16:960:665 - Time Series Analysis - Homework 4

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Problem (18)**.** Show that if $\phi(\cdot)$ and $\theta(\cdot)$ have no common zeros, and if $\phi(z) = 0$ for some $|z| = 1$, then the ARMA equations

$$\phi(B)X_t = \theta(B)Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$$

have no stationary solution. [Hint. Assume there is one, then use the relationship between the spectral distributions of $\{X_t\}$ and $\{Z_t\}$ to derive a contradiction. Also think how you would do it without using spectral distributions.]

Solution: Note that the spectral density of $\{Z_t\}$ is given by

$$f_Z(\lambda) = \frac{\sigma^2}{2\pi}, \quad -\pi \leq \lambda \leq \pi.$$

If we assume that there exists a stationary solution $\{X_t\}$ to the ARMA equations, then the spectral density of $\{X_t\}$ is given by

$$\begin{aligned} f_X(\lambda) &= \left| \frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})} \right|^2 f_Z(\lambda) \\ &= \frac{\sigma^2}{2\pi} \left| \frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})} \right|^2, \quad -\pi \leq \lambda \leq \pi. \end{aligned}$$

However, since $\phi(z) = 0$ for some $|z| = 1$, we have that $\phi(e^{-i\lambda_0}) = 0$ for some $\lambda_0 \in [-\pi, \pi]$. Now if we consider the autocovariance function of $\{X_t\}$, we have

$$\begin{aligned} \gamma_X(h) &= \int_{-\pi}^{\pi} e^{i\lambda h} f_X(\lambda) d\lambda \\ &= \int_{-\pi}^{\pi} e^{i\lambda h} \frac{\sigma^2}{2\pi} \left| \frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})} \right|^2 d\lambda. \\ &\text{diverges since } \phi(e^{-i\lambda_0}) = 0 \text{ at } \lambda_0. \end{aligned}$$

This is a contradiction since the autocovariance function of a stationary process must be finite for all lags h. Therefore, our assumption that there exists a stationary solution $\{X_t\}$ to the ARMA equations is false.

Problem (19)**.** Let $\{X_t\}$ and $\{Y_t\}$ be two stationary mean zero processes with spectral densities $f_X(\cdot)$ and $f_Y(\cdot)$. If $f_X(\cdot) \leq f_Y(\cdot)$ for all $\lambda \in [-\pi, \pi]$, show that $\Gamma_{n,Y} - \Gamma_{n,X}$ is a non-negative definite matrix, where $\Gamma_{n,X}$ and $\Gamma_{n,Y}$ are the covariance matrices of $(X_1, \ldots, X_n)'$ and $(Y_1, \ldots, Y_n)'$ respectively.

Solution: Remember that the covariance matrix $\Gamma_{n,X}$ of the vector $(X_1, \ldots, X_n)'$ is given by

$$\Gamma_{n,X} = \begin{pmatrix} \gamma_X(0) & \gamma_X(1) & \cdots & \gamma_X(n-1) \\ \gamma_X(1) & \gamma_X(0) & \cdots & \gamma_X(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_X(n-1) & \gamma_X(n-2) & \cdots & \gamma_X(0) \end{pmatrix},$$

for each $n \geq 1$, where $\gamma_X(h)$ is the autocovariance function of $\{X_t\}$. We can express the autocovariance function in terms of the spectral density as follows:

$$\gamma_X(h) = \int_{-\pi}^{\pi} e^{i\lambda h} f_X(\lambda) d\lambda.$$

Therefore, the covariance matrix $\Gamma_{n,X}$ can be expressed as

$$\Gamma_{n,X} = \int_{-\pi}^{\pi} \begin{pmatrix} 1 & e^{i\lambda} & \cdots & e^{i\lambda(n-1)} \\ e^{i\lambda} & 1 & \cdots & e^{i\lambda(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{i\lambda(n-1)} & e^{i\lambda(n-2)} & \cdots & 1 \end{pmatrix} f_X(\lambda) d\lambda.$$

Similarly, we can express the covariance matrix $\Gamma_{n,Y}$ of the vector $(Y_1, \ldots, Y_n)'$ as

$$\Gamma_{n,Y} = \int_{-\pi}^{\pi} \begin{pmatrix} 1 & e^{i\lambda} & \cdots & e^{i\lambda(n-1)} \\ e^{i\lambda} & 1 & \cdots & e^{i\lambda(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{i\lambda(n-1)} & e^{i\lambda(n-2)} & \cdots & 1 \end{pmatrix} f_Y(\lambda) d\lambda.$$

Now, we can compute the difference $\Gamma_{n,Y} - \Gamma_{n,X}$ as follows:

$$\Gamma_{n,Y} - \Gamma_{n,X} = \int_{-\pi}^{\pi} \begin{pmatrix} 1 & e^{i\lambda} & \cdots & e^{i\lambda(n-1)} \\ e^{i\lambda} & 1 & \cdots & e^{i\lambda(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{i\lambda(n-1)} & e^{i\lambda(n-2)} & \cdots & 1 \end{pmatrix} (f_Y(\lambda) - f_X(\lambda)) d\lambda.$$

Solution: (Continued from priot part) Since we are given that $f_X(\lambda) \leq f_Y(\lambda)$ for all $\lambda \in [-\pi, \pi]$, it follows that $f_Y(\lambda) - f_X(\lambda) \geq 0$ for all λ.
To show that $\Gamma_{n,Y} - \Gamma_{n,X}$ is a non-negative definite matrix, we need to show that for any non-zero vector $c \in \mathbb{R}^n$, the following holds:

$$a * (\Gamma_{n,Y} - \Gamma_{n,X})a \geq 0.$$

We can compute this as follows: with $a \in \mathbb{R}^n$, and $v(\lambda) = (1, e^{i\lambda}, \ldots, e^{i\lambda(n-1)})'$,

$$\begin{aligned} a * (\Gamma_{n,Y} - \Gamma_{n,X})a &= a * \left(\int_{-\pi}^{\pi} v(t)v(t)^*(f_Y(t) - f_X(t))dt \right) a \\ &= \int_{-\pi}^{\pi} a * v(t)v(t)^* a(f_Y(t) - f_X(t))dt \\ &= \int_{-\pi}^{\pi} |a * v(t)|^2 (f_Y(t) - f_X(t))dt \\ &\geq 0, \end{aligned}$$

since $|a * v(t)|^2 \geq 0$ and $f_Y(t) - f_X(t) \geq 0$ for all $t \in [-\pi, \pi]$.
Therefore, we have shown that $\Gamma_{n,Y} - \Gamma_{n,X}$ is a non-negative definite matrix.

Problem (20)**.** Assume $\{X_t\}$ is a mean zero stationary process with the spectral distribution function

$$F_X(\lambda) = \begin{cases} \pi + \lambda, & -\pi \leq \lambda < -\pi/6, \\ 3\pi + \lambda, & -\pi/6 \leq \lambda < \pi/6, \\ 5\pi + \lambda, & \pi/6 \leq \lambda \leq \pi. \end{cases}$$

For which value d does the differenced process $\Delta_d X_t := X_t - X_{t-d}$ have a spectral density?

Solution: Note that at the points $\lambda = -\pi/6$ and $\lambda = \pi/6$, the spectral distribution function $F_X(\lambda)$ has jumps of size 2π.
Note that the spectral density exists if the spectral distribution function is absolutely continuous. The differenced process $\Delta_d X_t$ has a spectral density if the differencing removes the jumps in the spectral distribution function.
The differenced process $\Delta_d X_t$ has a spectral distribution function given by

$$\begin{aligned} dF_{\Delta_d X}(\lambda) &= |1 - e^{-i\lambda d}|^2 dF_X(\lambda) \\ &= 4\sin^2\left(\frac{\lambda d}{2}\right) dF_X(\lambda). \end{aligned}$$

To remove the jumps at $\lambda = -\pi/6$ and $\lambda = \pi/6$, we need to choose d such that $4\sin^2\left(\frac{\lambda d}{2}\right)$ is zero at these points. This occurs when $\frac{\pi d}{12} = k\pi$ for some integer k. Thus, we have

$$d = 12k, \quad k \in \mathbb{Z}.$$

Therefore, the differenced process $\Delta_d X_t$ has a spectral density for $d = 12k$, where k is any integer. Thus the smallest positive integer value for d is $d = 12$.

Problem (21). Please install the `R` package `datasets`, and use `data(sunspot.year)` to load the Wölfer sunspot numbers from 1700 to 1988. Let $\{X_t\}$ denote the original data, and $\{Y_t\}$ denote the mean-corrected series, $Y_t = X_t - 49.13$. The following AR(2) model for $\{Y_t\}$ is obtained

$$Y_t = 1.389Y_{t-1} - .691Y_{t-2} + Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0, 273.6).$$

Determine and plot the spectral density of the fitted model and find the frequency at which it achieves its maximum value. What is the corresponding period?

Solution: We know that the spectral density of an AR(p) process is given by

$$f_Y(\lambda) = \frac{\sigma^2}{2\pi} \left| \frac{1}{\phi(e^{-i\lambda})} \right|^2,$$

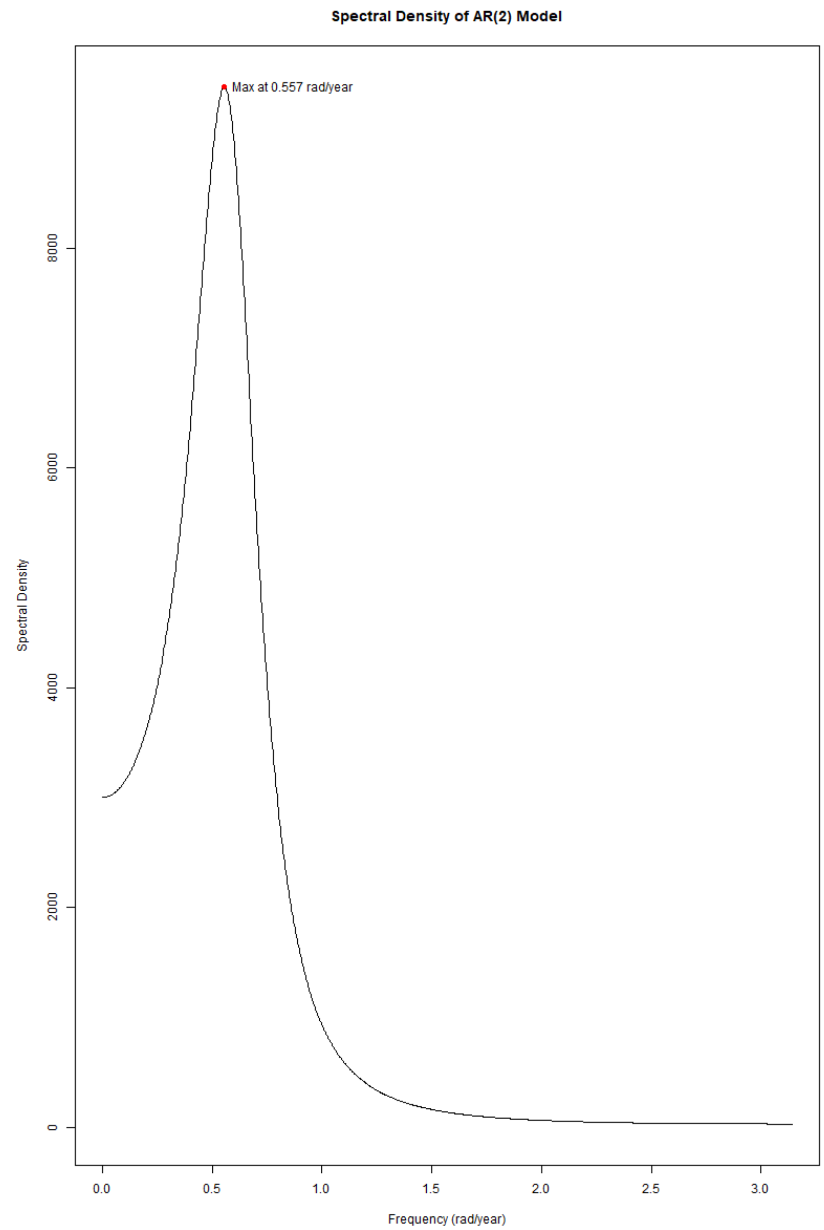
Thus for our AR(2) model, we have

$$\phi(B) = 1 - 1.389B + 0.691B^2,$$

and the spectral density is given by

$$f_Y(\lambda) = \frac{273.6}{2\pi} \left| \frac{1}{1 - 1.389e^{-i\lambda} + 0.691e^{-2i\lambda}} \right|^2.$$

The plot of the spectral density is shown below:



And we find that the frequency at which it achieves its maximum value is approximately $\lambda \approx .557$.
And thus the corresponding period is 11.28 years.

Problem (22)**.** Suppose $X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t$ is a causal AR(p) process, where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$, and let $\{\psi_n,\, n \geq 0\}$ be its memory function. Show that

$$\left\| X_{n+h} - \mathcal{P}_{\overline{\mathrm{sp}}\{X_1,\ldots,X_n\}} X_{n+h} \right\|^2 = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2, \quad \text{when } n \geq p.$$

Solution: We know that for a causal AR(p) there exists a WN process $\{Z_t\}$ such that

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \quad t \in \mathbb{Z},$$

In other words, we can express the AR(p) process $\{X_t\}$ as an infinite MA process with coefficients given by the memory function $\{\psi_j\}$.
We can represent X_{n+h} as

$$\begin{aligned} X_{n+h} &= \sum_{j=0}^{\infty} \psi_j Z_{n+h-j} \\ &= \sum_{j=0}^{h-1} \psi_j Z_{n+h-j} + \sum_{j=h}^{\infty} \psi_j Z_{n+h-j}. \end{aligned}$$

Note that the first sum $\sum_{j=0}^{h-1} \psi_j Z_{n+h-j}$ involves only the white noise terms $Z_{n+1}, Z_{n+2}, \ldots, Z_{n+h}$, which are not in the span of $\{X_1, \ldots, X_n\}$ since $n \geq p$. Therefore, this part of X_{n+h} is orthogonal to the space spanned by $\{X_1, \ldots, X_n\}$.
The second sum $\sum_{j=h}^{\infty} \psi_j Z_{n+h-j}$ can be expressed in terms of $\{X_1, \ldots, X_n\}$ since it involves only the white noise terms Z_t for $t \leq n$. Thus, this part lies in the span of $\{X_1, \ldots, X_n\}$.
Therefore, the projection of X_{n+h} onto the space spanned by $\{X_1, \ldots, X_n\}$ is given by

$$\mathcal{P}_{\overline{\text{sp}}\{X_1,\ldots,X_n\}} X_{n+h} = \sum_{j=h}^{\infty} \psi_j Z_{n+h-j}.$$

The prediction error is then

$$X_{n+h} - \mathcal{P}_{\overline{\text{sp}}\{X_1,\ldots,X_n\}} X_{n+h} = \sum_{j=0}^{h-1} \psi_j Z_{n+h-j}.$$

Taking the norm squared of the prediction error, we have

$$\begin{aligned} \left\| X_{n+h} - \mathcal{P}_{\overline{\text{sp}}\{X_1,\ldots,X_n\}} X_{n+h} \right\|^2 &= E\left[\left(\sum_{j=0}^{h-1} \psi_j Z_{n+h-j} \right)^2 \right] \\ &= \sum_{j=0}^{h-1} \psi_j^2 E[Z_{n+h-j}^2] \quad \text{(since } Z_t \text{ are uncorrelated)} \\ &= \sigma^2 \sum_{j=0}^{h-1} \psi_j^2. \end{aligned}$$

Thus we have shown that

$$\left\| X_{n+h} - \mathcal{P}_{\overline{\text{sp}}\{X_1,\dots,X_n\}} X_{n+h} \right\|^2 = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2, \quad \text{when } n \geq p.$$

Problem (23)**.** Suppose that

$$X_t = A\cos(\pi t/3) + B\sin(\pi t/3) + Z_t + .5Z_{t-1}, \qquad t \in \mathbb{Z},$$

where $\{Z_t\} \sim \text{WN}(0,1)$, A and B are uncorrelated random variables with mean 0 and variance 4, and $E(AZ_t) = E(BZ_t) = 0$ for all $t \in \mathbb{Z}$.

1. Find the best linear predictor of X_{t+1} based on X_t and X_{t-1}.

Solution: The goal is to find the best linear predictor of X_{t+1} based on X_t and X_{t-1}. We can express the predictor as

$$\hat{X}_{t+1} = \phi_1 X_t + \phi_2 X_{t-1},$$

where ϕ_1 and ϕ_2 are coefficients to be determined.
We can also define the autocovariance function $\gamma_X(h) = E[X_t X_{t+h}]$.

$$\gamma_X(h) = \gamma_S(h) + \gamma_M(h),$$

Where $\gamma_S(h)$ is the autocovariance function of the Periodic part and $\gamma_M(h)$ is the autocovariance function of the MA(1) part.
First, we compute $\gamma_S(h)$:

$$\begin{aligned}
\gamma_S(h) &= E[(A\cos(\pi t/3) + B\sin(\pi t/3))(A\cos(\pi(t+h)/3) + B\sin(\pi(t+h)/3))] \\
&= E[A^2]\cos(\pi t/3)\cos(\pi(t+h)/3) + E[B^2]\sin(\pi t/3)\sin(\pi(t+h)/3) \\
&= 4\cos(\pi t/3)\cos(\pi(t+h)/3) + 4\sin(\pi t/3)\sin(\pi(t+h)/3) \\
&= 4\cos(\pi h/3).
\end{aligned}$$

Next, we compute $\gamma_M(h)$:

$$\begin{aligned}
\gamma_M(0) &= E[(Z_t + 0.5Z_{t-1})^2] = E[Z_t^2] + 0.25E[Z_{t-1}^2] = 1 + 0.25 = 1.25, \\
\gamma_M(1) &= E[(Z_t + 0.5Z_{t-1})(Z_{t+1} + 0.5Z_t)] = 0.5E[Z_t^2] = 0.5, \\
\gamma_M(h) &= 0, \quad |h| > 1.
\end{aligned}$$

Therefore, the total autocovariance function is

$$\begin{aligned}
\gamma_X(0) &= 4 + 1.25 = 5.25, \\
\gamma_X(1) &= 2 + 0.5 = 2.5, \\
\gamma_X(2) &= -2 + 0 = -2, \\
\gamma_X(h) &= 4\cos(\pi h/3), \quad |h| > 2.
\end{aligned}$$

Now, we can set up the Yule-Walker equations to solve for ϕ_1 and ϕ_2:

$$\begin{pmatrix} \gamma_X(0) & \gamma_X(1) \\ \gamma_X(1) & \gamma_X(0) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \gamma_X(1) \\ \gamma_X(2) \end{pmatrix}.$$

Substituting the values, we have

$$\begin{pmatrix} 5.25 & 2.5 \\ 2.5 & 5.25 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 2.5 \\ -2 \end{pmatrix}.$$

Solving this system of equations, we find

$$\phi_1 = \frac{290}{341} \approx 0.8504,$$
$$\phi_2 = -\frac{268}{341} \approx -0.7859.$$

Therefore, the best linear predictor of X_{t+1} based on X_t and X_{t-1} is

$$\hat{X}_{t+1} = \frac{290}{341} X_t - \frac{268}{341} X_{t-1}.$$

2. What is the mean squared error of the best linear predictor of X_{t+1} based on $\{X_j,\, j \leq t\}$?

Solution: Note that when we are predicting X_{t+1} based on the entire past $\{X_j, j \leq t\}$, we can see that the periodic part of the process can be perfectly predicted since it is deterministic.
And for the MA(1) part, the best linear predictor of $Y_t = Z_t + 0.5Z_{t-1}$ based on $\{Y_j, j \leq t\}$ is given by $Y_{t+1|t} = 0.5Z_t$.
Therefore, the prediction error for the MA(1) part is

$$Y_{t+1} - Y_{t+1|\infty} = (Z_{t+1} + 0.5Z_t) - 0.5Z_t = Z_{t+1}.$$

And thus the prediction error for the entire process is

$$X_{t+1} - \hat{X}_{t+1|\infty} = Z_{t+1}.$$

The mean squared error (MSE) of the best linear predictor is then

$$E[(X_{t+1} - \hat{X}_{t+1|\infty})^2] = E[Z_{t+1}^2] = 1.$$

3. Show how A and B can be predicted by $\{X_j,\, j \leq n\}$.

Solution: We can use the orthogonalily of trigonometric functions to predict A and B. Note that the periodic part of the process has period 6.
Note that the average of $\cos^2(\pi t/3)$ over one period is given by $\frac{1}{6}\sum_{t=1}^{6}\cos^2(\pi t/3) = \frac{1}{2}$. Similarly, the average of $\sin^2(\pi t/3)$ over one period is also $\frac{1}{2}$. and the average of $\cos(\pi t/3)\sin(\pi t/3)$ over one period is 0.
Then we can define the sample projections:

$$C_n = \frac{2}{n}\sum_{t=1}^{n} X_t \cos(\pi t/3),$$
$$S_n = \frac{2}{n}\sum_{t=1}^{n} X_t \sin(\pi t/3).$$

As $n \to \infty$, by the law of large numbers, we have

$$C_n \to A,$$
$$S_n \to B.$$

Since Z_t and $cos(\pi t/3)$, $sin(\pi t/3)$ are orthogonal, the contributions from the noise terms vanish in the limit.
Therefore, we can predict A and B using the observations $\{X_j, j \leq n\}$ as n becomes large.