01:640:350H - Homework 12

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1. Section 6.2 Question 2(a)
   In each part apply Gram-Scmidt to the given subset S of the inner produt space V to obtain an orthogonal basis for span S.
   Then normaize the vectors in this basis to obtain an orthonormal basis β for span S.
   Then compute the fourier coefficients of the given vector relative to β
   Finally use theorem 6.5 to verify your answer.

$$V = R^3, S = \{[1,0,1].[0,1,1],[1,3,3]\}, x = [1,1,2]$$

Solution: (i) Gram-Schmidt
We can use the Gram-Schmidt process to find an orthogonal basis for S.
Let the orthogonal basis be $\gamma = \{w_1, w_2, w_3\}$

$$w_1 = v_1 = [1,0,1]$$

$$w_2 = v_2 - \frac{< v_2, w_1 >}{||w_1||^2} w_1 = [0,1,1] - \frac{1}{2}[1,0,1] = \frac{[-1,2,1]}{2}$$

$$w_3 = v_3 - \frac{< v_3, w_1 >}{||w_1||^2} w_1 - \frac{< v_3, w_2 >}{||w_2||^2} w_2 = [1,3,3] - \frac{4}{2}[1,0,1] - \frac{8}{6}[-1,2,1] = \frac{[1,1,-1]}{3}$$

(ii) Normalize
We can normalize the vectors in γ to get an orthonormal basis β

$$\beta = \{v_1, v_2, v_3\}$$

$$v_1 = \frac{[1,0,1]}{\sqrt{2}}$$

$$v_2 = \frac{[-1,2,1]}{2\sqrt{3/2}}$$

$$v_3 = \frac{[1,1,-1]}{3\sqrt{1/3}}$$

(iii) Fourier Coefficients
We can find the fourier coefficients of x relative to β

$$< x, v_1 > = \frac{[1,1,2]\cdot[1,0,1]}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$
$$< x, v_2 > = \frac{[1,1,2]\cdot[-1,2,1]}{2\sqrt{3/2}} = \frac{3}{2\sqrt{3/2}}$$
$$< x, v_3 > = \frac{[1,1,2]\cdot[1,1,-1]}{3\sqrt{1/3}} = \frac{0}{\sqrt{1}} = 0$$

Thus the fourier coefficients of x relative to β are $\{3/\sqrt{2}, 3/2\sqrt{3/2}, 0\}$

(iv) Verify
By theorem 6.5, we can verify that the fourier coefficients of x relative to β are correct.

$$x = \sum_{i=1}^{3} < x, v_i > v_i = \frac{3}{\sqrt{2}}\frac{[1,0,1]}{\sqrt{2}} + \frac{3}{2\sqrt{3/2}}\frac{[-1,2,1]}{2\sqrt{3/2}} + 0\frac{[1,1,-1]}{3\sqrt{1/3}} = [1,1,2]$$

2. Section 6.2 Question 2(c) In each part apply Gram-Scmidt to the given subset S of the inner produt space V to obtain an orthogonal basis for span S.
Then normaize the vectors in this basis to obtain an orthonormal basis β for span S.
Then compute the fourier coefficients of the given vector relative to β
Finally use theorem 6.5 to verify your answer.

$$V = P_2(R), < f, g >= \int_0^1 f(x)g(x)dx, S = \{1, x, x^2\}, h(x) = 1 + x$$

Solution: (i) Gram-Schmidt
We can use the Gram-Schmidt process to find an orthogonal basis for S.
Let the orthogonal basis be $\gamma = \{w_1, w_2, w_3\}$

$w_1 = 1$

$w_2 = x - \dfrac{< x, 1 >}{||1||^2} 1 = x - \dfrac{1/2}{1} = x - \dfrac{1}{2}$

$w_3 = x^2 - \dfrac{< x^2, 1 >}{||1||^2} 1 - \dfrac{< x^2, x - \frac{1}{2} >}{||x - \frac{1}{2}||^2}(x - \dfrac{1}{2}) = x^2 - \dfrac{1/3}{1} - \dfrac{1/12}{1/12}(x - \dfrac{1}{2}) = x^2 - x + \dfrac{1}{6}$

(ii) Normalize

We can normalize the vectors in γ to get an orthonormal basis β

$$\beta = \{v_1, v_2, v_3\}$$

$$v_1 = \frac{1}{\sqrt{1}} = 1$$

$$v_2 = \frac{x - \frac{1}{2}}{\frac{\sqrt{3}}{6}}$$

$$v_3 = \frac{x^2 - x + \frac{1}{6}}{\frac{\sqrt{5}}{30}}$$

(iii) Fourier Coefficients

We can find the fourier coefficients of $h(x)$ relative to β

$$< h(x), v_1 > = \int_0^1 (1 + x) dx = \frac{3}{2}$$

$$< h(x), v_2 > = 2\sqrt{3} \int_0^1 (1 + x)(x - \frac{1}{2}) dx = \sqrt{3}/6$$

$$< h(x), v_3 > = 6\sqrt{5} \int_0^1 (1 + x)(x^2 - x + \frac{1}{6}) dx = 0$$

Thus the fourier coefficients of $h(x)$ relative to β are $\{3/2, \sqrt{3}/6, 0\}$

(iv) Verify

By theorem 6.5, we can verify that the fourier coefficients of $h(x)$ relative to β are correct.

$$h(x) = \sum_{i=1}^{3} < h(x), v_i > v_i = \frac{3}{2} 1 + \frac{\sqrt{3}}{6} \frac{x - \frac{1}{2}}{\frac{\sqrt{3}}{6}} + 0 \frac{x^2 - x + \frac{1}{6}}{\frac{\sqrt{5}}{30}} = 1 + x$$

3. Section 6.2 Question 3 in R^2 let

$$\beta = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \right\}$$

Find the fourier coefficients of $(3, 4)$ relative to β.

Solution: We can find the fourier coefficients of $(3, 4)$ relative to β

$$< (3,4), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) > = \frac{3}{\sqrt{2}} + \frac{4}{\sqrt{2}} = 7/\sqrt{2}$$
$$< (3,4), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) > = \frac{3}{\sqrt{2}} - \frac{4}{\sqrt{2}} = -1/\sqrt{2}$$

Thus the fourier coefficients of $(3, 4)$ relative to β are $\{7/\sqrt{2}, -1/\sqrt{2}\}$

4. Section 6.5 Question 12 Let A be an $n \times n$ real symetric or complex nomral matrix. Prove that

$$det(A) = \prod_{i=1}^{n} \lambda_i$$

where the λ_i are the (not nessarily distinct) eigenvalues of A.

Solution: Since A is a real symmetric or complex normal matrix, it must be normal. Thus A can be diagonalized by a unitary matrix P

$$P^{-1}AP = D$$

where D is a diagonal matrix with the eigenvalues of A on the diagonal.
Since P is unitary, $det(P) = 1$
Thus

$$\begin{aligned} A &= PDP^{-1} \\ det(A) &= det(PDP^{-1}) \\ &= det(P)det(D)det(P^{-1}) \\ &= det(D) \\ &= \prod_{i=1}^{n} \lambda_i \end{aligned}$$

Therefore $det(A) = \prod_{i=1}^{n} \lambda_i$

5. Section 6.5 Question 17

   Prove that a matrix that is both unitary and upper triangular must be diagonal.

Solution: Let A be a unitary and upper triangular matrix.
Thus A can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Since A is unitary, $A^*A = I$
Thus

$$A^*A = \begin{bmatrix} \overline{a_{11}} & 0 & \cdots & 0 \\ \overline{a_{12}} & \overline{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & \cdots & \overline{a_{nn}} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = I$$

Notice that for all i , $|a_{ii}| \neq 0$ since that leads to the rank of A being less than n and thus violating the orthogonalilty of each of the columns of A.

$$A^*A = \begin{bmatrix} |a_{11}|^2 & \overline{a_{11}}a_{12} & \cdots & \overline{a_{11}}a_{1n} \\ a_{12}\overline{a_{11}} & |a_{12}|^2 + |a_{22}|^2 & \cdots & \overline{a_{22}}a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{11}\overline{a_{1n}} & a_{12}\overline{a_{1n}} + a_{22}\overline{a_{2n}} & \cdots & |a_{1n}|^2 + |a_{2n}|^2 + |a_{nn}|^2 \end{bmatrix} = I$$

The only way that the above matrix can be the identity matrix is if $a_{ij} = 0$ for all $i \neq j$
Thus A is diagonal.

6. Section 6.5 Question 18 Show that "is unitarily equivalent" is an equivalence relation on $M_{n\times n}(C)$

Solution: The relation "is unitarily equivalent" is an equivalence relation on $M_{n\times n}(C)$ if it is reflexive, symmetric, and transitive.
Also remember that a unitary matrix is a matrix U such that $U^*U = UU^* = I$

(i) Reflexive
Let A be a matrix in $M_{n\times n}(C)$.

A is unitarily equivalent to itself since $A = IAI^{-1}$

(ii) Symmetric
Let A and B be matrices in $M_{n\times n}(C)$ such that A is unitarily equivalent to B.
Need B is unitarily equivalent to A.
Since A is unitarily equivalent to B, there exists a unitary matrix U such that $B = UAU^{-1}$
Thus $A = U^{-1}BU$
Since if U is unitary, U^{-1} is also unitary, B is unitarily equivalent to A.

(iii) Transitive
Let A, B, and C be matrices in $M_{n\times n}(C)$ such that A is unitarily equivalent to B and B is unitarily equivalent to C.
Need A is unitarily equivalent to C.
Since A is unitarily equivalent to B, there exists a unitary matrix U such that $B = UAU^*$
Since B is unitarily equivalent to C, there exists a unitary matrix V such that $C = VBV^*$
Thus $C = V(UAU^*)V^* = (VU)A(VU)^*$
We can see that VU is unitary since $V^*V = VV^* = I$ and $UU^* = U^*U = I$ then $(VU)^*(VU) = U^*V^*VU = U^*U = I$ an $(VU)(VU)^* = VUU^*V^* = VV^* = I$
So the product of two unitary matrices is unitary.
Thus A is unitarily equivalent to C.