PDEs: Homework 2

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1.4 Problem 4

A rod occupying the interval $0 \leq x \leq l$ is subject to the heat source $f(x) = 0$ for $0 < x < l/2$, and $f(x) = H$ for $l/2 < x < l$ where $H > 0$. The rod has physical constants $c = \rho = k = 1$, and its ends are kept at zero temperature.

a

Find the steady-state temperature of the rod.
Solution:
$u(0,t) = u(l,t) = 0, \lim_{t\to\infty} u_t = 0$ and $u(l/2,t)$ is equal on both sides with $u(l/2,t)_x$ are also equal on both sides
The steady-state temperature $u(x)$ satisfies the equation

$$u_{xx} + f(x) = 0$$

We can then integrate this to get:

$$\begin{aligned}
u_{xx} &= -f(x) \\
u_x &= -\int f(x)dx \\
u_x &= \begin{cases} 0 + C_1(t) & \text{for } 0 < x < l/2 \\ -Hx + C_2(t) & \text{for } l/2 < x < l \end{cases} \\
u &= -\begin{cases} C_1(t)x + C_3(t) & \text{for } 0 < x < l/2 \\ -\frac{H}{2}x^2 + C_2(t)x + C_4(t) & \text{for } l/2 < x < l \end{cases}
\end{aligned}$$

Now we can apply the boundary conditions.

At $x = 0$:

$$u(0,t) = C_3(t) = 0$$

At $x = l$:

$$u(l,t) = -\frac{H}{2}l^2 + C_2(t)l + C_4(t) = 0$$

At $x = l/2$:

$$u(l/2,t) = -\frac{H}{2}\left(\frac{l}{2}\right)^2 + C_2(t)\left(\frac{l}{2}\right) + C_4(t) = C_1(t)\frac{l}{2} + C_3(t)$$

And the partial derivative of x at $x = l/2$:

$$u_x(l/2,t) = -H\left(\frac{l}{2}\right) + C_2(t) = C_1(t)$$

Solving the system of equations we get:

$$T(x) = \begin{cases} \frac{Hl}{8}x & \text{for } 0 < x < l/2 \\ -\frac{H}{8}(l-4x)(1-x) & \text{for } l/2 < x < l \end{cases}$$

b

Which point is the hottest, and what is the temperature there? **Answer:**
We can determine the hottest place by taking the derivative of the temperature function and setting it to 0.
We can notice that for $0 < x < l/2$ the temperature is linearly increating thus the ottest temperature at that interval will be at $x = l/2$ with a temp of $\frac{Hl^2}{16}$
For $l/2 < x < l$ the temperature derivative is

$$T'(x) = H(\frac{5l}{8} - x)$$

For $x = \frac{5l}{8}$ the derivative is 0 and thus the temperature is at a maximum on that interval.
Evaluating the temperature at that point we get:

$$T(\frac{5l}{8}) = \frac{9Hl^2}{128}$$

Since the temp at $x = \frac{5l}{8}$ is greater than the temp at $x = l/2$ the hottest point is at $x = 5l/8$ with a temperature of $\frac{9Hl^2}{128}$

1.4 Problem 6

Two homogeneous rods have the same cross section, specific heat c, and density ρ but different heat conductivities κ_1 and κ_2 and lengths L_1 and L_2. Let $k_j = \kappa_j/c\rho$ be their diffusion constants. They are welded together so that the temperature u and the heat flux κu_x at the weld are continuous. The left-hand rod has its left end maintained at temperature zero. The right-hand rod has its right end maintained at temperature T degrees.

a

Find the equilibrium temperature distribution in the composite rod. **Solution:** We can solve this problem by using the heat equation and applying the boundary conditions.
The heat equation is given by:

$$c\rho u_t = \nabla \cdot (\kappa \nabla u)$$

We can notice that since the rod has two seperate conductivities, the heat equation will be different for the two rods.
we can Consider the cases

$$\begin{cases} c\rho u_t = \kappa_1 u_{xx} & \text{for } 0 < x < L_1 \\ c\rho u_t = \kappa_2 u_{xx} & \text{for } L_1 < x < L_1 + L_2 \end{cases}$$

Since we want the equilibrium temperature we want the derivative of the temperature to be 0 as it approaches infinity.
We also can not that it will not be a function of time and thus we can solve the equation dividing both sizes by the respective κ by setting the right hand side to 0.
Thus we get the equations:

$$\begin{cases} u_{xx} = 0 & \text{for } 0 < x < L_1 \\ u_{xx} = 0 & \text{for } L_1 < x < L_1 + L_2 \end{cases}$$

We can solve these equations by integrating twice to get:

$$\begin{cases} u = C_1 x + C_2 & \text{for } 0 < x < L_1 \\ u = C_3 x + C_4 & \text{for } L_1 < x < L_1 + L_2 \end{cases}$$

The boundary conitions are that $u(0, t) = 0$ and $u(L_1 + L_2, t) = T$ and that the heat flux is continuous at the weld.
And at equilibrium $u(0) = 0$ and $u(L_1 + L_2) = T$
We also know that the heat flux is continuous at the weld and thus we can say that $\kappa_1 u_x(L_1) = \kappa_2 u_x(L_1)$
Thus we can solve the system of equations to get the equilibrium temperature distribution.

$$\begin{aligned} u(0) = 0 &\implies C_2 = 0 \\ u(L_1 + L_2) = T &\implies C_3(L_1 + L_2) + C_4 = T \\ \kappa_1 u(L_1) = \kappa_2 u(L_1) &\implies C_1 L_1 + C_2 = C_3 L_1 + C_4 \\ \kappa_1 u_x(L_1) = \kappa_2 u_x(L_1) &\implies \kappa_1 C_1 = \kappa_2 C_3 \end{aligned}$$

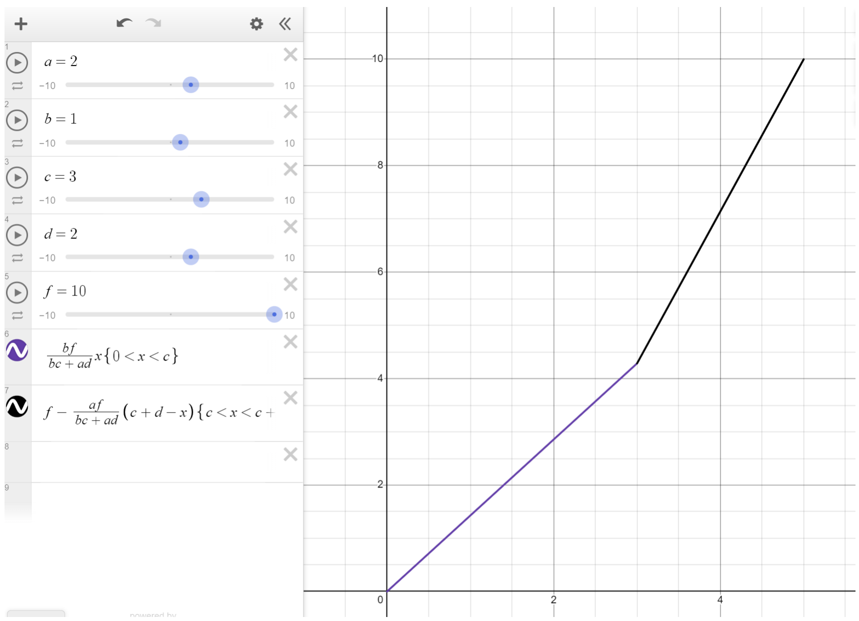
Solving the system of equations we get:

$$u(x) = \begin{cases} \frac{\kappa_2 T}{\kappa_2 L_1 + \kappa_1 L_2} x & \text{for } 0 < x < L_1 \\ T - \frac{\kappa_1 T}{\kappa_2 L_1 + \kappa_1 L_2}(L_1 + L_2 - x) & \text{for } L_1 < x < L_1 + L_2 \end{cases}$$

b

Sketch it as a function of x in case $k_1 = 2, k_2 = 1, L_1 = 3, L_2 = 2,$ and $T = 10$. (This exercise requires a lot of elementary algebra, but it's worth it.) **Solution:** Setting these values into the equation we get:

$$u(x) = \begin{cases} \frac{10}{7}x & \text{for } 0 < x < 3 \\ \frac{10}{7}(2x-3) & \text{for } 3 < x < 5 \end{cases}$$



1.5 Problem 2

Consider the problem

$$u''(x) + u'(x) = f(x)$$

$$u'(0) = u(0) = \frac{1}{2}[u'(l) + u(l)]$$

With $f(x)$ a given function.

a

Is the solution unique? Explain.
We can prove uniqueness by showing there are 2 solutions and then showing that they are equal.
We can let u_1 and u_2 be solutions to the problem.
Then we can let $w = u_1 - u_2$ and show that $w = 0$.

Since u_1, u_2 are solutions to the problem, we can substitute them into the equation to get:

$$u_1'' + u_1' = f(x)$$
$$u_2'' + u_2' = f(x)$$

Subtracting the two equations we get:

$$w'' + w' = 0$$

This is a second order linear equation and we can solve it by integrating factor of e^{-x}.
Thus we get the solution $w = C_1 e^{-x}$
Clearly $C_1 e^{-x} \neq 0$ for all x so $w \neq 0$ and thus the solution is unique.

b

Does a solution necessarily exist or is there a condition that $f(x)$ must satisfy for a solution to exist?
We can show exitance through the boundary conditions.
We can do this by integrating both sides of the equation and applying the boundary conditions.

$$\begin{aligned} \int_0^l u''(x) + u'(x) dx &= \int_0^l f(x) dx \\ [u'(l) + u(l)] - [u'(0) + u(0)] &= \int_0^l f(x) dx \\ 0 &= \int_0^l f(x) dx \end{aligned}$$

Thus we need to have that $f(x)$ is such that $\int_0^l f(x)dx = 0$ for a solution to exist.

1.5 Problem 6

Solve the equation

$$u_x + 2xy^2 u_y = 0$$

Solution:
We solve this by noticing the directional derivative in the direction of the vector $(1, 2xy^2)$ is 0. This means that the solution is constant along the lines $y^2 = x^2 + C$ for some constant C.
Thus we can also say that

$$\frac{dy}{dx} = \frac{2xy^2}{1} = 2xy^2$$

This is a separable differential equation and we can solve it by separating the variables and integrating.

$$\begin{aligned}
\frac{dy}{dx} &= 2xy^2 \\
\int y^{-2} dy &= \int 2x dx \\
-y^{-1} &= x^2 + C \\
y &= \frac{-1}{x^2 + C}
\end{aligned}$$

Thus we can say our solution will be in the form of $u(x, y) = u(x, \frac{-1}{x^2+C})$.
For any x, the characteristic curve only depends on C and not x We can see this if we take $x = 0$ as

$$u(0, \frac{1}{C}) = f(C)$$

For some arbitrary function $f(C)$
Since we can see that $C = x^2 + y^{-1}$ we can substitute this into the equation to get the solution.
Thus $u(x, y) = f(x^2 + y^{-1})$ for some arbitrary function f.

1.6 Problem 4

What is the type of the equation:

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0$$

Show that by direct substition that $u(x, y) = f(y+2x)+xg(y+2x)$ is a solution for arbitrary functions f and g.
Solution:
This is a parabolic equation due to $b^2 - 4ac = 16 - 16 = 0$
To prove that $u(x, y) = f(y + 2x) + xg(y + 2x)$ is a solution we can substitute it into the equation and show that it satisfies the equation.

$$\begin{aligned}
u_{xx} &= 4f''(y + 2x) + 2g'(y + 2x) + 2g'(y + 2x) + 4xg''(y + 2x) \\
u_{xy} &= 2f''(y + 2x) + g'(y + 2x) + 2xg''(y + 2x) \\
u_{yy} &= f''(y + 2x) + xg''(y + 2x) \\
u_{xx} - 4u_{xy} + 4u_{yy} &= 4f''(y + 2x) + 2g'(y + 2x) + 2g'(y + 2x) + 4xg''(y + 2x) \\
&- 8f''(y + 2x) - 4g'(y + 2x) - 8xg''(y + 2x) + 4f''(y + 2x) + 4xg''(y + 2x) \\
&= 0
\end{aligned}$$

1.6 Problem 6

Consider the equation

$$3u_y + u_{xy} = 0$$

a

What is its type?
Hyperbolic
Solution:
This is a hyperbolic equation due to $b^2 - 4ac = 1 - 0 = 1$

b

Find the general solution. Hint ($v = u_y$)
Solution:
We can preform a subsection of $v = u_y$ to get a first order linear equation.

$$3v + v_x = 0$$

This is a first order linear equation and we can solve it by separating the variables and integrating.
Thus results in the solution $v = C_1(y)e^{-3x}$
Where $C_1(y)$ is an arbitrary function of y.
Replacing v with u_y we get:

$$u_y = C_1(y)e^{-3x}$$

We can now solve this by separating the variables and integrating.

$$\begin{aligned} u_y = C_1(y)e^{-3x} \int du &= e^{-3x} \int C_1(y)dy \\ u &= e^{-3x}C_2(y) + C_3(x) \end{aligned}$$

Where $C_2(y) = \int C_1(y)dy$ which is arbitrary and $C_3(x)$ is an arbitrary function of x.

c

With auxiliary conditions $u(x,0) = e^{-3x}$ and $u_y(x,0) = 0$, does a solution exist? Is it unique?
We can determine if a solution exists and is unique by applying the boundary

conditions to the general solution.

$$u(x,0) = e^{-3x} = e^{-3x}C_2(0) + C_3(x)$$
$$u_y(x,0) = 0 = C_1(0)e^{-3x}$$

From the second equation we can see that $C_1(0) = 0$ and thus $u_y = 0$ for all x and y.
But this not nessarily mean that the solution is unique.
We can find two functions of $C_2(y)$ and $C_3(x)$ that satisfy the first equation.
For example we can let $C_2(y) = 1$ and $C_3(x) = 0$ and $C_2(y) = 2$ and $C_3(x) = -e^{-3x}$.
Thus the solution is not unique.