

Chapter 2

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2.1 Wave Equation

$$u_{tt} = c^2 u_{xx}, x \in \mathbb{R} \text{ ***}$$

Theorem 1: d'Alembert

Any solution to *** is of form

$$u(x, t) = f(x + ct) + g(x - ct)$$

where f and g are twice differentiable functions.

Proof: Method 1

$$(d_t^2 - c^2 d_x^2)u = 0$$

$$(d_t - cd_x)(d_t + cd_x)u = 0$$

Let $v = (d_t + cd_x)u$

Then $(d_t - cd_x)v = 0$ This is a transport equation. The general solution is

$$v(x, t) = h(x + ct)$$

for some function h .

$$u_t + cu_x = h(x + ct)$$

Inhomogeneous transport equation.

We can use linearity to find the general solution. Since it will be the sum of homogenous plus another function

$$u = u_p + g(x - ct)$$

Where u_p is a particular solution.

$$u_p = h(x + ct)$$

$$h'c + ch' = f$$

$$h' = \frac{f}{2c}$$

$$u = \frac{1}{2c} \int f(x+ct)dx + g(x-ct)$$

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$$

$$f(x+ct), g(x-ct)$$

Recall that $f(x+ct)$ is a wave traveling left at speed c and $g(x-ct)$ is a wave traveling right at speed c

Thus the solution is a wave traveling with speed c in both directions.

The wave equation is bidirectional. (unlike the transport equation)

Remark: u is a superposition of two waves traveling in opposite directions at fixed speed c .

Method 2:

Characteristic variable:

$$\xi = x + ct, \eta = x - ct$$

Then we can write

$$u(x, t) = u(\xi(x, t), \eta(x, t))$$

$$\partial_x = \partial_\xi + \partial_\eta$$

$$\partial_t = c\partial_\eta - c\partial_\xi$$

Factoring gives us

$$\partial_t^2 - c^2\partial_x^2 = (\partial_t - c\partial_x)(\partial_t + c\partial_x)$$

Plugging in we get

$$-2c\partial_\eta \cdot 2c\partial_\xi$$

$$-4c^2\partial_\xi\partial_\eta = 0$$

Thus we can differentiate with respect to ξ and η to get the general solution.

$$u_{\xi\eta} = 0$$

Integrating gives us

$$u = f(\xi) + g(\eta)$$

Theorem 2: (d'Alembert 1747)

$$\begin{cases} u_{tt} = c^2 u_{xx}, t > 0, x \in \mathbb{R} \\ u(x, 0) = \phi(x), x \in \mathbb{R} \\ u_t(x, 0) = \psi(x), x \in \mathbb{R} \end{cases}$$

Where ϕ is initial displacement and ψ is initial velocity.

$$u(x, t) = \frac{1}{2} (\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)ds$$

Proof: By theorem 1:

$$u(x, t) = f(x + ct) + g(x - ct)$$

Find f,g using ICs

$$u(x, 0) = f(x) + g(x) = \phi(x)$$

$$u_t(x, t) = cf'(x + ct) - cg'(x - ct)$$

$$u_t(x, 0) = cf'(x) - cg'(x) = \psi(x)$$

$$\begin{cases} cf' + cg' = c\phi' \\ cf' - cg' = \psi \end{cases}$$

Adding gives us

$$\begin{cases} 2cf' = c\phi' + \psi \\ 2cg' = c\phi' - \psi \end{cases}$$

$$\begin{cases} f' = \frac{1}{2} \left(\phi' + \frac{\psi}{c} \right) \\ g' = \frac{1}{2} \left(\phi' - \frac{\psi}{c} \right) \end{cases}$$

$$f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c} \int_0^s \psi(y)dy + A$$

$$g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c} \int_0^s \psi(y)dy + B$$

$$\begin{cases} f(0) = \frac{1}{2}\phi(0) + A \\ g(0) = \frac{1}{2}\phi(0) + B \end{cases}$$

$$f(0) + g(0) = \phi(0) + A + B$$

$f + g = \phi$ which cancels out

Thus we can set $A + B = 0$

$$\begin{aligned} u(x, t) &= f(\xi) + g(\eta) \\ &= \frac{1}{2}\phi(\xi) + \frac{1}{2c} \int_0^\xi \psi(y)dy + A \\ &\quad + \frac{1}{2}\phi(\eta) - \frac{1}{2c} \int_0^\eta \psi(y)dy + B \end{aligned}$$

$$\begin{aligned}
u(x, t) &= f(\xi) + g(\eta) \\
&= \frac{1}{2}\phi(\xi) + \frac{1}{2c} \int_0^\xi \psi(y) dy \\
&\quad + \frac{1}{2}\phi(\eta) - \frac{1}{2c} \int_0^\eta \psi(y) dy \\
&= \frac{\phi(\xi) + \psi(\eta)}{2} + \frac{1}{2c} \left(\int_0^\xi \psi(y) dy - \int_0^\eta \psi(y) dy \right) \\
&= \frac{\phi(\xi) + \psi(\eta)}{2} + \frac{1}{2c} \int_\eta^\xi \psi(y) dy
\end{aligned}$$

2 families of characteristics lines:

$$x \pm ct = \text{constant}$$

On one of these lines they are constant.

Example 1:

$$\psi(x) = 0$$

$$u(x, t) = \frac{1}{2} (\phi(x + ct) + \phi(x - ct))$$

Initial displacement ϕ splits into two parts. Each with half the amplitude. and one is opposite direction from the other

Example 2:

$$\psi(x) = 0$$

Interaction of "pulse".

They constructively interfere and they keep their shape.

Example 3:

$$\phi(x) = 0$$

$$u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy$$

If x is too large of magnitude it will be zero.

As we increase t , the integral will increase.

For larger t , I have more choices of x that make u nonzero.

Example 4:

$$\psi(x) = 0, \phi(s) = \sin(s)$$

$$u(x, t) = \frac{1}{2} (\sin(x + ct) + \sin(x - ct))$$

$$u(x, t) = \sin(x) \cos(ct)$$

This is a standing wave! **Remark 1:**

If $\psi = 0$ and ϕ is localized then the right and left moving waves will (in the long run) always separate.

If ϕ is not localized then they don't separate **Remark 2:** $\phi = 0$ and ψ is

localized, fix x_0 then $u(x_0, t) = 0$ for all $t > t_0$.

The initial disturbance is felt at x_0 and then the affect will be gone

2.2 Causality and Energy

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x, 0) = \phi(x) \\ u_t(x, 0) = \psi(x) \end{cases}$$

D'Alembert solution:

$$u(x, t) = \frac{1}{2} (\phi(x + ct) + \phi(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds$$

Lemma: If ϕ and ψ are compactly supported in $[a, b]$, (this means outside the interval they are zero) then $u(x, t) = 0$ outside of R where R is the "domain of influence of the interval $[a, b]$ "

The boundary lines are $x - ct = b$ and $x + ct = a$

This is like an open cone.

$R =$ region where disturbances are felt

Proof:

By assumption $\phi(s)$ and $\psi(s)$ are zero for $s < a$

Thus the outside area is $x + ct < a$

Also $x - ct \leq x + ct < a$

In that region $u(x, t) = 0$

Similarly we can do this for the region $x - ct > b$

Thus we can conclude that $u(x, t) = 0$ outside of the region R . Due to the fact that both $\phi = 0$ and the integral $\int_{x-ct}^{x+ct} \psi(s) ds = 0$

In the limiting case. This is a "light cone"

This is where we have x_0 being the point of disturbance.

The region of influence is the area where the disturbance is felt.

$$x + ct = x_0, x - ct = x_0$$

$$x \pm ct = x_0$$

$\phi(x_0)$ and $\psi(x_0)$ affect the values of $u(x, t)$ if $x \pm ct = x_0$ or $x_0 \in [x - ct, x + ct]$

Reverse Questions:

Fix value of u and see what initial conditions affect this.

We need initial values at $x_0 \pm ct_0$ and in $[x_0 - ct_0, x_0 + ct_0]$

This is like a triangle

This is called the interval/region of dependance of the point x_0 .

Remark:

$$u(x_0, t_0) = \frac{1}{2} (\phi(a) + \phi(b)) + \frac{1}{2c} \int_a^b \psi(s) ds$$

$$b - a = 2ct_0$$

$$u(x_0, t_0) = \frac{1}{2} (\phi(a) + \phi(b)) + t_0 \cdot \frac{1}{b-a} \int_a^b \psi(s) ds$$

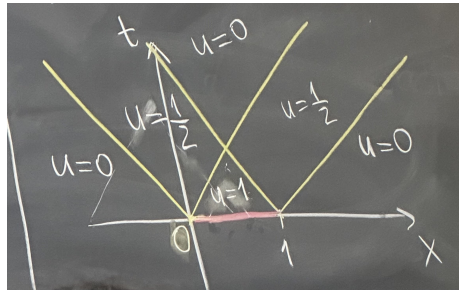
Note that the first term is average displacement and the second term is average velocity.

Example:

$$\psi = 0, \phi = \begin{cases} 1, & x \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

say $c = 1$

Now: add image of the region of influence.



We can view this as two Vs that intersect at a point and then separate.

Outside the two Vs it is 0, inside both it is 1, inside one it is 1/2

And if it is in the middle it is also 0

Finite speed of propagation:

Localized initial data ie compactly supported. (0 outside some interval)

This implies that $u(x, t)$ is also localize for any $t > 0$.

If ϕ, ψ are supported in $[a, b]$ then $u(x, t)$ is supported in $[a - ct, b + ct]$

when you are at position x you would feel the disturbance at time $t \geq \frac{x-b}{c}$ or distance from x to the interval

Remark:

The wave equation has no smoothing effect.

Singularity and discontinuities remain

Non-smooth initial data does not become smooth at any time.

Say ϕ has a discontinuity/ singularity at x_0 , then $u(x, t)$ is singular whenever $x \pm ct = x_0$

Singularity propagates along characteristics.

Causality:

Causality means the effect comes after the cause.

Cause = disturbance at x_0

Effect = disturbance at other points

Since there is a finite speed of propagation, the disturbance at x_0 can only affect points in the region of influence.

Huygen's Principle:

The disturbance in $[a, b]$ reaches x_1 at time t_1 and then it remains in the region of influence for all $t > t_1$.

If $\psi = 0$ region of influence of $[a, b]$ is just the effect is felt only at time t_1 and no lingering effects. In other words, just feel once then gone.

Kirkchoff's formula

In 3d the solution to the wave equation is given by Kirkchoff's formula.

it only uses ϕ and ψ on the boundary of the region of influence.

this is true for $n \geq 3$ and odd dimensions.

So in 2D this is not true!

Energy Conservation:

$$u_{tt} = c^2 u_{xx}$$

$$E(t) = \frac{1}{2} \int_{\mathbb{R}} u_t^2(t, x) + c^2 u_x^2(t, x) dx$$

Then $E(t)$ is constant in time. In particular $E(t) = E(0)$

Remark:

ϕ, ψ are localized, in other words, vanish outside of some interval. $[a, b]$

Thus the function $u(x, t)$ is also localized in $[a - ct, b + ct]$

Proof:

$$E'(t) = \int_{\mathbb{R}} u_t u_{tt} + c^2 u_x u_{xt} dx$$

$$E'(t) = \int_{\mathbb{R}} u_t c^2 u_{xx} + c^2 u_x u_{xt} dx$$

This motivates integrating by parts.

$$\int_{\mathbb{R}} u_x u_{tx} dx = [u_x u_t]_{-\infty}^{\infty} - \int_{\mathbb{R}} u_{xx} u_t dx$$

$$E'(t) = \int_{\mathbb{R}} u_t c^2 u_{xx} - c^2 u_{xx} u_t dx = 0$$

Remark:

In higher dimensions u_x^2 is replaced by $|\nabla u|^2$

0.1 2.3: Heat Equation

$$u_t = ku_{xx}$$

Now we consider this on a finite interval

$$t \in (0, T), x \in (0, L)$$

$$R = (0, T) \times (0, L)$$

$$\bar{R} = [0, T] \times [0, L]$$

This is including the boundary.

We also was all boundary without the top "t" side call it Γ

This is called the parabolic boundary.

Max Principle:

If $u(x, t)$ is a solution to the heat equation in R and it is continuous on \bar{R} Then
 $\max(u)_{\bar{R}} = \max(u)_{\Gamma}$

This is an extension of the extreme value theorem

Proof:

$$v(t, x) = u(t, x) + \epsilon x^2$$

Where $\epsilon > 0$

$$v_t = u_t = ku_{xx} = k(v_{xx} - 2\epsilon)$$

$$v_{xx} = u_{xx} + 2\epsilon$$

Thus we gain:

$$v_t = kv_{xx} - 2k\epsilon$$

$$v_t < kv_{xx}$$

v attains its maximum on the boundary.

Let $v(t_0, x_0) = \max(V)_{\bar{R}}$ if $(t_0, x_0) \in \bar{R}$ then $v_t \leq kv_{xx}$

$$v(t_0, x_0) = 0$$

$$f(x) = v(t_0, x) \text{ has max at } x_0, \text{ then}$$

$$f''(x) = 0$$

$$v_t(t_0, x_0) < kv_{xx}(t_0, x_0) \leq 0$$

This is contradiction.

if $(t_0, x_0) \in \text{top boundary}$ ie $t_0 = T, x \in (0, L)$

$$g(t) = v(t, x_0)$$

has max at T then $g'(T) \geq 0$

$$g'(T) = v_t(T, x_0) = u_t(T, x_0) \leq 0$$

$$0 \geq v_t < kv_{xx} \geq 0$$

This is a contradiction.

Consider

Why does it work on gamma?

Thus $\max(v)_\Gamma = \max(v)_{\bar{R}}$

Now we need to prove this for u

Let $M = \max(u)_\Gamma$

goal: $u(t, x) \leq M$ for all $(t, x) \in \bar{R}$

$$\begin{aligned} u + \epsilon x^2 &= V \\ &\leq \max(V)_\Gamma \\ &= \max(u + \epsilon x^2)_\Gamma \\ &\leq \max(u)_\Gamma + \epsilon L^2 \end{aligned}$$

Let $\epsilon \rightarrow 0$

$$u \leq \max(u)_\Gamma$$

Remark:

Strong max principle says that $\max(u)_{\bar{R}}$ can **only** be attained on Γ

Immediate consequence of this the solution to the heat equation is unique.

Corollary:

We can do this for minimum as well.

Proof:

Let $v = -u$

Then $v_t = -u_t = -ku_{xx} = kv_{xx}$

Then we can apply the max principle to v

Theorem: Uniqueness

The solution to the heat equation is unique.

Proof:

$$\begin{cases} u_t = ku_{xx} + f(t, x) \\ u = u_0 \end{cases}$$

The following are the initial/boundary conditions.

$$\begin{cases} u(t, 0) = a(t) \\ u(t, L) = b(t) \\ u(0, x) = \phi(x) \end{cases}$$

This has at most one solution.

Proof:

Suppose u_1, u_2 are two solutions.

Let $w = u_1 - u_2$

Then $w_t = kw_{xx}$

$$\begin{cases} w(t, 0) = 0 \\ w(t, L) = 0 \\ w(0, x) = 0 \end{cases}$$

$$\max(w)_{\bar{R}} = 0$$

$$\min(w)_{\bar{R}} = 0$$

Thus $w = 0$ ie $u_1 = u_2$ in R

Remark:

We cant prescribe data for u on top boundary. It is determined by the PDE and the initial data.

Proof 2:

$$E(t) = \int_0^L w^2(t, x) dx$$

$$E'(t) = 2 \int_0^L ww_t dx = 2 \int_0^L kw w_{xx} dx$$

$$= 2k [ww_x]_0^L - 2k \int_0^L w_x^2 dx$$

$w = 0$ on the boundary.

$$= -2k \int_0^L w_x^2 dx \leq 0$$

$$E'(t) \leq 0$$

ie heat dissipates over time.

$$E(t) \leq E(0)$$

$$\int_0^L w^2(0, x) dx = 0$$

Since we know that by our definition of the integral it is non negative.

Thus $0 \geq w^2(0, x) \geq 0$

Thus $w = 0$

Thus $u_1 = u_2$

Remark:

The heat equation is a smoothing effect.

$$u_{xx}(A) < 0 \implies u_t(A) < 0$$

Peaks go down and valleys go up.

Want to reach equilibrium.

Remark:

Uniqueness for heat IBVP holds in:

Stability estimates:

Lemma 1:

Consider R and Γ

$$\begin{cases} u_t = ku_{xx}, \in R \\ u = u_0 \in \Gamma \end{cases}$$

$$\max(u)_{\bar{R}} = \max(u)_{\Gamma}$$

Proof:

$$\begin{aligned} -|u_0| &\leq u_0 \leq |u_0| \\ u &\leq \max(u)_{\bar{R}} = \max(u)_{\Gamma} \leq \max(|u|)_{\Gamma} \\ -u &\geq \min(u)_{\bar{R}} = \min(u)_{\Gamma} \geq -\max(|u|)_{\Gamma} \\ |u| &\leq \max(|u|)_{\Gamma} \text{ everywhere} \end{aligned}$$

Initial/boundary data u_0 controls u everywhere.

Corollary:

$$\begin{cases} \partial_t u_1 = k\partial_{xx} u_1 u_1(x, 0) = \phi_1(x) \\ u_1(0, t) = u_1(l, t) = 0 \end{cases}$$

$$\begin{cases} \partial_t u_2 = k\partial_{xx} u_2 u_2(x, 0) = \phi_2(x) \\ u_2(0, t) = u_2(l, t) = 0 \end{cases}$$

We think u_1 as the true one and the u_2 is the experimentally derived one with perturbations.

$$\max|u_2 - u_1|_{\bar{R}} \leq \max|\phi_2 - \phi_1|_{[0, l]}$$

Proof:

Let $u = u_2 - u_1$

Let $\phi = \phi_2 - \phi_1$

u will solve the heat equation with initial condition ϕ

$$\begin{cases} u_t = ku_{xx} \\ u(x, 0) = \phi(x) \\ u(0, t) = u(l, t) = 0 \end{cases}$$

By lemma 1 we have that $\max(u)_{\bar{R}} \leq \max(u)_{[0, l]}$

Lemma 2:

$$\int_0^l u^2 dx \leq \int_0^l \phi^2 dx$$

$$\begin{cases} u_t = ku_{xx} \\ u(x, 0) = \phi(x) \\ u(0, t) = u(l, t) = 0 \end{cases}$$

Proof:

$$E(t) = \int_0^l u^2 dx$$

$$E'(t) \geq 0 \implies E(t) \leq E(0)$$

Corollary 2:

In the setting of Corollary 1:

$$\int_0^l |u_2 - u_1|^2 dx \leq \int_0^l |\phi_2 - \phi_1|^2 dx$$

$f = f(x)$ on I

$$\|f\|_\infty = \max |f|_I$$

$$\|f\|_\infty = 0 \implies f = 0$$

$$\|f\|_{L^2} = \left(\int_I |f|^2 dx \right)^{1/2}$$

This is known as the L^2 norm; mean-squared

$$|f(x)| \leq \|f(x)\|_\infty$$

$$|f(x)|^2 \leq \|f(x)\|_\infty^2$$

$$\int_I |f(x)|^2 dx \leq \|f(x)\|_\infty^2 \int_I dx$$

$$\|f\|_{L^2} \leq \|f\|_\infty \sqrt{|I|}$$

$$\|u_2(\star, t) - u_1(\star, t)\|_{L^2} \leq \|\phi_2 - \phi_1\|_{L^2}$$

The size at any time is controlled by the initial data.

2.4: Heat Equation on the Real Line

Theorem:

$$S(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

This is the fundamental solution to the heat equation.

1. $S_t = S_{xx}$
2. $\int_{-\infty}^{\infty} S(x, t) dx = 1, \forall t > 0$
3. $\lim_{t \rightarrow 0^+} \int_{\mathbb{R}} S(x, t) f(x) dx = f(0)$ for all continuous and bounded f on \mathbb{R}

$$\lim_{t \rightarrow 0^+} S(x, t) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \quad \textbf{Proof:}$$

Need to prove:

$$\lim_{t \rightarrow 0^+} \int_{\mathbb{R}} S(x, t) [f(x) - f(0)] dx = 0$$

Fix $\epsilon > 0$ then there exists $\delta > 0$ such that $|f(x) - f(0)| < \epsilon$ for $|x - 0| < \delta$

$$\begin{aligned} I &= \int_{|x| < \delta} + \int_{|x| \geq \delta} \\ |I_1| &\leq \int_{|x| < \delta} S(x, t) |f(x) - f(0)| dx \\ |f(x) - f(0)| &< \epsilon \\ |I_1| &\leq \epsilon \int_{|x| < \delta} S(x, t) dx \leq \epsilon \end{aligned}$$

This works for any t

$$\begin{aligned} |f(x)| &\leq M \\ |f(x) - f(0)| &\leq 2M \\ |I_2| &\leq 2M \int_{|x| \geq \delta} S(x, t) dx \end{aligned}$$

Let $y = \frac{x}{\sqrt{4t}}$
Then $y^2 = \frac{x^2}{4t}$

$$\begin{aligned} |I_2| &\leq 2M \int_{|y| \geq \frac{\delta}{\sqrt{4t}}} \frac{1}{\sqrt{4\pi t}} e^{-y^2} \sqrt{4t} dy \\ |I_2| &\leq \frac{2M}{\sqrt{\pi}} \int_{|y| \geq \frac{\delta}{\sqrt{4t}}} e^{-y^2} dy \end{aligned}$$

Notice that the integral goes to 0 as $t \rightarrow 0$

$$\begin{aligned}\int_R g(y)dy &= \lim_{a \rightarrow \infty} \int_{-c}^c g(y)dy \\ \int_R g(y)dy &= \lim_{t \rightarrow 0^+} \int_{-1/t}^{1/t} g(y)dy \\ \int_R g(y)dy &= \lim_{t \rightarrow 0^+} \int_{|y| \geq 1/t} g(y)dy\end{aligned}$$

We can see this goes to 0 as $t \rightarrow 0$

Definition

$$\lim_{t \rightarrow 0^+} S(x, t) = \delta_0(x)$$

which means (3).

Formally we can write this as:

$$\delta_0(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta_0(x) dx = 1$$

Corollary:

Take any y and consider $F(x, t) = S(x - y, t)$

then (1), (2) hold for F and

$$\lim_{t \rightarrow 0^+} *F(x, t) = \delta_y(x) \iff \lim_{t \rightarrow 0^+} \int_R F(x, t) f(x) dx = f(y)$$

$$u = u(t, y)$$

$$\begin{cases} u_t = k u_{yy} \\ u(0, y) = \phi(y) \end{cases}$$

$$\phi(x) = \int_{-\infty}^{\infty} \phi(y) \delta_x(y) dy$$

Theorem:

$$u(t, y) = \int_{-\infty}^{\infty} \phi(y) S(y, t) dy$$

This solves heat equation !!!

IC is considered as a limit

$$S(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

$$S_t = S_{xx}$$

$$\int_{-\infty}^{\infty} S(x, t) dx = 1$$

For any $y \in \mathbb{R}$

$$S_t(x - y, t) = S_{xx}(x - y, t)$$

$$\int_{-\infty}^{\infty} S(x - y, t) dy = 1$$

$$\int_{-\infty}^{\infty} S(x - y, t) f(y) dy = f(x)$$

Theorem:

Let f be bounded and continuous on \mathbb{R}

$$u(t, x) = \int_{-\infty}^{\infty} f(y) S(x - y, t) dy$$

Then $u(t, x)$ solves the heat equation.

In other words,

$$\begin{cases} u_t = k u_{xx} \\ u(0, x) = f(x) \end{cases}$$

Proof:

$$u_t = \int_{-\infty}^{\infty} f(y) S_t(x - y, t) dy$$

$$u_{xx} = \int_{-\infty}^{\infty} f(y) S_{xx}(x - y, t) dy$$

Since we know that S solves the heat equation we can say thus u solves the heat equation.

Also we know the limit of S is the delta function and thus must solve the initial condition.

Remark:

$$\begin{cases} u_t = \Delta u \\ u(0, x) = f(x) \end{cases}$$

$$u(t, x) = \int_{\mathbb{R}^n} f(y) S(x - y, t) dy$$

$$S(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

Remark 2:

If f is bounded and piecewise continuous then

$$\lim_{t \rightarrow 0^+} u(t, x) = \frac{f(x^-) + f(x^+)}{2}$$

Theorem: Smoothing Property

If f is bounded and piecewise continuous then

$$u(x, t) \text{ is infinitely differentiable in } x \text{ for } t > 0$$

This is unlike the wave equation.

Proof:

put all derivatives inside the integral and hit S

Clearly S is infinitely differentiable

Remark:

Convolution is smoothing.

$$u(x, t) = (S \star f)(x, t)$$

This is essentially a (normal) weighted average of f

Note:

1) The wave equation is time reversible.

$$\begin{aligned} &\begin{cases} u_{tt} = u_{xx} \\ u(x, 0) = \phi(x) \\ u_t(x, 0) = \psi(x) \end{cases} \\ &v(x, t) = u(x, -t) \\ &v_t = -u_t \\ &v_{tt} = u_{tt} \end{aligned}$$

Thus backwards and forwards are the same.

2) The heat equation is not time reversible.

$$\begin{aligned} &\begin{cases} u_t = u_{xx} \\ u(x, 0) = \phi(x) \end{cases} \\ &v(x, t) = u(x, -t) \\ &v_t = -u_t \\ &-v_t = v_{xx} \end{aligned}$$

This is different and is called the backwards heat equation.

This is an ill posed problem. Thus it has problems with Uniqueness, Existence, and Stability

Infinite speed of propagation

If initial $f(x)$ is localized and positive then the solution to the heat equation is $u(x, t) > 0, \forall x \in R, t > 0$

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} f(y) dy$$

This is the solution of the heat equation with initial condition $f(x)$

$$\begin{cases} u_t = u_{xx} \\ u(x, 0) = f(x) \end{cases}$$

Let $v(x, t) = u(x, t/k)$

For ease let us call $s = t/k$

$$\begin{aligned} v_t &= u_s \frac{1}{k} \\ v_{xx} &= u_{xx} \\ v_t &= v_{xx} \end{aligned}$$

This implies

$$\begin{cases} v_t = v_{xx} \\ v(x, 0) = f(x) \end{cases}$$

This can be rewritten as

$$\begin{aligned} v(x, t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} f(y) dy \\ v(x, kt) &= u(x, t) \end{aligned}$$

Thus

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} f(y) dy$$

Thus k only speeds up or slows down the solution.

Derivation of the Fundamental Solution:

$$u_t = u_{xx}$$

We can consider the symmetries of the equation!

Study the symmetries: Scale the variables

$$\begin{aligned} t &\rightarrow Bt \text{ and } x \rightarrow Ax \\ v(x, t) &= u(Ax, Bt) \\ v_t &= u_t(Ax, Bt) = Bu_t(Ax, Bt) \\ v_{xx} &= u_{xx}(Ax, Bt) = A^2 u_{xx}(Ax, Bt) \end{aligned}$$

$$v_t = \frac{B}{A^2} v_{xx}$$

We get the same equation if $B = A^2$

Let $\lambda = A$ If $u(x, t)$ solves the heat equation then $u(\lambda x, \lambda^2 t)$ also solves the heat equation for any λ .

Look for a solution that is unaffected by scaling. (scale invariant, self-similar)

$$u(x, t) = u(\lambda x, \lambda^2 t), \forall x \in R, t > 0, \forall \lambda \in R$$

We can choose a useful choice of λ

Choose lambda such that x or $t = 1$

Ex: $\lambda = \frac{1}{\sqrt{t}}$

$$u(x, t) = u(x/\sqrt{t}, 1)$$

Thus $u(x, t) = f(x/\sqrt{t})$

$$y = \frac{x}{\sqrt{t}}$$

Substitute y into the heat equation

$$u_t = f' \cdot x \cdot \frac{-1}{2t^{3/2}}$$

$$u_{xx} = f'' \cdot \frac{1}{t}$$

$$u_t = u_{xx}$$

$$f' \cdot x \cdot \frac{-1}{2t^{3/2}} = f'' \cdot \frac{1}{t}$$

$$\frac{-x}{2\sqrt{t}} f' = f''$$

$$f''(y) + \frac{1}{2} y f'(y) = 0$$

This is a second order ODE.

$$f''(y) = -\frac{1}{2} y f'(y)$$

$$\frac{f''}{f'} = -\frac{1}{2} y$$

$$\ln|f'| = -\frac{1}{4} y^2 + C$$

$$f' = C_1 e^{-\frac{1}{4} y^2}$$

$$\text{let } g(y) = \int e^{-\frac{1}{4} y^2} dy$$

$$f = C_1 g(y) + C_2$$

Need to solve

$$\begin{cases} S_t = S_{xx} \\ S(x, 0^+) = \delta_0(x) \end{cases}$$

Can we match this IC?

No, because of the δ_0

Which is not a function! to work with a function consider the integral of the delta function.

$$\int_{-\infty}^x \delta_0(x) dx = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

$$\int_{-\infty}^{\infty} f(y) \delta_0(y) dy = f(0)$$

This is called the Heaviside function.

$$H(x) = \int_{-\infty}^x \delta_0(y) dy$$

Solve

$$\begin{cases} F_t = F_{xx} \\ F(x, 0) = H(x) \end{cases}$$

We also know that the derivative of a solution of the heat equation also solves the heat equation.

$$u(x, t) = C_1 g(x/\sqrt{t}) + C_2$$

$$u(x, 0) = C_1 g(\infty) + C_2 = 1, \forall x > 0$$

$$u(x, 0) = C_1 g(\infty) + C_2 = 0, \forall x < 0$$

$$g(\pm\infty) = \sqrt{\pm\pi}$$

$$C_1 = \frac{1}{2\sqrt{\pi}}$$

$$C_2 = 1/2$$

$$S = F = \frac{1}{2\sqrt{\pi}} g'(x/\sqrt{t}) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

NEW SECTION

1. Heat + Dirichlet BC

$$\begin{cases} u_t = u_{xx} \\ u(x, 0) = f(x) \\ u(0, t) = 0 \\ u(\pi, t) = 0 \end{cases}$$

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t} \sin(nx)$$

Can we choose c_n such that $u(x, 0) = f(x)$

3. Heat + Neumann BC

$$\begin{cases} u_t = u_{xx} \\ u(x, 0) = f(x) \\ u_x(0, t) = 0 \\ u_x(\pi, t) = 0 \end{cases}$$

This is an insulated endpoint.

$$u(x, t) = X(x)T(t)$$

$$\frac{T'}{T} = \frac{X''}{X} = \alpha$$

$$T'(t) = \alpha T(t)$$

$$T(t) = e^{\alpha t}$$

$$X'' = \alpha X$$

We have BC: $u_x(0, t) = X'(0)T(t) = 0$ and $u_x(\pi, t) = X'(\pi)T(t) = 0$

Eigenval problem:

$$\begin{cases} X'' = \alpha X \\ X'(0) = 0 \\ X'(\pi) = 0 \end{cases}$$

if $\alpha > 0 \implies c_1, c_2 = 0, X = 0$

if $\alpha \leq 0 \implies X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$

$$X'(x) = -c_1 \lambda \sin(\lambda x) + c_2 \lambda \cos(\lambda x)$$

$$X'(0) = 0 \implies c_2 \lambda = 0$$

Thus $c_2 = 0$

$$X'(\pi) = -\lambda c_1 \sin(\lambda \pi) = 0$$

$$\lambda = n$$

Thus (1) has e-vals $\alpha_n = -n^2$

With corresponding e-funcs $X_n(x) = \cos(nx)$

$$\text{Thus } u_n(x, t) = e^{-n^2 t} \cos(nx)$$

Consider the linear combination of these solutions.

These solved the PDE and BC, but what about the IC? $f(x) = \sum_{n=1}^{\infty} c_n \cos(nx)$

3. Laplace's Equation in a Rectangle

$$\begin{cases} \Delta u = 0 \\ u = f \text{ on } \partial R \end{cases}$$

$$u_{xx} + u_{yy} = 0$$

$$\begin{cases} u(x, 0) = f_0(x) \\ u(x, 1) = f_1(x) \\ u(0, y) = g_0(y) \\ u(\pi, y) = g_1(y) \end{cases}$$

Linearity principle:

$$u = u_1 + u_2 + u_3 + u_4$$

It is enough to consider 1 piece.

Consider the "top" boundary

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0, y) = u(\pi, y) = 0 \\ u(x, 0) = 0 \\ u(x, 1) = f_1(x) \end{cases}$$

Seperate variables: $u(x, y) = X(x)Y(y)$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \alpha$$

$$u(0, y) = X(0)Y(y) = 0$$

$$\begin{cases} X'' = \alpha X \\ X(0) = X(\pi) = 0 \end{cases}$$

$$\alpha_n = -n^2$$

$$X_n(x) = \sin(nx)$$

$$\begin{cases} Y'' = -\alpha Y \\ Y(0) = 0 \end{cases}$$

$$Y = 2c_1 \sinh(ny)$$

Thus $u_n(x, y) = \sinh(ny) \sin(nx)$ plus the 3 BC on the top.

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh(ny) \sin(nx)$$

Reamains the 4th BC.

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(nx)$$

4. Laplace's Equation in a Disk

$$\begin{cases} \Delta u = 0 \\ u = f \text{ on } \partial D \end{cases}$$

Now we want to write in polar coordinates.

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

$$\Delta u = u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r$$

We have BC $u(1, \theta) = f(\theta)$, $u(r, \theta)$ is bounded as $r \rightarrow 0$ $u(r, \pi) = u(r, -\pi)$, $u_\theta(r, \pi) = u_\theta(r, -\pi)$

$$\begin{aligned} u(r, \theta) &= R(r)\Theta(\theta) \\ R''T + \frac{1}{r}R'T + \frac{1}{r^2}RT'' \\ \frac{r^2R'' + rR'}{R} &= -\frac{\Theta''}{\Theta} = \alpha \end{aligned}$$

Now we have 2 odes.

$$\begin{cases} T'' = -\alpha T \\ r^2R'' + rR' + \alpha R = \alpha R \end{cases}$$

$$\begin{aligned} RT(\pi) &= RT(-\pi) \\ R(T(\pi) - T(-\pi)) &= 0 \end{aligned}$$

Clearly $T(\pi) - T(-\pi) = 0$

We take $\alpha > 0$

$$\begin{aligned} T(\theta) &= c_1 \cos(\lambda\theta) + c_2 \sin(\lambda\theta) \\ -\lambda c_1 \sin(\lambda\pi) + \lambda c_2 \cos(\lambda\pi) &= \lambda c_1 \sin(\lambda\pi) + \lambda c_2 \cos(\lambda\pi) \\ \begin{cases} c_2 \sin(\lambda\pi) = 0 \\ c_1 \sin(\lambda\pi) = 0 \end{cases} \end{aligned}$$

Clearly $\sin(\lambda\pi) = 0$

$\lambda = n$

$\alpha = n^2$ has 2 LI eigenfunctions with dim 2 eigenspace. We can then go to our other equation

$$r^2R'' + rR' - n^2R = 0$$

If $n = 0$ then $R = c_1 + c_2 \ln(r)$

Thus we have 2 LI solutions: 1 and $\ln(r)$

If $n > 0$ then $r^2R'' + rR' - n^2R = 0$

This is Euler's ode: we guess $R = r^p$

$$\begin{aligned} r^2p(p-1)r^{p-2} + rpr^{p-1} - n^2r^p &= 0 \\ p(p-1) + p - n^2 &= 0 \\ p^2 - n^2 &= 0 \\ p &= \pm n \end{aligned}$$

Thus our solutions are r^n and r^{-n}

In summary:

If $n = 0$ then $T = 1$ and $R = 1, \ln(r)$ (nope)

If $n > 0$ then $T = \cos(n\theta), \sin(n\theta)$ and $R = r^n, r^{-n}$

$$u_n(r, \theta) = \frac{a_0}{2} \sum_{n=1}^{\infty} r^n c_n \cos(n\theta) + d_n \sin(n\theta)$$

That satisfies the PDE and the bc: bounded as $r \rightarrow 0$ and $u(r, \pi) = u(r, -\pi)$

Need: $f(\theta) = \sum_{n=1}^{\infty} c_n \cos(n\theta) + d_n \sin(n\theta)$

Dirichlet BC: given an f on $[0, \pi]$ find $f(\theta) = \sum b_n \sin(nx)$

Neumann BC: given an f on $[0, \pi]$ find $f(\theta) = a_0/2 + \sum b_n \cos(nx)$

Periodic BC: given an f on $[-\pi, \pi]$ find $f(\theta) = a_0/2 + \sum b_n \cos(nx) + c_n \sin(nx)$

Chapter 5: Fourier Series

Say f is given by a Fourier sin series.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

How to find b_n Notation: $\|f\|_{L^2(a,b)}^2 = \int_a^b |f(x)|^2 dx$

$$\langle f, g \rangle_{L^2(a,b)} = \int_a^b f(x)g(x)dx$$

$$\text{Kronecker Delta: } \delta_{nm} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

Orthogonality of Sine Functions:

$$\langle \sin(nx), \sin(mx) \rangle_{L^2(0,\pi)} = \int_0^\pi \sin(nx) \sin(mx) dx = \pi/2 \cdot \delta_{nm}$$

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) dx = \frac{2}{\pi} \langle f, \sin(nx) \rangle$$

This is cool because $\sin(nx)$ are eigenfunctions of $-\frac{d^2}{dx^2}$
eigenvectors of symmetric matrix A are orthogonal.

We can see that $\frac{d^2}{dx^2} \sim A$ and the matrix A are symmetric.

L is a symmetric linear operator.

This means that $Lu_1 = \lambda_1 u_1$ and $Lu_2 = \lambda_2 u_2$

In other words the eigenfunctions of a symmetric operator are orthogonal.

Suppose $L^T = L$ and $\lambda_1 \neq \lambda_2$

Then $\langle u_1, u_2 \rangle = 0$

Proof:

$$\langle Lu_1, u_2 \rangle = \langle u_1, L^T u_2 \rangle$$

$$\langle \lambda_1 u_1, u_2 \rangle = \langle u_1, \lambda_2 u_2 \rangle$$

$$\lambda_1 \langle u_1, u_2 \rangle = \lambda_2 \langle u_1, u_2 \rangle$$

$$(\lambda_1 - \lambda_2) \langle u_1, u_2 \rangle = 0$$

$$\lambda_1 \neq \lambda_2 \implies \langle u_1, u_2 \rangle = 0$$

Claim:

L is symmetric with respect to the inner product: $\langle f, g \rangle = \int_0^\pi f(x)g(x)dx$

For functions solving the DBC.

Taylor vs Fourier Series:

Taylor: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

Fourier: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin(nx)$

Taylor: always converges on an interval

Fourier: convergence is not so nice

Taylor: $f(x)$ is infinitely differentiable

Fourier: $f(x)$ is only continuous, sometimes discontinuous

Taylor: local information

Fourier: global information