# 01:640:350H - Homework 3

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## 2.1: Problem 2

**Prove that $T$ is a linear transformation.**

$$T : R^3 \to R^2 \text{ defined by } T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$$

We will verify that:

$$T = L_A \text{ where } A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Now let's check that $T = L_A$: For $a_1, a_2 \in \mathbb{R}$

$$L_A \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = A \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 \\ 2a_3 \end{pmatrix}$$

Clearly $T = L_A$. Thus $T$ is a linear transformation due to the fact that matrix multiplication is a linear operation.
**Find bases for both $N(T)$ and $R(T)$.**
Since we have that $T = L_A$, we can find the bases for $N(T)$ and $R(T)$ by finding the null space and column space of $A$.
First, we can bring $A$ to RREF form and then call it $R$:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}r_2 \to r_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can see that for a null space, any element $x \in N(t)$ must solve $Ax = 0$. Additionally any element $x \in N(t)$ that solves $Ax = 0$ must also solve $Rx = 0$. We can see that the null space is

$$N(T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

due to the fact that $a_2$ is a free variable. Thus we have that a basis for the null space is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.
For the column space, we can see that the column space is the span of the columns of $A$ that are pivot columns of $R$. Thus we have

$$R(T) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$$

Thus we have that a basis for the column space is $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$.

**Compute nullity and rank of $T$ and verify the dimension theorem**
The nullity of $T$ is the dimension of the null space of $T$. We have that the null space of $T$ is spanned by $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$. Thus the nullity of $T$ is 1.
The rank of $T$ is the dimension of the column space of $T$. We have that the column space of $T$ is spanned by $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$. Thus the rank of $T$ is 2.
The dimension theorem states that the rank of $T$ plus the nullity of $T$ is equal to the dimension of the domain of $T$. We have that the dimension of the domain of $T$ is 3. Thus we have that $2 + 1 = 3$ which verifies the dimension theorem.

**Use the approriate theorms in this sections to determine where T is one-to-one or onto.**
Clearly since $N(T) \neq \{0\}$, $T$ is not one-to-one.
We can also see that if $dim(R(T)) = dim(W)$, then $T$ is onto. We have that $dim(R(T)) = 2$ and $dim(W) = 2$. Thus $T$ is onto.

## 2.1: Problem 3

**Prove that $T$ is a linear transformation.**

$$T : R^2 \to R^3 \text{ defined by } T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$$

We will verify that:

$$T = L_A \text{ where } A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 2 & -1 \end{bmatrix}$$

Now let's check that $T = L_A$: For $a_1, a_2 \in \mathbb{R}$

$$L_A \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = A \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ 0 \\ 2a_1 - a_2 \end{pmatrix}$$

Clearly $T = L_A$. Thus $T$ is a linear transformation due to the fact that matrix multiplication is a linear operation.

**Find bases for both $N(T)$ and $R(T)$.**
Since we have that $T = L_A$, we can find the bases for $N(T)$ and $R(T)$ by finding the null space and column space of $A$.
First, we can bring $A$ to RREF form:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 2 & -1 \end{bmatrix} \xrightarrow{r_3 \leftrightarrow r_2} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 0 \end{bmatrix} \xrightarrow{r_2 - 2r_1 \to r_2} \begin{bmatrix} 1 & 1 \\ 0 & -3 \\ 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{3}r_2 \to r_2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{r_1 - r_2 \to r_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

We can see that for the null space, it must solve $Ax = 0$. Clealry since there are no free variables, we have that the null space is $\{0\}$.
Thus we have that the basis for the null space is $\{0\}$.
For the column space, we can see that the column space is the span of the columns of $A$ that are pivot columns. Thus we have

$$R(T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

**Compute nullity and rank of $T$ and verify the dimension theorem**
The nullity of $T$ is the dimension of the null space of $T$. We have that the null space of $T$ is $\{0\}$. Thus the nullity of $T$ is 0.
The rank of $T$ is the dimension of the column space of $T$. We have that the column space of $T$ is spanned by $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$. Thus the rank of $T$ is 2.
The dimension theorem states that the rank of $T$ plus the nullity of $T$ is equal to the dimension of the domain of $T$. We have that the dimension of the domain of $T$ is 2. Thus we have that $2 + 0 = 2$ which verifies the dimension theorem.

**Use the approriate theorms in this sections to determine where T is one-to-one or onto.**
Clearly since $N(T) = \{0\}$, $T$ is one-to-one.
We can also see that if $dim(R(T)) = dim(W)$, then $T$ is onto. We have that $dim(R(T)) = 2$ and $dim(W) = 3$. Thus $T$ is not onto.

# 2.1: Problem 9(a)

State why the transformation is not linear.

$$T : R^2 \to R^2 \text{ defined by } T(a_1, a_2) = (1, a_2)$$

We can see that for $a_1, a_2 \in \mathbb{R}$, and some scalar $c \in \mathbb{R}$, we have that

$$cT(a_1, a_2) = c(1, a_2) = (c, ca_2)$$

$$T(c(a_1, a_2)) = T(ca_1, ca_2) = (1, ca_2)$$

We can see that $c(1, a_2) \neq (1, ca_2)$. Thus $T$ is not a linear transformation.

# 2.1: Problem 9(b)

State why the transformation is not linear.

$$T : R^2 \to R^2 \text{ defined by } T(a_1, a_2) = (a_1, a_1^2)$$

We can see that for $a_1, a_2, b_1, b_2 \in \mathbb{R}$, define $x = (a_1, a_2)$ and $y = (b_1, b_2)$. We have that

$$T(x + y) = T(a_1 + b_1, a_2 + b_2) = (a_1 + b_1, (a_1 + b_1)^2)$$

$$T(x) + T(y) = T(a_1, a_2) + T(b_1, b_2) = (a_1, a_1^2) + (b_1, b_1^2) = (a_1 + b_1, a_1^2 + b_1^2)$$

We can see that $(a_1 + b_1, (a_1 + b_1)^2) \neq (a_1 + b_1, a_1^2 + b_1^2)$.

# 2.1: Problem 15

Recall the definition of $P(R)$ on page 11. Define $T : P(R) \to P(R)$ by $T(f(x)) = \int_0^x f(t)dt$. Prove that $T$ is linear and one to one but not onto.
We can consider $f(x), g(x) \in P(R)$ and some scalar $c \in R$. We have that $f(x) = \sum_{i=0}^n a_i x^i$ and $g(x) = \sum_{i=0}^m b_i x^i$.
We need to show that $T(f(x) + g(x)) = T(f(x)) + T(g(x))$ and $T(cf(x)) = cT(f(x))$.
Without loss of generality, we can say that $n \geq m$. We have that $g(x) = \sum_{i=0}^n b_i x^i$ and for $i \geq m + 1$ $b_i = 0$
Thus we have that

$$T(f(x)) = \int_0^x \sum_{i=0}^n a_i t^i dt = \sum_{i=0}^n \int_0^x a_i t^i dt = \sum_{i=0}^n \frac{a_i}{i+1} x^{i+1}$$

$$T(g(x)) = \int_0^x \sum_{i=0}^n b_i t^i dt = \sum_{i=0}^n \int_0^x b_i t^i dt = \sum_{i=0}^n \frac{b_i}{i+1} x^{i+1}$$

$$T(f(x)) + T(g(x)) = \sum_{i=0}^n \frac{a_i}{i+1} x^{i+1} + \sum_{i=0}^n \frac{b_i}{i+1} x^{i+1} = \sum_{i=0}^n \frac{a_i + b_i}{i+1} x^{i+1}$$

As well as:

$$(f + g)(x) = \sum_{i=0}^n (a_i + b_i) x^i$$

$$T(f+g)(x) = \int_0^x \left(\sum_{i=0}^{n}(a_i+b_i)t^i\right)dt = \sum_{i=0}^{n}\int_0^x (a_i+b_i)t^i dt$$
$$= \sum_{i=0}^{n}\frac{a_i+b_i}{i+1}x^{i+1}$$

We can see that $T(f(x)+g(x)) = T(f(x)) + T(g(x))$.
Now we can show that $T(cf(x)) = cT(f(x))$. We have that

$$T(cf(x)) = \int_0^x cf(t)dt = c\int_0^x f(t)dt = cT(f(x))$$

Thus we have that $T$ is a linear transformation.
Now we can show that $T$ is one-to-one. We can show this by showing that for any two arbitrary polynomials $h(x)$ and $k(x)$, if $T(h(x)) = T(k(x))$, then $h(x) = k(x)$.
Consider $h(x) = \sum_{i=0}^{n} a_i x^i$ and $k(x) = \sum_{i=0}^{m} b_i x^i$. We have that

$$T(h(x)) = T(k(x)) \implies \int_0^x h(t)dt = \int_0^x k(t)dt$$

$$\implies \sum_{i=0}^{n}\frac{a_i}{i+1}x^{i+1} = \sum_{i=0}^{m}\frac{b_i}{i+1}x^{i+1}$$

We can see that the only way this can be true is if $n = m$ and $a_i = b_i$ for all $i$. Thus we have that $h(x) = k(x)$.
Thus we have that $T$ is one-to-one.
Now we can show that $T$ is not onto.
We can do this by noting the polynomial of degree -1, which is the zero polynomial. We have that

$$T(0) = \int_0^x 0dt = 0$$

Every other polynomial that has a degree greater than -1 after its application of T will increase in degree, but not the zero polynomial. Thus we will not have an output of a polynomial of degree 0.
Thus we have that $T$ is not onto.

# 2.2: Problem 2(a)

Let $\beta, \gamma$ be the standard ordered bases for $R^n$ and $R^m$ respectively. For each linear transformation $T : R^n \to R^m$, compute $[T]_\beta^\gamma$

$$T : R^2 \to R^3 \text{ defined by } T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$$

We can see that

$$T(1,0) = (2,3,1)$$

$$T(0,1) = (-1,4,0)$$

Thus we have that

$$[T]_\beta^\gamma = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}$$

# 2.2: Problem 2(b)

Let $\beta, \gamma$ be the standard ordered bases for $R^n$ and $R^m$ respectively. For each linear transformation $T : R^n \to R^m$, compute $[T]_\beta^\gamma$

$$T : R^3 \to R^2 \text{ defined by } T(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$$

We can see that

$$T(1,0,0) = (2,1)$$

$$T(0,1,0) = (3,0)$$

$$T(0,0,1) = (-1,1)$$

Thus we have that

$$[T]_\beta^\gamma = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

# 2.2: Problem 2(c)

Let $\beta, \gamma$ be the standard ordered bases for $R^n$ and $R^m$ respectively. For each linear transformation $T : R^n \to R^m$, compute $[T]_\beta^\gamma$

$$T : R^3 \to R \text{ defined by } T(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$$

We can see that

$$T(1,0,0) = 2$$

$$T(0,1,0) = 1$$

$$T(0,0,1) = -3$$

Thus we have that

$$[T]_\beta^\gamma = \begin{bmatrix} 2 & 1 & -3 \end{bmatrix}$$

# 2.2: Problem 2 Extra Question

The way that our $[T]_\beta^\gamma$ is defined is the same way that we define the matrix $A$ to show that $T = L_A$. This makes sense as we are defining the transformation in terms of the standard basis.
We can see that rather than guessing the matrix (A) that corresponds to the transformation $L_A$ we can multiply each element of the standard ordered basis by the transformation to get the columns of the matrix.
The Jth column is the same as $T(e_j)$ where $e_j$ is the jth column of the standard ordered basis.

# 2.2: Problem 4

Define:

$$T : M_{2\times 2} \to P_2(R) \text{ by } T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b) + (2d)x + bx^2$$

Let

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ and } \gamma = \{1, x, x^2\}$$

Find $[T]_\beta^\gamma$.
We can see that

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1 + (2\cdot 0)x + 0\cdot x^2 = 1$$
$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 1 + (2\cdot 0)x + 1\cdot x^2 = 1 + x^2$$
$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = 0 + (2\cdot 0)x + 0\cdot x^2 = 0$$
$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0 + 2x + 0\cdot x^2 = 2x$$

Thus we have that

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

# 2.2: Problem 5(d)

Let:

$$\alpha = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ and } \beta = \{1, x, x^2\} \text{ and } \gamma = \{1\}$$

Define $T : P_2(R) \to R$ by $T(f(x)) = f(2)$ and find $[T]_\beta^\gamma$.
We can see that

$$T(1) = 1$$

$$T(x) = 2$$

$$T(x^2) = 4$$

Thus we have that

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$$