Chapter 2

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2.1 Wave Equation

$$u_{tt} = c^2 u_{xx}, x \in \mathbb{R} ***$$

Theorem 1: d'Alembert
Any solution to *** is of form

$$u(x,t) = f(x+ct) + g(x-ct)$$

where f and g are twice differentiable functions.
Proof: Method 1

$$(d_t^2 - c^2 d_x^2)u = 0$$

$$(d_t - cd_x)(d_t + cd_x)u = 0$$

Let $v = (d_t + cd_x)u$
Then $(d_t - cd_x)v = 0$ This is a transport equation. The general solution is

$$v(x,t) = h(x+ct)$$

for some function h.

$$u_t + cu_x = f(x+ct)$$

Inhomogeneous transport equation.
We can use linearity to find the general solution. Since it will be the sum of homogenous plus another function

$$u = u_p + g(x-ct)$$

Where u_p is a particular solution.

$$u_p = h(x+ct)$$

$$h'c + ch' = f$$

$$h' = \frac{f}{2c}$$

$$u = \frac{1}{2c}\int f(x+ct)dx + g(x-ct)$$

***

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$$

$$f(x+ct), g(x-ct)$$

Recall that $f(x+ct)$ is a wave traveling left at speed c and $g(x-ct)$ is a wave traveling right at speed c

Thus the solution is a wave traveling with speed c in both directions.

The wave equation is bidirectional.(unlike the transport equation)

Remark: u is a superposition of two waves traveling in opposite directions at fixed speed c.

Method 2:

Characteristic variable:

$$\xi = x + ct, \eta = x - ct$$

Then we can write

$$u(x,t) = u(\xi(x,t), \eta(x,t))$$

$$\partial_x = \partial_\xi + \partial_\eta$$

$$\partial_t = c\partial_\eta - c\partial_\xi$$

Factoring gives us

$$\partial_t^2 - c^2\partial_x^2 = (\partial_t - c\partial_x)(\partial_t + c\partial_x)$$

Plugging in we get

$$-2c\partial_\eta \cdot 2c\partial_\xi$$

$$-4c^2\partial_\xi\partial_\eta = 0$$

Thus we can differentiate with respect to ξ and η to get the general solution.

$$u_{\xi\eta} = 0$$

Integrating gives us

$$u = f(\xi) + g(\eta)$$

Theorem 2: (d'Alembert 1747)

$$\begin{cases} u_{tt} = c^2 u_{xx}, t > 0, x \in \mathbb{R} \\ u(x,0) = \phi(x), x \in \mathbb{R} \\ u_t(x,0) = \psi(x), x \in \mathbb{R} \end{cases}$$

Where ϕ is initial displacement and ψ is initial velocity.

$$u(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

Proof: By theorum 1:

$$u(x,t) = f(x+ct) + g(x-ct)$$

Find f,g using ICs

$$u(x,0) = f(x) + g(x) = \phi(x)$$
$$u_t(x,t) = cf'(x+ct) - cg'(x-ct)$$
$$u_t(x,0) = cf'(x) - cg'(x) = \psi(x)$$
$$\begin{cases} cf' + cg' = c\phi' \\ cf' - cg' = \psi \end{cases}$$

Adding gives us

$$\begin{cases} 2cf' = c\phi' + \psi \\ 2cg' = c\phi' - \psi \end{cases}$$
$$\begin{cases} f' = \frac{1}{2}\left(\phi' + \frac{\psi}{c}\right) \\ g' = \frac{1}{2}\left(\phi' - \frac{\psi}{c}\right) \end{cases}$$
$$f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \psi(y)dy + A$$
$$g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \psi(y)dy + B$$
$$\begin{cases} f(0) = \frac{1}{2}\phi(0) + A \\ g(0) = \frac{1}{2}\phi(0) + B \end{cases}$$
$$f(0) + g(0) = \phi(0) + A + B$$

$f + g = \phi$ which cancels outs
Thus we can set $A + B = 0$

$$\begin{aligned} u(x,t) &= f(\xi) + g(\eta) \\ &= \frac{1}{2}\phi(\xi) + \frac{1}{2c}\int_0^\xi \psi(y)dy + A \\ &+ \frac{1}{2}\phi(\eta) - \frac{1}{2c}\int_0^\eta \psi(y)dy + B \end{aligned}$$

$$
\begin{aligned}
u(x,t) &= f(\xi) + g(\eta) \\
&= \frac{1}{2}\phi(\xi) + \frac{1}{2c}\int_0^{\xi} \psi(y)dy \\
&+ \frac{1}{2}\phi(\eta) - \frac{1}{2c}\int_0^{\eta} \psi(y)dy \\
&= \frac{\phi(\xi)+\psi(\eta)}{2} + \frac{1}{2c}\left(\int_0^{\xi} \psi(y)dy - \int_0^{\eta} \psi(y)dy\right) \\
&= \frac{\phi(\xi)+\psi(\eta)}{2} + \frac{1}{2c}\int_{\eta}^{\xi} \psi(y)dy
\end{aligned}
$$

2 families of characteristics lines:

$$x \pm ct = constant$$

On one of these lines they are constant.
Example 1:
$\psi(x) = 0$

$$u(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right)$$

Initial displacement ϕ splits into two parts. Each with half the amplitude. and one is opposite direction from the other
Example 2:
$\psi(x) = 0$
Interaction of "pulse".
They constructively interfere and they keep their shape.
Example 3:
$\phi(x) = 0$

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy$$

If x is too large of magnititue it will be zero.
As we increase t, the integral will increase.
For largers t, I have more choses of x that make u nonzero.
Example 4:
$\psi(x) = 0, \phi(s) = sin(s)$

$$u(x,t) = \frac{1}{2}\left(sin(x+ct) + sin(x-ct)\right)$$

$$u(x,t) = sin(x)cos(ct)$$

This is a standing wave! **Remark 1:**
If $\psi = 0$ and ϕ is localized then the right and left moving waves will (in the long run) always separate.
If ϕ is not localized then then they don't separate **Remark 2:** $\phi = 0$ and ψ is

localized, fix x_0 then $u(x_0, t) = 0$ for all $t > t_0$.
The intial disturbance is felt at x_0 and then the affect will be gone

2.2 Causality and Energy

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

D'Alembert solution:

$$u(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

Lemma: If ϕ and ψ are compactly supported in $[a, b]$, (this means outside the interval they are zero) then $u(x, t) = 0$ outside of R where R is the "domain of influence of the interval $[a, b]$
The boundary lines are $x - ct = b$ and $x + ct = a$
This is like an open cone.

$$R = \text{ regieon where distubances are felt}$$

Proof:
By assumption $\phi(s)$ and $\psi(s)$ are zero for $s < a$
Thus the outside area is $x + ct < a$
Also $x - ct \leq x + ct < a$
In that region $u(x, t) = 0$
Similarly we can do this for the region $x - ct > b$
Thus we can conclude that $u(x, t) = 0$ outside of the region R. Due to the fact that both $\phi = 0$ and the integral $\int_{x-ct}^{x+ct} \psi(s)ds = 0$

In the limiting case. This is a "light cone"
This is where we have x_0 being the point of disturbance.
The region of influence is the area where the disturbance is felt.

$$x + ct = x_0, x - ct = x_0$$

$$x \pm ct = x_0$$

$\phi(x_0)$ and $\psi(x_0)$ affect the values of $u(x, t)$ if $x \pm ct = x_0$ or $x_0 \in [x - ct, x + ct]$

Reverse Questions:
Fix value of u and see what initial conditions affect this.
We need initial values at $x_0 \pm ct_0$ and in $[x_0 - ct_0, x_0 + ct_0]$
This is like a triangle
This is called the interval/region of dependance of the point x_0.

Remark:

$$u(x_0,t_0)=\frac{1}{2}\left(\phi(a)+\phi(b)\right)+\frac{1}{2c}\int_a^b\psi(s)ds$$

$$b-a=2ct_0$$

$$u(x_0,t_0)=\frac{1}{2}\left(\phi(a)+\phi(b)\right)+t_0\cdot\frac{1}{b-a}\int_a^b\psi(s)ds$$

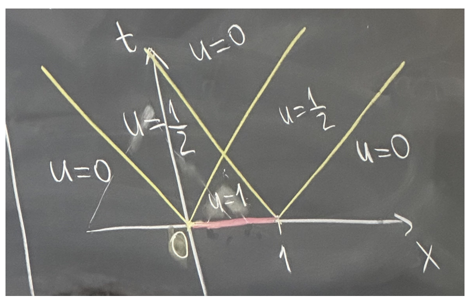
Note that the first term is average displacement and the second term is average velocity.

Example:

$$\psi=0,\phi=\begin{cases}1, & x\in[0,1]\\ 0, & \text{otherwise}\end{cases}$$

say $c=1$

Now: add image of the region of influence.



We can view this as two Vs that intersect at a point and then separate.
Outside the two Vs it is 0, inside both it is 1, inside one it is 1/2
And if it is in the middle it is also 0

Finite speed of propagation:

Localized initial data ie compactly supported. (0 outside some interval)
This implies that $u(x,t)$ is also localize for any $t>0$.
If ϕ,ψ are supported in $[a,b]$ then $u(x,t)$ is supported in $[a-ct,b+ct]$
when you are at position x you would feel the disturbance at time $t\geq\frac{x-b}{c}$ or distance from x to the interval

Remark:

The wave equation has no smoothing effect.
Singularity and discontinuities remain
Non-smooth initial data does not become smooth at any time.
Say ϕ has a discontinuity/ singularuty at x_0, then $u(x,t)$ is singular whenever $x\pm ct=x_0$
Singularity propogrates along characteristics.

Causality:
Causality means the effect comes after the cause.
Cause = distubance at x_0
Effect = disturbance at other points
Since there is a finite speed of propagation, the disturbance at x_0 can only affect points in the region of influence.

Huygen's Principle:
The disturbance in $[a, b]$ reaches x_1 at time t_1 and then it remains in the region of influence for all $t > t_1$.
If $\psi = 0$ region of influence of $[a, b]$ is just the effect is felt only at time t_1 and no lingering effects. In other words, just feel once then gone.
Kirkchoff's formula
In 3d the solution to the wave equation is given by Kirkchoff's formula.
it only uses ϕ and ψ on the boundary of the region of influence.
this is true for $n \geq 3$ and odd dimensions.
So in 2D this is not true!

Energy Conservation:

$$u_{tt} = c^2 u_{xx}$$
$$E(t) = \frac{1}{2} \int_{\mathbb{R}} u_t^2(t, x) + c^2 u_x^2(t, x) dx$$

Then $E(t)$ is constant in time. In particular $E(t) = E(0)$
Remark:
ϕ, ψ are localized, in other words, vanish outside of some interval. $[a, b]$
Thus the function $u(x, t)$ is also localized in $[a - ct, b + ct]$
Proof:

$$E'(t) = \int_{\mathbb{R}} u_t u_{tt} + c^2 u_x u_{xt} dx$$
$$E'(t) = \int_{\mathbb{R}} u_t c^2 u_{xx} + c^2 u_x u_{xt} dx$$

This motivates integrating by parts.

$$\int_{\mathbb{R}} u_x u_{tx} dx = [u_x u_t]_{-\infty}^{\infty} - \int_{\mathbb{R}} u_{xx} u_t dx$$
$$E'(t) = \int_{\mathbb{R}} u_t c^2 u_{xx} - c^2 u_{xx} u_t dx = 0$$

Remark:
In higher dimensions u_x^2 is replaced by $|\nabla u|^2$

0.1 2.3: Heat Equation

$$u_t = ku_{xx}$$

Now we consider this on a finite interval

$$t \in (0, T), x \in (0, L)$$

$$R = (0, T) \times (0, L)$$

$$\bar{R} = [0, T] \times [0, L]$$

This is inculding the boundary.
We also was all boundary without the top "t" side call it Γ
This is called the parabolic boundary.
Max Principle:
If $u(x, t)$ is a solution to the heat equation in R and it is continuous on $\bar{R}$ Then $max(u)_{\bar{R}} = max(u)_{\Gamma}$
This is an extension of the extreme value theorem
Proof:

$$v(t, x) = u(t, x) + \epsilon x^2$$

Where $\epsilon > 0$

$$v_t = u_t = ku_{xx} = k(v_{xx} - 2\epsilon)$$

$$v_{xx} = u_{xx} + 2\epsilon$$

Thus we gain:

$$v_t = kv_{xx} - 2k\epsilon$$

$$v_t < kv_{xx}$$

v attains its maximum on the boundary.
Let $v(t_0, x_0) = max(V)_{\bar{R}}$ if $(t_0, x_0) \in \bar{R}$ then $v_t \leq kv_{xx}$

$$v(t_0, x_0) = 0$$

$$f(x) = v(t_0, x) \text{ has max at } x_0\text{, then}$$

$$f''(x) = 0$$

$$v_t(t_0, x_0) < kv_{xx}(t_0, x_0) \leq 0$$

This is contradiction.
if $(t_0, x_0) \in$ top boundary ie $t_0 = T, x \in (0, L)$

$$g(t) = v(t, x_0)$$

has max at T then $g'(T) \geq 0$

$$g'(T) = v_t(T, x_0) = u_t(T, x_0) \leq 0$$

$$0 \geq v_t < kv_{xx} \geq 0$$

This is a contradiction.
Consider
Why does it work on gamma?
Thus $max(v)_\Gamma = max(v)_{\bar{R}}$

Now we need to prove this for u
Let $M = max(u)_\Gamma$
goal: $u(t,x) \leq M$ for all $(t,x) \in \bar{R}$

$$\begin{aligned} u + \epsilon x^2 &= V \\ &\leq max(V)_\Gamma \\ &= max(u + \epsilon x^2)_\Gamma \\ &\leq max(u)_\Gamma + \epsilon L^2 \end{aligned}$$

Let $\epsilon \to 0$

$$u \leq max(u)_\Gamma$$

Remark:
Strong max principle says that $max(u)_{\bar{R}}$ can **only** be attained on Γ
Immediate consequence of this the solution to the heat equation is unique.
Corollary:
We can do this for minimum as well.
Proof:
Let $v = -u$
Then $v_t = -u_t = -ku_{xx} = kv_{xx}$
Then we can apply the max principle to v
Theorem: Uniqueness
The solution to the heat equation is unique.
Proof:

$$\begin{cases} u_t = ku_{xx} + f(t,x) \\ u = u_0 \end{cases}$$

The following are the intital/boundary conditions.

$$\begin{cases} u(t,0) = a(t) \\ u(t,L) = b(t) \\ u(0,x) = \phi(x) \end{cases}$$

This has at most one solution.
Proof:
Suppose u_1, u_2 are two solutions.
Let $w = u_1 - u_2$

Then $w_t = kw_{xx}$

$$\begin{cases} w(t,0) = 0 \\ w(t,L) = 0 \\ w(0,x) = 0 \end{cases}$$

$$max(w)_{\bar{R}} = 0$$

$$min(w)_{\bar{R}} = 0$$

Thus $w = 0$ ie $u_1 = u_2$ in R
Remark:
We cant prescribe data for u on top boundary. It is determned by the PDE and the initial data.
Proof 2:

$$E(t) = \int_0^L w^2(t,x)dx$$

$$E'(t) = 2\int_0^L ww_t dx = 2\int_0^L wkw_{xx}dx$$

$$= 2k\left[ww_x\right]_0^L - 2k\int_0^L w_x^2 dx$$

$w = 0$ on the boundary.

$$= -2k\int_0^L w_x^2 dx \leq 0$$

$$E'(t) \leq 0$$

ie heat dissipates over time.

$$E(t) \leq E(0)$$

$$\int_0^L w^2(0,x)dx = 0$$

Since we know that by our definition of the integral it is non negative.
Thus $0 \geq w^2(0,x) \geq 0$
Thus $w = 0$
Thus $u_1 = u_2$
Remark:
The heat equation is a smoothing effect.

$$u_{xx}(A) < 0 \implies u_t(A) < 0$$

Peaks go down and valleys go up.
Want to reach equilibrium.
Remark:

Uniqueness for heat IBVP holds in:
Stability estimates:
Lemma 1:
Consider R and Γ

$$\begin{cases} u_t = ku_{xx}, \in R \\ u = u_0 \in \Gamma \end{cases}$$

$$max(u)_{\bar{R}} = max(u)_{\Gamma}$$

Proof:

$$-|u_0| \leq u_0 \leq |u_0|$$

$$u \leq max(u)_{\bar{R}} = max(u)_{\Gamma} \leq max(|u|)_{\Gamma}$$

$$-u \geq min(u)_{\bar{R}} = min(u)_{\Gamma} \geq -max(|u|)_{\Gamma}$$

$$|u| \leq max(|u|)_{\Gamma} \text{ everywhere}$$

Initial/boundary data u_0 controls u everywhere.
Corollary:

$$\begin{cases} \partial_t u_1 = k\partial_{xx} u_1 u_1(x, 0) = \phi_1(x) \\ u_1(0, t) = u_1(l, t) = 0 \end{cases}$$

$$\begin{cases} \partial_t u_2 = k\partial_{xx} u_2 u_2(x, 0) = \phi_2(x) \\ u_2(0, t) = u_2(l, t) = 0 \end{cases}$$

We think u_1 as the true one and the u_2 is the experimentally derived one with perturbations.

$$max|u_2 - u_1|_{\bar{R}} \leq max|\phi_2 - \phi_1|_{[0,l]}$$

Proof:
Let $u = u_2 - u_1$
Let $\phi = \phi_2 - \phi_1$
u will solve the heat equation with initial condition ϕ

$$\begin{cases} u_t = ku_{xx} \\ u(x, 0) = \phi(x) \\ u(0, t) = u(l, t) = 0 \end{cases}$$

By lemma 1 we have that $max(u)_{\bar{R}} \leq max(u)_{[0,l]}$

Lemma 2:

$$\int_0^l u^2 dx \leq \int_0^l \phi^2 dx$$

$$\begin{cases} u_t = ku_{xx} \\ u(x,0) = \phi(x) \\ u(0,t) = u(l,t) = 0 \end{cases}$$

Proof:

$$E(t) = \int_0^l u^2 dx$$

$$E'(t) \geq 0 \implies E(t) \leq E(0)$$

Corollary 2:
In the setting of Corollary 1:

$$\int_0^l |u_2 - u_1|^2 dx \leq \int_0^l |\phi_2 - \phi_1|^2 dx$$

$f = f(x)$ on I

$$||f||_\infty = max|f|_I$$

$$||f||_\infty = 0 \implies f = 0$$

$$||f||_{L^2} = \left(\int_I |f|^2 dx\right)^{1/2}$$

This is known as the L^2 norm; mean-squared

$$|f(x)| \leq ||f(x)||_\infty$$

$|f(x)|^2 \leq ||f(x)||_\infty^2$

$$\int_I |f(x)|^2 dx \leq ||f(x)||_\infty^2 \int_I dx$$

$$||f||_{L^2} \leq ||f||_\infty \sqrt{|I|}$$

$$||u_2(\star, t) - u_1(\star, t)||_{L^2} \leq ||\phi_2 - \phi_1||_{L^2}$$

The size at any time is controlled by the intial data.

2.4: Heat Equation on the Real Line

Theorem:

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

This is the fundamental solution to the heat equation.

1. $S_t = S_{xx}$
2. $\int_{-\infty}^{\infty} S(x,t)dx = 1, \forall t > 0$
3. $\lim_{t\to 0^+} \int_R S(x,t)f(x)dx = f(0)$ for all continuous and bounded f on $\mathbb{R}$

$\lim_{t\to 0^+} S(x,t) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$ **Proof:**

Need to prove:

$$\lim_{t\to 0^+} \int_R S(x,t)[f(x) - f(0)]dx = 0$$

Fix $\epsilon > 0$ then there exists $\delta > 0$ such that $|f(x) - f(0)| < \epsilon$ for $|x - 0| < \delta$

$$I = \int_{|x|<\delta} + \int_{|x|\geq\delta}$$

$$|I_1| \leq \int S(x,t)|f(x) - f(0)|dx$$

$$|f(x) - f(0)| < \epsilon$$

$$|I_1| \leq \epsilon \int_{x\leq\delta} S(x,t)dx \leq \epsilon$$

This works for any t

$$|f(x)| \leq M$$

$$|f(x) - f(0)| \leq 2M$$

$$|I_2| \leq 2M \int_{|x|\geq\delta} S(x,t)dx$$

Let $y = \frac{x}{\sqrt{4t}}$
Then $y^2 = \frac{x^2}{4t}$

$$|I_2| \leq 2M \int_{|y|\geq\frac{\delta}{\sqrt{4t}}} \frac{1}{\sqrt{4\pi t}} e^{-y^2}\sqrt{4t}dy$$

$$|I_2| \leq \frac{2M}{\sqrt{\pi}} \int_{|y|\geq\frac{\delta}{\sqrt{4t}}} e^{-y^2}dy$$

Notice that the integral goes to 0 as $t \to 0$

$$\int_R g(y)dy = \lim_{a\to\infty} \int_{-c}^{c} g(y)dy$$

$$\int_R g(y)dy = \lim_{t\to 0^+} \int_{-1/t}^{1/t} g(y)dy$$

$$\int_R g(y)dy = \lim_{t\to 0^+} \int_{|y|\geq 1/t} g(y)dy$$

We can see this goes to 0 as $t \to 0$
Definition

$$\lim_{t\to 0^+} S(x,t) = \delta_0(x)$$

which means (3).
Formally we can write this as:

$$\delta_0(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta_0(x)dx = 1$$

Corollary:
Take any y and consider $F(x,t) = S(x-y,t)$
then (1), (2) hold for F and
$\lim_{t\to 0^+} *F(x,t) = \delta_y(x) \iff \lim_{t\to 0^+} \int_R F(x,t)f(x)dx = f(y)$

$u = u(t,y)$

$$\begin{cases} u_t = ku_{yy} \\ u(0,y) = \phi(y) \end{cases}$$

$$\phi(x) = \int_{-\infty}^{\infty} \phi(y)\delta_x(y)dy$$

Theorem:

$$u(t,y) = \int_{-\infty}^{\infty} \phi(y)S(y,t)dy$$

This solves heat equation !!!
IC is considered as a limit