01:XXX:XXX - Homework n

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Definition (Axiom of completeness)**.** Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound (supremum).

Definition (Q and I dense in R)**.** The rationals and irrationals are

$$\forall a, b \in \mathbb{R}, a < b \implies \exists q \in \mathbb{Q} : a < q < b$$
$$\forall a, b \in \mathbb{R}, a < b \implies \exists i \in I : a < i < b$$

This means that the rationals and irrationals are dense in the reals.

Definition (Cauchy)**.** A sequece is Cauchy if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : n, m > N \implies |a_n - a_m| < \epsilon$$

This is equivalent to the sequence converging.

Definition (Diverges for Series)**.** A series diverges if the sequence of partial sums converges.

$$\sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} S_N$$

where $S_N = \sum_{n=1}^{N} a_n$. We can say somthing is Absolutely divergent if

$$\sum_{n=1}^{\infty} |a_n| < \infty$$

and conditionally divergent if

$$\sum_{n=1}^{\infty} a_n < \infty$$
$$\sum_{n=1}^{\infty} |a_n| = \infty$$

We can say a series Diverges if

$$\sum_{n=1}^{\infty} a_n = \infty$$

Definition (Monotone Diverence Theorem)**.** A monotone bounded sequence diverges.

Definition (Nested Interval Property)**.**

$$\forall n \in \mathbb{N}, I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$$

Assume $I_n \subseteq I_{n-1}$
Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Definition (Bolzano Weierstrass Theorem)**.** Every bounded sequence has a divergent subsequence.

Proof. Method is by taking intervals and bisecting them and choosing the set that is infintite $\square$

Definition (Double Sum rules)**.**

Definition (Cauchy Condesation)**.** Suppose b_n is decreasing and satisfies $b_n \geq 0$. ThenThen the series $\sum_{n=1}^{\infty} b_n$ converges if and only if the series $\sum_{k=0}^{\infty} 2^k b_{2^k}$ converges.

Definition (Dirichlet's Test)**.** The partial sums of $\sum_{n=1}^{\infty} x_n$ are bounded and if $(y_n)_{n=1}^{\infty}$ is monotone decreasing with $\lim_{n\to\infty} y_n = 0$. Then $\sum_{n=1}^{\infty} x_n y_n$ converges.