# PDEs

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## Introduction

**What is a PDE?**
Start with ODE: $u = u(x)$, equation involing indepednant variable $x$ and dependent variable $u$ as well as its derivatives.
**Example:** $u'' - xu = 0, x \in I$ (Airy Functions). Second order Linear ODE.
$Lu = u'' - xu$ Where L is an operator.
Linearity means 2 things: $L(u_1 + u_2) = Lu_1 + Lu_2$ and $L(cu) = cLu$
$\forall u_1, u_2 \in \mathcal{F}, \forall c \in \mathbb{F}$
PDE: $u = u(x, y, ...)$ equation involving independent variables $x, y, ...$ and Function $u$ as well as its partial derivatives $u_x, u_y, u_{xx}, u_{yy}, u_{xy}$
**Example:** $x^2u - sin(xy)u_{xxyy} + 3u_x = 0$ 4th order linear PDE of 2 vars.
Remark: Importance of linearity: say $u_1, u_2$ are solutions of a Linear PDE: $Lu_1 = 0, Lu_2 = 0$ then $c_1u_1 + c_2u_2, (\forall c_1, c_2 \in \mathbb{R})$ is also a solution of $Lu = 0$
More generally, if $u_1, ..., u_n$ are solutions, then $\sum_{j=1}^{n} c_ju_j$ is also a solution.
**Example:** $u = u(x, y)$, solve $u_{xx} = 0$. $u_x = f(y)$, $u = f(y)x + g(y)$ where $f, g$ are arbitrary functions. $(\forall f, g \in \mathcal{F})$
$Lu = 0$ is homogenous, $Lu = f$ is non-homogenous.

## 1.2 First Order PDE of x,y

$$x, y, u_x, u_y, u$$

**Generally:** $a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0$
**Example 1:** $u_x = 0$: $u = f(y)$ No change in the x direction, hence the function stays constant on all horizontal lines.
**Example 2: Geometric Method** $au_x + bu_y = 0, (a, b \in \mathbb{R})$
$\vec{v} = (a, b)$; $\nabla u =< u_x, u_y >$; $\nabla u \cdot \vec{v} = 0 \to D_{\vec{v}}u = 0$
No change in the "v" direction (say $|\vec{v}| = 1$)
$x = ta, y = tb \to ay - bx = 0$ On the lines $ay - bx = c$ where $c$ is a constant, the function $u$ is constant. Lets call its value $f(c)$
$u(x, y) = f(ay - bx)$ where $f$ is a function of a single variable
The lines where these are solutions/constant are called **characteristic lines**

Check: $u_x = -bf'(ay-bx)$, $u_y = af'(ay-bx)$
**Change of variable** Change our plane such that $\vec{v}$ is our "x" axis.
View $(x,y) = x+iy, (x',y') = x'+iy'$. Multiplying by $e^{i\alpha}$ rotates the plane ccw by $\alpha$
$x'+iy' = (x+iy)e^{i\alpha}$ where $\vec{v} = (a,b) = (cos(\alpha), sin(\alpha))$
$x' = xcos(\alpha) + ysin(\alpha)$, $y' = -xsin(\alpha) + ycos(\alpha)$
Rewrite PDE in our new system: $u = u(x',y') = u(x'(x,y), y'(x,y))$

$$u_x = u_{x'}\frac{\partial x'}{\partial x} + u_{y'}\frac{\partial y'}{\partial x}$$

$$u_x = au_{x'} - bu_{y'}$$

$$u_y = u_{x'}\frac{\partial x'}{\partial y} + u_{y'}\frac{\partial y'}{\partial y}$$

$$u_y = au_{y'} + bu_{x'}$$

$$au_x + bu_y = 0$$

$$a^2u_{x'} + b^2u_{y'} = 0$$

$$u_{x'} = 0$$

$$u = f(y') = f(ay-bx)$$

**Example 3:** $u_x + yu_y = 0$
$u$ doesnt change in the direction of $\vec{v} = (1,y)$ at the point $(x,y)$
Lets call $C$ the characteristic curve: $\begin{cases} x = x(t), y = y(t) \end{cases}$ tangent to $\vec{v}$ at any $(x,y)$

$$\frac{d}{dt}u(x(t),y(t)) = 0$$

$$\frac{dy}{dx} = y \to y(x) = ce^x, (\forall c \in \mathbb{R})$$

$$u(x, ce^x) = f(c)$$

$$u(x,y) = f(ye^{-x})$$

**Remark:** More generally $a(x,y)u_x + b(x,y)u_y = 0$

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}\text{: ODE for characteristic curves}$$

### 1.3.1 Mass flow/Transport Equation/Continuity Equation

Substance that flows in space. (eg. fluid)
$\rho = \rho(x, y, z, t)$ density of the substance at point $(x, y, z)$ at time $t$
$\vec{v} = \vec{v}(x, y, z, t)$ velocity of the substance at point $(x, y, z)$ at time $t$
Consider $R$ as an arbitrary region in space.
Conservation of mass: $m(t) = \int_R \rho(x, y, z, t)dV$ mass in R at time t
Consider $[t, t + \Delta t]$, $m(t + \Delta t) = \int_R \rho(x, y, z, t + \Delta t)dV$
Substance leaves/enters in $R$ through the boundary $\partial R$
Consider a small part of the boundary call it $\partial S$ and see how much mass has left through this boundary patch over time period $[t, t + \Delta t]$
We want to introdouce the "normal" $\vec{n}$ over the boundary $\partial S$

$$\text{height} = \vec{v}\Delta t \cdot \vec{n} \text{ and area of base} = dS \to \text{volume} = \Delta t\vec{v} \cdot \vec{n}dS$$

$$\rho = \text{mass/vol} \to \Delta t\rho\vec{v} \cdot \vec{n}dS$$

$$\Delta m = \Delta t \int_{\partial R} \rho\vec{v} \cdot \vec{n}dS$$

Mass Conservation: $m(t + \Delta t) = m(t) - \Delta m$

$$\frac{1}{\Delta t} \int_R \rho(x, y, z, t + \Delta t) - \rho(x, y, z, t)dV = \int_{\partial R} \rho\vec{v} \cdot \vec{n}dS$$

$$= \int_R div(\rho\vec{v})dV$$

Where $div(\vec{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$
Let $\Delta t \to 0$

$$\frac{\partial \rho}{\partial t} + div(\rho\vec{v}) = 0$$

This is the Transport Equation.
**Example:** $\vec{v} = c(1, 0)$ and $\rho = \rho(x, t)$

$$\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = 0$$

$$\rho(t, x) = f(x - ct)$$

$\rho_t + c\rho_x = 0, t > 0, x \in \mathbb{R}, \rho(0, x) = \rho_0(x), x \in \mathbb{R}$ Initial condition

$$\rho(t, x) = \rho_0(x - ct)$$

## 1.3.2 Heat Equation/Diffusion/Energy Flux

Flow of energy: $\vec{q}(x, y, z, t)$ energy flux at point $(x, y, z)$ at time $t$
During the time interval $[t, t + \Delta t]$ the energy $\Delta E = \Delta t \int_{\partial R} \vec{q} \cdot \vec{n} dS$ has left the test volume $R$ through the boundary $\partial R$
Consider the patch $\partial S$ of the boundary $\partial R$ and the normal $\vec{n}$

$$\Delta t \vec{q} \cdot \vec{n} dS \to e(\vec{n}) \Delta t dS$$

Cauchy tensor deformation.
To measure the energy inside $R$ we need the specific heat $c(x, y, z)$ and it measure the energy containing in 1 degree of temperature in 1 unit mass.

$$c = \frac{e}{T \cdot \text{mass}} = \frac{e}{T \cdot \rho \text{vol}}$$

$$T c \rho = \frac{e}{\text{vol}}$$

$$E(t) = \int_R T(\vec{x}, t) \rho(\vec{x}) c(\vec{x}) dV$$

This is the energy inside R at time t.

$$E(t + \Delta t) = E(t) - \Delta E$$

$$\int_R T_t(\vec{x}, t) \rho(\vec{x}) c(\vec{x}) + div \vec{q} dV = 0$$

$$T_t c \rho + div \vec{q} = 0$$

Incomplete: we need to know how $\vec{q}$ depends on $T$
Forier's law of heat conduction: $\vec{q} = -k \nabla T$
$k(\vec{x})$ = heat conductivity of the material
Heat flows from hot to cold.

$$grad(t)$$

is the direction of the greatest increase of $T$

$$c \rho T_t - div(k \nabla T) = 0$$

Specific case: Assume $c, \rho, k$ are constants.

$$\nabla T = \begin{pmatrix} T_x \\ T_y \\ T_z \end{pmatrix} \cdot \left( d_x, d_y, d_z \right)$$

$$div \nabla T = \nabla \cdot \nabla T = \nabla^2 T = T_{xx} + T_{yy} + T_{zz}$$

This is the laplacian of $T$

$$T_t = \mathbf{D} \nabla^2 T, \mathbf{D} = \frac{k}{c \rho}$$

Fick's law of Diffusion.
High density to low density.

## Wave Equation

Consider a string
We have $x$ and $u(x,t)$
Consider a small part of the string $[x, x+\Delta x]$ called $d\ell$
There is a tangent force $T(x+\Delta x, t)$
The mass of the string is $m(x)$ from origin to x
Newton's law: $F = ma$

$$T(x+\Delta x, t) - T(x,t) = (m(x+\Delta x) - m(x))\vec{r}_t t(x,t)$$

Divide by $\Delta x$ and let $\Delta x \to 0$

$$\vec{T}_x(x,t) = \rho(x)\vec{r}_t t(x,t)$$

Where $\rho$ is the linear density of the string
$T$ is tangent to the string: $T$ is parallel to $\vec{r}_x$
introduce $\vec{\tau} = \frac{\vec{r}_x}{|\vec{r}_x|}$

$$T = T(x,t)\vec{\tau}$$

Where $T$ is constant along the string.
Assume small vibration so that $|u_x|$ is small.

$$\vec{r} = (1, u_x) = (1, 0)$$