01:XXX:XXX - Homework n

Pranav Tikkawar

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Missed notes: Counting proesses
$\{N(t), t >= 0\}$ They follow 3 properties:

1. $N(t) \geq 0$
2. $N(t)$ is integer valued
3. $N(t)$ is monotone increasing

$$N(t) : R \to N$$

Monotone increasing function of t

$$N(t) - N(s) = \text{ Number of events in } (t, s]$$

Little o notation
A function f is said to be little o $o(h)$ if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0$$

eg: $f(h) = h^2$ is little $o(h)$
If u add two function in little $o(h)$ then it is still little $o(h)$
Definition: A counting process $\{N(t), t \geq 0\}$ is a Poisson process if:

1. $N(0) = 0$
2. The number of events in disjoint intervals are independent.
3. $P(N(t+h) - N(t) = 1) = \lambda h + o(h)$ where λ is the rate of the Poisson process. (this mean it is dependant on the length of the interval)
4. $P(N(t+h) - N(t) \geq 2) = o(h)$

Lemma 5.1:
Let $\{N(t), t \geq 0\}$ be a Poisson process. Define $\{N_s(t), t \geq 0\}$ by $N_s(t) = N(s+t) - N(s)$
Then $\{N_s(t), t \geq 0\}$ is a Poisson process with rate λ
Proof:

$$\begin{aligned}
N_s(0) = N(s+0) - N(s) = 0 \\
(a, b) \cap (c, d) = \emptyset \\
P(N_s(b) - N_s(a) = x, N_s(d) - N_s(c) = y) \\
P(N(b-s) - N(a-s) = x, N(d-s) - N(c-s) = y) \\
P(N(b-s) - N(a-s) = x) P(N(d-s) - N(c-s) = y) \\
P(N_s(b) - N_s(a) = x) P(N_s(d) - N_s(c) = y)
\end{aligned}$$

Thus disjoint intervals are independent.

$$P(N_s(t+h) - N_s(t) = 1) = P(N(s+t+h) - N(s+t) = 1)$$

We assume N has stationary increments.

$$P(N(s+t+h) - N(s+t) = 1) = P(N(t+h) - N(t) = 1) = \lambda h + o(h)$$

Lmma 5.2:
Let $T_1 = min(t > 0 : N(t) = 1)$
it is time of arrival
T_1 is exponentially distributed with rate λ
Proof:

$$P_0(t) = P(N(t) = 0)$$
$$P_0(t+h) = P(N(t) = 0, N(t+h) - N(t) = 0)$$
$$P_0(t+h) = P(N(t) = 0)P(N(t+h) - N(t) = 0)$$
$$P_0(t+h) = P_0(t)(1 - \lambda h - 2o(h))$$

note that $-2o(h) = o(h)$ cuz it basically 0

$$P_0(t+h) = P_0(t) - \lambda h P_0(t) + o(h)P_0(t)$$

$$\frac{dP_0(t)}{t} = -\lambda P_0(t) + 0$$

This solves to with IC $P_0(0) = 1$

$$P_0(t) = e^{-\lambda t}$$

Define:
$T_n for n \geq 1$ is the time between the $(n-1)^t h$ and $n^t h$ arrival.
Proposition 5.4:
$T_1, T_2, \dots$ are independent and exponentially distributed with rate λ
Proof:
Rea book.
Remark:
Define $S_n = \sum_{i=1}^{n} T_i$
From last time, S_n has a gamma distribution with parameters n and λ

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$$

Theoremm 5.1
If $\{N(t), t \geq 0\}$ is a Poisson process with parameter λ then $N(t)$ is a poisson random variable with parameter λt

Proof:

$$\begin{aligned} P(N(t) = n) &= \int_0^\infty P(N(t) = n | S_n = t) \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} dt \\ &= P(T_{n+1} = t - s | T_1 + T_2 + \cdots + T_n = s) \\ &= P(T_{n+1} = t - s) \\ &= \frac{(\lambda t)^n e^{-\lambda t}}{n!} \end{aligned}$$

Example
Let $\{N(t), t \geq 0\}$ be a Poisson process with rate $\lambda = \frac{1}{3}$
Find:
a) $P(N(5) > N(3))$
This means there are > 0 events in $(3, 5]$

$$\begin{aligned} P(N(5) > N(3)) &= 1 - P(N(5) - N(3) = 0) \\ &= 1 - P(N(2) = 0) \\ &= 1 - e^{-\frac{2}{3}} \end{aligned}$$

b) $P(\{N(4) = 1\}, \{N(5) = 3\})$

c) $E(N(5)|N(3) = 2)$
d) $E(T_b|N(3) = 4)$
Last time we finished 5.3.2 + examples
5.3.3 Further thinning of a poisson process.
Suppose $\{N(t), t \geq 0\}$ is a Poisson process with rate λ
There are events of 2 types: 1 w/ probability p and 2 w/ probability $1 - p$
Write $N_1(t)$ for the number of type 1 events in $(0, t]$
$N_2(t)$ for the number of type 2 events in $(0, t]$
textbfProposition 5.5
$\{N_1(t), t \geq 0\}$ is a Poisson process with rate $p\lambda$ and $\{N_2(t), t \geq 0\}$ Poisson process with rate $(1 - p)\lambda$
Compound Poisson process
Suppose random variables are iid with disstribution F with mean μ and variance σ^2
The non-negative integer valued random varianle $S = \sum_{i=1}^N X_i$ is called a compound Poisson random variable.
Conditional Variance formula

$$Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$$

If N is a poison random variable with parameter λ then:

$$Var(S) = \lambda\sigma^2 + \mu^2\lambda$$

Read example 5.27