Chapter 4

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Chapter 4

Markov Property

If the probability of the nest state only depends on the current state, it satisfies the "Markov Property".
Drunkards walk example

$$\mathbb{P}(x_{i+1} = x_i \pm 1) = \frac{1}{2}\mathbb{P}(x_{i+1} \neq x_i \pm 1) = 0$$

$$\mathbb{P}(x_{i+1} = x + 1 | x_i = x) = 1/2$$
$$\mathbb{P}(x_{i+1} = x - 1 | x_i = x) = 1/2$$

Formal Definition

Let $\{X_n, n \in \mathbb{N}\}$ be a stochastic process that takes discrete time values. Suppose $\mathbb{P}(X_{n+1} = j | X_n = i_n ... X_0 = i_0) = P_{i,j}$ Such a stochastic process is called a Markov Chain. P_{ij} is the transition probability from state i to state j.

Transition Probability Matrix

Let $i, j \in \mathbb{N}$ be possible states of the Markov Chain. The matrix $P = [P_{ij}]$ is called the transition probability matrix of the Markov Chain. Where $P_{ij} = \mathbb{P}(x_{n+1} = j | x_n = i)$. **Ex 4.1**

$$\mathbb{P}(\text{rain tomorrow}|\text{rain today}) = \alpha$$

$$\mathbb{P}(\text{rain tomorrow}|\text{no rain today}) = \beta$$

$$\text{Let} \begin{cases} 0 = \text{rain} \\ 1 = \text{no rain} \end{cases}$$

$$P = \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix}$$

Ex 4.4 Suppose whether it rains tomorrow or not depends on both todays and yesterdays weather.

Today's Weather	Yesterdays's Weather	Value
Rain	Rain	0
Rain	No Rain	1
No Rain	Rain	2
No Rain	No Rain	3

Suppose:
$\mathbb{P}$(rain tomorrow|rain today, rain yesterday) = .7
$\mathbb{P}$(rain tomorrow|rain today, no rain yesterday) = .5
$\mathbb{P}$(rain tomorrow|no rain today, rain yesterday) = .4
$\mathbb{P}$(rain tomorrow|no rain today, no rain yesterday) = .2

$$P = \begin{bmatrix} .7 & 0 & .3 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .4 & 0 & .6 \\ 0 & .2 & 0 & .8 \end{bmatrix}$$

4.2 Chapman-Kolmogorov Theorem

P_{ij} = probability of going from state i to state j
$P_{ij}^{(n)}$ = probability of going from state i to state j in n steps.
$P_{ij}^{(n+m)} = \sum_k P_{ik}^{(n)} P_{kj}^{(m)}$ (pg.197)
Look at example 4.10 for next class
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Proof of Chapman-Kolmogorov Theorem

Equation: $P_{ij}^{(n+m)} = \sum_k P_{ik}^{(n)} P_{kj}^{(m)}$
We can visualize this as a graph with n+m steps and we consider all the paths $i \to j$ and sum them with the law of total probability.
Proof:

$$P_{ij}^{(n+m)} = \mathbb{P}(X_{n+m} = j | X_0 = i)$$
$$= \sum_k \mathbb{P}(X_{n+m} = j, X_n = k | X_0 = i)$$
$$= \sum_k \mathbb{P}(X_{n+m} = j | X_n = k, X_0 = i)\mathbb{P}(X_n = k | X_0 = i)$$

Note that this is the probabilty of going from k to j in m steps(which doesn't depend on $x_0 = i$ due to the Markov Property) and from i to k in n steps. Homogeneity of a Markov Chain.
Example 4.10 An urn always contains 2 balls. Possible ball colors are red and

blue. Each stage of the process we pick a ball and randomly replace it with another ball. Replacement of the same color is .8 and replacement of a different color is .2.
If initially both the first balls are red, what is the probability that the 5th ball is red?

$$P = \begin{bmatrix} .8 & .2 & 0 \\ .1 & .8 & .1 \\ 0 & .2 & .8 \end{bmatrix}$$

Note: for a set up where the probability of changing colors is invariant of the color of the ball, the transition matrix will be visually "radially" symmetric***.

$$\begin{aligned} \mathbb{P}(X_5 = \text{red}) &= P_{22}^{(4)} + \frac{1}{2}P_{21}^{(4)} + 0P_{12}^{(4)} \\ &= 0.7048 \end{aligned}$$

Ask what are other Properties of stochastic matrix
$a_{i,j} = an - i, n - j$
Example 4.11
In a sequence of independent flips of a fair coint, let N denote the number of flips until there is a run of 3 heads.
Find (a) $P(N \leq 8)$ (b) $P(N = 8)$
Consider 4 states: 0,1,2,3. given by n = the number of consecutive heads

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) $= P_{03}^{(8)}$
(b) $= \frac{1}{2}P_{02}^{(7)}$

4.3 Classification of States

Definition: State j of is accessible from state i if $P_{ij}^{(n)} > 0$ for some $n \geq 0$. If the states are accessible from each other, they are said to communicate.
Communication is an equivalence relation.
Reflexive and symmetric are obvious.
Transitive is proven by the Chapman-Kolmogorov Theorem.
This relation divides the states into classes.

Reccurent and Transient States

Definition: A given state i of a Markov Chain let f_i denote the probability that the chain will eventually return to state i.

A state is called **Recurrent** if $f_i = 1$ and **Transient** if $f_i < 1$.
The expected number of revisits to a recurrent state is infinite.
For a transient state the probability of being in state i for exactly n times period is $f_i^n(1-f_i)$: Note that this is Geometric distribution
Lets notice state properties:

$$\begin{aligned} f_i &= \mathbb{P}(x_{n+N} = i | X_n = i) \\ &= \mathbb{P}(x_N = i | X_0 = i) \end{aligned}$$

A Recurrent state is revisited infinitely often after it is visited once it will be revisited by the markov properties, and it repeats.
A Transient state is revisited only a finite number of times.
Proof of Transitive state finite recurrence:
The probability a transient state is revisited exactly n times is $f_i^{n-1}(1-f_i)$

$$\begin{aligned} E(n) &= \sum_{n=1}^{\infty} n f_i^{n-1}(1-f_i) \\ &= (1-f_i) \sum_{n=1}^{\infty} \frac{d}{df_i} f_i^n \\ &= (1-f_i) \frac{d}{df_i} \sum_{n=1}^{\infty} f_i^n \\ &= (1-f_i) \frac{d}{df_i} \frac{f_i}{1-f_i} \\ &= \frac{1}{(1-f_i)} \end{aligned}$$

Proposition 4.1
A state is is

1. Recurrent if $\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$
2. Transient if $\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$

Proof:
Define $I_n = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{otherwise} \end{cases}$
The number of times period the process is in state i is $\sum_{n=0}^{\infty} I_n$

The expected value of the number of times the process is in state i is

$$
\begin{aligned}
\mathbb{E}(\sum_{n=0}^{\infty} I_n) &= \sum_{n=0}^{\infty} \mathbb{E}(I_n) \\
&= \sum_{n=0}^{\infty} \mathbb{P}(x_n = i | x_0 = i) \\
&= \sum_{n=0}^{\infty} P_{ii}^{(n)}
\end{aligned}
$$

Corallary 4.2 (pg 207)
If i is recurrent and i communicates with j, then j is recurrent.
Proof:

$$i \leftrightarrow j \rightarrow \exists k \text{ s.t. } P_{ij}^k > 0 \text{ and } P_{ji}^k$$

For any n,

$$P_{ij}^{(m+n+k)} \geq P_{ji}^m P_{ii}^n P_{ij}^k$$

$$\sum_{n=1}^{\infty} P_{ij}^{(m+n+k)} \geq \sum_{n=1}^{\infty} P_{ji}^m P_{ii}^n P_{ij}^k$$

$$\sum_{t=0}^{\infty} P_{jj}^t \geq \sum_{n=1}^{\infty} P_{ij}^{(m+n+k)} \geq P_{ji}^m P_{ij}^k sum_{n=1}^{\infty} P_{ii}^n \geq \infty$$

Thus if i is recurrent and i communicates with j, then j is recurrent.
Remark: If the state i is transient and if the state j communicates with i, then j is transient.
Proof: Assume the if, Suppose j is not transient. Then j is recurrent. Then i is recurrent. This is a contradiction. Thus j is transient.
Remark: Transience and Recurrence are class properties.
Remark: Suppose we have a Markov Chain with a finite number of states. Then at least one state is recurrent.
A Markov Chain with exactly one communication class is called irreducible.
A finite state irreducible Markov Chain must have all states recurrent.
Example 4.18 Consider a Markov Chain with 5 states.

$$P = \begin{bmatrix} .5 & .5 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 & 0 \\ 0 & 0 & .5 & .5 & 0 \\ 0 & 0 & .5 & .5 & 0 \\ .25 & .25 & 0 & 0 & .5 \end{bmatrix}$$

Find the equivalence classes, classify them as reccurent or transient.
Solution:
The equivalence classes are $\{0, 1\}$ and $\{2, 3\}$ and $\{4\}$
4 is its own class due to the fact the communication between 0 and 1 is not

symmetric.
Does number of nonzero eigenvectors equal the number of equivalence classes?
Example 4.19
Markov Chain wih states $(0, \pm 1 \dots)$
$P_{i,i+1} = p$ and $P_{i,i-1} = 1 - p$
pg 209-211
A random walk is symmetric if $p = 1/2$
Can prove that the random walk recurrent in that case **Remark:**
Definition of Reccurence:

$$f_i = P(\text{Ever coming back to state i — starting at state i})$$

$$f_i = P(\sum_n (x_n = i) | x_0 = i)$$

Random Walk
State space is $\mathscr{Z}$
TRansition probabilities are: $P_{i,i+1} = p$ and $P_{i,i-1} = 1 - p$
Note: when there is only one equivalence class, the Markov Chain is irreducible.
Find $f_0 = \beta = P(\text{ever returning to 0 — starting at 0})$
Condition the probability β on the next transtion.

$$\beta = (p)P(\text{ever returns to } 0 | x_1 = 1) + (1 - p)P(\text{returns to } 0 | x_1 = -1)$$

Let $\alpha = P(\text{ever returns to } 0 | x_1 = 1)$
$\alpha = (p)P(\text{ever returns to } 0 | x_1 = 1, x_2 = 2) + (1 - p)P(\text{ever returns to } 0 | x_1 = 1, x_2 = 0)$

$$\alpha = 1 - p + p\alpha^2$$

Solving gives

$$\alpha = \frac{1 - p}{p}$$

If the random walk is symetric then $\alpha = 1$ is the only Solution. Substitution in equation with beta gives $f_0 = 1$

4.4 Long Run Proportions and Limiting Probability

Let $i \neq j$ be states of a Markov Chain.
Define f_{ij} as the probability that the Markov chain, starting in state i, will ever reach state j.

$$f_{ij} = \sum_{n=1}^{\infty} P_{ij}^{(n)}$$

Proposition 4.3
If the state i is reccurent and i communicates with j, then probability of eventually reaching j is 1, $f_{ij} = 1$.
Proof:

$$i \leftrightarrow j \rightarrow \exists n > 0 \text{ s.t. } P_{ij}^{(n)} > 0$$

Assume n is the minimum such integer.
Since i is recurrent, the infinite sequence $0 = k_0 < k_1 < k_2 < ...$ exists such that $X_{k_r} = i$ for $r = 0, 1, 2, 3, ...$
Define $z = min(r > 0, X_{k_r + n} = j)$
Then $P(Z = z) = P_{ij}^{n}(1 - P_{ij}^{(n)})^{z-1}$
Thus $f_{ij} = 1$ by sandwich Theorem
Assume j is a recurrent state.
Define $N_j = min(n > 0 | X_n = j)$
Let $m_j = E[N_j | x_0 = j]$
It is the expextected number of steps to return to j.
Since we know that $P(N_j < \infty | x_0 = j) = 1$
Still it may happen that $m_j = E[N_j | x_0 = j] = \infty$
Definition: if $m_j < \infty$ then j is positive recurrent.
If $m_j = \infty$ then j is null recurrent.
We define π_j to be the long run proportion of time the Markoc chain is in state j.

$$\pi_j = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} I_k$$

Where $I_k = 1$ if $X_k = j$ and 0 otherwise.
Proposition 4.4 :
IF the markov chain is irreducible and reccurent then any intial state x_0 will have $\pi_j = \frac{1}{m_j}$
At time $T_0 + \sum T_k$ the chain enters state j for the (n+1)th time, the proporttion of th time the chain is in state j during this is $\frac{n+1}{T_0 + \sum T_k}$

$$\pi_j = \lim_{n \to \infty} \frac{n+1}{T_0 + \sum T_k}$$

Prop 4.4 if a MC is irreducible and recurrent, then $\pi_j = \frac{1}{m_j}$
Propr 4.5 if the state i is positive reccurent and if the state j communicates with i then the state j is also positive recurrent.
Proof: $i \Leftrightarrow j \rightarrow \exists n > 0$ such that $P_{ij}^{n} > 0, \pi_i P_{ij}^{n} =$ Proportion of times that the pricess wil be in state j, n steps after it was in step i. $< \pi_j$

$$\pi_i P_{ij}^{n} \leq \pi_j$$

$pi_i > 0$ since it is positive reccurent. since P is also finite thus π_j is positive reccurent.
Remark: null recurrence is also a class property.

Claim: an irreducible finite state markov chain is positive recurrent.
Proof: Let m_j be the expected return time to state j.
If you have a finite MC then the there is one EC, and if one is null recurrent then all are null recurrent.
Suppose that satte i in such MC is null recurrent.
Then $\pi_j = 0$ Since null reccurnece is a class property and there is only one class, thus all states are null recurrent.
$\pi_i = 0$ for all states.
$\sum pi_i = 0$ with probability one. This is a contradiction.
Theorum 4.1 Consider a irreducible Markov Chain. If the chain is positive reccurent then the long run proportions are unique colutions of the system of equations

$$\sum_i \pi_i P_{ij} = \pi_j$$

and $\sum_j \pi_j = 1$
Think of it like all way the ways the π go to state j.
Similar to conservation of flow.
Matrix Intutition:
Write $\vec{\pi} = \begin{bmatrix} \pi_0 & \pi_1 & \pi_2 & \dots \end{bmatrix}$
be the row vector with entris π_j
Then $\vec{\pi} P = \vec{\pi}$
And $\sum_j \pi_j = 1$
Example 1
Consider a two state Markov Chain with transition matrix

$$P = \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix}$$

Compute the long run proptions π_0 and π_1
Assume $\alpha, \beta \neq 0, 1$

$$\pi_0 P_{01} + \pi_1 P_{11} = \pi_1$$
$$\pi_0 P_{00} + \pi_1 P_{10} = \pi_0$$
$$\pi_0 + \pi_1 = 1$$

In matrix formulation we have $\begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix} = \begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix}$
Short cut to remember

$$\begin{bmatrix} & 0 & 1 \\ 0 & P_{00} & P_{01} \\ 1 & P_{10} & P_{11} \\ & \pi_0 & \pi_1 \end{bmatrix}$$

Note that $\pi_0 = \frac{\beta}{1-\alpha+\beta}$ and $\pi_1 = \frac{1-\alpha}{1-\alpha+\beta}$
Example 2
Doubly stochastic matrix. If the sum of the columns and rows are equal to 1.

If the transition matri of a Markov Chain with n-states is a doubly stochastic matrix , then the long run proportions are $\pi_j = 1/n$ for all j.
Proof:

$$[1/n, 1/n, 1/n, \dots]P = [1/n, 1/n, 1/n, \dots]$$

$$1/n \sum_j P_{ij} = 1/n$$

$$\sum_j P_{ij} = 1$$

Since solutions are unique as in Theroum 4.1, thus $\pi_j = 1/n$ for all j.
Example 3
Simple random symmetric walk. This is also reflectant at the edges.
States are $(0, 1, 2, \dots, L)$
$P_{01} = 1, P_{L,L-1} = 1$
$P_{i,i-1} = 1/2$ and $P_{i,i+1} = 1/2$
Simple case: L = 2

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\pi_0 = 0, \pi_1 = 1/2, \pi_2 = 1/2$$

Case L = 3

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\pi_0 = 1/6, \pi_1 = 1/3, \pi_2 = 1/3, \pi_3 = 1/6$$

Prove that for L = n, $\pi_i = \frac{1}{n}$ for all i other than $\pi_0, \pi_L = 1/2L$.
Example
For L = 1000, what is the probability of revisiting state 0?
it is 2000 as it is $\pi = 1/m$
If the system is inconsistent the MC is transient or null reccurent.
$\pi_j = 0$ for all j.
Example 4.26:
MC with acceptable staus in the set A and unacceptable status A^C
If $x_n \in A$ process is "up", if $x_n \in A^C$ process is "down".
Find:
i: Rate at which the process goes from up to down.
ii: Average length of time process remain down when it goes down.
iii: Average length of time process remains up when it goes up.
Solution:

Let $i \in A$ and $j \in A^C$
The rate at which the process enters j from i is

$$= \pi_i P_{ij}$$

Rare ar which process enters j from any acceptable state is

$$\sum_{i \in A} \pi_i P_{ij}$$

Rate at which the process goes from $A \to A^C$ is

$$\sum_{j \in A^C} \sum_{i \in A} \pi_i P_{ij}$$

Let u be the average time process stays up.
Let d be the average time process stays down.
rate at which a breakdown occurs is $\frac{1}{u+d}$
Therefore

$$\frac{1}{u+d} = \sum_{j \in A^C} \sum_{i \in A} \pi_i P_{ij}$$

Proportion the process is up is

$$\frac{u}{u+d} = \sum_{i \in A} \pi_i$$

Get u and d in terms of π

$$u = \frac{\sum_{i \in A} \pi_i}{\sum_{j \in A^C} \sum_{i \in A} \pi_i P_{ij}}$$
$$d = \frac{\sum_{j \in A^C} \sum_{i \in A} \pi_i P_{ij}}{\sum_{i \in A^C} \pi_i}$$

Stationary Probability
If the intial distribution of states is chosen accoring to long run propotions π_j, then the future distrubituion of the state of the system will be the same if $P(x_0 = j) = \pi_j$ then $P(x_n = j) = \pi_j$
Using induction we can see this true for all n. (4.4)

4.4.1: Limiting probabilities: **Example:** Consider a two state MC with

$$P = \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix}$$

Numericcal calculation show that $P^{(n)}$ converges to a limiting distribution

$$P^n = \begin{bmatrix} 4/7 & 3/7 \\ 4/7 & 3/7 \end{bmatrix}$$

Claim: The limiting probabilties $\lim_{n\to\infty} P(x_n = j)$ if they exist are equal to the long run proportions π_j
Proof: Assume $\alpha_j = \lim_{n\to\infty} P(x_n = j)$ exists.
Then $P(x_{n+1} = j) = \sum_i P(x_{n+1} = j|x_n = i)P(x_n = i)$
Gives $\lim_{n\to\infty} P(x_{n+1} = j) = \lim_{n\to\infty} \sum_i P_{ij} P(x_n = i)$
Thus $\alpha_j = \sum_i P_{ij}\alpha_i$ for all j
Also $\sum_j \alpha_j = 1$
Recall that pi_j are the unique solutions of the system of equations

$$\sum_i \pi_i P_{ij}, \sum_i \pi_j = 1$$

Therefore $\lim_{n\to\infty} \alpha_j = \pi_j$ if α_j exists
When do limits not exist? When $n \to \infty$ diverges or collates
A chain that can only return to a state a multiple of $d > 1$ times is called a periodic chain. And does not have limiting probabilties.
Definition:
An irrecucible, positive reccurent, aperiodic Markov Chain is said to be ergodic.
Branching Proess:
A branching process is a Markov Chain with time give vy generations in $0, 1, 2, 3, \ldots$ and states fiven by populations in $0, 1, 2, 3, \ldots$
Induviduals in each gen eration produce offspring

$$X_i = \#ofoffspringofindividualofthe(i-1)^t hgenerations$$

Remark
If 0 is a reccurent state because $P_{00} = 1$
Then it is an abosrbing state.
Proof is somewhat trivial using matrix multiplication.
Remark 2
Define $P_0 = P$[An individual produces 0 offspring]
If $P_0 > 0$ then all the states other than 0 are transient.
Proof:
Consider P_{i0}
it is th probability for going from state i to 0.

$$= P[EachoneoftheiIndividualsproduces0offspring]$$
$$= P_0^i$$
$$= P_0^i > 0$$

Thus the state i is transient for $i \neq 0$
Remark 3 If $P_0 > 0$ then the population eventually either becomes extinct or grows indefinitely.
Note:
We do not use transient probabilities to study branching processes. We mostly use the probability distribution of the number of offspring produced by an individuals.

Let $P_j = P$[An individual produces j offspring]
Compute the mean and variance of X_n
Mean
$X_0 = 1$

$$\mathbb{E}(X_n) = \mu = \sum_j jP_j$$

$$\mathbb{E}(X_n) = \mathbb{E}[\mathbb{E}[X_n|X_{n-1}]]$$

write $z_i, i = 1, 2, 3, \ldots$ for the number of offspring of the x_{n-1} individuals in the $(n-1)^th$ generation.
Then $X_n = \sum_{i=1}^{X_{n-1}} Z_i$

$$\mathbb{E}(X_n) = \mathbb{E}[\mathbb{E}[\sum_{i=1}^{X_{n-1}} Z_i|X_{n-1}]]$$

$$\mu = z_i|X_{n-1}$$

$$E[X_n] = \mathbb{E}(\mu X_{n-1})$$

$$E[X_n] = \mu\mathbb{E}(X_{n-1})$$

Since $X_0 = 1$
Thus $\mathbb{E}[X_n] = \mu^n$
Variance

$$\text{Var}(X_n) = \sigma^2 = \sum_j j^2 P_j - \mathbb{E}(X_n)^2$$

We can also note that

$$var(X_n) = \sigma^2(\mu^{n-1} + \mu^n + .. + \mu^{2n-2})$$

$$var(X_n) = \begin{cases} \sigma^2\mu^{n-1}\frac{1-\mu^{n-1}}{1-\mu} & \text{if } \mu \neq 1 \\ n\sigma^2 & \text{if } \mu = 1 \end{cases}$$

Probabilityt of extinction (of a population)

$$\pi_0 = P[\text{Population becomes extinct}]$$

$$\pi_0 = \lim_{n\to\infty} P(X_n = 0|X_0 = 1)$$

Case 1: if $\mu < 1$ then $\pi_0 = 1$
Proof:
We can see this that for each generation the population decreases.
Thus the long run proportion of the population being 0 is 1.
Case 2: if $\mu > 1$ then $\pi_0 < 1$
Proof:
This follows since the population grows indefinitely.
The equation for π_0 is

$$\pi_0 = \textbf{Population dies out}$$

$$= \sum_{j=0}^{\infty} P(\text{Population dies out}|x_1 = j)P_j$$

$$= \sum_{j=0}^{\infty} \pi_0^j P_j$$

For $\mu > 1$ it can be shown that π_0 is the smallest solution to the equation $\pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j$

Example 4.34
Suppose $P_0 = 1/2, P_1 = 1/4, P_2 = 1/4$
Compute π_0.
Find $\mu = 0(1/2) + 1(1/4) + 2(1/4) = .75$
Since $\mu < 1$ then $\pi_0 = 1$

Example 4.35
Suppose $P_0 = 1/4, P_1 = 1/4, P_2 = 1/2$

4.8 Time Reversible Markov Chains

Detour:
Better understand the concept of a stationary M.C.
It has a stationary distribution $\vec{\pi} = [\pi_1, \ldots, \pi_n]$ that satisfies $\vec{\pi}P = \vec{\pi}$
Comnsider a time series that is states at a time, as it propogrates, it reaches a stationary distribution. This implies that the distribution of the states at time n is the same as the distribution at time $n + 1$
Example: MC with TPM:

$$P = \begin{bmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{bmatrix}$$

We can calc the stationary distribution by solving the system of equations:

$$\pi_1 = \frac{1}{3}\pi_1 + \frac{1}{2}\pi_2$$

$$\pi_2 = \frac{2}{3}\pi_1 + \frac{1}{2}\pi_2$$

$$\pi_1 + \pi_2 = 1$$

This gives $\pi_1 = 3/7$ and $\pi_2 = 4/7$
Suppose at time $t = 0$ the probability distribution is:

$$P(x_0 = 0) = 3/7 \text{ and } P(x_0 = 1) = 4/7$$

Then at time $t = 1$ the distribution is the same as at time $t = 0$

Time Reversible Markov Chains
Consider an ergodic MC that has been running for a long time.
Consider the revise process $X_n, X_{n-1}, \ldots, X_0$ starting for some large n.
It satisfies the markov property, future given the present is independent of the past.
The Transition probabilties of the rversed chain are given by $Q_{ij} = P(X_{n-1} = j|X_n = i) = P(X_m = j|X_{m+1} = i)$
We can find it out using bayes formula

$$Q_{ij} = \frac{P(X_m = j)P(X_{m+1} = i|X_m = j)}{P(X_{m+1} = i)}$$

$$Q_{ij} = \frac{\pi_j P_{ji}}{\pi_i}$$

Definition: A Markov Chain is time reversible if $Q_{ij} = P_{ij}$
If the MC is time reversible

$$P_{ij} = \frac{\pi_j P_{ji}}{\pi_i}$$

$$\pi_i P_{ij} = \pi_j P_{ji}$$

This is saying the rate you go from i to j is the same as j to i
Verify reversible by computing $\vec{\pi} =$ and checking if $Q_{ij} = P_{ij}$
But more efficielty we can fin $x_i > 0$ such that $\sum x_i = 1$ and $x_i P_{ij} = x_j P_{ji}$
Proof:

$$\sum_i x_i P_{ij} = \sum_i x_j P_{ji}$$

$$\sum_i x_i P_{ij} = x_j \sum_i P_{ji}$$

$$\sum_i x_i P_{ij} = x_j$$

and $\sum x_i = 1$
By theorem 4.1 the solutions fo this sysmen are unique and are $\vec{\pi}$
Example 4.38
Consider an aribitrary connected graph accociate with a (+ve) number w_{ij} for each edge. Conside a particle moving from node to node such that the particle will move from node i to node j with probability $P_{ij} = \frac{w_{ij}}{d_i}$ where $d_i = \sum_j w_{ij}$
The TPM is:

$$P = \begin{bmatrix} w_{11}/d_1 & w_{12}/d_1 & \ldots \\ w_{21}/d_2 & w_{22}/d_2 & \ldots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

For this example the TPM is:

$$P = \begin{bmatrix} 0 & 1/2 & 0 & 1/6 & 1/3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1/5 & 0 & 0 & 0 & 4/5 \\ 1/6 & 0 & 1/6 & 2/3 & 0 \end{bmatrix}$$

Time reversibility for such an MC is given by:

$$\pi_i P_{ij} = \pi_j P_{ji}$$

$$\pi_i \frac{w_{ij}}{d_i} = \pi_j \frac{w_{ji}}{d_j}$$

$$\frac{\pi_i}{d_i} = \frac{\pi_j}{d_j} = c$$

$$\pi_i = c d_i$$

$$c = \frac{1}{\sum_i \sum_j w_{ij}}$$

Thus

$$\pi_i = \frac{\sum_j w_{ij}}{\sum_i \sum_j w_{ij}}$$

Note you need to calc twice for ij and ji
Note if we pick all the w_{ij} to be the same we get a random walk on a graph
Consider the MC with TPM: 2/3 going clockwise and 1/3 going counter clockwise with 3 states.

$$P = \begin{bmatrix} 1/3 & 2/3 & 0 \\ 0 & 1/3 & 2/3 \\ 2/3 & 0 & 1/3 \end{bmatrix}$$

This is doubly stochastic.
Argue that Q is the same as P^T
Then show $Q \neq P$ because P is not symmetric.
Missed notes: Counting proesses
$\{N(t), t >= 0\}$ They follow 3 properties:

1. $N(t) \geq 0$
2. $N(t)$ is integer valued
3. $N(t)$ is monotone increasing

$$N(t) : R \to N$$

Monotone increasing function of t

$$N(t) - N(s) = \text{ Number of events in } (t, s]$$

Little o notation
A function f is said to be little o $o(h)$ if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0$$

eg: $f(h) = h^2$ is little $o(h)$
If u add two function in little $o(h)$ then it is still little $o(h)$
Definition: A counting process $\{N(t), t \geq 0\}$ is a Poisson process if:

1. $N(0) = 0$
2. The number of events in disjoint intervals are independent.
3. $P(N(t+h) - N(t) = 1) = \lambda h + o(h)$ where λ is the rate of the Poisson process. (this mean it is dependant on the length of the interval)
4. $P(N(t+h) - N(t) \geq 2) = o(h)$

Lemma 5.1:
Let $\{N(t), t \geq 0\}$ be a Poisson process. Define $\{N_s(t), t \geq 0\}$ by $N_s(t) = N(s+t) - N(s)$
Then $\{N_s(t), t \geq 0\}$ is a Poisson process with rate λ
Proof:

$$\begin{aligned} N_s(0) = N(s+0) - N(s) = 0 \\ (a,b) \cap (c,d) = \emptyset \\ P(N_s(b) - N_s(a) = x, N_s(d) - N_s(c) = y) \\ P(N(b-s) - N(a-s) = x, N(d-s) - N(c-s) = y) \\ P(N(b-s) - N(a-s) = x)P(N(d-s) - N(c-s) = y) \\ P(N_s(b) - N_s(a) = x)P(N_s(d) - N_s(c) = y) \end{aligned}$$

Thus disjoint intervals are independent.

$$P(N_s(t+h) - N_s(t) = 1) = P(N(s+t+h) - N(s+t) = 1)$$

We assume N has stationary increments.

$$P(N(s+t+h) - N(s+t) = 1) = P(N(t+h) - N(t) = 1) = \lambda h + o(h)$$

Lmma 5.2:
Let $T_1 = min(t > 0 : N(t) = 1)$
it is time of arrival
T_1 is exponentially distributed with rate λ
Proof:

$$P_0(t) = P(N(t) = 0)$$

$$P_0(t+h) = P(N(t)=0, N(t+h)-N(t)=0)$$
$$P_0(t+h) = P(N(t)=0)P(N(t+h)-N(t)=0)$$
$$P_0(t+h) = P_0(t)(1-\lambda h - 2o(h))$$

note that $-2o(h) = o(h)$ cuz it basically 0

$$P_0(t+h) = P_0(t) - \lambda h P_0(t) + o(h)P_0(t)$$
$$\frac{dP_0(t)}{t} = -\lambda P_0(t) + 0$$

This solves to with IC $P_0(0) = 1$

$$P_0(t) = e^{-\lambda t}$$

Define:
$T_n for n \geq 1$ is the time between the $(n-1)^t h$ and $n^t h$ arrival.
Proposition 5.4:
$T_1, T_2, \ldots$ are independent and exponentially distributed with rate λ
Proof:
Rea book.
Remark:
Define $S_n = \sum_{i=1}^n T_i$
From last time, S_n has a gamma distribution with parameters n and λ

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$$

Theoremm 5.1
If $\{N(t), t \geq 0\}$ is a Poisson process with parameter λ then $N(t)$ is a poisson random variable with parameter λt
Proof:

$$P(N(t)=n) = \int_0^\infty P(N(t)=n|S_n=t)\frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}dt$$
$$= P(T_{n+1} = t-s | T_1 + T_2 + \cdots + T_n = s)$$
$$= P(T_{n+1} = t-s)$$
$$= \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

Example
Let $\{N(t), t \geq 0\}$ be a Poisson process with rate $\lambda = \frac{1}{3}$
Find:
a) $P(N(5) > N(3))$
This means there are > 0 events in $(3, 5]$

$$P(N(5) > N(3)) = 1 - P(N(5) - N(3) = 0)$$

$$= 1 - P(N(2) = 0)$$

$$= 1 - e^{-\frac{2}{3}}$$

b) $P(\{N(4) = 1\}, \{N(5) = 3\})$

c) $E(N(5)|N(3) = 2)$
d) $E(T_b|N(3) = 4)$