Math Theory of Probability

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1 Chapter 1: Combinatorial Analysis

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Basic Principle of Counting.
Suppose that 2 experiments are to be preformed. Then if exp 1 can result in any one of n_1 possible outcomes and for each of these outcomes, exp 2 can result in any one of n_2 possible outcomes, then the total number of possible outcomes for the 2 experiments is $n_1 \cdot n_2$.

Permutations.
How many ways are there of arranging n distinct things?
There are n ways to choose the first thing, $n-1$ ways to choose the second thing, $n-2$ ways to choose the third thing, and so on.
Thus, the total number of ways of arranging n distinct things is $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1 = n!$

Permutations with repeats.

$$\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_r!}$$

different permutation of n objects which any arbitrary n_i are alike.

Combinations.

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

How many ways are there of choosing r things from n distinct things?

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Example 4c: n items m are dysfunctional and $n-m$ are functional. What is the probability that no two dysfunctional items are adjacent?
Sol: There are $\binom{n-m+1}{m}$ ways. If we think of the functional (plus one for the before spot) we can put the dysfunctional items in. Thus resulting in $\binom{n-m+1}{m}$ ways.

Question Prove that $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$
It is shown is pascal's triangle.
Since the left is the number of Combinations of n things taken r at a time, and the right is the number of Combinations of $n-1$ things taken $r-1$ at a time and r at a time.
Thus the right side is the number of Combinations in which A is included and the number of Combinations in which A is not included.

Binomial Theorum

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

This gives the coefficients of the expansion of $(x+y)^n$

Multinomial Theorum

$$(x_1 + x_2 + \ldots + x_k)^n = \sum_{r_1+r_2+\ldots+r_k=n} \frac{n!}{r_1!r_2!\ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}$$

This gives the coefficients of the expansion of $(x_1 + x_2 + \ldots + x_k)^n$

Something cool $\binom{n}{r}\binom{r}{k} = \frac{n!}{r!k!(n-r-k)!}$

Example 5: 8 players, 4 matches (identitcal) played of 2 players. How many ways can the matches be played?
Sol: There are $\frac{8!}{2!2!2!2!4!} = 105$ ways.
How many ways can people win?
16 ways.

Class Activity: Consider the equation $x_1 + x_2 + \ldots + x_r = n$ where each x_i is non-negative. How many possible solutions are there to this equation.

2 Chapter 2: Axioms of Probability

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Axioms of probability
The probability of Something happening is the number of ways the thing happens dived by the possible outcomes.
Axiom 1: $0 \leq P(A) \leq 1$
Axiom 2: $P(S) = 1$ where S is whole same space
Axiom 3: If $A_1, A_2, \ldots$ are mutually exclusive events, then $P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$

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$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

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n ppl who throw the hats in what is the probablity that no one gets their own hat?
$\frac{1}{e}$ as $n \to \infty$
We want probability of $A_1 \cap A_2 \cap \ldots \cap A_n$
$P(A_1 \cap A_2 \cap \ldots \cap A_n) = 1 - P(A_1^c \cup A_2^c \cup \ldots \cup A_n^c)$
$P(A_1^c \cup A_2^c \cup \ldots \cup A_n^c) = 1 - P(A_1^c) - P(A_2^c) - \ldots - P(A_n^c) + P(A_1^c \cap A_2^c) + \ldots$
With a buch of work we get:

$$1 - \sum_{i=1}^{n} (-1)^{i+1} \frac{(x-i)!}{x!} \binom{x}{i}$$

This results in $1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots$

Example 50 on page 42 of 8th edition.

3 Chapter 3: Conditional Probability and Independence

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$$P(A \cap B) = P(A)P(B|A) = P(A)P(B)$$

iff E and F are independent.

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Bayes formula

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Odds

$$\frac{P(A)}{P(A^c)}$$

New Odds

$$\frac{P(A|B)}{P(A^c|B)} = \frac{P(B|A) \cdot P(A)}{P(B|A^c) \cdot P(A^c)}$$

4 Chapter 4: Random Variables

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Random Variable Info
Binomial Random Variable
$P[x = n] = \binom{N}{n} p^n (1-p)^{N-n}$
$\mathbb{E}[x] = Np$
$Var[x] = Np(1-p)$
Poisson Random Variable
$P[x = n] = \frac{e^{-\lambda}\lambda^n}{n!}$
$\mathbb{E}[x] = \lambda$
$Var[x] = \lambda$
Geometric Random Variable
$P[x = n] = (1-p)^{n-1} p$
$\mathbb{E}[x] = \frac{1}{p}$
$Var[x] = \frac{1-p}{p^2}$
Negative Binomial Random Variable
$P[x = n] = \binom{n-1}{r-1} p^r (1-p)^{n-r}$
$\mathbb{E}[x] = \frac{r}{p}$
$Var[x] = \frac{r(1-p)}{p^2}$
Hypergeometric Random Variable
$P[x = n] = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$
$\mathbb{E}[x] = \frac{Mk}{N}$
$Var[x] = \frac{N-M}{N-1} k \frac{M}{N} \frac{N-k}{N-1}$
$Var[x] = np(1-p)(1 - \frac{n-1}{N-1})$
Zeta Random Variable
$P[x = k] = \frac{c}{k^{\alpha+1}}$
$\mathbb{E}[x] = \frac{c}{\alpha-1}$

$Var[x] = \frac{c^2}{(\alpha-1)^2(\alpha-2)}$

5 Chapter 5: Continuous Random Variables

$pdf[a \leq x \leq b] = \int_a^b f(x)dx$
Where $f(x)$ is the probability density function.
A Cumulative Distribution Function is $cdf(x) = P[X \leq x]$
Also $cdf(x) = \int_{-\infty}^x f(x)dx$
Gaussian Random Variable
$P[x = n] = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
$\mathbb{E}[x] = \mu$
$Var[x] = \sigma^2$
Exponential Random Variable
$P[x = n] = \lambda e^{-\lambda x}$
$\mathbb{E}[x] = \frac{1}{\lambda}$
$Var[x] = \frac{1}{\lambda^2}$
A non negative Random variable c is memmoryless if $P[x > s + t | x > s] = P[x > t]$ **Gamma Random Variable**
$P[x = n] = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{(\alpha-1)!}$
$P[x = n] = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$
$\mathbb{E}[x] = \frac{\alpha}{\lambda}$
$Var[x] = \frac{\alpha}{\lambda^2}$
Weibull Random Variable
$P[x = n] = \frac{\alpha}{\beta}(\frac{x}{\beta})^{\alpha-1} e^{-(\frac{x}{\beta})^\alpha}$
$\mathbb{E}[x] = \beta\Gamma(1 + \frac{1}{\alpha})$
$Var[x] = \beta^2[\Gamma(1 + \frac{2}{\alpha}) - \Gamma^2(1 + \frac{1}{\alpha})]$