01:640:481 - Homework 7

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1. 12.14
   A single observation of a random variable having a geometric distribution is to be used to test the null hypothesis that its parameter equals θ_0 against the alternative that it equals $\theta_1 > \theta_0$. Use the Neyman-Pearson lemma to find the best critical region of size α.

   Solution: Let X be the random variable with the geometric distribution. The likelihood function is given by

   $$L(\theta) = \theta(1-\theta)^{x-1}$$

   By NPL the ratio of the likelihoods is bounded by a constant k. That is,

   $$\begin{aligned} \frac{L(\theta_0)}{L(\theta_1)} &= \left(\frac{\theta_0 \cdot (1-\theta_0)^{x-1}}{\theta_1 \cdot (1-\theta_1)^{x-1}}\right) \leq k \\ ln\left(\frac{\theta_0 \cdot (1-\theta_0)^{-1}}{\theta_1 \cdot (1-\theta_1)^{-1}}\right) + x \cdot ln\left(\frac{1-\theta_0}{1-\theta_1}\right) &\leq ln(k) \\ x &\leq \frac{ln(k) - ln\left(\frac{\theta_0 \cdot (1-\theta_0)^{-1}}{\theta_1 \cdot (1-\theta_1)^{-1}}\right)}{ln\left(\frac{1-\theta_0}{1-\theta_1}\right)} \end{aligned}$$

   Thus if we take the critical region to be $x \leq c$ where c is the above expression, we have the best critical region of size α.

2. 12.24
   Given a random sample of size n from a normal population with unknown mean and variance, find an expression for the likelihood ratio statistic for testing the null hypothesis $\sigma = \sigma_0$ against the alternative hypothesis $\sigma \neq \sigma_0$.

   Solution: Let X be the random variable with the normal distribution with a pdf of

   $$f(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

   The likelihood function is given by

   $$L(\mu, \sigma) = \frac{1}{(2\pi)^{n/2}\sigma^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

   The likelihood ratio statistic is given by $\Lambda = \frac{L(\mu,\sigma_0)}{L(\mu,\hat{\sigma}_{MLE})}$ where $\hat{\sigma}_{MLE}$ is the maximum likelihood estimator of σ. The MLE of σ is $\hat{\sigma}_{MLE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2}$ Thus

$$\begin{aligned}\Lambda &= \frac{\sigma_{MLE}}{\sigma_0}e^{-\frac{1}{2\sigma_{MLE}^2}\sum_{i=1}^{n}(x_i-\mu)^2+\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(x_i-\mu)^2} \\ &= (\frac{\sum_{i=1}^{n}(x_i-\bar{x})}{n\sigma_0^2})^{n/2}e^{\frac{1}{2}[\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sigma_0^2}-n]}\end{aligned}$$

3. 13.1

Given a random sample of size n from a normal population with the known variance σ^2, show that the null hypothesis $\mu = \mu_0$ can be tested against the alternative hypothesis $\mu \neq \mu_0$ with the use of a one-tailed criterion based on the chi-square distribution.

Solution: Let X be a random variable. The sample mean is given by $\bar{X}$ and $Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$ is a standard normal random variable under the null hypothesis.
$Z^2 \sim \chi_1^2$ under the null hypothesis. Thus by the 1 tail test

$$\frac{n(\bar{X}-\mu_0)^2}{\sigma^2} \geq \chi_{1,\alpha}^2$$

Where $\chi_{1,\alpha}^2$ is the critical value of the chi-square distribution with 1 degree of freedom at the α level of significance.