a basis of $\mathbb{S}_k^n$. Let $\mathcal{B}$ be the concatenation of all the $\mathcal{B}_k$. According to the preceding theorem, $\mathcal{B}$ contains

$$\sum_{k=0}^{n} \binom{n+1}{k} = 2^{n+1}$$

elements. Equation (56) shows that $\mathcal{B}$ spans $\mathbb{S}^n$. As the dimension of $\mathbb{S}^n$ is precisely 2^{n+1}, $\mathcal{B}$ is a basis for $\mathbb{S}^n$. Thus, the decomposition (56) is unique. □

The screw space is in fact the Grassmann algebra on the space of 1-screws, and the wedge product is precisely the exterior product of this algebra. A k-screw is an element of grade k, and each of them can be decomposed into a sum of k-blade screws, which are the monomials of the algebra. Therefore, any arbitrary screw can be written as a sum of blade screws, incitating us to further investigate the special properties of blade screws.

2.2. Blade screws as geometrical objects

2.2.1. Central affine subspace

Theorem. Let $\left|\mathscr{S}\right\}$ be a k-blade screw of resultant S and moment field M.Then:

1. M is uniform on all the $(k-1)$-dimensional affine subspaces of E^n that are parallel to S.

2. If S is non zero, there is a unique $(k-1)$-dimensional affine subspace $\mathcal{E}$ where $\forall A \in \mathcal{E}$, $S \rfloor M(A) = 0$. $\mathcal{E}$ is called the *central affine subspace* of $\left|\mathscr{S}\right\}$.

where $\rfloor$ denotes the *left contraction*, an alternate generalization of the inner product of vectors [25]. We use it here to prevent 1-screws from always having $\mathbb{E}^n$ as their central subspace. Indeed, if λ be any scalar and M any multivector, $\lambda \cdot M = 0$ whereas $\lambda \rfloor M = \lambda M$.

Proof. 1 follows directly from the Varignon relation.
We now need to prove 2. First, let us consider the vector:

$$\mathbf{r} \stackrel{\text{def}}{=} S^{-1}(S \rfloor M(A))S^{-1} \tag{57}$$

$$= S^{-1}\widehat{S}^{-1}(S \rfloor M(A)) \tag{58}$$

$$= \widehat{S}^{-1} \rfloor M(A) \tag{59}$$

The equality (58) holds because $S \rfloor M(A)$ is a vector orthogonal to S, and the equality (59) follows from Eqn.(30). Then we use it to decompose:

$$\left| \begin{matrix} S \\ M(A) \end{matrix} \right\}_A = \left| \begin{matrix} S \\ \mathbf{r} \wedge S \end{matrix} \right\}_A + \left| \begin{matrix} 0 \\ M(A) - \mathbf{r} \wedge S \end{matrix} \right\}_A \tag{60}$$

This is the canonical decomposition of $\left| \mathscr{S} \right\}$, where we call the first term a *slider*, and the second is a couple. We calculate the projection of S on the couple:

$$S \rfloor (M(A) - \mathbf{r} \wedge S) = S\mathbf{r}S - S \rfloor (\mathbf{r} \wedge S) \tag{61}$$

Yet, as $\mathbf{r}$ is orthogonal to S: $\mathbf{r} \wedge S = \mathbf{r}S$. Inserting this in Eqn.(61) yields:

$$S \rfloor (M(A) - \mathbf{r} \wedge S) = S\mathbf{r}S - S \rfloor (\mathbf{r}S) = S\mathbf{r}S - \langle S\mathbf{r}S \rangle_1 = 0 \tag{62}$$

as $S\mathbf{r}S = S \rfloor M(A)$ is indeed a vector.

Thus, because of the linearity of the left contraction, the central affine subspaces of $\left| \mathscr{S} \right\}$ are exactly the central affine subspaces of the associate slider. Let us compute the moment of the slider at $I \stackrel{\text{def}}{=} A + \mathbf{r}$:

$$\left| \begin{matrix} S \\ \mathbf{r} \wedge S \end{matrix} \right\}_A = \left| \begin{matrix} S \\ \mathbf{r} \wedge S - \mathbf{r} \wedge S \end{matrix} \right\}_I = \left| \begin{matrix} S \\ 0 \end{matrix} \right\}_I \tag{63}$$

This proves that the affine subspace parallel to S passing by $I = A + \mathbf{r}$ is a central affine subspace of the slider (noted $\mathcal{E}$), and therefore of $\left| \mathscr{S} \right\}$. Moreover, for any point B *outside* of $\mathcal{E}$ (which means $\overrightarrow{BI} \wedge S \neq 0$):

$$\left| \begin{matrix} S \\ 0 \end{matrix} \right\}_I = \left| \begin{matrix} S \\ \overrightarrow{BI} \wedge S \end{matrix} \right\}_B \tag{64}$$

Defining $\mathbf{u} = (\overrightarrow{BI} \wedge S)S^{-1}$ (the rejection of $\overrightarrow{BI}$ from S) allows us to write:

$$S \rfloor (\overrightarrow{BI} \wedge S) = S \rfloor (\mathbf{u}S) = \langle S\mathbf{u}S \rangle_1 = (-)^{k-1}S^2\mathbf{u} \tag{65}$$

This quantity is obviously non zero. Hence, $\mathcal{E}$ is the only central affine subspace of $\left| \mathscr{S} \right\}$. □

Definition. A **slider** is a non zero blade screw with zero moment at a point. A slider naturally has 0 moment on its whole central affine subspace. It corresponds to a "line vector", a screw with zero pitch.

2.2.2. 1-screws

The central affine subspace of a 1-blade screw is a point. A 1-blade screw itself is thus a point together with an amplitude (a 1-couple is a point at infinity). As was already pointed out by Bourguignon et Bamberger [7], 1-screws constitute the correct point of view on barycentration theory. Indeed, we can associate to any mechanical system Σ of mass m and centre of inertia G a unique 1-slider

$$\big|M_\Sigma\big\} = \begin{vmatrix} m \\ 0 \end{vmatrix}\Big\}_G \tag{66}$$

called the *barycentric screw* of Σ. Conversely, any 1-slider is a representation of a mechanical system: its resultant is its mass, and its central affine subspace is its centre of inertia.

Theorem. Let Σ_1 and Σ_2 be two mechanical systems, and $\big|M_1\big\}$, $\big|M_2\big\}$ the associate barycentric screws. The barycentric screw of the system $\Sigma_1 \cup \Sigma_2$ is:

$$\big|M_{1\cup 2}\big\} = \big|M_1\big\} + \big|M_2\big\} \tag{67}$$

Analogously, for any scalar distribution (of charge, for example) over a spatial domain, there is a barycentric screw with respect to this scalar distribution.

2.2.3. 2-screws

The central affine subspace of a 2-blade screw is a straight line. A 2-blade screw on $\mathbb{E}^3$ is thus a straight line together with an amplitude and a pitch p. Indeed, given I a point on the central axis of the screw, we have: $p = S^{-1}{\cdot}M(I)^*$. The first criterion, established in section 1.1, is thereby satisfied. Ball's screws can be identified with the 2-screws on $\mathbb{E}^3$ of our theory. In this sense, any other screw of our theory can be termed a *generalized* screw.

2.3. Coscrews and screw calculus

Definition. A **dual first moment field** M is a multivector field on $\mathbb{E}^n$ such that there exists a multivector S satisfying:

$$\forall A, B \in \mathbb{E}^n,\ M(A) = M(B) + S \cdot \overrightarrow{AB} \tag{68}$$

Equation (68) is called a *Varignon relation* for M, and S is a *resultant* of M.

Theorem. The resultant of a dual first moment field is unique up to a scalar.

The proof is the same as for the first moment field. Let us notice that a dual first moment field is a first moment field multiplied by I^{-1}.

Definition. A **coscrew** $\big\{\mathscr{S}\big|$ is a dual first moment field M with fixed resultant S. Let $A, B \in \mathbb{E}^n$. We write:

$$\{\mathscr{S}| = {}_A\begin{Bmatrix} S \\ M(A) \end{Bmatrix}\Bigg| = {}_B\begin{Bmatrix} S \\ M(A) + S \cdot \overrightarrow{BA} \end{Bmatrix}\Bigg| \tag{69}$$

$M(A)$ is called the *moment* of $\{\mathscr{S}|$ at the point A. The resultant and the moment at A are called the *reduction elements* of $\{\mathscr{S}|$ at A.

Equation (69) provides a unique characterization of $\{\mathscr{S}|$: if $(S_1, M_1) \neq (S_2, M_2)$, then:

$${}_A\begin{Bmatrix} S_1 \\ M_1 \end{Bmatrix}\Bigg| \neq {}_A\begin{Bmatrix} S_2 \\ M_2 \end{Bmatrix}\Bigg|$$

Definition. Let S_1, M_1, S_2, M_2 be four multivectors, and $A \in \mathbb{E}^n$. We define the **addition** of two coscrews and the **multiplication by a scalar** to be:

$${}_A\begin{Bmatrix} S_1 \\ M_1 \end{Bmatrix}\Bigg| + {}_A\begin{Bmatrix} S_2 \\ M_2 \end{Bmatrix}\Bigg| = {}_A\begin{Bmatrix} S_1 + S_2 \\ M_1 + M_2 \end{Bmatrix}\Bigg| \tag{70}$$

$$\lambda\, {}_A\begin{Bmatrix} S \\ M \end{Bmatrix}\Bigg| = {}_A\begin{Bmatrix} \lambda S \\ \lambda M \end{Bmatrix}\Bigg| \tag{71}$$

These operations are invariant by change of point. The proof is the same as for the screws in section 2.1. Moreover, under these operations, the set of coscrews is a linear space of dimension 2^{n+1} and has the same graded structure as the space of screws. We also note 0 the zero element of the coscrew space.

Definitions. Let $k \in \mathbb{N}$.

1. A **k-coscrew** is a coscrew whose moments are $(k-1)$-vectors and whose resultant is a k-vector. In particular, the resultant of a $(n+1)$-coscrew is zero. The moments of a 0-coscrew are zero.

2. A **couple** (coscrew) is a coscrew with zero resultant. Couples have uniform moment fields. A k-couple is a couple which is also a k-coscrew.

3. A **blade coscrew** is a coscrew whose reduction elements at any point are blades.

Definition. Let S_1, M_1, S_2, M_2 be four multivectors, and $A \in \mathbb{E}^n$. We define the **comoment** of a coscrew on a screw as:

$${}_A\begin{Bmatrix} S_1 \\ M_1 \end{Bmatrix|}\begin{matrix} S_2 \\ M_2 \end{matrix}\Bigg\}_A = -\widehat{S}_1 * \widehat{M}_2 + S_2 * M_1 \tag{72}$$

where $*$ denotes the *scalar product* defined as $U * V = \langle UV \rangle_0 = V * U$ [21, 25]. It is important to note that the multivectors U and V need to have at

least a component of the same grade for $U * V$ to be non zero.
Let us assume that $\left\{\mathscr{S}\right|$ and $\left|\mathscr{S}\right\}$ are respectively a blade coscrew and a blade screw. Then they must have equal grade for their comoment to be non zero ($\left\{\mathscr{S}\right|$ must be a k-coscrew and $\left|\mathscr{S}\right\}$ a k-screw). Indeed, were they not of the same grade, both the scalar products in Eqn.(72) would vanish.
If $\left\{\mathscr{S}\right|$ and $\left|\mathscr{S}\right\}$ are any coscrew and any screw, then they can be decomposed into parts of different grades:

$$\left\{\mathscr{S}\right| = \sum_{k=0}^{n+1} \left\{k\right| \tag{73}$$

$$\left|\mathscr{S}\right\} = \sum_{k=0}^{n+1} \left|k\right\} \tag{74}$$

And this, together with the previous discussion, leads to the important property of grade decomposition of the the comoment:

$$\left\{\mathscr{S}\middle|\mathscr{S}\right\} = \sum_{k=0}^{n+1} \left\{k\middle|k\right\} \tag{75}$$

This allows us to write most of our proofs and calculations of comoments for a k-blade coscrew applied to a k-blade screw only, and then generalize them naturally to any screw and coscrew.

Invariance of the comoment

Until now, we have only defined the comoment at a given point A. Let us prove that this definition is invariant by change of point for a k-blade coscrew applied to a k-blade screw:

$$\begin{aligned}
\left\{\mathscr{S}_1\middle|\mathscr{S}_2\right\}\Big|_B^A &= {}_A\left\{\begin{matrix} S_1 \\ M_1(A)\end{matrix}\middle|\begin{matrix} S_2 \\ M_2(A)\end{matrix}\right\}_A - {}_B\left\{\begin{matrix} S_1 \\ M_1(B)\end{matrix}\middle|\begin{matrix} S_2 \\ M_2(B)\end{matrix}\right\}_B \\
&= -\widehat{S}_1 * \widehat{M_2}(A) + S_2 * M_1(A) + \widehat{S}_1 * \widehat{M_2}(B) - S_2 * M_1(B) \\
&= -\widehat{S}_1 * \left(\widehat{\overrightarrow{AB} \wedge S_2}\right) + S_2 * (S_1 \cdot \overrightarrow{AB}) \\
&= -\widehat{S}_1 * (\overrightarrow{AB} \wedge (-\widehat{S}_2)) + S_2 * (S_1 \cdot \overrightarrow{AB})
\end{aligned}$$

from the Varignon relations and the definition of grade involution. As S_2 and $S_1 \cdot \overrightarrow{AB}$ are $(k-1)$-vectors, their scalar product is also their contraction. Hence $S_2 * (S_1 \cdot \overrightarrow{AB}) = S_2 \rfloor (\overrightarrow{AB} \rfloor (-\widehat{S}_1))$. We can use the properties of the contraction given in [25] to obtain:

$$\left\{\mathscr{S}_1\middle|\mathscr{S}_2\right\}\Big|_B^A = \widehat{S}_1 * (\overrightarrow{AB} \wedge \widehat{S}_2) - (S_2 \wedge \overrightarrow{AB})\rfloor \widehat{S}_1$$
$$= \widehat{S}_1 * (\overrightarrow{AB} \wedge \widehat{S}_2 - \overrightarrow{AB} \wedge \widehat{S}_2) = 0 \quad \square$$

Theorem. Endowed with the comoment, the coscrews are the linear forms on $\mathbb{S}^n$.

Proof. We denote by $(\mathbb{S}^n)^*$ the dual space of $\mathbb{S}^n$. Equation (72) shows that each coscrew is a linear form on $\mathbb{S}^n$. We call Φ the linear morphism which so associates a linear form to a coscrew. We prove that Φ is injective by assuming $\{\mathscr{S}| = {}_A\left\{\begin{matrix} S_1 \\ M_1 \end{matrix}\right|$ a coscrew such that $\Phi(\{\mathscr{S}|) = 0_{(\mathbb{S}^n)^*}$ and then proving that, under this hypothesis, $\{\mathscr{S}| = 0$. If S_1 is non zero, then we can find a screw $\left|\begin{matrix} S_2 \\ M_2 \end{matrix}\right\}_A$ such that $\widehat{S}_1 * \widehat{M_2} \neq 0$, which means (from Eqn.(72)):

$$M_1 * S_2 = \widehat{S}_1 * \widehat{M_2} \neq 0 \tag{76}$$

because $\Phi(\{\mathscr{S}|) = 0_{(\mathbb{S}^n)^*}$, so the comoment $M_1 * S_2 - \widehat{S}_1 * \widehat{M_2}$ is zero. Yet, S_2 can be taken arbitrarily in $\mathbb{G}^n$. There is no multivector M_1 such that Eqn.(76) holds for any value of S_2, hence our conjecture that $S_1 \neq 0$ leads to a contradiction.
Now, if the moment of $\{\mathscr{S}|$ at any point is non zero, an analogous argument leads to the same contradiction. Thus, both the resultant and the moment must be zero, i.e. $\{\mathscr{S}| = 0$.
Hence, the linear map Φ is injective. As the space of coscrews is isomorphic to $\mathbb{S}^n$ (because they have equal dimensions) and therefore to $(\mathbb{S}^n)^*$, Φ is an isomorphism. This proves the result. $\square$

Definition. Let $\left|\begin{matrix} S \\ M \end{matrix}\right\}_A$ be a screw. Its **dual coscrew**, noted $*{}_A\left\{\begin{matrix} S \\ M \end{matrix}\right|$ is defined as:

$$*{}_A\left\{\begin{matrix} S \\ M \end{matrix}\right| = {}_A\left\{\begin{matrix} -\widehat{S^*} \\ M^* \end{matrix}\right| \tag{77}$$

The dual coscrew of a k-screw is therefore a $(n-k+1)$-coscrew.

The dual coscrew is the dual field of a first moment field, whence the invariance by change of point:

$$M(A)^* - M(B)^* = (\overrightarrow{AB} \wedge S)^* = \overrightarrow{AB} \cdot S^*$$
$$= -\widehat{S^*} \cdot \overrightarrow{AB}$$

We can now define a *bilinear form* on $\mathbb{S}^n$:

$$\begin{array}{ccc} \mathbb{S}^n \times \mathbb{S}^n & \longrightarrow & \mathbb{R} \\ \left(\big|1\big\} \times \big|2\big\}\right) & \longmapsto & \underset{A}{*}\big\{1\big|2\big\}_A \end{array}$$

It can be easily verified that this map is indeed bilinear. We want to verify if it is symmetric (as usual, we suppose that the coscrew and the screw have the same grade k). For brevity, let us define the quantity

$$\Delta^* \overset{\text{def}}{=} \underset{A}{*}\left\{\begin{matrix} S_1 \\ M_1 \end{matrix}\middle|\begin{matrix} S_2 \\ M_2 \end{matrix}\right\}_A - \underset{A}{*}\left\{\begin{matrix} S_2 \\ M_2 \end{matrix}\middle|\begin{matrix} S_1 \\ M_1 \end{matrix}\right\}_A \tag{78}$$

and notice that the comoment is symmetric if, and only if, $\Delta = 0$. Hence, from the definitions:

$$\Delta^* = S_1^* * \widehat{M_2} + M_1^* * S_2 - S_2^* * \widehat{M_1} - M_2^* * S_1$$

Because of the equal-grade hypothesis, the scalar products are all contractions, and we can use the duality property of the contraction:

$$\begin{aligned} \Delta^* &= (\widehat{M_2} \wedge S_1 + S_2 \wedge M_1 - \widehat{M_1} \wedge S_2 - S_1 \wedge M_2)^* \\ \Delta &= (-)^k M_2 \wedge S_1 - S_1 \wedge M_2 + S_2 \wedge M_1 - (-)^{n-k+1} M_1 \wedge S_2 \\ &= (-)^k (-)^{(n-k)k} S_1 \wedge M_2 - S_1 \wedge M_2 + S_2 \wedge M_1 - (-)^{n-k+1} (-)^{(k-1)(n-k+1)} S_2 \wedge M_1 \end{aligned}$$

where we have used the commutation rule of the exterior product for two blades of grade u and v respectively:

$$U \wedge V = (-)^{uv} V \wedge U \tag{79}$$

Yet the parity of $k(n-k)$ is the same as the parity of $k(n-1)$ (check the different cases), whence $(-)^k(-)^{(n-k)k} = (-)^{kn}$. Writing $l \overset{\text{def}}{=} n-k+1$, we also find that

$$(-)^{n-k+1}(-)^{(k-1)(n-k+1)} = (-)^l(-)^{(n-l)l} = (-)^{ln} = (-)^{kn} \tag{80}$$

Thus:

$$\Delta = (-)^{kn} S_1 \wedge M_2 - S_1 \wedge M_2 + S_2 \wedge M_1 - (-)^{kn} S_2 \wedge M_1$$

Then, the comoment is symmetric if, and only if, kn is even. In particular, the comoment is always symmetric in the screw algebra of even-dimensional spaces. We, however, are more interested in the case of 2-screws over $\mathbb{E}^3$, where k is even, so the comoment is symmetric. We can simply rewrite it:

$${}_A\left\{\begin{matrix} S_1 \\ M_1 \end{matrix}\middle|\begin{matrix} S_2 \\ M_2 \end{matrix}\right\}_A = -S_1 * M_2 + S_2 * M_1 \tag{81}$$

The signature of the symmetric bilinear form it induces is $(3,3)$, and we notice that it coincides with the usual reciprocal product. Our second criterion is thus satisfied.

The topology that we will use for the screw algebra is however not the topology induced by the comoment, but the topology induced by the isomorphism with $\mathbb{R}^{2^{n+1}}$.

Let $t \longmapsto \big|t\big\} \quad ; \quad t \in [-1\,;\,1]$ be a curve in the screw space. We can differentiate this curve by taking the pointwise derivative of the associated moment field. The derivative being taken "pointwise", we naturally need to specify a frame of derivation $\mathcal{R}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\big|t\big\}\,\bigg|_{\mathcal{R}} = \left|\begin{matrix}\dot{S}(t)\\ \dot{M}(t)\end{matrix}\right\}_A \tag{82}$$

where A is fixed in $\mathcal{R}$, S and M are the reduction elements of $\big|t\big\}$ at A.

3. Kinematics and dynamics of a rigid body

In this section, we discuss how our new formalism can be used to model the kinematical and dynamical properties of a mechanical system, and namely of a rigid body. From now on, we work in the 3-dimensional space $\mathbb{E}^3$. Since every k-vector in $\mathbb{G}^3$ is a blade, every k-screw in $\mathbb{S}^3$ is a blade screw.

3.1. Main definitions and properties

The instantaneous motion of a rigid body $\mathcal{S}$ is described by a *twist*, i.e. its velocity field, which verifies (for any two points A and B):

$$\mathbf{V}_{B/\mathcal{R}} = \mathbf{V}_{A/\mathcal{R}} + \overset{\hookrightarrow}{\Omega}_{\mathcal{S}/\mathcal{R}} \cdot \overrightarrow{AB} \tag{83}$$

A twist is thus a 2-coscrew:

$$\big\{V_{\mathcal{S}/\mathcal{R}}\big| = {}_A\left\{\begin{matrix}\overset{\hookrightarrow}{\Omega}_{\mathcal{S}/\mathcal{R}}\\ \mathbf{V}_{A/\mathcal{R}}\end{matrix}\right| \tag{84}$$

It satisfies the *velocity composition law*. Three rigid bodies $1,2,3$ be given:

$$\big\{V_{1/3}\big| = \big\{V_{1/2}\big| + \big\{V_{2/3}\big| \tag{85}$$

Definition. The **volume element screw** is a field of infinitesimal 1-screws on $\mathbb{E}^3$:

$$P \mapsto \left|\begin{matrix}\mathrm{d}V(P)\\ 0\end{matrix}\right\}_P = \left|\begin{matrix}\mathrm{d}V(P)\\ \overrightarrow{OP}\,\mathrm{d}V(P)\end{matrix}\right\}_O \tag{86}$$

where $\mathrm{d}V(P)$ is the volume element around $P \in \mathbb{E}^3$. Integrating this field over a finite volume Σ yields the barycentric screw of Σ (with respect to the volume distribution, and not to the mass distribution).

Definition. The **mass element screw** is a field of infinitesimal 1-screws on $\mathbb{E}^3$:

$$P \mapsto \big|\mathrm{d}M(P)\big\} = \left|\begin{matrix}\mathrm{d}m(P)\\ 0\end{matrix}\right\}_P = \left|\begin{matrix}\mathrm{d}m(P)\\ \overrightarrow{OP}\,\mathrm{d}m(P)\end{matrix}\right\}_O \tag{87}$$

where $\mathrm{d}m(P) = \rho\,\mathrm{d}V(P)$ is the mass element around $P \in \mathbb{E}^3$ (ρ is the mass density field). The mass element screw is thus the product of the mass density by the volume element screw. (Multiplication of a scalar field by a screw field is naturally defined by pointwise multiplication of a scalar by a screw.)
The barycentric screw with respect to the mass distribution, which we discussed in section 2.2.2, is the integral of the mass element screw over the system Σ:

$$\big|M_\Sigma\big\} = \int_{P\in\Sigma} \big|\mathrm{d}M(P)\big\} \tag{88}$$

The force and torque exerted by E on a system Σ are represented by a *wrench*, i.e. the associated torque field, simply defined by:

$$\big|W_{E\to\Sigma}\big\} = \int_{P\in\Sigma} \left|\begin{matrix}\mathrm{d}V(P)\\ 0\end{matrix}\right\}_P \wedge \mathbf{f}_V(P) \tag{89}$$

with $\mathbf{f}_V$ the force density acting upon Σ. Equations (86) and (89) yield:

$$\begin{aligned}\big|W_{E\to\Sigma}\big\} &= \int_{P\in\Sigma} \left|\begin{matrix}\mathrm{d}V(P)\\ \overrightarrow{OP}\,\mathrm{d}V(P)\end{matrix}\right\}_O \wedge \mathbf{f}_V(P)\\ &= \left|\begin{matrix}\int_{P\in\Sigma}\mathrm{d}V(P)\,\mathbf{f}_V(P)\\ \int_{P\in\Sigma}\mathrm{d}V(P)\,\overrightarrow{OP}\wedge\mathbf{f}_V(P)\end{matrix}\right\}_O\\ &= \left|\begin{matrix}\mathbf{F}_{E\to\Sigma}\\ \overset{\hookrightarrow}{M}_{O,E\to\Sigma}\end{matrix}\right\}_O\end{aligned} \tag{90}$$

with $\mathbf{F}_{E\to\Sigma}$ the total force exerted upon Σ, and $\overset{\hookrightarrow}{M}_{O,E\to\Sigma}$ the total torque at O.
Quite naturally, if the forces at stake are surface forces, the integral in Eqn.(89) is to be taken over the surface of Σ, and $\mathbf{f}_V$ is to be replaced by the stress acting on the surface.

Hence, the power exerted upon a rigid body $\mathcal{S}$ submitted to a wrench and undergoing a twist can be written in a reference frame $\mathcal{R}$:

$$P_{E\to\mathcal{S}/\mathcal{R}} = \big\{V_{\mathcal{S}/\mathcal{R}}\big|W_{E\to\mathcal{S}}\big\} \tag{91}$$

Definition. The **kinetic screw** of a mechanical system Σ with respect to the frame $\mathcal{R}$ is the 2-screw

$$\left|C_{\Sigma/\mathcal{R}}\right\} = \int_{P\in\Sigma} \left|\mathrm{d}M(P)\right\} \wedge \mathbf{V}_{P/\mathcal{R}} \tag{92}$$

This screw verifies:

$$\begin{aligned}
\left|C_{\Sigma/\mathcal{R}}\right\} &= \int_{P\in\Sigma} \left|\begin{matrix}\mathrm{d}m(P) \\ \overrightarrow{OP}\,\mathrm{d}m(P)\end{matrix}\right\}_O \wedge \mathbf{V}_{P/\mathcal{R}} \\
&= \left|\begin{matrix}\int_{P\in\Sigma} \mathrm{d}m(P)\,\mathbf{V}_{P/\mathcal{R}} \\ \int_{P\in\Sigma} \mathrm{d}m(P)\,\overrightarrow{OP} \wedge \mathbf{V}_{P/\mathcal{R}}\end{matrix}\right\}_O \\
&= \left|\begin{matrix}\mathbf{p}_{\Sigma/\mathcal{R}} \\ \overset{\hookrightarrow}{L}_{O,\Sigma/\mathcal{R}}\end{matrix}\right\}_O
\end{aligned} \tag{93}$$

with $\mathbf{p}_{\Sigma/\mathcal{R}}$ the total linear momentum, and $\overset{\hookrightarrow}{L}_{O,\Sigma/\mathcal{R}}$ the total angular momentum of Σ with respect to $\mathcal{R}$.

Fundamental equation of dynamics

Let $\mathcal{R}$ be a galilean frame, and Σ a mechanical system.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|C_{\Sigma/\mathcal{R}}\right\}\Big|_{\mathcal{R}} = \sum_{\substack{i \\ \text{indexing exterior wrenches}}} \left|W_{i\to\Sigma}\right\} \tag{94}$$

3.2. The description of finite motions

Up to now, we have seen screws only as geometrical objects, using 1-screws and 2-screws, yet we have not fully harvested the potential of our formalism.

One of the main problems of Screw Theory is to concile the descriptions of instantaneous and finite motions. The matrix formalism introduces a somewhat cumbersome *finite twist* and a rather complicated *triangle product*. The Study-Kotelnikov formalism is the only one that considers screws and finite motions as the same kind of objects, as it describes both as particular species of dual quaternions, respectively with no scalar part and with norm 1.

A finite motion $\underline{\mathrm{D}}$ [3] is a transformation $\mathbb{E}^3 \to \mathbb{E}^3$ which associates a final position field $\underline{\mathrm{D}}(\mathbf{x})$ to an initial position field $\mathbf{x}$. The Mozzi-Chasles theorem shows that $\underline{\mathrm{D}}$ can be written:

$$\underline{\mathrm{D}}(\mathbf{x}) = R(\mathbf{x}-\mathbf{r}) + \mathbf{r} + \mathbf{t} \tag{95}$$

[3] The underbar is traditionally used in Geometric Algebra to denote a linear transformation.

with $\underline{\mathrm{R}}$ a rotation, $\mathbf{r}$ the vector pointing from the origin of the frame to the axis of rotation and perpendicular thereto, and $\mathbf{t}$ the pure translation vector, parallel to the axis of rotation. $\underline{\mathrm{D}}$ is thus completely characterized by the rotor R and the total translation $\mathbf{T} \stackrel{\text{def}}{=} -\underline{\mathrm{R}}(\mathbf{r}) + \mathbf{r} + \mathbf{t}$, which naturally depends on the choice of origin. The total translation field is however not a first moment field. We need to elaborate on it if we want to work only with generalized screws. In geometric algebra, a rotation of angle θ in the oriented plane of unit pseudoscalar i is represented by the *rotor* [9, 23, 24]:

$$R = \exp\left(i\frac{\theta}{2}\right) = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2} \tag{96}$$

A rotor is thus a unit spinor. The rotors in $\mathbb{G}^3$ are precisely the unit quaternions, and the rotors in $\mathbb{G}^2$ are precisely the unit complex numbers. Every result on the complex plane is applicable, with the reverse $R^\dagger$ being the complex conjugate of R.
$\underline{\mathrm{D}}$ is written in the language of geometric algebra:

$$\underline{\mathrm{D}}(\mathbf{x}) = R^\dagger\mathbf{x}R - R^\dagger\mathbf{r}R + \mathbf{r} + \mathbf{t} \tag{97}$$

$$= R^\dagger\mathbf{x}R + R^\dagger(R - R^\dagger)\mathbf{r} + \mathbf{t} \tag{98}$$

($\mathbf{r}R = R^\dagger\mathbf{r}$ because they are coplanar.)
Yet:

$$\mathbf{r} = \frac{i\sin\frac{\theta}{2}}{i\sin\frac{\theta}{2}}\mathbf{r} = 2\frac{1}{R - R^\dagger}(R\cdot\mathbf{r}) \tag{99}$$

because the scalar part of the rotor vanishes under the inner product and R is parallel to i, so $R\cdot\mathbf{r} = i\sin\frac{\theta}{2}\mathbf{r}$, with i the unit pseudoscalar of the plane of rotation. Thus:

$$\underline{\mathrm{D}}(\mathbf{x}) = R^\dagger\mathbf{x}R + 2R^\dagger\frac{R - R^\dagger}{R - R^\dagger}(R\cdot\mathbf{r}) + R^\dagger\mathbf{t}R \tag{100}$$

$$= R^\dagger\mathbf{x}R + 2R^\dagger(R\cdot\mathbf{r} + \frac{\mathbf{t}}{2}R) \tag{101}$$

where we used $\mathbf{t} = R^\dagger\mathbf{t}R$ in Eqn.(100) because $\mathbf{t}$ is othogonal to $R^\dagger$.

The expression between parentheses in Eqn.(101) is clearly the moment at the origin of a coscrew of resultant R and pure moment $\frac{\mathbf{t}}{2}R$.

Definition. A **motor**[4] is a coscrew describing the finite motion of a rigid

[4]In the early history of Screw Theory, the word "motor" was sometimes used as a contraction of "moment" and "vector" to denote what we call today a screw. We use it here in the sense of "something that generates a motion", which we borrowed from Projective Geometric Algebra (see section 4).

body $\mathcal{S}$ between t_0 and t. Let θ be the angle of rotation, i the unit pseudoscalar of the plane of rotation, $\mathbf{t}$ the pure translation and I a point on the axis. The associated motor is:

$$\left\{D_{\mathcal{S},t_0:t}\right| = {}_I\!\left\{\begin{matrix} \mathrm{e}^{i\theta/2} \\ \frac{\mathbf{t}}{2}\mathrm{e}^{i\theta/2} \end{matrix}\right| \tag{102}$$

Let A be any point in $\mathbb{E}^3$. For brevity , we adopt the following notation:

$$\left\{D_{\mathcal{S},t_0:t}\right| = {}_A\!\left\{\begin{matrix} R \\ T \end{matrix}\right|$$

We identify Eqn.(95) and Eqn.(101) so that we establish at the point A the following formulae to convert the moment of motion T into the total translation $\mathbf{T}$, and conversely:

$$\mathbf{T} = 2R^{\dagger}T \quad ; \quad T = \frac{1}{2}R\mathbf{T} \tag{103}$$

Let us assume two rigid motions $\underline{\mathrm{D}}_1(\underline{\mathrm{R}}_1, \mathbf{T}_1)$ and $\underline{\mathrm{D}}_2(\underline{\mathrm{R}}_2, \mathbf{T}_2)$ respectively described by the motors:

$${}_A\!\left\{\begin{matrix} R_1 \\ T_1 \end{matrix}\right| \quad \text{and} \quad {}_A\!\left\{\begin{matrix} R_2 \\ T_2 \end{matrix}\right| \tag{104}$$

The finite motion $\underline{\mathrm{D}}_3 = \underline{\mathrm{D}}_2 \circ \underline{\mathrm{D}}_1$ can be calculated as follows:

$$\begin{aligned} \underline{\mathrm{D}}_3(\mathbf{x}) &= R_2^{\dagger}(R_1^{\dagger}\mathbf{x}R_1 + \mathbf{T}_1)R_2 + \mathbf{T}_2 \\ &= \underbrace{R_2^{\dagger}R_1^{\dagger}}_{=R_3^{\dagger}}\mathbf{x}\underbrace{R_1R_2}_{=R_3} + \underbrace{R_2^{\dagger}\mathbf{T}_1R_2 + \mathbf{T}_2}_{=\mathbf{T}_3} \end{aligned} \tag{105}$$

From there, using the conversation formulae of Eqn.(103), we obtain the moment of the motor associated with $\underline{\mathrm{D}}_3$:

$$\begin{aligned} M_3 &= \frac{1}{2}R_3(R_2^{\dagger}\mathbf{T}_1R_2 + \mathbf{T}_2) = \frac{1}{2}(R_1\mathbf{T}_1R_2 + R_3\mathbf{T}_2) \\ &= T_1R_2 + R_1T_2 \end{aligned} \tag{106}$$

Whence:

Motion composition theorem. Let $\mathcal{S}$ be a rigid body which undergoes two successive motions ${}_A\!\left\{\begin{matrix} R_1 \\ T_1 \end{matrix}\right|$ then ${}_A\!\left\{\begin{matrix} R_2 \\ T_2 \end{matrix}\right|$. The total motion is described by:

$${}_A\!\left\{\begin{matrix} R_1R_2 \\ R_1T_2 + T_1R_2 \end{matrix}\right| \tag{107}$$

Coscrews with spinor resultant are called *even coscrews*. Motors are even coscrews.

Definition. Let S_1, S_2 be two spinors, M_1, M_2 two multivectors, and A a point in $\mathbb{E}^3$. The **composition** of two even coscrews is defined by:

$$ {}_A\!\left\{\begin{matrix} S_1 \\ M_1 \end{matrix}\right| \circ {}_A\!\left\{\begin{matrix} S_2 \\ M_2 \end{matrix}\right| = {}_A\!\left\{\begin{matrix} S_1 S_2 \\ S_1 M_2 + M_1 S_2 \end{matrix}\right| \tag{108} $$

Invariance of the composition.

$$ \begin{aligned} M_3(P)\Big|_{P=B}^{P=A} &= S_1 M_2(A) + M_1(A) S_2 - S_1 M_2(B) - M_1(B) S_2 \\ &= S_1(S_2 \cdot \overrightarrow{AB}) + (S_1 \cdot \overrightarrow{AB}) S_2 \\ &= \frac{1}{2}(S_1 S_2 \overrightarrow{AB} - S_1 \overrightarrow{AB} S_2) + \frac{1}{2}(S_1 \overrightarrow{AB} S_2 - \overrightarrow{AB} S_1 S_2) \\ &= (S_1 S_2) \cdot \overrightarrow{AB} \quad \square \end{aligned} \tag{109} $$

We note Mot(3) the set of motors.

Theorem. Mot(3) is a double covering of the Special Euclidian Group SE(3).

Proof. We note Φ the map:

$$ \begin{array}{ccc} \mathrm{Mot}(3) & \longrightarrow & \mathrm{SE}(3) \\ {}_A\!\left\{\begin{matrix} R \\ T \end{matrix}\right| & \longmapsto & \underline{\mathrm{D}}(\underline{\mathrm{R}}, \mathbf{T}) \end{array} \tag{110} $$

It is easy to verify that Φ is a local homeomorphism between Mot(3) and SE(3), so Mot(3) is indeed a covering space of SE(3).

Let us consider the following motors:

$$ \left\{D_1\right| = {}_A\!\left\{\begin{matrix} \mathrm{e}^{i\frac{\theta}{2}} \\ \frac{\mathbf{t}}{2}\mathrm{e}^{i\frac{\theta}{2}} \end{matrix}\right| \quad ; \quad \left\{D_2\right| = {}_A\!\left\{\begin{matrix} \mathrm{e}^{i\frac{\theta+2\pi}{2}} \\ \frac{\mathbf{t}}{2}\mathrm{e}^{i\frac{\theta+2\pi}{2}} \end{matrix}\right| $$

It is quite clear that $\left\{D_1\right| \neq \left\{D_2\right|$, although they represent the same motion, i.e. $\Phi(\left\{D_1\right|) = \Phi(\left\{D_2\right|)$, as these motors only differ by a rotation of 2π.

As a rotation $\underline{\mathrm{R}}$ can be described by two (and only two) different rotors $\mathrm{e}^{i\frac{\theta}{2}}$ and $\mathrm{e}^{i(\frac{\theta}{2}+\pi)}$, we conclude that a finite motion $\underline{\mathrm{D}}$ can be described by two (and only two) motors, $\left\{D_1\right|$ and $\left\{D_2\right|$. Hence, $\Phi^{-1}[\{\underline{\mathrm{D}}\}]$ contains exactly two motors, and the result follows. $\square$

Theorem. $(\mathrm{Mot}(3), \circ)$ is a Lie group.

Proof. Composition is indeed associative. The identity element is the coscrew $\left\{\begin{matrix}1\\0\end{matrix}\right|$. A motor ${}_A\left\{\begin{matrix}R\\T\end{matrix}\right|$ be given, ${}_A\left\{\begin{matrix}R^\dagger\\-R^\dagger T R^\dagger\end{matrix}\right|$ is its inverse.

Therefore, $(\mathrm{Mot}(3), \circ)$ is a group.

Mot(3) is path-connected, and therefore connected. Moreover, it is a double covering of SE(3), which is a Lie group, and the map that accomplishes this covering (defined in Eqn.(110)) is a group morphism. Hence, Mot(3) is a Lie group. $\square$

Let us introduce a curve in Mot(3)

$$t \mapsto \left\{t\right| \overset{\text{def}}{=} {}_A\left\{\begin{matrix}R(t)\\T(t)\end{matrix}\right| \quad ; \quad t \in [-1\,;1]$$

such that $\left\{t=0\right|$ is the identity motor. We differentiate it with respect to the reference frame $\mathcal{R}$ to obtain the associated Lie algebra:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{t\right|\Bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\,{}_A\left\{\begin{matrix}R(t)\\ \frac{\mathbf{t}(t)}{2}R(t) + R(t)\cdot\mathbf{r}(t)\end{matrix}\right|\Bigg|_{t=0}$$

with A fixed in $\mathcal{R}$. There is a bivector Ω such that: $\dot{R} = \frac{1}{2}\Omega R$ [9, 23], so:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left\{t\right|\Bigg|_{t=0} &= {}_A\left\{\begin{matrix}\frac{1}{2}\Omega R\\ \frac{\dot{\mathbf{t}}}{2}R + \frac{\mathbf{t}}{4}\Omega R + (\frac{1}{2}\Omega R)\cdot\mathbf{r} + R\cdot\dot{\mathbf{r}}\end{matrix}\right|\Bigg|_{t=0} \\
&= {}_A\left\{\begin{matrix}\frac{1}{2}\Omega\\ \frac{\dot{\mathbf{t}}}{2} + \frac{1}{2}\Omega\cdot\mathbf{r}\end{matrix}\right| \\
&= \frac{1}{2}\,{}_A\left\{\begin{matrix}\Omega\\ \mathbf{V}\end{matrix}\right|
\end{aligned} \tag{111}$$

where $\mathbf{V}$ is the velocity of the point A', fixed in $\mathcal{S}$, that coincides at t with A, fixed in $\mathcal{R}$, and this allows us to define the Lie exponential:

$$\left\{D_{\mathcal{S},0:t}\right| = {}_A\left\{\begin{matrix}R(t)\\T(t)\end{matrix}\right| \overset{\text{def}}{=} \exp\left(\frac{1}{2}\,{}_A\left\{\begin{matrix}\overleftrightarrow{\Omega}_{\mathcal{S}/\mathcal{R}}\\ \mathbf{V}_{A/\mathcal{R}}\end{matrix}\right|\right) = \exp\left(\frac{1}{2}\left\{V_{\mathcal{S}/\mathcal{R}}\right|\right) \tag{112}$$

So the Lie algebra associated to Mot(3) is the space of 2-coscrews (physically, the space of instantaneous twists). We are now willing to find an expression of the Lie bracket. Let ${}_A\left\{\begin{matrix}S\\M\end{matrix}\right|$ be an even coscrew, and let us determine the Lie bracket:

$$
\begin{aligned}
&\frac{1}{2}{}_A\begin{Bmatrix}\Omega\\ \mathbf{V}\end{vmatrix} \times {}_A\begin{Bmatrix}S\\ M\end{vmatrix} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\{t| \circ {}_A\begin{Bmatrix}S\\ M\end{vmatrix} \circ \{t|^{-1}\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}{}_A\begin{Bmatrix}R\\ T\end{vmatrix} \circ {}_A\begin{Bmatrix}SR^\dagger\\ -SR^\dagger TR^\dagger + MR^\dagger\end{vmatrix}\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}{}_A\begin{Bmatrix}RSR^\dagger\\ R(-SR^\dagger T + M)R^\dagger + TSR^\dagger\end{vmatrix}\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}{}_A\begin{Bmatrix}RSR^\dagger\\ R\big([R^\dagger T, S] + M\big)R^\dagger\end{vmatrix}\Big|_{t=0}
\end{aligned}
\tag{113}
$$

where $[\cdot,\cdot]$ in the last line denotes the commutator of GA: $[U, V] = UV - VU$. For clarity, we compute separately the different derivatives:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}RSR^\dagger &= \frac{1}{2}\big(\Omega RSR^\dagger + RSR^\dagger\Omega^\dagger\big) \\
&= \frac{1}{2}\big(\Omega RSR^\dagger - RSR^\dagger\Omega\big)
\end{aligned}
\tag{114}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\big([R^\dagger T, S] + M\big) = -\frac{1}{2}\big[R^\dagger\Omega T, S\big] + \Big[R^\dagger\dot{T}, S\Big] \tag{115}
$$

From Eqn.(115) comes:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}R\big([R^\dagger T, S] + M\big)R^\dagger &= \frac{1}{2}\Omega R\big([R^\dagger T, S] + M\big)R^\dagger \\
&+ \frac{1}{2}R\Big(-\big[R^\dagger\Omega T, S\big] + \Big[R^\dagger\dot{T}, S\Big]\Big)R^\dagger \\
&- \frac{1}{2}R\big([R^\dagger T, S] + M\big)R^\dagger\Omega
\end{aligned}
$$

Evaluating these quantities at $t = 0$ (where $R = 1$, $T = 0$ and $\dot{T} = \frac{1}{2}\mathbf{V}$) finally yields:

$$
{}_A\begin{Bmatrix}\Omega\\ \mathbf{V}\end{vmatrix} \times {}_A\begin{Bmatrix}S\\ M\end{vmatrix} = {}_A\begin{Bmatrix}[\Omega, S]\\ [\Omega, M] + [\mathbf{V}, S]\end{vmatrix} \tag{116}
$$

This completes our study on the use of this new formalism, deminstrating that the third criterion is also completely satisfied for 2-screws.

4. Discussion

The issue at stake here is in fact one of the most ancient problems of mechanics: we want to deal with *linear* algebra, but the physical world is an *affine* space. So we need a way to connect affine and linear concepts. The usual method is to impose a coordinate frame, but this leads to difficulties when treating affine transformations, and of course it is conceptually suboptimal because the reference frame is arbitrary chosen. The second usual solution consists in embedding the 3-dimensional affine space into a larger 4-dimensional linear space. This way is more efficient but is also conceptually suboptimal because it introduces a supplementary dimension to space, which we should try to avoid. As our formalism does not require similar arbitrary choices, we believe it to be the correct point of view on affine geometry.

The formalism we proposed and developed in this paper is nothing new from the mathematical point of view. The screw space, endowed with the wedge product, becomes a mere Grassmann algebra, and the composition of section 3.2 is very similar to a geometric product. If it could be generalized to the whole algebra, our screw algebra would be identical to the Projective Geometric Algebra (PGA) $\mathbb{G}^{3,0,1}$ [26], from which we indeed took the term "motor".

The comparative disadvantage of PGA is precisely that it introduces a fourth (and degenerate!) dimension to space, which obscures the geometrical significance of the objects and operations of PGA. We might say that, by doing so, PGA loses some of the very spirit of GA. On the contrary, we only ever worked with $\mathbb{G}^3$ multivectors, objects whose geometrical significance is straightforward. In addition, PGA is not quite origin-independent, as the multivectors used to represent motions or subspaces indeed depend on the choice of origin. The emphasis we placed on the origin-independance explains why we cannot generalize the composition to the whole algebra (it would lose the invariance by change of point), and therefore why the screw algebra is somewhat different from the PGA. We could say that we restricted PGA to cases which the operations are meaningful from the affine point of view.

Note that PGA, because its geometric and inner products are always well defined, probably remains a better choice for applications in computing. The value of our formalism mainly lies in the insight it can bring into rigid body mechanics. In particular, we believe it to be a powerful pedagogical tool.

Our treatment of finite motions is strictly equivalent to dual quaternions, the real part of the dual quaternion being the resultant of the motor, and the imaginary part being the moment. But dual quaternions suffer from the same problems as PGA: they are not origin-independent and need the introduction of the imaginary unit ϵ, which is physically awkward.

The idea that screws allow for an easy treatment of barycentration was taken from Bourguignon and Bamberger [7]. It also follows from the conception that screws are the correct way to treat affine geometry: as 2-screws are weighted lines, 1-screws are weighted points. However, Bourguignon and Bamberger's approach was purely mathematical and highly abstract, so it needed a transposition before it could be used in engineering. We believe that this issue was successfully addressed by this very article.

An original interpretation of Screw Theory was proposed by Géry de Saxcé, who defines screws as antisymmetric affine tensors [8]. This is an accidental feature of our formalism, since our 2-screws are exterior products (wedge products) of 1-screws (and not of vectors, whence the idea of an "affine" tensor), and the exterior product is rigourously defined as the antisymmetrization of the tensor product.

In a recent article [27], a formulation of Screw Theory very analogous to ours was briefly exposed, to be used for robot modelling and control. It employs the matrix formalism, but uses geometric algebra to compute the moment and resultant of the screw. Our formalism is recovered by simply replacing the moment of a screw, or the resultant of a coscrew, by its dual. It can easily be viewed as a direct application of the theoretical framework we developed here.

Conclusion

The first aim of this paper was to provide new insight in Screw Theory by reviewing five different formulations thereof and by showing that each of these has original features that can prove useful to the practice of rigid body mechanics. We inferred from them three criteria that any good theory of screws must satisfy, and a variety of additional requirements that we believe to be useful but are not met by all formulations. This overview led us to introduce Geometric Algebra and to notice how insightful and computationally powerful it was.

From there, we built a new formalism based on multivector fields (for the origin-independance) and GA (for the coordinate-free property, the computational efficiency and the natural geometrical interpretation of the objects). Not only does it embed the usual properties and operations on screws, but it also generalizes them to embed more general affine geometry, points as barycentres, and dual quaternions. In this respect, our description of finite motions is both clear and elegant, and their composition obeys a simple algebraic rule, to be compared with the traditional "triangle product". Furthermore, these results were achieved without introducing any "ungeometrical" concept such as an imaginary unit or supplementary spatial dimensions.

For all those reasons, we believe our formalism to be a correct and powerful synthesis of most formulations of Screw Theory.

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