01:640:481 - Neyman-Pearson Lemma

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1. Recall Exp(λ) population has PDF: $f(x) = \frac{1}{\lambda}e^{-\frac{x}{\lambda}}, x > 0$. The null hypothesis $\lambda = 10$ is to be tested against the alternative $\lambda = 5$ using observed sample data $x_1, x_2, \ldots, x_n$. Use the Neyman-Pearson Lemma to obtain a test with the most power when the size of the critical region is to be a fixed α.

Solution: The likelihood function is given by

$$L(\lambda) = \frac{1}{\lambda^n}e^{-\frac{\sum x_i}{\lambda}}.$$

The likelihood ratio is given by

$$\Lambda = \frac{L_0}{L_1} \leq k \quad \in C.$$

Where C is the critical region and k is a constant determined by the size of the critical region. The Neyman-Pearson Lemma states that the most powerful test is given by the likelihood ratio test.

$$\begin{aligned}
\Lambda = \frac{L(10)}{L(5)} &= \frac{5}{10}^n e^{-\frac{\sum x_i}{10}+\frac{\sum x_i}{5}} \leq k \\
&= \frac{1}{2}^n e^{\frac{\sum x_i}{10}} \leq k \\
ln(\Lambda) &= nln(\frac{1}{2}) + \frac{\sum x_i}{10} \leq ln(k) \\
\sum x_i &\leq 10nln(2) + 10ln(k).
\end{aligned}$$

Therefore the most powerful test rejects the null hypothesis if $\sum x_i < 10nln(2) + 10ln(k)$. and the size of the critical region is α.

2. In the previous question suppose $n = 1$ (sample size of one) and the probability of type 1 error $\alpha = 0.05$. What is the critical region in this case?

Solution: When $n = 1$ and $\alpha = 0.05$, we have

$$\begin{aligned}
P(X < c|\lambda = 10) &= 0.05 \\
P(X < c) = 1 - e^{-\frac{c}{10}} &= 0.05 \\
e^{-\frac{c}{10}} &= 0.95 \\
c = -10ln(0.95) &\approx 0.0513.
\end{aligned}$$

Therefore the critical region is $x < 0.0513$.