# HW Math 350H

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## Section 1.2

## Question 7

### Let $S = \{0, 1\}$ and $F = R$. In $\mathcal{F}(S, F)$, show that $f = g$ and $f + g = h$ where $f(t) = 2t + 1$, $g(t) = 1 + 4t - 2t^2$, and $h(t) = 5^t + 1$

We can prove this by proving each case separately.
**Proof of f = g**

$$\text{Let } t = 0 \implies f(0) = g(0) \implies 2(0) + 1 = 1 + 4(0) - 2(0)^2 \implies 1 = 1$$
$$\text{Let } t = 1 \implies f(1) = g(1) \implies = 2(1) + 1 = 1 + 4(1) - 2(1)^2 \implies 3 = 3$$

**Proof of f + g = h**

$$\text{Let } t = 0 \implies f(0) + g(0) = h(0) \implies 2(0) + 1 + 1 + 4(0) - 2(0)^2 = 5^0 + 1 \implies 2 = 2$$
$$\text{Let } t = 1 \implies f(1) + g(1) = h(1) \implies 2(1) + 1 + 1 + 4(1) - 2(1)^2 = 5^1 + 1 \implies 6 = 6$$

## Question 8

### In any vector space $V$, show that $(a+b)(x+y) = ax+ay+bx+by$ for any $x, y \in V$ and $a, b \in F$

We can initially treat $x + y$ as a single vector and thus use (VS 7) to distribute the scalars then use (VS 8) on each of the resulting vectors to lead to 4 vectors.

$$\begin{aligned}
&(a + b)(x + y) \\
&= a(x + y) + b(x + y), \text{(VS 7)} \\
&= (ax + ay) + (bx + by), \text{(VS 8)}
\end{aligned}$$

Note that the there are multiple parentesis representaions for this sum as follows:

$$\begin{aligned}
&= ((ax + ay) + bx) + by \\
&= (ax + ay) + (bx + by) \\
&= (ax + (ay + bx)) + by \\
&= ax + (ay + (bx + by)) \\
&= ax + ((ay + bx) + by)
\end{aligned}$$

These solutions are equivalent as by (VS 2) we can rearrange the order of parentesis along vector addition:

$$\begin{aligned}
((ax + ay) + bx) + by &= (ax + (ay + bx)) + by (VS2) \\
(ax + (ay + bx)) + by &= ax + ((ay + bx) + by) (VS2) \\
ax + ((ay + bx) + by) &= ax + (ay + (bx + by)) (VS2) \\
ax + (ay + (bx + by)) &= (ax + ay) + (bx + by) (VS2)
\end{aligned}$$

Finally we can see that $((ax+ay)+(bx+by))$ is equivalent to $ax+ay+bx+by$ By the transitive property of equality, we can conclude that $(a + b)(x + y) = ax + ay + bx + by$ regardless of the parenthesis representations we choose to distribute the term out by.

# Question 9

## Prove Corollaries 1 and 2 of Theorem 1.1 and Theorem 1.2 (c)

**Corollary 1:** The vector $\underline{0}$ in (VS 3)is unique.
**Proof:** Assume there are two zero vectors $\underline{0}$ and $\underline{0}'$ in $V$.
Thus $\forall x \in V$ we have $x + \underline{0} = x$ and $x + \underline{0}' = x$
Thus by transitiviy we have $x + \underline{0} = x + \underline{0}'$
By theorem 1.1 we have $\underline{0} = \underline{0}'$
This is a contradiction as we assumed there were two distinct zero vectors.
Thus the zero vector is unique.
**Corollary 2:** The vector y in (VS 4) is unique.
**Proof:** Assume there are two vectors $y$ and $y'$ in $V$ such that $x + y = 0$ and $x + y' = 0$
Thus by transitiviy we have $x + y = x + y'$
Thus by theorem 1.1 we have $y = y'$
This is a contradiction as we assumed there were two distinct vectors.
Thus the vector y is unique.
**Theorem 1.2 (c):** $a\underline{0} = \underline{0}, \forall a \in F$
**Proof:** Let $a \in F$, Need $a\underline{0} = \underline{0}$

Consider $a\underline{0} + a\underline{0}$
Thus by (VS 7) we have $a\underline{0} + a\underline{0} = a(\underline{0} + \underline{0})$
Thus by (VS 3) we have $a(\underline{0} + a\underline{0}) = a\underline{0}$
By (VS 4) We let $y = -a\underline{0}$ and add y to both sided
Thus we have $a\underline{0} = \underline{0}$ as desired

# Question 11

## Let $V = \{0\}$ consit of a single vector **0** an define $0 + 0 = 0$ and $c0 = 0$ for each $c \in F$. Prove that $V$ is a vector space over $F$

Note that $\forall x \in V \implies x = 0$ Since V has only one element.
Thus for the following proofs I will simply state what needs to be proven and then implicitly let $x, y, z$ be arbitrary elements of $V$ which implies $x, y, z = 0$
**Proof of VS 1:**
Let $x, y \in V$ Nee $x + y = y + x \in V$
$0 + 0 = 0 + 0$
$0 = 0$
**Proof of VS 2:**
Let $x, y, z \in V$ Need $(x + y) + z = x + (y + z)$
$(0 + 0) + 0 = 0 + (0 + 0)$
$0 + 0 = 0 + 0$
$0 = 0$
**Proof of VS 3:**
Let $x \in V$ Need $\exists \underline{0} \in V, x + \underline{0} = x$
$0 + \underline{0} = 0$
Since 0 is the only element in $V$, $\underline{0} = 0$
Thus $0 + 0 = 0$
$0 = 0$
**Proof of VS 4:**
Let $x \in V$ Need $\exists y \in V, x + y = 0$
Let $y = 0$
$x + y = 0 + 0 = 0$ as desired
**Proof of VS 5:**
Let $x \in V$ Need $1x = x$
$1 * 0 = 0$
$0 = 0$
**Proof of VS 6:**
Let $a, b \in F, x \in V$ Need $(ab)x = a(bx)$
$ab0 = a(b0)$
$0 = a0$
$0 = 0$
**Proof of VS 7:**

Let $a \in F, x, y \in V$ Need $a(x+y) = ax + ay$
$a(0+0) = a0 + a0$
$0 = 0$
**Proof of VS 8:**
Let $a, b \in F, x \in V$ Need $(a+b)x = ax + bx$
$(a+b)0 = a0 + b0$
$0 = 0$

## Question 12

### A real-valued function $f$ is defined on the set of all real numbers is called and even function if $f(-t) = f(t)$ for all $t \in \mathbb{R}$. Prove that the set of even functions defined on the real line with the operations of addition and scalar multiplication defined by $(f+g)(t) = f(t)+g(t)$ and $(cf)(t) = c[f(t)]$ is a vector space over $\mathbb{R}$

Note that any for 2 even functions $f, g$ their sum $f + g$ is also even as:

$$\begin{aligned}(f+g)(-t) &= f(-t) + g(-t) \\ &= f(t) + g(t) \\ &= (f+g)(t)\end{aligned}$$

As well as for any scalar $c$:

$$\begin{aligned}(cf)(-t) &= c[f(-t)] \\ &= c[f(t)] \\ &= (cf)(t)\end{aligned}$$

**Proof of VS 1:**
Let $f, g \in V$, Need $f + g = g + f$
$(f+g)(t) = f(t) + g(t) = g(t) + f(t)$
$f(t) + g(t) = f(t) + g(t)$
$f + g = g + f$ as desired
**Proof of VS 2:**
Let $f, g, h \in V$ Need $(f+g) + h = f + (g+h)$
$((f+g)+h)(t) = (f+g)(t) + h(t) = f(t) + g(t) + h(t)$
$(f+(g+h))(t) = f(t) + (g+h)(t) = f(t) + g(t) + h(t)$
Thus $(f+g) + h = f + (g+h)$ as desired
**Proof of VS 3:**
$\exists \underline{0} \in V, \forall f \in V$ Need $f + \underline{0} = f$
$(f + \underline{0})(t) = f(t) + \underline{0}(t) = f(t)$
Let $\underline{0}(t) = 0$
$f(t) + \underline{0} = f(t)$ as desired

Note that $\underline{0} \in V$ as 0 is even since $0(t) = 0(-t) = 0$
**Proof of VS 4:**
Let $f \in V$ Need $\exists g \in V$ s.t. $f + g = \underline{0}$
Let $g = -1f$
$(f + -1f)(t) = f(t) + -1f(t) = \underline{0}$ as desired
Note that $g$ is even as $g(t) = -1f(t) = -1f(-t) = g(-t)$
**Proof of VS 5:**
Let $f \in V$ Need $1f = f$
$(1f)(t) = 1f(t)$
$1f(t) = f(t)$
Thus $1f = f$ as desired
**Proof of VS 6:**
Let $a, b \in \mathbb{R}, f \in V$ Need $(ab)f = a(bf)$
$(ab)f(t) = a(bf(t))$
$abf(t) = abf(t)$
$(ab)f = a(bf)$ as desired
**Proof of VS 7:**
Let $a \in \mathbb{R}, f, g \in V$ Need $a(f + g) = af + ag$
$a(f + g)(t) = a(f(t) + g(t)) = af(t) + ag(t) = af + ag$ as desired
**Proof of VS 8:**
$\forall a, b \in \mathbb{R}, \forall f \in V, (a + b)f = af + bf$
$(a + b)f(t) = af(t) + bf(t) = af + bf$ as desired

## Question 17

### Let $V = \{(a_1, a_2) : a_1, a_2 \in F\}$. where F is a field. Define addition of elements of V coordinatewise, and for $c \in F$ and $(a_1, a_2) \in V$ define $c(a_1, a_2) = (a_1, 0)$ Is V a vector space over F with operations?

V is not a vector space as (VS 5) does not hold.
Let $x = (0, 1), c = 1$.
Thus $1x = 1(0, 1) = (0, 0)$
Clearly $(0, 0) \neq (0, 1)$

## Section 1.3

## Question 5

### Prove that $A + A^t$ is symmetric for any square matrix A

Let $A$ be an $n$ by $n$ matrix with each entry $a_{i,j}$ corresponding to entry in the ith row and jth column.

Clearly $A^t$ has the values of $a_{ij}$ in the entries in the ith column and jth row. In other words, its values of $a_{ji}$ in the ith row and jth column
Thus $A + A^t$ has entries of $a_{ij} + a_{ji}$ in the ith row and jth column.
This would be symmetric as for every symmetric matrix the for each entry $a_{ij} = a_{ji}$
Clearly $a_{ij} + a_{ji} = a_{ji} + a_{ij}$
Thus $A + A^t$ is symmetric

## Question 8a

### Determine if the following sets are subspaces of $\mathbb{R}^3$ under the operation of adition and scalar multiplication defined on $\mathbb{R}^3$ Justify your answers: $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2$ and $a_3 = -a_2\}$

This is a subspace as the since $W$ is a subset of a vector space ($\mathbb{R}^3$) it must satisfy the 8 properties of a vector space.
It also satisfies $0 \in W$ as for $a_2 = 0, (0, 0, 0) \in W$
It satisfies the closure property of addition as:
Let $x = (a_1, a_2, a_3)$ and $y = (b_1, b_2, b_3)$
Thus $x = (3a_2, a_2, -a_2)$ and $y = (3b_2, b_2, -b_2)$
$x + y = (3a_2 + 3b_2, a_2 + b_2, -a_2 - b_2)$
$x + y = (3(a_2 + b_2), a_2 + b_2, -(a_2 + b_2))$
Let $c = a_2 + b_2$
$x + y = (3c, c, -c)$
This clearly is also in $W$ thus it satisfies closure.
It satisfies the closure property of scalar multiplication as:
Let $x = (a_1, a_2, a_3)$
$x = (3a_2, a_2, -a_2)$
$cx = (3ca_2, ca_2, -ca_2)$
$cx = (3c(a_2), c(a_2), -c(a_2))$
Let $d = ca_2$
$cx = (3d, d, -d)$
This clearly is also in $W$ thus it satisfies closure.

## Question 8b

### Determine if the following sets are subspaces of $\mathbb{R}^3$ under the operation of adition and scalar multiplication defined on $\mathbb{R}^3$ Justify your answers: $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$

This is not a subspace as there is no element in the set that satisfies the condition $0 \in W$

Let $x = (a_1, a_2, a_3)$
Thus $x = (a_3 + 2, a_2, a_3)$
Clearly $0 \notin W$ since $a_3 + 2 \neq a_3, \forall a_3 \in \mathbb{R}$

## Question 11

### Is the set $W = \{f \in P(F) : f(x) = 0$ or $f(x)$ has degree $n\}$ a subspace of $P(F)$ if $n \geq 1$? Justify your answer.

Yes, this is a subspece of $P(F)$ as it satisfies the closure properties of addition and scalar multiplication as well has the 0 polynomial. Firstly: $0 \in W$ as $f(x) = 0$
Thus the zero polynomial is in $W$
Secondly: Let $f, g \in W$
Then $f(x) = 0$ or $f(x)$ has degree $n$ and can be represented as $\sum_{i=0}^{n} a_i x^i$
Then $g(x) = 0$ or $g(x)$ has degree $n$ and can be represented as $\sum_{i=0}^{n} b_i x^i$
Thus $f + g = \sum_{i=0}^{n} (a_i + b_i) x^i$
If $f(x) = 0$ and $g(x) = 0$ then $f + g = 0$
If $f(x)$ or $g(x)$ has degree $n$ and the other is 0, then $f + g$ is either $f$ or $g$ (whichever one is of degree $n$) has degree $n$
If both $f(x)$ and $g(x)$ have degree $n$, then $f + g = \sum_{i=0}^{n} (a_i + b_i) x^i$ which also has degree $n$
Thus it is closed under addition

Thirdly: Let $f \in W, c \in \mathbb{R}$
Then $f(x) = 0$ or $f(x)$ has degree $n$ and be represented as $\sum_{i=0}^{n} a_i x^i$
Then $cf = c \sum_{i=0}^{n} a_i x^i$
Thus $cf \in W$ as $cf = 0$ or $cf = c \sum_{i=0}^{n} a_i x^i$ which has degree $n$
Thus it is closed under scalar multiplication