01:640:350H - Homework 9

Pranav Tikkawar

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1. Sec. 5.4 Problem 2(b)
   For each of the following linear operators T on the vector space V, and if V is the determine wheter the given subspace W is a T-invariant subspace of V.

$$V = P(R), \quad T(f(x)) = xf(x), \quad W = P_2(R)$$

> **Solution:** Clearly W is not a T-invariant subspace of V since if we take $f(x) = x^2$, then $T(f(x)) = x^3$ which is not in W.

2. Sec. 5.4 Problem 3
   Let T be a linear operator on a finite dimensional bector space V. Prove that the following subspaces are T-invariant subspaces of V.

   (a) $\{0\}$ and V

   (b) $N(T)$ and $R(T)$

   (c) E_λ for any eigenvalue λ of T

> **Solution: Case:** $\{0\}$
> This is trivial since $T(0) = 0$ and $0 \in \{0\}$.
> **Case:** V
> This is also trivial since $T(v) = w$ for any $v \in V$ and $w \in V$.
> **Case:** $N(T)$
> Let $v \in N(T)$, then $T(v) = 0$. Since T is a linear operator, $T(0) = 0$ and $0 \in N(T)$.

Thus $N(T)$ is a T-invariant subspace of V.
Case: $R(T)$
Let $v \in R(T)$, then $T(v) = w$. $w \in R(t)$ by definition of $R(T)$. Thus $R(T)$ is a T-invariant subspace of V.
Case: E_λ
Let $v \in E_\lambda$, then $T(v) = \lambda v$. Since T is a linear operator, $T(\lambda v) = \lambda T(v) = \lambda^2 v$. Thus E_λ is a T-invariant subspace of V.

3. Sec. 5.4 Problem 6(a) For each of the linear operator T on the vector space V, find an ordered basis for the T-cyclic subspace generated by the vector z.

$$V = R^4 \quad T(a,b,c,d) = (a+b, b-c, a+c, a+d) \quad z = e_1$$

Solution: Since T is a linear operator we can say $T = L_A$ for

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Let W be the T-cyclic subspace generated by z and γ be the basis of W. We know that the generating set of W is $\{z, T(z), T^2(z), \dots, T^{n-1}(z)\}$. Thus we need the longest LI set of vectors from this set.
Thus

$$T(z) = (1,0,1,1)$$
$$T^2(z) = (1,-1,2,2)$$
$$T^3(z) = (0,-3,3,3)$$

Clearly $T^3(z) = -3T(z) + 3T^2(z)$. Thus $\gamma = \{z, T(z), T^2(z)\}$ is a basis for W.
Now we can see that for any $v \in W$, $v = a_1 z + a_2 T(z) + a_3 T^2(z)$. Which implies $T(v) = a_1 T(z) + a_2 T^2(z) + a_3 T^3(z) = (a_1 - 3a_3)T(z) + (a_2 + 3a_3)T^2(z)$ which is in W.
Thus we can see that W is a T-invariant subspace of V.

4. Sec. 5.4 Problem 6(b) For each of the linear operator T on the vector space V, find an ordered basis for the T-cyclic subspace generated by the vector z.

$$V = P_3(R) \quad T(f(x)) = f''(x) \quad z = x^3$$

Solution: We can see that $T = L_A$ for

$$A = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let W be the T-cyclic subspace generated by z and γ be the basis of W. We know that the generating set of W is $\{z, T(z), T^2(z), \ldots, T^{n-1}(z)\}$. Thus we need the longest LI set of vectors from this set to be the basis for W
Thus

$$T(z) = 6x$$
$$T^2(z) = 0$$

We can see that for any $k > 2$, $T^k(z) = 0$. Thus $\gamma = \{z, T(z)\} = \{x^3, x\}$ is a basis for W.

5. Sec. 5.4 Problem 9 (for 6(a),(b)) For each Linear operator T and cyclic subspace W in Exercise 6, compute the characteristic polynomial of T_W in two ways as in Example 6.

Solution: Case: 6(a)
By means of Theorem 5.21 we can see that $T^3(z) = -3T(z) + 3T^2(z)$. Hence

$$0z + 3T(z) - 3T^2(z) + T^3(z) = 0$$

Therefore by Theorem 5.21, the characteristic polynomial of T_W is

$$f(t) = (-1)^3(0 + 3t - 3t^2 + t^3) = -t^3 + 3t^2 - 3t$$

By means of determinants we can see that $\gamma = \{z, T(z), T^2(z)\}$ is a basis for W. $T(z) = (1, 0, 1, 1)$, $T^2(z) = (1, -1, 2, 2)$, $T^3(z) = (0, -3, 3, 3)$.

$$T(z) = (1, 0, 1, 1) \implies [(0, 1, 0)]_\gamma$$
$$T^2(z) = (1, -1, 2, 2) \implies [(0, 0, 1)]_\gamma$$
$$T^3(z) = (0, -3, 3, 3) \implies [(0, -3, 3)]_\gamma$$

Thus $[T_W]_\gamma = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix}$. Thus the characteristic polynomial of T_W is

$$f(t) = det(A - tI) = \begin{vmatrix} -t & 0 & 0 \\ 1 & -t & -3 \\ 0 & 1 & 3-t \end{vmatrix}$$

$$= -t \begin{vmatrix} -t & -3 \\ 1 & 3-t \end{vmatrix} = -t(t^2 - 3t + 3) = -t^3 + 3t^2 - 3t$$

Case: 6(b)
By means of Theorem 5.21: we can see that $T^2(z) = 0$. Hence

$$0z + 0T(z) + T^2(z) = 0$$

Therefore by Theorem 5.21, the characteristic polynomial of T_W is

$$f(t) = (-1)^2(0 + 0t + t^2) = t^2$$

By means of determinants we can see that $\gamma = \{z, T(z)\} = \{x^3, x\}$ is a basis for W. $T(z) = 6x$, $T^2(z) = 0$.

$$T(z) = 6x \implies [(0, 6)]_\gamma$$
$$T^2(z) = 0 \implies [(0, 0)]_\gamma$$

Thus $[T_W]_\gamma = \begin{bmatrix} 0 & 0 \\ 6 & 0 \end{bmatrix}$. Thus the characteristic polynomial of T_W is

$$f(t) = det(A - tI) = \begin{vmatrix} -t & 0 \\ 6 & -t \end{vmatrix} = t^2$$

6. Sec. 5.4 Problem 10 (for 6(a),(b))
For each linear operator in Exercise 6, find the characteristic polynomial $f(t)$ of T, and verify that the characteristic polynomial of T_W divides $f(t)$.

Solution: Case: 6(a)
We can see that $T = L_A$ for

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

The characteristic polynomial of T is

$$f(t) = det(A - tI) = \begin{vmatrix} 1-t & 1 & 0 & 0 \\ 0 & 1-t & -1 & 0 \\ 1 & 0 & 1-t & 0 \\ 1 & 0 & 0 & 1-t \end{vmatrix} = t^4 - 4t^3 + 6t^2 - 3t$$

The characteristic polynomial of T_W is

$$f(t) = -t^3 + 3t^2 - 3t$$

We can see that $f(t) = (1-t)(t^3 - 3t^2 + 3t)$. Thus $f_W(t)$ divides $f(t)$.

Case: 6(b)

We can see that $T = L_A$ for

$$A = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial of T is

$$f(t) = det(A - tI) = \begin{vmatrix} -t & 0 & 2 & 0 \\ 0 & -t & 0 & 6 \\ 0 & 0 & -t & 0 \\ 0 & 0 & 0 & -t \end{vmatrix} = t^4$$

The characteristic polynomial of T_W

$$f(t) = t^2$$

We can see that $f(t) = t^2(t^2)$. Thus $f_W(t)$ divides $f(t)$.

7. Sec. 5.4 Problem 16

   Let T be a linear operator on a finite-dimensional vector space V

   (a) Prove that if the characteristic polynomial of T splits, then so oes the characteristic polynomial of the restriction of T to any T-invariant subspace of V.

   (b) Deduce if the characteristic polynomial of T splits, then any non trivial T-invariant subspace of V contains an eigenvector of T.

Solution: Part (a)

Assume the characteristic polynomial of T splits. Let W be a T-invariant subspace of V. Let T_W be the restriction of T to W. Let γ be a basis for W. Since T and T_W is a linear operator, $T = L_A$ for some A and $T_W = L_B$ for some B. We can see that B is a submatrix of A. Thus the characteristic polynomial of T_W is a factor of the characteristic polynomial of T.

Thus if the characteristic polynomial of T splits, then so does the characteristic polynomial of the restriction of T to any T-invariant subspace of V.

Part (b)

Assume the characteristic polynomial of T splits. Let W be a non-trivial T-invariant subspace of V. By part (a) we know that the characteristic polynomial of T_W is a factor of the characteristic polynomial of T. Since the roots of the characteristic polynomial are eigenvalues of a Linear operator, there must be some eigenvector of T in W.

8. Sec. 5.4 Problem 18
Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$$

(a) Prove that A is invertable iff $a_0 \neq 0$.

(b) Prove that of A is invertable, then $A^-1 = (-1/a_0)[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I]$.

(c) Use (b) to compute A^{-1} for

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

Solution: Part (a)
We can see that $a_0 = det(A)$ as the characteristic polynomial is generated by $det(A - tI)$ and if we take $t = 0$ we get $det(A)$ and all elements of $f(t)$ (the characteristic polynomial) go to zero except the constant a_0 term. Thus $a_0 = det(A)$.
Thus A is invertible iff $det(A) \neq 0$ iff $a_0 \neq 0$.

Part (b)
From the Cayley Hamilton theorem we know that a matrix A satisfies its characteristic polynomial. Thus $f(A) = 0$. Thus we can see that

$$(-1)^n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = A_0$$

were A_0 is the zero matrix. Thus we can see that

$$A((-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I) = -a_0 I$$

Thus $A^{-1} = (-1/a_0)[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I]$.

Part (c)
First we can compute the characteristic polynomial of A as

$$det(A - tI) = \begin{vmatrix} 1-t & 2 & 1 \\ 0 & 2-t & 3 \\ 0 & 0 & -1-t \end{vmatrix} = (1-t)(2-t)(-1-t) = -t^3 + 2t + t - 2$$

Thus $a_0 = -2, a_1 = 1, a_2 = 2$ We also can see that $A^2 = \begin{bmatrix} 1 & 6 & 6 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ Thus

$$A^{-1} = (1/2)\left(A^2 + 2A + I\right)$$

$$= (1/2)\left(-\begin{bmatrix}1 & 6 & 6\\0 & 4 & 3\\0 & 0 & 1\end{bmatrix} + 2\begin{bmatrix}1 & 2 & 1\\0 & 2 & 3\\0 & 0 & -1\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right)$$

$$= \begin{bmatrix}1 & -1 & -2\\0 & 1/2 & -3/2\\0 & 0 & -1\end{bmatrix}$$

Clearly we can see that this is the inverse of A.

9. Sec. 5.4 Problem 21
   Let T be a linear operator on a two-dimensional vector space V. Prove that either V is a T-cyclic subspace of itself or $T = cI$ for some scalar c.

Solution: Let T be a linear operator on a two-dimensional vector space V. Let z be a vector in V.
We need to prove that either V is a T-cyclic subspace of itself or $T = cI$ for some scalar c.
Let us consider two cases $T(z) = cIz$ and $T(z) \neq cIz$.
In other words we will consider if z is an eigenvector of T or not.
Case 1: $T(z) = cIz$
Clearly we get the our second condition that we need to prove of $T = cI$ for some scalar c.
Case 2: $T(z) \neq cIz$
Let us consider the set $\{z, T(z)\}$. Since $T(z) \neq cIz$, we can see that z and $T(z)$ are linearly independent. Thus $\{z, T(z)\}$ is a basis for V since it is a linearly independent set of vectors in a two-dimensional vector space.