01:640:311H - Homework 2

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1. Prove the following statements using the ϵ–N definition of the limit:
   (a) $\lim_{n\to\infty} \frac{n-4}{n+7} = 1$
   (b) $\lim_{n\to\infty} \frac{2n-3}{n+5} = 2$

Solution: (a): We want to show that for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$,

$$\left|\frac{n-4}{n+7} - 1\right| < \epsilon$$

Take $N = \frac{11}{\epsilon} - 7$.

$$n > \frac{11}{\epsilon} - 7$$
$$\frac{11}{n+7} < \epsilon$$
$$\left|\frac{-11}{n+7}\right| < \epsilon$$
$$\left|\frac{n-4}{n+7} - 1\right| < \epsilon$$

(b): We want to show that for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$,

$$\left|\frac{2n-3}{n+5} - 2\right| < \epsilon$$

Take $N = \frac{13}{\epsilon} - 5$.

$$\begin{aligned} n &> \frac{13}{\epsilon} - 5 \\ \frac{13}{n+5} &< \epsilon \\ \left|\frac{-13}{n+5}\right| &< \epsilon \\ \left|\frac{2n-3}{n+5} - 2\right| &< \epsilon \end{aligned}$$

2. (a) We say that a function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous if there exists an L such that for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq L|x - y|$$

Show that if $\{x_n\}_{n=1}^{\infty} \to x$ and $f(x)$ is Lipschitz, then $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x)$.

(b) We say that a function $g : \mathbb{R} \to \mathbb{R}$ is α-Hölder continuous if there exists an $L \in \mathbb{R}$ such that for every $x, y \in \mathbb{R}$,

$$|g(x) - g(y)| \leq L|x - y|^{\alpha}$$

(In particular, Lipschitz functions are Hölder continuous with Hölder exponent $\alpha = 1$). Prove that if $x_n \to x$ and g is α-Hölder continuous for $\alpha \in (0, 1)$, then $g(x_n) \to g(x)$.

Solution: **(a)**: Suppose that $\{x_n\}_{n=1}^{\infty} \to x$ and f is Lipschitz continuous. Need to show that $\{f(x_n)\}_{n=1}^{\infty} \to f(x)$. Fix $L > 0$ from the Lipschitz condition. Then if we take $y = x$ and $x = x_n$, in the Lipschitz definition, we have

$$|f(x_n) - f(x)| \leq L|x_n - x|$$

We also know that there exists an $N \in \mathbb{N}$ such that for all $n > N$, we have

$$|x_n - x| < \frac{\epsilon}{L}$$

Thus, we have

$$|f(x_n) - f(x)| \leq L|x_n - x| < L\frac{\epsilon}{L} = \epsilon$$

Thus, We have $\{f(x_n)\}_{n=1}^{\infty} \to f(x)$. **b**: Suppose that $\{x_n\}_{n=1}^{\infty} \to x$ and g is α-Hölder continuous for $\alpha \in (0,1)$. Need to show that $\{g(x_n)\}_{n=1}^{\infty} \to g(x)$ Fix $L > 0$ from the Hölder condition. Then if we take $y = x$ and $x = x_n$, in the Hölder definition, we have

$$|g(x_n) - g(x)| \leq L|x_n - x|^{\alpha}$$

We also know that there exists an $N \in \mathbb{N}$ such that for all $n > N$, we have

$$|x_n - x| < (\frac{\epsilon}{L})^{\frac{1}{\alpha}}$$

Thus, we have

$$|g(x_n) - g(x)| \leq L|x_n - x|^{\alpha} < \epsilon$$

Thus, We have $\{g(x_n)\}_{n=1}^{\infty} \to g(x)$.

3. Let $x_{n_{n=1}}^{\infty}$ and $y_{n_{n=1}}^{\infty}$ be sequences and define

$$\{z_n\}_{n=1}^{\infty} = \{x_1, y_1, x_2, y_2, \ldots\}_{n=1}^{\infty}$$

Show that if $z_n \to x$ then $\{x_n\}_{n=1}^{\infty} \to x$ and $\{y_n\}_{n=1}^{\infty} \to x$.

Solution: Suppose that $\{z_n\}_{n=1}^{\infty} \to x$. We want to show that $\{x_n\}_{n=1}^{\infty} \to x$ and $\{y_n\}_{n=1}^{\infty} \to x$. We can see that the subsequence $\{z_{2n}\}_{n=1}^{\infty} = \{x_n\}_{n=1}^{\infty}$ and $\{z_{2n-1}\}_{n=1}^{\infty} = \{y_n\}_{n=1}^{\infty}$. We also know that $2n - 1 > 2n > n$ for all $n \in \mathbb{N}$ and thus must converge if our original sequence converges. Thus, we have $\{x_n\}_{n=1}^{\infty} \to x$ and $\{y_n\}_{n=1}^{\infty} \to x$.

4. Let $T : \mathbb{N} \to \mathbb{N}$ be an injective function. Prove that if $\{x_n\}_{n=1}^{\infty}$ converges to x, then $\{x_{T(n)}\}_{n=1}^{\infty}$ also converges to x.

Solution: Suppose that $\{x_n\}_{n=1}^{\infty} \to x$. We want to show that $\left\{x_{T(n)}\right\}_{n=1}^{\infty} \to x$. We know that if $x_n \to x$, in a topological sense, a sequence converges to x if and only if any given neighborhood of x contains all but finitely many terms of the sequence. Thus the set $S = \{n : n \in N \text{ and } x_n \notin V_\epsilon(x)\}$ is a finite set. Aslo since T is injective the set $S' = \left\{n : n \in N \text{ and } x_{T(n)} \notin V_\epsilon(x)\right\}$ is also finite since as each element of the image of T has a unique inverse. Thus we have that for all $\epsilon > 0$ there exists a finite set of elements not in the neighborhood of x. Thus, we have $\left\{x_{T(n)}\right\}_{n=1}^{\infty} \to x$.

5. Let a be a positive real number.
   (a) Assuming $a > 1$, write an $\epsilon - n$ proof that $a^{\frac{1}{n}} \to 1$. (Hint: Bernoulli's inequality, which you proved in HW 1, may be helpful.)
   (b) Explain how your answer for part (a) can be used to prove $a^{\frac{1}{n}} \to 1$ for all positive real numbers a.

Solution: (a): Suppose $a > 1$. We want to show that $a^{\frac{1}{n}} \to 1$. For all $\epsilon > 0$. We need to show there exists an $N \in \mathbb{N}$ such that for all $n > N$, the following holds:

$$|a^{\frac{1}{n}} - 1| < \epsilon$$

We can take $N = \frac{a-1}{\epsilon}$ ie $a = 1 + \epsilon N$ and. Then for all $n > N$,

$$\begin{aligned} a &< 1 + \epsilon n \\ a &< (1+\epsilon)^n \\ a^{\frac{1}{n}} &< 1 + \epsilon \\ a^{\frac{1}{n}} - 1 &< \epsilon \\ |a^{\frac{1}{n}} - 1| &< \epsilon \end{aligned}$$

(b):
Case: 1 $a > 1$ then the proof follows from part (a)
Case: 2 $a = 1$ then $a^{\frac{1}{n}} = 1$ for all $n \in \mathbb{N}$
Case: 3 $0 < a < 1$ Then we know that $\left\{\frac{1}{a^{\frac{1}{n}}}\right\}_{n=1}^{\infty} \to 1$ by part (a) and the algebraic limit theorem. Thus, we have $a^{\frac{1}{n}} \to 1$.

6. Given a sequence $\{x_n\}_{n=1}^{\infty}$, define the sequence $\{s_n\}_{n=1}^{\infty}$ with general term

$$s_n = \frac{1}{n}\sum_{k=1}^{n} x_k$$

Prove that if $\{s_n\}_{n=1}^{\infty}$ is convergent, then

$$\lim_{n\to\infty} \frac{x_n}{n} = 0$$

(Hint: Try to express $\frac{x_n}{n}$ in terms of s_n.)

Solution: Suppose that $\{s_n\}_{n=1}^{\infty}$ is convergent. We want to show that $\lim_{n\to\infty} \frac{x_n}{n} = 0$.

We can first notice what $\frac{x_n}{n}$ is in terms of s_n. We can see that

$$\begin{aligned} s_n &= \frac{1}{n}\sum_{k=1}^{n} x_k \\ ns_n &= \sum_{k=1}^{n} x_k \\ ns_n - (n-1)s_{n-1} &= x_n \\ \frac{x_n}{n} &= s_n - s_{n-1} + \frac{1}{n}s_{n-1} \end{aligned}$$

We can use the algebraic limit theorem to consider each term in the limit.

$$\begin{aligned} \lim_{n\to\infty} \frac{x_n}{n} &= \lim_{n\to\infty} s_n - s_{n-1} + \frac{1}{n}s_{n-1} \\ \lim_{n\to\infty} s_n &- \lim_{n\to\infty} s_{n-1} + \lim_{n\to\infty} \frac{1}{n}s_{n-1} \end{aligned}$$

Since we know that $\{s_n\}_{n=1}^{\infty}$ is convergent, let us call the value it converges to s thus we have

$$\begin{aligned} \lim_{n\to\infty} s_n &= s \\ \lim_{n\to\infty} s_{n-1} &= s \end{aligned}$$

Thus clearly

$$\lim_{n\to\infty} \frac{1}{n}s_{n-1} = 0$$

Thus, we have

$$\lim_{n\to\infty} \frac{x_n}{n} = s - s + 0 = 0$$

Thus, we have $\lim_{n\to\infty} \frac{x_n}{n} = 0$.