01:640:311H - Homework 1

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February 4, 2025

1. If $x \geq 0$ and $y \geq 0$, prove that $\sqrt{xy} \leq \frac{x+y}{2}$. (Hint: Use the fact that $(\sqrt{x} - \sqrt{y})^2 \geq 0$)

Solution: Suppose $x \geq 0$ and $y \geq 0$.
Then, we have

$$\begin{aligned}
(\sqrt{x} - \sqrt{y})^2 &\geq 0 \\
\sqrt{x}^2 - 2\sqrt{xy} + \sqrt{y}^2 &\geq 0 \\
x - 2\sqrt{xy} + y &\geq 0 \\
x + y &\geq 2\sqrt{xy} \\
\sqrt{xy} &\leq \frac{x+y}{2}
\end{aligned}$$

2. Bernoulli's inequality states that for every integer $n \geq 0$ and real numbers $x \geq -1$, $(1+x)^n \geq 1 + nx$. Use induction to prove this inequality.

Solution: We will prove this by induction on n.
Suppose $x \geq -1$.
Base Case: $n = 0$
Then, we have

$$\begin{aligned}
(1+x)^0 &\geq 1 + 0 \\
1 &\geq 1
\end{aligned}$$

inductive hypothesis: Suppose that for some $k \geq 0$, $(1+x)^k \geq 1 + kx$.
Inductive Step: We want to show that $(1+x)^{k+1} \geq 1 + (k+1)x$.
Then, we have

$$\begin{aligned}
(1+x)^{k+1} &\geq 1 + (k+1)x \\
(1+x)(1+x)^k &\geq 1 + (k+1)x \\
(1+x)(1+kx) &\geq 1 + (k+1)x \\
1 + kx + x + kx^2 &\geq 1 + (k+1)x \\
kx^2 &\geq 0
\end{aligned}$$

Since $x \geq -1$, we have $kx^2 \geq 0$.
Thus, by induction, we have $(1+x)^n \geq 1 + nx$ for all $n \geq 0$.

3. Let $S = \mathbb{Q} \cap [a, b]$. Prove that $\sup S = b$. (Note that b could be rational or irrational).

Solution: Suppose $S = \mathbb{Q} \cap [a, b]$.
Clealry b is an upper bound of S.
Suppose b' is another upper bound of S such that $b' < b$.
By the density of the rationals in the reals, we can find a c such that $b' < c < b$.
Clealry this c is in S. and hence b' is not an upper bound of S which leads to a contradiction.
Thus, b is the least upper bound of S.

4. Recall that in class, we defined the set $-A = \{-a : a \in A\}$

(a) Prove that if x is a lower bound of $-A$, then $-x$ is an upper bound of A. (Note: We proved the opposite implication in class.)

Solution: Let x be a lower bound of $-A$.
Then for all $a \in -A$, $x \leq a$.
Thus $-x \geq -a$.
Thus, $-x$ is an upper bound of A.

(b) Prove that if A is a set of real numbers that is bounded above, then $\inf(-A) = -\sup A$.

Solution: Suppose A is a set of real numbers that is bounded above.
Then $-A = \{-a : a \in A\}$.
Suppose $M = \inf(-A)$.
Then M is a lower bound of $-A$.
Thus, $-M$ is an upper bound of A.
Also M is greater than any lower bound of $-A$.
We need that $-M$ is less than any upper bound of A.
We can see that for all lower bounds of $-A$ (call it L), we have $M \geq L$.
Thus, $-M \leq -L$
We can see that $-L$ is an upper bound of A.
and hence $-M$ is less than any upper bound of A.
Thus, $-M$ is the least upper bound of A.
Thus, we have $M = -\sup A$.

(c) Prove that if A is a nonempty set of real numbers that is bounded below, then A has a greatest lower bound. (In other words, the completeness axiom also holds for inf's.)

Solution: We can see that it holds by considering the set $-A$.
We can see that $-A$ is bounded above.
Thus, by the completeness axiom, $-A$ has a least upper bound.

Now considering $--A$ we can see that it has a greatest lower bound by the earlier part.
Clearly $--A = A$ and thus A has a greatest lower bound.

5. The Cut Property of the real numbers states that if A and B are disjoint sets with $A \cup B = \mathbb{R}$ such that for all $a \in A$ and $b \in B$, $a < b$, then there exists a $c \in \mathbb{R}$ such that $a \leq c$ for all $a \in A$ and $c \leq b$ for all $b \in B$.

   (a) Use the Axiom of Completeness to prove the Cut Property.

   Solution: Suppose A and B are disjoint sets with $A \cup B = \mathbb{R}$ such that for all $a \in A$ and $b \in B$, $a < b$.
   Clealry A is bounded above and B is bounded below.
   Thus, by the completeness axiom, A has a least upper bound $supA$ and B has a greatest lower bound $infB$.
   We need to show that $supA \leq infB$.
   Suppose not.
   Then, we have $supA > infB$.
   Then we have that the the infimum of B is less than the supremum of A and thus is in A which leads to a contradiction or $infB < a$ for all $a \in A$ which is also a contradiction.
   Thus, we have $supA \leq infB$.
   Since $A \cup B = \mathbb{R}$ we must have that $supA = infB$.
   We can then take that to be our c.

   (b) Show that the implication goes the other way: that is, assume that $\mathbb{R}$ has the Cut Property and $E \subset \mathbb{R}$ is bounded above, and prove that $\sup E$ exists.

   Solution: Suppose E is nonempty and bounded above.
   Let B be the set of all upper bounds of E.
   Let $A = \mathbb{R} \setminus B$.
   Thus they are disjoint sets with $A \cup B = \mathbb{R}$ and for all $a \in A$ and $b \in B$, $a < b$.
   Thus, by the cut property, there exists a $c \in \mathbb{R}$ such that $a \leq c$ for all $a \in A$ and $c \leq b$ for all $b \in B$.
   Clealry c is an upper bound of A and since $E \subset A$ then c is an upper bound of E.
   We need to show that c is the least upper bound of E.
   Consider another upper bound c' of E.
   Then, we have $c' \in B$ by definition of B.
   Thus, we have $c \leq c'$ and hence c is the least upper bound of E. Thus, we have that $\sup E$ exists.

6. Remember that in class we said that a set S was dense in $\mathbb{R}$ if for every $a, b \in \mathbb{R}$ with

$a < b$, there existed an element $s \in S \cap (a, b)$. Prove that a set S is dense iff for every $a, b \in \mathbb{R}$ with $a < b$, the set $S \cap (a, b)$ is infinite. (You may freely use the fact that a finite set has a minimum element and a maximum element).

Solution: $\Longrightarrow$ Suppose S is dense.
Prove for every $a, b \in \mathbb{R}$ with $a < b$, there exists an element $s \in S \cap (a, b)$.
Assume for contradiction that $S \cap (a, b)$ is finite.
Then, we can see that $S \cap (a, b)$ has a minimum element and a maximum element.
Let M be the maximum element of $S \cap (a, b)$.
Then we have $M \in \mathbb{R}$ and by density of S, we can find an s such that $M < s < b$.
But then M is not the maximum element of $S \cap (a, b)$ which leads to a contradiction.
Thus, we have that $S \cap (a, b)$ is infinite.
$\Longleftarrow$ Suppose for every $a, b \in \mathbb{R}$ with $a < b$, the set $S \cap (a, b)$ is infinite.
Prove that S is dense.
Assume for contradiction that S is not dense.
Then, there exists an $a, b \in \mathbb{R}$ with $a < b$ such that there does not exist an element $s \in S \cap (a, b)$.
But then $S \cap (a, b)$ is finite which leads to a contradiction.