01:640:423 - Chapter 5

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Introducing Fourier Series

$f(x)$ is a 2π periodic function. ie $f(x) = f(2\pi + x)$
Goal: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(nx) + b_n sin(nx)$
Some underlying assumptions:
$f(x)$ is integrable on a finite interval. eg bounded; continous on R except for finitly many points in each bounded interval.
Also consider complex form:

$$\begin{aligned} cos(nx) &= \frac{e^{inx} + e^{-inx}}{2} \\ sin(nx) &= \frac{e^{inx} - e^{-inx}}{2i} \\ e^{inx} &= cos(nx) + isin(nx) \end{aligned}$$

$$\begin{aligned} S_N(x) &= \frac{a_0}{2} + \sum_{n=1}^{N} a_n cos(nx) + b_n sin(nx) \\ &= \frac{a_0}{2} + \sum_{n=1}^{N} \left(\frac{a_n}{2} + \frac{b_n}{2i}\right) e^{inx} + \left(\frac{a_n}{2} - \frac{b_n}{2i}\right) e^{-inx} \end{aligned}$$

Rename: $c_0 = \frac{a_0}{2}, c_n = \frac{a_n}{2} + \frac{b_n}{2i}, c_{-n} = \frac{a_n}{2} - \frac{b_n}{2i}$

$$\begin{aligned} S_N(x) &= c_0 + \sum_{n=1}^{N} c_n e^{inx} + \sum_{n=1}^{N} c_{-n} e^{-inx} \\ &= c_0 + \sum_{n=1}^{N} c_n e^{inx} + \sum_{n=N}^{1} c_n e^{inx} \\ &= \sum_{n=-N}^{N} c_n e^{inx} \end{aligned}$$

$c_n = \frac{a_n - ib_n}{2}$
$c_{-n} = \frac{a_n + ib_n}{2}$
$c_0 = \frac{a_0}{2}$

$$a_n = c_n + c_{-n}$$
$$b_n = i(c_n - c_{-n})$$

Assume $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$. How to find c_n?
Recall $< f, g >_{L^2(-\pi,\pi)} = \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$
Lemma for orthogonality: $\{e^{inx}\}_{n=-\infty}^{\infty}$ is orthogonal with respect to $< \cdot, \cdot >_{L^2(-\pi,\pi)}$

$$< e^{inx}, e^{imx} >_{L^2(-\pi,\pi)} = \int_{-\pi}^{\pi} e^{inx}\overline{e^{imx}}dx = 2\pi\delta_{nm}$$

So if we consider $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$, then

$$< f, e^{imx} >_{L^2(-\pi,\pi)} = \sum_{-\infty}^{\infty} c_n < e^{inx}, e^{imx} >_{L^2(-\pi,\pi)} = 2\pi c_m$$

$$c_m = \frac{1}{2\pi} < f, e^{imx} >_{L^2(-\pi,\pi)}$$

More explicitly,

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}dx$$

Now we can solve for a_n and b_n using c_n

$$\begin{aligned}
a_n &= c_n + c_{-n} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(e^{-inx} + e^{inx})dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx
\end{aligned}$$

Similarly,

$$\begin{aligned}
b_n &= i(c_n - c_{-n}) \\
&= i\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx\right) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(e^{-inx} - e^{inx})dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx)dx
\end{aligned}$$

Note that the interval of integration is $(-\pi, \pi)$ because $f(x)$ is 2π periodic.

Lemma 1. *Let $F(x)$ be a 2π periodic function. Then $\int_a^{a+2\pi} F(x)dx$ doesnt depend on a.*

Proof.

$$I(a) = \int_0^{a+2\pi} F(x)dx - \int_0^a F(x)dx$$
$$I'(a) = F(a+2\pi) - F(a) = 0$$

□

Remark. $a_0 =$ cos stuff with $n = 0$
That is why we have $\frac{a_0}{2}$

Remark. c_0 is the average of the function on the interval

Definition. $f(x)$ 2π periodic function and integrable on $(-\pi, \pi)$. then the Fourier series of $f(x)$ is

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$
or

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(nx) + b_n sin(nx)$$

where $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) cos(nx) dx$ and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) sin(nx) dx$

Remark. $\sim$ means correspondence as we dont know if the FS converges to $f(x)$ if at all.

Observations: if $f(x)$ is even, then $b_n = 0$
if $f(x)$ is odd, then $a_n = 0$

Remark. $a_n, b_n \to 0$ as $n \to \infty$
and $c_n \to 0$ as $n \to \pm\infty$
this is due to osculations and cancellations

Proof. Assume f is differentiable.

$$\pi a_n = \int_{-\pi}^{\pi} f(x) \frac{sin(nx)'}{n} dx$$
$$= \frac{1}{n} \left[f(x) sin(nx) \right]_{-\pi}^{\pi} - \frac{1}{n} \int_{-\pi}^{\pi} f'(x) sin(nx) dx$$

The first item is 0 because $f(x)$ is 2π periodic.

$$|a_n| = \frac{1}{\pi n}\int_{-\pi}^{\pi} |f'(x)|dx \to 0$$

□

Example. $f(x) = x$ on $(-\pi, \pi]$
Extend it to R periodically.
The function $f(x)$ is odd. So $a_n = 0$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x sin(nx)dx \\
&= \frac{1}{\pi}\left[-\frac{x cos(nx)}{n}\right]_{-\pi}^{\pi} + \frac{1}{\pi n}\int_{-\pi}^{\pi} cos(nx)dx \\
&= \frac{1}{\pi}\left[-\frac{\pi sin(n\pi)}{n} + \frac{\pi sin(n\pi)}{n}\right] \\
&= \frac{2}{n}(-1)^{n+1}
\end{aligned}$$

Thus the Fourier series of $f(x)$ is

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} sin(nx)$$

$$f(x) = 2(sin(x) - \frac{sin(2x)}{2} + \frac{sin(3x)}{3} - \frac{sin(4x)}{4} + \ldots)$$

Note. No convergence test from calc 2 applies to this

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n+1} sin(nx)$$

$$S_N(x) = \sum_{n=1}^{N} \frac{1}{n}(-1)^{n+1} sin(nx)$$
$$f(x) = \lim_{N\to\infty} S_N(x)$$

Example.

$$f(x) = |x| \text{ on } (-\pi, \pi]$$
$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2 cos(nx)}$$

Remark. Limit and convergence are not easy as we see in ex 1. but convergence can be easy, but limit is not easy

Remark. Decay of fouir coefficients: Ex 1: $\frac{1}{n}$, Ex 2: $\frac{1}{n^2}$ This gives faster decay for ex 2 over ex 1
Fast decay of coefficients $\implies$ faster convergence $\implies$ better approximation with less terms

1 5.3 and 5.4

Definition. f is piecwise continous on $[a, b]$ if it is continous on $[a, b]$ except for finitely many points where it has finite jumps ie $p_1, p_2, \dots, p_n$
and $f(p_i^{\pm})$ exists for all i

Remark. if a or b is one of the excepitonal points we only require existence of $f(a^+)$ or $f(b^-)$

Definition. $f \in p.wC^1[a, b]$ if f and f' are piecewise continous on $[a, b]$
Whats allowed? Finitly many jumps (discontinotinouties of f) and finitely many corners or cusps (discontinuities of f')

Definition. $f \in p.wC(R)$ if f is piecewise continous on (a, b) for any $a, b \in R$

Theorem 1. *If f is 2π periodic and $\in p.w.C^1(R)$ then*

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(nx) + b_n sin(nx) \qquad = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \frac{f(x^-) + f(x^+)}{2}$$

where $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}dx$ $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)cos(nx)dx$ $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)sin(nx)dx$

Remark. if f is continous at x then $f(x^-) = f(x^+) = f(x)$ and the sum of the Fourier series at x is $f(x)$

$$S_n(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n cos(nx) + b_n sin(nx)$$
$$= \sum_{n=-N}^{N} c_n e^{inx}$$

Goal: $S_n(x) \to \frac{f(x^-)+f(x^+)}{2}$ as $N \to \infty$
Take x fixed

$$\begin{aligned} S_N(x) &= \sum_{n=-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} dy \cdot e^{inx} \\ &= \int_{-\pi}^{\pi} f(y) \left(\sum_{n=-N}^{N} \frac{1}{2\pi} e^{in(x-y)} \right) dy \end{aligned}$$

The item in the parenthesis is the Dirichlet kernel $D_N(x-y)$
$D_N(z) = \sum_{n=-N}^{N} \frac{1}{2\pi} e^{inz}$

$$S_N(x) = \int_{-\pi}^{\pi} f(y) D_N(x-y) dy$$

Note that $D_N(z) = D_N(-z)$
Change of variables: $z = x - y$

$$S_N(x) = \int_{-\pi-x}^{\pi-x} f(x+z) D_N(z) dz = \int_{-\pi}^{\pi} f(x+z) D_N(z) dz$$

Lemma 2.

$$D_N(z) = \frac{sin((N+\frac{1}{2})z)}{sin(z/2)}$$

Proof.

$$\begin{aligned} 2\pi D_N(z) &= e^{-inz} \sum_{n=1}^{2N} e^{inz} \\ &= e^{-inz} \frac{e^{i(2N+1)z} - 1}{e^{iz} - 1} \\ &= \frac{e^{i(N+1)z} - e^{-i(N)z}}{e^{iz} - 1} \cdot \frac{e^{-iz/2}}{e^{-iz/2}} \\ &= \frac{e^{i(N+1/2)z} - e^{i(N+1/2)z}}{e^{iz/2} - e^{-iz/2}} \end{aligned}$$

Note that $sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$
Note $2\pi D_N(0) = 2N + 1$
$2\pi D_N(\pm\pi) = (-1)^N$

$\square$

pick up $f(x+z)$ at $z = 0$ like dirac delta function

Theorem 2. *f is 2π periodic and $\in p.w.C^1(R)$ then $\lim_{N\to\infty} S_N(x) = \frac{f(x^-)+f(x^+)}{2}$ for all x*

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{inx} = \frac{a_0}{2} + \sum_{n=1}^{N} a_n cos(nx) + b_n sin(nx)$$

where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}dy$

We can also write $S_N(x)$ as

$$S_N(x) = \int_{-\pi}^{\pi} f(x+z)D_N(z)dz$$

If f is not continous at x then

$$S_N(x) = \int_{-\pi}^{0} f(x+z)D_N(z)dz + \int_{0}^{\pi} f(x+z)D_N(z)dz$$

$$\frac{f(x^-)+f(x^+)}{2} = \int_{-\pi}^{0} f(x^-)D_N(z)dz + \int_{0}^{\pi} f(x^+)D_N(z)dz = \frac{1}{2}f(x^-) + \frac{1}{2}f(x^+)$$

$$S_N(x) - \frac{f(x^-)+f(x^+)}{2} = \int_{-\pi}^{0}(f(x+z) - f(x^-))D_N(z)dz + \int_{0}^{\pi}(f(x+z) - f(x^+))D_N(z)dz$$
$$= \to 0 \text{ as } N \to \infty$$

Corollary. f, g are 2π periodic and $\in p.w.C^1(R)$.
If f, g have the same Fourier coefficients then $\frac{f(x^-)+f(x^+)}{2} = \frac{g(x^-)+g(x^+)}{2}$ for all x
In particular $f(x) = g(x)$ for all x in which f and g are continous

Functions on $[\pi, \pi]$ f is piecewise continous on $[-\pi, \pi]$; extend f to R periodically
Use f on $(-\pi, \pi]$ to contruct $\tilde{f}$ on R
now $\tilde{f}$ is 2π periodic and $\in p.w.C^1(R)$
Now we can see that $c_n = \frac{\tilde{f}(x^-+)\tilde{f}(x^+)}{2}$
Clealry $f(x^-) = \tilde{f}(x^-)$ and with more work (noticing we can go to the next period) we can show $f(-x^+) = \tilde{f}(x^+)$

Functions on $[0, \pi]$
f is piecewise continous on $[0, \pi]$; extend f to R periodically
We have a two stage extension process: f on $[0, \pi]$ to f on $[-\pi, \pi]$ to $\tilde{f}$ on R
When we do our first extensin we can do even or odd extensions:

$$f_{even}(x) \begin{cases} f(x) \text{ for } x \in [0, \pi] \\ f(-x) \text{ for } x \in [-\pi, 0) \end{cases}$$

$$f_{odd}(x)\begin{cases} f(x) \text{ for } x \in (0,\pi] \\ 0 \text{ for } x = 0 \\ -f(-x) \text{ for } x \in [-\pi, 0) \end{cases}$$

For f_{even} we have the Fourier series be

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(nx)$$

$$= \frac{f_{even}(x^-) + f_{even}(x^+)}{2}$$

$$\begin{cases} f(0^+) at x = 0 \\ f(\pi^-) at x = \pi \end{cases}$$

For f_{odd} we have the Fourier series be

$$\sum_{n=1}^{\infty} b_n sin(nx)$$

$$= \frac{f_{odd}(x^-) + f_{odd}(x^+)}{2}$$

$$\begin{cases} 0 at x = 0 \\ 0 at x = \pi \end{cases}$$

Example. $f(x) = x$ on $[0, \pi]$

$f_{even}(x) = \begin{cases} x \text{ for } x \in [0,\pi] \\ -x \text{ for } x \in [-\pi, 0) \end{cases}$

$f_{odd}(x) = \begin{cases} x \text{ for } x \in (0,\pi] \\ 0 \text{ for } x = 0 \\ x \text{ for } x \in [-\pi, 0) \end{cases}$

We know $x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\in odd} \frac{1}{n^2} cos(nx)$ and $x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{sin(nx)}{n}$

Functions of $[-l, l]$

f is piecewise continous on $[-l, l]$; extend f to R periodically

$$g(x) = f(lx/\pi) x \in [-\pi, \pi]$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(lx/\pi)e^{-inx}dx = \frac{1}{2l}\int_{-l}^{l} f(y)e^{-iny\pi/l}dy$$

2 L^2 Theory for Fourier series

$L^2 := L^2(-\pi, \pi) = \{f : [-\pi, \pi] \to C : \int_{-\pi}^{\pi} f(x)^2 < \infty\}$
Ex: any continous f is in L^2

$$||f||^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)^2 dx$$

$$< f, g >= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)} dx$$

$$< f, f >= ||f||^2$$

Example.

$$e^{inx} \in L^2$$

$$< e^{inx}, e^{imx} >= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx}\overline{e^{imx}} dx = \delta_{nm}$$

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} dx$$

$$\phi_n(x) = e^{inx}$$

$$c_n =< f, \phi_n >$$

$$f(x) = \sum_{n=-\infty}^{\infty} < f, \phi_n > \phi_n(x)$$

3 L2 theory for Fourier series

$$f : [-\pi, \pi] \to C$$

$$||f||^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

$$L^2 = \{f : [-\pi, \pi] \to C : ||f|| < \infty\}$$

$$< f, g >= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)} dx$$

We can prove this be $a * b \leq |a|^2/2 + |b|^2/2$ since $0 \leq (a - b)^2$

Proof. $|f \cdot g| = |f||g| \leq \frac{|f|^2}{2} + \frac{|g|^2}{2}$
Integreate by x

$$\int_{-\pi}^{\pi} |f(x)g(x)|dx \leq \int_{-\pi}^{\pi} \frac{|f(x)|^2}{2}dx + \int_{-\pi}^{\pi} \frac{|g(x)|^2}{2}dx$$

We know the RHS is finite so the LHS is finite

□

We can also do this by Cauchy Schwarz inequality

Theorem 3.

$$|| < f, g > || \leq ||f|| \cdot ||g||$$

Proof. Consider $f - tg$ where t is a parameter

$$0 \leq ||f - tg||^2 =< f - tg, f - tg >= ||f||^2 - 2t < f, g > +t^2||g||^2$$

Properties

- $< f, g_1 + g_2 >=< f, g_1 > + < f, g_2 >$
- $< f_1 + f_2, g >=< f_1, g > + < f_2, g >$
- $< af, g >= a < f, g >$
- $< f, ag >= \overline{a} < f, g >$

$$||f - tg||^2 = ||f||^2 - t < g, f > -\overline{t < g, f >} + |t|^2||g||^2$$

$-t < g, f > -\overline{t < g, f >} = 2Re[t < g, f >]||f - tg||^2$ $= ||f||^2 - 2Re[t < g, f >] + |t|^2||g||^2$

Let $t \geq 0$

$$0 \leq ||f||^2 - 2tRe(< g, f >) + t^2||g||^2$$

minimize in t for a critical point:

$$0 = -2Re(< g, f >) + 2t||g||^2$$

$$t = \frac{Re(< g, f >)}{||g||^2}$$

Now we have

$$0 \leq ||f||^2 - 2Re[< g, f >]\frac{Re(< g, f >)}{||g||^2} + \frac{| < g, f > |^2}{||g||^2}||g||^2$$

$$\leq ||f||^2 - \frac{| < g, f > |^2}{||g||^2}(Re(< g, f >))^2 \leq ||f||^2||g||^2|Re(< g, f >)| \leq ||f||||g||$$

Thus $t = \frac{\overline{<g,f>}}{||g||^2}$

□

$\phi_n(x) = e^{inx}$

$$< \phi_n, \phi_m >= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} \overline{e^{imx}} dx = \delta_{nm}$$

Thus $\{\phi_n\}$ is an orthonormal set in L^2
Which means that it is orthogonal and $||\phi_n|| = 1$
Fourier series $f \sim \sum_{n=-\infty}^{\infty} < f, \phi_n > \phi_n$

Lemma 3. *Fourier sum as best approximation in L^2*
$f \in L^2$, N is a fixed integer.
***Goal** approximate f with $S_N(x) = \sum_{n=-N}^{N} a_n e^{inx}$ in the L^2 sense*
***Claim** $||f - \sum_{n=-N}^{N} a_n \phi_n||$ is the minimized for $a_n =< f, \phi_n >$*

Proof.

$$\begin{aligned} ||f - \sum_{n=-N}^{N} a_n\phi_n||^2 &= ||f||^2 - 2Re[< f, \sum_{n=-N}^{N} a_n\phi_n >] + || \sum_{n=-N}^{N} a_n\phi_n||^2 \\ &= ||f||^2 - 2 \sum_{n=-N}^{N} Re[\overline{a_n}, < f, \phi_n >] + \sum_{n=-N}^{N} |a_n|^2||\phi_n||^2 \end{aligned}$$

Also consider

$$< \sum_n a_n\phi_n, \sum_m a_m\phi_m >= \sum_n \sum_m a_n\overline{a_m} < \phi_n, \phi_m >= \sum_n |a_n|^2||\phi_n||^2$$

Aka pythogeran theorem: $|u + v|^2 = |u|^2 + |v|^2$ if u, v orthogonal

$$\begin{aligned} ||f - \sum_{n=-N}^{N} a_n\phi_n||^2 &= ||f||^2 - 2 \sum_{n=-N}^{N} Re[\overline{a_n}c_n] + \sum_{n=-N}^{N} |a_n|^2 \\ &= ||f||^2 + \sum_{n=-N}^{N} |a_n|^2 - 2 \sum_{n=-N}^{N} Re[\overline{a_n}c_n] + c_n^2 - c_n^2 \\ &= ||f||^2 + \sum_{n=-N}^{N} |a_n - c_n|^2 - \sum_{n=-N}^{N} |c_n|^2 \end{aligned}$$

We can see that this is minimize if $a_n = c_n$ for all n □

Definition. distance between f, g is

$$d(f, g) = ||f - g||$$

Definition. $f_n \to f$ in L^2 if $d(f_n, f) \to 0$ as $n \to \infty$

$$||f_n - f|| \to 0$$

This is called Mean Square Convergence

Example. L^2 convergence is different from pointwise convergence
$f_n(x) = \begin{cases} n^p \text{ for } x \in [0, 1/n] \\ 0 \text{ for } x \in (1/n, 1] \end{cases}$, $p > 0$

$$||f_n - 0||^2 = \int_0^1 f_n(x)^2 dx = n^{2p-1} \to 0 \text{ if } 2p - 1 < 0 \implies p < \frac{1}{2}$$

$$f_n(x) \not\to 0 \text{ pointwise}$$

Metric space X with a notion of distance $d(x, y) \forall x, y \in X$

Example.

$$\mathbb{R} with d(x, y) = |x - y|$$
$$L^2 with d(f, g) = ||f - g||$$

Definition. Hueristic definition:
X is complete if it has no holes

Example. $\mathbb{Q}$ is not complete because $\sqrt{2} \notin \mathbb{Q}$

$\mathbb{R} = \overline{\mathbb{Q}}$ completenes of $\mathbb{Q}$ fill in the holes

Definition. $\{x_n\}$ is a Cauchy if its terms get arbitrarily close to each other as $n \to \infty$

$$\forall \epsilon > 0, \exists N > 0, \text{ such that } d(x_n, x_m) < \epsilon, \forall n, m > N$$

Definition. (X, d) is called complete if any Cauchy sequence in X converges to a point in X

$$L^2 \leftrightarrow \textit{Lebesguq space}$$

L^2-integral is lebesgue integral, it generalize the reiman integral and improper integral

Theorem 4. *L^2 is complete*

Theorem 5. *Bessel inequality*
$f \in L^2$ and $c_n = < f, \phi_n >$ then

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq ||f||^2$$

Proof.

$$S_N(x) = \sum_{n=-N}^{N} c_n \phi_n(x)$$

$$\begin{aligned} ||f - S_N||^2 &= ||f||^2 - 2\sum Re(< f, c_n\phi_n >) + \sum |c_n|^2 \\ &= ||f||^2 - 2\sum |c_n|^2 + \sum |c_n|^2 \\ &= ||f||^2 - \sum |c_n|^2 \sum |c_n^2| \leq ||f||^2 \end{aligned}$$

□

Corollary. Let $f \in L^2$ then $c_n =< f, \phi_n >= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}$
$c_n \to 0$ as $n \to \pm\infty$

Proof.

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq ||f||^2 < \infty$$

$$|c_n|^2 \to 0$$

□

Theorem 6. *$f \in L^2$ and $c_n =< f, \phi_n >$ and $S_N(x) = \sum_{n=-N}^{N} c_n\phi_n(x)$*
Then $\{S_N\}$ converg in L^2.
ie there exist $s \in L^2$ such that $S_N \to s$ in L^2 as $N \to \infty$

Quick Review

- $L^2 = \{f : [-\pi, \pi] \to C : ||f|| < \infty\}$
- $||f||^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$
- $< f, g >= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$
- $\phi_n(x) = e^{inx}$
- $< \phi_n, \phi_m >= \delta_{nm}$
- $f(x) = \sum_{n=-\infty}^{\infty} < f, \phi_n > \phi_n(x)$
- $c_n =< f, \phi_n >$

Theorem 7. *$f \in L^2$ and $c_n =< f, \phi_n >$ and $S_N(x) = \sum_{n=-N}^{N} c_n\phi_n(x)$*
Then $\{S_N\}$ converg in L^2.
ie there exist $s \in L^2$ such that $S_N - s \to 0$ in L^2 as $N \to \infty$

Proof. We will show that $\{S_N\}$ is a Cauchy sequence in L^2
We mean closeness with respect to the L^2 norm/distance
Additionally since L^2 is complete, we know that the sequence converges to a point in L^2
Cauchy: $||S_N - S_M|| \to 0$ as $N, M \to \infty$

$$S_N - S_M = \sum_{n=-N}^{N} c_n\phi_n - \sum_{n=-M}^{M} c_n\phi_n = \sum_{n=-M-1}^{-N} c_n\phi_n + \sum_{n=M+1}^{N} c_n\phi_n$$

$$||S_N - S_M||^2 =< S_N - S_M, S_N - S_M >= \sum_{-M-1}^{-N} |c_n|^2 + \sum_{M+1}^{N} |c_n|^2$$

because these are tails of a convergent series, they go to 0 as $N, M \to \infty$
We know thus by Bessel's inequality: $\sum_{n=-\infty}^{\infty} |c_n|^2 \leq ||f||^2$

$$\sum_{0}^{N} - \sum_{0}^{M} = \sum_{M+1}^{N} |c_n|^2 \to c - c = 0$$

□

Remark. Notation:

$$S_N(x) = \sum_{n=-N}^{N} c_n\phi_n(x) \xrightarrow[N\to\infty]{L^2} s(x)$$

$$s = \sum_{n=-\infty, L^2}^{\infty} c_n\phi_n$$

We can call this L^2 convergence or sum

Theorem 8. *$f \in L^2$ and $c_n =< f, \phi_n >$ then*

$$f = \sum_{n=-\infty, L^2}^{\infty} c_n\phi_n$$

$$\int_{-\pi}^{\pi} |f(x) - \sum_{n=-N}^{N} c_n\phi_n(x)|^2 dx \xrightarrow{N\to\infty} 0$$

Fourier series converges to f "on average" in the L^2 sense

Remark. 1. Assume f is very nice and prove the theorem for such f
This relies on a notion of uniform convergence
uniform convergence $\implies$ pointwise convergence & L^2 convergence
$f_n \to f$ uniformly on I if $max_{x\in I}|f_n(x) - f(x)| \to 0$ as $n \to \infty$
We can see that $||f_n - f||^2 = \int_I |f_n(x) - f(x)|^2 dx \leq \int_I max_{x\in I}|f_n(x) - f(x)|^2 dx \xrightarrow{N\to\infty} 0$

2 For general $f \in L^2$ we can approximate f a sequene of $f_n \in L^2$ that are very nice
Then we can see that 1 applies to f_n

Thus $\{e^{inx}\}$ is a basis for L^2
In otherwords it is a complete system
$L^2 \infty$-dimensional space, all $\{e^{inx}\}$ are linearly independent

Remark. A hermitian matrix is a matrix that is equal to its conjugate transpose
If A is hermitian then A has a complex orthogonal basis of eigenvectors
$A = \frac{d^2}{dx^2}$ with BC periodicities of u and u'. ie $\begin{cases} u'' = \lambda u \\ u(-\pi) = u(\pi) \\ u'(-\pi) = u'(\pi) \end{cases}$ The generalization is the sturm lousiville problem