

01:640:350 - Final Exam

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Contents

1	Notes	2
1.0.1	Normal, Self-Adjoint, Unitary, and Orthogonal Operators	2

Topics

- Abstract Vector Spaces
- Span, Linear Independence, Basis, and Dimension
- Linear Transformations and Matrix Representations
- Change of Coordinate Matrix
- Determinants, Eigenvalues, and Eigenvectors
- Characteristic Polynomial and Diagonalization
- Invariant Subspaces and Cayley-Hamilton Theorem
- Jordan Canonical Form
- Real and Complex Inner Product Spaces
- Normal, Self-Adjoint, Unitary, and Orthogonal Operators

Guaranteed questions:

Formulate and prove a theorem that asserts that the reduced row echelon form of a given $m \times n$ matrix over a field F is unique. (Do not spend time defining RREF or "row-equivalence of matrices" or "pivot columns." Use such concepts in your proof as appropriate.)

1 Notes

1.0.1 Normal, Self-Adjoint, Unitary, and Orthogonal Operators

Definition (Symmetric Matrix). A matrix A is said to be symmetric if $A = A^T$.

Definition (Orthogonal). A matrix A is said to be orthogonal if $A^T A = I$.

This is clear as each of the columns are orthogonal to each other.

Definition (Orthonormal Matrix). A matrix A is said to be orthonormal if $A^T A = I$ and $A^T = A^{-1}$.

This is clear as each of the columns are orthogonal to each other and have a magnitude of 1.

Theorem 1 (Symmetric Matrix, Diagonalization). *If A is a symmetric matrix, There exists an orthogonal matrix P and a diagonal matrix D such that $P^T A P = D$.*

Definition (Unitary Matrix). A matrix A is said to be unitary if $A^* A = A A^* = I$.

This is the analog of orthogonal matrices in complex vector spaces. This is clear as each of the columns are orthogonal to each other.

We need to remember the definition of inner product over C to be $\langle x, y \rangle = y^* x$.

Definition (Hermitian Matrix). A matrix A is said to be Hermitian if $A = A^*$.

This is the analog of symmetric matrices in complex vector spaces. Where A^* is the conjugate transpose of A .

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$$A^* = \overline{A^T}$$

Definition (Normal Matrix). A matrix A is said to be normal if $AA^* = A^*A$.

This basically means that A and A^* commute.

Theorem 2 (Unitary Diagonalization). *If A is a Normal Matrix then there exists a unitary matrix U and a diagonal matrix D such that $U^*AU = D$.*

And the converse

*If U is a unitary matrix and D is a diagonal matrix such that $U^*AU = D$ then A is a normal matrix.*

Problems

1. Formulate and prove a theorem that asserts that the reduced row echelon form of a given $m \times n$ matrix over a field F is unique. (Do not spend time defining RREF or "row-equivalence of matrices" or "pivot columns." Use such concepts in your proof as appropriate.)

Solution: Suppose M is an $n \times m$ matrix over a field F .

Then the reduced row echelon form of M is unique.

In other words, If R and S are both reduced row echelon forms of M then $R = S$.

Let us denote $R = [r_1 \dots r_m]$ and $S = [s_1 \dots s_m]$. We need to show that $r_i = s_i$ for all i . Note that if r_i is a LC of prior rows then s_i is also a LC of prior rows with same coefficients by the CCP.

Similarly if r_i is a pivot column then s_i is also a pivot column with a one at the same index. We can prove this by induction over the number of columns of M .

Base Case: $r_1 = 0 \leftrightarrow s_1 = 0$ since $\{0\}$ is Linearly Dependant set. Similarly, if $r_1 \neq 0$ then since it is in RREF it must be $[1, 0, 0, \dots]$ and s_1 must also be $[1, 0, 0, \dots]$ as $r_1 \neq 0 \leftrightarrow s_1 \neq 0$.

Inductive Hypothesis: Suppose $r_i = s_i$ for all $i \leq k$.

Inductive Step: We need to show that $r_{k+1} = s_{k+1}$.

We can either have r_{k+1} a linear combination of prior pivot columns or not a linear combination of prior pivot columns.

Suppose there are p prior pivot columns in r_{k+1}

If r_{k+1} is a linear combination of prior pivot columns then s_{k+1} is also a linear combination of prior pivot columns with same coefficients by the CCP since R and S are both Row equivalent.

If r_{k+1} is not a linear combination of prior pivot columns then it must be of form $[0, 0, \dots, 1, 0, \dots]$ where the 1 is in the $p+1$ th index and s_{k+1} must also be of the same form by the CCP.

Therefore $r_{k+1} = s_{k+1}$ and by induction $r_i = s_i$ for all i .

2. In \mathbb{F}^n , every finite linearly independent set has at most n elements, and if it has exactly n elements, then it is a basis of \mathbb{F}^n .

Solution: Suppose $S = \{v_1, v_2, \dots, v_k\}$ is a linearly independent set in \mathbb{F}^n .

Consider $A = [v_1 v_2 \dots v_k]$ which is an $n \times k$ size matrix.

Let $R = \text{RREF}(A)$.

Since A is a linearly independent set, $Rx = 0$ has only the trivial solution.

Thus the nullity of R is 0.

Thus R must be of form $\begin{bmatrix} I_k \\ 0 \end{bmatrix}$. Since there are n rows, $k \leq n$.

3. In \mathbb{F}^n , every spanning set has at least n elements, and if it has exactly n elements, then it is a basis of \mathbb{F}^n .

Solution: Suppose $S = \{v_1, v_2, \dots, v_k\}$ is a spanning set in \mathbb{F}^n .

Suppose $A = [v_1 v_2 \dots v_k]$ is an $n \times k$ matrix.

Let $R = \text{RREF}(A)$.

Then $\text{Col}(A) = F^n$ implies $\text{Col}(R) = F^n$.

Thus $Ax = b$ is consistent and $Rx = c$ is consistent for all $b, c \in F^n$ and there are no zero rows in R .

Thus $k \geq n$.