Function Problems

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Problem 1

a

If f and g are decreasing functions on $\mathbb{R}$ then thier composition $g \circ f$ is not necessarily decreasing. For example, let $f(x) = -x$ and $g(x) = -x$. Then f and g are decreasing functions, but $g \circ f = -(-x) = x$ is not decreasing.

b

If f and g are decreasing functions on $\mathbb{R}$ then their composition is always increasing as if we consider $x_1, x_2 \in \mathbb{R}$ such that $x_1 < x_2$, then $f(x_1) > f(x_2)$ and $g(f(x_1)) < g(f(x_2))$ due to the fact that g is decreasing. Hence, $g \circ f$ is increasing.

c

If f and g are increasing functions on $\mathbb{R}$ then thier pointwise sum $f + g$ is always increasing as if we consider $x_1, x_2 \in \mathbb{R}$ such that $x_1 < x_2$, then $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$ due to the fact that f and g are increasing. Hence, $f(x_1) + g(x_1) < f(x_2) + g(x_2)$ and $f + g$ is increasing.

d

If f and g are increasing functions on $\mathbb{R}$ then thier pointwise product $f \cdot g$ is not necessarily increasing. For example, let $f(x) = x$ and $g(x) = x$. Then f and g are increasing functions, but $f \cdot g = x^2$ is not increasing for all $x \in \mathbb{R}$.

Problem 2

a

Let $r : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be given by the rule $r(a, b) = 2^{a-1}(2b - 1)$. Prove that r is one-to-one and onto N (a bijection).

One-to-one

Need: $(\forall a_1, a_2, b_1, b_2 \in \mathbb{N})\ [r(a_1, b_1) = r(a_2, b_2) \Rightarrow (a_1, b_1) = (a_2, b_2)]$
Let $a_1, a_2, b_1, b_2 \in \mathbb{N}$ such that $r(a_1, b_1) = r(a_2, b_2)$. Then $2^{a_1-1}(2b_1 - 1) = 2^{a_2-1}(2b_2 - 1)$. Since 2^{a_1-1} and 2^{a_2-1} are both powers of 2, they are both positive and non-zero. Hence, we can divide both sides of the equation by 2^{a_1-1} to get $(2b_1-1) = 2^{a_2-a_1}(2b_2-1)$. Since $2b_1-1$ and $2b_2-1$ are both odd and non-zero, we can divide both sides of the equation by $2b_2 - 1$ to get $\frac{2b_1-1}{2b_2-1} = 2^{a_2-a_1}$. Since the left side is a fraction with odd numerator and denominator, it must also be odd. But the right side is a power of 2, so the only way for the equation to hold is if $a_2 - a_1 = 0$ and $2b_1 - 1 = 2b_2 - 1$. Hence, $a_1 = a_2$ and $b_1 = b_2$ and r is one-to-one.

Onto

Need: $(\forall n \in \mathbb{N})\ (\exists a, b \in \mathbb{N})\ r(a, b) = n$
Let $n \in \mathbb{N}$. Then n be written in prime facortization form. That is, n is the product of some powers of primes. The even primes, which is only 2, can be contributed by the term 2^{a-1} and all the other odd primes can be contributed by the term $2b - 1$ as the product of odd numbers is always odd. Hence, r is onto.

b

Let $g : \mathbb{N} \times \mathbb{N} \to 8\mathbb{N}$ be given by the rule $g(m, n) = 2^{m+2}(2n - 1)$. Prove that r is one-to-one and onto N (a bijection).

One-to-One

Need: $(\forall m_1, m_2, n_1, n_2 \in \mathbb{N})\ [g(m_1, n_1) = g(m_2, n_2) \Rightarrow (m_1, n_1) = (m_2, n_2)]$
Let $m_1, m_2, n_1, n_2 \in \mathbb{N}$ such that $g(m_1, n_1) = g(m_2, n_2)$. Then $2^{m_1+2}(2n_1-1) = 2^{m_2+2}(2n_2 - 1)$. We can then divide both sides by 8 to get $2^{m_1-1}(2n_1 - 1) = 2^{m_2-1}(2n_2 - 1)$. This leads to a proof that is identical to the one in part a, so g is one-to-one.

Onto

Need: $(\forall k \in 8\mathbb{N})\ (\exists m, n \in \mathbb{N})\ g(m, n) = k$
Let $n \in 8\mathbb{N}$. Then k be written in prime facortization form times 8. That is, k is a product of 8 times a series of primes. After factoring out 8 from 2^{m+2} we get 2^{m-1} thus resulting in a identical proof to the one in part a, so g is onto.

Problem 3

Let $A = \{1, 2, 3, 4\}$ For each subproblem, describe a codomain B and a function $f : A \to B$

a

one-to-one but not onto
Let $B = \{1, 2, 3, 4, 5\}$ and $f : A \to B$ be given by the rule $f(x) = x$. Then f is one-to-one but not onto.

b

onto B but not one-to-one Let $B = \{0\}$ and $f : A \to B$ be given by the rule $f(x) = 0$. Then f is onto but not one-to-one.

c

both one-to-one and onto Let $B = \{1, 2, 3, 4\}$ and $f : A \to B$ be given by the rule $f(x) = x$. Then f is both one-to-one and onto.

d

neither one-to-one nor onto Let $B = \{0, 1\}$ and $f : A \to B$ be given by the rule $f(x) = 0$ Then f is neither one-to-one nor onto.

Problem 4

Find nonempty sets A, B, C and functions $f : A \to B$ and $g : B \to C$ for the following questions

a

f is onto but $g \circ f$ is not onto.
$A = \{1, 2\}, B = \{1, 2\}, C = \{1, 2\}$ where $f : A \to B$ is given by the rule $f(x) = x$ and $g : B \to C$ is given by the rule $g(x) = 1$. Then f is onto but $g \circ f$ is not onto.

b

g is onto but $g \circ f$ is not onto.
$A = \{1, 2\}, B = \{1, 2\}, C = \{1, 2\}$ where $f : A \to B$ is given by the rule $f(x) = 1$ and $g : B \to C$ is given by the rule $g(x) = x$. Then g is onto but $g \circ f$ is not onto.

c

$g \circ f$ is onto but f is not onto.
$A = \{1, 2\}, B = \{1, 2, 3\}, C = \{1\}$ where $f : A \to B$ is given by the rule $f(x) = x$ and $g : B \to C$ is given by the rule $g(x) = 1$ Then $g \circ f$ is onto but f is not onto.

d

f is 1-1 but $g \circ f$ is not 1-1.
$A = \{1, 2\}, B = \{1, 2\}, C = \{1\}$ where $f : A \to B$ is given by the rule $f(x) = x$ and $g : B \to C$ is given by the rule $g(x) = 1$ Then f is 1-1 but $g \circ f$ is not 1-1.

e

g is 1-1 but $g \circ f$ is not 1-1.
$A = \{1, 2\}, B = \{1, 2\}, C = \{1, 2\}$ where $f : A \to B$ is given by the rule $f(x) = 1$ and $g : B \to C$ is given by the rule $g(x) = x$ Then g is 1-1 but $g \circ f$ is not 1-1.

f

$g \circ f$ is 1-1 but g is not 1-1.
$A = \{1, 2\}, B = \{1, 2, 3\}, C = \{1, 2\}$ where $f : A \to B$ is given by the rule $f(x) = x$ and $g : B \to C$ is given by the rule $g(x) = x$ for $x \in \{1, 2\}$ and $g(x) = 1$ for $x = 3$ Then $g \circ f$ is 1-1 but g is not 1-1.

Problem 5

Let $f : A \to B$ and $g : B \to C$ be 1-1 functions. Prove that $g \circ f$ is also 1-1.

Proof

Suppose: $f : A \to B$ and $g : B \to A$ are 1-1 functions.
Need: $g \circ f$ is 1-1.
In other words: $(\forall a_1, a_2 \in A)[g(f(a_1)) = g(f(a_2)) \to a_1 = a_2]$
Proof: Let $a_1, a_2 \in A$ such that $g(f(a_1)) = g(f(a_2))$. Since g is 1-1 then $f(a_1) = f(a_2)$. Since f is 1-1 then $a_1 = a_2$ as desired. Hence, $g \circ f$ is 1-1.

Problem 6

Let $f : A \to B$ and $g : B \to C$ be onto functions. Prove that $g \circ f$ is also onto.

Proof

Suppose: $f : A \to B$ and $g : B \to A$ are onto functions.
Need: $g \circ f$ is onto.
In other words: $(\forall z \in C)(\exists x \in A)[g(f(x)) = z]$
Proof: Let $z \in C$, since g is onto C then $(\exists y \in B)[g(y) = z]$. Since f is onto B then $(\exists x \in A)[f(x) = y]$ as desired. Hence, $g(f(x)) = z$ and $g \circ f$ is onto.

Problem 7

Let $f : A \to B$ and $g : B \to C$ be functions. Assume $g \circ f$ is one to one. Prove that f is one to one.

Proof

Suppose $f : A \to B$ and $g : B \to C$ are functions.
Assume $g \circ f$ is one to one.
Need: f is one to one.
In other words: $(\forall a_1, a_2 \in A)[f(a_1) = f(a_2) \to a_1 = a_2]$
Proof: Let $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. Composing g to both sides gives $g(f(a_1)) = g(f(a_2))$ Since $g \circ f$ is one to one, then $a_1 = a_2$ as desired.

Problem 8

Let $f : A \to B$ and $g : B \to C$ be functions. Assume $g \circ f$ is onto. Prove that g is onto.

Proof

Suppose $f : A \to B$ and $g : B \to C$ Assume: $g \circ f$ is onto.
Need: g is onto.
In other words: $(\forall z \in C)(\exists y \in B)[g(y) = z]$
Proof: Let $z \in C$ then since $g \circ f$ is onto then $\exists x \in A$ such that $g(f(x)) = z$. Let $y := f(x)$, clearly $y \in B$ and $g(y) = g(f(x)) = z$ as desired. Hence, g is onto.

Problem 9

Let $f : A \to B$ and $g : B \to A$ be functions satsifying $g \circ f = I_A$.

a

Prove that if f is onto then $f \circ g = I_B$.

Proof

Suppose: $f : A \to B$ and $g : B \to A$ Assume $g \circ f = I_A$. and f is onto
Need $f \circ g = I_B$
In other words: $(\forall b \in B)[f(g(b)) = b]$
Proof: since $g \circ f = I_A$ we can say that f is one to one. Since f is onto and one to one then f is invertible (Main Theorem). Since f is invertible then it is both right and left invertable. By defintion of right invertibility, $f \circ g = I_B$ as desired.

b

Prove that if g is 1-1 then $f \circ g = I_B$.

Proof

Suppose: $f : A \to B$ and $g : B \to A$ Assume $g \circ f = I_A$. and g is 1-1
Need $f \circ g = I_B$
Proof: Since $g \circ f = I_A$ then g is onto. Since g is 1-1 and onto then g is invertible (Main Theorem). Since g is invertible then it is both right and left invertable. By defintion of left invertibility, $f \circ g = I_B$ as desired.

c

Prove by example that $g \circ f = I_A$ alone doesnt imply $f \circ g = I_B$

Proof

Let $A = \mathbb{R}_{\geq 0}$ and $B = \mathbb{R}_{\leq 0}$ and $f : A \to B$ be given by the rule $f(x) = -\sqrt{x}$ and $g : B \to A$ be given by the rule $g(x) = x^2$. Then $g \circ f = I_A$ but $f \circ g = I_B$ is not true.

d

(Optional) Can you fine such an example with $A = B$

Proof

It is not possible as if $A = B$ that means A and B have the same elements. And since $f \circ g \neq I_B$ that imply that f is not onto but for two sets to be equal, the functions must be 1-1 and onto (defintion of Cardinality). Hence, it is not possible to find such an example with $A = B$.

Problem 10

Suppose A, B, C, D are nonempty sets and $g : B \to C$ is 1-1 function. For each of the following two claims, prove it or give a specific counterexample (In a counterexample you may choose your A, B, C, D, g.)

a

For any two function $f_1 : A \to B$ and $f_2 : A \to B$, if $g \circ f_1 = g \circ f_2$ then $f_1 = f_2$.

Proof

Suppose: A, B are notempty sets and $f_1 : A \to B$ and $f_2 : A \to B$ are functions.
Assume $g \circ f_1 = g \circ f_2$
Need $f_1 = f_2$
This is not true for any two functions as we can take $A = \mathbb{R}$, $B = \mathbb{R}$, and $C = \mathbb{R}$ and $g(x) = x^2$ and $f_1(x) = x$ and $f_2(x) = -x$. Then $g \circ f_1 = g \circ f_2$ but $f_1 \neq f_2$.

b

For any two function $h_1 : C \to D$ and $h_2 : C \to D$, if $h_1 \circ g = h_2 \circ g$ then $h_1 = h_2$.

Proof

Suppose C, D are nonempty sets and $h_1 : C \to D$ and $h_2 : C \to D$ are functions.
Assume $h_1 \circ g = h_2 \circ g$
Need $h_1 = h_2$
Proof: This is not true as we can take $B = \{1\}, C = \{1, 2, 3\}$ and $D = \{1, 2, 3\}$ and $g : B \to C$ be given by the rule $g(x) = 1$ and $h_1 : C \to D$ be given by the rule $h_1(x) = x$ and $h_2 : C \to D$ be given by the rule $h_2(x) = x$ for $x \in \{1, 2\}$, $h_2(3) = 4$ and $h_2(4) = 3$. Then $h_1 \circ g = h_2 \circ g$ but $h_1 \neq h_2$.

Problem 11

a

Let $f : A \to B$ be 1-1 and onto, and let $g : B \to A$ be a function. Prove that g is the inverse of f iff $f \circ g = I_B$

Proof

Suppose A, B are nonempty sets and $f : A \to B$ is 1-1 and onto and $g : B \to A$ is a function.
Need: g is the inverse of f iff $f \circ g = I_B$

Part I

Need: g is the inverse of f implies $f \circ g = I_B$
Suppose g is the inverse of f. Need $f \circ g = I_B$
Proof: Since g is the inverse of f then $f \circ g = I_B$ by defintion of inverse.

Part II

Need: $f \circ g = I_A$ implies g is the inverse of f
Suppose $f \circ g = I_B$

Need: g is the inverse of f
Proof: Since f is 1-1 and onto then f is invertible (Main Theorem). Since $f \circ g = I_B$ then g is an inverse of f . Since f is invertible then there is only one inverse of f and since g is an inverse of f then g is the inverse of f as desired.

b

Problem 12

Let A and B be nonempty sets and $f : A \to B$ a function. Assume there exists a function $g : B \to A$ such that $g \circ f = I_A$. Prove that f is 1-1.

Proof

Suppose: A and B are nonempty sets and $f : A \to B$ a function.
Assume there exists a function $g : B \to A$ such that $g \circ f = I_A$
Need f is 1-1
In other words: $(\forall a_1, a_2 \in A)[f(a_1) = f(a_2) \to a_1 = a_2]$
Proof: Let $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. Then id we compose g with both sides of the equation we get $g(f(a_1)) = g(f(a_2))$. Since $g \circ f = I_A$ then $a_1 = a_2$ as desired. Hence, f is 1-1.

Problem 13

Let A and B be nonempty sets and $f : A \to B$ a function. Assume there exists a function $g : B \to A$ such that $f \circ g = I_B$. Prove that f is onto.

Proof

Suppose: A and B are nonempty sets and $f : A \to B$ a function.
Assume there exists a function $g : B \to A$ such that $f \circ g = I_B$
Need f is onto
That is: $(\forall y \in B)(\exists x \in A)[f(x) = y]$
Proof:Let $y \in B$ and define $x := g(y)$ then $f(x) = f(g(y)) = (f \circ g)(y) = I_B(y) = y$ as desired. Hence, f is onto.

Problem 14

Find an example of a function which has more than one left inverse. Do the same for right inverses.

a

Find two nonempty sets A and B and a function $f : A \to B$ which has more than one left inverse.

Proof

Need: $f : A \to B$ which has more than one left inverse.
In other words: Need two functions $g : B \to A$ and $h : B \to A$ such that $g \circ f = I_A$ and $h \circ f = I_A$ and $g \neq h$
Proof: Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ and $f : A \to B$ be given by the rule $f(x) = x$ and $g : B \to A$ be given by the rule $g(x) = x$ for $x \in \{1, 2\}$ and $g(3) = 1$ and $h : B \to A$ be given by the rule $h(x) = x$ for $x \in \{1, 2\}$ and $h(3) = 2$. Then $g \circ f = I_A$ and $h \circ f = I_A$ and $g \neq h$ as desired.

b

Find two nonempty sets A and B and a function $f : A \to B$ which has more than one right inverse.

Proof

Let $A = \mathbb{R}$ and $B = \mathbb{R}_{\geq 0}$ and $f : A \to B$ be given by the rule $f(x) = x^2$. and $g : B \to A$ be given by the rule $g(x) = \sqrt{x}$ and $h : B \to A$ be given by the rule $h(x) = -\sqrt{x}$. Then $f \circ g = I_B$ and $f \circ h = I_B$ and $g \neq h$ as desired.

Problem 15

Let A and B be nonempty sets and $f : A \to B$ and $g : B \to A$ be functions. Prove that if $g \circ f = I_A$ is equivalant to $(\forall x \in A)(\forall y \in B)[f(x) = y \to g(y) = x]$

Proof

Part I

Need: $g \circ f = I_A$ implies $(\forall x \in A)(\forall y \in B)[f(x) = y \to g(y) = x]$
Suppose: A and B are nonempty sets and $f : A \to B$ and $g : B \to A$ be functions.
Assume: $g \circ f = I_A$
Need: $(\forall x \in A)(\forall y \in B)[f(x) = y \to g(y) = x]$
Assume: $f(x) = y$
Proof: Let $x \in A$ then $f(x) = y$, then $g(y) = g(f(x)) = I_A(x) = x$ as desired.

Part II

Need $(\forall x \in A)(\forall y \in B)[f(x) = y \to g(y) = x]$ implies $g \circ f = I_A$
Suppose: A and B are nonempty sets and $f : A \to B$ and $g : B \to A$ be functions.
Assume: $(\forall x \in A)(\forall y \in B)[f(x) = y \to g(y) = x]$
Need: $g \circ f = I_A$
In other words $(\forall x \in A)[g(f(x)) = x]$

Proof: Let $x \in A$ then $g(y) = x$ then substituting $f(x) = y$ then $g(f(x)) = x$ as desired.

Problem 16

Let A and B be nonempty sets and $f : A \to B$ and $g : B \to A$ be functions. Prove that $f \circ g = I_B$ is equivalent to $(\forall x \in A)(\forall y \in B)[f(x) = y \leftarrow g(y) = x]$

Proof

Part I

Need: $f \circ g = I_B$ implies $(\forall x \in A)(\forall y \in B)[f(x) = y \leftarrow g(y) = x]$
Suppose: A and B are nonempty sets and $f : A \to B$ and $g : B \to A$ be functions.
Assume $f \circ g = I_B$
Need $(\forall x \in A)(\forall y \in B)[f(x) = y \leftarrow g(y) = x]$
Proof: Let $y \in B$ then $g(y) = x$ then $f(x) = f(g(y)) = I_B(y) = y$ as desired.

Part II

Need $(\forall x \in A)(\forall y \in B)[f(x) = y \leftarrow g(y) = x]$ implies $f \circ g = I_B$
Suppose: A and B are nonempty sets and $f : A \to B$ and $g : B \to A$ be functions.
Assume: $(\forall x \in A)(\forall y \in B)[f(x) = y \leftarrow g(y) = x]$
Need: $f \circ g = I_B$
In other words: $(\forall y \in B)[f(g(y)) = y]$
Proof: Let $y \in B$ then $f(x) = y$ then substituting $g(y) = x$ then $f(g(y)) = y$ as desired.

Problem 17

Let A, B, C be nonempty sets, and let $f : A \to B$ and $g : B \to C$ be invertible functions. Show that $g \circ f : A \to C$ is also invertible and that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof

Suppose: A, B, C are nonempty sets and $f : A \to B$ and $g : B \to C$ are invertible functions.
Need to show that: $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$
That is we need to show that $(g \circ f) \circ (g \circ f)^{-1} = I_C$ and $(g \circ f)^{-1} \circ (g \circ f) = I_A$

Part I

Need: $(g \circ f) \circ (g \circ f)^{-1} = I_C$
Proof: $(g \circ f) \circ (g \circ f)^{-1} = (g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ I_B \circ g^{-1} = g \circ g^{-1} = I_C$ as desired.

Part II

Need: $(g \circ f)^{-1} \circ (g \circ f) = I_A$
Proof: $(g \circ f)^{-1} \circ (g \circ f) = (f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ I_B \circ f = f^{-1} \circ f = I_A$ as desired.

Problem 18

Let A, B, C, D be nonempty sets, and let $f : A \to B$ and $g : B \to C$ and $h : C \to D$ be functions. Show that $h \circ g \circ f : A \to D$ is also invertable and that $(h \circ g \circ f)^{-1} = f^{-1} \circ g^{-1} \circ h^{-1}$.

Proof

Suppose: A, B, C, D are nonempty sets, and $f : A \to B$ and $g : B \to C$ and $h : C \to D$ are functions that are all invertable.
Need to show that: $h \circ g \circ f$ is invertable and $(h \circ g \circ f)^{-1} = f^{-1} \circ g^{-1} \circ h^{-1}$

Part I

Need: $h \circ g \circ f$ is invertable
Proof: Since f, g, h are all invertable then we can compose to create the functions to get $h \circ g \circ f$. Then we can compose the inverses of the functions "in reverse order" $(f^{-1} \circ g^{-1} \circ h^{-1})$ to get the inverse of $h \circ g \circ f$ as desired as $f^{-1} \circ g^{-1} \circ h^{-1} \circ h \circ g \circ f = I_A$ and $h \circ g \circ f \circ f^{-1} \circ g^{-1} \circ h^{-1} = I_D$.

Part II

Given: $(h \circ g \circ f)^{-1} = f^{-1} \circ g^{-1} \circ h^{-1}$
We need to show that $(h \circ g \circ f) \circ (h \circ g \circ f)^{-1} = I_D$ and $(h \circ g \circ f)^{-1} \circ (h \circ g \circ f) = I_A$ in order to prove that $(h \circ g \circ f)^{-1} = f^{-1} \circ g^{-1} \circ h^{-1}$ and it is invertable.

Subpart I

Need $(h \circ g \circ f) \circ (h \circ g \circ f)^{-1} = I_D$

Subpart II

Need $(h \circ g \circ f)^{-1} \circ (h \circ g \circ f) = I_A$