01:640:423 - Chapter 7

Pranav Tikkawar

November 14, 2024

Greeen's identities

Works in $2D$ and $3D$
We can consider a body D and an $\vec{n}$ which is the normal to the boundary of D.

$$\partial_n u = \nabla u \cdot n$$

u, v are nice functions on $\bar{D} = Dv\partial D$
Now consider integration by parts and divergence theorm:

$$\int_D v\Delta u d\bar{x} = \int_D \nabla \cdot (v\nabla u) d\bar{x} - \int_D \nabla v \cdot \nabla u d\bar{x}$$

Theorem 1 (Green's First Identity)**.** *Since* $div(v\nabla u) = div(< vu_x, vu_y >) = \partial_x(vu_x) + \partial_y(vu_y)$
$= v_x u_x + vu_{xx} + v_y u_y + vu_{yy}$
$v\Delta u = div(v\nabla u) - \nabla v \cdot \nabla u$

$$\int_D v\Delta u d\bar{x} = \int_{\partial D} v\nabla u \cdot n d\bar{s} - \int_D \nabla v \cdot \nabla u d\bar{x}$$

We can then consider that $\int_{\partial D} v\nabla u \cdot n d\bar{s} = \int_D v\nabla u + \nabla v \cdot \nabla u d\bar{x}$
This gives us Green's first identity.

Theorem 2 (Green's second Theorem)**.** *We can do similar stuff by swithcing u and v*

$$\int_{\partial D} u\partial_n v d\bar{s} = \int_D u\Delta v + \nabla u \cdot \nabla v d\bar{x}$$

We can take the difference between the above and Green's first identity to get:

$$\int_{\partial D} (u\partial_n v - v\partial_n u) d\bar{x} = \int_D u\Delta v - v\Delta u d\bar{x}$$

This is Green's second identity.

Example. Take $v = 1$

$$\int_{\partial D} \partial_n u ds = \int_D \Delta u d\bar{x}$$

This is just the divergence theorem.

Example (Neumann problem)**.**

$$(1) \text{ is } \begin{cases} \Delta u = 0 & \text{in } D \\ \partial_n u = g & \text{on } \partial D \end{cases}$$

Need $\int_{\partial D} g ds = 0$
for the solvability of (1)

Remark.

$$\begin{cases} \Delta u = F & \text{in } D \\ u = g & \text{on } \partial D \end{cases}$$

We need

$$\int_D F d\bar{x} = \int_{\partial D} g ds$$

This is called the compatibility condition.

Remark. If u solves (1) then $u + c$ also solves (1)
Thus no uniqueness.
This is the only obstruction to uniqueness
IE if u, v solve (1) then $u - v =$ constant. This is because $w = u - v$ solves $\Delta w = 0$ and $\partial_n w = 0$
The only way to have uniqueness is to have a normalization condition ie

$$\int_{\partial D} u ds = 1$$

Theorem 3 (Mean Value Property)**.** *This was prved in 2D using the Poisson kernel. But the following is an alternative way to prove it.*

$B_r = \{\bar{x} : |\bar{x}| < r\}$
$u \in C^2(\bar{B}_r)$ *and* $\Delta u = 0$ *in* B_r
We want to prove that average of the sphere is the value at the center.

$$u(0) = (ave) \int_{\partial B_r} u ds$$

We can consider $f(r) = (ave) \int_{\partial B_r} u ds = \frac{1}{4\pi r^2} \int_{\partial B_r} u(\bar{x}) ds$
We want to rescale, since we know $|x| = r$

$$|\bar{y}| = \frac{|\bar{x}|}{r} = 1$$

Thus we get

$$f(r) = \frac{1}{4\pi r^2}\int_{\partial B_1} u(r\bar{y})r^2 ds(\bar{y}) = \frac{1}{4\pi}\int_{\partial B_1} u(r\bar{y})ds(\bar{y})$$

Now we can differntiate $f(r)$ with respect to r

$$f'(r) = \frac{1}{4\pi}\int_{\partial B_1} \nabla u(r\bar{y}) \cdot \bar{y} ds(\bar{y})$$

Clealry $\bar{y}$ is the norm to the sphere.
We can also first scale back

$$\begin{aligned} f'(r) &= \frac{1}{4\pi}\int_{\partial B_r} \nabla u(\bar{x}) \cdot \bar{x}/r ds(\bar{x})/r^2 = \\ &= \frac{1}{4\pi r^2}\int_{\partial B_r} \partial_n u ds \end{aligned}$$

We can now apply Green's identity to get

$$f'(r) = \frac{1}{4\pi r^2}\int_{B_r} \Delta u d\bar{x} = 0$$

Thus $f(r)$ is constant. for any $r \leq 1$
We can now take

$$f(1) = \lim_{r\to 0} f(r) = \lim_{r\to 0} \frac{1}{4\pi r^2}\int_{\partial B_r} u ds$$

We can see that this is $0/0$ but we can use some dirac delta type moment to get $u(0)$

$$\begin{aligned} f(1) &= \lim_{r\to 0} \frac{1}{4\pi r^2}\int_{\partial B_r} u ds \\ &= \lim_{r\to 0} \frac{1}{4\pi}\int_{\partial B_r} u(r\bar{y}) ds(\bar{y}) \\ &= u(0) \end{aligned}$$

Theorem 4 (Uniqueness for Dirichlet Problems)**.**

$$\begin{cases} \Delta u = 0 & in\ D \\ u = 0 & on\ \partial D \end{cases} \implies u = 0$$

We can prove this by maximum principle.
Alternative proof by green's first identity.
If we take $v = u$ then we get

$$\int_{\partial D} u\partial_n n ds = \int_D u\Delta u + |\nabla u|^2 d\bar{x}$$

We know $\Delta u = 0$ *and* $u = 0$ *on* ∂D

$$\int_{\partial D} 0\partial_n n ds = \int_D u0 + |\nabla u|^2 d\bar{x}$$
$$0 = \int_D |\nabla u|^2 d\bar{x}$$

This implies that $|\nabla u| = 0$ *an thus* $u =$ *constant.*
Thus by the boundary condition $u = 0$*.*

Theorem 5 (Dirichlet principle)**.**

$$\begin{cases} \Delta u = 0 & in\ D \\ u = f & on\ \partial D \end{cases}$$

Admissible set A: is the set of all functions $\in C^2(D)$ *with the same BC*

$$A = \left\{ w \in C^2(\bar{D}) : w = f \ on\ \partial D \right\}$$

We can introduce the energy functional

$$E(w) = \frac{1}{2} \int_D |\nabla w|^2 d\bar{x}$$

This is called the energy functional.
This is kinda like potential energy.
Assume $u \in A$ *then*

$$\Delta u = 0 \ in\ D \leftrightarrow E[u] = \min_{w \in A} E[w]$$

Ie the minimum energy is harmonic in D