16:960:665 - Time Series Analysis - Homework 2

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Problem (6). (a) Suppose $\mathscr{H}$ is a separable Hilbert space and $\mathscr{H} = \overline{\text{sp}}\{x_i,\ i = 1, 2, \infty\}$.Let x be an element of $\mathscr{H}$. Show that

$$\mathcal{P}_{\overline{\text{sp}}\{x_1, x_2, \ldots, x_n\}}(x) \to x \quad \text{as } n \to \infty.$$

Solution: Let $V_n = \overline{\text{sp}}\{x_1, x_2, \ldots, x_n\}$. Since $V_n \subseteq V_{n+1}$, we have a nested sequence of closed subspaces. Since $\mathscr{H}$ is separable, then $\bigcup_{n=1}^{\infty} V_n$ is dense in $\mathscr{H}$. Therefore, for any $x \in \mathscr{H}$ and any $\epsilon > 0$, there exists an N such that for all $n \geq N$, there exists a $y_n \in V_n$ with $\|x - y_n\| < \epsilon$.
Since $\mathcal{P}_{V_n}(x)$ is the orthogonal projection of x onto V_n, it minimizes the distance from x to any point in V_n. Thus, we have:

$$\|x - \mathcal{P}_{V_n}(x)\| \leq \|x - y_n\| < \epsilon \quad \text{for all } n \geq N.$$

This shows that $\|x - \mathcal{P}_{V_n}(x)\| \to 0$ as $n \to \infty$, which implies that $\mathcal{P}_{V_n}(x) \to x$ in the norm of $\mathscr{H}$. Hence, we conclude that:

$$\mathcal{P}_{\overline{\text{sp}}\{x_1, x_2, \ldots, x_n\}}(x) \to x \quad \text{as } n \to \infty.$$

(b) Suppose $\{X_t,\ t \in \mathbb{Z}\}$ is a stationary process. Show that

$$\mathcal{P}_{\overline{\text{sp}}\{X_{n-j},\ 1 \leq j \leq \infty\}}(X_n) = \lim_{r \to \infty} \mathcal{P}_{\overline{\text{sp}}\{X_{n-1}, X_{n-2}, \ldots, X_{n-r}\}}(X_n).$$

Solution: Let $V_r = \overline{\text{sp}}\{X_{n-1}, X_{n-2}, \ldots, X_{n-r}\}$. Since $V_r \subseteq V_{r+1}$, we have a nested sequence of closed subspaces. The union $\bigcup_{r=1}^{\infty} V_r$ is dense in $V_\infty := \overline{\text{sp}}\{X_{n-j},\ 1 \leq j \leq \infty\}$ because it includes all finite linear combinations of the X_{n-j}'s.
For any $X_n \in \mathscr{H}$, and any $\epsilon > 0$, there exists an R such that for all $r \geq R$, there exists a $Y_r \in V_r$ with $\|X_n - Y_r\| < \epsilon$. Since $\mathcal{P}_{V_r}(X_n)$ is the orthogonal projection of X_n onto V_r, it minimizes the distance from X_n to any point in V_r. Thus, we have:

$$\|X_n - \mathcal{P}_{V_r}(X_n)\| \leq \|X_n - Y_r\| < \epsilon \quad \text{for all } r \geq R.$$

This shows that $\|X_n - \mathcal{P}_{V_r}(X_n)\| \to 0$ as $r \to \infty$, which implies that $\mathcal{P}_{V_r}(X_n) \to X_n$ in the norm of $\mathscr{H}$. Hence, we conclude that:

$$\mathcal{P}_{\overline{\text{sp}}\{X_{n-j},\ 1\leq j\leq\infty\}}(X_n) = \lim_{r\to\infty} \mathcal{P}_{\overline{\text{sp}}\{X_{n-1}, X_{n-2}, \ldots, X_{n-r}\}}(X_n).$$

Problem (7)**.** Consider the following ARMA processes.

(i) AR(3): $r_t = 0.3 + 0.8r_{t-1} - .5r_{t-2} - .2r_{t-3} + a_t$.

(ii) MA(3): $r_t = 0.3 + a_t + 0.8a_{t-1} - .5a_{t-2} - .2a_{t-3}$.

(iii) ARMA(3,2): $r_t = 0.3 + 0.8r_{t-1} - .5r_{t-2} - .2r_{t-3} + a_t + 0.5a_{t-1} + 0.3a_{t-2}$.

Assume all a_t are i.i.d $N(0, 4)$. For each of the three preceding process, do the following:

(a) Calculate the ACF up to lag 12. [Hint. You may need to read Section 3.3 before trying (iii).]

Solution:

(i) AR(3):

(ii) MA(3):

(iii) ARMA(3,2):

(b) Simulate a series of length $T = 250$, give the time series plot.

Solution:

(c) Compare the true ACF plot (plot what you obtained in Part (a)) with the sample ACF plot (use the R function `acf()`).

Solution:

Problem (8)**.** Consider the AR(1) process $X_t = 2X_{t-1} + Z_t$, where $Z_t \sim \text{WN}(0, \sigma^2)$. Define

$$Z_t^* := .25Z_t - \frac{3}{4}\sum_{j=1}^{\infty} 2^{-j} Z_{t+j}$$

(a) Express the unique stationary solution X_t in terms of Z_t.

Solution:

(b) Prove that $\{Z_t^*\}$ is a white noise. What is its variance?

Solution: Mean:

$$\begin{aligned} E[Z_t^*] &= .25E[Z_t] - \frac{3}{4}\sum_{j=1}^{\infty} 2^{-j} E[Z_{t+j}] \\ &= 0 - 0 = 0 \end{aligned}$$

Variance:

$$
\begin{aligned}
\operatorname{Var}(Z_t^*) &= E[(Z_t^*)^2] \\
&= E\left[\left(.25Z_t - \frac{3}{4}\sum_{j=1}^{\infty} 2^{-j} Z_{t+j}\right)^2\right] \\
&= E\left[\frac{1}{16}Z_t^2 - \frac{3}{8}Z_t\sum_{j=1}^{\infty} 2^{-j} Z_{t+j} + \frac{9}{16}\left(\sum_{j=1}^{\infty} 2^{-j} Z_{t+j}\right)^2\right] \\
&= \frac{1}{16}E[Z_t^2] + \frac{3}{8}E\left[Z_t\sum_{j=1}^{\infty} 2^{-j} Z_{t+j}\right] + \frac{9}{16}E\left[\left(\sum_{j=1}^{\infty} 2^{-j} Z_{t+j}\right)^2\right] \\
&= \frac{1}{16}\sigma^2 + 0 + \frac{9}{16}E\left[\sum_{j=1}^{\infty} 4^{-j} Z_{t+j}^2\right] \\
&= \frac{1}{16}\sigma^2 + \frac{9}{16}\sum_{j=1}^{\infty} 4^{-j} E[Z_{t+j}^2] \\
&= \frac{1}{16}\sigma^2 + \frac{9}{16}\sum_{j=1}^{\infty} 4^{-j} \sigma^2 \\
&= \frac{1}{16}\sigma^2 + \frac{3}{16}\sigma^2 \\
&= \frac{1}{4}\sigma^2
\end{aligned}
$$

(c) Prove that $X_t = .5X_{t-1} + Z_t^*$.

Solution:

Problem (9)**.** Suppose that $\{X_t\}$ and $\{Y_t\}$ are two zero-mean stationary processes with the same autocovariance function, and that Y_t is an ARMA(p, q) process.

(a) If $\phi_1, \ldots, \phi_p$ are the AR coefficients for Y_t, define $W_t := X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p}$. Show that $\{W_t\}$ has an autocovariance function which is zero for lags $|h| > q$.

Solution:

(b) Apply Proposition 3.2.1 to $\{W_t\}$ to conclude that $\{X_t\}$ is also an ARMA(p, q) process.

Solution:

Problem (10). Read Proposition 5.1.1 and its proof (a very nice one!) before you work on this problem. Suppose there are n observations $X_1, X_2, \ldots, X_n$ of a stationary time series. Define

$$\hat{\gamma}(h) = \begin{cases} n^{-1} \sum_{t=1}^{n-|h|} (X_{t+h} - \bar{X})(X_t - \bar{X}) & \text{if } |h| < n, \\ 0 & \text{if } |h| > n. \end{cases}$$

Note that although the sample autocovariannces are usually only defined for lags $|h| < n$, here $\hat{\gamma}(\cdot)$ is defined as a function on all integers, where it takes value 0 when $|h| \geq n$.

(a) Show that the function $\hat{\gamma}(\cdot)$ is non-negative definite.

Solution:

(b) There is nothing you need to do for this part. But observe that (i) by Theorem 1.5.1, there exists some stationary process $\{Y_t\}$ of which $\hat{\gamma}(\cdot)$ is the autocovariance function; and (ii) from Proposition 3.2.1 it then follows that $\{Y_t\}$ is an MA$(n-1)$ process.

Solution:

(c) Prove that if $\hat{\gamma}(0) > 0$, then $\hat{\Gamma}_n$ is non-singular. (In the last Homework, you showed that $\hat{\Gamma}_n$ is non-negative definite, and now you know that it is also strictly positive-definite unless the n observations are all equal.)

Solution:

Problem (11).

(a) Consider a MA(∞) process $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$, where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$, and $\sum_{j=0}^{\infty} |\psi_j| < \infty$. Show that the autocovariance function $\gamma(\cdot)$ of $\{X_t\}$ satisfies $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$.

Solution:

(b) Let $\{X_t\}$ be a causal ARMA process with autocovariance function $\gamma(\cdot)$. Show that there exist a constant $C > 0$ and another constant $s \in (0, 1)$ such that $|\gamma(h)| \leq C s^{|h|}$ for all $h \in \mathbb{Z}$, and hence $\sum_h |\gamma(h)| < \infty$.

Solution:

Problem (12)**.** The process $X_t = Z_t - Z_{t-1}$, where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$, is not invertible according to Definition 3.1.4. Show however that $Z_t \in \overline{\mathrm{sp}}\{X_j, -\infty < j \leq t\}$ by considering the mean square limit of the sequence $\sum_{j=0}^{n}(1 - j/n)X_{t-j}$ as $n \to \infty$.

Solution: