Stat 958:587 - Homework 3

Due: 11/25/2025

This notebook contains four problems covering Bayesian hierarchical modeling, Gibbs sampling, stochastic differential equations, and Monte Carlo methods.

Problem 1: Bayesian Hierarchical Model for Power Plant Pump Failures

The following table gives for ten power plant pumps the recorded number of failures Y_i and the length t_i of operation time (in 1000s of hours):

Pump i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
t_i	94.3	15.7	62.9	126	5.24	31.4	1.05	1.05	2.1	10.5
Y_i	5	1	5	14	3	19	1	1	4	22

The number of failures is assumed to follow a Poisson distribution:

$$Y_i \sim \text{Poisson}(\theta_i t_i), \quad i = 1, \ldots, 10$$

where θ_i is the failure rate for pump i. For Bayesian inference we adopt a conjugate gamma prior distribution for the failure rates:

$$\theta_i \sim \Gamma(\alpha, \beta), \quad i = 1, \ldots, 10$$

Instead of specifying particular values for both hyperparameters α and β, we assume the following hierarchical prior specification:

$$\alpha = 1$$
$$\beta \sim \Gamma(1/5, 1)$$

Part (a): Compute the full conditional posterior distributions

Compute the full conditional posterior distributions for θ_i and β.

Hint: Use the trick of "$\propto$" in calculating Bayesian posteriors.

Your answer:

Part (a): Full Conditional Posterior Distributions

Given Model

- $Y_i \mid \theta_i \sim \text{Poisson}(\theta_i t_i)$
- $\theta_i \mid \beta \sim \Gamma(\alpha, \beta)$ with $\alpha = 1$
- $\beta \sim \Gamma\left(\frac{1}{5}, 1\right)$

Derivation of Full Conditional for $\theta_i \mid \beta, Y$

Using the proportionality trick:

$$p(\theta_i \mid \beta, Y_i) \propto p(Y_i \mid \theta_i) \cdot p(\theta_i \mid \beta)$$

$$\propto \left[e^{-\theta_i t_i}(\theta_i t_i)^{Y_i}\right] \cdot \left[\beta^\alpha \theta_i^{\alpha-1} e^{-\beta\theta_i}\right]$$

$$\propto e^{-\theta_i t_i} \cdot \theta_i^{Y_i} \cdot \theta_i^{1-1} \cdot e^{-\beta\theta_i}$$

$$\propto \theta_i^{Y_i} \cdot e^{-(t_i+\beta)\theta_i}$$

This is the kernel of a Gamma distribution.

$$\boxed{\theta_i \mid \beta, Y_i \sim \Gamma(Y_i + 1,\, t_i + \beta)}$$

Derivation of Full Conditional for $\beta \mid \boldsymbol{\theta}, Y$

$$p(\beta \mid \boldsymbol{\theta}, Y) \propto p(\boldsymbol{\theta} \mid \beta) \cdot p(\beta)$$

$$\propto \left[\prod_{i=1}^{10} \beta^\alpha \theta_i^{\alpha-1} e^{-\beta\theta_i}\right] \cdot \left[\beta^{1/5-1} e^{-\beta}\right]$$

$$\propto \beta^{n\alpha} \cdot e^{-\beta\sum_{i=1}^{10}\theta_i} \cdot \beta^{1/5-1} \cdot e^{-\beta}$$

With $n = 10$ and $\alpha = 1$:

$$\propto \beta^{10} \cdot e^{-\beta\sum\theta_i} \cdot \beta^{-4/5} \cdot e^{-\beta}$$

$$\propto \beta^{10-4/5} \cdot e^{-\beta(1+\sum\theta_i)}$$

$$\propto \beta^{46/5} \cdot e^{-\beta(1+\sum\theta_i)}$$

This is the kernel of a Gamma distribution.

$$\boxed{\beta \mid \boldsymbol{\theta} \sim \Gamma\left(\frac{51}{5},\, 1 + \sum_{i=1}^{10}\theta_i\right)}$$

Summary of Full Conditional Distributions

Parameter	Full Conditional Distribution
θ_i	$\Gamma\left(Y_i + 1,\, t_i + \beta\right)$
β	$\Gamma\left(\frac{51}{5},\, 1 + \sum_{i=1}^{10} \theta_i\right)$

Note: The Gamma distribution is parameterized as $\Gamma(\text{shape}, \text{rate})$.

Part (b): Formulate the Gibbs sampler

Based on the results from part (a), formulate the Gibbs sampler for this problem.

Your answer:

Part (b): Gibbs Sampler Algorithm

Overview

The Gibbs sampler is an MCMC algorithm that alternately samples from the full conditional distributions derived in Part (a). For this hierarchical model, we iteratively sample from:

- $\theta_i \mid \beta, Y_i \sim \Gamma(Y_i + 1,\, t_i + \beta)$
- $\beta \mid \boldsymbol{\theta} \sim \Gamma\left(\frac{51}{5},\, 1 + \sum_{i=1}^{10} \theta_i\right)$

Gibbs Sampling Algorithm

Algorithm Steps

Initialization:

1. Set $\beta^{(0)}$ (e.g., $\beta^{(0)} = 1$)
2. Initialize $\theta_i^{(0)}$ for $i = 1, \ldots, 10$ (e.g., $\theta_i^{(0)} = Y_i/t_i$)

For each iteration $m = 1, 2, \ldots, M$:

Step 1: Update θ_i for each pump

For $i = 1, 2, \ldots, 10$:

$$\theta_i^{(m)} \sim \Gamma\left(Y_i + 1,\, t_i + \beta^{(m-1)}\right)$$

Step 2: Update β

$$\beta^{(m)} \sim \Gamma\left(\frac{51}{5}, 1 + \sum_{i=1}^{10} \theta_i^{(m)}\right)$$

Pseudocode

```
Algorithm: Gibbs Sampler

Input: Y, t, M (number of iterations), B (burn-in)

Initialize:
  beta[0] = 1.0
  theta[0, :] = Y / t

For m = 1 to M:
    For i = 1 to 10:
        shape = Y[i] + 1
        rate = t[i] + beta[m-1]
        theta[m, i] ~ Gamma(shape, rate)

    shape_beta = 51/5
    rate_beta = 1 + sum(theta[m, :])
    beta[m] ~ Gamma(shape_beta, rate_beta)

Return: theta[B+1:M], beta[B+1:M]
```

Convergence Recommendations

- **Burn-in:** Remove first 1,000 iterations
- **Total iterations:** Run 10,000-50,000 iterations
- **Check trace plots** for convergence and mixing
- **Plot autocorrelation** to assess correlation between samples

Part (c): Implement the Gibbs sampler and plot marginal posteriors

Implement the Gibbs sampler and apply it to the pump data. Plot the marginal posterior densities of β, θ_1 and θ_6.

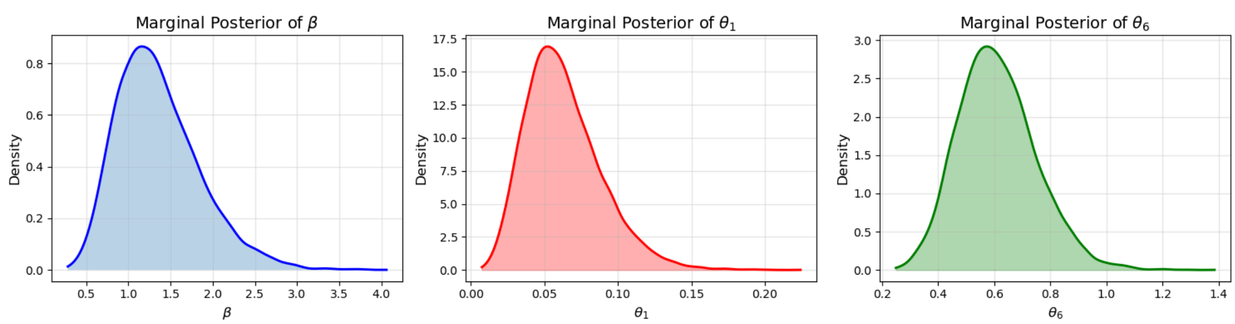
Required: 3 plots

Your solution:

In [19]:
```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import gaussian_kde

# Set random seed for reproducibility
```

```python
np.random.seed(42)

# Pump data
t = np.array([94.3, 15.7, 62.9, 126, 5.24, 31.4, 1.05, 1.05, 2.1, 10.5])
y = np.array([5, 1, 5, 14, 3, 19, 1, 1, 4, 22])

# Hyperparameters
alpha = 1
n_pumps = 10

# Gibbs sampler parameters
M = 10000  # Total iterations
burn_in = 1000  # Burn-in period

# Storage for samples
beta_samples = np.zeros(M)
theta_samples = np.zeros((M, n_pumps))

# Initialize
beta_samples[0] = 1.0
theta_samples[0, :] = y / t  # MLE initialization

# Gibbs sampler
for m in range(1, M):
    # Step 1: Sample theta_i from Gamma(Y_i + 1, t_i + beta)
    for i in range(n_pumps):
        shape_theta = y[i] + alpha
        rate_theta = t[i] + beta_samples[m-1]
        theta_samples[m, i] = np.random.gamma(shape_theta, 1/rate_theta)

    # Step 2: Sample beta from Gamma(51/5, 1 + sum(theta))
    shape_beta = alpha + n_pumps * alpha - 1 + 1/5  # = 51/5
    rate_beta = 1 + np.sum(theta_samples[m, :])
    beta_samples[m] = np.random.gamma(shape_beta, 1/rate_beta)

# Remove burn-in
beta_post = beta_samples[burn_in:]
theta1_post = theta_samples[burn_in:, 0]  # theta_1
theta6_post = theta_samples[burn_in:, 5]  # theta_6 (index 5)

# Create plots
fig, axes = plt.subplots(1, 3, figsize=(15, 4))

# Plot 1: Marginal posterior of beta
ax1 = axes[0]
kde_beta = gaussian_kde(beta_post)
x_beta = np.linspace(beta_post.min(), beta_post.max(), 200)
ax1.plot(x_beta, kde_beta(x_beta), 'b-', linewidth=2)
ax1.fill_between(x_beta, kde_beta(x_beta), alpha=0.3)
ax1.set_xlabel(r'$\beta$', fontsize=12)
ax1.set_ylabel('Density', fontsize=12)
ax1.set_title(r'Marginal Posterior of $\beta$', fontsize=14)
ax1.grid(True, alpha=0.3)

# Plot 2: Marginal posterior of theta_1
ax2 = axes[1]
kde_theta1 = gaussian_kde(theta1_post)
x_theta1 = np.linspace(theta1_post.min(), theta1_post.max(), 200)
ax2.plot(x_theta1, kde_theta1(x_theta1), 'r-', linewidth=2)
ax2.fill_between(x_theta1, kde_theta1(x_theta1), alpha=0.3, color='red')
```

```python
ax2.set_xlabel(r'$\theta_1$', fontsize=12)
ax2.set_ylabel('Density', fontsize=12)
ax2.set_title(r'Marginal Posterior of $\theta_1$', fontsize=14)
ax2.grid(True, alpha=0.3)

# Plot 3: Marginal posterior of theta_6
ax3 = axes[2]
kde_theta6 = gaussian_kde(theta6_post)
x_theta6 = np.linspace(theta6_post.min(), theta6_post.max(), 200)
ax3.plot(x_theta6, kde_theta6(x_theta6), 'g-', linewidth=2)
ax3.fill_between(x_theta6, kde_theta6(x_theta6), alpha=0.3, color='green')
ax3.set_xlabel(r'$\theta_6$', fontsize=12)
ax3.set_ylabel('Density', fontsize=12)
ax3.set_title(r'Marginal Posterior of $\theta_6$', fontsize=14)
ax3.grid(True, alpha=0.3)

plt.tight_layout()
plt.savefig('problem1c_posteriors.png', dpi=150, bbox_inches='tight')
plt.show()

# Print summary statistics
print(f"beta:    Mean = {beta_post.mean():.4f}, Std = {beta_post.std():.4f}")
print(f"theta_1: Mean = {theta1_post.mean():.4f}, Std = {theta1_post.std():.4f}")
print(f"theta_6: Mean = {theta6_post.mean():.4f}, Std = {theta6_post.std():.4f}")
```




```
beta:    Mean = 1.3582, Std = 0.4940
theta_1: Mean = 0.0631, Std = 0.0256
theta_6: Mean = 0.6112, Std = 0.1382
```

Problem 2: Hierarchical Poisson Model with Gibbs Sampling

Let $Y \mid \lambda, \beta \sim \text{Pois}(\lambda)$, $\lambda \mid \beta \sim \text{gamma}(2, \beta)$, and $\beta \sim \text{gamma}(a, b)$ where a, b are two fixed constants.

This is a hierarchical model where the data Y have a Poisson distribution with mean λ. We use a gamma prior for λ which depends on an unknown parameter β. Finally, we put a gamma 'hyperprior' with known constants (a, b) on β.

Part (a): Find the full conditional distributions

Find the full conditional distributions $\lambda \mid \beta, Y$ and $\beta \mid \lambda, Y$.

Your answer:

Part (a): Full Conditional Distributions

Given Model

- $Y \mid \lambda, \beta \sim \text{Poisson}(\lambda)$
- $\lambda \mid \beta \sim \text{Gamma}(2, \beta)$
- $\beta \sim \text{Gamma}(a, b)$ where a, b are fixed constants

---

Derivation of Full Conditional for $\lambda \mid \beta, Y$

Using the proportionality trick:

$$p(\lambda \mid \beta, Y) \propto p(Y \mid \lambda) \cdot p(\lambda \mid \beta)$$

$$\propto \left[e^{-\lambda} \cdot \lambda^Y\right] \cdot \left[\beta^2 \cdot \lambda^{2-1} \cdot e^{-\beta\lambda}\right]$$

$$\propto e^{-\lambda} \cdot \lambda^Y \cdot \lambda \cdot e^{-\beta\lambda}$$

$$\propto \lambda^{Y+1} \cdot e^{-(1+\beta)\lambda}$$

This is the kernel of a Gamma distribution.

$$\boxed{\lambda \mid \beta, Y \sim \text{Gamma}(Y+2,\, 1+\beta)}$$

Derivation of Full Conditional for $\beta \mid \lambda, Y$

Using the proportionality trick:

$$p(\beta \mid \lambda, Y) \propto p(\lambda \mid \beta) \cdot p(\beta)$$

Note: Y does not depend on β directly given λ.

$$\propto \left[\beta^2 \cdot \lambda \cdot e^{-\beta\lambda}\right] \cdot \left[b^a \cdot \beta^{a-1} \cdot e^{-b\beta}\right]$$

$$\propto \beta^2 \cdot e^{-\beta\lambda} \cdot \beta^{a-1} \cdot e^{-b\beta}$$

$$\propto \beta^{a+1} \cdot e^{-\beta(b+\lambda)}$$

This is the kernel of a Gamma distribution.

$$\boxed{\beta \mid \lambda \sim \text{Gamma}(a+2,\, b+\lambda)}$$

Summary of Full Conditional Distributions

Parameter	Full Conditional Distribution
λ	$\text{Gamma}(Y+2,\, 1+\beta)$
β	$\text{Gamma}(a+2,\, b+\lambda)$

Note: The Gamma distribution is parameterized as $\text{Gamma}(\text{shape}, \text{rate})$.

Part (b): Gibbs sampling algorithm and posterior analysis

Based on the full conditional distributions from part (a):

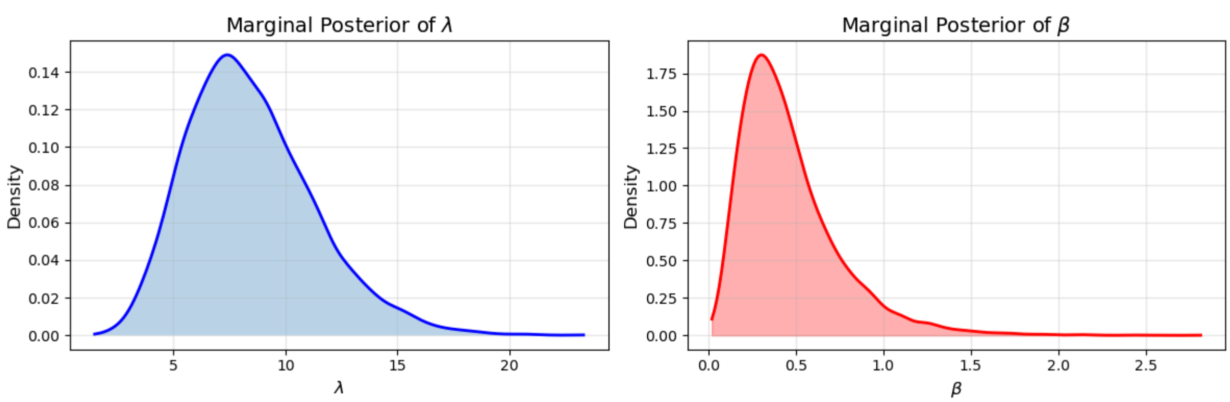
1. Write a Gibbs sampling algorithm to get the plot of the posterior distribution
2. Pick your own observed data Y and known constants (a, b)
3. Report the posterior mean and 95% credible interval

Your solution:

In [22]:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import gaussian_kde

# Set random seed for reproducibility
np.random.seed(42)

# Choose observed data Y and hyperparameters (a, b)
Y = 10  # Observed count data
a = 2   # Hyperparameter for beta prior
b = 1   # Hyperparameter for beta prior

# Gibbs sampler parameters
M = 10000  # Total iterations
burn_in = 1000  # Burn-in period

# Storage for samples
lambda_samples = np.zeros(M)
beta_samples = np.zeros(M)

# Initialize
lambda_samples[0] = 1.0
beta_samples[0] = 1.0

# Gibbs sampler
for m in range(1, M):
    # Step 1: Sample lambda from Gamma(Y + 2, 1 + beta)
    shape_lambda = Y + 2
    rate_lambda = 1 + beta_samples[m-1]
    lambda_samples[m] = np.random.gamma(shape_lambda, 1/rate_lambda)

    # Step 2: Sample beta from Gamma(a + 2, b + lambda)
    shape_beta = a + 2
    rate_beta = b + lambda_samples[m]
    beta_samples[m] = np.random.gamma(shape_beta, 1/rate_beta)

# Remove burn-in
lambda_post = lambda_samples[burn_in:]
```

```
beta_post = beta_samples[burn_in:]

# Create plots
fig, axes = plt.subplots(1, 2, figsize=(12, 4))

# Plot 1: Marginal posterior of lambda
ax1 = axes[0]
kde_lambda = gaussian_kde(lambda_post)
x_lambda = np.linspace(lambda_post.min(), lambda_post.max(), 200)
ax1.plot(x_lambda, kde_lambda(x_lambda), 'b-', linewidth=2)
ax1.fill_between(x_lambda, kde_lambda(x_lambda), alpha=0.3)
ax1.set_xlabel(r'$\lambda$', fontsize=12)
ax1.set_ylabel('Density', fontsize=12)
ax1.set_title(r'Marginal Posterior of $\lambda$', fontsize=14)
ax1.grid(True, alpha=0.3)

# Plot 2: Marginal posterior of beta
ax2 = axes[1]
kde_beta = gaussian_kde(beta_post)
x_beta = np.linspace(beta_post.min(), beta_post.max(), 200)
ax2.plot(x_beta, kde_beta(x_beta), 'r-', linewidth=2)
ax2.fill_between(x_beta, kde_beta(x_beta), alpha=0.3, color='red')
ax2.set_xlabel(r'$\beta$', fontsize=12)
ax2.set_ylabel('Density', fontsize=12)
ax2.set_title(r'Marginal Posterior of $\beta$', fontsize=14)
ax2.grid(True, alpha=0.3)

plt.tight_layout()
plt.savefig('problem2b_posteriors.png', dpi=150, bbox_inches='tight')
plt.show()

# Compute posterior mean and 95% credible interval
lambda_mean = np.mean(lambda_post)
lambda_ci = np.percentile(lambda_post, [2.5, 97.5])

beta_mean = np.mean(beta_post)
beta_ci = np.percentile(beta_post, [2.5, 97.5])

# Print results
print(f"\nChosen values: Y = {Y}, a = {a}, b = {b}")
print(f"Lambda:  Posterior Mean: {lambda_mean:.4f}")
print(f"95% Credible Interval: [{lambda_ci[0]:.4f}, {lambda_ci[1]:.4f}]")
print(f"Beta:  Posterior Mean: {beta_mean:.4f}")
print(f"95% Credible Interval: [{beta_ci[0]:.4f}, {beta_ci[1]:.4f}]")
```



```
Chosen values: Y = 10, a = 2, b = 1
Lambda:  Posterior Mean: 8.4551
95% Credible Interval: [3.8920, 14.9486]
Beta:  Posterior Mean: 0.4647
95% Credible Interval: [0.1098, 1.1898]
```

Problem 3: Stochastic Differential Equation for Stock Price Simulation

Assume that a stock price, say s_t, at time t is a smooth function of time. Denote this smooth function as $g(t)$. Suppose that $g(t)$ satisfies the following stochastic differential equation:

$$\frac{d}{dt}g(t) = \mu g(t) + \sigma g(t)\frac{d}{dt}W(t)$$

where both the drift μ and the volatility σ are constants and the noise $W(t)$ is the standard Brownian motion.

Generate and plot a path of s_t on $t \in [0, 1]$ using the following inputs:

(i) $s_0 = 0.35$, $\mu = 3$, $\sigma = 0.1$, $\Delta t = 0.05$

(ii) $s_0 = 0.5$, $\mu = -2.5$, $\sigma = 0.1$, $\Delta t = 0.1$

For each case, use:

1. Random walk to simulate the Brownian motions
2. Brownian bridge to simulate the Brownian motions

Required: 4 plots total

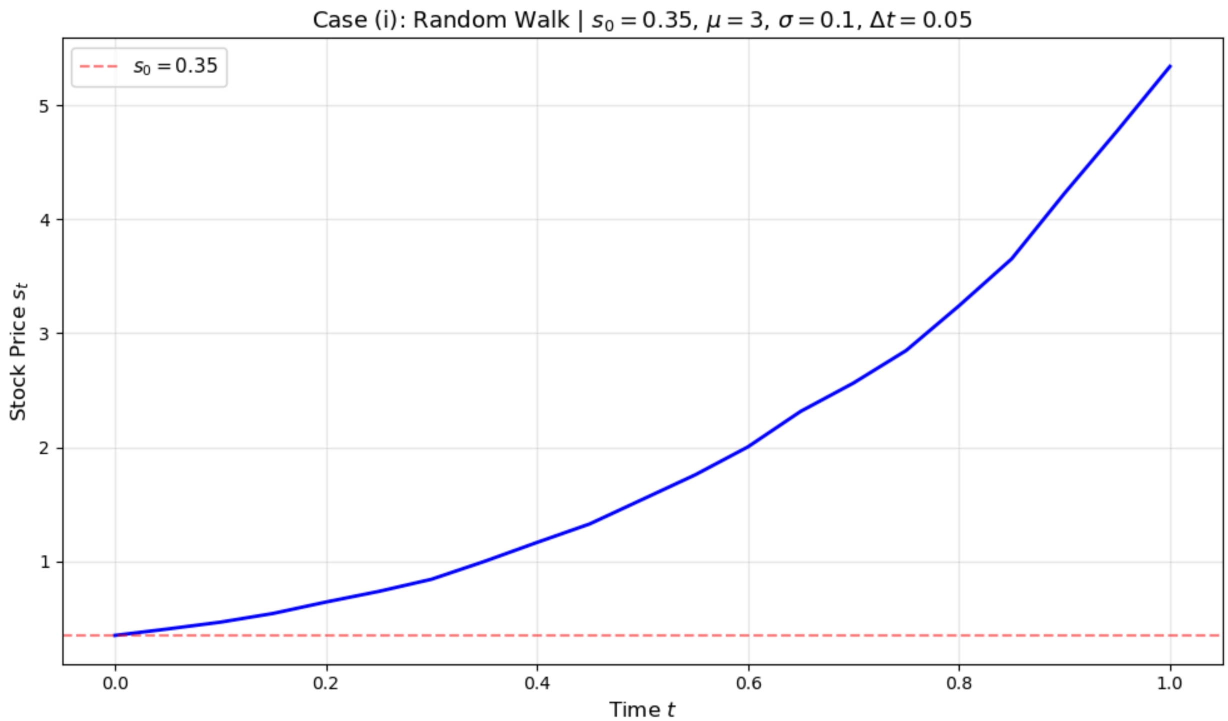
Your solution:

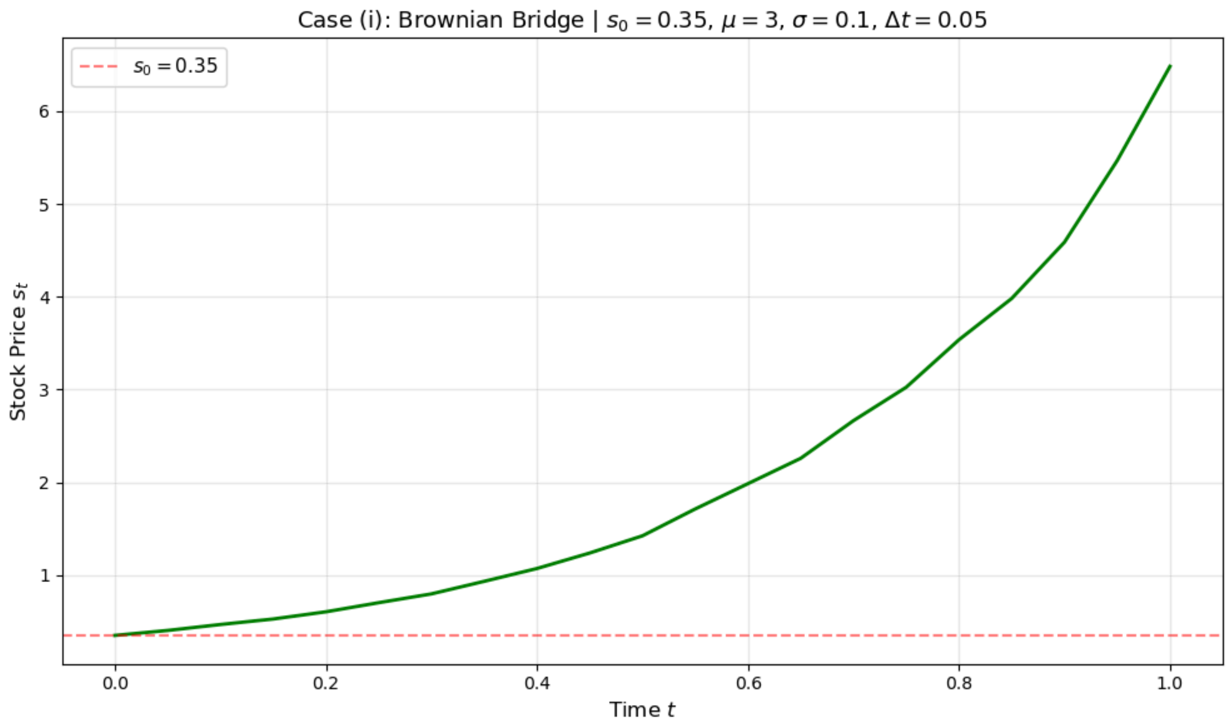
In [ ]:
```python
import numpy as np
import matplotlib.pyplot as plt

# ========== HELPER FUNCTIONS ==========

def simulate_gbm_random_walk(s0, mu, sigma, T, dt):
    """
    Simulate GBM using random walk for Brownian motion.
    ds = mu*s*dt + sigma*s*dW
    dW = sqrt(dt) * Z, where Z ~ N(0,1)
    """
    n_steps = int(T / dt)
    t = np.linspace(0, T, n_steps + 1)
    s = np.zeros(n_steps + 1)
    s[0] = s0

    for i in range(n_steps):
        Z = np.random.randn()
        dW = np.sqrt(dt) * Z
        s[i+1] = s[i] + mu * s[i] * dt + sigma * s[i] * dW
```

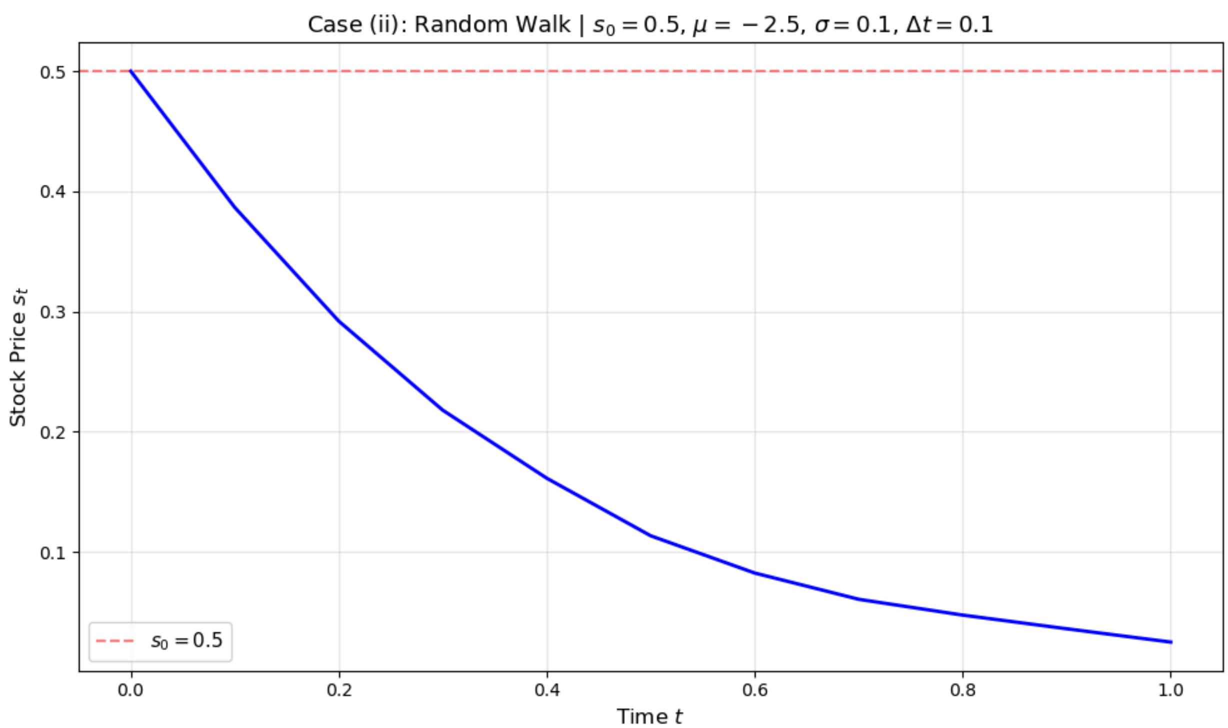
```python
    return t, s


def simulate_gbm_brownian_bridge(s0, mu, sigma, T, dt):
    """
    Simulate GBM using Brownian bridge for Brownian motion.

    First, generate W(T) ~ N(0, T).
    Then, for each t_i, sample W(t_i) given W(0)=0 and W(T):
    W(t_i) ~ N( (t_i/T)*W(T), t_i*(T-t_i)/T )

    Finally, update s using the SDE.
    """
    n_steps = int(T / dt)
    t = np.linspace(0, T, n_steps + 1)

    # Generate Brownian bridge
    W = np.zeros(n_steps + 1)
    W[0] = 0
    WT = np.sqrt(T) * np.random.randn()  # W(T) ~ N(0, T)

    # Sample intermediate points using Brownian bridge
    for i in range(1, n_steps):  # Note: not n_steps + 1
        ti = t[i]
        t_prev = t[i-1]
        # Condition on W(t_{i-1}) and W(T)
        tau = (ti - t_prev) / (T - t_prev)
        mean_W = W[i-1] + tau * (WT - W[i-1])
        var_W = (ti - t_prev) * (T - ti) / (T - t_prev)
        W[i] = mean_W + np.sqrt(var_W) * np.random.randn()
    W[-1] = WT  # Ensure W(T) = WT


    # Compute stock price from Brownian motion path
    s = np.zeros(n_steps + 1)
    s[0] = s0
    for i in range(n_steps):
        dW = W[i+1] - W[i]
        s[i+1] = s[i] + mu * s[i] * dt + sigma * s[i] * dW

    return t, s


# ========== PARAMETERS ==========
np.random.seed(42)

# Case (i): s0=0.35, mu=3, sigma=0.1, dt=0.05
s0_i = 0.35
mu_i = 3
sigma_i = 0.1
dt_i = 0.05
T = 1.0

# Case (ii): s0=0.5, mu=-2.5, sigma=0.1, dt=0.1
s0_ii = 0.5
mu_ii = -2.5
sigma_ii = 0.1
dt_ii = 0.1

# ========== PLOT 1: CASE (i) - RANDOM WALK ==========
```

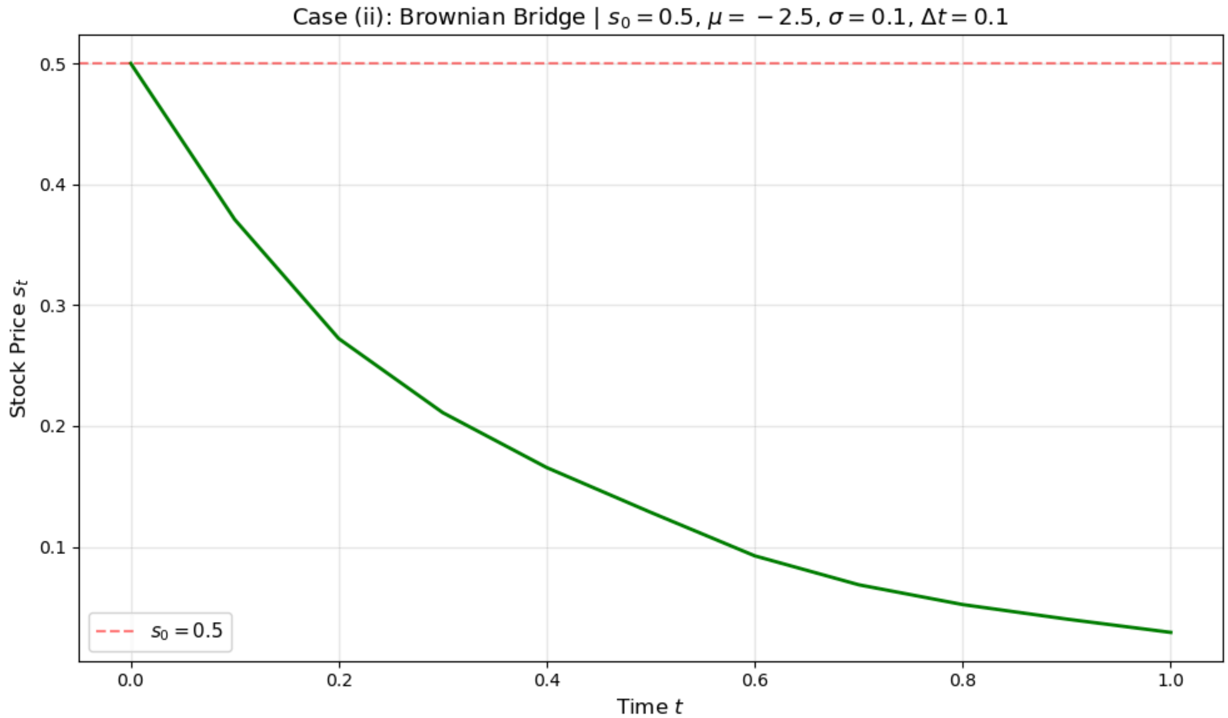
```
t1, s1 = simulate_gbm_random_walk(s0_i, mu_i, sigma_i, T, dt_i)

plt.figure(figsize=(10, 6))
plt.plot(t1, s1, 'b-', linewidth=2)
plt.axhline(s0_i, color='red', linestyle='--', alpha=0.5, label=f'$s_0={s0_i}$')
plt.xlabel('Time $t$', fontsize=12)
plt.ylabel('Stock Price $s_t$', fontsize=12)
plt.title(f'Case (i): Random Walk | $s_0={s0_i}$, $\mu={mu_i}$, $\sigma={sigma_i}$, $\
          fontsize=13)
plt.grid(True, alpha=0.3)
plt.legend(fontsize=11)
plt.tight_layout()
plt.savefig('problem3_case_i_random_walk.png', dpi=150, bbox_inches='tight')
plt.show()

# ========== PLOT 2: CASE (i) - BROWNIAN BRIDGE ==========
t2, s2 = simulate_gbm_brownian_bridge(s0_i, mu_i, sigma_i, T, dt_i)

plt.figure(figsize=(10, 6))
plt.plot(t2, s2, 'g-', linewidth=2)
plt.axhline(s0_i, color='red', linestyle='--', alpha=0.5, label=f'$s_0={s0_i}$')
plt.xlabel('Time $t$', fontsize=12)
plt.ylabel('Stock Price $s_t$', fontsize=12)
plt.title(f'Case (i): Brownian Bridge | $s_0={s0_i}$, $\mu={mu_i}$, $\sigma={sigma_i}$
          fontsize=13)
plt.grid(True, alpha=0.3)
plt.legend(fontsize=11)
plt.tight_layout()
plt.savefig('problem3_case_i_brownian_bridge.png', dpi=150, bbox_inches='tight')
plt.show()

# ========== PLOT 3: CASE (ii) - RANDOM WALK ==========
t3, s3 = simulate_gbm_random_walk(s0_ii, mu_ii, sigma_ii, T, dt_ii)

plt.figure(figsize=(10, 6))
plt.plot(t3, s3, 'b-', linewidth=2)
plt.axhline(s0_ii, color='red', linestyle='--', alpha=0.5, label=f'$s_0={s0_ii}$')
plt.xlabel('Time $t$', fontsize=12)
plt.ylabel('Stock Price $s_t$', fontsize=12)
plt.title(f'Case (ii): Random Walk | $s_0={s0_ii}$, $\mu={mu_ii}$, $\sigma={sigma_ii}$
          fontsize=13)
plt.grid(True, alpha=0.3)
plt.legend(fontsize=11)
plt.tight_layout()
plt.savefig('problem3_case_ii_random_walk.png', dpi=150, bbox_inches='tight')
plt.show()

# ========== PLOT 4: CASE (ii) - BROWNIAN BRIDGE ==========
t4, s4 = simulate_gbm_brownian_bridge(s0_ii, mu_ii, sigma_ii, T, dt_ii)

plt.figure(figsize=(10, 6))
plt.plot(t4, s4, 'g-', linewidth=2)
plt.axhline(s0_ii, color='red', linestyle='--', alpha=0.5, label=f'$s_0={s0_ii}$')
plt.xlabel('Time $t$', fontsize=12)
plt.ylabel('Stock Price $s_t$', fontsize=12)
plt.title(f'Case (ii): Brownian Bridge | $s_0={s0_ii}$, $\mu={mu_ii}$, $\sigma={sigma_
          fontsize=13)
plt.grid(True, alpha=0.3)
plt.legend(fontsize=11)
plt.tight_layout()
```

```
plt.savefig('problem3_case_ii_brownian_bridge.png', dpi=150, bbox_inches='tight')
plt.show()

# ========== SUMMARY STATISTICS ==========

print("PROBLEM 3: SDE STOCK PRICE SIMULATION - SUMMARY")

print("\nCASE (i): sₒ=0.35, μ=3, σ=0.1, Δt=0.05")
print(f"  Random Walk:       s(0)={s1[0]:.4f}, s(1)={s1[-1]:.4f}")
print(f"  Brownian Bridge:   s(0)={s2[0]:.4f}, s(1)={s2[-1]:.4f}")

print("\nCASE (ii): sₒ=0.5, μ=-2.5, σ=0.1, Δt=0.1")
print(f"  Random Walk:       s(0)={s3[0]:.4f}, s(1)={s3[-1]:.4f}")
print(f"  Brownian Bridge:   s(0)={s4[0]:.4f}, s(1)={s4[-1]:.4f}")

print("\n" + "="*70)
```




Case (i): Brownian Bridge | $s_0 = 0.35$, $\mu = 3$, $\sigma = 0.1$, $\Delta t = 0.05$
$s_0 = 0.35$
Stock Price s_t
Time t
0.0
0.2
0.4
0.6
0.8
1.0
1
2
3
4
5
6


Case (ii): Random Walk | $s_0 = 0.5$, $\mu = -2.5$, $\sigma = 0.1$, $\Delta t = 0.1$
$s_0 = 0.5$
Stock Price s_t
Time t
0.0
0.2
0.4
0.6
0.8
1.0
0.1
0.2
0.3
0.4
0.5




```
======================================================================
PROBLEM 3: SDE STOCK PRICE SIMULATION - SUMMARY
======================================================================

CASE (i): s₀=0.35, μ=3, σ=0.1, Δt=0.05
  Random Walk:       s(0)=0.3500, s(1)=5.3407
  Brownian Bridge:   s(0)=0.3500, s(1)=6.4837

CASE (ii): s₀=0.5, μ=-2.5, σ=0.1, Δt=0.1
  Random Walk:       s(0)=0.5000, s(1)=0.0251
  Brownian Bridge:   s(0)=0.5000, s(1)=0.0293

======================================================================
```

Problem 4: Spread Option Pricing with Monte Carlo

Suppose S_1 and S_2 are prices of two stocks and follow a correlated GBM(μ, Σ) process under the risk-neutral measure, where:

$$\mu = \begin{pmatrix} r \\ r \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

A spread option has payoff $(|S_1(T) - S_2(T)| - K)^+$ at time T. Use the Monte Carlo method to get the option price with $N = 5000$ simulations.

Parameters:

- $\sigma_1 = 0.04$
- $\sigma_2 = 0.06$
- $\rho = 0.6$
- $S_1(0) = 35$

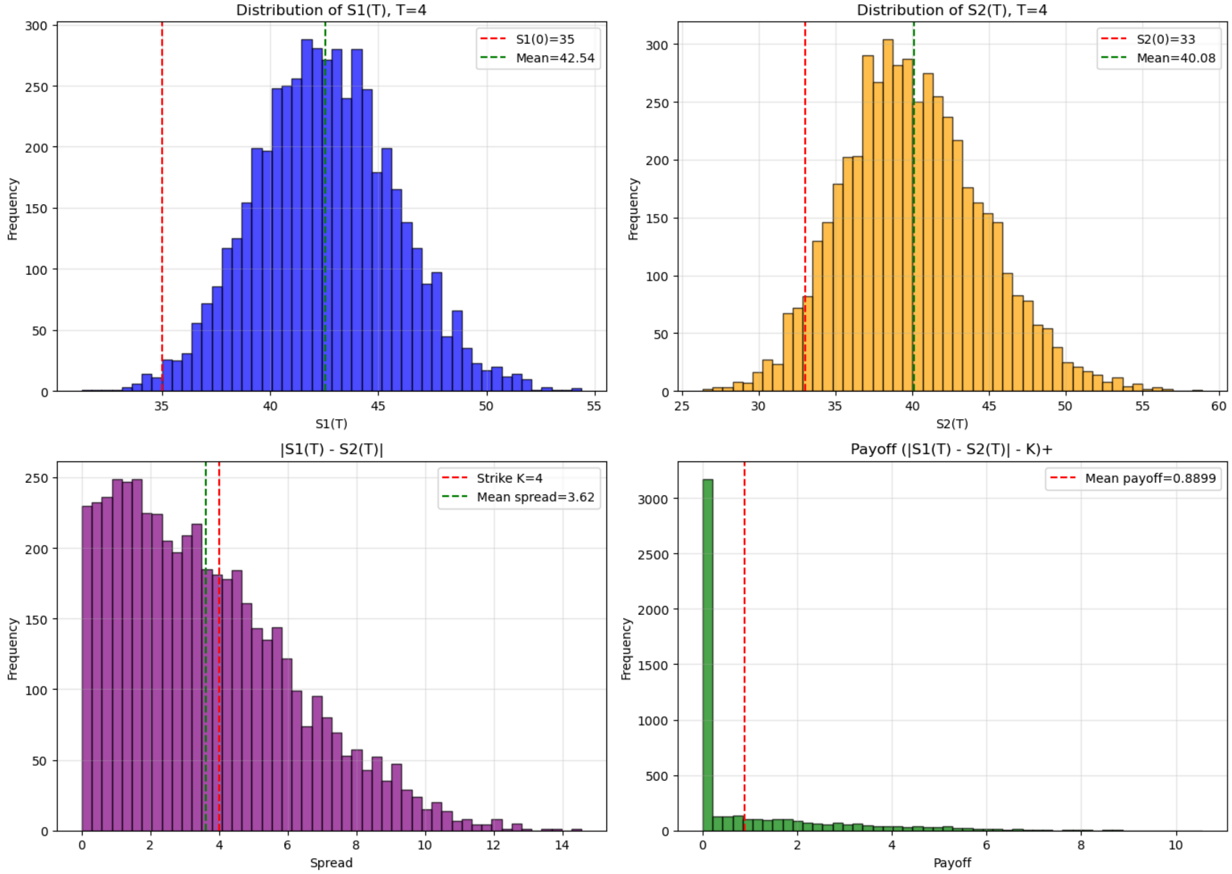
- $S_2(0) = 33$
- Risk-free rate $r = 0.05$
- Strike price $K = 4$
- Time to maturity $T = 4$
- Time step $dt = 1$

Your solution:

In [29]:
```
import numpy as np
import matplotlib.pyplot as plt

# ---------- PARAMETERS ----------
np.random.seed(42)

sigma1 = 0.04
sigma2 = 0.06
rho    = 0.6
S1_0   = 35
S2_0   = 33
r      = 0.05
K      = 4
T      = 4
dt     = 1.0
N      = 5000  # number of Monte Carlo paths

n_steps = int(T / dt)

# ---------- STORAGE ----------
S1_final = np.zeros(N)
S2_final = np.zeros(N)

# ---------- CORRELATED BROWNIAN INCREMENTS ----------
# Use CORRELATION matrix (unit variances), not covariance
corr_matrix = np.array([[1.0, rho],
                        [rho, 1.0]])
L = np.linalg.cholesky(corr_matrix)

# ---------- MONTE CARLO SIMULATION ----------
for sim in range(N):
    S1 = S1_0
    S2 = S2_0

    for _ in range(n_steps):
        # independent standard normals
        Z = np.random.randn(2)

        # correlated standard normals
        Z_corr = L @ Z

        # Brownian increments with correct volatility scaling
        dW1 = sigma1 * np.sqrt(dt) * Z_corr[0]
        dW2 = sigma2 * np.sqrt(dt) * Z_corr[1]

        # Euler-Maruyama update under risk-neutral measure
        S1 = S1 + r * S1 * dt + S1 * dW1
        S2 = S2 + r * S2 * dt + S2 * dW2
```

```
    S1_final[sim] = S1
    S2_final[sim] = S2

# ---------- PAYOFFS & PRICING ----------
spreads = np.abs(S1_final - S2_final)
payoffs = np.maximum(spreads - K, 0.0)

discount_factor = np.exp(-r * T)
option_price = discount_factor * np.mean(payoffs)

std_error = discount_factor * np.std(payoffs, ddof=1) / np.sqrt(N)
ci_low  = option_price - 1.96 * std_error
ci_high = option_price + 1.96 * std_error

# ---------- PLOTS ----------
fig, axes = plt.subplots(2, 2, figsize=(14, 10))

# S1(T)
ax = axes[0, 0]
ax.hist(S1_final, bins=50, color='blue', alpha=0.7, edgecolor='black')
ax.axvline(S1_0, color='red', linestyle='--', label=f'S1(0)={S1_0}')
ax.axvline(S1_final.mean(), color='green', linestyle='--',
           label=f'Mean={S1_final.mean():.2f}')
ax.set_title(f'Distribution of S1(T), T={T}')
ax.set_xlabel('S1(T)')
ax.set_ylabel('Frequency')
ax.legend()
ax.grid(alpha=0.3)

# S2(T)
ax = axes[0, 1]
ax.hist(S2_final, bins=50, color='orange', alpha=0.7, edgecolor='black')
ax.axvline(S2_0, color='red', linestyle='--', label=f'S2(0)={S2_0}')
ax.axvline(S2_final.mean(), color='green', linestyle='--',
           label=f'Mean={S2_final.mean():.2f}')
ax.set_title(f'Distribution of S2(T), T={T}')
ax.set_xlabel('S2(T)')
ax.set_ylabel('Frequency')
ax.legend()
ax.grid(alpha=0.3)

# spreads
ax = axes[1, 0]
ax.hist(spreads, bins=50, color='purple', alpha=0.7, edgecolor='black')
ax.axvline(K, color='red', linestyle='--', label=f'Strike K={K}')
ax.axvline(spreads.mean(), color='green', linestyle='--',
           label=f'Mean spread={spreads.mean():.2f}')
ax.set_title('|S1(T) - S2(T)|')
ax.set_xlabel('Spread')
ax.set_ylabel('Frequency')
ax.legend()
ax.grid(alpha=0.3)

# payoffs
ax = axes[1, 1]
ax.hist(payoffs, bins=50, color='green', alpha=0.7, edgecolor='black')
ax.axvline(payoffs.mean(), color='red', linestyle='--',
           label=f'Mean payoff={payoffs.mean():.4f}')
ax.set_title('Payoff (|S1(T) - S2(T)| - K)+')
ax.set_xlabel('Payoff')
```

```python
ax.set_ylabel('Frequency')
ax.legend()
ax.grid(alpha=0.3)

plt.tight_layout()
plt.show()

# ---------- TEXT OUTPUT ----------
print("PROBLEM 4: SPREAD OPTION PRICING - MONTE CARLO RESULTS (CORRECTED)")

print("\nParameters:")
print(f"  σ₁ = {sigma1}")
print(f"  σ₂ = {sigma2}")
print(f"  ρ = {rho}")
print(f"  S₁(0) = {S1_0}")
print(f"  S₂(0) = {S2_0}")
print(f"  r = {r}")
print(f"  K = {K}")
print(f"  T = {T}")
print(f"  dt = {dt}")
print(f"  N = {N}")

print("\nFinal Stock Price Statistics:")
print(f"  S₁(T): Mean = {S1_final.mean():.4f}, Std = {S1_final.std(ddof=1):.4f}")
print(f"  S₂(T): Mean = {S2_final.mean():.4f}, Std = {S2_final.std(ddof=1):.4f}")
print(f"  Empirical Correlation: {np.corrcoef(S1_final, S2_final)[0, 1]:.4f}")

print("\nSpread and Payoff Statistics:")
print(f"  Mean spread = {spreads.mean():.4f}")
print(f"  Mean payoff (undiscounted) = {payoffs.mean():.4f}")
print(f"  % ITM = {100 * (payoffs > 0).mean():.2f}%")

print("\nSPREAD OPTION PRICE:")
print(f"  Price = {option_price:.4f}")
print(f"  Std. error = {std_error:.4f}")
print(f"  95% CI = [{ci_low:.4f}, {ci_high:.4f}]")
print("="*70 + "\n")
```




```
PROBLEM 4: SPREAD OPTION PRICING - MONTE CARLO RESULTS (CORRECTED)

Parameters:
  σ₁ = 0.04
  σ₂ = 0.06
  ρ = 0.6
  S₁(0) = 35
  S₂(0) = 33
  r = 0.05
  K = 4
  T = 4
  dt = 1.0
  N = 5000

Final Stock Price Statistics:
  S₁(T): Mean = 42.5373, Std = 3.2421
  S₂(T): Mean = 40.0759, Std = 4.5519
  Empirical Correlation: 0.5949

Spread and Payoff Statistics:
  Mean spread = 3.6217
  Mean payoff (undiscounted) = 0.8899
  % ITM = 39.18%

SPREAD OPTION PRICE:
  Price = 0.7286
  Std. error = 0.0184
  95% CI = [0.6924, 0.7647]
======================================================================
```