Probability

Simple probability

Suppose an unbiased coin is tossed: the result is equally likely to be a head as a tail. The **sample space** or **possibility space** is the set S of all the possible outcomes (in this case H or T). The total number of possible outcomes of a single toss is *2*, ie. a head or a tail. We say that the probability of getting a head is $\frac{1}{2}$, or

P(head) = $\frac{1}{2}$. Similarly, P(tail) = $\frac{1}{2}$.

The probability of a particular outcome of a trial (ie. an attempt at something) is expressed as a fraction $\frac{a}{b}$, where a is the number of ways in which the particular outcome can occur and b is the total number of possible outcomes of the trial.

<u>Example 4</u>

1. A six-sided die is to be thrown. Calculate the probability that the result will be (a) an odd number, (b) a number less than 3, (c) a multiple of 3.
2. One card is to be drawn from a pack of 52 cards. Calculate the probability that the card will be (a) a diamond, (b) a court card, (c) a black card (ie. a club or a spade).
3. A bag consists of 8 discs of which 4 are red, 3 are blue and 1 is yellow. Calculate the probability that when one disc is drawn from the bag it will be (a) red, (b) yellow, (c) blue, (d) not blue.
4. Four cards are drawn at random from a pack of playing cards. What is the probability that the four cards drawn are (a) all spades (b) three clubs and one diamond?
5. Five letters are chosen at random from word WONDERFUL. Find the probability that exactly three consonants are chosen.

Exercise 1

Homework: Book 2B, page 395, ex. 9.1 (all).

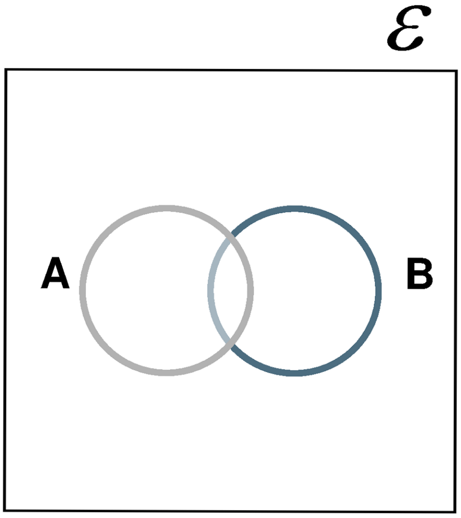
Set notation

Set notation can be useful when considering a number of different events that may occur in a given trial. Consider the set of numbers {1, 2, 3, 4, 5, 6, 7, 8, 9}. This set is called the **universal set**, $(\mathcal{E})$.

Also $\mathcal{E}$ contains a number of subsets $(\subset)$.

For example, the subset of odd numbers = {1, 3, 5, 7, 9}, the subset of multiples of three = {3, 6, 9}

If A is the event of the selected number being odd and B is the event of the selected number being a multiple of three, we can show the sets of outcomes as a Venn diagram:



Note

$A \cup B$ (A union B) = {1, 3, 5, 6, 7, 9}

$A \cap B$ (A intersection B) = {3, 9}

$\mathscr{C}(A) = A'$ (complement of A) = {2, 4, 6, 8}

$\mathscr{C}(B) = B'$ = {1, 2, 4, 5, 7, 8}

$n(A)$ (number of elements in set A) = 5.

<u>Also</u>

P(odd number) = $\frac{5}{9}$, P(multiple of 3) = $\frac{1}{3}$,

P(odd number and multiple of 3) = $\frac{2}{9}$, P(not a multiple of 3) = $\frac{2}{3}$

Example 5

1. Sets A and B are subsets of the universal set and n(A) = 15, n(B) = 10 and $n(A \cap B) = 7$. If *P(A)* is the probability of the selected element belonging to set A, find (a) *P(A)*, (b) $P(A \cap B)$, (c) $P(A')$, (d) $P(A \cup B)$.
2. A universal set has 24 elements and A and B are subsets of the universal set such that $n(A) = 14$, $n(B) = 9$ and $n(A \cap B) = 6$. If P(A) is the probability of selecting an element belonging to set A, calculate (a) P(A), (b) $P(A \cap B)$, (c) $P(B')$, $P(A \cup B)'$.

3. Let A and B be two events. Find an expression and give the Venn diagram for event A that occurs but not event B.
4. If A, B and C are three events, find an expression and give Venn diagram for the case that events A, B and C occur.
5. A die is rolled and the number that appears on top is noted. Let A be the event that an odd number occurs, B be the event that an even number occurs and C that a number greater than three occurs. Find the sample space S, $A \cup C$, $B \cap C$ and C'.

Exercise 5

1. One element is selected at random from a universal set of 15 elements. R and Q are subsets of the universal set and $n(R) = 6$, $n(Q) = 8$ and $n(R \cap Q) = 2$. Calculate (a) P(Q), (b) $P(R \cup Q)$, (c) $P(R')$, (d) $P(R \cap Q)$.
2. A and B are subsets of the universal set and $n(A) = 25$, $n(B) = 20$, $[n(A \cup B)'] = 20$ and there are 50 elements in the universal set. When one element is selected at random, calculate (a) P(A), (b) $P(A \cup B)$, (c) $P(B')$, (d) $P(A \cap B)$.

Mutually Exclusive events

Two events A a
nd B are called mutually exclusive if they <u>canno</u>t occur simultaneously ie. either event A occurs or event B occurs. Therefore for mutually exclusive events $A \cap B = \varnothing$ (empty set) and $P(A \cup B) = P(A) + P(B)$.

Example 6

1. In a class of 30 students, event A is the set of those students who chose Mathematics as one of their 'A' level subjects. There are 18 students. Event B is the set of those students who chose Physics as one of their 'A' level subjects. There are 15 students. Are events A and B mutually exclusive or not?
2. Consider the sample space $S = \{3,4,5,6,7,8,9\}$. Let A be the event that an odd number occurs, B be the event that an even number occurs and C be the event that a prime number occurs. Show that the events A and B and B and C are mutually exclusive.

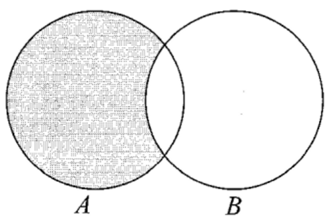
Complementary Events

The complement of A is *sample S without set A*: $P(A') = 1 - P(A)$

Example 7

1. If A' is the complement of an event A, show that $P(A) = 1 - P(A')$.
2. Four letters are taken at random from the word LEAVING. What is the probability that at least one vowel is taken?
3. If five cards are drawn at random from a pack of playing cards, find the probability that at least one of them is a face card.
4. Four balls are drawn at random from a bag containing five red and four blue balls. What is the probability that at least two blue balls are drawn?

Relative Complement and Addition Rule



Consider two events A and B. Then the relative complement of B with respect to A – denoted by A\B – is the event that occurs if event A occurs but event B does not occur.

Example 8

1. If A and B are any two events, then show that $P(A \setminus B) = P(A) - P(A \cap B)$.
2. If A and B are any two events, show that $P(A \cup B) = P(A) - P(A \cap B)$.
3. If A and B are any two events such that $P(A) = \frac{1}{6}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$ find $P(A \cup B)$.
4. If A and B are any two events such that $P(A \cup B) = \frac{4}{5}$, $P(A \cap B) = \frac{1}{3}$ and $P(A') = \frac{1}{4}$, find (i) $P(A)$, (ii) $P(B)$, (iii) $P(A \setminus B)$.
5. In a survey, it was found that the probability that a family owns a television is 0.74, while the probability that a family owns a DVD player is 0.42. The probability that a family owns a television and a DVD player is 0.26. What is the probability that a family owns a television, or a DVD player or both?

6. 40% of the interviewed students said that Mathematics is his/her favourite subject. Also 20% of those interviewed said that they prefer mobile phones of type Z. 50% said that Mathematics is their favourite subject or prefer mobile phones of type Z. Find the probability that a randomly selected student will (a) have Mathematics as his/her favourite subject and prefer mobile phones of type Z, (b) favour only one thing.

Independent events

Two events are said to be independent if the outcome of one event has no bearing upon the outcome of the other. These events are called **independent**. For two independent events A and B
$P(A \text{ and } B) = P(A)\ x\ P(B)$ or, in set notation,
$P(A \cap B) = P(A)\ x\ P(B)$.

Example 9

1. A fair coin is tossed three times. Show that the events of getting heads at the first toss and heads at the second toss are independent.
2. Box A contains 10 items of which 8 are non defective, and box B contains 15 items of which 11 are non defective. An item is drawn at random from each box. Find the probability that both items are defective.
3. Three persons Lawrence, Mario and Nicholas play a game by drawing a card from a pack of playing cards. The first to draw an ace wins. The cards drawn are put back in the pack. They play in the order Lawrence, then Mario and then Nichols. Find the probability that (a) Mario wins on his first attempt, (b) Lawrence wins on his third attempt, (c) Nicholas wins.

Conditional Probability

The probability that another event B occurs, once event A has already occurred is called the conditional probability of B given A has occurred. It is denoted by $P(B \mid A)$.

Example 10

1. A fair die is rolled once. If A is the event that the number obtained is less than 4, while B is the event that an even number is rolled, find the probability of rolling an even number given it is less than 4.
2. A local market-research study showed that 30% of the businesses surveyed advertised on television and that 70% showed a profit. It also showed that 20% advertised on television and showed a profit. Find the probability that a randomly selected business (a) showed a profit, given it advertised on television, (b) advertised on television, given it showed profit.
3. If X, Y and Z are any three events, prove that $P(X \cap Y \mid Z) \text{ x } P(Z) = P(X \mid Y \cap Z) \text{ x } P(Y \cap Z)$.

Tree diagrams

A tree diagram can be used to show the probabilities of certain outcomes occurring when two or more trials take place in succession. The outcome is written at the end of the branch gives the probability of the outcome occurring.

Example 11

1. Edward and Marlene are to play a tennis tournament. The first player to win two games in succession or a total of three games wins the tournament. Draw a decision diagram to find all the possible outcomes of this tournament.
2. A box contains three coins: one coin is fair, another one is weighted so that the probability of getting a head is $\frac{1}{4}$, and the last coin is two-headed. A coin is selected at random and tossed once. Find the probability of obtaining a head.

ANSWERS

Example 1

1. 8	2. 192	3. i) 6	ii) 3	4. 75600
6. 120	7. a) 120	b) 24	8. 40320	9. 15
10.243				

Example 2

1. a) 56 b) 1008 c) 5 d) 4
2. a) 5! b) 3(5!) c) 5! d) 5!÷5 e) 5(5!) f) 9(5!) g) 2(5!) h) 36(5!) i) 55(5!)
3. 18 4. 10! 5. a) 81 b) 256 6. 40320
7. 5040; 720 8. 24 9. 48 10. 10080
11. 80640 12. a) 60 b) 5 c) 36 d) 3 e) 36
13. 20 14. 59049 15. 315 16. 2520
17. a) 479001600 b) 79833600 c) 7257600
18. a) 360 b) 180 c) 180
19. 24 20. i) 40320 ii) 5040 21. 360
22. 30240; 5040

Example 3

1. 15 2. 350 3. a) 126 b) 15 4. 20790
5. a) 21 b) 2940 c) 210 d) 29 6. 120
7. 126 8. 10 9. 330 10. 10 11. 4845
12. 210 13. 150 14. 2010 15. a) 126 b) 105
16. a) 15 b) 1 17. a) 20 b) 4 18. a) combination b) permutation c) permutation 19. a) 720 b) 1800
20. 120 21. 8 22. 59875200; 43545600 23. 630
24. a) 105 b) 3 c) 99

Exercise 3

1. 5005 2. 1001 3. 40640 4. i) 70 ii) 56 5. 715
6. i) 120 ii) 81

Example 4

1. a) $\frac{1}{2}$ b) $\frac{1}{3}$ c) $\frac{1}{3}$ 2. a) $\frac{1}{4}$ b) $\frac{3}{13}$ c) $\frac{1}{2}$ 3. a) $\frac{1}{2}$ b) $\frac{1}{8}$ c) $\frac{3}{8}$ d) 518 4. a) 0.0026 b) 0.0137 5. 0.476

Example 5

1. a) $\frac{5}{6}$ b) $\frac{7}{18}$ c) $\frac{1}{6}$ d) 1 2. a) $\frac{7}{12}$ b) $\frac{1}{4}$ c) $\frac{1}{3}$ d) $\frac{7}{24}$
5. {1,2,3,4,5,6}; {1,3,4,5,6}; {4,6}; {1,2,3}

Exercise 5

1. a) $\frac{8}{15}$ b) $\frac{4}{5}$ c) $\frac{3}{5}$ d) $\frac{2}{15}$ 2. a) $\frac{1}{2}$ b) $\frac{3}{5}$ c) $\frac{3}{5}$ d) $\frac{3}{10}$

Example 6

1. No

Example 7

2. 0.97 3. 0.745 4. 0.643

Example 8

3. $\frac{1}{4}$ 4. i) $\frac{3}{4}$ ii) $\frac{23}{60}$ 5. 0.9 6. a) 0.1 b) 0.4

Example 9

2. $\frac{7}{45}$ 3. a) $\frac{12}{169}$ b) 0.0476 c) 0.307

Example 10

1. $\frac{1}{3}$ 2. a) $\frac{2}{3}$ b) $\frac{2}{7}$

Example 11

2. $\frac{7}{12}$