TODO

Pranav Tikkawar

September 10, 2024

1 Probability Review

Moment Generating Functions

Suppose X is a random variable. The rth moment of X about the origin is defined as

$$\mu_r' := \mathbb{E}(X^r) = \int x^r f(x)dx$$

where $f(x)$ is the PDF.
The first moment is the mean indicated by μ
The rth moment about the mean is defined as

$$\mu_r := \mathbb{E}((X-\mu)^r) = \int (x-\mu)^r f(x)dx$$

μ_2 is the variance of X indicated by σ^2 and is always non-negative
$\mathrm{Var}(x) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$
A random variable X taking values in $\mathbb{R}$ is said to be norm with parameter μ and σ^2 if its PDF is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Case: $\mu = 0$ and $\sigma^2 = 1$ is called the standard normal distribution.
Moment Generating Function
The moment generating function of a random variable X is defined as

$$M_X(t) = \mathbb{E}(e^{tX}) = \int e^{tx} f(x)dx$$

Note $e^{tx} = 1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + ...$
Can also be considered as

$$M_X(t) = \sum_{n=0}^{\infty} \frac{t^n \mu_n'}{n!}$$

where μ_n is the nth moment of X about the origin.

$$M_X(t) = \mathbb{E}(e^{tX}) = \mathbb{E}(1 + tX + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + ...)$$

$$M_X(t)' = \mathbb{E}(Xe^{tX}) = \mathbb{E}(X) + \mathbb{E}(X^2)t + \mathbb{E}(X^3)\frac{t^2}{2!} + ...$$

$$M_X(0)' = \mathbb{E}(X)$$

$$M_X(0)^{(n)} = \mu_n(x) = \mathbb{E}(X^n)$$

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$Var(X) = M''_X(0) - M'_X(0)^2$$

Properties of MGF:

- MGF is unique to the distribution, ie, eg: If MGF is a distribution of $\frac{1}{1-3t}$ then the distribution is exponential with parameter $\lambda = 3$
- $M_{x+a}(t) = e^{at}M_X(t)$
- $M_{bX}(t) = M_X(bt)$
- $M_{x+y}(t) = \mathbb{E}(e^{tx}e^{ty})$
- $M_{x+y}(t) = M_x(t)M_y(t)$ if X and Y are independent

Why? If X and Y are independent:

$$f(x, y) = f_x(x)f_y(y)$$

where $f(x, y)$ is the joint PDF of X and Y

Example

$$x - -\text{Exp}(\lambda)$$

$$Y = 3X \to M_Y(t) = M_X(3t) = \frac{1}{1 - \lambda 3t}$$

$$y - -Exp(3\lambda)$$

MGF of the normal

$$\begin{aligned}
M_X(t) = \mathbb{E}(e^{tX}) &= \int e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int e^{tx - \frac{x^2 - 2x\mu + \mu^2}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int e^{-\frac{(x-(\mu+t\sigma^2))^2 + (\mu+t\sigma^2)^2 - \mu^2}{2\sigma^2}} dx \\
&= e^{\frac{t^2\sigma^4 + 2\mu t\sigma^2}{2\sigma^2}} \int \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-(\mu+t\sigma^2))^2}{2\sigma^2}} dx \\
&= e^{\frac{t^2\sigma^2}{2} + \mu t}
\end{aligned}$$

Q:
Suppose $X = N(\mu_1, \sigma_1^2)$ and $Y = N(\mu_2, \sigma_2^2)$ are independent. Show that $Z = X + Y$ is $N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$
Using independance of X and Y we show that $M_Z(t) = M_X(t)M_Y(t)$