Workshop 8: 292H

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Question 6

Harmonic Oscillator W/ Friction: $mx'' = -kx - ax'(t)$

a

$y = x'$, $y' = x''$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $g(t) = \begin{bmatrix} 0 \\ \frac{f(t)}{m} \end{bmatrix}$

$$X'(t) = BX(t)$$

$$B = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{a}{m} \end{bmatrix}$$

b

Compute e^{tB} for the matrix $B = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{a}{m} \end{bmatrix}$ For sake of ease I will take $m = 1/2$.
This the equation will be $x'' = -2kx - 2ay'$
The matrix $B = \begin{bmatrix} 0 & 1 \\ -2k & -2a \end{bmatrix}$

Case 1: $a^2 > 2k$

The characteristic equation is $\mu^2 + 2a\mu + 2k = 0$
The roots are $\mu_1 = -a + \sqrt{a^2 - 2k}$ and $\mu_2 = -a - \sqrt{a^2 - 2k}$
The eigenvectors are $v_1 = \begin{bmatrix} 1 \\ \mu_1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ \mu_2 \end{bmatrix}$

$$e^{tB} = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix} \begin{bmatrix} e^{\mu_1 t} & 0 \\ 0 & e^{\mu_2 t} \end{bmatrix} \begin{bmatrix} \mu_2 & -1 \\ -\mu_1 & 1 \end{bmatrix} * \frac{1}{\mu_2 - \mu_1}$$

$$\frac{1}{\mu_2 - \mu_1} \begin{bmatrix} \mu_1 e^{\mu_2 t} + \mu_2 e^{\mu_1 t} & e^{\mu_2 t} - e^{\mu_1 t} \\ \mu_1\mu_2 e^{\mu_1 t} + \mu_1\mu_2 e^{\mu_2 t} & \mu_2 e^{\mu_2 t} - \mu_1 e^{\mu_1 t} \end{bmatrix}$$

Case 2: $a^2 = 2k$

Here there will be only one root of the CP and thus only one eigenvector.
The eigenvalue is $\mu = -a$ and the eigenvector is $v = \begin{bmatrix} 1 \\ -a \end{bmatrix}$
We can get a generalized eigenvector by solving $(B - \mu I)w = v$

$$\begin{bmatrix} a & 1 & 1 \\ -2k & -a & -a \end{bmatrix}, \begin{bmatrix} a & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus our generalized eigenvector is $w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$e^{tB} = \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix} \begin{bmatrix} e^{-at} & 1 \\ 0 & e^{-at} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$$

Case 3: $a^2 < 2k$

In this case the roots are complex.
The roots are $\mu_1 = -a + i\sqrt{2k - a^2}$ and $\mu_2 = -a - i\sqrt{2k - a^2}$
The eigenvectors are $v_1 = \begin{bmatrix} 1 \\ -\mu_1 \end{bmatrix}$ We can split this exponetial into real and imaginary parts then multiply by the eigenvalue to get 2 distinct LI solutions

$$e^{-at}(cos(\sqrt{2k - a^2}t) + isin(\sqrt{2k - a^2}t)) \begin{bmatrix} 1 \\ a - i\sqrt{2k - a^2} \end{bmatrix}$$

$$e^{-at} \begin{bmatrix} cos(\sqrt{2k - a^2}t) \\ acos(\sqrt{2k - a^2}t) + sin(\sqrt{2k - a^2}t)\sqrt{2k - a^2} \end{bmatrix} + e^{-at} \begin{bmatrix} sin(\sqrt{2k - a^2}t) \\ asin(\sqrt{2k - a^2}t) - cos(\sqrt{2k - a^2}t)\sqrt{2k - a^2} \end{bmatrix}$$

Thus we have our Matrix Exponential.

$$M(t) = \begin{bmatrix} cos(\sqrt{2k - a^2}t) & sin(\sqrt{2k - a^2}t) \\ acos(\sqrt{2k - a^2}t) + sin(\sqrt{2k - a^2}t)\sqrt{2k - a^2} & asin(\sqrt{2k - a^2}t) - cos(\sqrt{2k - a^2}t)\sqrt{2k - a^2} \end{bmatrix}$$

$$M(0) = \begin{bmatrix} 1 & 0 \\ a & \sqrt{2k - a^2} \end{bmatrix}$$

$$M(0)^{-1} = \frac{1}{\sqrt{2k - a^2}} \begin{bmatrix} \sqrt{2k - a^2} & 0 \\ -a & 1 \end{bmatrix}$$

$$e^{tB} = M(t)M(0)^{-1}$$

$$e^{tB} = \frac{1}{\sqrt{2k - a^2}} \begin{bmatrix} cos(\sqrt{2k - a^2}t) & sin(\sqrt{2k - a^2}t) \\ acos(\sqrt{2k - a^2}t) + sin(\sqrt{2k - a^2}t)\sqrt{2k - a^2} & asin(\sqrt{2k - a^2}t) - cos(\sqrt{2k - a^2}t)\sqrt{2k - a^2} \end{bmatrix}$$

$$* \begin{bmatrix} \sqrt{2k - a^2} & 0 \\ -a & 1 \end{bmatrix}$$

(multiply the two matrices)

Question 7

Harmonic Oscillator w/ other stuff: $mx'' = -kx - ax' + f(t)$

a

Find integral forumlas for the three prior cases.
Duhamel's formula is $x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}f(s)ds$

Case 1: $(a/m)^2 > 4k/m$

Using the same idea as before we can find the CP:

$$\mu^2 + \frac{a}{m}\mu + \frac{k}{m}$$

The eigenvalues: $\mu_1 = (-\frac{a}{m} + \sqrt{\frac{a}{m}^2 - 4\frac{k}{m}})/2, \mu_2 = (-\frac{a}{m} - \sqrt{\frac{a}{m}^2 - 4\frac{k}{m}})/2$ The eigenvectors: $v_1 = \begin{bmatrix} 1 \\ -\mu_1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ -\mu_2 \end{bmatrix}$
Following the same logic as prior we can find the matrix exponential.

$$e^{tA} = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix} \begin{bmatrix} e^{\mu_1 t} & 0 \\ 0 & e^{\mu_2 t} \end{bmatrix} \begin{bmatrix} \mu_2 & -1 \\ -\mu_1 & 1 \end{bmatrix} * \frac{1}{\mu_2 - \mu_1}$$

Now the integral formula is:

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}f(s)ds$$

The integral part is:

$$\int_0^t e^{(t-s)A}f(s)ds = \int_0^t \frac{1}{\mu_1 - \mu_2} \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix} \begin{bmatrix} e^{\mu_1 (t-s)} & 0 \\ 0 & e^{\mu_2 (t-s)} \end{bmatrix} \begin{bmatrix} \mu_2 & -1 \\ \mu_1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{f(s)}{m} \end{bmatrix} ds$$

$$\int_0^t \frac{1}{\mu_2 - \mu_1} \begin{bmatrix} \mu_1 e^{\mu_2 t} + \mu_2 e^{\mu_1 t} & e^{\mu_2 t} - e^{\mu_1 t} \\ \mu_1\mu_2 e^{\mu_1 t} + \mu_1\mu_2 e^{\mu_2 t} & \mu_2 e^{\mu_2 t} - \mu_1 e^{\mu_1 t} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{f(s)}{m} \end{bmatrix} ds$$

$$\int_0^t \begin{bmatrix} \frac{f(t)}{m} e^{\mu_2 t} - e^{\mu_1 t} \\ \frac{f(t)}{m} \mu_2 e^{\mu_2 t} - \mu_1 e^{\mu_1 t} \end{bmatrix} ds$$

Case 2: $(a/m)^2 = 4k/m$

The eigenvalue is $\mu = -\frac{a}{2m}$ and the eigenvector is $v = \begin{bmatrix} 1 \\ -\frac{a}{2m} \end{bmatrix}$

The generalized eigenvector is $\begin{bmatrix} \frac{a}{2m} & 1 & 1 \\ -\frac{k}{m} & -\frac{a}{2m} & -\frac{a}{2m} \end{bmatrix}, \begin{bmatrix} \frac{a}{2m} & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Thus the generalized eigenvector is $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ Thus the matrix exponential is:

$$e^{tA} = \begin{bmatrix} 1 & 0 \\ -\frac{a}{2m} & 1 \end{bmatrix} \begin{bmatrix} e^{-\frac{a}{2m}t} & 1 \\ 0 & e^{-\frac{a}{2m}t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{a}{2m} & 1 \end{bmatrix}$$

The integral part is:

$$\int_0^t e^{(t-s)A} f(s) ds$$

$$\int_0^t \begin{bmatrix} e^{-\frac{a}{2m}(t-s)} + \frac{a}{2m} & 1 \\ \frac{a}{2m}(e^{-\frac{a}{2m}(t-s)} - b) - \frac{a}{2m}e^{-\frac{a}{2m}(t-s)} & e^{-\frac{a}{2m}(t-s)} - \frac{a}{2m} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{f(s)}{m} \end{bmatrix} ds$$

$$\int_0^t \begin{bmatrix} \frac{f(s)}{m} \\ \frac{f(s)}{m}(e^{-\frac{a}{2m}(t-s)} - \frac{a}{2m}) \end{bmatrix} ds$$

Case 3: $(a/m)^2 < 4k/m$

This is the same as the previous case.
The roots are $\mu_1 = -\frac{a}{2m} + i\sqrt{\frac{4k}{m} - \frac{a}{m}^2}$ and $\mu_2 = -\frac{a}{2m} - i\sqrt{\frac{4k}{m} - \frac{a}{m}^2}$
The eigenvectors are $v_1 = \begin{bmatrix} 1 \\ -\mu_1 \end{bmatrix}$
The matrix exponential is:

$$e^{tA} = e^{-\frac{a}{2m}t} *$$

$$\begin{bmatrix} cos(\sqrt{\frac{4k}{m} - \frac{a}{m}^2}t) & sin(\sqrt{\frac{4k}{m} - \frac{a}{m}^2}t) \\ \frac{a}{2m}cos(\sqrt{\frac{4k}{m} - \frac{a}{m}^2}t) + sin(\sqrt{\frac{4k}{m} - \frac{a}{m}^2}t)\sqrt{\frac{4k}{m} - \frac{a}{m}^2} & \frac{a}{2m}sin(\sqrt{\frac{4k}{m} - \frac{a}{m}^2}t) - cos(\sqrt{\frac{4k}{m} - \frac{a}{m}^2}t)\sqrt{\frac{4k}{m} - \frac{a}{m}^2} \end{bmatrix}$$

The integral part is:

$$\int_0^t e^{(t-s)A} f(s) ds$$

$$\int_0^t \begin{bmatrix} e^{(t-s)A} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{f(s)}{m} \end{bmatrix} ds$$

$$\int_0^t \begin{bmatrix} \frac{f(s)}{m}(sin(\sqrt{\frac{4k}{m} - \frac{a}{m}^2}(t-s))) \\ \frac{f(s)}{m}\frac{a}{2m}sin(\sqrt{\frac{4k}{m} - \frac{a}{m}^2}(t-s)) - cos(\sqrt{\frac{4k}{m} - \frac{a}{m}^2}(t-s))\sqrt{\frac{4k}{m} - \frac{a}{m}^2} \end{bmatrix} ds$$

b

Find the solution to the IVP $mx'' = -kx - ax' + f(t)$, $x(0) = 0$, $x'(0) = 0$, $f(t) = cos(t)$, $m = 1$, $a = 1$, and $k = 5/4$
Checking the case: $(1/1)^2 < 5/1$

Thus this is case 3 and the eigenvalues are $\mu_1 = -1/2 + 2i$ and $\mu_2 = -1/2 - 2i$
The eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 1/2 - 2i \end{bmatrix}$ The matrix exponential is:

$$e^{tA} = e^{-t/2}(cos(2t) + isin(2t)) \begin{bmatrix} 1 \\ 1/2 - 2i \end{bmatrix}$$

$$e^{tA} = e^{-t/2}(\begin{bmatrix} cos(2t) \\ cos(2t)/2 + 2sin(2t) \end{bmatrix} + \begin{bmatrix} sin(2t) \\ sin(2t)/2 - 2cos(2t) \end{bmatrix})$$

$$e^{tA} = \begin{bmatrix} cos(2t)e^{-t/2} & sin(2t)e^{-t/2} \\ (cos(2t)e^{-t/2} + 2sin(2t))e^{-t/2} & (sin(2t)e^{-t/2} - 2cos(2t))e^{-t/2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/4 & 1/2 \end{bmatrix}$$

$$= \frac{e^{-t/2}}{4} \begin{bmatrix} 4cos(2t) - sin(2t) & 2sin(2t) \\ 7sin(2t) + 6cos(2t) & 2sin(2t) - 4cos(2t) \end{bmatrix}$$

The integral part is:

$$\int_0^t e^{(t-s)A} f(s) ds$$

$$\int_0^t \left[e^{(t-s)A}\right] \begin{bmatrix} 0 \\ cos(s) \end{bmatrix} ds$$

$$\int_0^t \begin{bmatrix} 2sin(2(t-s))cos(s)e^{-(t-s)/2} \\ (2sin(2t) - 4cos(2t))cos(s)e^{-(t-s)/2} \end{bmatrix} ds$$

Since we only care what $x(t)$ is we can focus on the first term of the matrix and integrate that

$$\int_0^t 2sin(2(t-s))cos(s)e^{-(t-s)/2} ds$$

We can rewrite this as a sum of trig functions as $2sin(2t-2s)cos(s) = sin(\frac{2t-s}{2}) + sin(\frac{2t-3s}{2})$

$$\int_0^t sin(\frac{2t-s}{2})e^{-t+s/2} ds + \int_0^t sin(\frac{2t-3s}{2})e^{-t+s/2} ds$$

Using integration by parts we can solve this integral

$$(-e^{-t/2}(sin(t)+cos(t))+sin(t/2)+cos(t/2))+(-e^{-t/2}(sin(t)+3cos(t))/5-(sin(t/2)-cos(t/2))/5)$$

This plus the matrix exponential is the solution to the IVP, but since $x(0) = 0$ and $x'(0) = 0$ the solution is just the integral.
Therefore the solution is

$$x(t) = (-e^{-t/2}(sin(t)+cos(t))+sin(t/2)+cos(t/2))+(-e^{-t/2}(sin(t)+3cos(t))/5-(sin(t/2)-cos(t/2))/5)$$