01:640:423 - Homework 3

Pranav Tikkawar

October 5, 2024

1. Section 2.1 Problem 5
(The hammer blow) Let $\phi(x) \equiv 0$ and $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the string profile (u vs x) at each of the sucesses instances: $t = a/2c, a/c, 3a/2c, 2a/c,$ and $5a/c$. Hint:[Calculate

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = \frac{1}{2c} \text{ length of } (x-ct, x+ct) \cap (-a,a)$$

Then $u(x, a/2c) = (1/2c)$ length of $(x-ct, x+ct) \cap (-a,a)$ This takes on different values for $|x| < a/2$ for $a/2 < x < 3a/2$ and for $x > 3a/2$. Continue this for the other times.]
Solution:

$$u_{tt} = c^2 u_{xx}$$
$$u(x,0) = 0$$
$$u_t(x,0) = \begin{cases} 0 & \text{if } |x| \geq a \\ 1 & \text{if } |x| < a \end{cases}$$

We can use D'Alemberts formula to solve this problem. We have

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

Since $\phi(x) = 0$, we have

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

Clealry since $\psi(x) = 1$ for $|x| < a$ We only need to consider:

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} ds = \frac{1}{2c} \text{ length of } (x-ct, x+ct) \cap (-a,a)$$

Since we know that the wave function is even, ie $u(x,t) = u(-x,t)$, we only need to consider the case when $x > 0$ and for any interval that a function of x will need to have a negative x for the corresponding interval.

Case 1: $t = a/2c$
We can see that $x \pm a/2$ becomes the boundary of the interval.
We can consider the following cases:

- $x \in (-a/2, a/2)$
- $x \in (a/2, 3a/2)$
- $x \in (3a/2, \infty)$

Subcase 1: $x \in (-a/2, a/2)$

$$(x - a/2, x + a/2) \cap (-a, a) = (x - a/2, x + a/2)$$
$$u(x, a/2c) = \frac{1}{2c} \int_{x-a/2}^{x+a/2} ds = \frac{a}{2c}$$

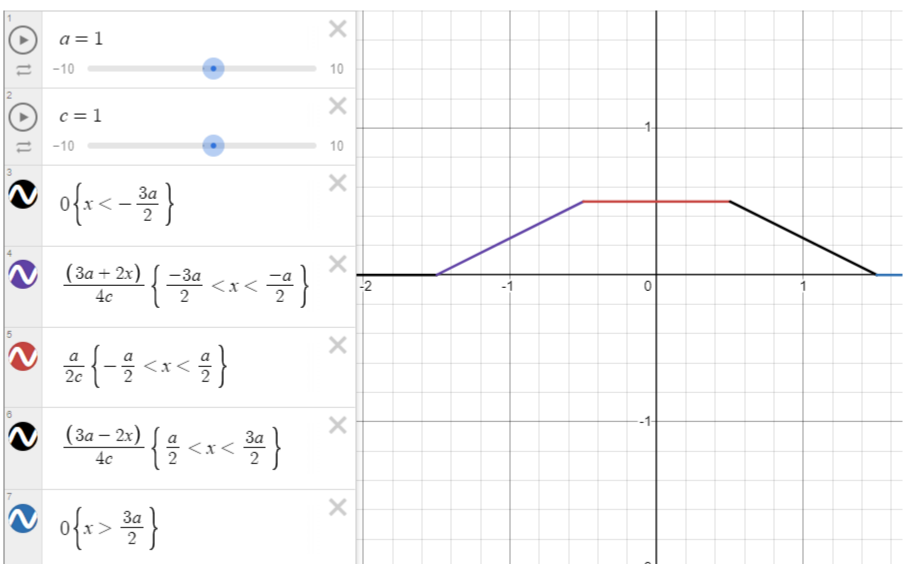
Subcase 2: $x \in (a/2, 3a/2)$

$$(x - a/2, x + a/2) \cap (-a, a) = (x - a/2, a)$$
$$u(x, a/2c) = \frac{1}{2c} \int_{x-a/2}^{a} ds = \frac{3a - 2x}{4c}$$

Subcase 3: $x \in (3a/2, \infty)$

$$(x - a/2, x + a/2) \cap (-a, a) = \emptyset$$
$$u(x, a/2c) = \frac{1}{2c} \int_{a}^{a} ds = 0$$

Plotting the graph



Case 2: $t = a/c$
We can see that $x \pm a$ becomes the boundary of the interval.
We can consider the following cases:

- $x \in (0, 2a)$
- $x \in (2a, \infty)$

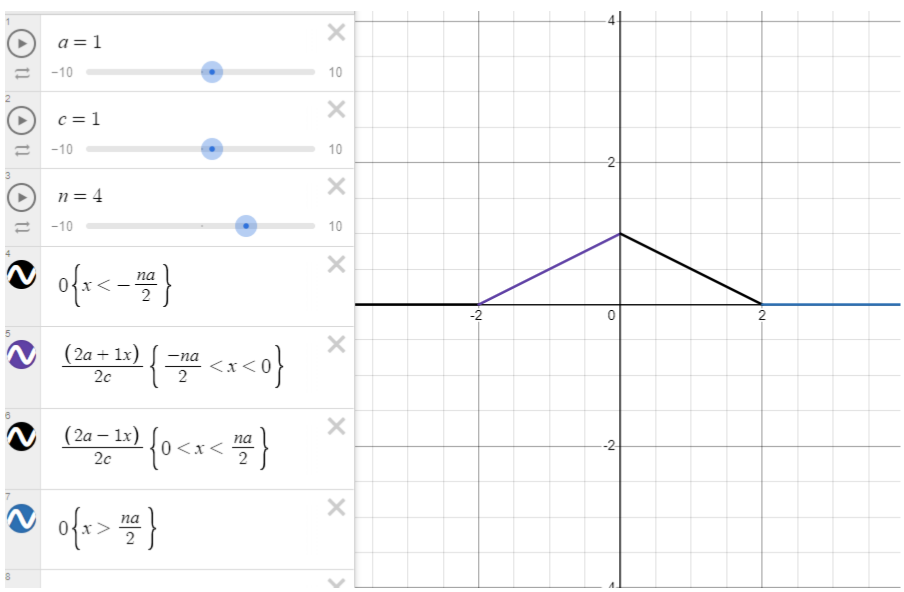
Subcase 1: $x \in (0, 2a)$

$$(x-a, x+a) \cap (-a, a) = (x-a, a)$$
$$u(x, a/c) = \frac{1}{2c}\int_{x-a}^{a} ds = \frac{2a-x}{2c}$$

Subcase 2: $x \in (2a, \infty)$

$$(x-a, x+a) \cap (-a, a) = \emptyset$$
$$u(x, a/c) = \frac{1}{2c}\int_{a}^{a} ds = 0$$

Plotting the graph



Case 3: $t = 3a/2c$
We can see that $x \pm 3a/2$ becomes the boundary of the interval.
We can consider the following cases:

- $x \in (-a/2, a/2)$
- $x \in (a/2, 5a/2)$

- $x \in (5a/2, \infty)$

Subcase 1: $x \in (-a/2, a/2)$

$$(x - 3a/2, x + 3a/2) \cap (-a, a) = (-a, a)$$
$$u(x, 3a/2c) = \frac{1}{2c}\int_{-a}^{a} ds = \frac{a}{c}$$

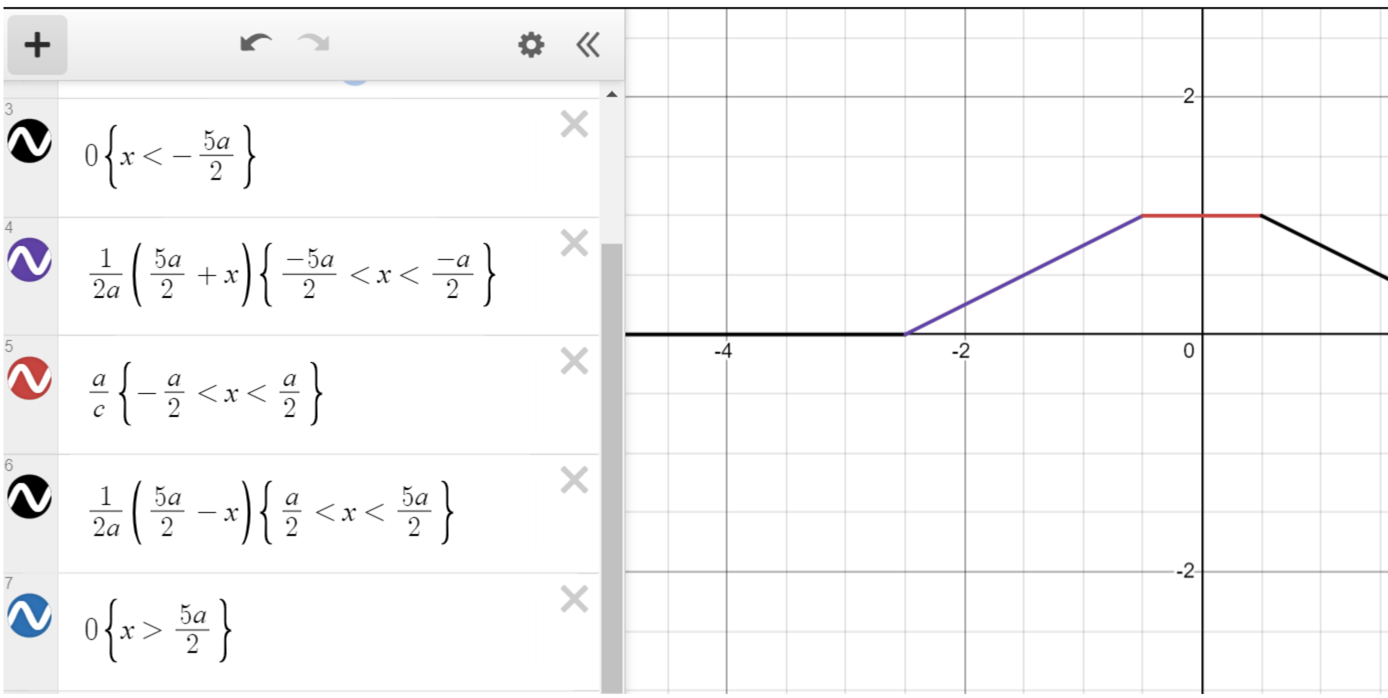
Subcase 2: $x \in (a/2, 5a/2)$

$$(x - 3a/2, x + 3a/2) \cap (-a, a) = (x - 3a/2, a)$$
$$u(x, 3a/2c) = \frac{1}{2c}\int_{x-3a/2}^{a} ds = \frac{1}{2c}(\frac{5a}{2} - x)$$

Subcase 3: $x \in (5a/2, \infty)$

$$(x - 3a/2, x + 3a/2) \cap (-a, a) = \emptyset$$
$$u(x, 3a/2c) = \frac{1}{2c}\int_{a}^{a} ds = 0$$

Plotting the graph



Case 4: $t = 2a/c$
We can see that $x \pm 2a$ becomes the boundary of the interval.
We can consider the following cases:

- $x \in (-a, a)$
- $x \in (a, 3a)$
- $x \in (3a, \infty)$

Subcase 1: $x \in (-a, a)$

$$(x - 2a, x + 2a) \cap (-a, a) = (-a, a)$$
$$u(x, 2a/c) = \frac{1}{2c} \int_{-a}^{a} ds = \frac{a}{c}$$

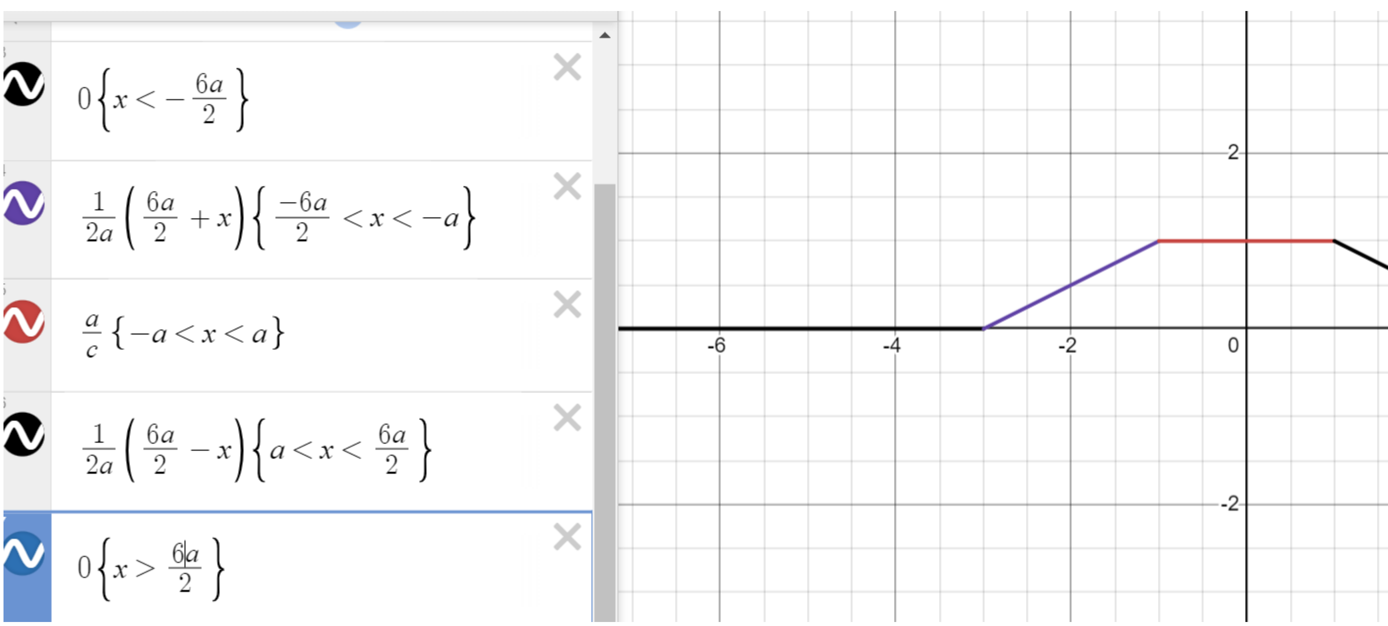
Subcase 2: $x \in (a, 3a)$

$$(x - 2a, x + 2a) \cap (-a, a) = (x - 2a, a)$$
$$u(x, 2a/c) = \frac{1}{2c} \int_{x-2a}^{a} ds = \frac{1}{2c}(3a - x)$$

Subcase 3: $x \in (3a, \infty)$

$$(x - 2a, x + 2a) \cap (-a, a) = \emptyset$$
$$u(x, 2a/c) = \frac{1}{2c} \int_{a}^{a} ds = 0$$

Plotting the graph



Case 5: $t = 5a/c$
We can see that $x \pm 5a$ becomes the boundary of the interval.
We can consider the following cases:

- $x \in (-4a, 4a)$
- $x \in (4a, 6a)$
- $x \in (6a, \infty)$

Subcase 1: $x \in (-4a, 4a)$

$$(x - 5a, x + 5a) \cap (-a, a) = (-a, a)$$
$$u(x, 5a/c) = \frac{1}{2c} \int_{-a}^{a} ds = \frac{a}{c}$$

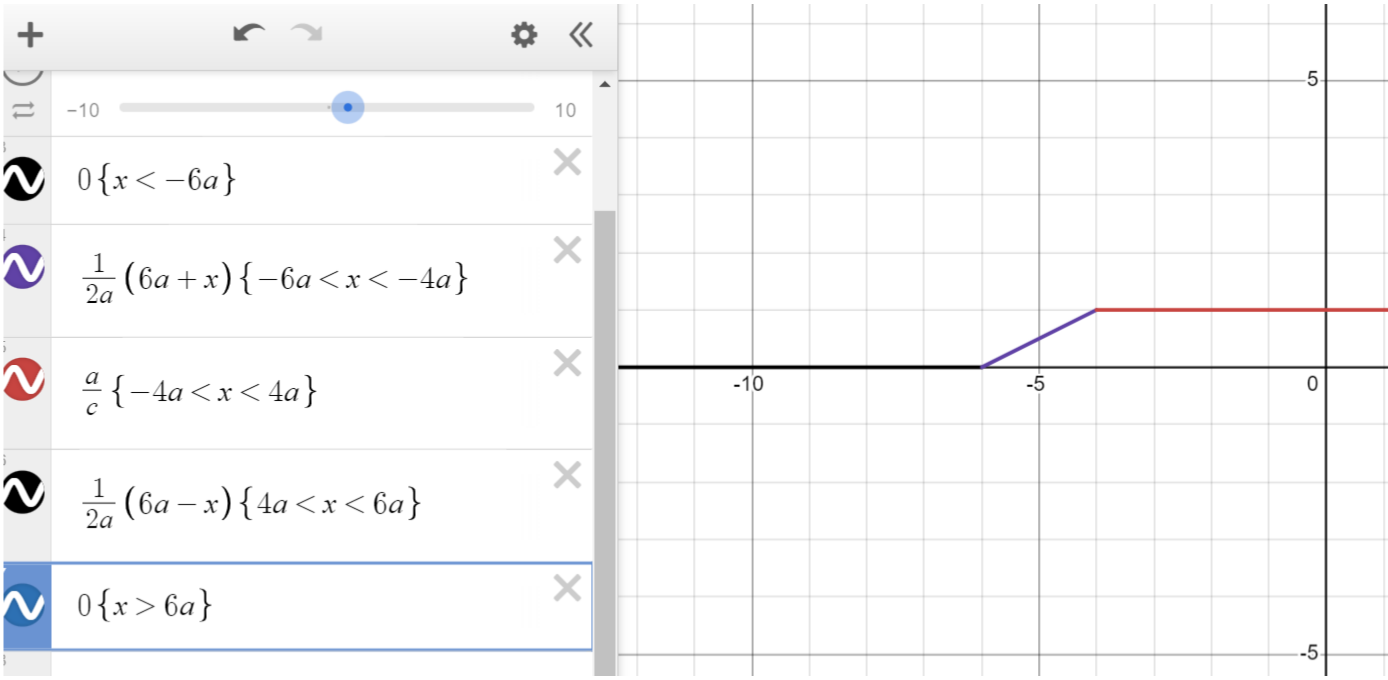
Subcase 2: $x \in (4a, 6a)$

$$(x - 5a, x + 5a) \cap (-a, a) = (x - 5a, a)$$
$$u(x, 5a/c) = \frac{1}{2c} \int_{x-5a}^{a} ds = \frac{1}{2c}(6a - x)$$

Subcase 3: $x \in (6a, \infty)$

$$(x - 5a, x + 5a) \cap (-a, a) = \emptyset$$
$$u(x, 5a/c) = \frac{1}{2c} \int_{a}^{a} ds = 0$$

Plotting the graph



2. Section 2.1 Problem 9 Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0$ for $u(x,0) = x^2$ and $u_t(x,0) = e^t$.
Solution:
To solve this we can use the factorization method. We can write the equation as:

$$\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) u = 0$$

Let $v = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}$. We can write the equation as:

$$\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right) v = 0$$

We now have a system of first second order odes:

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = v \\ -\frac{1}{4}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} = 0 \end{cases}$$

We can see that alond the curves of the tx plane:

$$\frac{dx}{dt} = -\frac{1}{4}$$

Thus we can rewrite the solution v as a function of $x + \frac{1}{4}t$ and $x(\xi, 0) = \xi$ as we know this is the solution on this characteristic curve.
We can also plug this back into the first equation to get:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = v = f(x + \frac{1}{4}t)$$

We can now solve this equation using the method of characteristics. We have:

$$\frac{dx}{dt} = 1$$

Thus $x(\eta, 0) = \eta$ Thus the equation becomes:

$$\frac{du}{dt} = f(x + \frac{1}{4}t)$$

We also know that $\eta = x - t$ Thus:

$$\frac{du}{dt} = f(\eta + \frac{5}{4}t)$$

Integraring both sides get

$$u = \int f(\eta + \frac{5}{4}t)dt + g(\eta)$$

For arbitray functions f and g. Now converting back to the original variables we get:

$$u(x,t) = \int f(x + \frac{1}{4}t)dt + g(x - t)$$

We can now use the initial conditions to solve for f and g. We have:

$$u(x,0) = x^2 = f(x)dt + g(x)$$

$$u_t(x,0) = e^t = \frac{1}{4}f'(x) + g'(x)$$

We can now solve for f and g to get the solution.
Solving for f and g
We can see that $f' + g' = 2x$ and $\frac{1}{4}f' + g' = e^t$
We can see that $f' = \frac{8}{5}x + \frac{4}{5}e^t$ and $g' = \frac{2}{5}x - \frac{4}{5}e^t$
Thus

$$f = \frac{4}{5}x^2 + \frac{4}{5}e^t$$

$$g = \frac{1}{5}x^2 - \frac{4}{5}e^t$$

Now pluggin back in the $x + \frac{1}{4}t$ and $x - t$ we get:

$$u(x,t) = x^2 + \frac{1}{4}x^2 + \frac{4}{5}e^{x-t}(e^{5t/4} - 1)$$

3. Section 2.2 Problem 2 for a solution $u(x,t)$ of the wave equation with $\rho = T = C = 1$, the enrgy density is defined as $e = \frac{1}{2}(u_t^2 + u_x^2)$ and the momentum density as $p = u_t u_x$.
a) Show that $\frac{\partial e}{\partial t} = \frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial t} = \frac{\partial e}{\partial x}$
b) Show that both $e(x,t)$ and $p(x,t)$ satisfy the wave equation.
Solution:

a:
Since we know that u solves the wave equation we have that:

$$u_{tt} = u_{xx}$$

We can now calculate the partial derivatives of e and p:

$$\frac{\partial e}{\partial t} = u_t u_{tt} + u_x u_{xt}$$

$$\frac{\partial p}{\partial x} = u_{tx} u_x + u_t u_{xx}$$

We can sub $u_{tt} = u_{xx}$ to get:

$$\frac{\partial e}{\partial t} = u_t u_{xx} + u_x u_{xt} = u_t u_{xx} + u_x u_{tx} = u_t u_{xx} + u_x u_{tt} = u_t u_{xx} + u_x u_{xx} = (u_t u_x)_x = \frac{\partial p}{\partial x}$$

Similarly we can calculate the other partial derivative:

$$\frac{\partial p}{\partial t} = u_{tt} u_x + u_t u_{xt} = u_{xx} u_x + u_t u_{xt} = u_{xx} u_x + u_{tx} u_t = u_{xx} u_x + u_{xt} u_t = (u_x u_t)_x = \frac{\partial e}{\partial x}$$

b:
Since we know from part a that

$$\begin{cases} p_t = e_x \\ e_t = p_x \end{cases}$$

We can take the t and x derivative of both sides for the top and bottom respectively

$$\begin{cases} p_{tt} = e_{xt} \\ e_{tx} = p_{xx} \end{cases}$$

Clearly p solves the wave equation.
Now if we switch the derivative to take the x and t derivatives for top and bottom respectively we get:

$$\begin{cases} p_{xt} = e_{xx} \\ e_{tt} = p_{tx} \end{cases}$$

Thus e also solves the wave equation.

4. Section 2.2 Problem 5 For a damped string, equation (1.3.3), show that the energy decreases.
The equation is defined by $u_{tt} - c^2 u_{xx} + r u_t = 0$
Solution:
We need to show that the t derivative of the $KE + PE$ is negative.
We have that the $KE = \frac{1}{2} u_t^2 = \frac{1}{2}\rho \int_R u_t^2 dx$

$$KE_t = \frac{1}{2}\rho \int_R 2 u_t u_{tt} dx$$

Upon subbing in the wave equation we get:

$$KE_t = \rho \int_R u_t(c^2 u_{xx} - ru_t)dx = \rho c^2 \int_R u_t u_{xx} dx - \rho r \int_R u_t^2 dx$$

Once we integrate by parts and consider that $c^2 = T/\rho$ we get

$$KE_t = T(u_t u_x)|_R - T \int_R u_{tx} u_x dx - \rho r \int_R u_t^2 dx$$

$$KE_t = -\frac{1}{2} T \frac{d}{dt} \int_R u_x^2 dx - \rho r \int_R u_t^2 dx$$

Potential energy is defined as

$$PE = \frac{1}{2} T \int_R u_x^2 dx$$

Thus the time derivative total energy is:

$$KE_t + PE_t = -\rho r \int_R u_t^2 dx < 0$$

Thus the energy decreases.

5. Section 2.3 Problem 4 Consider the diffusion equation $u_t = u_{xx}$ in $x \in (0,1)$ and $t \in (0,\infty)$ with $u(0,t) = u(1,t) = 0$ and $u(x,0) = 4x(1-x)$.
a) Show that $0 < u(x,t) < 1$ for all $x \in (0,1)$ and $t > 0$.
b) Show that $u(x,t) = u(1-x,t)$ for all $x \in (0,1)$ and $t > 0$.
c) Use the energy method to show that $\int_0^1 u^2 dt$ is a strictly decreasing function of t.
Solution:
a:
We can utilize the minimum and maximum principles
For the bottom bound we can use the minimum principle and see that on the Γ boundary we have $u(0,t) = u(1,t) = 0$ and $u(x,0) = 4x(1-x)$ which is clearly positive. Thus the minimum value of $u(x,t)$ is 0.
For the upper bound we can use the maximum principle and see that the maximum value on the boundary is at $x = \frac{1}{2}$ and $t = 0$ which is 1. Thus the maximum value of $u(x,t)$ is 1.
Thus $0 < u(x,t) < 1$ for all $x \in (0,1)$ and $t > 0$.
b:
We can clearly see that $u(1-x,t)$ solves the diffusion equation as

$$\frac{\partial}{\partial t} u(1-x,t) = u_t$$

$$\frac{\partial}{\partial x} u(1-x,t) = -u_x$$

$$\frac{\partial^2}{\partial x^2} u(1-x,t) = u_{xx}$$

Since $u_t = u_{xx}$ we have that $u(1-x,t)$ solves the diffusion equation.
Additionally
The range of $u(x,t)$ is $(0,1)$ and the range of $u(1-x,t)$ is also $(0,1)$
Additionally

$$u(0,t) = u(1,t) = 0 \implies u(1,t) = u(0,t) = 0$$

As well as

$$u(x,0) = 4x(1-x) \implies u(1-x,0) = 4(1-x)x$$

c:
We can first consider the diffusion equation in the form of $u_t = u_{xx}$
Then we can multiply by u on both sides to get $uu_t = uu_{xx}$
We can rewrite to get $\frac{1}{2}\frac{\partial}{\partial t}u^2 = (u_x u)_x - u_x^2$
Now integraring over the interval $(0,1)$ we get:

$$\frac{1}{2}\int_0^1 \frac{\partial}{\partial t}u^2 dx = \int_0^1 (u_x u)_x dx - \int_0^1 u_x^2 dx$$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2 dx = (u_x u)|_0^1 - \int_0^1 u_x^2 dx$$

$$\frac{d}{dt}\int_0^1 u^2 dx = -2\int_0^1 u_x^2 dx$$

Clearly since the RHS is always negative, the LHS is always negative. Thus the integral is a strictly decreasing function of t.

6. Section 2.3 Problem 6 Prove the comparison principle for the diffusion equation: If u and v are two solutions, and if $u \leq v$ for $t = 0$, for $x = 0$, and for $x = l$, then $u \leq v$ for $0 \leq t < \infty, 0 \leq x \leq l$. **Solution:**
Since we know that u and v are solutions to the diffusion equation we have that $u_t = u_{xx}$ and $v_t = v_{xx}$
Also we can consider the function $w = u - v$
Since we know that $u \leq v$ for $t = 0$, for $x = 0$, and for $x = l$, we have that $w \leq 0$ for $t = 0$, for $x = 0$, and for $x = l$
We can also consider that $w_t = u_t - v_t = u_{xx} - v_{xx} = w_{xx}$
The middle terms of the equation can factor to get: $(u-v)_t = (u-v)_{xx}$
Since clearly w solves the diffusion equation and $w \leq 0$ for $t = 0$, for $x = 0$, and for $x = l$, we have that $w \leq 0$ for $0 \leq t < \infty, 0 \leq x \leq l$
Thus we can say tat $u \leq v$ for $0 \leq t < \infty, 0 \leq x \leq l$ due to the minimum principle of $u - v$

7. Section 2.3 Problem 8 Consider the diffusion equation on $(0,l)$ with the Robin BC $u_x(0,t) - a_0 u(0,t) = 0$ and $u_x(l,t) + a_l u(l,t) = 0$ If $a_0 > 0$ and $a_l > 0$, use the energy method to show that the endpoints contribute to the decrease of $\int_0^l u^2 dx$

Solution:
The boundary conditions are:

$$u_x(0,t) - a_0 u(0,t) = 0 \to u_x(0,t) = a_0 u(0,t)$$

$$u_x(l,t) + a_l u(l,t) = 0 \to u_x(l,t) = -a_l u(l,t)$$

We can then consider the diffusion equation: $u_t = k u_{xx}$
We can multiply by u on both sides to get $uu_t = kuu_{xx}$
We can rewrite to get $\frac{1}{2}\frac{\partial}{\partial t}u^2 = (u_x u)_x - u_x^2$
Then integrating over the interval $(0,l)$ we get:

$$\frac{1}{2}\int_0^l \frac{\partial}{\partial t}u^2 dx = k\int_0^l (u_x u)_x dx - k\int_0^l u_x^2 dx$$

$$\frac{1}{2}\frac{d}{dt}\int_0^l u^2 dx = k(u_x u)|_0^l - k\int_0^l u_x^2 dx$$

$$\frac{1}{2}\frac{d}{dt}\int_0^l u^2 dx = k[u_x(l,t)u(l,t) - u_x(0,t)u(0,t)] - k\int_0^l u_x^2 dx$$

$$\frac{d}{dt}\int_0^l u^2 dx = 2k[-a_l u^2(l,t) - a_0 u^2(0,t)] - 2k\int_0^l u_x^2 dx$$

Since $a_0 > 0$ and $a_l > 0$ the lhs is all negative. Thus the endpoints contribute to the decrease of $\int_0^l u^2 dx$