01:XXX:XXX - Homework n

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1.1 Study Guide Concepts

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1.1.1 2.3 - The Diffusion Equation

Definition (Max Principle (weak))**.** If $u(x,t)$ is a solution to the Diffusion Equation in a rectangle $0 \leq x \leq L$, $0 \leq t \leq T$, then the maximum of $u(x,t)$ occurs on the boundary of the rectangle. In other words on $x = 0, x = L, t = 0$.
The minimum is similar as we can show that $-u(x,t)$ satisfies the same equation.
The natrual interpretaion of this is that if you have a rod with no internal heat sourse, the hottest or coldest spot can only occour at $t = 0$ or on the edges.

Definition (Uniqueness)**.** There is uniqueness for the Dirichlet problem for the Diffusion Equation. That means there is at most one solution of

$$\begin{cases} u_t - ku_{xx} = f(x,t) \text{ for } 0 < x < L, t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$$

For any given $f(x,t), \phi(x), g(t), h(t)$
We can do proof by max principle.

Proof. We want to show that for all u_1, u_2 that satisfy the above conditions, $u_1 = u_2$.
Let $w = u_1 - u_2$. Then w satisfies the following:

$$\begin{cases} w_t - kw_{xx} = 0 \text{ for } 0 < x < L, t > 0 \\ w(x,0) = 0 \\ w(0,t) = 0 \\ w(L,t) = 0 \end{cases}$$

By max prinicple $w(x,t)$ has a maximum on its boundary. Also it must have a minimum on its boundary. Since $w(x,0) = 0$, the minimum and the maximum must be 0. Thus $w(x,t) = 0$ for all x, t.
Thus $u_1 = u_2$. □

Now we can do a proof by energy.

Proof. We know that $w = u_1 - u_2$

$$\begin{aligned} 0 &= 0 \cdot w && (1) \\ &= (w_t - kw_{xx})w && (2) \\ &= (1/2w^2)_t + (-kww_x)_x + kw_x^2 && (3) \end{aligned}$$

We can now integrat about $0 < x < L$

$$\begin{aligned} 0 = \int_0^L (1/2w^2)_t dx - kw_x w|_{0\ goesto0}^L + k \int_0^L w_x^2 dx && (4) \\ \frac{d}{dt}\int_0^L 1/2w^2 dx = -k \int_0^L w_x^2 dx && (5) \\ && (6) \end{aligned}$$

Clealrly the derivative of $\int_0^L w^2 dx$ is decreasing

$$\int_0^L w^2 dx \leq \int_0^L w(x,0)^2 dx$$

The RHS is 0, so the LHS is 0. Thus $w = 0$. □

Definition (Stablitity)**.** The solution to the Diffusion Equation is stable. That means that if you have a small perturbation in the initial conditions, the solution will not change much. In other words they functions are "bounded" by initial conditions.
This is in a L_2 sense.

$$\int_0^l [u_1(x,t) - u_2(x,t)]^2 dx \leq \int_0^l [u_1(x,0) - u_2(x,0)]^2 dx$$

1.1.2 2.4 - Diffusion on the Whole Line

Definition (Invariance Properties)**.** We have 5 basic invariance properties of the Diffusion Equation.

- Translation $u(x-y,t)$ is a solution if $u(x,t)$ is a solution.
- Any derivative of $u(x,t)$ is a solution.
- A linear combination of solutions is a solution.
- An integral of a solution is a solution. Thus if $S(x,t)$ is a solution then so is $S(x-y,t)$ and so is $v(x,t) = \int_{-\infty}^{x} S(x-y,t)g(y)dy$. for any $g(y)$.
- Dilation. If $u(x,t)$ is a solution then so is $u(\sqrt{a}x, at)$ for any $a > 0$.

Definition (Fundamental Solution to the Diffusion Equation)**.** The fundamental solution to the Diffusion Equation is

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}$$

This is a solution to the Diffusion Equation with $f(x,t) = 0$ and $u(x,0) = \delta(x)$. We can derive this by utilizing the invariance properties.

1.1.3 2.5 - Comparison of Waves and Diffusion

Property	**Waves**	**Diffusion**
Speed of Propogation	c	Infinite
Singulatities for $t > 0$	Transported along characteristics with speed c	Lost immediately
Well posed for $t > 0$	Yes	Yes for bounded
Well posed for $t < 0$	Yes	No
Max Principle	No	Yes
Behavior at infinity	Energy is constant so it doesn't decay	Decays to zero
Information	Transported	Lost gradually

Table 1: Comparison of Waves and Diffusion

1.1.4 4.1 - Seperation of Variables

2 Problems

1. Question 1.