Homework 1: Math 481

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Problem 4.8

Find the expected value of the random variable X whose probability density is given by

$$f(x) = \begin{cases} x \text{ for } 0 < x < 1 \\ 2 - x \text{ for } 1 < x < 2 \\ 0 \text{ otherwise} \end{cases}$$

Solution

To find the expected value of the random variable X, we can use the formula for the expected value of a continuous random variable:

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Since $f(x)$ is non-zero only in the interval $(0, 2)$, we can split the integral into two parts:

$$E[X] = \int_0^1 x f(x) dx + \int_1^2 x f(x) dx$$

Substituting the expression for $f(x)$, we have:

$$E[X] = \int_0^1 x^2 dx + \int_1^2 x(2 - x) dx$$

Now we can evaluate each integral separately.

$$E[X] = \frac{1}{3} + \frac{2}{3} = 1$$

Problem 4.19

Find μ, μ_2' and σ^2 for the random variable X with the following probability density function:

$$f(x) = \begin{cases} x/2 \text{ for } 0 < x < 2 \\ 0 \text{ otherwise} \end{cases}$$

Solution

To find the μ which is the 1st momenet about the mean, we can say that it is by definition: 0. This is because it is $E[X - \mu] = E[X] - E[\mu] = \mu - \mu = 0$.
If in this case μ denotes the 1st moment about the origin, we can find it as follows:

$$\mu = E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_0^2 x \cdot \frac{x}{2}dx = \int_0^2 \frac{x^2}{2}dx = \left[\frac{x^3}{6}\right]_0^2 = \frac{8}{6} = \frac{4}{3}$$

Next, we can find μ_2' which is the 2nd moment about the origin:

$$\mu_2' = E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^2 x^2 \cdot \frac{x}{2}dx = \int_0^2 \frac{x^3}{2}dx = \left[\frac{x^4}{8}\right]_0^2 = \frac{16}{8} = 2$$

Finally, we can find the variance σ^2:

$$\sigma^2 = \mu_2' - \mu^2 = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \frac{18}{9} - \frac{16}{9} = \frac{2}{9}$$

Problem 4.23

If the random varibale X has the mean μ and standard deviation σ, show that the random varibale Z who's values are related to those of X by the equation $Z = \frac{X-\mu}{\sigma}$ has $E[Z] = 0$ and $var[Z] = 1$.
A distribution that has the mean 0 and the variance 1 is said to be in standard form, and when we perform the above change of variable, we are said to be standardizing the distribution of X

Solution

To find the expected value of Z, we can use the linearity of expectation:

$$E[Z] = E\left[\frac{X - \mu}{\sigma}\right] = \frac{1}{\sigma}E[X - \mu] = \frac{1}{\sigma}(E[X] - \mu) = \frac{1}{\sigma}(\mu - \mu) = 0$$

Next, we can find the variance of Z:

$$var[Z] = var\left[\frac{X - \mu}{\sigma}\right] = \frac{1}{\sigma^2}var[X] = \frac{1}{\sigma^2}\sigma^2 = 1$$

Thus, we have shown that $E[Z] = 0$ and $var[Z] = 1$.

Problem 4.33

Fin the moment generating function of the continuous random variable X with the following probability density function:

$$f(x) = \begin{cases} 1 \text{ for } 0 < x < 1 \\ 0 \text{ otherwise} \end{cases}$$

and use it to find μ'_1, μ'_2, and σ^2.

Solution

The moment generating function $M_X(t)$ is defined as:

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx = \int_0^1 e^{tx} dx$$

Now we can evaluate the integral:

$$M_X(t) = \left[\frac{e^{tx}}{t} \right]_0^1 = \frac{e^t - 1}{t}$$

Next, we can find the first moment about the origin μ'_1:

$$\mu'_1 = M'_X(0) = \frac{d}{dt} \left(\frac{e^t - 1}{t} \right) \Big|_{t=0} = \frac{(t-1)e^t - 1}{t^2} = \frac{1}{2}$$

Next, we can find the second moment about the origin μ'_2:

$$\mu'_2 = M''_X(0) = \frac{d^2}{dt^2} \left(\frac{e^t - 1}{t} \right) \Big|_{t=0} = \frac{(t^2 - 2t + 2)e^t - 2}{t^3} \Big|_{t=0} = \frac{1}{3}$$

Finally, we can find the variance σ^2:

$$\sigma^2 = \mu'_2 - \mu'^2_1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{12}$$

Problem 4.40

Given the moment generating function $M_X(t) = e^{3t+8t^2}$, find the moment generating function of the random variable $Z = \frac{1}{4}(X - 3)$ an use it to determine the mean and variance of Z.

Solution

To find the moment generating function of Z, we can use the following relationship:

$$M_Z(t) = M_{\frac{1}{4}(X-3)}(t)$$

Using the property of moment generating functions, we have:

$$M_Z(t) = e^{-3t/4} M_X(t/4) = e^{-3t/4} e^{3(t/4)+8(t/4)^2} = e^{t^2/2}$$

Next, we can find the mean and variance of Z using the moment generating function:

$$E[Z] = M'_Z(0) = \frac{d}{dt} e^{t^2/2} \Big|_{t=0} = t e^{t^2/2} \Big|_{t=0} = 0$$

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$$M_Z''(0) = \frac{d^2}{dt^2}e^{t^2/2}\bigg|_{t=0} = (t^2+1)e^{t^2/2}\bigg|_{t=0} = 1$$

Finally, we can find the variance σ^2:

$$\sigma^2 = M_Z''(0) - (M_Z'(0))^2 = 1 - 0^2 = 1$$

Problem 4.54

If $var(X_1) = 5, var(X_2) = 4, var(X_3) = 7, cov(X_1, X_2) = 3, cov(X_1, X_3) = -2$
and X_2, X_3 are independent, find the covariance of $Y_1 = X_1 - 2X_2 + 3X_3$ and
$Y_2 = -2X_1 + 3X_2 + 4X_3$.

Solution

$$
\begin{aligned}
cov(Y_1, Y_2) &= cov(X_1 - 2X_2 + 3X_3, -2X_1 + 3X_2 + 4X_3) \\
&= -2var(X_1) - 6var(X_2) + 12var(X_3) + 7cov(X_1, X_2) - 2cov(X_1, X_3) + cov(X_2, X_3) \\
&= -2(5) - 6(4) + 12(7) + 7(3) - 2(-2) + 0 \\
&= -10 - 24 + 84 + 21 + 4 + 0 \\
&= 75
\end{aligned}
$$

Problem 8.1

Cov of sample mean with sample mean minus a random var is 0. Hint: Use
covariance properties and connection to variance. Statement use corollary of
theorem 4.15 on page 136 to show that if $X_1, X_2, \ldots, X_n$ constitute a random
sample from an infinite population, then $cov(X_r - \bar{X}, \bar{X}) = 0$ for all i.

Solution

The covariance of $X_r - \bar{X}$ and $\bar{X}$ can be expressed as:

$$cov(X_r - \bar{X}, \bar{X}) = -var(\bar{X}) + cov(X_r, \bar{X})$$

Since $\bar{X}$ is the sample mean, we have:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Thus, we can express the covariance as:

$$cov(X_r, \bar{X}) = cov\left(X_r, \frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} cov(X_r, X_i)$$

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Using corollary 4 we can see that:

$$Cov(X_r, 1/n \sum X_i) - Var(\bar{X}) = \frac{1}{n} cov(X_r, X_r) - Var(\bar{X})$$
$$= \frac{1}{n} Var(X_r) - Var(\bar{X})$$

Assuming IID, we can see that:

$$Var(\bar{X}) = cov(1/n \sum X_i, 1/n \sum X_i)$$
$$= \frac{1}{n^2} \sum cov(X_r, X_r) \qquad = \frac{1}{n^2} nVar(X) = \frac{Var(X)}{n}$$

Thus we can see that:
$$var(X_r)/n - var(\bar{X}) = 0$$

Thus we have shown that $cov(X_r - \bar{X}, \bar{X}) = 0$.

Problem 8.2

Independent samples from two different populations. a) is straightforward. Why? b) Use covariance connection and then properties of covariance and independence.

Use Theorem 4.14 on page 135 and its corollary to show that if $X_{11}, X_{12}, \ldots, X_{1n_1}$ and $X_{21}, X_{22}, \ldots, X_{2n_2}$ are independent random variables, with the first n_1 constituting a random sample from and infinite population with the mean μ_1 and variance σ_1^2, and the second n_2 constituting a random sample from an infinite population with mean μ_2 and variance σ_2^2, Then show that

a) $E[\bar{X}_1 - \bar{X}_2] = \mu_1 - \mu_2$

b) $var(\bar{X}_1 - \bar{X}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$

Solution

a

By the linearity of expectation, and the fact that we have:

$$E[\bar{X}_1 - \bar{X}_2] = E[\bar{X}_1] - E[\bar{X}_2]$$
$$= E[\sum x_{1i}/n_1] - E[\sum x_{2i}/n_2]$$
$$= \frac{1}{n_1} \sum E[x_{1i}] - \frac{1}{n_2} \sum E[x_{2i}]$$
$$= \frac{1}{n_1} n_1 \mu_1 - \frac{1}{n_2} n_2 \mu_2$$
$$= \mu_1 - \mu_2$$

b

To find the variance of the difference of the two sample means, we can use the properties of variance and covariance. Since the two samples are independent, we have:

$$var(\bar{X}_1 - \bar{X}_2) = var(\bar{X}_1) + var(\bar{X}_2)$$
$$= var\frac{(\sum X_1)}{n_1} + var\frac{(\sum X_2)}{n_2}$$
$$= \frac{\sum var(X_1)}{n_1^2} + \frac{\sum var(X_2)}{n_2^2}$$
$$= n_1\frac{\sigma_1^2}{n_1^2} + n_2\frac{\sigma_2^2}{n_2^2}$$
$$= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

Problem 8.3

With reference to Exercise 2, show that if the two samples come from normal populations, then $\bar{X}_1 - \bar{X}_2$ is a random variable having a normal distribution with mean $\mu_1 - \mu_2$ and variance $\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$.

Solution

We can consider the MGF of these two independent random variables. The MGF of the difference of two independent normal random variables is given by:

$$M_{\bar{X}_1 - \bar{X}_2}(t) = E[e^{t(\bar{X}_1 - \bar{X}_2)}]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t(\bar{X}_1 - \bar{X}_2)} f_{\bar{X}_1}(x_1)f_{\bar{X}_2}(x_2)dx_1dx_2$$
$$= \int_{-\infty}^{\infty} e^{t\bar{X}_1} f_{\bar{X}_1}(x_1)dx_1 \cdot \int_{-\infty}^{\infty} e^{-t\bar{X}_2} f_{\bar{X}_2}(x_2)dx_2$$
$$= M_{\bar{X}_1}(t) \cdot M_{\bar{X}_2}(-t)$$

Since we know that both $\bar{X}_1$ and $\bar{X}_2$ are normally distributed, we can substitute in their MGFs:

$$M_{\bar{X}_1 - \bar{X}_2}(t) = e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} \cdot e^{-\mu_2 t + \frac{\sigma_2^2 t^2}{2}}$$
$$= e^{(\mu_1 - \mu_2)t + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}}$$

This clearly is also the MGF of a normal distribution with mean $\mu_1 - \mu_2$ and variance $\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$.