Workshop 11

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1 Question 1

Suppose $p(t), q(t)$ are continuous functions on (a, b) Consider $x'' + px' + qx = 0$
Given any two differentiable functions $x_1(t), x_2(t)$, define the function $W(t) = x_1(t)x_2'(t) - x_1'(t)x_2(t)$ Called the Wronskian
Assumimg these are two solutions of the DE for t in (a, b) show that $W'(t) = -p(t)W(t)$. Use this to show $W(t) = 0 or W(t) \neq 0$ for all t in (a, b)
Solution:

$$W(t) = x_1(t)x_2'(t) - x_1'(t)x_2(t)$$
$$W'(t) = x_1'x_2' + x_1x_2'' - x_1'x_2' - x_1''x_2$$
$$W'(t) = x_1x_2'' - x_1''x_2$$
$$W'(t) = x_1(-px_2' - qx_2) - x_2(-px_1' - qx_1)$$
$$W'(t) = -px_1x_2' - qx_1x_2 + px_1x_2' + qx_1x_2$$
$$W'(t) = -p(x_1x_2' - x_1'x_2)$$
$$W'(t) = -pW(t)$$

Now, we know that $W'(t) = -pW(t)$ we can "solve" for $W(t)$ by integrating both sides.

$$\int \frac{dW(t)}{W(t)} = \int -p(t)dt$$
$$\ln|W(t)| = -\int p(t)dt + C$$
$$W(t) = e^{-\int p(t)dt}C$$

We see from prior experience that this solution is existant and unique for all time t in (a, b)
Thus, $W(t) = 0$(an equilibrium solution) or $W(t) \neq 0$ for all t in (a, b)

2 Question 2

Consider the DE $x'' + px' + qx = r$ where $p(t), q(t), r(t)$ are continuous functions on (a, b)
Let $x_1(t), x_2(t)$ be two solutions of the DE for t in (a, b) such that $W(t) =\neq 0$ for all t in (a, b)
Let $Y(t) = c_1(t)x_1(t) + c_2(t)x_2(t)$
Assume $c_1'x_1 + c_2'x_2 = 0$. Show $c_1'x_1' + c_2'x_2' = r(t)$
Conclude that we get a formula for $c_1'(t), c_2'(t)$
Solution:

$$Y(t) = c_1(t)x_1(t) + c_2(t)x_2(t)$$
$$Y'(t) = c_1x_1' + c_2x_2'$$
$$Y''(t) = c_1'x_1' + c_1x_1'' + c_2'x_2' + c_2x_2''$$
$$Y''(t) + pY'(t) + qY(t) = r$$
$$c_1'x_1' + c_1x_1'' + c_2'x_2' + c_2x_2'' + pc_1x_1' + pc_2x_2' + qc_1x_1 + qc_2x_2 = r$$
$$c_1'x_1' + c_2'x_2' + c_1(x_1'' + px_1' + qx_1) + c_2(x_2'' + px_2' + qx_2) = r$$
$$c_1'x_1' + c_2'x_2' = r$$

As desired.
Now, we can solve for $c_1'(t), c_2'(t)$ by considering the matrix representaion of the above equations.

$$\begin{bmatrix} x_1 & x_2 \\ x_1' & x_2' \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$
$$\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} x_2' & -x_2 \\ -x_1' & x_1 \end{bmatrix} \begin{bmatrix} 0 \\ r \end{bmatrix} \frac{1}{W(t)}$$
$$\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} -x_2r \\ x_1r \end{bmatrix} \frac{1}{W(t)}$$

As desired.

3 Question 3

Set up the ODE as $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ r \end{bmatrix}$
Notice $\begin{bmatrix} x_1 \\ x_1' \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ x_2' \end{bmatrix}$ are two LI solution of the homogenous part. Do the integral in the Duhamels fomular and extract first component. Compare to the Previous Question.
Solution:

From Duhamels formula in terms of flow we get:

$$X(t) = \Phi_{t,0}X_0 + \int_0^t \Phi_{t,s} r(s) ds$$

$$X(t) = \Phi_{t,0}X_0 + M(t)\int_0^t M(s)^{-1} \begin{bmatrix} 0 \\ r(s) \end{bmatrix} ds$$

$$X(t) = \Phi_{t,0}X_0 + \begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix} \int_0^t \begin{bmatrix} x_2(s) & -x_2(s) \\ -x_1(s) & x_1(s) \end{bmatrix} \begin{bmatrix} 0 \\ r(s) \end{bmatrix} ds$$

$$X(t) = \Phi_{t,0}X_0 + \begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix} \int_0^t \begin{bmatrix} c_1'(s) \\ c_2'(s) \end{bmatrix} ds$$

$$X(t) = \Phi_{t,0}X_0 + \begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix} \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}$$

$$X(t) = \Phi_{t,0}X_0 + \begin{bmatrix} c_1(t)x_1(t) + c_2(t)x_2(t) \\ c_1(t)x_1'(t) + c_2(t)x_2'(t) \end{bmatrix}$$

$$X(t) = \Phi_{t,0}X_0 + Y(t)$$