01:640:481 - Homework 3

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1. Question 1. 8.23
Variance of sample variance of a normal family. (Hint: See the text for a theorem that tells you about the distribution of sample variance of a normal family. We discussed the variance of that kind of distributions.) [3O1]
Use Theorem 11 to show that, for random samples of size n from a normal population with the variance σ^2, the sampling distribution of S^2 has the mean σ^2 and the variance $\frac{2\sigma^4}{n-1}$. (A general formula for the variance of S2 for random samples from any population with finite second and fourth moments may be found in the book by H. Cramer listed among the references at the end of ´ this chapter.)
Solution:
We know that the sampling distribution of $(n-1)S^2/\sigma^2$ is χ^2_{n-1}.
We can clearly see that the mean of $S_2 = \sigma^2$ due to properties of expectaion and the chi-square distribution as

$$E((n-1)S^2/\sigma^2) = (n-1)$$
$$E(S^2) = \sigma^2$$

Now, we can find the variance of S^2 as follows:

$$Var((n-1)S^2/\sigma^2) = 2(n-1)Var(S^2) \qquad = \frac{2\sigma^4}{n-1}$$

Solution: Clearly the mean of S^2 is σ^2 and the variance is $\frac{2\sigma^4}{n-1}$.

2. Question 2. 6.7
Prove the important recursive property of the gamma function integral.
Use integration by parts to show that $\Gamma(\alpha) = (\alpha - 1) \cdot \Gamma(\alpha - 1)$ for $\alpha > 1$.
Solution:
We know that the gamma function is defined as:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$$

Now, we can use integration by parts to prove the recursive property of the gamma function integral.

$$\begin{aligned}\Gamma(\alpha) &= \int_0^\infty x^{\alpha-1}e^{-x}dx \\ &= \left[-x^{\alpha-1}e^{-x}\right]_0^\infty + (\alpha-1)\int_0^\infty x^{\alpha-2}e^{-x}dx \\ &= 0 + (\alpha-1)\int_0^\infty x^{\alpha-2}e^{-x}dx \\ &= (\alpha-1)\Gamma(\alpha-1)\end{aligned}$$

Solution: We have proved that $\Gamma(\alpha) = (\alpha - 1) \cdot \Gamma(\alpha - 1)$ for $\alpha > 1$.

3. Question 3. 6.11
   Understanding the gamma distribution's pdf.
   Show that a gamma distribution with $\alpha > 1$ has a relative maximum at $x = \beta(\alpha - 1)$. What happens when $0 < \alpha < 1$ and when $\alpha = 1$?
   Solution:
   The probability density function (pdf) of a gamma distribution is given by:

$$f(x; \alpha, \beta) = \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}$$

   To find the relative maximum, we take the derivative of the pdf with respect to x and set it to zero:

$$\begin{aligned}
\frac{d}{dx} f(x; \alpha, \beta) &= \frac{d}{dx} \left(\frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)} \right) \\
&= \frac{d}{dx} \left(x^{\alpha-1} e^{-x/\beta} \right) \cdot \frac{1}{\beta^\alpha \Gamma(\alpha)} \\
&= \left((\alpha - 1) x^{\alpha-2} e^{-x/\beta} - \frac{1}{\beta} x^{\alpha-1} e^{-x/\beta} \right) \cdot \frac{1}{\beta^\alpha \Gamma(\alpha)} \\
&= x^{\alpha-2} e^{-x/\beta} \left((\alpha - 1) - \frac{x}{\beta} \right) \cdot \frac{1}{\beta^\alpha \Gamma(\alpha)}
\end{aligned}$$

   Clearly this is only zero when $\alpha - 1 - x/\beta = 0$

Solution:

$$x = \beta(\alpha - 1)$$

   We know that for $0 < \alpha < 1$ the gamma distribution is increasing, and for $\alpha < 1$ the gamma distribution is decreasing. And for $\alpha = 1$, the gamma distribution is an exponential distribution.

4. Question 4. 8.45
   Verify the results of Example 4, that is, the sampling distributions of Y_1, Y_n, and $\tilde{X}$ shown there for random samples from an exponential population.

Solution:
The exponential distribution is given by:

$$f(x;\lambda) = \frac{1}{\lambda}e^{-x/\lambda}$$

The mean and variance of the exponential distribution are given by:

$$E(X) = 1/\lambda$$
$$Var(X) = 1/\lambda^2$$

We want to verify the sample distribution of Y_1, Y_n, and $\tilde{X}$ and we can utilize theorem 16 to do so.
Theorem 16 for an exponential distribution states that:

$$g_r(y_r) = \frac{n!}{(r-1)!(n-r)!}[\int_0^{y_r} \frac{1}{\lambda}e^{-x/\lambda}dx]^{r-1}\frac{1}{\lambda}e^{-x/\lambda}[\int_{y_r}^{\infty} \frac{1}{\lambda}e^{-x/\lambda}dx]^{n-r}$$

We can use this theorem to verify the results of Example 4.
The pdf of Y_1 is given by:

$$f(y_1) = \frac{n}{\lambda}[e^{-y_1/\lambda} * (e^{-y_1/\lambda})^{n-1}]$$
$$= \frac{n}{\lambda}[e^{-ny_r/\lambda}]$$

The pdf of Y_n is given by:

$$f(y_n) = \frac{n}{\lambda}[(1-(e^{-y_n/\lambda})^{n-1}] * (e^{-y_n/\lambda})]$$

The pdf of $\tilde{X}$ (where $n = 2m+1$) is given by:

$$f(\tilde{x}) = \frac{(2m+1)!}{m!m!}\frac{1}{\lambda}e^{-\tilde{x}/\lambda}\left[\int_0^{\tilde{x}} \frac{1}{\lambda}e^{-x/\lambda}dx\right]^m \left[\int_{\tilde{x}}^{\infty} \frac{1}{\lambda}e^{-x/\lambda}dx\right]^m$$
$$= \frac{(2m+1)!}{m!m!}\frac{1}{\lambda}e^{-\tilde{x}/\lambda}\left[1-e^{-\tilde{x}/\lambda}\right]^m \left[e^{-\tilde{x}/\lambda}\right]^m$$
$$= \frac{(2m+1)!}{m!m!}\frac{1}{\lambda}\left[1-e^{-\tilde{x}/\lambda}\right]^m \left[e^{-\tilde{x}/\lambda}\right]^{m+1}$$

5. Question 5. 8.48
   Find the mean and the variance of the sampling distribution of Y1 for random samples of size n from the population of Exercise 46
   We are considering the distribution
   Solution:
   The sample distribution of Y_1:

$$f(y_1) = n(1-y_1)^{n-1}$$

The mean of Y_1 is given by:

$$\begin{aligned}
E(Y_1) &= \int_0^1 y_1 n(1-y_1)^{n-1} dy_1 \\
&= \int_0^1 n(1-y_1)^{n-1} dy_1 \\
&= (-y(1-y)^n)\Big|_0^1 + \int_0^1 (1-y)^n dy \\
&= \frac{1}{n+1}(1-y)^{n+1}\Big|_0^1 \\
&= \frac{1}{n+1}
\end{aligned}$$

The variance of Y_1 is given by:

$$\begin{aligned}
Var(Y_1) &= E(Y_1^2) - E(Y_1)^2 \\
&= \int_0^1 y_1^2 n(1-y_1)^{n-1} dy_1 - \left(\frac{1}{n+1}\right)^2
\end{aligned}$$

We know that

$$\int_0^1 y_1^2 n(1-y_1)^{n-1} dy_1 = \frac{2}{(n+1)(n+2)}$$

Therefore, the variance of Y_1 is given by:

$$\begin{aligned}
Var(Y_1) &= \frac{2}{(n+1)(n+2)} - \left(\frac{1}{n+1}\right)^2 \\
&= \frac{n}{(n+1)^2(n+2)}
\end{aligned}$$

Solution: The mean of Y_1 is $\frac{1}{n+1}$ and the variance of Y_1 is $\frac{n}{(n+1)^2(n+2)}$.