Time Series Analysis - Homework 5

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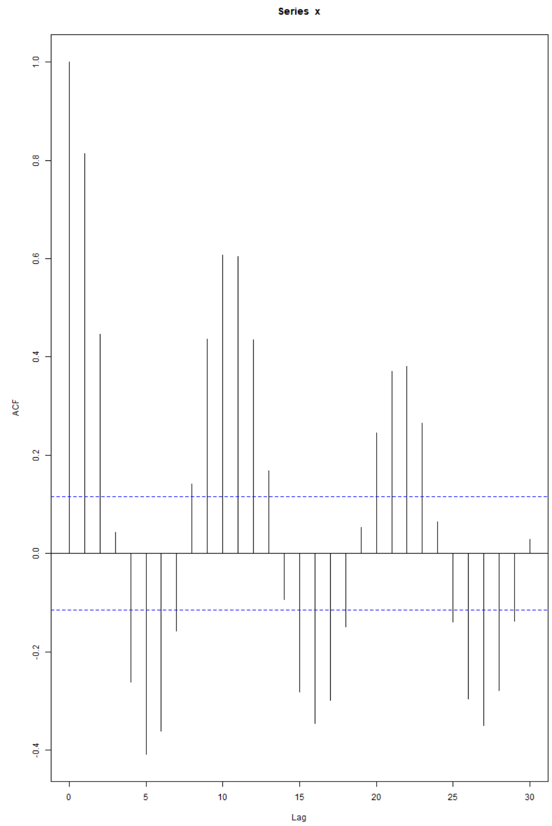
Problem (24)**.** Use the Wölfer sunspot number introduced in Problem 21.

1. Calculate the sample autocovariances up to lag 3. Also plot the sample ACF using the `R` function `acf()`.

Solution: The sample autocovariances up to lag 3 are as follows:

$$\hat{\gamma}(0) = 1552.81307$$
$$\hat{\gamma}(1) = 1264.19939$$
$$\hat{\gamma}(2) = 693.89068$$
$$\hat{\gamma}(3) = 66.49035$$

The sample ACF plot up to 30 lags is shown below:



2. Calculate the Yule-Walker estimators of ϕ_1, ϕ_2, σ^2 in the AR(2) model

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + Z_t, \quad \{Z_t\} \sim \mathrm{IID}(0, \sigma^2),$$

for the mean corrected series $Y_t = X_t - 49.13$.

Solution: To calculate the Yule-Walker estimators for the AR(2) model, we use the sample autocovariances calculated in part (a). The Yule-Walker equations for an AR(2) model are given by:

$$\hat{\gamma}(1) = \phi_1 \hat{\gamma}(0) + \phi_2 \hat{\gamma}(1)$$
$$\hat{\gamma}(2) = \phi_1 \hat{\gamma}(1) + \phi_2 \hat{\gamma}(0)$$

Plugging in the values from part (a):

$$\phi_1 = 1.33556, \quad \phi_2 = -0.64047$$

The estimator for σ^2 is given by:

$$\begin{aligned}\hat{\sigma}^2 &= \hat{\gamma}(0) - \phi_1 \hat{\gamma}(1) - \phi_2 \hat{\gamma}(2) \\ &= 1552.81307 - 1.33556 \times 1264.19939 - (-0.64047) \times 693.89068 \\ &= 308.81509\end{aligned}$$

Thus, the Yule-Walker estimators are:

$$\hat{\phi}_1 = 1.33556$$
$$\hat{\phi}_2 = -0.64047$$
$$\hat{\sigma}^2 = 308.81509$$

3. Calculate the sample PACF up to lag 3. Also plot the sample PACF using the `R` function `pacf()`.

Solution: The way to calculate the sample PACF up to lag 3 is as follows: PACF at lag 1:

$$\hat{\phi}_{1,1} = \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)} = \frac{1264.19939}{1552.81307} = 0.8141$$

PACF at lag 2:

$$\Gamma_2 = \begin{pmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) \end{pmatrix} = \begin{pmatrix} 1552.81307 & 1264.19939 \\ 1264.19939 & 1552.81307 \end{pmatrix}$$
$$\gamma_2 = \begin{pmatrix} \hat{\gamma}(1) \\ \hat{\gamma}(2) \end{pmatrix} = \begin{pmatrix} 1264.19939 \\ 693.89068 \end{pmatrix}$$
$$\begin{bmatrix} \hat{\phi}_{2,1} \\ \hat{\phi}_{2,2} \end{bmatrix} = \Gamma_2^{-1}\gamma_2$$
$$\hat{\phi}_{2,1} = 1.33556, \hat{\phi}_{2,2} = -0.6405$$

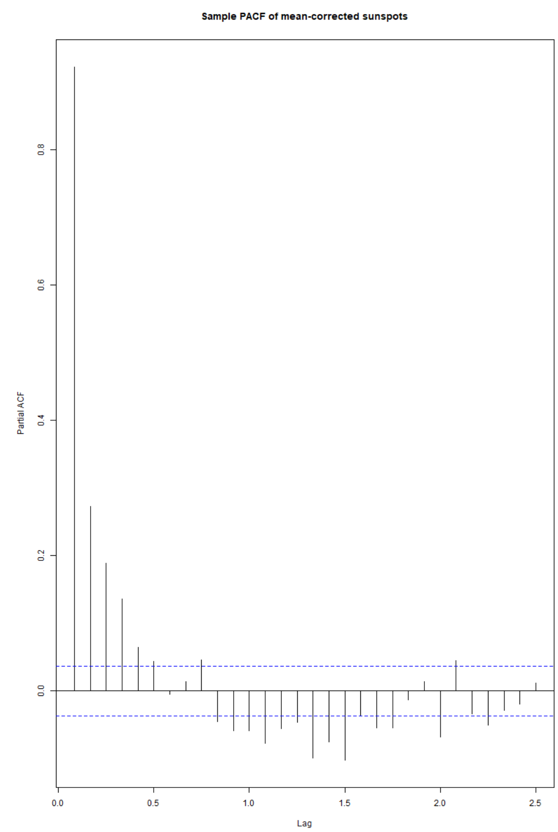
PACF at lag 3: (ommitting detailed calculations for brevity)

$$\hat{\phi}_{3,1} = 1.2307$$
$$\hat{\phi}_{3,2} = -0.4217784\hat{\phi}_{3,3} \qquad = -0.1637426$$

Thus the sample PACF values up to lag 3 are:

$$\hat{\phi}_{1,1} = 0.8141$$
$$\hat{\phi}_{2,2} = -0.6405$$
$$\hat{\phi}_{3,3} = -0.1637$$

The sample PACF plot up to 30 lags is shown below:



Problem (25)**.** Let $\gamma(\cdot)$ be an autocovariance function such that the autocovariance matrix Γ_k is non-singular for every $k \geq 1$. Define $\phi_k = (\phi_{k1}, \phi_{k2}, \ldots, \phi_{kk})' := \Gamma_k^{-1}\gamma_k$.

1. Show that $\phi_k(z) := 1 - \phi_{k1}z - \phi_{k2}z^2 - \cdots - \phi_{kk}z^k \neq 0$ for $|z| \leq 1$.

Solution: These coefficients ϕ_{kj} are given by the Yule-Walker equations for the best linear predictor of X_t based on its past k values.

Note that Γ_k is positive definite since it is non-singular and symmetric.

Let us consider the Durbin–Levinson algorithm, which provides a recursive method to compute the coefficients ϕ_{kj}. Let $\kappa_k = \phi_{kk}$:

$$\begin{aligned} \phi_{j+1,i} &= \phi_{j,i} - \kappa_{j+1}\phi_{j,j+1-i}, \quad i = 1, 2, \ldots, j \\ \phi_{j+1,j+1} &= \kappa_{j+1} \end{aligned}$$

Note that, when Γ_k is positive definite, κ_k satisfies $|\kappa_k| < 1$ for all k, since $\sigma_k^2 = \sigma_{k-1}^2(1 - \kappa_k^2)$ and $\sigma_k^2 > 0$.

Let $A_j(z) = 1 - \sum_{i=1}^{j} \phi_{ji} z^i$ and the reversed polynomial $A_j^*(z) = z^j A_j(1/z) = z^j - \sum_{i=1}^{j} \phi_{ji} z^{j-i}$.

Using the Durbin–Levinson recursions, we can derive the relation:

$$A_{j+1}(z) = A_j(z) - \kappa_{j+1} z A_j^*(z)$$

Lemma. If all zeros of $A_j(z)$ lie outside the unit circle, then

$$|A_j(z)| \geq |A_j^*(z)| \quad \text{for } |z| \leq 1,$$

with strict inequality for $|z| < 1$.

Proof of Lemma. Let the zeros of $A_j(z)$ be $z_1, \ldots, z_j$ with $|z_i| > 1$ for all i. Then

$$A_j(z) = \prod_{i=1}^{j} \big(1 - z/z_i\big), \qquad A_j^*(z) = z^j A_j(1/z) = \prod_{i=1}^{j} (z - 1/z_i).$$

Hence

$$\frac{|A_j(z)|}{|A_j^*(z)|} = \prod_{i=1}^{j} \frac{|1 - z/z_i|}{|z - 1/z_i|} = \prod_{i=1}^{j} \frac{|z_i - z|}{|z_i z - 1|}.$$

Fix i and set $w = z_i$. For each factor we have

$$|w - z|^2 - |wz - 1|^2 = (|w|^2 - 1)(1 - |z|^2).$$

This identity is obtained by expanding both sides:

$$|w - z|^2 = |w|^2 - w\bar{z} - \bar{w}z + |z|^2,$$
$$|wz - 1|^2 = |w|^2|z|^2 - wz - \bar{w}\bar{z} + 1,$$

and subtracting. Since $|w| > 1$ and $|z| \leq 1$, we have $(|w|^2 - 1) \geq 0$ and $(1 - |z|^2) \geq 0$, so

$$|w - z|^2 \geq |wz - 1|^2 \quad \Rightarrow \quad |w - z| \geq |wz - 1|.$$

If $|z| < 1$, then $(|w|^2 - 1)(1 - |z|^2) > 0$, so the inequality is strict: $|w - z| > |wz - 1|$.

Applying this to each $w = z_i$ gives, for $|z| \leq 1$,

$$\frac{|A_j(z)|}{|A_j^*(z)|} = \prod_{i=1}^{j} \frac{|z_i - z|}{|z_i z - 1|} \geq 1,$$

and for $|z| < 1$ each factor is > 1, so the product is > 1. Thus $|A_j(z)| \geq |A_j^*(z)|$ for $|z| \leq 1$, with strict inequality when $|z| < 1$.

Now we can induct on the degree j to show that $A_j(z) \neq 0$ for $|z| \leq 1$.

Base Case($j = 1$): $A_1(z) = 1 - \phi_{11}z$. Since $|\phi_{11}| < 1$, the zero of $A_1(z)$ is at $z = 1/\phi_{11}$ with $|1/\phi_{11}| > 1$. Thus $A_1(z) \neq 0$ for $|z| \leq 1$.

Inductive step: Assume $A_j(z) \neq 0$ for $|z| \leq 1$. By the Key Lemma, $|A_j(z)| \geq |A_j^*(z)|$ for $|z| \leq 1$.

For $A_{j+1}(z) = A_j(z) - \kappa_{j+1}zA_j^*(z)$, we compute:

$$\begin{aligned} |A_{j+1}(z)| &= |A_j(z) - \kappa_{j+1}zA_j^*(z)| \\ &\geq |A_j(z)| - |\kappa_{j+1}||z||A_j^*(z)|. \end{aligned}$$

For $|z| = 1$: Since $|A_j(z)| = |A_j^*(z)|$ and $|\kappa_{j+1}| < 1$,

$$|A_{j+1}(z)| \geq |A_j(z)|(1 - |\kappa_{j+1}|) > 0.$$

For $|z| < 1$: Since $|A_j(z)| > |A_j^*(z)|$ and $|z| < 1$,

$$|A_{j+1}(z)| \geq |A_j(z)| - |\kappa_{j+1}||z||A_j^*(z)| > |A_j(z)| - |A_j^*(z)| \geq 0.$$

More rigorously: $|A_{j+1}(z)| > 0$ follows because if $|A_{j+1}(z_0)| = 0$ for some $|z_0| < 1$, then

$$|A_j(z_0)| = |\kappa_{j+1}z_0A_j^*(z_0)|,$$

which contradicts $|A_j(z_0)| > |A_j^*(z_0)|$ and $|\kappa_{j+1}||z_0| < 1$.

Thus $A_{j+1}(z) \neq 0$ for $|z| \leq 1$.

By induction, $A_k(z) \neq 0$ for $|z| \leq 1$, which implies that $\phi_k(z) \neq 0$ for $|z| \leq 1$.

2. Conclude that if $\hat{\gamma}(0) > 0$, then the AR(p) model given by Yule-Walker estimators must be causal.

Solution: Note that an AR(p) model is causal if the roots of the characteristic polynomial lie outside the unit circle.

Consider the Yule-Walker Estimatiors for the AR(p) model:

$$\hat{\phi}_p = \hat{\Gamma}_p^{-1}\hat{\gamma}_p$$

where $\hat{\Gamma}_p$ is the sample autocovariance matrix and $\hat{\gamma}_p$ is the vector of sample autocovariances. $\hat{\Gamma}_p$ is non-negative definite by construction If $\hat{\gamma}(0) > 0$, then $\hat{\Gamma}_p$ is strictly positive definite since the variance is positive and data is not constant.

Since $\hat{\gamma}(0) > 0$, by part (a), the polynomial

$$\hat{\phi}_p(z) = 1 - \hat{\phi}_{p1}z - \hat{\phi}_{p2}z^2 - \cdots - \hat{\phi}_{pp}z^p$$

has no zeros inside or on the unit circle. Therefore, all roots of the characteristic polynomial lie outside the unit circle, which implies that the AR(p) model is causal.

Problem (26)**.** For an causal AR(p) process, show that $\det \Gamma_m = (\det \Gamma_p)\sigma^{2(m-p)}$ for all $m > p$. Conclude that the (m, m)-th entry of Γ_m^{-1} is σ^{-2}.

Solution: Let $\{X_t\}$ be a causal AR(p) process defined by:

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t, \qquad \{Z_t\} \sim \mathrm{IID}(0, \sigma^2).$$

Let Γ_m be the $m \times m$ autocovariance matrix with (i, j)-entry $\gamma(i - j)$. We want to show $\det(\Gamma_m) = (\det \Gamma_p)\sigma^{2(m-p)}$ for all $m > p$, and that $(\Gamma_m^{-1})_{mm} = \sigma^{-2}$.

Step 1: One-step prediction error and Schur complement.

For $n \geq 1$, let $\hat{X}_n^{n+1}$ be the best linear predictor of X_{n+1} based on $(X_1, \ldots, X_n)$ and define the one-step prediction error variance

$$P_n^{n+1} := \mathbb{E}\big[(X_{n+1} - \hat{X}_n^{n+1})^2\big].$$

Let

$$\Gamma_n = \begin{pmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(n-2) \\ \vdots & \vdots & & \vdots \\ \gamma(n-1) & \gamma(n-2) & \cdots & \gamma(0) \end{pmatrix}, \qquad \gamma_n = \begin{pmatrix} \gamma(1) \\ \vdots \\ \gamma(n) \end{pmatrix}.$$

The Yule–Walker equations for the optimal linear predictor give $\Gamma_n \phi_n = \gamma_n$, and the orthogonality principle yields

$$P_n^{n+1} = \gamma(0) - \gamma_n' \Gamma_n^{-1} \gamma_n.$$

Now write Γ_{n+1} in block form:

$$\Gamma_{n+1} = \begin{pmatrix} \Gamma_n & \gamma_n \\ \gamma_n' & \gamma(0) \end{pmatrix}.$$

The Schur complement of Γ_n in Γ_{n+1} is

$$S_{n+1} = \gamma(0) - \gamma_n' \Gamma_n^{-1} \gamma_n = P_n^{n+1}.$$

Hence, by the block determinant formula,

$$\det(\Gamma_{n+1}) = \det(\Gamma_n)\,\det(S_{n+1}) = \det(\Gamma_n)\,P_n^{n+1}.$$

Step 2: For a causal AR(p), $P_n^{n+1} = \sigma^2$ for $n \geq p$.

Since $\{X_t\}$ is a causal AR(p) process,

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t.$$

For any $n \geq p$, the best linear predictor of X_{n+1} based on $(X_1, \ldots, X_n)$ is exactly

$$\hat{X}_n^{n+1} = \phi_1 X_n + \phi_2 X_{n-1} + \cdots + \phi_p X_{n+1-p},$$

since these p past values appear in the defining equation and are all observed.

The prediction error is therefore

$$\begin{aligned} X_{n+1} - \hat{X}_n^{n+1} &= \big(\phi_1 X_n + \cdots + \phi_p X_{n+1-p} + Z_{n+1}\big) - \big(\phi_1 X_n + \cdots + \phi_p X_{n+1-p}\big) \\ &= Z_{n+1}. \end{aligned}$$

Hence

$$P_n^{n+1} = \mathbb{E}[Z_{n+1}^2] = \sigma^2, \qquad \text{for all } n \geq p.$$

Step 3: Determinant formula.

Apply the recursion $\det(\Gamma_{n+1}) = \det(\Gamma_n) P_n^{n+1}$ from $n = p$ up to $n = m-1$:

$$\begin{aligned} \det(\Gamma_{p+1}) &= \det(\Gamma_p)\,P_p^{p+1}, \\ \det(\Gamma_{p+2}) &= \det(\Gamma_{p+1})\,P_{p+1}^{p+2}, \\ &\vdots \\ \det(\Gamma_m) &= \det(\Gamma_{m-1})\,P_{m-1}^{m}. \end{aligned}$$

Multiplying these equalities gives

$$\det(\Gamma_m) = \det(\Gamma_p) \prod_{n=p}^{m-1} P_n^{n+1}.$$

Since $P_n^{n+1} = \sigma^2$ for all $n \geq p$, we obtain

$$\det(\Gamma_m) = \det(\Gamma_p) \prod_{n=p}^{m-1} \sigma^2 = \det(\Gamma_p)\,\sigma^{2(m-p)}.$$

Step 4: The (m, m)-th entry of Γ_m^{-1}.

Finally, write Γ_m in the 2×2 block form

$$\Gamma_m = \begin{pmatrix} \Gamma_{m-1} & \gamma_{m-1} \\ \gamma'_{m-1} & \gamma(0) \end{pmatrix},$$

where $\gamma_{m-1} = (\gamma(1), \ldots, \gamma(m-1))'$. The Schur complement of Γ_{m-1} is

$$S_m = \gamma(0) - \gamma'_{m-1}\Gamma_{m-1}^{-1}\gamma_{m-1} = P_{m-1}^m.$$

The block inverse formula yields

$$\Gamma_m^{-1} = \begin{pmatrix} * & * \\ * & S_m^{-1} \end{pmatrix},$$

so the (m, m)-th entry of Γ_m^{-1} is S_m^{-1}:

$$(\Gamma_m^{-1})_{mm} = S_m^{-1} = \frac{1}{P_{m-1}^m}.$$

For $m > p$ we have $m - 1 \geq p$ and hence $P_{m-1}^m = \sigma^2$, so

$$(\Gamma_m^{-1})_{mm} = \frac{1}{\sigma^2} = \sigma^{-2}.$$

This proves both claims.