01:640:311 - Homework n

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This is a set of all of the theorems talked in class and in the book numbered.

Theorem 1 (0.0.0: Theorem Name)**.** *This is a theorem. and a teplate for theorems..*

Proof. This is a proof.

$$\text{This is a proof.} \quad e = mc^2$$

$\square$

Theorem 2 (Nested Interval Property: (s 1.4))**.** *If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ is a sequence of closed intervals in $\mathbb{R}$ then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.*

Proof. Let $A = \{a_1, a_2, a_3, \dots\}$ be the set of left endpoints of the intervals.
Now since the I_ns are nested, $I_n \subseteq I_1$ for all n.
Thus each $a_n \in I_1$ for all n.
so $a_n \leq b_1$.
It follows that b_1 is an upper bound for A so sup A exists.
Now we need to prove that $x \in \bigcap_{n=1}^{\infty} I_n$.
To do thi we need to how that $x \in I_n$ for all n.
This mean that $a_n \leq x \leq b_n$ for all n.
Step 1 $a_n \leq x$ for all n.
Remember that $x = \sup A$.
So $a_n \leq x$ for all n.
Step 2 $x \leq b_n$ for all n.
Since $x = \sup A$, x i less than very upper bound of A so it i enough to show that b_n is an upper bound of A.
$b_n \geq a_m$ for all m.
Case 1 $n > m$.
Then $I_n \subset I_m$ so $b_n \in [a_m, b_m] = I_m$.
Case 2 $n \leq m$.
Then $I_m \subset I_n$ so $a_m \in [a_n, b_n] = I_n$.
so $a_m \leq b_n$.
This b_n is an upper bound of A.
Thus $x \leq b_n$ for all n.
Thus $x \in I_n$ for all n.
Thus $x \in \bigcap_{n=1}^{\infty} I_n$. which means the intersection is not empty. $\square$

Theorem 3 (Archimedan Property)**.** *The set $\mathbb{N}$ is not bounded above.*

Proof. Suppose (by contradiction) $\mathbb{N}$ is bounded above.
Then by the least upper bound property, sup $\mathbb{N}$ exists.
Let us call $\alpha = \sup \mathbb{N}$ and it is a real number.
Thus $\alpha - 1 < \alpha$ so $\alpha - 1$ is not an upper bound of $\mathbb{N}$.
So we can fine an $n \in \mathbb{N}$ such that $\alpha - 1 < n$.
Thus $\alpha < n + 1$.
But $n + 1 \in \mathbb{N}$ so α is not an upper bound of $\mathbb{N}$.

$\square$

Theorem 4 (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *$\forall a < b \in \mathbb{R}$ there exists $q \in \mathbb{Q}$ such that $a < q < b$.*

Definition (Open Set)**.** An open set is a set S that for all $x \in S$ for all epsilon neighborhoods $V_\epsilon(x)$ of x, $V_\epsilon(x) \subseteq S$.
In other words, any point has a circle that can be drawn around it that is completely contained in the set.
The union of open sets is open.
The intersection of finitely many open sets is open.
A set is open iff its compliment is closed.

Definition (Closed Set)**.** A closed set is a set S it contains all of its limit points.
A set is closed iff every cauchy sequence contained in S has limit in S.
The intersection of closed sets is closed.
The union of finitely many closed sets is closed.
A set is closed iff its compliment is open.

Definition (Limit Point)**.** A point x is a limit point of a set S if every epsilon neighborhood $V_\epsilon(x)$ of x intersects the set S at a point other than x.
In other words x is a limit point if there is a sequence of points in S that converges to x where the sequence does not contain x.

Definition (Isolated Point)**.** An isolated point is a point x in a set S that is not a limit point of S.
In other words, there exists an epsilon neighborhood $V_\epsilon(x)$ of x such that $V_\epsilon(x) \cap S = \{x\}$.

Definition (Closure and Interior)**.** The closure of a set S denoted by $\overline{S}$ is the union of S and all of its limit points.
The interior of a set S denoted by S° is the collection of all points $x \in S$ such that there exists an epsilon neighborhood $V_\epsilon(x)$ of x that is completely contained in S.
Let S be a set then:
S is closed iff $S = \overline{S}$.
S is open iff $S = S^\circ$.

Definition (Compact Set)**.** A compact set is a set that every Sequence in K has a subsequence that converges to a point in K.
A set is compact if and only if it is closed and bounded.
A set is compact if and only if every open cover has a finite subcover.

Definition (Open Cover)**.** An open cover of a set S is a collection of open sets $\{U_\alpha\}$ such that union of all the open sets contains S.
In other words, $S \subseteq \bigcup_\alpha U_\alpha$.

Theorem 5 (Hiene-Borel Theorem)**.** *A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.*

Definition (Perfect Set)**.** A Perfect set is a set S thtat is closed and contains no isolated points.
A nonempty perfect set is uncountable.
Examples are Cantor Set and the set of all real numbers.

Definition (Seperated, Disconnected, and Connected Set)**.** Two sets A and B are seperated if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.
A set S is disconnected if it can be written as the union of two nonempty seperated sets.
A set S is connected if it is not disconnected.

Definition (F_σ set)**.** A set S is an F_σ set if it is the countable union of closed sets. A set is F_σ if and only if its compliment is G_δ.

Definition (G_δ set)**.** A set S is an G_δ set if it is the countable intersection of open sets. A set is G_δ if and only if its compliment is F_σ.

Definition (Dense and Nowhere-Dense Set)**.** We say a set S is dense in X if $\overline{S} = X$.
A set S is nowhere-dense if $\overline{S}$ contains no open interval. ie $\overline{S}^\circ = \emptyset$.
A set E is nowhere-dense in R iff $\overline{E}^c$ is dense in R.

Definition (Baire's Theorem)**.** The set of Real numbers R cannot be written as the countable union of nowhere-dense sets.

Definition (Functional Limit)**.** Suppose $f : A \to \mathbb{R}$ and $c \in \mathbb{R}$ is a limit point of A.
We say that $\lim_{x \to c} f(x) = L$ if

$$\begin{aligned} \forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - c| < \delta &\implies |f(x) - L| < \epsilon \\ \forall V_\epsilon(L) \text{ there exists a } V_\delta(c) \text{ such that } \forall x \in V_\delta(c) &\implies f(x) \in V_\epsilon(L) \end{aligned}$$