TODO

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Question 1

Consider $\begin{bmatrix} x \\ y \end{bmatrix}'(t) = v(x, y, t), x(0) = x_0, y(0) = y_0$
Apply Picard iteration scheme starting with inital function the constant function $X(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ Then compare the examples

a

$v(x, y, t) = \begin{bmatrix} t \\ x^2 \end{bmatrix}$
The Picard iteration scheme is given

$$x_0(t) = x_0$$
$$x_1(t) = x_0 + \int_0^t \begin{bmatrix} s \\ x_0^2 \end{bmatrix} ds = \begin{bmatrix} x_0 + \frac{t^2}{2} \\ y_0 + x_0^2 t \end{bmatrix}$$
$$x_2(t) = x_0 + \int_0^t \begin{bmatrix} s \\ (x_0 + \frac{s^2}{2})^2 \end{bmatrix} ds = \begin{bmatrix} x_0 + \frac{t^2}{2} \\ y_0 + x_0^2 t + x_0 \frac{t^3}{3} + \frac{x^5}{20} \end{bmatrix}$$
$$x_3(t) = x_0 + \int_0^t \begin{bmatrix} s \\ (x_0 + \frac{s^2}{2})^2 \end{bmatrix} ds = \begin{bmatrix} x_0 + \frac{t^2}{2} \\ y_0 + x_0^2 t + x_0 \frac{t^3}{3} + \frac{x^5}{20} \end{bmatrix}$$

We can see from here that for all X_n for $n \geq 2$ they will be eqaul meaning that

$$X(t) = \begin{bmatrix} x_0 + \frac{t^2}{2} \\ y_0 + x_0^2 t + x_0 \frac{t^3}{3} + \frac{x^5}{20} \end{bmatrix}$$

b

$v(x, y, t) = \begin{bmatrix} -y \\ x \end{bmatrix}$ with specific intial condition $x(0) = 1, y(0) = 0$ The Picard iteration scheme is given

$$x_0(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$x_1(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \int_0^t \begin{bmatrix} -y_0 \\ x_0 \end{bmatrix} ds = \begin{bmatrix} x_0 - y_0 t \\ y_0 + x_0 t \end{bmatrix}$$

$$x_2(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \int_0^t \begin{bmatrix} -y_0 - x_0 t \\ x_0 - y_0 t \end{bmatrix} ds = \begin{bmatrix} x_0 - y_0 t - x_0 \frac{t^2}{2} \\ y_0 + x_0 t - y_0 \frac{t^2}{2} \end{bmatrix}$$

$$x_3(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \int_0^t \begin{bmatrix} -y_0 - x_0 t + y_0 \frac{t^2}{2} \\ x_0 - y_0 t - x_0 \frac{t^2}{2} \end{bmatrix} ds = \begin{bmatrix} x_0 - y_0 t - x_0 \frac{t^2}{2} + y_0 \frac{t^3}{6} \\ y_0 + x_0 t - y_0 \frac{t^2}{2} - x_0 \frac{t^3}{6} \end{bmatrix}$$

After plugging in for $x_0 = 1, y_0 = 0$ we get

$$x_3(t) = \begin{bmatrix} 1 - \frac{t^2}{2} \\ t - \frac{t^3}{6} \end{bmatrix}$$

As we continue to iterate we will notice a common taylor series expansion of $sin(t)$ and $cos(t)$

Thus:

$$X(t) = \begin{bmatrix} cos(t) \\ sin(t) \end{bmatrix}$$

Question 2

A function that obeys the properties of a contraction maps must obey 2 properties.

The function decreases by a constant factor $0 < k < 1$ for all $x, y \in \mathbb{R}^n$ and $f(x) - f(y) \leq k(x - y)$

This means that a function in the form of $f(x) = kx$ will satisfy this property.

For example $f(x) = .5x$ will satisfy this property as at each natural number n the function will decrease by a factor of .5.

We can continue this to that it will go to 0 as $n \to \infty$

Thus the function $f(x) = .5x$ is a contraction map.