

Chapter 2

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2.1 Wave Equation

$$u_{tt} = c^2 u_{xx}, x \in \mathbb{R} \text{ ***}$$

Theorem 1: d'Alembert

Any solution to *** is of form

$$u(x, t) = f(x + ct) + g(x - ct)$$

where f and g are twice differentiable functions.

Proof: Method 1

$$(d_t^2 - c^2 d_x^2)u = 0$$

$$(d_t - cd_x)(d_t + cd_x)u = 0$$

Let $v = (d_t + cd_x)u$

Then $(d_t - cd_x)v = 0$ This is a transport equation. The general solution is

$$v(x, t) = h(x + ct)$$

for some function h .

$$u_t + cu_x = f(x + ct)$$

Inhomogeneous transport equation.

We can use linearity to find the general solution. Since it will be the sum of homogenous plus another function

$$u = u_p + g(x - ct)$$

Where u_p is a particular solution.

$$u_p = h(x + ct)$$

$$h'c + ch' = f$$

$$h' = \frac{f}{2c}$$

$$u = \frac{1}{2c} \int f(x+ct)dx + g(x-ct)$$

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$$

$$f(x+ct), g(x-ct)$$

Recall that $f(x+ct)$ is a wave traveling left at speed c and $g(x-ct)$ is a wave traveling right at speed c

Thus the solution is a wave traveling with speed c in both directions.

The wave equation is bidirectional. (unlike the transport equation)

Remark: u is a superposition of two waves traveling in opposite directions at fixed speed c .

Method 2:

Characteristic variable:

$$\xi = x + ct, \eta = x - ct$$

Then we can write

$$u(x, t) = u(\xi(x, t), \eta(x, t))$$

$$\partial_x = \partial_\xi + \partial_\eta$$

$$\partial_t = c\partial_\eta - c\partial_\xi$$

Factoring gives us

$$\partial_t^2 - c^2\partial_x^2 = (\partial_t - c\partial_x)(\partial_t + c\partial_x)$$

Plugging in we get

$$-2c\partial_\eta \cdot 2c\partial_\xi$$

$$-4c^2\partial_\xi\partial_\eta = 0$$

Thus we can differentiate with respect to ξ and η to get the general solution.

$$u_{\xi\eta} = 0$$

Integrating gives us

$$u = f(\xi) + g(\eta)$$

Theorem 2: (d'Alembert 1747)

$$\begin{cases} u_{tt} = c^2 u_{xx}, t > 0, x \in \mathbb{R} \\ u(x, 0) = \phi(x), x \in \mathbb{R} \\ u_t(x, 0) = \psi(x), x \in \mathbb{R} \end{cases}$$

Where ϕ is initial displacement and ψ is initial velocity.

$$u(x, t) = \frac{1}{2} (\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)ds$$

Proof: By theorem 1:

$$u(x, t) = f(x + ct) + g(x - ct)$$

Find f,g using ICs

$$u(x, 0) = f(x) + g(x) = \phi(x)$$

$$u_t(x, t) = cf'(x + ct) - cg'(x - ct)$$

$$u_t(x, 0) = cf'(x) - cg'(x) = \psi(x)$$

$$\begin{cases} cf' + cg' = c\phi' \\ cf' - cg' = \psi \end{cases}$$

Adding gives us

$$\begin{cases} 2cf' = c\phi' + \psi \\ 2cg' = c\phi' - \psi \end{cases}$$

$$\begin{cases} f' = \frac{1}{2} \left(\phi' + \frac{\psi}{c} \right) \\ g' = \frac{1}{2} \left(\phi' - \frac{\psi}{c} \right) \end{cases}$$

$$f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c} \int_0^s \psi(y)dy + A$$

$$g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c} \int_0^s \psi(y)dy + B$$

$$\begin{cases} f(0) = \frac{1}{2}\phi(0) + A \\ g(0) = \frac{1}{2}\phi(0) + B \end{cases}$$

$$f(0) + g(0) = \phi(0) + A + B$$

$f + g = \phi$ which cancels out

Thus we can set $A + B = 0$

$$\begin{aligned} u(x, t) &= f(\xi) + g(\eta) \\ &= \frac{1}{2}\phi(\xi) + \frac{1}{2c} \int_0^\xi \psi(y)dy + A \\ &\quad + \frac{1}{2}\phi(\eta) - \frac{1}{2c} \int_0^\eta \psi(y)dy + B \end{aligned}$$

$$\begin{aligned}
u(x, t) &= f(\xi) + g(\eta) \\
&= \frac{1}{2}\phi(\xi) + \frac{1}{2c} \int_0^\xi \psi(y) dy \\
&\quad + \frac{1}{2}\phi(\eta) - \frac{1}{2c} \int_0^\eta \psi(y) dy \\
&= \frac{\phi(\xi) + \psi(\eta)}{2} + \frac{1}{2c} \left(\int_0^\xi \psi(y) dy - \int_0^\eta \psi(y) dy \right) \\
&= \frac{\phi(\xi) + \psi(\eta)}{2} + \frac{1}{2c} \int_\eta^\xi \psi(y) dy
\end{aligned}$$

2 families of characteristics lines:

$$x \pm ct = \text{constant}$$

On one of these lines they are constant.

Example 1:

$$\psi(x) = 0$$

$$u(x, t) = \frac{1}{2} (\phi(x + ct) + \phi(x - ct))$$

Initial displacement ϕ splits into two parts. Each with half the amplitude. and one is opposite direction from the other

Example 2:

$$\psi(x) = 0$$

Interaction of "pulse".

They constructively interfere and they keep their shape.

Example 3:

$$\phi(x) = 0$$

$$u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy$$

If x is too large of magnitude it will be zero.

As we increase t , the integral will increase.

For larger t , I have more choices of x that make u nonzero.

Example 4:

$$\psi(x) = 0, \phi(s) = \sin(s)$$

$$u(x, t) = \frac{1}{2} (\sin(x + ct) + \sin(x - ct))$$

$$u(x, t) = \sin(x) \cos(ct)$$

This is a standing wave! **Remark 1:**

If $\psi = 0$ and ϕ is localized then the right and left moving waves will (in the long run) always separate.

If ϕ is not localized then they don't separate **Remark 2:** $\phi = 0$ and ψ is

localized, fix x_0 then $u(x_0, t) = 0$ for all $t > t_0$.

The initial disturbance is felt at x_0 and then the affect will be gone

2.2 Causality and Energy

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x, 0) = \phi(x) \\ u_t(x, 0) = \psi(x) \end{cases}$$

D'Alembert solution:

$$u(x, t) = \frac{1}{2} (\phi(x + ct) + \phi(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds$$

Lemma: If ϕ and ψ are compactly supported in $[a, b]$, (this means outside the interval they are zero) then $u(x, t) = 0$ outside of R where R is the "domain of influence of the interval $[a, b]$ "

The boundary lines are $x - ct = b$ and $x + ct = a$

This is like an open cone.

$R =$ region where disturbances are felt

Proof:

By assumption $\phi(s)$ and $\psi(s)$ are zero for $s < a$

Thus the outside area is $x + ct < a$

Also $x - ct \leq x + ct < a$

In that region $u(x, t) = 0$

Similarly we can do this for the region $x - ct > b$

Thus we can conclude that $u(x, t) = 0$ outside of the region R . Due to the fact that both $\phi = 0$ and the integral $\int_{x-ct}^{x+ct} \psi(s) ds = 0$

In the limiting case. This is a "light cone"

This is where we have x_0 being the point of disturbance.

The region of influence is the area where the disturbance is felt.

$$x + ct = x_0, x - ct = x_0$$

$$x \pm ct = x_0$$

$\phi(x_0)$ and $\psi(x_0)$ affect the values of $u(x, t)$ if $x \pm ct = x_0$ or $x_0 \in [x - ct, x + ct]$

Reverse Questions:

Fix value of u and see what initial conditions affect this.

We need initial values at $x_0 \pm ct_0$ and in $[x_0 - ct_0, x_0 + ct_0]$

This is like a triangle

This is called the interval/region of dependance of the point x_0 .

Remark:

$$u(x_0, t_0) = \frac{1}{2} (\phi(a) + \phi(b)) + \frac{1}{2c} \int_a^b \psi(s) ds$$

$$b - a = 2ct_0$$

$$u(x_0, t_0) = \frac{1}{2} (\phi(a) + \phi(b)) + t_0 \cdot \frac{1}{b - a} \int_a^b \psi(s) ds$$

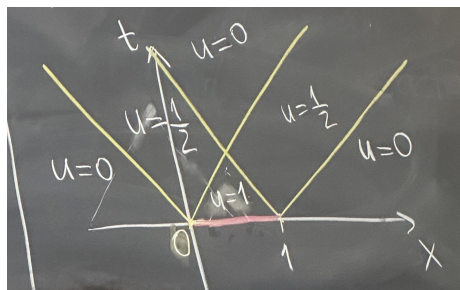
Note that the first term is average displacement and the second term is average velocity.

Example:

$$\psi = 0, \phi = \begin{cases} 1, & x \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

say $c = 1$

Now: add image of the region of influence.



We can view this as two Vs that intersect at a point and then separate.

Outside the two Vs it is 0, inside both it is 1, inside one it is 1/2

And if it is in the middle it is also 0

Finite speed of propagation:

Localized initial data ie compactly supported. (0 outside some interval)

This implies that $u(x, t)$ is also localize for any $t > 0$.

If ϕ, ψ are supported in $[a, b]$ then $u(x, t)$ is supported in $[a - ct, b + ct]$

when you are at position x you would feel the disturbance at time $t \geq \frac{x-b}{c}$ or distance from x to the interval

Remark:

The wave equation has no smoothing effect.

Singularity and discontinuities remain

Non-smooth initial data does not become smooth at any time.

Say ϕ has a discontinuity/ singularity at x_0 , then $u(x, t)$ is singular whenever $x \pm ct = x_0$

Singularity propagates along characteristics.

Causality:

Causality means the effect comes after the cause.

Cause = disturbance at x_0

Effect = disturbance at other points

Since there is a finite speed of propagation, the disturbance at x_0 can only affect points in the region of influence.

Huygen's Principle:

The disturbance in $[a, b]$ reaches x_1 at time t_1 and then it remains in the region of influence for all $t > t_1$.

If $\psi = 0$ region of influence of $[a, b]$ is just the effect is felt only at time t_1 and no lingering effects. In other words, just feel once then gone.

Kirkchoff's formula

In 3d the solution to the wave equation is given by Kirkchoff's formula.

it only uses ϕ and ψ on the boundary of the region of influence.

this is true for $n \geq 3$ and odd dimensions.

So in 2D this is not true!