01:XXX:XXX - Homework n

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Definition 1. Sample Variance
The sample variance is defined as $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. If $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with mean μ and variance σ^2, then the sample variance S^2 is an unbiased estimator of the population variance σ^2. That is $E(S^2) = \sigma^2$. It also has a chi-squared distribution with $n-1$ degrees of freedom. That is $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{\nu=n-1}$.
Important identity: $\sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n}(X_i - \mu)^2 - n(\bar{X} - \mu)^2$.
Sample variance is an unbiased estimator of the population variance. That is $E(S^2) = \sigma^2$.

Definition 2. Chebyshev's Theorem
If X is a random variable with mean μ and variance σ^2, then for any $k > 0$, $P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$. Applying Chebyshev to a sample mean we get the weak law of large numbers. That is for a sample mean $\bar{X}$, $P(|\bar{X} - \mu| \geq k) \leq \frac{\sigma^2}{nk^2}$. Example question:
How large should n so that the $\bar{X}$ approximates μ within ϵ with probability at least $1 - \delta$ with population $\sigma^2_{pop} = \sigma^2$? **Sol:**

$$P(|\bar{X} - \mu| < \epsilon) \geq 1 - \frac{\sigma^2}{n\epsilon^2} \geq 1 - \delta$$
$$n \geq \frac{\sigma^2}{\epsilon^2\delta}$$

Definition 3. Chi-Squared Distribution
Parameters: ν degrees of freedom
MGF: $\frac{1}{(1-2t)^{\nu/2}}$
Mean: ν
Variance: 2ν
If $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with mean μ and variance σ^2, then the sum of squares of these random variables is a chi-squared random variable with n degrees of freedom. That is $Y = X_1^2 + X_2^2 + \ldots + X_n^2 \sim \chi^2_{\nu=n}$.

Definition 4. Moment Generating Function
The moment generating function of a random variable X is defined as $M_X(t) = E(e^{tX})$.

Some properties of the moment generating function are:

$$\begin{aligned} M_X(0) &= 1 \\ M'_X(0) &= E(X) \\ M''_X(0) &= E(X^2) \\ M_{aX+b}(t) &= e^{bt}M_X(at) \\ M_{X+Y}(t) &= M_X(t)M_Y(t) \text{ if } X \text{ and } Y \text{ are independent} \end{aligned}$$

Definition 5. Central Limit Theorem
Suppose $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with well definied mgf. Then the sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ approaches standard normal

$$P(a \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq b)$$

as $n \to \infty$:

$$\lim_{n\to\infty} P(a \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Definition 6. Gamma Distribution
Parameters: α, β
PDF: $f(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}$
MGF: $(1 - \beta t)^{-\alpha}$
Mean: $\alpha\beta$
Variance: $\alpha\beta^2$
We know that chi-squared distribution is a special case of gamma distribution with $\alpha = \nu/2$ and $\beta = 2$.
We know that the exponential distribution is a special case of gamma distribution with $\alpha = 1$ and $\beta = \lambda$.

Definition 7. Rth order statistic
The rth order statistic of a random sample $X_1, X_2, \ldots, X_n$ is the rth smallest value in the sample. That is $X_{(r)}$ is the rth order statistic.
The pdf of the rth order statistic is given by $f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!}F(x)^{r-1}(1-F(x))^{n-r}f(x)$.
We can clearly see that this is the probability of $r - 1$ values being less than x and $n - r$ values being greater than x and 1 being exactly x.

1 Textbook:

Exam topics:
MGFs
Chapter 6.3 Gamma distribution pg(178)

Definition 8. Gamma Distribution
Parameters: α, β
PDF: $f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$ for $x > 0$
MGF: $(1 - \beta t)^{-\alpha}$
Mean: $\alpha\beta$
Variance: $\alpha\beta^2$

Definition 9. exponential distribution
Parameters: λ
PDF: $f(x) = \frac{e^{-x/\lambda}}{\lambda}$ for $x > 0$
MGF: $(1 - \lambda t)^{-1}$
Mean: λ
Variance: λ^2
Note that this is a special case of the gamma distribution with $\alpha = 1$ and $\beta = \lambda$.

Definition 10. Chi-Squared Distribution
Parameters: ν degrees of freedom
PDF: $f(x) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2}$ for $x > 0$
MGF: $\frac{1}{(1-2t)^{\nu/2}}$
Mean: ν
Variance: 2ν
Note that this is a special case of the gamma distribution with $\alpha = \nu/2$ and $\beta = 2$.
If X is the standard normal distribution, then X^2 is a chi-squared distribution with 1 degree of freedom.
More generally, if $X_1, X_2, \ldots, X_n$ are independent and identically distributed standard normal random variables, then $X_1^2 + X_2^2 + \ldots + X_n^2$ is a chi-squared distribution with n degrees of freedom.
If $X_1, X_2 \ldots, X_n$ are independent and identically distributed chi-squared random variables with $\nu_1, \nu_2, \ldots, \nu_n$ degrees of freedom, then $X_1 + X_2 + \ldots + X_n$ is a chi-squared distribution with $\nu_1 + \nu_2 + \ldots + \nu_n$ degrees of freedom.

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Definition 11. Random Sample:
A random sample is a set of independent and identically distributed random variables.
$X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables they consitute a random sample of size n from the population.

Definition 12. Sample Mean:
The sample mean is defined as $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$. $E[\bar{X}] = \mu$ and $Var[\bar{X}] = \frac{\sigma^2}{n}$. If $\bar{X}$ is from a normal population of μ, σ^2, then $\bar{X}$ is normally distributed with mean μ and variance $\frac{\sigma^2}{n}$.

Definition 13. Sample Variance:
The sample variance is defined as $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$.

Chapter 8.2 sample mean pg(235)

Definition 14. Law of Large Numbers:
For any positive constant c, the proability that $\bar{X}$ will take a value between $\mu \pm c$ is at least $1 - \frac{\sigma^2}{nc^2}$. When $n \to \infty$ the probability approaches 1.
In other words, the sample mean $\bar{X}$ approaches the population mean μ as the sample size n increases.

Definition 15. Central Limit Theorem:
Suppose $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables from an infinite population with a mean μ and variance σ^2 and an MGF $M_X(t)$. Then the limiting distribution of

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

is the standard normal distribution as $n \to \infty$.

Chapter 8.4 Chi-Squared pg(242)

Theorem 1. *If $\bar{X}$ and S^2 are the sample mean and sample variance of a random sample of size n from a normal population with mean μ and variance σ^2, then*

1. *$\bar{X}$ and S^2 are independent random variables.*
2. *The random variable $\frac{(n-1)S^2}{\sigma^2}$ has a chi-squared distribution with $n-1$ degrees of freedom.*

Chapter 8.7 Order Statistic pg(252)

Definition 16. Order Statistic:
Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a population with CDF $F(x)$. The order statistics are the random variables $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ defined as follows:

$$\begin{aligned} X_{(1)} &= \min(X_1, X_2, \ldots, X_n) \\ X_{(2)} &= \text{second smallest value in the sample} \\ &\vdots \\ X_{(n)} &= \max(X_1, X_2, \ldots, X_n) \end{aligned}$$

The pdf of the rth order statistic is given by $f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} F(x)^{r-1}(1-F(x))^{n-r} f(x)$. Clearly this is the probability that there are $r-1$ values less than x, $n-r$ values greater than x, and exactly 1 value equal to x.
Another form of the pdf is

$$g_r(y_r) = \frac{n!}{(r-1)!(n-r)!} f(y_r)[\int_{-\infty}^{y_r} f(y)dy]^{r-1}[\int_{y_r}^{\infty} f(y)dy]^{n-r}$$

Common order statistics are the minimum $Y_{(1)}$, the maximum $Y_{(n)}$, and the median $Y_{(m+1)}$ for $n = 2m+1$.

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Definition 17. Point Estimator:
Using the vaule of a sample statistic to estimate the value of a population parameter is called point estimation. We refer to the value of the statsitc as a point estimate.
A point estimator is unbiased if $E[\hat{\theta}] = \theta$.

Chapter 10.2 Point estimator, unbiased estimators pg(284)

Definition 18. Unbiased Estimator:
A point estimator $\hat{\theta}$ of a parameter θ is said to be unbiased if $E[\hat{\theta}] = \theta$.

Definition 19. Bias:
The bias of an estimator $\hat{\theta}$ of a parameter θ is defined as $Bias(\hat{\theta}) = E[\hat{\theta}] - \theta$. An estimator is unbiased if $Bias(\hat{\theta}) = 0$.

Definition 20. Asymtotically Unbiased:
An estimator $\hat{\theta}$ of a parameter θ is said to be asymptotically unbiased if $lim_{n\to\infty} Bias(\hat{\theta}) = 0$.

Chapter 10.8 Method of Max liklyhood pg(301)

Definition 21. Method of Maximum Likelihood:
The method of maximum likelihood is a method of estimating the value of a parameter by maximizing the likelihood function. The likelihood function is defined as $L(\theta) = \prod_{i=1}^{n} f(x_i|\theta)$.
We also consider the log-likelihood function $l(\theta) = \ln(L(\theta)) = \sum_{i=1}^{n} \ln(f(x_i|\theta))$.
The maximum likelihood estimator $\hat{\theta}$ is the value of θ that maximizes the likelihood function.