# Workshop 7

Pranav Tikkawar

March 17, 2024

## Question 2

Find the motion through the intial point $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ if $A = \begin{bmatrix} 6 & -8 & -7 \\ 3 & -3 & -3 \\ 3 & -2 & -4 \end{bmatrix}$

Eigenvalues of the matrix is given by $A - \lambda I = \begin{bmatrix} 6-\lambda & -8 & -7 \\ 3 & -3-\lambda & -3 \\ 3 & -2 & -4-\lambda \end{bmatrix}$ The charecteristic polynomial of the matrix is given by

$$(-6-\lambda)[(-3-\lambda)(-4-\lambda)-6] + 8[3(-4-\lambda)+9] - 7[-3(-3-\lambda)-6]$$

The charecteristic polynomial simplifies to

$$-\lambda^3 - \lambda^2 - 9\lambda - 9$$

$$(-1-\lambda)(\lambda^2+9)$$

Thus the eigenvalues of the matrix is given by $\mu_1 = -1, \mu_2 = 3i, \mu_3 = -3i$. The eigenvector for the eigenvalue $\mu_1 = -1$ is given by solving the equation $(A - \mu_1 I)X = 0$. Here the matrix $[A - \mu_1 I|0]$ is given by

$$\begin{bmatrix} 7 & -8 & -7 & 0 \\ 3 & -2 & -3 & 0 \\ 3 & -2 & -3 & 0 \end{bmatrix}$$

The reduced row echelon form of the matrix is

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the eigenvector for the eigenvalue $\mu_1 = -1$ is given by

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The eigenvector for the eigenvalue $\mu_2 = 3i$ is given by solving the equation $(A - \mu_2 I)X = 0$. Here the matrix $[A - \mu_2 I|0]$ is given by

$$\begin{bmatrix} 6-3i & -8 & -7 & 0 \\ 3 & -3-3i & -3 & 0 \\ 3 & -2 & -4-3i & 0 \end{bmatrix}$$

The reduced row echelon form of the matrix is

$$\begin{bmatrix} 0 & 1+3i & -1-3i & 0 \\ 1 & -1-i & -1 & 0 \\ 0 & 1+3i & -1-3i & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1-i & -1 & 0 \\ 0 & 1+3i & -1-3i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -2-i & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the eigenvector for the eigenvalue $\mu_2 = 3i$ is given by

$$\begin{bmatrix} 2+i \\ 1 \\ 1 \end{bmatrix}$$

Therefore we can split up this eigenvector into two vectors, one for the real part and one for the imaginary part. Thus the eigenvector for the eigenvalue $\mu_2 = 3i$ is given by

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

And $x = e^{3it} x_0$ is given by $[cos(3t) + isin(3t)] \begin{bmatrix} 2+i \\ 1 \\ 1 \end{bmatrix}$ for the first 2 rows

Thus the solution to the system of differential equations is given by

$$\begin{bmatrix} 2cos(3t) - sin(3t) \\ cos(3t) \\ cos(3t) \end{bmatrix} + \begin{bmatrix} cos(3t) + 2sin(3t) \\ sin(3t) \\ sin(3t) \end{bmatrix}$$

Therefore our $M(t)$ is given by

$$\begin{bmatrix} 2cos(3t) - sin(3t) & cos(3t) + 2sin(3t) & e^{-t} \\ cos(3t) & sin(3t) & 0 \\ cos(3t) & sin(3t) & e^{-t} \end{bmatrix}$$

Our $M(0)$ is given by

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The inverse $M(0)^{-1}$ is given by:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Thus the new matrix $G(t)$ which solves $Y(t) = G(t)Y^{(0)}$ is given by

$$\begin{bmatrix} 2cos(3t) - sin(3t) & cos(3t) + 2sin(3t) & e^{-t} \\ cos(3t) & sin(3t) & 0 \\ cos(3t) & sin(3t) & e^{-t} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

When considering the intial condition: $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, we get the solution to the system of differential equations as

$$\begin{bmatrix} 2cos(3t) - sin(3t) & cos(3t) + 2sin(3t) & e^{-t} \\ cos(3t) & sin(3t) & 0 \\ cos(3t) & sin(3t) & e^{-t} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2cos(3t) - sin(3t) & cos(3t) + 2sin(3t) & e^{-t} \\ cos(3t) & sin(3t) & 0 \\ cos(3t) & sin(3t) & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2cos(3t) - sin(3t) - 2cos(3t) - 4sin(3t) + e^{-t} \\ cos(3t) - 2sin(3t) \\ cos(3t) - 2sin(3t) + e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} -5sin(3t) + e^{-t} \\ cos(3t) - 2sin(3t) \\ cos(3t) - 2sin(3t) + e^{-t} \end{bmatrix}$$

Given as a sum of Periodic and non-Periodic terms is

$$\begin{bmatrix} -5sin(3t) \\ cos(3t) - 2sin(3t) \\ cos(3t) - 2sin(3t) \end{bmatrix} + \begin{bmatrix} e^{-t} \\ 0 \\ e^{-t} \end{bmatrix}$$

The limit as $t \to \infty$ is going to be a Periodic motion. since the non-periodic term will tend to 0 as $t \to \infty$ and the periodic term will remain and continue to oscillate.