01:XXX:XXX - Homework n

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$$|+\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$
$$|-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$
$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$X = |+\rangle\langle+| - |-\rangle\langle-|$$
$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

L4

Communation
A communation of $[A, B]$ is $\lambda AB - \lambda BA$
Simelanousl Digonalization
Iff $[AB] = 0$ thenthere is a baiss sutch that A and B are both digonal in that baiss.

0.1 Expection of an operator

The expectation value of an operator $\hat{O}$ in a state ψ is given by:

$$\langle\hat{O}\rangle = \langle\psi|\hat{O}|\psi\rangle$$

Also

$$\langle A\rangle_\psi = \langle\psi|A|\psi\rangle$$

$$\begin{aligned} \langle A\rangle_\psi &= \langle\psi|A|\psi\rangle \\ &= \sum a_i \langle\psi|i\rangle\langle i|\psi\rangle \\ &= \sum a_i |\langle\psi|i\rangle|^2 \end{aligned}$$

Example.

$$\begin{aligned}
Z|0\rangle &= +1|0\rangle \\
Z|1\rangle &= -1|1\rangle \\
Z &= |0\rangle\langle 0| - |1\rangle\langle 1|
\end{aligned}$$

$$\begin{aligned}
\langle Z\rangle_\psi &= \langle\psi|0\rangle\langle 0|\psi\rangle - \langle\psi|1\rangle\langle 1|\psi\rangle \\
&= |\langle\psi|0\rangle|^2 - |\langle\psi|1\rangle|^2 \\
&= |\alpha|^2 - |\beta|^2 \\
&= p(0) - p(1)
\end{aligned}$$

Example.

$$\begin{aligned}
|\psi\rangle &= |0\rangle \\
\langle Z\rangle_\psi &= 1 \\
\langle X\rangle_\psi &= 0
\end{aligned}$$

0.2 Variance/Uncertainty of an operators

Given an operator A define the operator
$(\Delta A)_\psi = A - \langle A\rangle_\psi$

$$\begin{aligned}
\langle(\Delta A)_\psi\rangle_\psi &= \langle(\Delta A)\rangle \\
&= \langle(A - \langle A\rangle)\rangle \\
&= \langle A^2 - 2A\langle A\rangle + \langle A\rangle^2\rangle \\
&= \langle A^2\rangle - \langle A\rangle^2
\end{aligned}$$

0.3 Heisenburg Uncertainty Principle

For any two Hermitian operators A and B we have:

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A,B]\rangle|^2$$

$$\geq \frac{1}{4}|\langle AB - BA\rangle|^2$$

When A and B commutes then the lower bound is 0. We can also use CS-inequality to show that:

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$$

$\langle\psi|A|\psi\rangle$ is real if $A = A^\dagger$
$\langle\psi|A|\psi\rangle$ is imaginary if $A = -A^\dagger$

Example.

$$\begin{aligned}
|\alpha\rangle &= \Delta A|\psi\rangle \\
|\beta\rangle &= \Delta B|\psi\rangle \\
\Delta A &= A - \langle A\rangle \\
\Delta B &= B - \langle B\rangle \\
\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle &= \langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \\
&\geq |\langle\psi|\Delta A\Delta B|\psi\rangle|^2 \\
(\Delta A)(\Delta B) &= \frac{1}{2}[A, B] + \frac{1}{2}\{A, B\}
\end{aligned}$$

Where $\{A, B\} = AB + BA$ is the anti-commutator of A and B.
Now we can see that

$$\begin{aligned}
\langle\psi|\Delta A\Delta B|\psi\rangle &= \frac{1}{2}\langle\psi|[A, B]|\psi\rangle + \frac{1}{2}\langle\psi|\{A, B\}|\psi\rangle \\
&= \frac{1}{2}(c_1 + ic_2)
\end{aligned}$$

We can square it then see

$$\begin{gathered}
\frac{c_1^2 + c_2^2}{4} \\
\geq \frac{c_2 or c_1}{4}
\end{gathered}$$

Example.