01:XXX:XXX - Homework n

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1.1.1 2.3 - The Diffusion Equation

Definition (Max Principle (weak))**.** If $u(x,t)$ is a solution to the Diffusion Equation in a rectangle $0 \leq x \leq L$, $0 \leq t \leq T$, then the maximum of $u(x,t)$ occurs on the boundary of the rectangle. In other words on $x = 0, x = L, t = 0$.
The minimum is similar as we can show that $-u(x,t)$ satisfies the same equation.
The natrual interpretaion of this is that if you have a rod with no internal heat sourse, the hottest or coldest spot can only occour at $t = 0$ or on the edges.

Definition (Uniqueness)**.** There is uniqueness for the Dirichlet problem for the Diffusion Equation. That means there is at most one solution of

$$\begin{cases} u_t - ku_{xx} = f(x,t) \text{ for } 0 < x < L, t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$$

For any given $f(x,t), \phi(x), g(t), h(t)$
We can do proof by max principle.

Proof. We want to show that for all u_1, u_2 that satisfy the above conditions, $u_1 = u_2$.
Let $w = u_1 - u_2$. Then w satisfies the following:

$$\begin{cases} w_t - kw_{xx} = 0 \text{ for } 0 < x < L, t > 0 \\ w(x,0) = 0 \\ w(0,t) = 0 \\ w(L,t) = 0 \end{cases}$$

By max prinicple $w(x,t)$ has a maximum on its boundary. Also it must have a minimum on its boundary. Since $w(x,0) = 0$, the minimum and the maximum must be 0. Thus $w(x,t) = 0$ for all x, t.
Thus $u_1 = u_2$. □

Now we can do a proof by energy.

Proof. We know that $w = u_1 - u_2$

$$\begin{aligned} 0 &= 0 \cdot w && (1) \\ &= (w_t - kw_{xx})w && (2) \\ &= (1/2w^2)_t + (-kww_x)_x + kw_x^2 && (3) \end{aligned}$$

We can now integrat about $0 < x < L$

$$\begin{aligned} 0 &= \int_0^L (1/2w^2)_t dx - kw_x w|_0^L{}_{goesto0} + k\int_0^L w_x^2 dx && (4) \\ \frac{d}{dt}\int_0^L 1/2w^2 dx &= -k\int_0^L w_x^2 dx && (5) \\ & && (6) \end{aligned}$$

Clealrly the derivative of $\int_0^L w^2 dx$ is decreasing

$$\int_0^L w^2 dx \leq \int_0^L w(x,0)^2 dx$$

The RHS is 0, so the LHS is 0. Thus $w = 0$. □

Definition (Stablitity)**.** The solution to the Diffusion Equation is stable. That means that if you have a small perturbation in the initial conditions, the solution will not change much. In other words they functions are "bounded" by initial conditions.
This is in a L_2 sense.

$$\int_0^l [u_1(x,t) - u_2(x,t)]^2 dx \leq \int_0^l [u_1(x,0) - u_2(x,0)]^2 dx$$

1.1.2 2.4 - Diffusion on the Whole Line

Definition (Invariance Properties)**.** We have 5 basic invariance properties of the Diffusion Equation.

- Translation $u(x-y,t)$ is a solution if $u(x,t)$ is a solution.
- Any derivative of $u(x,t)$ is a solution.
- A linear combination of solutions is a solution.
- An integral of a solution is a solution. Thus if $S(x,t)$ is a solution then so is $S(x-y,t)$ and so is $v(x,t)=\int_{-\infty}^{\infty} S(x-y,t)g(y)dy$. for any $g(y)$.
- Dilation. If $u(x,t)$ is a solution then so is $u(\sqrt{a}x, at)$ for any $a>0$.

Definition (Fundamental Solution to the Diffusion Equation)**.** The fundamental solution to the Diffusion Equation is

$$S(x,t)=\frac{1}{\sqrt{4\pi kt}}e^{-x^2/4kt}$$

This is a solution to the Diffusion Equation with $f(x,t)=0$ and $u(x,0)=\delta(x)$. We can derive this by utilizing the invariance properties.

1.1.3 2.5 - Comparison of Waves and Diffusion

Property	**Waves**	**Diffusion**
Speed of Propogation	c	Infinite
Singulatities for $t>0$	Transported along characteristics with speed c	Lost immediately
Well posed for $t>0$	Yes	Yes for bounded
Well posed for $t<0$	Yes	No
Max Principle	No	Yes
Behavior at infinity	Energy is constant so it doesn't decay	Decays to zero
Information	Transported	Lost gradually

Table 1: Comparison of Waves and Diffusion

1.1.4 4.1 - Seperation of Variables

Definition (Seperation Solution Process for waves)**.** We can consider the homogeneous Dirichlet problem for wave Equation. Due to linearity we can see that we have a seperated solution of the form $u(x,t)=X(x)T(t)$.

$$\begin{cases} u_{tt}=c^2u_{xx} \\ u(0,t)=u(L,t)=0 \\ u(x,0)=\phi(x) \\ u_t(x,0)=\psi(x) \end{cases}$$

Thus we can see the following ratios:

$$-\frac{T''}{cT} = -\frac{X''}{X} = \lambda$$

We know this must be a consant since it doesnt depend on x or t.
We can now do our test cases:

$$\begin{cases} \lambda = \beta^2 \\ \lambda = 0 \\ \lambda = -\beta^2 \end{cases}$$

We can Clearly see that this doesn't make sense for $\lambda = 0$ and $\lambda = -\beta^2$. they lead to solutions that are trivial and solutions that do not follow the boundary conditions.
Thus for $\lambda = \beta^2$ we have the following:

$$\begin{cases} X'' + \beta^2 X = 0 \\ T'' + c^2\beta^2 T = 0 \end{cases} \quad \begin{cases} X(x) = A\cos(\beta x) + B\sin(\beta x) \\ T(t) = C\cos(\beta ct) + D\sin(\beta ct) \end{cases}$$

Thus by imposing the BC we see that $A = 0$ and $Bsin(\beta l) = 0$. Thus for non trivial solutions we have $\beta = n\pi/l$. Thus our $\lambda = (n\pi/l)^2$ and our particualr eigenfunction correspodning to this eigenvalue is $X_n(x) = \sin(n\pi x/l)$ and $T_n(t) = \cos(n\pi ct/l) + \sin(n\pi ct/t)$.
When we take our particualr solutions $u_n(x,t) = (A_n\cos(n\pi ct/l) + B_n\sin(n\pi ct/l))\sin(n\pi x/l)$ we can see that we can form any solution as a linear combination of these solutions.

$$u(x,t) = \sum_{n=1}^{\infty}(A_n\cos(n\pi ct/l) + B_n\sin(n\pi ct/l))\sin(n\pi x/l)$$

We also require our IC to be satisfied.

$$\phi(x) = \sum_{n=1}^{\infty} A_n\sin(n\pi x/l)$$

$$\psi(x) = \sum_{n=1}^{\infty} B_n\sin(n\pi x/l)$$

Definition (Seperation Solution Process for Diffusion)**.** We can consider the homogeneous Dirichlet problem for Diffusion Equation.

$$\begin{cases} u_t = ku_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

We can see that we have a seperated solution of the form $u(x,t) = X(x)T(t)$.

$$\begin{cases} -\frac{T'}{kT} = -\frac{X''}{X} = \lambda \end{cases}$$

We can now do our test cases:

$$\begin{cases} \lambda = \beta^2 \\ \lambda = 0 \\ \lambda = -\beta^2 \end{cases}$$

We can see that this is the same X''/X as the wave equation. Thus we have the same solutions. Thus our $\lambda = (n\pi/l)^2$ For $n \in \mathbb{Z}$ thus our $T_n(t) = e^{-k(n\pi/l)^2 t}$ and $X_n(x) = \sin(n\pi x/l)$.
Thus our particualr solutions $u_n(x,t) = A_n e^{-k(n\pi/l)^2 t} \sin(n\pi x/l)$ and we can form any solution as a linear combination of these solutions.

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k(n\pi/l)^2 t} \sin(n\pi x/l)$$

We also require our IC to be satisfied.

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/l)$$

1.1.5 4.2 - The Neumann Condition

Definition (Neumann Condition for Diffusion)**.** The Neumann Condition is the following:

$$\begin{cases} u_t = k u_{xx} \\ u_x(0,t) = u_x(L,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

We can see that we can reach the same eigenvalues but we need to check the eigenfunctions for solving the IC.
Thus we can see that $X'(l) = 0 = -C\beta \sin(\beta l)$. Thus $\beta = n\pi/l$ and $X_n(x) = \cos(n\pi x/l)$ and $T_n(t) = e^{-k(n\pi/l)^2 t}$.
Additionally 0 is an eigenvalue and $X_0(x) = 1$ and $T_0(t) = 1$. (which makes it a consntant)
Thus our particualr solutions $u_n(x,t) = A_n e^{-k(n\pi/l)^2 t} \cos(n\pi x/l)$ and we can form any solution as a linear combination of these solutions.

$$u(x,t) = A_0/2 + \sum_{n=1}^{\infty} A_n e^{-k(n\pi/l)^2 t} \cos(n\pi x/l)$$

We also require our IC to be satisfied.

$$\phi(x) = A_0/2 + \sum_{n=1}^{\infty} A_n \cos(n\pi x/l)$$

Definition (Neumann Condition for Waves)**.**

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u_x(0,t) = u_x(L,t) = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

We can see that we can reach the same eigenvalues but we also see that 0 is an eigenvalue with $X_0(x) = 1$ and $T_0(t) = A + Bt$.
Thus we can say the LC of our particulars

$$u(x,t) = A_0/2 + B_0 t/2 + \sum_{n=1}^{\infty} (A_n \cos(n\pi ct/l) + B_n \sin(n\pi ct/l)) \cos(n\pi x/l)$$

We also require our IC to be satisfied.

$$\phi(x) = A_0/2 + \sum_{n=1}^{\infty} A_n \cos(n\pi x/l)$$

$$\psi(x) = B_0/2 + \sum_{n=1}^{\infty} (n\pi c/l) B_n \cos(n\pi x/l)$$

1.1.6 4.3 - The Robin Condition

Definition (Robin Condition for Diffusion)**.** The Robin Condition is the following:

$$\begin{cases} u_t = k u_{xx} \\ u_x(0,t) - a_0 u(0,t) = 0 \\ u_x(L,t) + a_L u(L,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

THIS IS EXCESSIVE JUST DO ALGEBRA

1.1.7 6.1 - Laplace's Equation

Definition (Laplace's Equation)**.** We define Laplace's Equation (homogeneous) as the following:

$$\begin{cases} \Delta u = 0 \end{cases}$$

And the inhomogeneous Laplace's Equation as the following:

$$\begin{cases} \Delta u = f(x) \end{cases}$$

Definition (Max Principle)**.** The Max is on the boundary of the region. The max can be inside the region if the region if the solution is constant.

Definition (Invariance Properties)**.** We say Laplace's Equation is invariant under all rigid motions.
A rigid motion in the plane consists of tranlations and rotations.
IE

$$x' = x + a, y' = y + b$$

and

$$x' = x\cos(\theta) - y\sin(\theta), y' = x\sin(\theta) + y\cos(\theta)$$

Definition (Laplacian in Polar)**.** We can define the 2-D Laplacian in polar coordinates as the following:

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

We can prove that if we take the following change of variables:

$$x = r\cos(\theta), y = r\sin(\theta)$$

That the Laplacian in polar coordinates is the following:

$$\Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Note: $log(r)$ is the fundamental solution to the Laplacian in 2-D as it solves the rotationally invariant polar Laplacian.

1.1.8 6.2 - Rectangles and Cubes

This is a bitch to do, just know that we seperate it into a LC of each BC and solve each one in order of homogeneous then inhomogeneous

1.1.9 6.4 - Circles, Wedges and Annuli

Definition (Wedge)**.** We can take a wegde being

$$\{0 < \theta < \theta_0, 0 < r < a\}$$

Where our BC are

$$u(r, 0) = u(r, \beta) = 0, u_r(a, \theta) = h(\theta)$$

We can seperate the variables into

$$u(r, \theta) = R(r)\Theta(\theta)$$
$$\Theta'' + \lambda\Theta = 0$$
$$r^2R'' + rR' - \lambda R = 0$$

$$\Theta(\theta) = \sin\frac{n\pi\theta}{\beta}$$
$$R(r) = r^\alpha \text{ for } \alpha = \pm\sqrt{\lambda} = \pm\frac{n\pi}{\beta}$$

Thus the LC of our solution is

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{n\pi/\beta} \sin\frac{n\pi\theta}{\beta}$$

We can solve the inhomogeneous BC

$$h(\theta) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{\beta} a^{n\pi/B-1} \sin\frac{n\pi\theta}{\beta}$$

We can sove for A_n by recognizing that the RHS is the Fourier Sine Series of $h(\theta)$.

$$A_n = a^{1-n\pi/\beta}\frac{2}{n\pi}\int_0^{\beta} h(\theta)\sin\frac{n\pi\theta}{\beta}d\theta$$

Definition (Annulus)**.**

Definition (Exterior of a Circle)**.**

1.1.10 5.1 - The Fourier Coefficients

Definition (Fourier Sine Series)**.** These following integrals show the orthogonality of *sin* and *cos* functions.

$$\int_0^l \sin(n\pi x/l)\sin(m\pi x/l)dx = \begin{cases} 0 & \text{if } n \neq m \\ l/2 & \text{if } n = m \end{cases}$$

Thus if we consider the following:

$$\begin{aligned}
\phi(x) &= \sum_{n=1}^{\infty} A_n \sin(n\pi x/l) \\
\phi(x)\sin(m\pi x/l) &= \sum_{n=1}^{\infty} A_n \sin(n\pi x/l)\sin(m\pi x/l) \\
\int_0^l \phi(x)\sin(m\pi x/l)dx &= \sum_{n=1}^{\infty} A_n \int_0^l \sin(n\pi x/l)\sin(m\pi x/l)dx \\
\int_0^l \phi(x)\sin(m\pi x/l)dx &= A_m l/2 \\
A_m &= \frac{2}{l}\int_0^l \phi(x)\sin(m\pi x/l)dx
\end{aligned}$$

Thus we can see that the A_n are the Fourier Sine Coefficients. We can continue for all values of n to get the entire series.

Definition (Fourier Cosine Series)**.** These following integrals show the orthogonality of *sin* and *cos* functions.

$$\int_0^l \cos(n\pi x/l)\cos(m\pi x/l)dx = \begin{cases} 0 & \text{if } n \neq m \\ l/2 & \text{if } n = m \end{cases}$$

Thus if we consider the following:

$$\phi(x) = A_0/2 + \sum_{n=1}^{\infty} A_n \cos(n\pi x/l)$$

$$\phi(x)\cos(m\pi x/l) = A_0/2\cos(m\pi x/l) + \sum_{n=1}^{\infty} A_n \cos(n\pi x/l)\cos(m\pi x/l)$$

$$\int_0^l \phi(x)\cos(m\pi x/l)dx = A_0/2\int_0^l \cos(m\pi x/l)dx + \sum_{n=1}^{\infty} A_n \int_0^l \cos(n\pi x/l)\cos(m\pi x/l)dx$$

$$\int_0^l \phi(x)\cos(m\pi x/l)dx = A_m l/2$$

$$A_m = \frac{2}{l}\int_0^l \phi(x)\cos(m\pi x/l)dx$$

We can see that the A_0 has $n = 0$ and the *cos* term is 1. Thus $A_0 = 2/l\int_0^l \phi(x)dx$.
Thus we can see that the A_n are the Fourier Cosine Coefficients. We can continue for all values of n to get the entire series.

Definition (Full Fourier Series)**.** We can see that the full Fourier Series is the sum of the Sine and Cosine Series.

$$\phi(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/l) + \sum_{n=1}^{\infty} b_n \sin(n\pi x/l)$$

We can see that the interval is $[-l, l]$ and our eigenfunctions are $\{1, \cos(n\pi x/l), \sin(n\pi x/l)\}$.Thus we can mulitply any two of these and integrate to get the orthogonality.
Thus we can see that

$$\begin{cases} A_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\cos(n\pi x/l)dx \\ B_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\sin(n\pi x/l)dx \end{cases}$$

Definition (c_n, a_n, b_n)**.** .
$a_n = c_n + c_{-n}$
$b_n = i(c_n - c_{-n})$

2 Content to Review

4.1 - Seperation of Variables for waves
4.3 - Robin Bounding Coundition 5.1 - Recognizing Fourier Series

3 Problems

1. Pg(110):
   Solve the following problem:

$$\begin{aligned} u_t t &= c^2 u_x x \\ u(0,t) &= u(L,t) = 0 \\ u(x,0) &= x \\ u_t(x,0) &= 0 \end{aligned}$$

   We know that for the wave equation

$$\begin{aligned} u(x,t) &= \sum_1^\infty (A_n \cos(n\pi ct/l) + B_n \sin(n\pi ct/l)) \sin(n\pi x/l) \\ u_t(x,t) &= \sum_1^\infty (-A_n \sin(n\pi ct/l) + B_n \cos(n\pi ct/l)) \frac{n\pi c}{l} \sin(n\pi x/l) \end{aligned}$$

   Thus from our IC

$$0 = \sum_1^\infty \frac{n\pi c}{l} B_n \sin(n\pi x/l)$$

   Thus $B_n = 0$ for all n.
   Now for our other IC

$$x = \sum_1^\infty A_n \sin(n\pi x/l)$$

   We can see that $\{A_i\}$ is the Fourier Sine Coefficients of x.

4 Review after exam

1. 3

$$\begin{cases} \Delta u = 0 \text{ in } x^2 + y^2 > 1 \\ u = y^2 \text{ on } x^2 + y^2 = 1 u \text{ is bounded as } x^2 + y^2 \to \infty \end{cases}$$

Solution: Sol in form of

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}(a_n \cos(n\theta) + b_n \sin(n\theta))$$

We can see that the BC is $y^2 = r^2 \sin^2(\theta)$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n}(a_n \cos(n\theta) + b_n \sin(n\theta)) = sin^2(\theta)$$

$$sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$$

$$\begin{cases} a_0 = 1 \\ b_n = 0 \\ a_2 = -1/2 \end{cases}$$

So our solution is

$$u(r, \theta) = 1/2 - \frac{1}{2r^2} cos(2\theta)$$

2. 4

$$\begin{cases} \Delta u = \lambda u \text{ in } D \\ u = 0 \text{ on } \partial D \end{cases}$$

Solution: Separating of variables

$$u(x, y) = X(x)Y(y)$$
$$X''Y + XY'' = \lambda XY$$
$$\frac{X''}{X} = -\frac{Y''}{Y} + \lambda = \alpha$$

We solve in X

$$\begin{cases} X'' - \alpha X = 0 \\ X(0) = X(\pi) = 0 \end{cases}$$

$\alpha = -n^2$ and $X_n = \sin(nx)$
We solve in Y

$$\begin{cases} Y'' - (\lambda - \alpha)Y = 0 \\ Y(0) = Y(\pi) = 0 \end{cases}$$

$\lambda - \alpha = -m^2$ and $Y_m = \sin(mx)$
Thus $\lambda = -m^2 - n^2$.

$$u_{n,m}(x,y) = \sin(nx)\sin(my)$$

for $n, m \in \mathbb{N}$ Notice that each lamda is not uniquely detemrined

3. 5

$$\partial_{xx}u + \partial_{yy}u + \partial_{xy}u = 0$$

u is in form of $u(x,y) = X(x)Y(y)$

Solution:

$$\begin{aligned} X''Y + XY'' + X'Y' &= 0 \\ \frac{X''}{X} + \frac{X'Y'}{XY} &= -\frac{Y''}{Y} \\ \text{Take partial x} & \\ \frac{\partial}{\partial x}\left[(\frac{X''}{X})' + \frac{X'}{X}'\frac{Y'}{Y}\right] &= 0 \\ \frac{Y'}{Y} &= -\frac{\frac{X''}{X}'}{\frac{X'}{X}'} = \alpha \\ Y' = \alpha Y &\implies Y(y) = e^{\alpha y} \\ \frac{X''}{X} + \frac{X'}{X}\alpha = -\alpha^2 &\implies X'' + \alpha X' + \alpha^2 X = 0 \\ \lambda &= \frac{-1 \pm i\sqrt{3}}{2}\alpha \\ X_1(x) &= e^{\lambda_0 \alpha x} \\ X_2(x) &= e^{\overline{\lambda_0}\alpha x} \end{aligned}$$

Transport method

$$(\partial_x^2 + \partial_y^2 + \partial_x \partial_y)u = 0$$

Factor as

$$(x - \lambda y)(x - \overline{\lambda} y)$$

Thus our operator is

$$(\partial_x - \lambda \partial_y)(\partial_x - \overline{\lambda} \partial_y)u = 0$$

$$u(x, y) = f(\lambda x + y) + g(\overline{\lambda} x + y)$$

If we take f, g to be exponential then we can see that the solution is sepeble

$$e^{\lambda x} e^y + e^{\overline{\lambda} x} e^y$$

Take $f(z) = e^{\alpha z} = g(z)$