16:960:665 - Homework 3

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Problem (13)**.** Assume that $K(\cdot)$ is a complex-valued function defined on $\mathbb{Z}$, and that $K(\cdot)$ is non-negative definite.

1. Prove that $K(\cdot)$ is Hermitian, *i.e.* $K(h) = \overline{K(-h)}$.

Solution: We know that since $K(\cdot)$ is non-negative definite, thus

$$\sum_{j=1}^{n}\sum_{k=1}^{n} a_j \overline{a_k} K(j-k) \geq 0$$

for any complex numbers $a_1, a_2, \ldots, a_n$ and any positive integer n.
Let the matrix Γ be defined as

$$\Gamma_{j,k} = K(j-k)$$

for $1 \leq j, k \leq n$.
Since we know that Γ is non-negative definite, thus

$$a^* \Gamma a \geq 0$$

Then Γ is also Hermitian, which means that

$$\Gamma = \overline{\Gamma}^T$$

Thus by matching the elements of the matrices, we have

$$K(j-k) = \overline{K(k-j)}$$

for all $j, k \in \mathbb{Z}$.
Let $h = j - k$, then we have

$$K(h) = \overline{K(-h)}$$

for all $h \in \mathbb{Z}$ as desired.

2. Let $K_1(\cdot)$ and $K_2(\cdot)$ be the real and imaginary part of $K(\cdot)$, *i.e.* $K(h) = K_1(h) + iK_2(h)$ for all $h \in \mathbb{Z}$. According to Part (a), we know that $K_1(\cdot)$ is even and $K_2(\cdot)$ is odd. For any positive integer n, define the $(2n) \times (2n)$ matrix

$$L^{(n)} = \frac{1}{2}\begin{pmatrix} K_1^{(n)} & -K_2^{(n)} \\ K_2^{(n)} & K_1^{(n)} \end{pmatrix}, \quad \text{where } K_1^{(n)} := [K_1(j-k)]_{j,k=1}^n \text{ and } K_2^{(n)} := [K_2(j-k)]_{j,k=1}^n.$$

Prove that $L^{(n)}$ is symmetric and non-negative definite. [Hint. Here you need to use the non-negative definiteness of $K(\cdot)$.]

Solution: Define the vector $z = u + iv$, where $u, v \in \mathbb{R}^n$.
Note that matrix $K_1^{(n)}$ is symmetric since $K_1(h)$ is even, and matrix $K_2^{(n)}$ is skew-symmetric since $K_2(h)$ is odd.
Thus, the matrix $L^{(n)}$ is symmetric:

$$\begin{aligned}(L^{(n)})^T &= \frac{1}{2}\begin{pmatrix} (K_1^{(n)})^T & (K_2^{(n)})^T \\ -(K_2^{(n)})^T & (K_1^{(n)})^T \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} K_1^{(n)} & -K_2^{(n)} \\ K_2^{(n)} & K_1^{(n)} \end{pmatrix} = L^{(n)}.\end{aligned}$$

Now we need to show that $L^{(n)}$ is non-negative definite.
For any vector $x = [u^T, v^T]^T \in \mathbb{R}^{2n}$, define $z = u + iv \in \mathbb{C}^n$.
Then we have

$$\begin{aligned} x' L^{(n)} x &= \frac{1}{2}\sum_{jk} z_j \overline{z_k}[K_1(j-k) + iK_2(j-k)] \\ &= \frac{1}{2}\sum_{jk} z_j \overline{z_k} K(j-k) \geq 0, \end{aligned}$$

Since we know that $K(\cdot)$ is non-negative definite, the sum is non-negative for any choice of z.
Thus $L^{(n)}$ is non-negative definite as desired.

3. Let $(Y_1, \ldots, Y_n, Z_1, \ldots, Z_n)'$ be a random vector which has a multivariate normal distribution with mean zero and covariance matrix $L^{(n)}$. Define $W_t = Y_t + iZ_t$ for $1 \leq t \leq n$. Show that the covariance matrix of $(W_1, \ldots, W_n)'$ is given by $K^{(n)} := [K(j-k)]_{j,k=1}^n$.

Solution:

Let $(Y_1, \ldots, Y_n, Z_1, \ldots, Z_n)'$ be a mean-zero multivariate normal vector with covariance matrix $L^{(n)}$. Define for each $1 \le t \le n$,

$$W_t = Y_t + iZ_t.$$

The mean of W_t is zero:

$$\mathbb{E}[W_t] = \mathbb{E}[Y_t] + i\mathbb{E}[Z_t] = 0.$$

Consider the covariance between W_j and W_k:

$$\mathrm{Cov}(W_j, W_k) = \mathbb{E}[W_j \overline{W_k}].$$

Expanding W_j and $\overline{W_k}$ gives

$$W_j = Y_j + iZ_j, \quad \overline{W_k} = Y_k - iZ_k.$$

Multiplying and taking expectation:

$$\begin{aligned}
\mathbb{E}[W_j \overline{W_k}] &= \mathbb{E}[(Y_j + iZ_j)(Y_k - iZ_k)] \\
&= \mathbb{E}[Y_j Y_k] + \mathbb{E}[Z_j Z_k] + i\big(\mathbb{E}[Z_j Y_k] - \mathbb{E}[Y_j Z_k]\big).
\end{aligned}$$

Define

$$K_1(j-k) = \mathbb{E}[Y_j Y_k] + \mathbb{E}[Z_j Z_k], \qquad K_2(j-k) = \mathbb{E}[Z_j Y_k] - \mathbb{E}[Y_j Z_k].$$

Therefore,

$$\mathrm{Cov}(W_j, W_k) = K(j-k) := K_1(j-k) + iK_2(j-k).$$

The covariance matrix $K^{(n)}$ of $(W_1, \ldots, W_n)'$ is the Hermitian matrix

$$K^{(n)} = \begin{bmatrix} K(1-1) & K(1-2) & \cdots & K(1-n) \\ K(2-1) & K(2-2) & \cdots & K(2-n) \\ \vdots & \vdots & \ddots & \vdots \\ K(n-1) & K(n-2) & \cdots & K(n-n) \end{bmatrix}$$

where $K(j-k)$ is as computed above. $K^{(n)}$ is Hermitian since $K(k-j) = \overline{K(j-k)}$.

4. Apply the Kolmogorov's Existence Theorem to deduce that there exist a bivariate mean zero Gaussian process $(Y_t, Z_t)'$ such that

$$\begin{aligned}
\mathbb{E}(Y_{t+h} Y_t) = \mathbb{E}(Z_{t+h} Z_t) &= \tfrac{1}{2} K_1(h) \\
\mathbb{E}(Z_{t+h} Y_t) = -\mathbb{E}(Y_{t+h} Z_t) &= \tfrac{1}{2} K_2(h).
\end{aligned}$$

Solution: By the construction in part (c), for every n the joint distribution of $(Y_1, \dots, Y_n, Z_1, \dots, Z_n)'$ is multivariate normal with mean zero and covariance matrix $L^{(n)}$.
For any finite collection of time points $t_1, t_2, \dots, t_m$, the finite-dimensional distributions of $(Y_{t_1}, \dots, Y_{t_m}, Z_{t_1}, \dots, Z_{t_m})'$ are multivariate normal with mean zero and covariance matrix constructed similarly to $L^{(n)}$.
These finite-dimensional distributions are consistent, so by the Kolmogorov Existence Theorem, there exists a bivariate mean zero Gaussian process $(Y_t, Z_t)'$ with the specified covariance structure.

5. Show that $\{X_t = Y_t + iZ_t,\ t \in \mathbb{Z}\}$ is a complex-valued process with autocovariance function $K(\cdot)$.

Solution: We define the complex-valued process

$$X_t = Y_t + iZ_t.$$

The mean of X_t is zero:

$$\mathbb{E}[X_t] = \mathbb{E}[Y_t] + i\mathbb{E}[Z_t] = 0.$$

The autocovariance function of X_t is given by

$$\begin{aligned}
\gamma_X(h) = \operatorname{Cov}(X_{t+h}, X_t) &= \mathbb{E}[X_{t+h}\overline{X_t}] \\
&= \mathbb{E}[(Y_{t+h} + iZ_{t+h})(Y_t - iZ_t)] \\
&= \mathbb{E}[Y_{t+h}Y_t] + \mathbb{E}[Z_{t+h}Z_t] + i\big(\mathbb{E}[Z_{t+h}Y_t] - \mathbb{E}[Y_{t+h}Z_t]\big) \\
&= K_1(h) + iK_2(h) \\
&= K(h).
\end{aligned}$$

Thus, the process $\{X_t, t \in \mathbb{Z}\}$ has autocovariance function $K(\cdot)$ as desired.

Problem (14)**.** Consider n frequencies $-\pi < \lambda_1 < \lambda_2 < \cdots < \lambda_n = \pi$.

1. Let $a_1, a_2, \dots, a_n$ be complex numbers. Prove that if

$$\sum_{j=1}^{n} a_j e^{it\lambda_j} = 0 \quad \text{for all } t \in \mathbb{Z}$$

then it must hold that $a_1 = a_2 = \cdots = a_n = 0$.

Solution: Assume that

$$\sum_{j=1}^{n} a_j e^{it\lambda_j} = 0 \quad \text{for all } t \in \mathbb{Z}.$$

Now consider the function evaluated at $t = 0, 1, \dots, n-1$:

$$\begin{aligned} \sum_{j=1}^{n} a_j &= 0 \\ \sum_{j=1}^{n} a_j e^{i1\lambda_j} &= 0 \\ \sum_{j=1}^{n} a_j e^{i2\lambda_j} &= 0 \\ &\vdots \\ \sum_{j=1}^{n} a_j e^{i(n-1)\lambda_j} &= 0 \end{aligned}$$

Converting this to a matrix equation, we have

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{i\lambda_1} & e^{i\lambda_2} & \cdots & e^{i\lambda_n} \\ e^{i2\lambda_1} & e^{i2\lambda_2} & \cdots & e^{i2\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ e^{i(n-1)\lambda_1} & e^{i(n-1)\lambda_2} & \cdots & e^{i(n-1)\lambda_n} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

This is the form of a Vandermonde matrix multiplied by the vector of coefficients a_j.
It is known that the Vandermonde matrix is invertible if and only if the λ_j are distinct.
Since the λ_j are given to be distinct, the only solution to this equation is the trivial solution: $a_1 = a_2 = \cdots = a_n = 0$.

2. Let $A_1, A_2, \dots A_n$ be complex random variables, and define $X_t = \sum_{j=1}^{n} A_j e^{it\lambda_j}$. Show that the process $\{X_t,\, t \in \mathbb{Z}\}$ is real-valued if and only if $\lambda_j = -\lambda_{n-j}$ and $A_j = \bar{A}_{n-j}$ for $1 \le j < n$, and A_n is real.

Solution:

Proof. $\implies$ Assume that the process $\{X_t, t \in \mathbb{Z}\}$ is real-valued. Then for each t. Thus we know that $X_t = \overline{X_t}$.

$$X_t = \sum_{j=1}^{n} A_j e^{it\lambda_j}$$
$$\overline{X_t} = \sum_{j=1}^{n} \overline{A_j} e^{-it\lambda_j}$$
$$\sum_{j=1}^{n} A_j e^{it\lambda_j} - \sum_{j=1}^{n} \overline{A_j} e^{-it\lambda_j} = 0$$

Since $\lambda_n = \pi$, we have $e^{it\lambda_n} = e^{it\pi} = e^{-it\pi} = e^{-it\lambda_n}$.
Thus we can rewrite the above equation as

$$\sum_{j=1}^{n-1} A_j e^{it\lambda_j} - \sum_{j=1}^{n-1} \overline{A_j} e^{-it\lambda_j} + (A_n - \overline{A_n}) e^{it\pi} = 0$$

Since this holds for all t, we must have $A_n = \overline{A_n}$, i.e., A_n is real, and for $1 \leq j < n$, $A_j = \overline{A_{n-j}}$ and $\lambda_j = -\lambda_{n-j}$.

$\impliedby$ Assume that $\lambda_j = -\lambda_{n-j}$ and $A_j = \overline{A_{n-j}}$ for $1 \leq j < n$, and A_n is real. Then we can write

$$X_t = \sum_{j=1}^{n} A_j e^{it\lambda_j}$$
$$\overline{X_t} = \sum_{j=1}^{n} \overline{A_j} e^{-it\lambda_j}$$

Since $\lambda_j = -\lambda_{n-j}$ and $A_j = \overline{A_{n-j}}$ for $1 \leq j < n$, and A_n is real, we have

$$\begin{aligned}\overline{X_t} &= \sum_{j=1}^{n-1} A_{n-j} e^{it\lambda_{n-j}} + A_n e^{-it\lambda_n} \\ &= \sum_{j=1}^{n-1} A_j e^{it\lambda_j} + A_n e^{it\lambda_n} \quad \text{By Reindexing } k = n - j \\ &= X_t\end{aligned}$$

Since $X_t = \overline{X_t}$ for all t, the process $\{X_t, t \in \mathbb{Z}\}$ is real-valued.

$\square$

Problem (15)**.** Prove that if $\gamma(\cdot)$ is real, then its spectral distribution $F(\cdot)$ is symmetric in the sense

$$F(\lambda) = F(\pi^-) - F(-\lambda^-), \quad -\pi < \lambda < \pi.$$

Solution: Note that $\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} dF(\lambda)$. Since $\gamma(h)$ is real, we have $\gamma(h) = \overline{\gamma(h)} = \int_{-\pi}^{\pi} e^{-ih\lambda} dF(\lambda)$. By changing the variable $\lambda \to -\lambda$, we get $\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} dF(-\lambda)$. Comparing the two expressions, we obtain $dF(\lambda) = dF(-\lambda)$, which implies that $F(\lambda)$ is symmetric in the sense that

$$F(\lambda) = F(\pi^-) - F(-\lambda^-), \quad -\pi < \lambda < \pi.$$

Problem (16)**.** Give an expression and a plot for the spectral density of each of the following processes. [Try to plot many more for fun!]

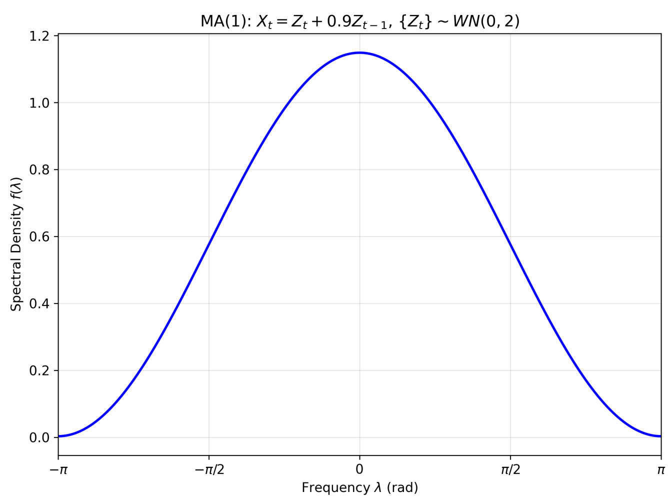
1. MA(1). $X_t = Z_t \pm 0.9 Z_{t-1}$, where $\{Z_t\} \sim \text{WN}(0, 2)$.

Solution: For a MA(1) process defined as $X_t = Z_t + \theta Z_{t-1}$, where $\{Z_t\} \sim \text{WN}(0, \sigma^2)$, the spectral density function is given by

$$f(\lambda) = \frac{\sigma^2}{2\pi} \left|1 + \theta e^{-i\lambda}\right|^2 = \frac{\sigma^2}{2\pi} \left(1 + \theta^2 + 2\theta\cos(\lambda)\right).$$

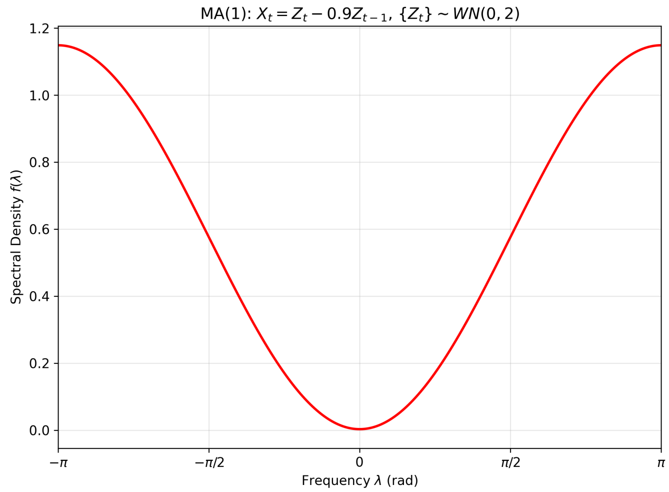
When $\theta = 0.9$ and $\sigma^2 = 2$, we have

$$f(\lambda) = \frac{2}{2\pi} \left(1 + 0.9^2 + 2 \cdot 0.9 \cos(\lambda)\right) = \frac{1}{\pi} \left(1.81 + 1.8\cos(\lambda)\right).$$



When $\theta = -0.9$ and $\sigma^2 = 2$, we have

$$f(\lambda) = \frac{2}{2\pi}\left(1 + (-0.9)^2 + 2 \cdot (-0.9)\cos(\lambda)\right) = \frac{1}{\pi}\left(1.81 - 1.8\cos(\lambda)\right).$$



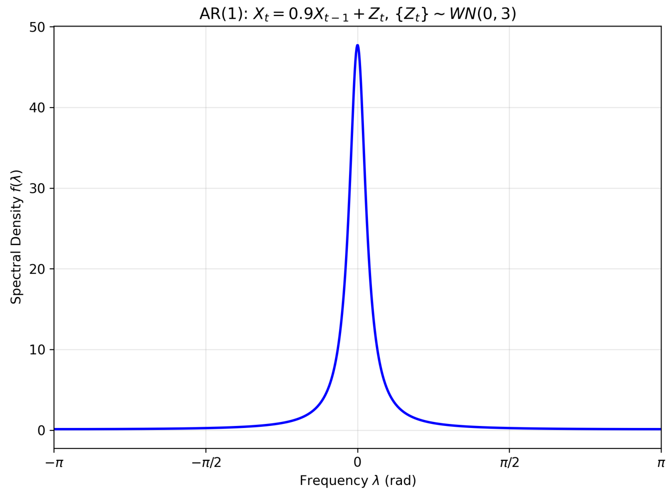
2. AR(1). $X_t = \pm 0.9X_{t-1} + Z_t$, where $\{Z_t\} \sim \text{WN}(0, 3)$.

Solution: For an AR(1) process defined as $X_t = \phi X_{t-1} + Z_t$, where $\{Z_t\} \sim \text{WN}(0, \sigma^2)$, the spectral density function is given by

$$f(\lambda) = \frac{\sigma^2}{2\pi}\frac{1}{|1 - \phi e^{-i\lambda}|^2} = \frac{\sigma^2}{2\pi}\frac{1}{1 + \phi^2 - 2\phi\cos(\lambda)}.$$

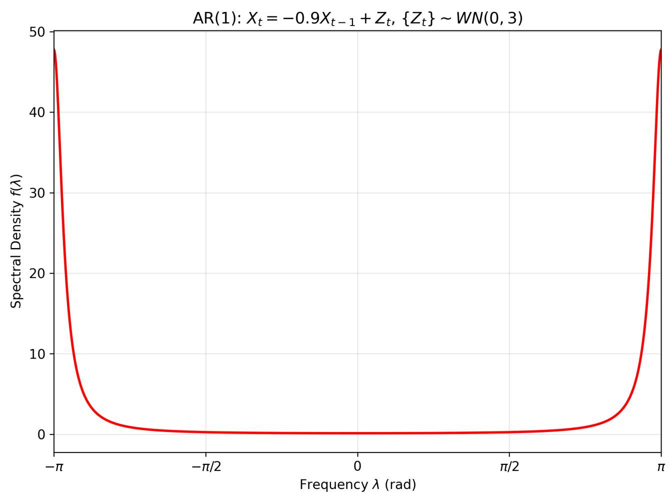
When $\phi = 0.9$ and $\sigma^2 = 3$, we have

$$f(\lambda) = \frac{3}{2\pi}\frac{1}{1 + 0.9^2 - 2 \cdot 0.9\cos(\lambda)} = \frac{3}{2\pi}\frac{1}{1.81 - 1.8\cos(\lambda)}.$$



When $\phi = -0.9$ and $\sigma^2 = 3$, we have

$$f(\lambda) = \frac{3}{2\pi}\frac{1}{1 + (-0.9)^2 - 2\cdot(-0.9)\cos(\lambda)} = \frac{3}{2\pi}\frac{1}{1.81 + 1.8\cos(\lambda)}.$$



3. Each of the processes in Problem 7:

   (i) AR(3): $r_t = 0.3 + 0.8r_{t-1} - .5r_{t-2} - .2r_{t-3} + a_t$.

   (ii) MA(3): $r_t = 0.3 + a_t + 0.8a_{t-1} - .5a_{t-2} - .2a_{t-3}$.

   (iii) ARMA(3,2): $r_t = 0.3 + 0.8r_{t-1} - .5r_{t-2} - .2r_{t-3} + a_t + 0.5a_{t-1} + 0.3a_{t-2}$.

   with $a_t \sim N(0, 4)$

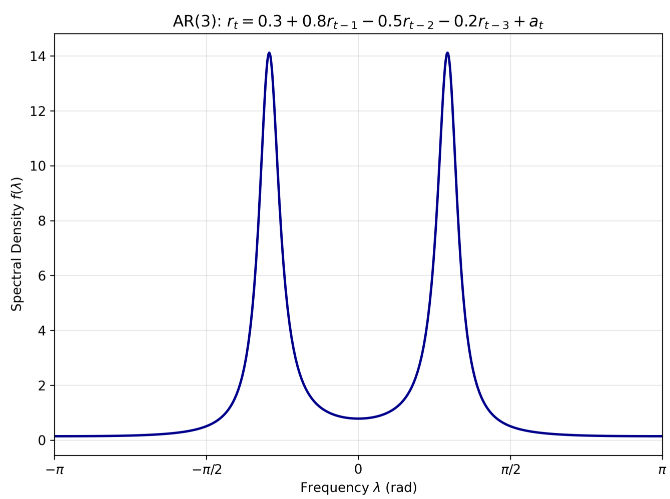
Solution:

(i) For the AR(3) process defined as $r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \phi_3 r_{t-3} + a_t$, where $\{a_t\} \sim \mathrm{WN}(0, \sigma^2)$, the spectral density function is given by

$$f(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{|1 - \phi_1 e^{-i\lambda} - \phi_2 e^{-2i\lambda} - \phi_3 e^{-3i\lambda}|^2}.$$

Substituting $\phi_1 = 0.8$, $\phi_2 = -0.5$, $\phi_3 = -0.2$, and assuming $\sigma^2 = 4$, we have

$$f(\lambda) = \frac{4}{2\pi} \frac{1}{|1 - 0.8e^{-i\lambda} + 0.5e^{-2i\lambda} + 0.2e^{-3i\lambda}|^2}.$$

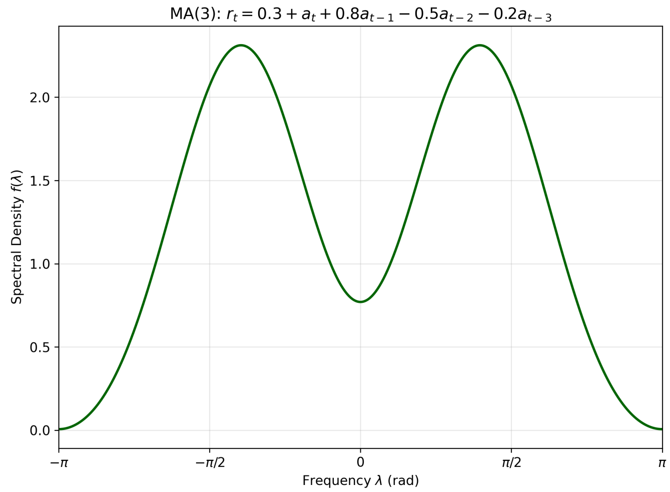


(ii) For the MA(3) process defined as $r_t = \mu + a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2} + \theta_3 a_{t-3}$, where $\{a_t\} \sim \mathrm{WN}(0, \sigma^2)$, the spectral density function is given by

$$f(\lambda) = \frac{\sigma^2}{2\pi} |1 + \theta_1 e^{-i\lambda} + \theta_2 e^{-2i\lambda} + \theta_3 e^{-3i\lambda}|^2.$$

Substituting $\theta_1 = 0.8$, $\theta_2 = -0.5$, $\theta_3 = -0.2$, and assuming $\sigma^2 = 4$, we have

$$f(\lambda) = \frac{4}{2\pi} |1 + 0.8e^{-i\lambda} - 0.5e^{-2i\lambda} - 0.2e^{-3i\lambda}|^2.$$

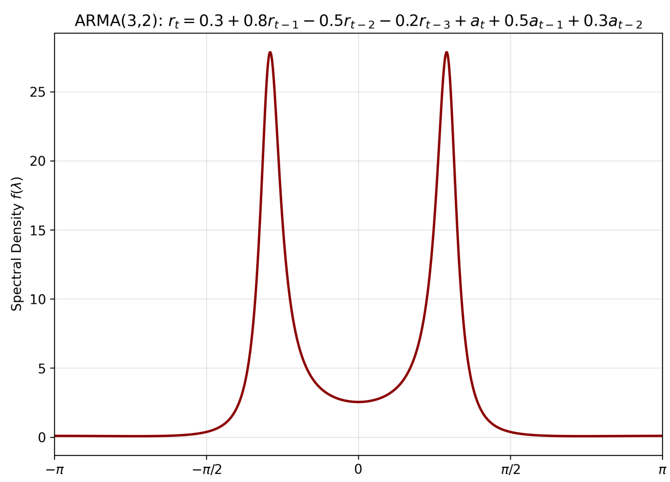


(iii) For the ARMA(3,2) process defined as $r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \phi_3 r_{t-3} + a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2}$, where $\{a_t\} \sim \mathrm{WN}(0, \sigma^2)$, the spectral density function is given by

$$f(\lambda) = \frac{\sigma^2}{2\pi} \frac{|1 + \theta_1 e^{-i\lambda} + \theta_2 e^{-2i\lambda}|^2}{|1 - \phi_1 e^{-i\lambda} - \phi_2 e^{-2i\lambda} - \phi_3 e^{-3i\lambda}|^2}.$$

Substituting $\phi_1 = 0.8$, $\phi_2 = -0.5$, $\phi_3 = -0.2$, $\theta_1 = 0.5$, $\theta_2 = 0.3$, and assuming $\sigma^2 = 4$, we have

$$f(\lambda) = \frac{4}{2\pi} \frac{|1 + 0.5e^{-i\lambda} + 0.3e^{-2i\lambda}|^2}{|1 - 0.8e^{-i\lambda} + 0.5e^{-2i\lambda} + 0.2e^{-3i\lambda}|^2}.$$



Problem (17)**.** Suppose $\gamma(\cdot)$ is a real-valued autocovariance function such that $\gamma(0) > 0$, and the covariance matrix Γ_n is singular for some $n > 1$. Find out the spectral distribution of $\gamma(\cdot)$.

Solution: Since Γ_n is singular, there exists a non-zero vector $a = (a_1, a_2, \ldots, a_n)'$ such that $\Gamma_n a = 0$. Consider the quadratic form

$$a'\Gamma_n a = \sum_{j=1}^{n}\sum_{k=1}^{n} a_j a_k \gamma(j-k) = 0.$$

Since $\gamma(\cdot)$ is an autocovariance function, it is non-negative definite. By the spectral representation theorem, we have

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} dF(\lambda)$$

for some spectral distribution function $F(\cdot)$. Substituting this into the quadratic form, we get

$$\begin{aligned} 0 &= \sum_{j=1}^{n}\sum_{k=1}^{n} a_j a_k \int_{-\pi}^{\pi} e^{i(j-k)\lambda} dF(\lambda) \\ &= \int_{-\pi}^{\pi} \left| \sum_{j=1}^{n} a_j e^{ij\lambda} \right|^2 dF(\lambda). \end{aligned}$$

Now consider

$$P(z) = \sum_{j=1}^{n} a_j z^j.$$

Since $|P(z)|^2 \geq 0$ and $dF(\lambda) \geq 0$, for the integral to be zero, $P(e^{i\lambda})$ must be zero for all λ in the support of F. Since $P(z)$ is a polynomial of degree at most n, it can have at most n distinct roots. Thus the support of F must be contained in the finite set of roots of $P(e^{i\lambda})$. Therefore, the spectral distribution $F(\cdot)$ is discrete and concentrated on the finite set of points corresponding to the roots of $P(e^{i\lambda})$.