APPM 4390: Homework 1

Omkar H. Ramachandran

Phew! That was possibly the longest homework set I've ever done in my time here in Boulder. Had a lot of fun doing it though!

1 Four different behaviors of the Hassell model

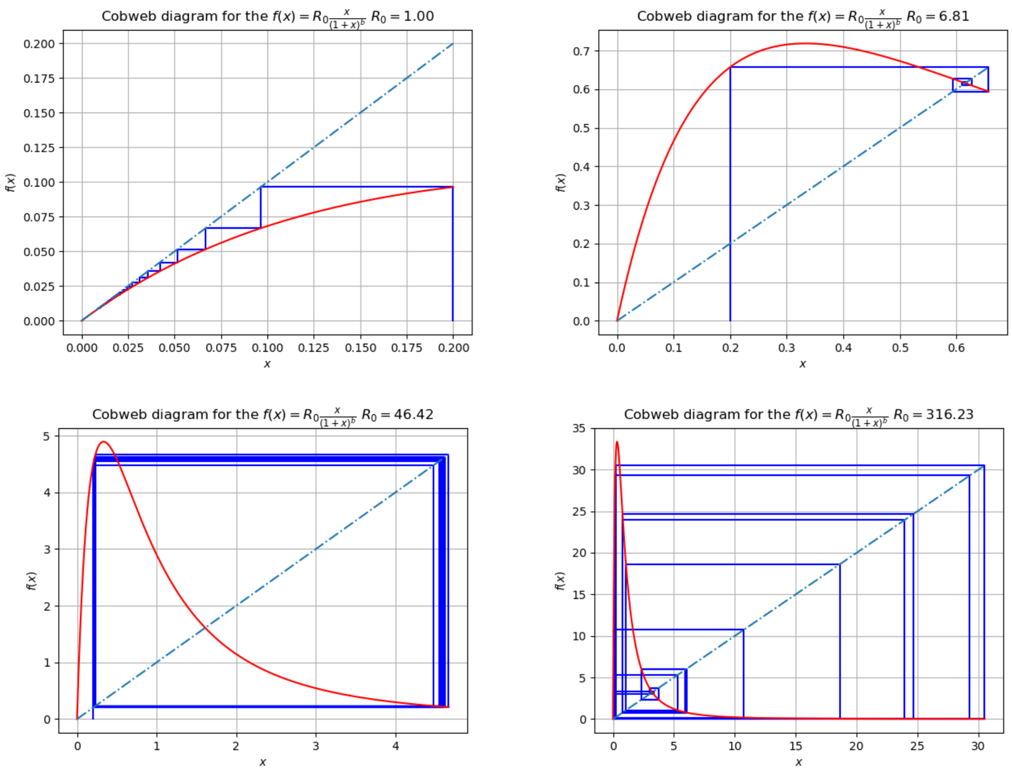

Figure 1: Cobweb plot for the Hassel model at 1. (Top Left) $R_0 = 1$, 2. (Top Right) $R_0 = 6.81$, 3. (Bottom Left) $R_0 = 46.42$ and 4. (Bottom Right) $R_0 = 316.23$

Looking at Figure 1, it is clear that for $b = 4$, choosing R_0 such that $\log(R_0) \sim \{0, 1, 2, 3\}$ will capture the behavior of the system. On choosing an extremely small value for R_0 (Figure 1.1), we see that $x_{n+1} < x_n \forall n$, thus demonstrating monotonically stable behavior. Choosing a slightly higher value of R_0 still keeps things stable (1.2), but $x_n < x_{n+2} \iff (n+1) \bmod (4) \neq 0, n > 0$, thus showing oscillatory behavior that collapses – eventually – to a stable fixed point. Going slightly higher, takes us to the oscillatory unstable regime (Figure 1.3) where we see the same oscillatory behavior as in 1.2, but without the function collapsing to a stable fixed point. We even see a drift in the oscillatory path, signifying that the fixed point is unstable even in $f^{(4)}$. Finally, in 1.4, we see chaos, with the trajectory following an aperiodic orbit.

2 Britton 1.1: Nonlinear population growth

Given the following equation:

$$x_{n+1} = \frac{\lambda x_n}{1 + x_n} \tag{1}$$

a: Does this equation over or undercompensate?

We know from Equation (1.2.4) and the explanation that immediately follows in EMB that the exponent in the denominator b is one when the equation exactly compensates between births and deaths.

b: Does it have a nontrivial steady state?

Steady states – or fixed points – occur when $x_{n+1} = x_n$. Looking at (1), we see that at a fixed point,

$$x_n = \frac{x_n}{1 + x_n}$$

Thus,

$$x_n^2 + x_n = \lambda x_n$$

Hence,

$$x_n^2 = (\lambda - 1)x_n$$

Clearly, a nontrivial positive fixed point exists iff $\lambda > 1$. If this condition is satisfied,

$$x^* = \lambda - 1$$

is a nontrivial steady state.

c and d: Discuss stability and bifurcations

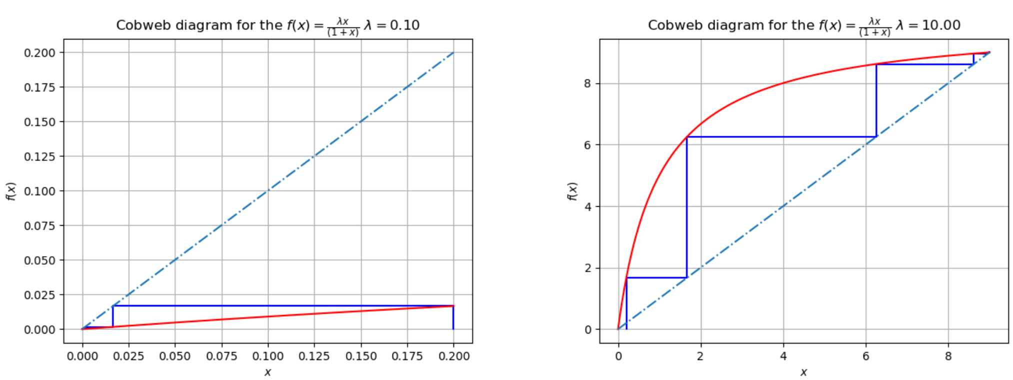

Figure 2: Cobweb plot of Equation 1. 1. (Left) $\lambda = 0.01$, 2. (Right) $\lambda = 10$

When $\lambda < 1$, the only fixed point is $x^* = 0$. At this point,

$$f'(x^*) = \frac{\lambda\left[(1 + x^*) - x^*\right]}{(1 + x^*)^2}$$

$$= \frac{\lambda}{(1 + x^*)^2} \leq 1$$

Since $\lambda \leq 1$, this fraction as a whole has to be less than or equal to 1. Thus, this fixed point is always stable. When $\lambda > 1$, we have two fixed points. The derivative is obviously the same. Thus, we have $f'(0) = \lambda > 1$, implying that 0 is an unstable fixed point in this regime. On the other hand, $f'(\lambda - 1) = 1/\lambda < 1$ which is always stable for $\lambda > 1$. Both these behaviors are captured in Figure 2. There is a saddle-node bifurcation that occurs at $\lambda = 1$. This is evidenced by the fact that a single stable fixed point is replaced by one stable and one unstable FP – and the second derivative of f vanishes at $\lambda = 1$

e: Solve the equation exactly

Suppose $y_n = 1/x_n$. We know that,

$$\frac{1}{y_{n+1}} = \frac{\lambda}{1 + y_n}$$

$$\lambda y_{n+1} = y_n + 1$$

Consider a solution that looks like $y_n = Ar^n + B$. We now have,

$$\lambda A r^{n+1} = Ar^n + 1$$

This has to hold $\forall n$. Thus, choose two particular points $n = 0$ and $n = 1$, to get

$$\lambda A r^2 = Ar + 1$$

$$r = \frac{1}{2\lambda} \pm \sqrt{\frac{1}{4\lambda} + \frac{1}{A\lambda}}$$

Choosing only the positive root and substituting the equation for $n = 0$, we have

$$\sqrt{A^2 + 4A\lambda} = A + 2$$

Squaring both sides,

$$4A\lambda = 4A + 4$$

Thus,

$$A = \frac{1}{\lambda - 1}$$

Finally,

$$r = \frac{1}{2\lambda} + \sqrt{\frac{1}{4\lambda} + \frac{\lambda - 1}{\lambda}}$$

If $0 < \lambda < 1$, $A < -1$, which implies that $r > 1$ (Since the term inside the square root is large and positive). Thus $y_n \sim (> 1)^n$, implying that $x_n \sim 1/(> 1)^n$ and thus decays monotonically to zero. On the other hand, if $\lambda \geq 1$, $r < 1$ (Both terms in $r < 1$ for $\lambda \geq 1$). As a result, y_n decreases monotonically to 1, implying that x_n increases with n. In fact, since

$$\lim_{n \to \infty} y_n = 1$$

ignoring B, we have

$$\lim_{n \to \infty} x_n = \frac{1}{A} = \lambda - 1$$

Thus, the solution monotonically increases to $\lambda - 1$, agreeing with my result in the previous section.

3 Britton 1.4: Sterile insects

$$N_{n+1} = f(N_n) = R_0 N_n \frac{N_n}{N_n + S} \frac{1}{1 + aN_n} \tag{2}$$

a: What does the $\frac{N_n}{N_n+S}$ do?

When N_n is very large,

$$\lim_{n\to\infty} \frac{N_n}{N_n + S} = 1$$

Demonstrating that for a huge initial insect population, sterilizing a few insects – in comparison – is of no use (i.e the members we control cannot affect the population in a meaningful way). If, on the other hand N_n is small, then

$$\frac{N_n}{N_n + S} \sim \frac{1}{S}$$

Thus, the population we control can reduce the number of insects in the next timestep by a factor of S (i.e, we have enough sterile insects to meaningfully control the system)

b: N^* vs S

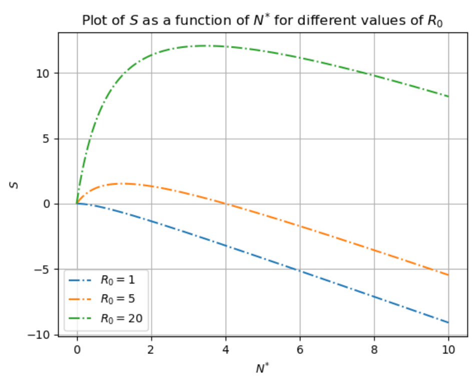

Figure 3: Plot of S as a function of the fixed point for different growth rates

From equation (2), we see that at fixed points,

$$1 = R_0 \frac{N^*}{N^* + S} \frac{1}{1 + aN^*}$$

Thus,

$$R_0 N^* = (N^* + S)(1 + aN^*)$$

Hence, denoting N^* as N

$$aN^2 + aSN + N + S = R_0 N$$

Thus,

$$S = \frac{R_0 N - N - aN^2}{aN + 1}$$

Assuming $a = 1$, the plot of S as a function of N is shown in Figure 3. In essence, the larger the growth constant, the larger – initially – the required S to curb the population. However, after a while, the required S drops, as the population is large enough for competition to become significant. For small R_0, of course, no control is necessary, since the only stable fixed point is $N = 0$

c: Critical value of S

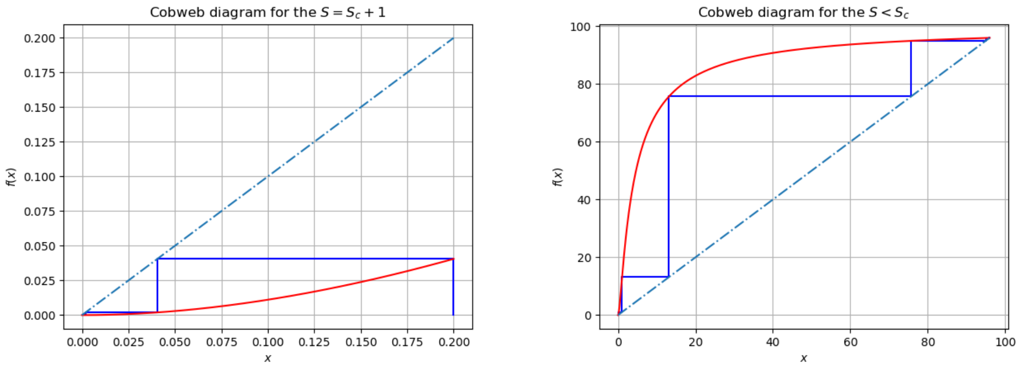

Figure 4: Cobweb plot for $S < S_c$ and $S > S_c$

Note: I'm assuming the question is asking for the least value of S that guarentees extinction. Given free reign of N, the lowest possible value of S is 0 We can find minimum values for S as a function N by looking for extremum points. Thus,

$$(aN + 1)(R_0 - 1 - 2aN) - (R_0 N - N - aN^2)a = 0$$

Hence,

$$a^2 N^2 - (R_0 - 1) + 2aN = 0$$

$$N_{max} = \frac{-2a \pm \sqrt{4a^2 + 4a^2(R_0 - 1)}}{2a^2}$$

Looking at the cobweb plots in Figure 4 that the effect of adding S is that it makes the curve dip under the line $y = x$, thus causing the fixed point at 0 to be stable.

4 Britton 1.11: Fisheries

Since the harvest rate is constant at qEN, the overall equation is the following:

$$\dot{N} = rN(1 - N/k) - qEN$$

a: Steady State equation

Steady states occur when $\dot{N} = 0$. Thus,

$$r(1 - N^*/k) = qE$$

Hence, we get

$$N^* = k(1 - qE/r)$$

b: Steady state yeild effort

At steady state, the overall yeild is

$$Y = qEN^* = qEk(1 - qE/r)$$

constituting the steady state yeild-effort relationship

c: Maximum sustainable yeild

Maximum yeild is obtained by finding the extremum point in $Y(E)$. Thus,

$$Y_{(E)} = qk(1 - qE/r) - \frac{q^2 Ek}{r} = 0$$

Thus,

$$qk(1 - qE_{max}/r) = \frac{q^2 E_{max} k}{r}$$

Rearranging and crossing out terms, we have

$$(r - qE_{max}) = qE_{max}$$

Therefore,

$$E_{max} = \frac{r}{2q}$$

Going back to our result in the previous section, the maximum sustainable yeild $Y_{max} = rk/4$

5 Britton 1.14: Hutchinson's Equation

Steady states occur when $\dot{N} = 0$. Thus,

$$(1 - N(t - \tau)/k) = 0 \rightarrow N(t - \tau) = k$$

and $N(t) = 0$ are steady states for the system. Taylor expanding about k, we get the following:

$$\dot{N} = -rN(t - \tau)$$

Guessing $N(t) = \exp(-\lambda\tau)$, we have

$$\lambda = -r \exp(-\lambda\tau)$$

Taking the derivative of λ wrt τ, we see that

$$\frac{d\lambda}{d\tau} = \frac{\lambda r}{\exp(\lambda\tau) - r\tau}$$

If the real part of $\frac{d\lambda}{d\tau} > 0$ for some $\lambda = i\beta$ – we can throw away the real part, since it's just a multiplicative factor in the end –, then instabilities can form. In this case, we're interested in $Re(\frac{i\beta r}{\exp(i\beta\tau) - r\tau})$. If this term is nonzero, so is it's reciprocal. Thus,

$$Re\left(\frac{i\beta r \exp(i\beta\tau) + i\beta r^2\tau}{\beta^2 r^2}\right)$$

$$= Re\left(\frac{r\sin(\beta\tau) - ir\cos(\beta\tau) + i\beta r^2\tau}{\beta^2 r^2}\right) = \frac{r\sin\beta\tau}{\beta^2 r^2} > 0$$

as long as $\beta\tau \neq n\pi$. Thus, in most cases, instabilities can form in the system. Looking at the fraction, as τ increases, the fraction can go from the left of the complex plane to the right ($i\beta r^2\tau$), so you can expect to see Hopf bifurcations.

6 More delay differential equations

The given equation is the following:

$$\dot{x} = -ax(t) - bx(t - \tau)$$

Let us define a new function $y = x - x^*$. Thus,

$$\dot{y} = -ay(t) - by(t - \tau) - C$$

a: Unstable equilibrium

First, let us linearize the equation around a fixed point of $x(t)$, x^*.

$$\frac{dy}{dt} = \frac{\partial g(y(t), y(t+\tau))}{\partial(t+\tau)} y(t+\tau)$$

This gives us the following:

$$\frac{dy}{dt} = -by(t-\tau)$$

Now, guess that the solution is $y(t) = \exp(\lambda t)$. Doing this yeilds the following:

$$\lambda = -b\exp(\lambda\tau)$$

Now, guess that $\lambda = i\beta$. Looking at the derivative wrt τ, we get the following:

$$\frac{d\lambda}{d\tau} = fracb\lambda\exp(-\lambda\tau)1 - b\tau\exp(-\lambda\tau)$$

Substituting $\lambda = i\beta$, we have

$$\begin{aligned} \left[\frac{d\lambda}{d\tau}\right]^{-1} &= frac1 - b\tau\exp(-\lambda\tau)b\lambda\exp(-\lambda\tau) \\ &= \frac{1}{b\lambda\exp(-\lambda\tau)} - \frac{\tau}{\lambda} \\ &= \frac{1}{ib\beta\exp(-i\beta\tau)} + \frac{i\tau}{\lambda} \end{aligned}$$

Since $\exp(-i\beta\tau) \sim -i\beta$, the derivative goes as $\frac{1}{\beta^2 b} > 0$. In addition, the complex term rotates with tau, thus passing from the left to the right in the complex plane. Thus, the system is likely to show Hopf bifurcations and instabilities.

b: Solve using `lambertw`

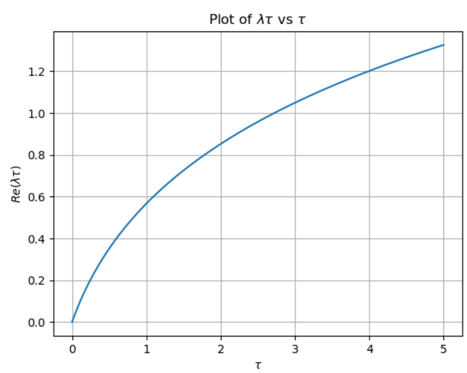

Figure 5: Plot of the dominant root obtained from the Lambertw function in scipy as a function of τ

First, we need to get the λ equation to the form $\lambda = w\exp(w)$. Thus, we have

$$\lambda\tau\exp(\lambda\tau) = -b\tau$$

Where, $w = \lambda\tau$. By computing $\lambda\tau$ using the Lambert-W function in, the dominant root was found to be greater than 1 when $\tau > 2.7$, which reprasents the advent of instability.