driftsim: Simulation of radial drift in PPDs

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The implementation of the method presented below can be found at `https://github.com/ShadowWarden/drift_sim.git`

Mathematical Formulation

Lagrangian formulation without dissipation

For a dust particle with mass m at a position (r, θ) from the central star of mass M encountering no dissipation, the Lagrangian can be written as follows:

$$L = T - U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{GMm}{r} \tag{1}$$

We know from the Euler-Lagrange equation that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \tag{2}$$

where q_i refers to the ith generalized coordinate. Substituting equation 1 into 2, we get the following:

$$m\ddot{r} = -\frac{GMm}{r^2} + mr\dot{\theta}^2 \tag{3}$$

$$mr^2\ddot{\theta} = 0 \tag{4}$$

Naive implementation of dissipation due to gas

Assuming that no energy dissipation occurs along r, the only damping factor is caused by the difference in relative velocities between gas and dust. This leaves equation 3 unchanged while altering 4 with the high Reynolds number drag equation gives the following:

$$mr^2\ddot{\theta} = -\frac{1}{2}C_D A\rho r^2\dot{\theta}^2_{rel}$$

where A is the cross-sectional area of the dust particle, ρ the density of the fluid and C_D the drag coefficient which is roughly 0.48 for uneven spheres. I had initially used Stoke's Law to compute drag - and got a far faster rate of drift -

but this form of drag is present only in laminar flows with very low R_e. For low viscosity fluids - like PPD gas - $R_e \propto 1/\nu$ is very small, thus making the high turbulence drag more likely. Thus, we get

$$\ddot{\theta} = -\frac{\alpha}{m}\dot{\theta}^2 \tag{5}$$

where $\alpha = \frac{1}{2}C_D A\rho(1-\beta^2)$ where $\beta \in [0,1]$ is the ratio of velocities between gas and dust. Note that equation 5 depends on the mass of the dust object, thus implying that more massive dust particles will drift slower than less massive ones at the same r.

Numerical formulation of Equations 3 and 5

Since the governing EOMs only depend on r and $\dot{\theta}$, we have:

$$\dot{v}_r = -\frac{GM}{r^2} + rv_\theta^2$$

$$\dot{r} = v_r$$

for the radial EOM and

$$\dot{v}_\theta = -\frac{\alpha}{m}v_\theta^2$$

$$\dot{\theta} = v_\theta$$

for the angular EOM. Discretizing the governing angular equation explicitly (should eventually make it RK4), we have:

$$v_\theta^{t+1} - v_\theta^t = -\frac{\alpha}{m}\left(v_\theta^t\right)^2 \Delta t$$

which implies that numerics for the angular equation becomes

$$v_\theta^{t+1} = v_\theta^t - \frac{\alpha}{m}\left(v_\theta^t\right)^2 \Delta t \tag{6}$$

$$\theta^{t+1} = \theta^t + v_\theta^t \Delta t$$

Similarly, for the radial equation, we have the following (No obvious implicit formulation):

$$v_r^{t+1} = v_r^t + \left(-\frac{GM}{(r^t)^2} + r^t(v_\theta^t)^2\right)\Delta t \tag{7}$$

$$r^{t+1} = r^t + v_r^t \Delta t$$

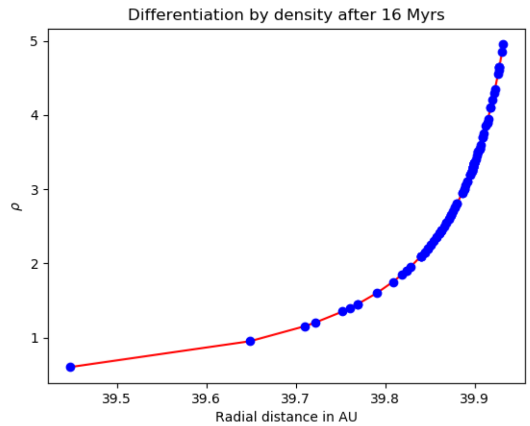

Figure 1: Plot of density of dust particles with radius. Clearly, the particles with the smallest density drifted further inward, but the overal extent of the drift is negligible ($0.5\,AU$ in $\sim 15\,Myrs$)

Results

For 1000 particles placed randomly along a circle at 40 AU starting with the circular orbit corresponding to said circle, the density distribution after $15\,Myrs$ is shown in Figure . Analysing the results from the simulation, we see the following:

- There is obervable differentiation, but the timescale over which this occurs is way too long - especially for dust particles at high radial distances away from the central star. In essence, accretion and coagulation are more dominant interactions for the parameters used.
- The rate of radial drift is very slow. Mathematically, this is because the drag force depends directly on the density of the fluid, which is tiny in the nebula (I used $\rho_s = 2 \times 10^{-8} kg/m^3$). This would also suggest that radial drift is not a dominant effect at high radial distances (dust at low radii have much smaller drift timescales)
- The simulation is an overestimate of the extent of drift, as the system is assumed to collisionless. If collisions were enabled, the drag acceleration would reduce even further - since it goes as $1/m$.
- The densities were sampled from a normal distribution with a $\mu = 3\,g/cm^3$ and $\sigma = 1\,g/cm^3$.