Workshop 6: Math 292

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Question 1

a)

We can computer the charecteristic polynomial of the matrix A by computing the determinant of the matrix $A - \lambda I$. Here the $A - \lambda I$ is given by

$$\begin{bmatrix} -1-\lambda & -2 & 2 \\ -2 & 3-\lambda & -6 \\ 0 & 2 & -3-\lambda \end{bmatrix}$$

The charecteristic polynomial by taking the determinant is

$$(-3-\lambda)[(-1-\lambda)(3-\lambda)-4]-2[6(1+\lambda)+4]$$

Thus the charecteristic polynomial of the matrix A simplifies to

$$(-\lambda^3-\lambda^2+\lambda+1)$$

Thus the eigenvalues for this matrix is $\mu_1 = -1, \mu_2 = -1, \mu_3 = 1$.

b)

The eigenvectors for the eigenvalues $\mu_1 = -1$ is given by solving the equation $(A - \mu_1 I)X = 0$. Here the matrix $[A - \mu_1 I|0]$ is given by

$$\begin{bmatrix} 0 & -2 & 2 & 0 \\ -2 & 4 & -6 & 0 \\ 0 & 2 & -2 & 0 \end{bmatrix}$$

The reduced row echelon form of the matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the eigenvector for the eigenvalue $\mu_1 = -1$ is given by

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Notice that even through the eigenvalue $\mu_1 = -1$ has a multiplicity of 2, there is only one linearly independent eigenvector, thus the geometric multiplicity of the eigenvalue $\mu_1 = -1$ is 1.
The eigenvector for the eigenvalue $\mu_3 = 1$ is given by solving the equation $(A - \mu_3 I)X = 0$. Here the matrix $[A - \mu_3 I|0]$ is given by

$$\begin{bmatrix} -2 & -2 & 2 & 0 \\ -2 & 2 & -6 & 0 \\ 0 & 2 & -4 & 0 \end{bmatrix}$$

The reduced row echelon form of the matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the eigenvector for the eigenvalue $\mu_3 = 1$ is given by

$$\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

c)

The matrix A can be rewritten as PUP^{-1} where P is the matrix of eigenvectors and U is a upper triangular matrix with eigenvalues on the diagonal since the matrix A isnt actually diagonalizable. The matrix P is given by where the question marks indicate unkown values.

$$\begin{bmatrix} -1 & ? & -1 \\ 1 & ? & 2 \\ 1 & ? & 1 \end{bmatrix}$$

The matrix U is given by

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is quite similar to the diagonalization of the matrix A but the matrix P is not invertible if we only had the two eigenvectors. Thus we need to find another vector for the middle row of P which can be done by solving the equation $(A - \mu_2 I)X = aP_1$ with $a = 1$ Here the matrix $[A - \mu_2 I|P_1]$ is given by

$$\begin{bmatrix} 0 & -2 & 2 & -1 \\ -2 & 4 & -6 & 1 \\ 0 & 2 & -2 & 1 \end{bmatrix}$$

The reduced row echelon form of the matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 1/2 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Giving us a vector to put in the middle row of the matrix P

$$\begin{bmatrix} -1 & 1/2 & -1 \\ 1 & 1/2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Inverting the matix P we get the matrix P^{-1}

$$\begin{bmatrix} -1 & 1/2 & -1 & 1 & 0 & 0 \\ 1 & 1/2 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$

d)

Solving the system $Y'(t) = UY(t)$ as a system of partially coupled DEs:

$$y_1' = -y_1 + y_2$$

$$y_2' = -y_2$$

$$y_3' = y_3$$

For y_2 and y_3 are obvious solutions:

$$y_2(t) = c_2 e^{-t}$$

$$y_3(t) = c_3 e^{t}$$

where $c_2 = y_2^{(0)}$ and $c_3 = y_3^{(0)}$
For y_1 we can solve the equation $y_1' = -y_1 + c_2 e^{-t}$

$$y_1' + y_1 = c_2 e^{-t}$$

With the integrating factor e^t we can solve the equation

$$e^t y_1' + e^t y_1 = c_2$$

$$y_1 = c_2 e^{-t} + c_1 t e^{-t}$$

where $c_1 = y_1^{(0)}$
Thus the general solution to the system of DEs is

$$\begin{cases} y_1(t) = c_2 e^{-t} + c_1 t e^{-t} \\ y_2(t) = c_2 e^{-t} \\ y_3(t) = c_3 e^{t} \end{cases}$$

Thus the new matrix $G(t)$ which solves $Y(t) = G(t)Y^{(0)}$ is given by

$$\begin{bmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix}$$

e)

The solution for the DE $X'(t) = AX(t)$ is given by $X(t) = e^{tA}X^{(0)}$

$$e^{tA} = e^{tPUP^{-1}} = Pe^{tU}P^{-1}$$

e^{tU} can be rewritten as $e^{t(D+N)}$ where D is the diagonal matrix of eigenvalues and N is simply $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Since D and N commute we can rewrite the equation as

$$e^{tU} = e^{tD}e^{tN}$$

The matrix e^{tD} is given by

$$\begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix}$$

The matrix e^{tN} is more interesting as $k \geq 2N^k, = 0$ So then the taylor series for e^{tN} is given by 1 + tN which is simply

$$\begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus the matrix e^{tU} is given by

$$\begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix} \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix}$$

Finally the matrix $e^{tA} = Pe^{tD}e^{tN}P^{-}1$ is

$$\begin{bmatrix} -1 & 1/2 & -1 \\ 1 & 1/2 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$

Attached below is the work for my calculations for the inverse of P and the matrix e^{tA}