01:XXX:XXX - Homework n

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1. (10 points) Section 5.4 Problem 5

Let $\phi(x) = \begin{cases} 0 & \text{for } 0 < x < 1 \\ 1 & \text{for } 1 < x < 3 \end{cases}$.

(a) Find the first four nonzero terms of its Fourier cosine series explicitly.

(b) For each x $(0 \le x \le 3)$, what is the sum of this series?

(c) Does it converge to $\phi(x)$ in the L^2 sense? Why?

(d) Put $x = 0$ to find the sum

$$1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \cdots .$$

Solution: Part a:

We can write the Fourier cosine series of $\phi(x)$ as

$$\phi(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right).$$

We can first solve for a_0:

$$\begin{aligned} a_0 &= \frac{2}{l}\int_0^l \phi(x)\,dx \\ &= \frac{2}{3}\int_0^1 0\,dx + \frac{2}{3}\int_1^3 1\,dx \\ &= \frac{2}{3}\,[x]_1^3 \\ &= \frac{2}{3}(3-1) \\ &= \frac{4}{3}. \end{aligned}$$

Next, we solve for a_n:

$$\begin{aligned}
a_n &= \frac{2}{l}\int_0^l \phi(x)\cos\left(\frac{n\pi x}{l}\right)\,dx \\
&= \frac{2}{3}\int_0^1 0\cos\left(\frac{n\pi x}{3}\right)\,dx + \frac{2}{3}\int_1^3 1\cos\left(\frac{n\pi x}{3}\right)\,dx \\
&= \frac{2}{3}\int_1^3 \cos\left(\frac{n\pi x}{3}\right)\,dx \\
&= \frac{2}{3}\left[\frac{3}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\right]_1^3 \\
&= \frac{2}{3}\left[\frac{3}{n\pi}\sin(n\pi) - \frac{3}{n\pi}\sin\left(\frac{n\pi}{3}\right)\right] \\
&= \frac{-2}{3}\left[\frac{3}{n\pi}\cdot\alpha\right]
\end{aligned}$$

Where $\alpha = \begin{cases} 0 & \text{if } n \bmod 6 = 0 \\ \frac{\sqrt{3}}{2} & \text{if } n \bmod 6 = 1 \\ \frac{\sqrt{3}}{2} & \text{if } n \bmod 6 = 2 \\ 0 & \text{if } n \bmod 6 = 3 \\ -\frac{\sqrt{3}}{2} & \text{if } n \bmod 6 = 4 \\ -\frac{\sqrt{3}}{2} & \text{if } n \bmod 6 = 5 \end{cases}$ Thus the first four nonzero terms of the Fourier cosine series are

$$\frac{2}{3} - \frac{\sqrt{3}}{\pi}\cos\left(\frac{\pi x}{3}\right) - \frac{\sqrt{3}}{\pi}\cdot\frac{1}{2}\cos\left(\frac{2\pi x}{3}\right) + \frac{\sqrt{3}}{\pi}\cdot\frac{1}{4}\cos\left(\frac{4\pi x}{3}\right).$$

Part b:
We can see that for $x \in [0,1)$, the sum of the series is 0 and for $x = 1$ the sum of the series is $1/2$. For $x \in (1,3]$, the sum of the series is 1.

Part c:
We want to see if

$$\int_0^3 [\phi(x) - \sum_1^N b_n \sin(\frac{n\pi x}{3})]^2 dx = 0$$

as $N \to \infty$.
We can see that for the intevals $[0,1) \cup (1,3]$, the sum of the series is 0 and 1 respectively.
Thus if we seperate the integral into these two regions we can see each region does converge to 0 and 1 respectively.

Thus the series converges to $\phi(x)$ in the L^2 sense.

Part d:
If we take $x = 0$ and let the sum we are looking for as S

$$0 = \frac{2}{3} - \frac{\sqrt{3}}{2\pi} - \frac{\sqrt{3}}{4\pi} + \frac{\sqrt{3}}{8\pi} - \cdots$$

$$-\frac{2}{3} = \frac{\sqrt{3}}{\pi}[S]$$

Thus

$$S = \frac{2\pi}{3\sqrt{3}}$$

2. (10 points) Section 5.4 Problem 6
Find the sine series of the function $\cos x$ on the interval $(0, \pi)$. For each x satisfying $-\pi \leq x \leq \pi$, what is the sum of the series?

Solution: The sine series of the function $\cos x$ on the interval $(0, \pi)$ is

$$\cos(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{nx}{\pi})$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} \cos(x) \sin(\frac{nx}{\pi})\, dx$$

$$= \frac{(-1)^n + 1}{\pi(n+1)} + \frac{(-1)^n + 1}{\pi(n-1)}$$

Which simplifies to

$$b_n = \frac{4n}{\pi(n^2 - 1)}$$

for n even Thus the sine series of the function $\cos x$ on the interval $(0, \pi)$ is

$$\cos(x) = \sum_{n=1}^{\infty} \frac{4(2n-1)}{\pi((2n-1)^2 - 1)} \sin(\frac{(2n-1)x}{\pi})$$

For each x satisfying $-\pi \leq x \leq \pi$, the sum of the series is $\cos(x)$.

3. (10 points) Section 5.4 Problem 9
Let $f(x)$ be a function on $(-l, l)$ that has a continuous derivative and satisfies the periodic

boundary conditions. Let a_n and b_n be the Fourier coefficients of $f(x)$, and let a_n' and b_n' be the Fourier coefficients of its derivative $f'(x)$. Show that

$$a_n' = \frac{n\pi b_n}{l} \quad \text{and} \quad b_n' = -\frac{n\pi a_n}{l} \quad \text{for } n \neq 0.$$

(Hint: Write the formulas for a_n' and b_n' and integrate by parts.) This means that the Fourier series of $f'(x)$ is what you'd obtain as if you differentiated term by term. It does not mean that the differentiated series converges.

Solution: We can see that the fomula for a_n' and b_n' are

$$\begin{aligned} a_n' &= \frac{2}{l}\int_{-l}^{l} f'(x)\cos\left(\frac{n\pi x}{l}\right)\,dx \\ b_n' &= \frac{2}{l}\int_{-l}^{l} f'(x)\sin\left(\frac{n\pi x}{l}\right)\,dx. \end{aligned}$$

We can integrate by parts to get

$$\begin{aligned} a_n' &= \frac{2}{l}\left[f(x)\cos\left(\frac{n\pi x}{l}\right)\right]_{-l}^{l} - \frac{2n\pi}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)\,dx \\ &= \frac{2}{l}\left[f(l)\cos(n\pi) - f(-l)\cos(-n\pi)\right] - \frac{2n\pi}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)\,dx \\ &= \frac{2}{l}\left[f(l) - f(-l)\right] - \frac{2n\pi}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)\,dx \\ &= \frac{2}{l}\left[f(l) - f(-l)\right] - \frac{2n\pi}{l}b_n. \\ &= \frac{2n\pi b_n}{l}. \end{aligned}$$

and

$$\begin{aligned} b_n' &= \frac{2}{l}\left[f(x)\sin\left(\frac{n\pi x}{l}\right)\right]_{-l}^{l} + \frac{2n\pi}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)\,dx \\ &= \frac{2}{l}\left[f(l)\sin(n\pi) - f(-l)\sin(-n\pi)\right] + \frac{2n\pi}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)\,dx \\ &= \frac{2}{l}\left[f(l) - f(-l)\right] + \frac{2n\pi}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)\,dx \\ &= \frac{2}{l}\left[f(l) - f(-l)\right] + \frac{2n\pi}{l}a_n. \\ &= -\frac{2n\pi a_n}{l}. \end{aligned}$$

4. (10 points) Section 5.4 Problem 10
Deduce from Exercise 9 that there is a constant k so that

$$|a_n| + |b_n| \le \frac{k}{n} \quad \text{for all } n.$$

Solution: We can see that for a_n and b_n we have

$$a_n = \frac{2}{l}\frac{f(x)}{n}sin(\frac{n\pi x}{l})|_{-l}^{l} - \frac{2}{nl}\int_{-l}^{l} f'(x)sin(n\pi x/l)dx$$
$$b_n = \frac{2}{l}\frac{f(x)}{n}cos(\frac{n\pi x}{l})|_{-l}^{l} + \frac{2}{nl}\int_{-l}^{l} f'(x)cos(n\pi x/l)dx$$

Thus we can see that

$$|a_n| \le |\frac{2}{nl}\int_{-l}^{l} f'(x)cos(n\pi x/l)|dx \le |\frac{2}{nl}\int_{-l}^{l} |f'(x)|dx$$
$$|b_n| \le |\frac{2}{nl}\int_{-l}^{l} f'(x)sin(n\pi x/l)|dx \le |\frac{2}{nl}\int_{-l}^{l} |f'(x)|dx$$

Let M be the maximum value of $|f'(x)|$ on $[-l, l]$. Then we have

$$|a_n| \le \frac{2M}{nl}\int_{-l}^{l} dx = \frac{4Ml}{nl} = \frac{4M}{n}$$
$$|b_n| \le \frac{2M}{nl}\int_{-l}^{l} dx = \frac{4Ml}{nl} = \frac{4M}{n}$$

Thus we can see that $|a_n| + |b_n| \le \frac{8M}{n}$ for all n.

5. (10 points) Section 5.4 Problem 12
Start with the Fourier sine series of $f(x) = x$ on the interval $(0, l)$. Apply Parseval's equality. Find the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Solution: We know from that the Fourier sine series of $f(x) = x$ on the interval

$(0, l)$ is

$$x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$
$$b_n = \frac{2}{l}\int_0^l x \sin\left(\frac{n\pi x}{l}\right)\, dx$$
$$b_n = -x\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)|_0^l + \frac{l}{n\pi}^2 \sin(\frac{n\pi x}{l})|_0^l$$
$$b_n = \frac{(-1)^{n+1} l}{n\pi}$$

We know that Parseval's equality is

$$\int_0^l |f|^2(x)\, dx = \sum_{n=1}^{\infty} |b_n|^2 \int_0^l ||X_n||^2$$

Thus we can see that

$$\int_0^l x^2 dx = \sum_{n=1}^{\infty} \frac{l^2}{n^2\pi^2} \int_0^l \sin^2(n\pi x/l)$$

$$\frac{l^3}{3} = \sum_{n=1}^{\infty} \frac{l^2}{n^2\pi^2}\frac{l}{2}$$

$$\frac{2l^2}{3} = \sum_{n=1}^{\infty} \frac{l^2}{n^2\pi^2}$$

$$\frac{2}{3} = \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}$$

$$\frac{2\pi}{3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

6. (10 points) Section 5.5 Problem 3 Prove the inequality

$$l\int_0^l (f'(x))^2 dx \geq [f(l) - f(0)]^2$$

for any real function $f(x)$ whose derivative $f'(x)$ is continuous. [Hint: Use Schwarz's inequality with the pair $f'(x)$ and 1.]

Solution: By Schwarz's inequality for g and h we have

$$(\int_0^l g(x)h(x)dx)^2 \leq \int_0^l g(x)^2dx \int_0^l h(x)^2dx$$

Applyng this with $g(x) = f'(x)$ and $h(x) = 1$ we get

$$(\int_0^l f'(x)dx)^2 \leq \int_0^l (f'(x))^2dx \int_0^l 1dx$$

$$(f(l) - f(0))^2 \leq l \int_0^l (f'(x))^2dx$$

$$l \int_0^l (f'(x))^2dx \geq [f(l) - f(0)]^2$$

7. (10 points) Section 6.3 Problem 1 Suppose that u is a harmonic function in the disk $D = \{r < 2\}$ and that $u = 3\sin 2\theta + 1$ for $r = 2$. Without finding the solution, answer the following questions.
   (a) Find the maximum value of u in D.
   (b) Calculate the value of u at the origin.

Solution: Part a:
We know that u is harmonic in D and that $u = 3\sin 2\theta + 1$ for $r = 2$. Thus we can see that the maximum value of u in D is 4 by the maximum principle.
Part b:
We know that u is harmonic in D and that $u = 3\sin 2\theta + 1$ for $r = 2$. Thus we can see that the value of u at the origin is 1 due to the fact that value at the origin is the average of the boundary values.

8. (10 points) Section 6.3 Problem 3 Solve $u_{xx} + u_{yy} = 0$ in the disk of $r < a$ with the following boundary conditions: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

Solution: We can rewrite our BC as $u = \frac{3sin(\theta) - 3sin(3\theta)}{4}$ on the boundary.
We can use the seperation of variables method to solve this.
We know the solution to be of the form of

$$u(r, \theta) = R(r)\Theta(\theta)$$

Thus we can see that

$$\frac{R''}{R} + \frac{R'}{rR} = -\frac{\Theta''}{\Theta} = -\lambda$$

Thus we have

$$\Theta'' + \lambda\Theta = 0$$

$$R'' + \frac{R'}{r} - \lambda R = 0$$

We can see that the solution to the first equation is

$$\Theta(\theta) = A\cos(\sqrt{\lambda}\theta) + B\sin(\sqrt{\lambda}\theta)$$

We can see that the solution to the second equation is

$$R(r) = Cr^{\sqrt{\lambda}} + Dr^{-\sqrt{\lambda}}$$

For the R equation we can see that $D = 0$ as the solution must be bounded at the origin.

Thus we have our complete solution as

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n[A_n\cos(n\theta) + B_n\sin(n\theta)]$$

Solving for A_0 we get

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{3sin(\theta) - 3sin(3\theta)}{4} d\theta = 0$$

For A_n we get

$$A_n = \frac{1}{\pi} \cdot \frac{1}{a^n}\int_{-\pi}^{\pi} \frac{3sin(\theta) - 3sin(3\theta)}{4}\cos(n\theta) d\theta = 0$$

For B_1 we get

$$B_1 = \frac{1}{\pi} \cdot \frac{1}{a}\int_{-\pi}^{\pi} \frac{3sin(\theta) - 3sin(3\theta)}{4}\sin(\theta) d\theta = \frac{3}{4a}$$

For B_2 we get

$$B_2 = \frac{1}{\pi} \cdot \frac{1}{a^2}\int_{-\pi}^{\pi} \frac{3sin(\theta) - 3sin(3\theta)}{4}\sin(2\theta) d\theta = 0$$

For B_3 we get

$$B_3 = \frac{1}{\pi} \cdot \frac{1}{a^3}\int_{-\pi}^{\pi} \frac{3sin(\theta) - 3sin(3\theta)}{4}\sin(3\theta) d\theta = -\frac{1}{4a^3}$$

For all other B_n we get

$$B_n = 0$$

Thus we have our solution as

$$u(r,\theta) = \frac{3}{4a} r\sin(\theta) - \frac{1}{4a^3} r^3\sin(3\theta)$$