

01:640:311 - Homework n

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This is a set of all of the theorems talked in class and in the book numbered.

Theorem 1 (0.0.0: Theorem Name). *This is a theorem. and a teplate for theorems..*

Proof. This is a proof.

This is a proof. $e = mc^2$

□

Theorem 2 (Nested Interval Property: (s 1.4)). *If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ is a sequence of closed intervals in \mathbb{R} then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.*

Proof. Let $A = \{a_1, a_2, a_3, \dots\}$ be the set of left endpoints of the intervals.

Now since the I_n s are nested, $I_n \subseteq I_1$ for all n .

Thus each $a_n \in I_1$ for all n .

so $a_n \leq b_1$.

It follows that b_1 is an upper bound for A so $\sup A$ exists.

Now we need to prove that $x \in \bigcap_{n=1}^{\infty} I_n$.

To do thi we need to how that $x \in I_n$ for all n .

This mean that $a_n \leq x \leq b_n$ for all n .

Step 1 $a_n \leq x$ for all n .

Remember that $x = \sup A$.

So $a_n \leq x$ for all n .

Step 2 $x \leq b_n$ for all n .

Since $x = \sup A$, x i less than very upper bound of A so it i enough to show that b_n is an upper bound of A .

$b_n \geq a_m$ for all m .

Case 1 $n > m$.

Then $I_n \subset I_m$ so $b_n \in [a_m, b_m] = I_m$.

Case 2 $n \leq m$.

Then $I_m \subset I_n$ so $a_m \in [a_n, b_n] = I_n$.

so $a_m \leq b_n$.

This b_n is an upper bound of A .

Thus $x \leq b_n$ for all n .

Thus $x \in I_n$ for all n .

Thus $x \in \bigcap_{n=1}^{\infty} I_n$. which means the intersection is not empty. □

Theorem 3 (Archimedean Property). *The set \mathbb{N} is not bounded above.*

Proof. Suppose (by contradiction) \mathbb{N} is bounded above.

Then by the least upper bound property, $\sup \mathbb{N}$ exists.

Let us call $\alpha = \sup \mathbb{N}$ and it is a real number.

Thus $\alpha - 1 < \alpha$ so $\alpha - 1$ is not an upper bound of \mathbb{N} .

So we can fine an $n \in \mathbb{N}$ such that $\alpha - 1 < n$.

Thus $\alpha < n + 1$.

But $n + 1 \in \mathbb{N}$ so α is not an upper bound of \mathbb{N} . □

Theorem 4 (Density of \mathbb{Q} in \mathbb{R}). *$\forall a < b \in \mathbb{R}$ there exists $q \in \mathbb{Q}$ such that $a < q < b$.*

Definition (Open Set). An open set is a set S that for all $x \in S$ for all epsilon neighborhoods $V_\epsilon(x)$ of x , $V_\epsilon(x) \subseteq S$.

In other words, any point has a circle that can be drawn around it that is completely contained in the set.

The union of open sets is open.

The intersection of finitely many open sets is open.

A set is open iff its complement is closed.

Definition (Closed Set). A closed set is a set S it contains all of its limit points.

A set is closed iff every Cauchy sequence contained in S has limit in S .

The intersection of closed sets is closed.

The union of finitely many closed sets is closed.

A set is closed iff its complement is open.

Definition (Limit Point). A point x is a limit point of a set S if every epsilon neighborhood $V_\epsilon(x)$ of x intersects the set S at a point other than x .

In other words x is a limit point if there is a sequence of points in S that converges to x where the sequence does not contain x .

Definition (Isolated Point). An isolated point is a point x in a set S that is not a limit point of S .

In other words, there exists an epsilon neighborhood $V_\epsilon(x)$ of x such that $V_\epsilon(x) \cap S = \{x\}$.

Definition (Closure and Interior). The closure of a set S denoted by \bar{S} is the union of S and all of its limit points.

The interior of a set S denoted by S° is the collection of all points $x \in S$ such that there exists an epsilon neighborhood $V_\epsilon(x)$ of x that is completely contained in S .

Let S be a set then:

S is closed iff $S = \bar{S}$.

S is open iff $S = S^\circ$.

Definition (Compact Set). A compact set is a set that every Sequence in K has a subsequence that converges to a point in K .

A set is compact if and only if it is closed and bounded.

A set is compact if and only if every open cover has a finite subcover.

Definition (Open Cover). An open cover of a set S is a collection of open sets $\{U_\alpha\}$ such that union of all the open sets contains S .

In other words, $S \subseteq \bigcup_\alpha U_\alpha$.

Theorem 5 (Heine-Borel Theorem). *A subset of \mathbb{R} is compact if and only if it is closed and bounded.*

Definition (Perfect Set). A Perfect set is a set S that is closed and contains no isolated points.

A nonempty perfect set is uncountable.

Examples are Cantor Set and the set of all real numbers.

Definition (Separated, Disconnected, and Connected Set). Two sets A and B are separated if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

A set S is disconnected if it can be written as the union of two nonempty separated sets.

A set S is connected if it is not disconnected.

Definition (F_σ set). A set S is an F_σ set if it is the countable union of closed sets. A set is F_σ if and only if its complement is G_δ .

Definition (G_δ set). A set S is an G_δ set if it is the countable intersection of open sets. A set is G_δ if and only if its complement is F_σ .

Definition (Dense and Nowhere-Dense Set). We say a set S is dense in X if $\overline{S} = X$.

A set S is nowhere-dense if \overline{S} contains no open interval. ie $\overline{S}^\circ = \emptyset$.

A set E is nowhere-dense in R iff \overline{E}^c is dense in R .

Definition (Baire's Theorem). The set of Real numbers R cannot be written as the countable union of nowhere-dense sets.