

# Homework 1: Math 481

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## Problem 4.8

Find the expected value of the random variable  $X$  whose probability density is given by

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

## Solution

To find the expected value of the random variable  $X$ , we can use the formula for the expected value of a continuous random variable:

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

Since  $f(x)$  is non-zero only in the interval  $(0, 2)$ , we can split the integral into two parts:

$$E[X] = \int_0^1 xf(x)dx + \int_1^2 xf(x)dx$$

Substituting the expression for  $f(x)$ , we have:

$$E[X] = \int_0^1 x^2 dx + \int_1^2 x(2 - x)dx$$

Now we can evaluate each integral separately.

$$E[X] = \frac{1}{3} + \frac{2}{3} = 1$$

## Problem 4.19

Find  $\mu, \mu'_2$  and  $\sigma^2$  for the random variable  $X$  with the following probability density function:

$$f(x) = \begin{cases} x/2 & \text{for } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

## Solution

To find the  $\mu$  which is the 1st moment about the mean, we can say that it is by definition: 0. This is because it is  $E[X - \mu] = E[X] - E[\mu] = \mu - \mu = 0$ .

If in this case  $\mu$  denotes the 1st moment about the origin, we can find it as follows:

$$\mu = E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_0^2 x \cdot \frac{x}{2} dx = \int_0^2 \frac{x^2}{2} dx = \left[ \frac{x^3}{6} \right]_0^2 = \frac{8}{6} = \frac{4}{3}$$

Next, we can find  $\mu'_2$  which is the 2nd moment about the origin:

$$\mu'_2 = E[X^2] = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^2 x^2 \cdot \frac{x}{2} dx = \int_0^2 \frac{x^3}{2} dx = \left[ \frac{x^4}{8} \right]_0^2 = \frac{16}{8} = 2$$

Finally, we can find the variance  $\sigma^2$ :

$$\sigma^2 = \mu'_2 - \mu^2 = 2 - \left( \frac{4}{3} \right)^2 = 2 - \frac{16}{9} = \frac{18}{9} - \frac{16}{9} = \frac{2}{9}$$

## Problem 4.23

If the random variable  $X$  has the mean  $\mu$  and standard deviation  $\sigma$ , show that the random variable  $Z$  whose values are related to those of  $X$  by the equation  $Z = \frac{X - \mu}{\sigma}$  has  $E[Z] = 0$  and  $\text{var}[Z] = 1$ .

A distribution that has the mean 0 and the variance 1 is said to be in standard form, and when we perform the above change of variable, we are said to be standardizing the distribution of  $X$ .

## Solution

To find the expected value of  $Z$ , we can use the linearity of expectation:

$$E[Z] = E\left[\frac{X - \mu}{\sigma}\right] = \frac{1}{\sigma} E[X - \mu] = \frac{1}{\sigma} (E[X] - \mu) = \frac{1}{\sigma} (\mu - \mu) = 0$$

Next, we can find the variance of  $Z$ :

$$\text{var}[Z] = \text{var}\left[\frac{X - \mu}{\sigma}\right] = \frac{1}{\sigma^2} \text{var}[X] = \frac{1}{\sigma^2} \sigma^2 = 1$$

Thus, we have shown that  $E[Z] = 0$  and  $\text{var}[Z] = 1$ .

## Problem 4.33

Find the moment generating function of the continuous random variable  $X$  with the following probability density function:

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and use it to find  $\mu'_1, \mu'_2$ , and  $\sigma^2$ .

### Solution

The moment generating function  $M_X(t)$  is defined as:

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx = \int_0^1 e^{tx} dx$$

Now we can evaluate the integral:

$$M_X(t) = \left[ \frac{e^{tx}}{t} \right]_0^1 = \frac{e^t - 1}{t}$$

Next, we can find the first moment about the origin  $\mu'_1$ :

$$\mu'_1 = M'_X(0) = \frac{d}{dt} \left( \frac{e^t - 1}{t} \right) \Big|_{t=0} = \frac{(t-1)e^t - 1}{t^2} \Big|_{t=0} = \frac{1}{2}$$

Next, we can find the second moment about the origin  $\mu'_2$ :

$$\mu'_2 = M''_X(0) = \frac{d^2}{dt^2} \left( \frac{e^t - 1}{t} \right) \Big|_{t=0} = \frac{(t^2 - 2t + 2)e^t - 2}{t^3} \Big|_{t=0} = \frac{1}{3}$$

Finally, we can find the variance  $\sigma^2$ :

$$\sigma^2 = \mu'_2 - \mu'^2_1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

### Problem 4.40

Given the moment generating function  $M_X(t) = e^{3t+8t^2}$ , find the moment generating function of the random variable  $Z = \frac{1}{4}(X - 3)$  and use it to determine the mean and variance of  $Z$ .

### Solution

To find the moment generating function of  $Z$ , we can use the following relationship:

$$M_Z(t) = M_{\frac{1}{4}(X-3)}(t)$$

Using the property of moment generating functions, we have:

$$M_Z(t) = e^{-3t/4} M_X(t/4) = e^{-3t/4} e^{3(t/4) + 8(t/4)^2} = e^{t^2/2}$$

Next, we can find the mean and variance of  $Z$  using the moment generating function:

$$E[Z] = M'_Z(0) = \frac{d}{dt} e^{t^2/2} \Big|_{t=0} = te^{t^2/2} \Big|_{t=0} = 0$$

$$M_Z''(0) = \left. \frac{d^2}{dt^2} e^{t^2/2} \right|_{t=0} = (t^2 + 1) e^{t^2/2} \Big|_{t=0} = 1$$

Finally, we can find the variance  $\sigma^2$ :

$$\sigma^2 = M_Z''(0) - (M_Z'(0))^2 = 1 - 0^2 = 1$$

## Problem 4.54

If  $\text{var}(X_1) = 5, \text{var}(X_2) = 4, \text{var}(X_3) = 7, \text{cov}(X_1, X_2) = 3, \text{cov}(X_1, X_3) = -2$  and  $X_2, X_3$  are independent, find the covariance of  $Y_1 = X_1 - 2X_2 + 3X_3$  and  $Y_2 = -2X_1 + 3X_2 + 4X_3$ .

### Solution

$$\begin{aligned} \text{cov}(Y_1, Y_2) &= \text{cov}(X_1 - 2X_2 + 3X_3, -2X_1 + 3X_2 + 4X_3) \\ &= -2\text{var}(X_1) - 6\text{var}(X_2) + 12\text{var}(X_3) + 7\text{cov}(X_1, X_2) - 2\text{cov}(X_1, X_3) + \text{cov}(X_2, X_3) \\ &= -2(5) - 6(4) + 12(7) + 7(3) - 2(-2) + 0 \\ &= -10 - 24 + 84 + 21 + 4 + 0 \\ &= 75 \end{aligned}$$

## Problem 8.1

Cov of sample mean with sample mean minus a random var is 0. Hint: Use covariance properties and connection to variance. Statement use corollary of theorem 4.15 on page 136 to show that if  $X_1, X_2, \dots, X_n$  constitute a random sample from an infinite population, then  $\text{cov}(X_r - \bar{X}, \bar{X}) = 0$  for all  $i$ .

### Solution

The covariance of  $X_r - \bar{X}$  and  $\bar{X}$  can be expressed as:

$$\text{cov}(X_r - \bar{X}, \bar{X}) = -\text{var}(\bar{X}) + \text{cov}(X_r, \bar{X})$$

Since  $\bar{X}$  is the sample mean, we have:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

Thus, we can express the covariance as:

$$\text{cov}(X_r, \bar{X}) = \text{cov}\left(X_r, \frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n} \sum_{i=1}^n \text{cov}(X_r, X_i)$$

Using corollary 4 we can see that:

$$\begin{aligned} \text{Cov}(X_r, 1/n \sum X_i) - \text{Var}(\bar{X}) &= \frac{1}{n} \text{cov}(X_r, X_r) - \text{Var}(\bar{X}) \\ &= \frac{1}{n} \text{Var}(X_r) - \text{Var}(\bar{X}) \end{aligned}$$

Assuming IID, we can see that:

$$\begin{aligned} \text{Var}(\bar{X}) &= \text{cov}(1/n \sum X_i, 1/n \sum X_i) \\ &= \frac{1}{n^2} \sum \text{cov}(X_r, X_r) = \frac{1}{n^2} n \text{Var}(X) = \frac{\text{Var}(X)}{n} \end{aligned}$$

Thus we can see that:

$$\text{var}(X_r)/n - \text{var}(\bar{X}) = 0$$

Thus we have shown that  $\text{cov}(X_r - \bar{X}, \bar{X}) = 0$ .

## Problem 8.2

Independent samples from two different populations. a) is straightforward. Why? b) Use covariance connection and then properties of covariance and independence.

Use Theorem 4.14 on page 135 and its corollary to show that if  $X_{11}, X_{12}, \dots, X_{1n_1}$  and  $X_{21}, X_{22}, \dots, X_{2n_2}$  are independent random variables, with the first  $n_1$  constituting a random sample from an infinite population with the mean  $\mu_1$  and variance  $\sigma_1^2$ , and the second  $n_2$  constituting a random sample from an infinite population with mean  $\mu_2$  and variance  $\sigma_2^2$ , Then show that

- a)  $E[\bar{X}_1 - \bar{X}_2] = \mu_1 - \mu_2$   
b)  $\text{var}(\bar{X}_1 - \bar{X}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$

## Solution

**a**

By the linearity of expectation, and the fact that we have:

$$\begin{aligned} E[\bar{X}_1 - \bar{X}_2] &= E[\bar{X}_1] - E[\bar{X}_2] \\ &= E[\sum x_{1i}/n_1] - E[\sum x_{2i}/n_2] \\ &= \frac{1}{n_1} \sum E[x_{1i}] - \frac{1}{n_2} \sum E[x_{2i}] \\ &= \frac{1}{n_1} n_1 \mu_1 - \frac{1}{n_2} n_2 \mu_2 \\ &= \mu_1 - \mu_2 \end{aligned}$$

**b**

To find the variance of the difference of the two sample means, we can use the properties of variance and covariance. Since the two samples are independent, we have:

$$\begin{aligned}
 \text{var}(\bar{X}_1 - \bar{X}_2) &= \text{var}(\bar{X}_1) + \text{var}(\bar{X}_2) \\
 &= \text{var}\frac{(\sum X_1)}{n_1} + \text{var}\frac{(\sum X_2)}{n_2} \\
 &= \frac{\sum \text{var}(X_1)}{n_1^2} + \frac{\sum \text{var}(X_2)}{n_2^2} \\
 &= n_1 \frac{\sigma_1^2}{n_1^2} + n_2 \frac{\sigma_2^2}{n_2^2} \\
 &= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}
 \end{aligned}$$

### Problem 8.3

With reference to Exercise 2, show that if the two samples come from normal populations, then  $\bar{X}_1 - \bar{X}_2$  is a random variable having a normal distribution with mean  $\mu_1 - \mu_2$  and variance  $\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$ .

#### Solution

We can consider the MGF of these two independent random variables. The MGF of the difference of two independent normal random variables is given by:

$$\begin{aligned}
 M_{\bar{X}_1 - \bar{X}_2}(t) &= E[e^{t(\bar{X}_1 - \bar{X}_2)}] \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t(\bar{X}_1 - \bar{X}_2)} f_{\bar{X}_1}(x_1) f_{\bar{X}_2}(x_2) dx_1 dx_2 \\
 &= \int_{-\infty}^{\infty} e^{t\bar{X}_1} f_{\bar{X}_1}(x_1) dx_1 \cdot \int_{-\infty}^{\infty} e^{-t\bar{X}_2} f_{\bar{X}_2}(x_2) dx_2 \\
 &= M_{\bar{X}_1}(t) \cdot M_{\bar{X}_2}(-t)
 \end{aligned}$$

Since we know that both  $\bar{X}_1$  and  $\bar{X}_2$  are normally distributed, we can substitute in their MGFs:

$$\begin{aligned}
 M_{\bar{X}_1 - \bar{X}_2}(t) &= e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} \cdot e^{-\mu_2 t + \frac{\sigma_2^2 t^2}{2}} \\
 &= e^{(\mu_1 - \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}}
 \end{aligned}$$

This clearly is also the MGF of a normal distribution with mean  $\mu_1 - \mu_2$  and variance  $\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$ .