HW2 - Math 481

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Problem 8.6

Look at the binomial random variables as on page 226, that is, as sums of iid Bernoulli random variables, and using central limit Theorem, prove Theorem 6.8 on page 191.
If X is a random random variable having a binomial distribution with parameters n and θ then the moment generating dunction of

$$Z = \frac{X - n\theta}{\sqrt{n\theta(1-\theta)}}$$

approaches the standard normal distribution as $n \to \infty$.

Solution:
Since we can consider a binomial random variable as a sum of iid Bernoulli random variables, we can write X as:

$$\sum_{i=1}^{n} Y_i$$

where Y_i are iid Bernoulli random variables with parameter θ.
Then we can consider Z as:

$$Z = \frac{\sum_{i=1}^n Y_i - n\theta}{\sqrt{n\theta(1-\theta)}}$$

$$Z = \frac{\sum_{i=1}^n Y_i/n - \theta}{\sqrt{\theta(1-\theta)/n}}$$

$$Z = \frac{\bar{Y} - \theta}{\sqrt{\theta(1-\theta)/n}}$$

where $\bar{Y} = \sum_{i=1}^n Y_i/n$ is the sample mean of the Bernoulli random variables.
By the Central Limit Theorem, the sample mean $\bar{Y}$ approaches a normal distribution as $n \to \infty$. with $\mu = \theta$ and $\sigma = \sqrt{\theta(1-\theta)}$ Therefore, Z also approaches a normal distribution as $n \to \infty$.

Problem 8.20

Prove Theorem 9
If $X_1, X_2, \ldots, X_n$ are independent random variables having Chi-Squared distributions with $v_1, v_2, \ldots, v_n$ degrees of freedom, then

$$Y = \sum_{i=1}^{n} X_i$$

has the Chi-Squared distribution with $v = \sum_{i=1}^{n} v_i$ degrees of freedom.
Solution:
To prove this using the moment generating function (MGF), we start by recalling the MGF of a Chi-Squared distribution with v_i degrees of freedom. The MGF of a Chi-Squared random variable X_i with v_i degrees of freedom is given by:

$$M_{X_i}(t) = (1 - 2t)^{-v_i/2}$$

Since $X_1, X_2, \ldots, X_n$ are independent Chi-Squared random variables, the MGF of their sum $Y = \sum_{i=1}^{n} X_i$ is the product of their individual MGFs:

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} (1 - 2t)^{-v_i/2}$$

This can be simplified as:

$$M_Y(t) = (1 - 2t)^{-\sum_{i=1}^{n} v_i/2}$$

Let $v = \sum_{i=1}^{n} v_i$. Then we have:

$$M_Y(t) = (1 - 2t)^{-v/2}$$

This is the MGF of a Chi-Squared distribution with v degrees of freedom. Therefore, Y has a Chi-Squared distribution with $v = \sum_{i=1}^{n} v_i$ degrees of freedom.

Problem 8.24

Show that if $X_1, X_2, \ldots, X_n$ are independent random variables, g=hazing the chi-squared distribution with $v = 1$ and $Y_n = \sum_{i=1}^{n} X_i$, then the limiting distribution of

$$Z_n = \frac{Y_n/n - 1}{\sqrt{2/n}}$$

as $n \to \infty$ is the standard normal distribution.
Solution:
We can clealry see that Y_n/n is the sample mean of the X_i's. Therefore, Y_n/n approaches a normal distribution by the central limit theorem as $n \to \infty$. with $\mu = 1$ and $\sigma = \sqrt{2}$.

Problem 8.26

Use the method of Excercise 25 to find the approximate values of the probably that a random variable having the chi-squared distribution with $\nu = 50$ will take on a value greater than 68.0

Solution:

We know from the previous exercise that

$$\frac{X - \nu}{\sqrt{2\nu}}$$

is an approximation of the chi-squared distribution as a standard normal distribution. Therefore, we can write:

$$\frac{X - 50}{100}$$

We can then calculate:

$$P((68 - 50)/100 < Z) = P(Z > 1.8) = 1 - P(Z < 1.8) = 1 - .9641 = .0359$$

Thus our approximation is that the probability that a random variable having the chi-squared distribution with $\nu = 50$ will take on a value greater than 68.0 is .0359.

Problem 4.31

What is the smallest value of k in Chebyshevs theorem for which the probability that a random variable will take on a value between $\mu - k\sigma$ and $\mu + k\sigma$ is:

a) at least .95

We can appply Chebyshev's theorem to get:

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

We can then solve for k:

$$1 - \frac{1}{k^2} \geq .95$$

$$\frac{1}{k^2} \leq .05$$

$$k^2 \geq 20$$

$$k \geq \sqrt{20}$$

b) at least .99

We can appply Chebyshev's theorem to get:

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

We can then solve for k:

$$1 - \frac{1}{k^2} \geq .99$$

$$\frac{1}{k^2} \leq .01$$

$$k^2 \geq 100$$

$$k \geq \sqrt{100}$$

$$k \geq 10$$