01:640:481 - Lecture 14 Workshop

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1. Consider the functions $f(x) = p^x(1-p)^{1-x}$. What are the values of $f(0)$ and $f(1)$? Notice that f is the pmf for the Bernoulli distribution with parameter p. What is the CRLB? Base in this what can you say about the unbiased estimator $\bar{X}$

Solution: We have $f(0) = p^0(1-p)^{1-0} = 1-p$ and $f(1) = p^1(1-p)^{1-1} = p$.
By the Cramer-Rao inequality we have that the variance of any unbiased estimator is

$$var(\hat{\Theta}) = \frac{1}{nE[(\frac{\partial}{\partial\Theta}\log f(X))^2]}$$

For the Bernoulli distribution, we have that the log-likelihood is given by

$$\ln f(X) = x \cdot ln(p) + (1-x) \cdot ln(1-p)$$

Taking the derivative with respect to p we get

$$\frac{\partial}{\partial p}\ln f(X) = \frac{x}{p} - \frac{1-x}{1-p}$$

Taking the square of this derivative results in

$$\left(\frac{x}{p} - \frac{1-x}{1-p}\right)^2 = \frac{x^2}{p^2} + \frac{(1-x)^2}{(1-p)^2} - 2\frac{x(1-x)}{p(1-p)}$$

Simplified we get

$$\frac{(x-p)^2}{p^2(1-p)^2}$$

We then apply the expectation operator to this expression to get

$$E\left[\frac{(x-p)^2}{p^2(1-p)^2}\right] = \frac{1}{p(1-p)}$$

Therefore, the CRLB is $\frac{p(1-p)}{n}$. Since the variance of the sample mean is $\frac{p(1-p)}{n}$, we see that the sample mean is the unbiased estimator that achieves the CRLB.