

HOMEWORK 4

1. 证明题 (请提交 PDF 格式)

(1) Prove that the OLS estimator $\hat{\beta}$ is the same as the maximum likelihood estimator.

(2) Prove the Gauss-Markov Theorem.

(3) Prove $E(\hat{\sigma}^2) = \sigma^2$.

(4) Given conditions:

(A1) The relationship between response (\mathbf{y}) and covariates (\mathbf{X}) is linear;

(A2) \mathbf{X} is a non-stochastic matrix and $\text{rank}(\mathbf{X}) = p$;

(A3) $E(\varepsilon) = \mathbf{0}$. This implies $E(\mathbf{y}) = \mathbf{X}\beta$;

(A4) $\text{cov}(\varepsilon) = E(\varepsilon\varepsilon^\top) = \sigma^2 I_N$; (Homoscedasticity)

(A5) ε follows multivariate normal distribution $N(\mathbf{0}, \sigma^2 I_N)$ (Normality)

Prove the following results:

$$\hat{\beta} \sim N(\beta, \sigma^2(\mathbf{X}^\top \mathbf{X})^{-1}) \quad (0.1)$$

$$(N - p)\hat{\sigma}^2 \sim \sigma^2 \chi_{N-p}^2 \quad (0.2)$$

(5) Suppose y follows the log-linear regression relationship with $x \in \mathbb{R}^p$, i.e.,

$$\log(y) = x^\top \beta + \epsilon, \quad (0.3)$$

where ϵ follows normal distribution $N(0, \sigma^2)$. Please calculate $E(y)$.

(6) Define $\hat{y}_i = x_i^\top \beta$. Let the intercept be included in the regression model. Define the total sum of squares (TSS) and explained sum of squares (ESS) as follows

$$\text{TSS} = \sum_i (y_i - \bar{y})^2, \quad \text{ESS} = \sum_i (\hat{y}_i - \bar{y})^2.$$

Please prove:

$$TSS = ESS + RSS.$$

提交时间：10 月 10 日，晚 20:00 之前。请预留一定的时间，迟交作业扣 3 分，作业抄袭 0 分。