01:640:311H - Homework n

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1. Write an ε-δ proof that the function $f(x) = x^2 - 2x$ is continuous at $x = 2$.

Solution: Let $\epsilon > 0$. Take $\delta = \min\left(1, \frac{\epsilon}{3}\right)$. Since $|x-2| < 1 \implies |x| < |x-2| + 2 < 1 + 2 = 3$. Also not that $f(2) = 0$. Then, we have

$$\begin{aligned} |x-2| < \delta \implies |f(x)| &= |x^2 - 2x| = |x(x-2)| \\ &= |x| \cdot |x-2| < |x| \cdot \delta \\ &= 3 \cdot \frac{\epsilon}{3} = \epsilon \end{aligned}$$

2. Suppose $f : A \to \mathbb{R}$ is continuous at a point $c \in A$. Prove that if $f(c) \neq 0$, then there exists a $\delta > 0$ such that $f(x) \neq 0$ for any $x \in V_\delta(c) \cap A$.

Solution: Let $\epsilon = |f(c) - 0| > 0$. Since f is continuous at c, there exists $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for all $x \in V_\epsilon(c) \cap A$.
Then, we have

$$\begin{aligned} |f(x) - f(c)| &< \epsilon \\ -\epsilon < f(x) - f(c) &< \epsilon \\ -|f(c)| < f(x) - f(c) &< |f(c)| \end{aligned}$$

Either $f(x) < 0$ or $f(x) > 0$.
If $f(x) < 0$, then $2f(c) < f(x) < 0$
If $f(x) > 0$, then $0 < f(x) < 2f(c)$
Thus, $f(x) \neq 0$ for all $x \in V_\delta(c) \cap A$.

3. Using only the ε-δ definition of continuity, prove that if $f, g : A \to \mathbb{R}$ are continuous at a point $c \in A$, then $f + g$ is also continuous at c.

Solution: Suppose f and g are continuous at c. Then, for any $\epsilon > 0$, there exists $\delta_1 > 0$ such that $|f(x) - f(c)| < \frac{\epsilon}{2}$ for all $x \in V_{\delta_1}(c) \cap A$ and there exists $\delta_2 > 0$ such that $|g(x) - g(c)| < \frac{\epsilon}{2}$ for all $x \in V_{\delta_2}(c) \cap A$.
Let $\delta = \min(\delta_1, \delta_2)$. Then, for all $x \in V_\delta(c) \cap A$, we have

$$\begin{aligned}
|(f+g)(x) - (f+g)(c)| &= |f(x) + g(x) - f(c) - g(c)| \\
&= |(f(x) - f(c)) + (g(x) - g(c))| \\
&\leq |f(x) - f(c)| + |g(x) - g(c)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Thus, $f + g$ is continuous at c.

4. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and U is open, then $f^{-1}(U) = \{x \in \mathbb{R} : f(x) \in U\}$ is open.

Solution: Suppose f is continuous and U is open. Let $x \in f^{-1}(U)$ then $f(x) \in U$.
Since U is open there exists an $\epsilon > 0$ such that there exist $V_\epsilon(f(x)) \subset U$
Since f is continous we know there exists a $\delta > 0$ such that $|y - x| < \delta \implies |f(y) - f(x)| < \epsilon$ for all $y \in \mathbb{R}$.
Thus for an arbitrary $y \in V_\delta(x)$ we have $f(y) \in V_\epsilon(f(x)) \subset U$
Thus $y \in f^{-1}(U)$ and hence $V_\delta(x) \subset f^{-1}(U)$
Thus $f^{-1}(U)$ is open.

5. Suppose $f, g : [a, b] \to \mathbb{R}$ are two continuous functions. Prove that the set $T = \{x : f(x) = g(x)\}$ is a closed set.

Solution: Suppose x is a limit point of T. Then there exists a sequence $\{x_n\} \subset T$ such that $x_n \to x$ and $x_n \neq x$ for all n.
Since f and g are continuous, we have $\lim_{n\to\infty} f(x_n) = f(x)$ and $\lim_{n\to\infty} g(x_n) = g(x)$.
Since we have that $f(x_n) = g(x_n)$ for all n, we have $f(x) = g(x)$.
Thus, $x \in T$ and hence T is closed.

6. Let $f : [a, b] \to \mathbb{R}$ be increasing. Prove that $\lim_{x\to a^+} f(x)$ exists.

Solution: Since f is increasing on $[a, b]$, the set $S = \{f(x) : x \in (a, b]\}$ is bounded below by $f(a)$. By the completeness of $\mathbb{R}$, S has a greatest lower bound (infimum). Let:

$$L = \inf S = \inf\{f(x) : x \in (a, b]\}.$$

We will show that $\lim_{x \to a^+} f(x) = L$.

1. Let $\epsilon > 0$. By the definition of infimum, there exists $y \in (a, b]$ such that:

$$f(y) < L + \epsilon.$$

2. Let $\delta = y - a > 0$. For all x satisfying $a < x < a + \delta$, we have:

$$a < x < y \leq b.$$

3. Since f is increasing:

$$L \leq f(x) \leq f(y) < L + \epsilon.$$

4. Combining these inequalities:

$$|f(x) - L| = f(x) - L < \epsilon \quad \text{for all } x \in (a, a + \delta).$$

By the definition of a right-hand limit, we conclude:

$$\lim_{x \to a^+} f(x) = L.$$