01:XXX:XXX - Homework n

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1. Question 1.
   Suppsoe $X_1 \sim Exp(\lambda = 2)$ and $X_2 \sim Exp(\lambda = 3)$ are independant. Use the tranformaiton theorem we learned to find the PDF of $Y = X_1 + X_2$.

   Solution: We can can see that $y = x_1 + x_2$ $f(x_1, x_2) = f(x_1)f(x_2) = \frac{1}{6}e^{-\frac{x_1}{2} - \frac{x_2}{3}}$
   Thus if we let $y = x_1 + x_2$ then $x_1 = y - x_2$ and $x_2 = y - x_1$
   $g(y, x_2) = f(x_1, x_2)|J|$
   Where $|J|$ is $\frac{\partial x_1}{\partial y}$ We can clealry see that $g(y, x_2) = \frac{1}{6}e^{-\frac{x_1}{2} - \frac{x_2}{3}}$
   Since we can see that $x_1 = y - x_2$ thus $y > x_2$ and $x_2 > 0$

   $$f_Y(t) = \int_0^y \frac{1}{6}e^{-\frac{y-x_2}{2} - \frac{x_2}{3}}\, dx_2$$

   $$f_Y(t) = e^{-\frac{y}{2}} + e^{-\frac{y}{3}}$$

   Thus the PDF of $Y = X_1 + X_2$ is $f_Y(t) = e^{-\frac{y}{2}} + e^{-\frac{y}{3}}$

2. Write a series of algebraic manipulations that converts

   $$P(-t_{\alpha/2,n-1} \leq T \leq t_{\alpha/2,n-1}) = 1 - \alpha$$

   to

   $$P(\bar{X} - t_{\alpha/2,n-1}\frac{S}{\sqrt{n}} \leq \mu \leq \bar{X} + t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}) = 1 - \alpha$$

   Solution: We know that $T \sim \frac{\bar{X}-\mu}{S/\sqrt{n}}$ where $\bar{X}$ is the sample mean and S is the sample standard deviation.

   $$P(-t_{\alpha/2,n-1} \leq T \leq t_{\alpha/2,n-1}) = 1 - \alpha$$
   $$P(-t_{\alpha/2,n-1} \leq \frac{\bar{X} - \mu}{S/\sqrt{n}} \leq t_{\alpha/2,n-1}) = 1 - \alpha$$
   $$P(-t_{\alpha/2,n-1}(S/\sqrt{n}) \leq \bar{X} - \mu \leq t_{\alpha/2,n-1}(S/\sqrt{n})) = 1 - \alpha$$
   $$P(\bar{X} - t_{\alpha/2,n-1}\frac{S}{\sqrt{n}} \leq \mu \leq \bar{X} + t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}) = 1 - \alpha$$

3. Question 3.
   Use the PDF we derived of t_ν (note: $\nu > 0$) distribution to obtain a formula for the number

   $$\int_0^\infty \frac{1}{(1 + m^2)^p}\, dm$$

   in terms of p. State the range of p for which this formula is valid.

(a) Directly integrate $\int_0^\infty \frac{1}{1+m^2}\,dm$ and also use your formula. Do you get the same answer? You may use $\Gamma(1/2)=\sqrt{\pi}$.

(b) Determine $\int_0^\infty \frac{1}{(1+m^2)^{7/2}}\,dm$. Simplify completely. You may use $\Gamma(1/2)=\sqrt{\pi}$.

(c) Determine $\int_0^\infty \frac{1}{(1+m^2)^{7}}\,dm$. Simplify completely. You may use $\Gamma(1/2)=\sqrt{\pi}$.

(d) (Challenge) Determine $\int_0^\infty \frac{1}{(1+m^2)^{p}}\,dm$ when either p is a positive half-integer or positive integer.

Solution: Part a: Let $G \sim \Gamma_\nu$ thus $G(x) = \frac{1}{\sqrt{\nu}}\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}\Gamma(\frac{\nu}{2})}(1+\frac{x^2}{\nu})^{-\frac{\nu+1}{2}}$
And since G is even

$$\int_0^\infty G(x) = .5$$

$$\int_0^\infty \left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2} dt = \frac{1}{2}\frac{\sqrt{\pi\nu}\Gamma(\nu/2)}{\Gamma(\frac{\nu+1}{2})}$$

We can sub $m = t\sqrt{\nu}$ and $\sqrt{\nu}dm = dt$
We can take $p = (\nu+1)/2$ and thus $\nu = 2p-1$

$$\int_0^\infty \frac{1}{(1+m^2)^p}dm = \frac{1}{2}\frac{\sqrt{\pi}\Gamma((2p-1)/2)}{\Gamma(p)}$$

This is valid for $p > 1/2$
Part b: We can direclty apply to see that $p = 1$ and thus

$$\frac{1}{2}\frac{\sqrt{\pi}^2}{1}$$

$$\frac{\pi}{2}$$

Part c: We can direclty apply to see that $p = 7/2$ and thus

$$\frac{\sqrt{\pi}}{2}\frac{\Gamma(3)}{\Gamma(7/2)}$$

$$\frac{\sqrt{\pi}}{2}\frac{8}{15\sqrt{\pi}}$$

$$4/15$$

Part d: We can direclty apply to see that $p = 7$ and thus

$$\frac{\sqrt{\pi}}{2}\frac{\Gamma(13/2)}{\Gamma(7)}$$

$$\frac{\sqrt{\pi}}{2}\frac{11\cdot 9\cdot 7\cdot 5\cdot 3\cdot \sqrt{\pi}}{2^5 6!}$$

$$\frac{231}{256}\pi$$