01:640:481 - Likelihood Ratio Test

Pranav Tikkawar

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1. Recall Exp(θ) population has PDF: $f(x) = \frac{1}{\theta}e^{-\frac{x}{\theta}}$. Suppose $x_1, x_2, \dots, x_n$ are observed sample values. Do the computation to find the value of θ (in terms of $x_1, x_2, \dots, x_n$) where the likelihood function is maximized.

   (a) Remember, this value is the MLE estimator $\hat{\theta}_{MLE}$ and this is a computation we have done earlier, and the answer is $\bar{x}$. We are doing it again to review it. Steps: Write $L(\theta)$ and take ln then differentiate with respect to θ, set to 0.

Solution:

$$
\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x_i}{\theta}} \\
&= \frac{1}{\theta^n} e^{-\frac{\sum_{i=1}^{n} x_i}{\theta}} \\
\ln L(\theta) &= -n \ln \theta - \frac{\sum_{i=1}^{n} x_i}{\theta} \\
\frac{d}{d\theta} \ln L(\theta) &= -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} x_i}{\theta^2} = 0 \\
\frac{n}{\theta} &= \frac{\sum_{i=1}^{n} x_i}{\theta^2} \\
\theta &= \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}
\end{aligned}
$$

2. This continues the previous question. A random sample of size n is used to test the null hypothesis that the parameter $\lambda = \theta_0$ against the alternative that it doesn't equal θ_0.

   (a) Here, the likelihood function $L(\theta) =$

   (b) Here, max of likelihood function over parameters that are in the null hypothesis, $L_\omega =$

   (c) Here, max of likelihood function over all parameters (i.e., that in the null and alternative hypothesis), $L_\Omega =$

   (d) Using the above, determine the likelihood ratio statistic $\lambda(x_1, x_2, \dots, x_n)$.

   (e) Use the previous part to show that the critical region of LRT has the form $\bar{x}e^{-\frac{\bar{x}}{\theta_0}} \leq K$

Solution: (a)

$$
\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x_i}{\theta}} \\
&= \frac{1}{\theta^n} e^{-\frac{\sum_{i=1}^{n} x_i}{\theta}}
\end{aligned}
$$

(b)

$$L_\omega = \frac{1}{\theta_0^n} e^{-\frac{\sum_{i=1}^n x_i}{\theta_0}}$$

(c)

$$L_\Omega = \frac{1}{\bar{x}^n} e^{-n}$$

(d)

$$\begin{aligned} \lambda(x_1, x_2, \ldots, x_n) &= \frac{L_\omega}{L_\Omega} \\ &= \frac{\frac{1}{\theta_0^n} e^{-\frac{\sum_{i=1}^n x_i}{\theta_0}}}{\frac{1}{\bar{x}^n} e^{-n}} \\ &= \left(\frac{\bar{x}}{\theta_0}\right)^n e^{n - \frac{n\bar{x}}{\theta_0}} \end{aligned}$$

(e) To show that the critical region of LRT has the form $\bar{x} e^{-\frac{\bar{x}}{\theta_0}} \leq K$, we can see that

$$\begin{aligned} \left(\frac{\bar{x}}{\theta_0}\right)^n e^{n - \frac{n\bar{x}}{\theta_0}} &\leq k \\ (\bar{x} e^{1 - \frac{\bar{x}}{\theta_0}})^n &< \theta_0^n k \\ \bar{x} e^{1 - \frac{\bar{x}}{\theta_0}} &< \theta_0 k^{1/n} \\ \bar{x} e^{-\frac{\bar{x}}{\theta_0}} &< \theta_0 k^{1/n} e^{-1} \end{aligned}$$

Thus if we take $K = \theta_0 k^{1/n} e^{-1}$, we get the desired form of $\bar{x} e^{-\frac{\bar{x}}{\theta_0}} \leq K$.