HW 3: Math 292

Pranav Tikkawar

February 23, 2024

1 -

a : Let $v(x) = (1-x^4)^{1/2}$, consider the sol of $x'(t) = v(x(t))$ with $x(0) = 0$

- The fucntion is not Lipschitz on the interval
- $\lim_{x_0 \to -1} |v'(x)| = \lim_{x_0 \to -1} |-2(1-x^4)^{\frac{-1}{2}} x^3| = \infty$
- $\lim_{x_0 \to 1} |v'(x)| = \lim_{x_0 \to 1} |-2(1-x^4)^{\frac{-1}{2}} x^3| = \infty$
- We see that $\frac{1}{\sqrt{1-x^4}} \leq \frac{1}{\sqrt{1-x^2}}$ and so are the integral over the same bounds
- We see that $\int_{x_0}^{1} \frac{1}{(1-x^2)^{1/2}} = sin^{-1}(1) - sin^{-1}(x_0)$
- Since $x_0 \in (-1, 1)$ the integral evaluates to a finite values as $sin^-1(x_0)$ is defined and finite
- Since the integral from barrows formula is finite, then the solution doesnt exist for all time.

b : Let $v(x) = (1-x^4)^2$, consider the sol of $x'(t) = v(x(t))$ with $x(0) = 0$

- The sol does exist within the interval as the function is Lipschitz all through the interval
- $\lim_{x_0 \to -1} |v'(x)| = \lim_{x_0 \to -1} |-8(1-x^4)x^3| = 0$
- $\lim_{x_0 \to 1} |v'(x)| = \lim_{x_0 \to 1} |-8(1-x^4)x^3| = 0$
- Also for every since $v'()$ is well defined $\forall x \in (-1, 1)$ we can say the function v is bounded by an L therefore making it Lipschitz over the interval.
- This ensures and unique and existant solution to the DE

2 -

a : Prove $\lim_{t \to \infty} |\frac{d}{dx} \Psi_t(x)| = \infty$

- $\lim_{t \to \infty} |\frac{V(\Psi_t(x))}{V(x)}| = \frac{1}{V(x)} \lim_{t \to \infty} V(\Psi_t(x))$
- Since $V(x)$ is Lipschitz then $\lim_{x \to \infty} \Psi_t(x) = \infty$
- Then $\lim_{t \to \infty} |V(\Psi_t(x))| = \infty$ as desired

b : $|x_2(t) - x_1(t)| = |\int_{x_1}^{x_2} \frac{d}{dx} \Psi(x) dx|$

- $\lim_{t\to\infty} \frac{d}{dx}\Psi(x) = \infty$ from the previous part
- $\int_{x_1}^{x_2} \lim_{t\to\infty} \frac{d}{dx}\Psi(x)$ (Source: Trust me bro)
  - Through some formuala that was talked about in the Recitation we know that $(\lim \int) \geq (\int \lim)$ and since the line above diverges that means that $\lim_{t\to\infty} |\int_{x_1}^{x_2} \frac{d}{dx}\Psi(x)dx|$ also diverges
- This limit evaluates to ∞

3 -

a - $x' = Ax$ where $A = \begin{bmatrix} -4 & 2 \\ 5 & -1 \end{bmatrix}$

- Eigenvalues are $\lambda = -6, 1$
- Corresponding Eigenvectors are $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$
- $M(t) = \begin{bmatrix} 2e^t & -e^{-6t} \\ 5e^t & e^{-6t} \end{bmatrix}$
- $M(0)^{-1} = \frac{1}{7}\begin{bmatrix} 1 & 1 \\ -5 & 2 \end{bmatrix}$
- $M(t)M(0)^{-1} = e^{tA} = \frac{1}{7}\begin{bmatrix} 2e^t + 5e^{-6t} & 2e^t - 2e^{-6t} \\ 5e^t - 5e^{-6t} & 5e^t + 2e^{-6t} \end{bmatrix}$

b - Find all x_0 such that $\lim_{x\to\infty} x(t) = 0$

- $\lim_{x\to\infty}(\frac{1}{7}\begin{bmatrix} 2e^t + 5e^{-6t} & 2e^t - 2e^{-6t} \\ 5e^t - 5e^{-6t} & 5e^t + 2e^{-6t} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- While it would possible to Gaussian Elimination, I simply solved it as a system of equations
- $x_1[2e^t + 5e^{-6t}] = x_2[2e^t - 2e^{-6t}]$ & $x_1[5e^t - 5e^{-6t}] = x_2[5e^t + 2e^{-6t}]$
- The aim is to find x_1 and x_2 so that all the terms with e^t cancel so we are only left with e^{-6t} terms, which go to 0 as t apporaches ∞
- we get $c\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ as the solution for all real numbers c, as well as the trivial solution $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

4 $x'(t) = Ax, x(0) = x_0, A = \begin{bmatrix} 2 & 9 & -3 \\ 6 & -1 & 3 \\ 6 & -9 & 11 \end{bmatrix}$ solve for e^{tA}

- Eigenvalues: $\lambda = 8, 8, 4$
- Corresponding Eigenvectors: $v_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ $v_2 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

- $M(t) = \begin{bmatrix} -e^{8t} & 3e^{8t} & -e^{-4t} \\ 0 & 2e^{8t} & e^{-4t} \\ 2e^{8t} & 0 & e^{-4t} \end{bmatrix}$
- $M(0) = \begin{bmatrix} -1 & 3 & -1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}$
- $M(0)^{-1} = \frac{1}{8} \begin{bmatrix} 2 & -3 & 5 \\ 2 & 1 & 1 \\ -4 & 6 & -2 \end{bmatrix}$
- $e^{tA} = M(t)M(0)^{-1} = \frac{1}{8} \begin{bmatrix} 4e^{8t} + 4e^{-4t} & 6e^{8t} - 6e^{-4t} & -2e^{8t} + 2e^{-4t} \\ 4e^{8t} - 4e^{-4t} & 2e^{8t} + 6e^{-4t} & 2e^{8t} - 2e^{-4t} \\ 4e^{8t} - 4e^{-4t} & -6e^{8t} + 6e^{-4t} & 10e^{8t} - 2e^{-4t} \end{bmatrix}$