

16:960:665 - Time Series Analysis - Homework 2

Pranav Tikkawar

November 13, 2025

Problem (6). (a) Suppose \mathcal{H} is a separable Hilbert space and $\mathcal{H} = \overline{\text{sp}}\{x_i, i = 1, 2, \infty\}$. Let x be an element of \mathcal{H} . Show that

$$\mathcal{P}_{\overline{\text{sp}}\{x_1, x_2, \dots, x_n\}}(x) \rightarrow x \quad \text{as } n \rightarrow \infty.$$

Solution: Let $V_n = \overline{\text{sp}}\{x_1, x_2, \dots, x_n\}$. Since $V_n \subseteq V_{n+1}$, we have a nested sequence of closed subspaces. Since \mathcal{H} is separable, then $\bigcup_{n=1}^{\infty} V_n$ is dense in \mathcal{H} . Therefore, for any $x \in \mathcal{H}$ and any $\epsilon > 0$, there exists an N such that for all $n \geq N$, there exists a $y_n \in V_n$ with $\|x - y_n\| < \epsilon$.

Since $\mathcal{P}_{V_n}(x)$ is the orthogonal projection of x onto V_n , it minimizes the distance from x to any point in V_n . Thus, we have:

$$\|x - \mathcal{P}_{V_n}(x)\| \leq \|x - y_n\| < \epsilon \quad \text{for all } n \geq N.$$

This shows that $\|x - \mathcal{P}_{V_n}(x)\| \rightarrow 0$ as $n \rightarrow \infty$, which implies that $\mathcal{P}_{V_n}(x) \rightarrow x$ in the norm of \mathcal{H} . Hence, we conclude that:

$$\mathcal{P}_{\overline{\text{sp}}\{x_1, x_2, \dots, x_n\}}(x) \rightarrow x \quad \text{as } n \rightarrow \infty.$$

(b) Suppose $\{X_t, t \in \mathbb{Z}\}$ is a stationary process. Show that

$$\mathcal{P}_{\overline{\text{sp}}\{X_{n-j}, 1 \leq j \leq \infty\}}(X_n) = \lim_{r \rightarrow \infty} \mathcal{P}_{\overline{\text{sp}}\{X_{n-1}, X_{n-2}, \dots, X_{n-r}\}}(X_n).$$

Solution: Let $V_r = \overline{\text{sp}}\{X_{n-1}, X_{n-2}, \dots, X_{n-r}\}$. Since $V_r \subseteq V_{r+1}$, we have a nested sequence of closed subspaces. The union $\bigcup_{r=1}^{\infty} V_r$ is dense in $V_{\infty} := \overline{\text{sp}}\{X_{n-j}, 1 \leq j \leq \infty\}$ because it includes all finite linear combinations of the X_{n-j} 's.

For any $X_n \in \mathcal{H}$, and any $\epsilon > 0$, there exists an R such that for all $r \geq R$, there exists a $Y_r \in V_r$ with $\|X_n - Y_r\| < \epsilon$. Since $\mathcal{P}_{V_r}(X_n)$ is the orthogonal projection of X_n onto V_r , it minimizes the distance from X_n to any point in V_r . Thus, we have:

$$\|X_n - \mathcal{P}_{V_r}(X_n)\| \leq \|X_n - Y_r\| < \epsilon \quad \text{for all } r \geq R.$$

This shows that $\|X_n - \mathcal{P}_{V_r}(X_n)\| \rightarrow 0$ as $r \rightarrow \infty$, which implies that $\mathcal{P}_{V_r}(X_n) \rightarrow X_n$ in the norm of \mathcal{H} . Hence, we conclude that:

$$\mathcal{P}_{\overline{\text{sp}}\{X_{n-j}, 1 \leq j \leq \infty\}}(X_n) = \lim_{r \rightarrow \infty} \mathcal{P}_{\overline{\text{sp}}\{X_{n-1}, X_{n-2}, \dots, X_{n-r}\}}(X_n).$$

Problem (7). Consider the following ARMA processes.

- (i) AR(3): $r_t = 0.3 + 0.8r_{t-1} - .5r_{t-2} - .2r_{t-3} + a_t$.
- (ii) MA(3): $r_t = 0.3 + a_t + 0.8a_{t-1} - .5a_{t-2} - .2a_{t-3}$.
- (iii) ARMA(3,2): $r_t = 0.3 + 0.8r_{t-1} - .5r_{t-2} - .2r_{t-3} + a_t + 0.5a_{t-1} + 0.3a_{t-2}$.

Assume all a_t are i.i.d $N(0, 4)$. For each of the three preceding process, do the following:

- (a) Calculate the ACF up to lag 12. [Hint. You may need to read Section 3.3 before trying (iii).]

Solution: We can approach this by using the

- (i) AR(3): $r_t = 0.3 + 0.8r_{t-1} - .5r_{t-2} - .2r_{t-3} + a_t$.

Write it in the form of: $\phi(B)r_t = a_t$ where $\phi(B) = 1 - 0.8B + 0.5B^2 + 0.2B^3$.

We can then write the system of equations for the ACF $\rho(h)$ as follows:

$$\begin{aligned} \rho(0) &= 1 \\ \rho(1) &= 0.8 - 0.5\rho(1) - 0.2\rho(2) \\ \rho(2) &= 0.8\rho(1) - 0.5 - 0.2\rho(1) \\ \rho(3) &= 0.8\rho(2) - 0.5\rho(1) - 0.2 \end{aligned}$$

We can see that by solving this we get $\rho(1) = .556$, $\rho(2) = -.167$, $\rho(3) = -.611$. For $h > 3$, we can use the recursive relation:

$$\rho(h) = 0.8\rho(h-1) - 0.5\rho(h-2) - 0.2\rho(h-3)$$

Thus we get the values:

$$\begin{aligned}
 \rho(4) &= -.517 \\
 \rho(5) &= -.074 \\
 \rho(6) &= -.321 \\
 \rho(7) &= .397 \\
 \rho(8) &= .172 \\
 \rho(9) &= -.125 \\
 \rho(10) &= -.266 \\
 \rho(11) &= -.184 \\
 \rho(12) &= -.010
 \end{aligned}$$

- (ii) MA(3): $r_t = 0.3 + a_t + 0.8a_{t-1} - .5a_{t-2} - .2a_{t-3}$.

We have $\theta(B) = 1 + 0.8B - 0.5B^2 - 0.2B^3$. The ACF for an MA(q) process is given by:

$$\begin{aligned}
 \gamma(h) &= \sigma^2 \sum_{j=0}^{q-h} \theta_j \theta_{j+h} \quad \text{for } h = 0, 1, \dots, q \\
 \gamma(h) &= 0 \quad \text{for } h > q
 \end{aligned}$$

Thus we can calculate:

$$\begin{aligned}
 \gamma(0) &= 4(1^2 + 0.8^2 + (-0.5)^2 + (-0.2)^2) = 4(1 + 0.64 + 0.25 + 0.04) = 4(1.93) = 7.72 \\
 \gamma(1) &= 4(1 * 0.8 + 0.8 * (-0.5) + (-0.5) * (-0.2)) = 4(0.8 - 0.4 + 0.1) = 4(0.5) = 2 \\
 \gamma(2) &= 4(1 * (-0.5) + 0.8 * (-0.2)) = 4(-0.5 - 0.16) = 4(-0.66) = -2.64 \\
 \gamma(3) &= 4(1 * (-0.2)) = -0.8 \\
 \gamma(h) &= 0 \quad \text{for } h > 3
 \end{aligned}$$

Now we can divide by $\gamma(0)$ to get the ACF:

$$\begin{aligned}
 \rho(0) &= 1 \\
 \rho(1) &= \frac{2}{7.72} \approx 0.259 \\
 \rho(2) &= \frac{-2.64}{7.72} \approx -0.342 \\
 \rho(3) &= \frac{-0.8}{7.72} \approx -0.104 \\
 \rho(h) &= 0 \quad \text{for } h > 3
 \end{aligned}$$

- (iii) ARMA(3,2): $r_t = 0.3 + 0.8r_{t-1} - .5r_{t-2} - .2r_{t-3} + a_t + 0.5a_{t-1} + 0.3a_{t-2}$.

We can write it in the form of: $\phi(B)r_t = c + \theta(B)a_t$ where $\phi(B) = 1 - 0.8B +$

$0.5B^2 + 0.2B^3$ and $\theta(B) = 1 + 0.5B + 0.3B^2$.

To find the ACF, we can use the formula for ARMA processes:

$$\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}$$

where $\psi(z) = \frac{\theta(z)}{\phi(z)}$.

The solution for the ACF is given by

$$\psi_0 = \theta_0 = 1$$

$$\psi_1 = \theta_1 + \phi_1 \psi_0 = 0.5 + 0.8 * 1 = 1.3$$

$$\psi_2 = \theta_2 + \phi_1 \psi_1 + \phi_2 \psi_0 = \theta_2 + \phi_2 + \theta_1 \phi_1 + \phi_1^2 = .3 + .5 + .4 + .64 = 1.84$$

$$\psi_3 = \theta_3 + \phi_1 \psi_2 + \phi_2 \psi_1 + \phi_3 \psi_0 = 0.622$$

$$\psi_4 = \phi_1 \psi_3 + \phi_2 \psi_2 + \phi_3 \psi_1 = -0.682$$

$$\psi_n = \phi_1 \psi_{n-1} + \phi_2 \psi_{n-2} + \phi_3 \psi_{n-3} \quad \text{for } n > 4$$

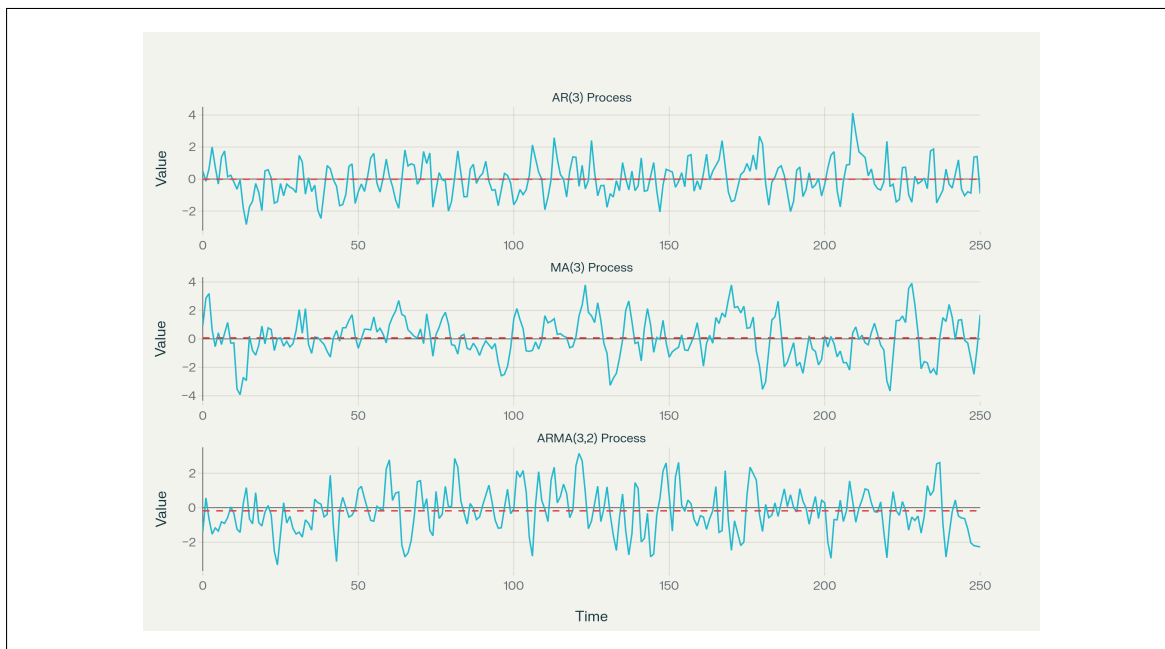
I have omitted the full calculations for brevity, but following this method we can compute the ACF values up to lag 12.

Similarly, to calculate $\rho(h)$ we can divide $\gamma(h)$ by $\gamma(0)$.

I will be omitting the full calculations for brevity as well.

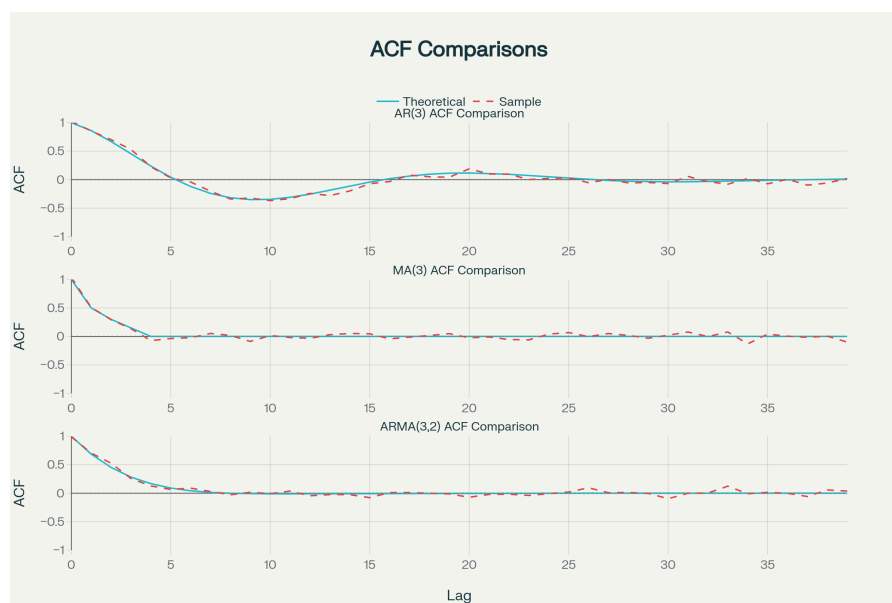
- (b) Simulate a series of length $T = 250$, give the time series plot.

Solution:



- (c) Compare the true ACF plot (plot what you obtained in Part (a)) with the sample ACF plot (use the R function `acf()`).

Solution:



The graph differences between the true and observed realization of the plots are quite significant. The main note is that the sample ACF has much more variability

due to the finite sample size of 250. The true ACF is smooth and follows the theoretical values closely, while the sample ACF shows fluctuations around the true values. This is expected as the sample ACF is an estimate based on a finite number of observations, leading to sampling variability.

Problem (8). Consider the AR(1) process $X_t = 2X_{t-1} + Z_t$, where $Z_t \sim \text{WN}(0, \sigma^2)$. Define

$$Z_t^* := .25Z_t - \frac{3}{4} \sum_{j=1}^{\infty} 2^{-j} Z_{t+j}$$

(a) Express the unique stationary solution X_t in terms of Z_t .

Solution: We can write the AR(1) process as:

$$(1 - 2B)X_t = Z_t$$

The unique stationary solution is given by:

$$\begin{aligned} X_t &= \frac{1}{1 - 2B} Z_t \\ &= -\frac{1}{2B} \frac{1}{1 - \frac{1}{2B}} Z_t \\ &= -\sum_{j=1}^{\infty} 2^{-j} Z_{t+j} \end{aligned}$$

This is the unique stationary solution for X_t in terms of Z_t . Note this is not causal.

(b) Prove that $\{Z_t^*\}$ is a white noise. What is its variance?

Solution: Mean:

$$\begin{aligned} E[Z_t^*] &= .25E[Z_t] - \frac{3}{4} \sum_{j=1}^{\infty} 2^{-j} E[Z_{t+j}] \\ &= 0 - 0 = 0 \end{aligned}$$

Variance:

$$\begin{aligned}
\text{Var}(Z_t^*) &= E[(Z_t^*)^2] \\
&= E \left[\left(.25Z_t - \frac{3}{4} \sum_{j=1}^{\infty} 2^{-j} Z_{t+j} \right)^2 \right] \\
&= E \left[\frac{1}{16} Z_t^2 - \frac{3}{8} Z_t \sum_{j=1}^{\infty} 2^{-j} Z_{t+j} + \frac{9}{16} \left(\sum_{j=1}^{\infty} 2^{-j} Z_{t+j} \right)^2 \right] \\
&= \frac{1}{16} E[Z_t^2] + \frac{3}{8} E \left[Z_t \sum_{j=1}^{\infty} 2^{-j} Z_{t+j} \right] + \frac{9}{16} E \left[\left(\sum_{j=1}^{\infty} 2^{-j} Z_{t+j} \right)^2 \right] \\
&= \frac{1}{16} \sigma^2 + 0 + \frac{9}{16} E \left[\sum_{j=1}^{\infty} 4^{-j} Z_{t+j}^2 \right] \\
&= \frac{1}{16} \sigma^2 + \frac{9}{16} \sum_{j=1}^{\infty} 4^{-j} E[Z_{t+j}^2] \\
&= \frac{1}{16} \sigma^2 + \frac{9}{16} \sum_{j=1}^{\infty} 4^{-j} \sigma^2 \\
&= \frac{1}{16} \sigma^2 + \frac{3}{16} \sigma^2 \\
&= \frac{1}{4} \sigma^2
\end{aligned}$$

(c) Prove that $X_t = .5X_{t-1} + Z_t^*$.

Solution: Note the noncausal solution for X_t from part (a):

$$X_t = - \sum_{j=1}^{\infty} 2^{-j} Z_{t+j}$$

Computing $.5 * X_{t-1}$:

$$\begin{aligned} X_{t-1} &= - \sum_{j=1}^{\infty} 2^{-j} Z_{t-1+j} \\ &= -\frac{1}{2} Z_t - \sum_{j=1}^{\infty} 2^{-j+1} Z_{t+j} \\ .5X_{t-1} &= -\frac{1}{4} Z_t - \frac{1}{4} \sum_{j=1}^{\infty} 2^{-j} Z_{t+j} \end{aligned}$$

Then $X_t - .5X_{t-1}$:

$$\begin{aligned} X_t - .5X_{t-1} &= - \sum_{j=1}^{\infty} 2^{-j} Z_{t+j} + \frac{1}{4} Z_t + \frac{1}{4} \sum_{j=1}^{\infty} 2^{-j} Z_{t+j} \\ X_t - .5X_{t-1} &= .25 Z_t - \frac{3}{4} \sum_{j=1}^{\infty} 2^{-j} Z_{t+j} \\ 'X_t - .5X_{t-1} &= Z_t^* \end{aligned}$$

Problem (9). Suppose that $\{X_t\}$ and $\{Y_t\}$ are two zero-mean stationary processes with the same autocovariance function, and that Y_t is an ARMA(p, q) process.

- (a) If ϕ_1, \dots, ϕ_p are the AR coefficients for Y_t , define $W_t := X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p}$. Show that $\{W_t\}$ has an autocovariance function which is zero for lags $|h| > q$.

Solution: Since $\{Y_t\}$ and $\{X_t\}$ have the same autocovariance function $\gamma(h)$, we can express the autocovariance function of $\{W_t\}$ as follows:

$$\begin{aligned} \gamma_W(h) &= \text{Cov}(\phi(B)X_t, \phi(B)X_{t+h}) \\ &= \sum_{i,j=0}^p \phi_i \phi_j \gamma(h+j-i) \end{aligned}$$

Now since $\{Y_t\}$ is an ARMA(p, q) process, we have $\phi(B)Y_t = \theta(B)Z_t$ where $\theta(B)$ is a polynomial of degree q .

Thus define

$$\begin{aligned} \gamma_{W'}(h) &:= \text{Cov}(\phi(B)Y_t, \phi(B)Y_{t+h}) \\ &= \text{Cov}(\theta(B)Z_t, \theta(B)Z_{t+h}) \\ &= 0 \quad \text{for } |h| > q \end{aligned}$$

Since $\{X_t\}$ and $\{Y_t\}$ have the same autocovariance function, we have $\gamma_W(h) = \gamma_{W'}(h)$. Therefore, we conclude that $\gamma_W(h) = 0$ for $|h| > q$.

(b) Apply Proposition 3.2.1 to $\{W_t\}$ to conclude that $\{X_t\}$ is also an ARMA(p, q) process.

Solution: Note that Proposition 3.2.1 states: If $\{X_t\}$ is a zero-mean stationary process with an autocovariance function $\gamma(\cdot)$ such that $\gamma(h) = 0$ for $|h| > q$, then $\{X_t\}$ is an MA(q) process.

From part (a), we have shown that $\{W_t\}$ has an autocovariance function $\gamma_W(h)$ such that $\gamma_W(h) = 0$ for $|h| > q$. Thus by Proposition 3.2.1, $\{W_t\}$ is an MA(q) process. IE it can be written as:

$$W_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \dots + \theta_q Z_{t-q}$$

where $Z_t \sim WN(0, \sigma^2)$

Now, recall the definition of W_t :

$$W_t = X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p}$$

Equating the two expressions for W_t , we have:

$$\begin{aligned} X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p} &= Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q} \\ \Rightarrow X_t &= \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q} \end{aligned}$$

Thus we have expressed X_t as an ARMA(p, q) process. Hence, we conclude that $\{X_t\}$ is also an ARMA(p, q) process.

Problem (10). Read Proposition 5.1.1 and its proof (a very nice one!) before you work on this problem. Suppose there are n observations X_1, X_2, \dots, X_n of a stationary time series. Define

$$\hat{\gamma}(h) = \begin{cases} n^{-1} \sum_{t=1}^{n-|h|} (X_{t+h} - \bar{X})(X_t - \bar{X}) & \text{if } |h| < n, \\ 0 & \text{if } |h| \geq n. \end{cases}$$

Note that although the sample autocovariannces are usually only defined for lags $|h| < n$, here $\hat{\gamma}(\cdot)$ is defined as a function on all integers, where it takes value 0 when $|h| \geq n$.

Proposition 1 (5.1.1). *If $\gamma(0) > 0$ and $\gamma(h) \rightarrow 0$ as $|h| \rightarrow \infty$, then the Covariance Matrix Γ_n is non-singular for all n .*

(a) Show that the function $\hat{\gamma}(\cdot)$ is non-negative definite.

Solution: To show that $\hat{\gamma}(\cdot)$ is non-negative definite, we need $\sum_{i=1}^m \sum_{j=1}^m a_i a_j \hat{\gamma}(i-j) \geq 0$ for any finite set of real numbers a_1, a_2, \dots, a_m . Consider:

$$Q = \sum_{i=1}^m \sum_{j=1}^m a_i a_j \hat{\gamma}(i-j)$$

$$\text{By definition} = \sum_{i=1}^m \sum_{j=1}^m a_i a_j \left(n^{-1} \sum_{t=1}^{n-|i-j|} (X_{t+i-j} - \bar{X})(X_t - \bar{X}) \right)$$

$$\text{rearranging the sums} = n^{-1} \sum_{t=1}^n \left(\sum_{i=1}^m a_i (X_t - \bar{X}) \right)^2$$

This is a sum of squares, and thus is always non-negative. Therefore, we conclude that $\hat{\gamma}(\cdot)$ is non-negative definite.

- (b) There is nothing you need to do for this part. But observe that (i) by Theorem 1.5.1, there exists some stationary process $\{Y_t\}$ of which $\hat{\gamma}(\cdot)$ is the autocovariance function; and (ii) from Proposition 3.2.1 it then follows that $\{Y_t\}$ is an $\text{MA}(n-1)$ process.

Solution: Nice!

- (c) Prove that if $\hat{\gamma}(0) > 0$, then $\hat{\Gamma}_n$ is non-singular. (In the last Homework, you showed that $\hat{\Gamma}_n$ is non-negative definite, and now you know that it is also strictly positive-definite unless the n observations are all equal.)

Solution: from part (a), we know that

$$a^T \hat{\Gamma}_n a = n^{-1} \sum_{t=1}^n \left(\sum_{i=1}^n a_i (X_t - \bar{X}) \right)^2$$

We know that since $\gamma(0) > 0$, not all X_t are equal. Therefore, there exists at least one t such that $X_t - \bar{X} \neq 0$. Thus, for any non-zero vector a , the term $\left(\sum_{i=1}^n a_i (X_t - \bar{X}) \right)^2$ will be positive for at least one t . Hence, we have:

$$a^T \hat{\Gamma}_n a > 0 \quad \text{for all non-zero } a$$

This implies that $\hat{\Gamma}_n$ is strictly positive-definite, and therefore non singular.

Problem (11).

- (a) Consider a $\text{MA}(\infty)$ process $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$, where $\{Z_t\} \sim \text{WN}(0, \sigma^2)$, and $\sum_{j=0}^{\infty} |\psi_j| < \infty$. Show that the autocovariance function $\gamma(\cdot)$ of $\{X_t\}$ satisfies $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$.

Solution: We know that the autocovariance function for an $\text{MA}(\infty)$ process is given by:

$$\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}$$

Now we can compute the sum of absolute values of the autocovariances:

$$\begin{aligned} |\gamma(h)| &= \sigma^2 \left| \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|} \right| \\ &\leq \sigma^2 \sum_{j=0}^{\infty} |\psi_j| |\psi_{j+|h|}| \quad (\text{by triangle inequality}) \\ \sum_{h=-\infty}^{\infty} |\gamma(h)| &= |\gamma(0)| + 2 \sum_{h=1}^{\infty} |\gamma(h)| \\ \sum_{h=0}^{\infty} |\gamma(h)| &\leq \sigma^2 \sum_{h=0}^{\infty} \sum_{j=0}^{\infty} |\psi_j| |\psi_{j+h}| \\ &= \sigma^2 \left(\sum_{j=0}^{\infty} |\psi_j| \right) \left(\sum_{k=0}^{\infty} |\psi_k| \right) \quad (\text{by changing index}) \\ &= \sigma^2 \left(\sum_{j=0}^{\infty} |\psi_j| \right)^2 < \infty \quad (\text{by our assumption}) \end{aligned}$$

- (b) Let $\{X_t\}$ be a causal ARMA process with autocovariance function $\gamma(\cdot)$. Show that there exist a constant $C > 0$ and another constant $s \in (0, 1)$ such that $|\gamma(h)| \leq Cs^{|h|}$ for all $h \in \mathbb{Z}$, and hence $\sum_h |\gamma(h)| < \infty$.

Solution: We know that for a causal ARMA process, the autocovariance function $\gamma(h)$ it can be expressed as an $\text{MA}(\infty)$ process:

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

where ψ_j are the coefficients of the $\text{MA}(\infty)$ representation.
The acf of this process is given by:

$$\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}$$

We know that for $h > \max(p, q)$ the acf satisfies the recursive relation:

$$\gamma(h) = \phi_1 \gamma(h-1) + \phi_2 \gamma(h-2) + \dots + \phi_p \gamma(h-p)$$

And the general solution to this is of the form:

$$\gamma(h) = \sum_{i=1}^k C_i r_i^{|h|}$$

Thus we can bound $|\gamma(h)|$ as follows:

$$\begin{aligned} |\gamma(h)| &\leq \sum_{i=1}^k |C_i| |r_i|^{|h|} \\ &\leq C s^{|h|} \quad \text{where } C = \sum_{i=1}^k |C_i| \text{ and } s = \max_i |r_i| < 1 \end{aligned}$$

Since $s \in (0, 1)$, we have:

$$\begin{aligned} \sum_{h=-\infty}^{\infty} |\gamma(h)| &\leq \sum_{h=-\infty}^{\infty} C s^{|h|} \\ &= C \left(1 + 2 \sum_{h=1}^{\infty} s^h \right) \\ &= C \left(1 + 2 \frac{s}{1-s} \right) < \infty \end{aligned}$$

Problem (12). The process $X_t = Z_t - Z_{t-1}$, where $\{Z_t\} \sim \text{WN}(0, \sigma^2)$, is not invertible according to Definition 3.1.4. Show however that $Z_t \in \overline{\text{sp}}\{X_j, -\infty < j \leq t\}$ by considering the mean square limit of the sequence $\sum_{j=0}^n (1 - j/n) X_{t-j}$ as $n \rightarrow \infty$.

Solution:

Definition (3.1.4). Suppose $\{X_t\}$ is a stationary solution of $\phi(B)X_t = \theta(B)Z_t$, it is said to be invertible if $\exists \pi_j$ such that $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and $Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$ for all $t \in \mathbb{Z}$.

We have $X_t = Z_t - Z_{t-1}$. Rearranging, we get $Z_t = X_t + Z_{t-1}$. Iterating this, we have:

$$\begin{aligned} Z_t &= X_t + X_{t-1} + Z_{t-2} \\ &= X_t + X_{t-1} + X_{t-2} + Z_{t-3} \\ &\vdots \\ &= \sum_{j=0}^n X_{t-j} + Z_{t-n-1} \end{aligned}$$

Now, consider the sequence $\sum_{j=0}^n (1 - j/n) X_{t-j}$:

$$\begin{aligned} S_n &= \sum_{j=0}^n (1 - j/n) X_{t-j} \\ &= \sum_{j=0}^n (1 - j/n) (Z_{t-j} - Z_{t-j-1}) \\ &= \sum_{j=0}^n (1 - j/n) Z_{t-j} - \sum_{j=0}^n (1 - j/n) Z_{t-j-1} \\ &= Z_t - \frac{1}{n} \sum_{j=1}^n Z_{t-j} - \left(1 - \frac{n+1}{n}\right) Z_{t-n-1} + \frac{1}{n} \sum_{j=0}^{n-1} Z_{t-j-1} \\ &= Z_t - \frac{1}{n} Z_{t-n-1} \end{aligned}$$

As $n \rightarrow \infty$, the term $\frac{1}{n} Z_{t-n-1} \rightarrow 0$ in mean square since Z_t is white noise with finite variance. Therefore, we have:

$$\lim_{n \rightarrow \infty} S_n = Z_t$$

This shows that Z_t can be expressed as the mean square limit of a sequence of linear combinations of X_j 's for $j \leq t$. Hence, we conclude that:

$$Z_t \in \overline{\text{sp}}\{X_j, -\infty < j \leq t\}$$