01:640:350H - Homework 5

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1. Question 1.3 23
   Let W_1 and W_2 be subspaces of a vector space V

   (a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

   (b) Prove that any subspace of V that contains both W_1 and W_2 must contain $W_1 + W_2$.

   Note that $\forall x \in W_1$ and $\forall y \in W_2$, $x + y \in V$ since W_1, W_2 are subspaces of V.

Solution: Part (a):
To show that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2, we need the following properties to hold:

(a) $W_1 + W_2 \subseteq V$

(b) $\underline{0} \in W_1 + W_2$

(c) $\forall x, y \in W_1 + W_2$, $x + y \in W_1 + W_2$

(d) $\forall x \in W_1 + W_2$ and $\forall c \in \mathbb{R}$, $cx \in W_1 + W_2$

First to show that $W_1 + W_2 \subseteq V$.
Suppose $z \in W_1 + W_2$. Then $z = x + y$ for some $x \in W_1$ and $y \in W_2$.
Since W_1, W_2 are subspaces of V, $x + y \in V$. Therefore $W_1 + W_2 \subseteq V$.
Next to show that $\underline{0} \in W_1 + W_2$.
Since W_1, W_2 are subspaces of V, $\underline{0} \in W_1$ and $\underline{0} \in W_2$.
Therefore $\underline{0} + \underline{0} \in W_1 + W_2 \implies \underline{0} \in W_1 + W_2$.
Next to show that $\forall x, y \in W_1 + W_2$, $x + y \in W_1 + W_2$.
Suppose $z_1, z_2 \in W_1 + W_2$. Then $z_1 = x_1 + y_1$ and $z_2 = x_2 + y_2$ for some $x_1, x_2 \in W_1$ and $y_1, y_2 \in W_2$.
Then $z_1 + z_2 = (x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2)$.

Since W_1, W_2 are subspaces of V, $x_1 + x_2 \in W_1$ and $y_1 + y_2 \in W_2$. Therefore $z_1 + z_2 \in W_1 + W_2$.
Finally to show that $\forall x \in W_1 + W_2$ and $\forall c \in \mathbb{R}$, $cx \in W_1 + W_2$.
Suppose $z \in W_1 + W_2$. Then $z = x + y$ for some $x \in W_1$ and $y \in W_2$.
Then $cz = c(x + y) = cx + cy$. Since W_1, W_2 are subspaces of V, $cx \in W_1$ and $cy \in W_2$. Therefore $cz \in W_1 + W_2$.
Therefore $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

Solution: Part (b):
Let W be a subspace of V that contains both W_1 and W_2.
We can see that $W_1 \subseteq W$ and $W_2 \subseteq W$.
Thus if we consider $x \in W_1$ and $y \in W_2$, then $x + y \in W$ since W is a subspace of V.
Therefore $W_1 + W_2 \subseteq W$.

2. Question 1.3 24 Show that F^n is the direct sum of the subspaces
$W_1 = \{(a_1, a_2, \dots, a_n) \in F^n : a_n = 0\}$ and $W_2 = \{(a_1, a_2, \dots, a_n) \in F^n : a_1 = a_2 = \cdots = a_{n-1} = 0\}$.

Solution: We can already see that $W_1 \subseteq F^n$ and $W_2 \subseteq F^n$.
To show that F^n is the direct sum of W_1 and W_2, we need to show that $F^n = W_1 + W_2$ and $W_1 \cap W_2 = \{\underline{0}\}$.
First to show that $W_1 \cap W_2 = \{\underline{0}\}$.
Suppose $x \in W_1 \cap W_2$. Then $x = (a_1, a_2, \dots, a_n)$ for some $a_1, a_2, \dots, a_n \in F$.
Since $x \in W_1$, $a_n = 0$. Since $x \in W_2$, $a_1 = a_2 = \cdots = a_{n-1} = 0$.
Therefore $x = (0, 0, \dots, 0) = \underline{0}$.
Next to show that $F^n = W_1 + W_2$.
First to show that $F_n \subseteq W_1 + W_2$.
Suppose $x \in F^n$. Then $x = (a_1, a_2, \dots, a_n)$ for some $a_1, a_2, \dots, a_n \in F$.
Let $y = (0, 0, \dots, a_n) \in W_1$ and $z = (a_1, a_2, \dots, a_{n-1}, 0) \in W_2$.
Then $y + z = (0, 0, \dots, a_n) + (a_1, a_2, \dots, a_{n-1}, 0) = (a_1, a_2, \dots, a_n) = x$.
Next show that $W_1 + W_2 \subseteq F^n$.
Suppose $x \in W_1 + W_2$. Then $x = y + z$ for some $y \in W_1$ and $z \in W_2$.
Then $y = (0, 0, \dots, a_n)$ and $z = (a_1, a_2, \dots, a_{n-1}, 0)$.
Then $y + z = (0, 0, \dots, a_n) + (a_1, a_2, \dots, a_{n-1}, 0) = (a_1, a_2, \dots, a_n) = x$.
Therefore $F^n = W_1 + W_2$ and $W_1 \cap W_2 = \{\underline{0}\}$.

3. Question 1.3 25
   Let W_1 denote the set of polynomials $f(x)$ in $P(F)$ suchthat in the representation

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

we have $a_i = 0$ when i is even. Likewise let W_2 denote the set of all polynomials $g(x)$ in $P(F)$ such that in the representation

$$g(x) = \sum_{i=0}^{n} b_i x^i$$

we have $b_i = 0$ when i is odd. Show that $P(F)$ is the direct sum of W_1 and W_2.

Solution: We can already see that $W_1 \subseteq P(F)$ and $W_2 \subseteq P(F)$.
To show that $P(F)$ is the direct sum of W_1 and W_2, we need to show that $P(F) = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$.

First to show that $W_1 \cap W_2 = \{0\}$.
Suppose $z(x) \in W_1 \cap W_2$. Then $z(x) = \sum_{i=0}^{n} c_i x^i$ for some $c_i \in F$.
Since $z(x) \in W_1$, $c_i = 0$ when i is even. Since $z(x) \in W_2$, $c_i = 0$ when i is odd.
Therefore there are no non-zero terms in $z(x)$ and $z(x) = 0$.

Next to show that $P(F) = W_1 + W_2$.
First to show that $P(F) \subseteq W_1 + W_2$.
Suppose $f(x) \in P(F)$. Then $f(x) = \sum_{i=0}^{n} a_i x^i$ for some $a_n \in F$.
Let $g(x) = \sum_{i=0}^{n} a_{2i} x^{2i} \in W_1$ and $h(x) = \sum_{i=0}^{n} a_{2i+1} x^{2i+1} \in W_2$.
Then $g(x) + h(x) = \sum_{i=0}^{n} a_{2i} x^{2i} + \sum_{i=0}^{n} a_{2i+1} x^{2i+1} = \sum_{i=0}^{n} a_i x^i = f(x)$.
Therefore $P(F) \subseteq W_1 + W_2$.

Next to show that $W_1 + W_2 \subseteq P(F)$.
Suppose $f(x) \in W_1 + W_2$. Then $f(x) = g(x) + h(x)$ for some $g(x) \in W_1$ and $h(x) \in W_2$.
Then $g(x) = \sum_{i=0}^{n} a_{2i} x^{2i}$ and $h(x) = \sum_{i=0}^{n} a_{2i+1} x^{2i+1}$.
Then $g(x) + h(x) = \sum_{i=0}^{n} a_{2i} x^{2i} + \sum_{i=0}^{n} a_{2i+1} x^{2i+1} = \sum_{i=0}^{n} a_i x^i = f(x)$.
Therefore $W_1 + W_2 \subseteq P(F)$.

Therefore $P(F)$ is the direct sum of W_1 and W_2.

4. Question 1.3 30
   Let W_1 and W_2 be subspaces of a vector space V. Prove that V is the direct sum of W_1 and W_2 iff each vector in v can be uniquely expressed as the sum of a vector in W_1 and a vector in W_2.

Solution: Proof of $\implies$:
We can do this by contradiction.
Suppose V is the direct sum of W_1 and W_2.
Then $V = W_1 + W_2$ and $W_1 \cap W_2 = \{\underline{0}\}$.
Then we can consider $v \in V$.
We can assume there is **not** a unique way to represent this as a sum of vectors in W_1 and W_2 i.e. $v = w_1 + w_2 = w_1' + w_2'$ for some $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$ where $w_1 \neq w_1'$ or $w_2 \neq w_2'$.
Then $w_1 - w_1' = w_2' - w_2$.
Since $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$, $w_1 - w_1' \in W_1$ and $w_2' - w_2 \in W_2$.
The only vector they can have in common is $\underline{0}$.
Then $w_1 - w_1' \in W_1 \cap W_2 = \{\underline{0}\}$.
Therefore $w_1 - w_1' = \underline{0} \implies w_1 = w_1'$.
Similarly $w_2 = w_2'$.
Therefore v can be uniquely expressed as the sum of a vector in W_1 and a vector in W_2.

Proof of $\impliedby$:
Suppose each vector in v can be uniquely expressed as the sum of a vector in W_1 and a vector in W_2.
Need to show that $V = W_1 + W_2$ and $W_1 \cap W_2 = \{\underline{0}\}$.
First we can show that $W_1 \cap W_2 = \{\underline{0}\}$.
We can do this by contradiction.
Suppose $v \in W_1 \cap W_2$ and $v \neq \underline{0}$.
Then $v = w_1 = w_2$ for some $w_1 \in W_1$ and $w_2 \in W_2$.
Then $v = w_1 + w_2$ where $w_1, w_2 \in W_1$ and W_2.
Since v can be uniquely expressed as the sum of a vector in W_1 and a vector in W_2, $w_1 = w_2 = \underline{0}$.
Therefore $v = \underline{0}$.
Therefore $W_1 \cap W_2 = \{\underline{0}\}$.

Next to show that $V = W_1 + W_2$.
First to show that $V \subseteq W_1 + W_2$.
Suppose $v \in V$.
Then $v = w_1 + w_2$ for some $w_1 \in W_1$ and $w_2 \in W_2$.
Then $v \in W_1 + W_2$.
Next to show that $W_1 + W_2 \subseteq V$.
Suppose $v \in W_1 + W_2$.
Then $v = w_1 + w_2$ for some $w_1 \in W_1$ and $w_2 \in W_2$.
Then $v \in V$.
Therefore $V = W_1 + W_2$.