HW1: Math 423

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1.1 Problem 11

Verify that $u(x,y) = f(x)g(y)$ is a solution of the PDE $uu_{xy} = u_x u_y$ for all pairs of functions f and g of one variable.
Solution:

$$\begin{aligned}
u_x &= f'(x)g(y) \\
u_y &= f(x)g'(y) \\
u_{xy} &= f'(x)g'(y) \\
u_x u_y &= f'(x)g(y)f(x)g'(y) \\
uu_{xy} &= f(x)g(y)f'(x)g'(y) \\
uu_{xy} &= u_x u_y
\end{aligned}$$

Thus clearly $u(x,y) = f(x)g(y)$ is a solution of the PDE $uu_{xy} = u_x u_y$ for all pairs of functions f and g of one variable.

1.1 Problem 12

Verify by direct substitution that the function $u(x,y) = sin(nx)sinh(ny)$ is a solution of the PDE $u_{xx} + u_{yy} = 0$ for all positive integers n.
Solution:

$$\begin{aligned}
u_x &= ncos(nx)sinh(ny) \\
u_y &= nsin(nx)cosh(ny) \\
u_{xx} &= -n^2 sin(nx)sinh(ny) \\
u_{yy} &= n^2 sin(nx)sinh(ny) \\
u_{xx} + u_{yy} &= -n^2 sin(nx)sinh(ny) + n^2 sin(nx)sinh(ny) \\
u_{xx} + u_{yy} &= 0
\end{aligned}$$

Thus clearly $u(x,y) = sin(nx)sinh(ny)$ is a solution of the PDE $u_{xx} + u_{yy} = 0$ for all positive integers n.

1.2 Problem 2

Solve the equation $3u_y + u_{xy} = 0$
Solution:
Let $v = u_y$
Then $v_x = u_{xy}$
Thus the equation becomes $3v + v_x = 0$
This is a first order linear ODE.
The integrating factor is e^{3x}
Multiplying both sides by the integrating factor we get

$$\begin{aligned} 3e^{3x}v + e^{3x}v_x &= 0 \\ \frac{d}{dx}(e^{3x}v) &= 0 \\ e^{3x}v &= C \\ v &= C_1 e^{-3x} \end{aligned}$$

Where C is a constant $\in \mathbb{R}$.
Now we have $v = u_y$
Integrating v with respect to y we get

$$\begin{aligned} u &= \int v dy \\ u &= \int Ce^{-3x} dy \\ u &= C_1 e^{-3x} y + C_2(x) \end{aligned}$$

Where C_1 is a constant s $\in \mathbb{R}$ and C_2 is function of x.

1.2 Problem 6

Solve the equation $\sqrt{1-x^2}u_x + u_y = 0$ with the condition that $u(0, y) = y$
Solution:
Noticing that the directional derivative in the direction of $(\sqrt{1-x^2}, 1)$ is 0. The function is constant along these curves. We can find the characteristic curves by solving the ODE $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} \\ y &= arcsin(x) + C \end{aligned}$$

Now we have $u(x, y(x)) = u(x, arcsin(x) + C)$
Let $x = 0$ as an arbitrary value so we have $u(0, y(0)) = u(0, arcsin(0) + C) = u(0, C) = f(C)$
Thus $C = y - arcsin(x)$
Thus $u(x, y) = f(y - arcsin(x))$
Now we can apply the initial condition $u(0, y) = y$
$u(0, y) = f(y - arcsin(0)) = f(y) = y$
Thus $f(y) = y$
Thus $u(x, y) = y - arcsin(x)$

1.2 Problem 10

Solve $u_x + u_y + u = e^{x+2y}$ with $u(x, 0) = 0$
Solution:

$$x' = x + y, y' = x - y$$
$$x = \frac{x' + y'}{2}, y = \frac{x' - y'}{2}$$
$$u_x = u_{x'} + u_{y'}$$
$$u_y = u_{x'} - u_{y'}$$
$$2u_{x'} + u(x', y') = e^{\frac{3x' - y'}{2}}$$

This is now a first order linear ODE.
The integrating factor is $e^{2x'}$
Multiplying both sides by the integrating factor we get

$$2e^{2x'}u_{x'} + e^{2x'}u = e^{\frac{7x' - y'}{2}}$$
$$\frac{d}{dx'}(e^{2x'}u) = e^{\frac{7x' - y'}{2}}$$
$$e^{2x'}u = \frac{2}{7}e^{\frac{7x' - y'}{2}} + C(y)$$
$$u = \frac{2}{7}e^{-\frac{3x' - y'}{2}} + C(y')e^{-2x'}$$

Now we have $u(x', y')$ Now we need to convert back into $u(x, y)$

$$u(x, y) = \frac{2}{7}e^{x+2y} + C(x - y)e^{-2x-2y}$$

1.3 Problem 6

Consider heat flow in a long circular cylinder where the tempurature only depends on t and r. From the 3d heat equation derivae the equation $u_t = k(u_{rr} + \frac{u_r}{r})$

Solution:

The heat equation in 3D is given by

$$c\rho \frac{\partial u}{\partial t} = \nabla \cdot (k \nabla u)$$

In cylindrical coordinates we can derive the Laplacian by rewretiting the divergence in cylindrical coordinates.

Consider that for cylindrical coordinates we have

$$r = \sqrt{x^2 + y^2}$$
$$\theta = tan^{-1}(\frac{y}{x})$$

Since we know the Laplacian is in the form of:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

We can write each of these terms in cylindrical coordinates.

Thus we have

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2 r}{\partial x^2}\frac{\partial}{\partial r} + (\frac{\partial r}{\partial x})^2 \frac{\partial^2}{\partial r^2} + \frac{\partial^2 \theta}{\partial x^2}\frac{\partial}{\partial \theta} + (\frac{\partial \theta}{\partial x})^2 \frac{\partial^2}{\partial \theta^2}$$
$$\frac{\partial^2}{\partial y^2} = \frac{\partial^2 r}{\partial y^2}\frac{\partial}{\partial r} + (\frac{\partial r}{\partial y})^2 \frac{\partial^2}{\partial r^2} + \frac{\partial^2 \theta}{\partial y^2}\frac{\partial}{\partial \theta} + (\frac{\partial \theta}{\partial y})^2 \frac{\partial^2}{\partial \theta^2}$$

Thus writing out all the partial derivatives we have

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$
$$\frac{\partial r}{\partial y} = \frac{y}{r}$$
$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2}$$
$$\frac{\partial \theta}{\partial y} = \frac{x}{r^2}$$
$$\frac{\partial^2 r}{\partial x^2} = \frac{y^2}{r^3}$$
$$\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$$
$$\frac{\partial^2 \theta}{\partial x^2} = \frac{2xy}{r^4}$$
$$\frac{\partial^2 \theta}{\partial y^2} = -\frac{2xy}{r^4}$$

Notice that $\frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial z^2}$
Also not that many terms cancel out like the $\frac{\partial^2 \theta}{\partial x^2}$ and $\frac{\partial^2 \theta}{\partial y^2}$ terms.
As well as many other terms sum to nice things Thus we can write the Laplacian in cylindrical coordinates as

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Thus we can note that since we are only interested in the radial and time dependence we can ignore the angular and z dependence.
Thus it follows that we have

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

Thus we can write the heat equation as

$$u_t = ku_{rr} + \frac{u_r}{r}$$

with $k = \frac{k}{c\rho}$

1.3 Problem 10

If $f(x)$ is continus and $|f(x)| \leq \frac{1}{|x|^3+1}$ for all x show that

$$\int_D \nabla \cdot f dx = 0$$

Solution:
Let D be a larger ball with boundary ∂D We can use the divergence theorem to write

$$|\int_D \nabla \cdot f dx| = |\int_{\partial D} f \cdot nd(\partial D)|$$

By cauchy schwartz we have
that $|f \cdot n| \leq |f||n|$
Since $|n| = 1$ we have

$$|\int_{\partial D} f \cdot nd(\partial D)| \leq \int_{\partial D} |f| d(\partial D)$$

It hs been given that $|f(x)| \leq \frac{1}{|x|^3+1}$.
since we have taken D to be a ball of radius R we can do a change of variable to polar, spherical coordinates with $d(\partial D) = R^2 sin(\theta) d\theta d\phi$, with integrating from $\theta \in (0, 2\pi), \phi \in (0, \pi)$
We can also note that $|x| = R$ on the boundary of the ball.
Thus we have

$$\int_{\partial D} |f| \cdot nd(\partial D)| \leq \int_{\partial D} \frac{1}{R^3+1} d(\partial D) = \int_0^{\pi} \int_0^{2\pi} \frac{R^2}{R^3+1} sin(\theta) d\theta d\phi$$

This integral evaluates to

$$\frac{R^2}{R^3+1}(4\pi)$$

We have now shown that

$$|\int_D \nabla \cdot f dx| \leq \frac{4\pi R^2}{R^3+1}$$

Now we can take the limit as $R \to \infty$

$$\lim_{R \to \infty} \frac{4\pi R^2}{R^3+1} = 0$$

Note that it goes to 0 similar to $\frac{1}{R}$
Therefore we have shown that

$$\int_D \nabla \cdot f dx = 0$$

Where D is all of space