Advanced Simulation Methods for Finance

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Cheatsheet

Monte Carlo Integration (MCI)

$$E[g(X)] = \int g(x)f(x)dx \approx \frac{1}{N}\sum_{i=1}^{N} g(x_i) \quad x_i \sim f(x), \quad \text{MSE} = \frac{1}{N}\,\text{var}\{g(X)\}$$

Law of Large Numbers (LLN): $\bar{X}_N \to \mu$ as $N \to \infty$

Simulation From Distributions

Inverse Transform Method: $U \sim \text{Unif}(0,1)$, $X = F^{-1}(U) \sim F$

Box-Muller for $N(0,1)$:

$$U_1, U_2 \sim \text{Unif}(0,1)$$
$$\theta = 2\pi U_1, \quad r = \sqrt{-2\ln(U_2)}$$
$$X = r\cos\theta, \quad Y = r\sin\theta$$

Exponential: $X = -(1/\lambda)\ln U$

Binomial: $Y = \sum_{i=1}^{n} 1\{U_i < p\}$

Sampling Techniques

Rejection Sampling:

$$\text{Sample } Z \sim g(x), U \sim \text{Unif}(0,1)$$
$$\text{Accept } Z \text{ if } U \leq \frac{f(Z)}{cg(Z)}$$

Efficiency $= 1/c$. Choose c small.

Importance Sampling:

$$\hat{E}[a(X)] = \frac{1}{N}\sum_{j=1}^{N} a(z_j)\omega_j, \ \omega_j = \frac{f(z_j)}{g(z_j)}, \ z_j \sim g(x)$$

Gibbs Sampling (MCMC):

$$(x^{(0)}, y^{(0)}, z^{(0)})$$
$$x^{(k+1)} \sim f(x|y^{(k)}, z^{(k)})$$
$$y^{(k+1)} \sim f(y|x^{(k+1)}, z^{(k)})$$
$$z^{(k+1)} \sim f(z|x^{(k+1)}, y^{(k+1)})$$

Burn-in: Discard first N iterations.

Metropolis-Hastings:

$$z \sim g(\cdot|x^{(k)})$$
$$\alpha = \min\left(1, \frac{f(z)g(x^{(k)}|z)}{f(x^{(k)})g(z|x^{(k)})}\right)$$
$$x^{(k+1)} = z \text{ with prob } \alpha; \ x^{(k+1)} = x^{(k)} \text{ otherwise}$$

Correlation Measures

Pearson:

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}$$

Spearman: Based on ranks, $\rho_S(X,Y) = \rho(F_1(X), F_2(Y))$

Kendall: $\tau_K = P[(X-X')(Y-Y') > 0] - P[(X-X')(Y-Y') < 0]$

Copulas

Definition: $F(x,y) = C(F_1(x), F_2(y))$

Gaussian:

$$\text{Simulate } z \sim N(0,I) \to z^* = Az$$
$$u_i = \Phi(z_i^*/\sigma_i)$$
$$x_i = F_i^{-1}(u_i)$$

Cholesky: $\Sigma = AA^T$

Archimedean: $C(u,v) = h^{-1}(h(u) + h(v))$

Student t Copula: Use scale from χ^2 variable.

Bootstrap

Algorithm: Resample with replacement from $\{x_1, ..., x_n\}$, create B samples, compute θ^*.

Standard Error: $SE_B = \sqrt{(1/B)\sum_{i=1}^{B}(\theta_i^* - \bar{\theta}^*)^2}$

Bias: $Bias_B = (1/B)\sum \theta_i^* - \hat{\theta}$

Aliases: CI symmetric $[\theta^*_{\alpha/2}, \theta^*_{1-\alpha/2}]$; Asymmetric $[2\hat{\theta} - \theta^*_{1-\alpha/2}, 2\hat{\theta} - \theta^*_{\alpha/2}]$

Bayesian Inference

Bayes Formula:

$$p(\theta|y) = \frac{p(y|\theta)\pi(\theta)}{\int p(y|\theta)\pi(\theta)d\theta}$$

Posterior $\propto$ Likelihood $\times$ Prior

Prior types: **Conjugate** — **Elicited** — **Noninformative** Credible intervals: $P(\theta \in C|y) \geq 1 - \alpha$ Bayes factor: $BF = p(y|M_1)/p(y|M_2)$ (Jeffreys scale interpretation)

Bayesian computing: Sample from $p(\theta|y)$ using MCMC/Gibbs/Metropolis-Hastings

Practical Tips

- Monte Carlo: Larger N yields better results; $\sqrt{N}$ convergence rate
- Rejection sampling: Pick g close to f; minimize c
- Always check MCMC convergence; discard burn-in
- Bootstrap: $B = 1000-5000$ typically
- Gaussian copula for normal-like marginals; t copula for tail dependence
- Use informative priors in Bayesian models when available

Computational Complexity

$$\text{MCI: } O(N)$$
$$\text{Rejection: } O(N \cdot c)$$
$$\text{Gibbs: } O(Kp) \text{ (dimensions } p\text{, iterations } K\text{)}$$
$$\text{Bootstrap: } O(B \cdot cost(\hat{\theta}))$$

Pitfalls

- Curse of Dimensionality: MC poorly scales in high dimensions
- MCMC: Not discarding burn-in increases bias
- Bootstrap: May break dependence structure
- Rejection: Large c makes the algorithm inefficient
- Importance: Poor g increases estimator variance
- Bayesian: Improper priors can yield improper posteriors

Key Formulas

Multivariate Normal:

$$X \sim N(\mu, \Sigma)$$

Simulate: $X = \mu + Az$, $z \sim N(0, I)$, $\Sigma = AA^T$

MCMC Convergence

$$\text{Invariant: } f(x_{k+1}) = \int p(x_{k+1}|x_k) f(x_k) dx_k$$
$$\text{Required: } \pi\text{-irreducible, aperiodic;}$$

Converges: $X_k \to f(x)$ as $k \to \infty$

Bootstrap CLT: $(\theta^* - \hat{\theta})/SE_{\hat{\theta}} \sim (\hat{\theta} - \theta_0)/SE_{\theta_0}$

End of Cheatsheet · Advanced Simulation Methods for Finance