Math 300: Midterm 3 Review

Pranav Tikkawar

April 24, 2024

Question 1

Let $A = \{1, 2, 3\}$. Give a relation on A that is For all these relations, consider that $R \subset A \times A$.

a

Reflexive, symmetric, and transitive.
Solution:
Let $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2), (1, 3), (3, 1)\}$.

b

Reflexive, symmetric, but not transitive.
Solution:
Let $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$.

c

Reflexive, not symmetric, and transitive.
Solution:
Let $R = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$.

d

Reflexive, not symmetric, and not transitive.
Solution:
Let $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$.

e

Not reflexive, symmetric, and transitive.
Solution:
Let $R = \emptyset$.

f

Not reflexive, symmetric, and not transitive.
Solution:
Let $R = \{(1,2),(2,1)\}$.

g

Not reflexive, not symmetric, and transitive.
Solution:
Let $R = \{(1,2),(2,3),(1,3)\}$.

h

Not reflexive, not symmetric, and not transitive.
Solution:
Let $R = \{(1,2),(2,3)\}$.

Question 2

a

Let $A = \{1,2\}$. All the relations on A which are symmetric and transitive, but not reflexive **Solution:**
$R = \emptyset, \{(1,1)\}\{(2,2)\}$

b

Let $A = \{1,2,3,4,5\}$. How many relations which are both symmetric and antisymmetric
Solution:
There are 32 such relations. If we consider the powerset of A then see that every single subset of A can be a relation that is symetric and antisymmetric if the relation is the identy relation. So there are $2^5 = 32$ such relations.

Question 3

Let $A = \{1,2,3\}$ For each of the following relations on A, determine whether it is reflexive, symmetric, antisymmetric, and/or transitive.

a

$R = \{(1,2)\}$
Solution:
Reflexive: No. $(1,1)$ is not in R.
Symmetric: No. $(2,1)$ is not in R.

Antisymmetric: Yes.
Transitive: Yes.

b

$S = \{(1,2),(1,3)\}$ **Solution:**
Reflexive: No. $(1,1)$ is not in S.
Symmetric: No. $(2,1)$ is not in S.
Antisymmetric: Yes.
Transitive: Yes.

c

$T = \{(1,2),(2,1),(1,1)\}$ **Solution:**
Reflexive: No. $(2,2)$ is not in T.
Symmetric: Yes Antisymmetric: No. $(1,2)$ and $(2,1)$ are in T but $1 \neq 2$.
Transitive: No. $(1,2)$ and $(2,1)$ are in T but $(2,2)$ is not in T.

Question 4

Let $A = \{1,2,3\}$. Size of relations:

- Min Reflexive: $\{(1,1),(2,2),(3,3)\}$
- Min symmetric: $\emptyset$
- Min antisymmetric: $\emptyset$
- Min transitive: $\emptyset$
- Min equivalence: $\{(1,1),(2,2),(3,3)\}$
- Min partial order: $\{(1,1),(2,2),(3,3)\}$
- Max symmetric: $\{(1,1),(2,2),(3,3),(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$
- Max antisymmetric: $\{(1,1),(2,2),(3,3),(1,2),(1,3),(2,3)\}$
- Max equivalence: $A \times A$
- Max partial: $\{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$

Question 5

Let S be the relation on $\mathbb{R}$ defined by $xSy : x < y + 1$. Determine whether S is reflexive, symmetric, antisymmetric, transitive.

Reflexive:

Need $xSx : x < x + 1$. This is true for all $x \in \mathbb{R}$. So S is reflexive.

Symmetric:

Need $xSy \Rightarrow ySx$ Counterexample: $x = 1$, $y = 100$. $1 < 100 + 1$ but $100 \not< 1 + 1$. So S is not symmetric.

Antisymmetric:

Need $xSy \wedge ySx \Rightarrow x = y$ Counterexample: $x = 1$, $y = 1.5$ $1 < 1.5 + 1$ and $1.5 < 1 + 1$ but $1 \neq 1.5$. So S is not antisymmetric.

Transitive:

Need $xSy \wedge ySz \Rightarrow xSz$ Counterexample: $x = 5$, $y = 4.3$, and $z = 3.5$. $5 < 4.3+1$ and $4.3 < 3.5 + 1$ but $5 \not< 3.5 + 1$. So S is not transitive.

Question 6

Let $E \subset \mathbb{N} \times \mathbb{N}$ be the relation defined as $xEy : xy \leq x + y$. Determine whether E is reflexive, symmetric, antisymmetric, transitive.

Reflexive:

$xEx : x \cdot x \leq x + x$. This is not true for values of 3 or greater. So E is not reflexive.

Symmetric:

if $xEy : xy \leq x + y$ then $yEx : yx \leq y + x$. This is true as multiplication and addition is commutative. So E is symmetric.

Antisymmetric:

if $xEy : xy \leq x + y$ and $yEx : yx \leq y + x$ then $x = y$. This is not true as $x = 2$ and $y = 3$ is a counterexample. So E is not antisymmetric.

Transitive:

if $xEy : xy \leq x + y$ and $yEz : yz \leq y + z$ then xEz would be $xz \leq x + z$. This is not true for $x = 2$, $y = 1$, and $z = 3$. So E is not transitive.

Problem 7

Let D be the relation on $\mathbb{N}$ defined as: xDy iff $x^2|y$. Determine whether D is reflexive, symmetric, antisymmetric, and transitive.

Reflexive

Need: xDx

$x^2|x$

Counter: 2. 2^2 does not divide 2. So D is not reflexive.

Symmetric

Need: $xDy \Rightarrow yDx$
$x^2|y \Rightarrow y^2|x$
This is not true in general. For example, $2^2|4$ but $4^2\nmid 2$. So D is not symmetric.

Antisymmetric

Need: $xDy \wedge yDx \Rightarrow x = y$
$x^2|y \wedge y^2|x \Rightarrow x = y$
$y = kx^2$ and $x = qy^2$ for some $k, q \in \mathbb{N}$.
Substitute $y = kx^2$ into $x = qy^2$ to get $x = q(kx^2)^2 = qk^2x^4$.
Divide by x (as it is $\neq 0$) to get $1 = qk^2x^3$.
This is only true if $x = 1$ and $q = k = 1$. So $x = y$ and D is antisymmetric

Transitive

Need: $xDy \wedge yDz \Rightarrow xDz$
$x^2|y \wedge y^2|z \Rightarrow x^2|z$
Suppose $k, q \in \mathbb{N}$ such that $y = kx^2 \wedge z = qy^2$
Then $z = q(kx^2)^2 = qk^2x^4$.
Since $k, q, x \in \mathbb{N}$, $qk^2x^2 \in \mathbb{N}$. We can call this r.
Thus $z = rx^2$ and $x^2|z$. So D is transitive.

Problem 8

Let S be the relation on $\mathbb{N}$ defined as: xSy iff $x|y^2$ Determine whether S is reflexive, symmetric, antisymmetric, and transitive.

Reflexive

Need: xSx
$x|x^2$
$(\exists q \in \mathbb{N})x^2 = qx$
This is true for all $x \in \mathbb{N}$. as the "q" will be x to satisfy the equation So S is reflexive.

Symmetric

Need: $xSy \Rightarrow ySx$
$x|y^2 \Rightarrow y|x^2$
$(\exists k, q \in \mathbb{N})y^2 = kx \Rightarrow x^2 = qy$
This is not true in general. For example, $x = 3$ and $y = 6$. $3|36$ but $6\nmid 9$. So S is not symmetric.

Antisymmetric

Need: $xSy \wedge ySx \Rightarrow x = y$
$x|y^2 \wedge y|x^2 \Rightarrow x = y$
$(\exists k, q \in \mathbb{N})y^2 = kx \wedge x^2 = qy \rightarrow x = y$
Counter: $x = 2$ and $y = 4$. $2|16$ and $4|4$ but $2 \neq 4$. So S is not antisymmetric

Transitive

Need: $xSy \wedge ySz \Rightarrow xSz$
$x|y^2 \wedge y|z^2 \Rightarrow x|z^2$
$(\exists k, q, r \in \mathbb{N})y^2 = kx \wedge z^2 = qy \rightarrow z^2 = rx$
Counter example is $x = 8$, $y = 4$, and $z = 2$. $8|16$ and $4|4$ but $8 \nmid 4$. So S is not transitive.

Problem 9

Let $S = \mathbb{R} \times \mathbb{R}$ be define as follows: for $(x_1, y_1) \in S$ and $(x_2, y_2) \in S$. We have $(x_1, y_1)P(x_2, y_2)$ iff $x_1 \leq x_2$ and $y_1 > y_2$. Determine whether P is reflexive, symmetric, antisymmetric, and transitive.

Reflexive

Need: $(x, y)P(x, y)$
$x \leq x$ and $y > y$
This is not true for all $(x, y) \in S$. So P is not reflexive. Counter: $(1, 2)$

Symmetric

Need: $(x_1, y_1)P(x_2, y_2) \Rightarrow (x_2, y_2)P(x_1, y_1)$
$x_1 \leq x_2$ and $y_1 > y_2 \Rightarrow x_2 \leq x_1$ and $y_2 > y_1$
This is not true in general. For example, $(1, 2)P(2, 1)$ but $(2, 1)\not{P}(1, 2)$. So P is not symmetric.

Antisymmetric

Need: $(x_1, y_1)P(x_2, y_2) \wedge (x_2, y_2)P(x_1, y_1) \Rightarrow (x_1, y_1) = (x_2, y_2)$
$x_1 \leq x_2$ and $y_1 > y_2 \wedge x_2 \leq x_1$ and $y_2 > y_1 \Rightarrow (x_1, y_1) = (x_2, y_2)$
This is vacuously true as there is no $(x_1, y_1)and(x_2, y_2)$ that satisfy the conditions. So P is antisymmetric

Transitive

Need: $(x_1, y_1)P(x_2, y_2) \wedge (x_2, y_2)P(x_3, y_3) \Rightarrow (x_1, y_1)P(x_3, y_3)$
$x_1 \leq x_2$ and $y_1 > y_2 \wedge x_2 \leq x_3$ and $y_2 > y_3 \Rightarrow x_1 \leq x_3$ and $y_1 > y_3$
This is true due to the transitive property of the inequalities. So P is transitive.

Problem 10

The properties of reflexivity, symmetry, antisymmetry, and transitivity are related to the identity relation and the operations of inversion and composition. Let $R \subset A \times A$ Prove:

a

R is reflexive iff $I_A \subset R$

Proof

Forward

Suppose $x \in A$. Assume R is reflexive. Need $I_A \subseteq R$.
Since R is reflexive, $\forall x \in A, xRx$. This means $(x, x) \in R$. Since $I_A = \{(x, x)|x \in A\}$, $I_A \subseteq R$.

Backward

Suppose $x \in A$ Assume $I_A \subseteq R$. Need R is reflexive.
Since $I_A \subseteq R$, $\forall x \in A, (x, x) \in R$. This means that $\forall x \in A, xRx$. So R is reflexive.

b

R is symmetric iff $R = \overleftarrow{R}$

Proof

Forward

Suppose $a, b \in A$. Assume R is symmetric. Need $R = \overleftarrow{R}$.
In other words, $\{(a, b)|a, b \in A \wedge aRb\} = \{(b, a)|a, b \in A \wedge aRb\}$.
Since R is symmetric, $\forall a, b \in A, aRb \Rightarrow bRa$. This means that $(a, b) \in R \Rightarrow (b, a) \in R$. So $R \subseteq \overleftarrow{R}$. Also since R is symmetric $\forall a, b \in A, bRa \Rightarrow aRb$. This means that $(b, a) \in R \Rightarrow (a, b) \in R$. So $\overleftarrow{R} \subseteq R$. So $R = \overleftarrow{R}$.

Backward

Suppose $R = \overleftarrow{R}$. Then $\forall x, y \in A, (x, y) \in R \Rightarrow (y, x) \in R$. This means that $\forall x, y \in A, xRy \Rightarrow yRx$. So R is symmetric.,

c

R is antisymmetric iff $R \cap \overleftarrow{R} \subset I_A$

Proof

Forward

Suppose R is antisymmetric. Then $\forall x, y \in A, xRy \land yRx \Rightarrow x = y$. This means that $(x,y) \in R \land (y,x) \in R \Rightarrow x = y$. Since $(x,y) \in R$ means xRy, $(y,x) \in R$ means $x\overleftarrow{R}y$ and $x = y$ means $(x,y) \in I_A$. Thus $R \cap \overleftarrow{R} \subset I_A$.

Backward

Suppose $R \cap \overleftarrow{R} \subset I_A$. Then $\forall x, y \in A, (x,y) \in R \land (y,x) \in R \Rightarrow (x,y) \in I_A$. This means that $\forall x, y \in A, xRy \land yRx \Rightarrow x = y$. Which is the definiton of antisymmetry, so R is antisymmetric.

d

R is transitive iff $R \circ R \subset R$

Proof

Forward

Suppose $x, z \in A$. Assume R is transitive. Need $R \circ R \subset R$.
Let $(x,z) \in R \circ R$ then $\exists y \in A$ such that $xRy \land yRz$. Since R is transitive and xRz. So $(x,z) \in R$. So $R \circ R \subseteq R$.

Backward

Suppose $R \circ R \subset R$. Then $\forall x, y, z \in A, (x,y) \in R \land (y,z) \in R \Rightarrow (x,z) \in R$. This means that $\forall x, y, z \in A, xRy \land yRz \Rightarrow xRz$. Which is the definition of transitivity, so R is transitive.

Problem 11

A relation V on $\mathbb{R}$ is given by xVy iff $x = y$ or $xy = 1$.

a

Prove that V is an equivalence relation.

Reflexive

Need: xVx
$x = x$ or $x \cdot x = 1$
This is true for all $x \in \mathbb{R}$. So V is reflexive.

Symmetric

Need: $xVy \Rightarrow yVx$
$x = y$ or $xy = 1 \Rightarrow y = x$ or $yx = 1$
This is true for all $x, y \in \mathbb{R}$. So V is symmetric

Transitive

Need: $xVy \wedge yVz \Rightarrow xVz$
$x = y$ or $xy = 1 \wedge y = z$ or $yz = 1 \Rightarrow x = z$ or $xz = 1$
Case 1: $x = y$ and $y = z$
Then $x = z$ and xVz. So V is transitive.
Case 2: $x = y$ and $yz = 1$
Then $xz = 1$ and xVz. So V is transitive.
Case 3: $xy = 1$ and $y = z$
Then $xz = 1$ and xVz. So V is transitive.
Case 4: $xy = 1$ and $yz = 1$
Then $1/x = 1/z$ and $x = z$ and xVz. So V is transitive.
Thus V is an equivalence relation.

b

Describe the equivalence classes of $3, -2/3, and 0$.
The equivalence class of 3 is $\{3, 1/3\}$
The equivalence class of $-2/3$ is $\{-2/3, -3/2\}$
The equivalence class of 0 is $\{0\}$

Problem 12

Let T be the relation on $\mathbb{Z}$ defined as: aTb iff there exists nonzero integers r and s such that $ar^2 = bs^2$. Prove that T is an equivalence relation.

Reflexive

Need: aTa
There exists $r = s = 1$ such that $a = a$. So T is reflexive.

Symmetric

Need: $aTb \Rightarrow bTa$
Suppose aTb. Then there exists $r, s \neq 0$ such that $ar^2 = bs^2$. This means that $bs^2 = ar^2$. So bTa. So T is symmetric

Transitive

Need: $aTb \wedge bTc \Rightarrow aTc$
Suppose aTb and bTc. Then there exists $r, s, t, u \neq 0$ such that $ar^2 = bs^2$ and $bt^2 = cu^2$. This means that $ar^2/s^2 = cu^2/t^2$. $ar^2t^2 = bs^2u^2$. So aTc. So T is transitive.

Problem 13

Let R be the relation on $\mathbb{N}$ defined as: aRb iff there exists odd integers k and l such that $ak = bl$.

a

$20R12$ is in this relation as $k = 3, l = 5$. to make $20 * 3 = 60 = 12 * 5$.

b

$7R10$ is not in this relation as there are no odd integers k to multiply to 7 to make an even number.

c

$20R10$ is not in the relation as there are no odd integers k to multiply to 20 to make an "odd" multiple of 10.

d

Reflexive

Need aRa
There exists $k = l = 1$ such that $ak = al$. So R is reflexive.

Symmetric

Need $aRb \Rightarrow bRa$
Suppose aRb. Then there exists k, l such that $ak = bl$. This means that $bl = ak$. So bRa. So R is symmetric

Transitive

Need $aRb \wedge bRc \Rightarrow aRc$
Suppose aRb and bRc. Then there exists k, l, m, n such that $ak = bl$ and $bm = cn$. This means that $akm = bln$. Since 2 odd integers multiplied is also odd then aRc. So R is transitive.

e

Describe the equivalence claseses:
The equivalence class of 2^0 is $\{2n+1 : n \in \mathbb{N}\}$
The equivalence class of 2^1 is $\{2(2n+1) : n \in \mathbb{N}\}$
The equivalence class of 2^2 is $\{2^2(2n+1) : n \in \mathbb{N}\}$
The equivalence class of 2^k $\forall k \geq 0$ is $\{2^k(2n+1) : n \in \mathbb{N}\}$
In other words, The equiliances classes are represented as 2^k where $k \in \mathbb{Z}$ and $k \geq 0$. And all the elements of the equivalence class is all the numbers $x \in \mathbb{N}$ which have the same power of 2 in thier prime factorization.

Problem 14

On $\mathbb{N}$ a relation P is given by aPb iff the prime factorization of a and b have the same power of 2.

a

Reflexive

Need aPa
The prime factorization of a is the same as the prime factorization of a. So P is reflexive.

Symmetric

Need $aPb \Rightarrow bPa$
Suppose aPb. Then the prime factorization of a and b have the same power of 2. This means that the prime factorization of b and a have the same power of 2. So P is symmetric

Transitive

Need $aPb \wedge bPc \Rightarrow aPc$
Suppose aPb and bPc. Then the prime factorization of a and b have the same power of 2 and the prime factorization of b and c have the same power of 2. This means that the prime factorization of a and c have the same power

b

$1/P$: $\{1, 3, 5\}$
$4/P$: $\{4, 12, 20\}$
$72/P$: $\{8, 24, 60\}$

Problem 15

a

Let P and Q be equivalence relations on a set A. Prove that $R := P \cap Q$ is an equivalence relation on A.

Reflexive

Suppose $a \in A$. Need aRa
Since P and Q are equivalence relations, aPa and aQa. This means that $(a, a) \in P$ and $(a, a) \in Q$. So $a \in P \cap Q$. So aRa.

Symmetric

Suppose $a, b \in A$ Assume aRb. Need bRa
Since aRb, $(a, a) \in P \cap q$ so $(a, b) \in P$ and $(a, b) \in Q$ Since P and Q are both equivalence relations they are symmetric. So $(b, a) \in P$ and $(b, a) \in Q$. So bRa. So R is symmetric.

Transitive

Suppose $a, b, c \in A$ Assume aRb and bRc. Need aRc
Since aRb and bRc, $(a, b) \in P \cap Q$ and $(b, c) \in P \cap Q$. This means that $(a, b) \in P$ and $(a, b) \in Q$ and $(b, c) \in P$ and $(b, c) \in Q$. Since P and Q are equivalence relations, they are transitive. So $(a, c) \in P$ and $(a, c) \in Q$. So aRc. So R is transitive.

b

Give an exaple of two equivalence relations P and Q on $A = \{1, 2, 3\}$ such that $T := P \cup Q$ is not an equivalence relation on A.
Solution:
Let $A = \{1, 2, 3\}$,
$P = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$,
$Q = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$.
Then $T = P \cup Q = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1), (1, 2), (2, 1)\}$.
T is not an equivalence relation as it is not transitive. For example, $(3, 1) \in T$

and $(1,2) \in T$ but $(3,2) \notin T$.

Problem 16

Let P be the relation on $\mathbb{N}$ defined as: aPb iff $b = 2^k a$ for some integers $k \geq 0$. Prove that P is a partial order.

Reflexive

Need: aPa
$a = 2^0 a$. So aPa.

Antisymmetric

Need: $aPb \wedge bPa \Rightarrow a = b$
Suppose aPb and bPa. Then $b = 2^k a$ and $a = 2^l b$ for some $k, l \geq 0$. This means that $b = 2^k 2^l b$. So $b = 2^{k+l} b$. This means $k + l = 0$ and $k = l = 0$. So $a = b$

Transitive

Need: $aPb \wedge bPc \Rightarrow aPc$
Suppose aPb and bPc. Then $b = 2^k a$ and $c = 2^l b$ for some $k, l \geq 0$. This means that $c = 2^k 2^l a$. So $c = 2^{k+l} a$. So aPc.

Problem 17

Let A an arbitrary nonempty set, and let P be a partial order on A. Define a new relation $<$ on A as follows $x < y$ iff xPy and $x \neq y$.

a

Prove that there are no $x, y \in A$ such that $x < y$ and $y < x$.
Suppose there exists $x, y \in A$ such that $x < y$ and $y < x$. Then xPy and yPx. Due to P being Partial Order and thus antisymmetric, $x = y$. This is a contradiction. So there are no $x, y \in A$ such that $x < y$ and $y < x$.

b

Prove that $<$ is transitive.
Assume that $<$ is not transitive. Then there exists $x, y, z \in A$ such that $x < y$ and $y < z$ but $x \not< z$. If $x < y$ then xPy and if $y < z$ then yPz. Since P is

a partial order, it is transitive. So xPz. But $x \not< z$ is a contradiction. So $<$ is transitive.

Problem 18

Let A be a nonempty set with partial order P. for each $t \in A$ define $S_t := \{x \in A : xPt\}$
Let $\mathscr{F} = \{S_t : t \in A\}$ then $\mathscr{F}$ is a subset of $\mathscr{P}(A)$ [since for every $t \in A$, $S_t \subset A$]. and thus can be partially orded by $\subseteq$. Let $a, v \in A$ be arbitrary.

i

Prove that if aPb then $S_a \subseteq S_b$
Suppose aPb. Let $x \in S_a$. Then xPa. Since aPb, xPb. So $x \in S_b$. So $S_a \subseteq S_b$.

Suppose $a, b \in A$. Assume aPb. Need $S_a \subseteq S_b$. Let $x \in S_a$. Then xPa. Since aPb, xPb by the transitivity of P. So $x \in S_b$. So $S_a \subseteq S_b$.

ii

Prove that if $S_a \subseteq S_b$ then aPb
Suppose $a, b \in A$. Assume $S_a \subseteq S_b$. Need aPb. Let $x \in S_a$. Then xPa. Since $S_a \subseteq S_b$, $x \in S_b$. Since we know that P is a partial order, it is also Reflexive, so a is in the set S_a. So there is an element $x \in S_b$ where $x = a$ so aPb.

Problem 19

a

Let P and Q be partial orders on the same nonempty set A. Prove that $P \cap Q$ is a partial order on A.
For sake of ease: Let $R := P \cap Q$

Reflexive

Suppose $a \in A$. Need aRa
Since P and Q are partial orders and thus reflexive, aPa and aQa. This means that $(a, a) \in P$ and $(a, a) \in Q$. So $(a, a) \in P \cap Q$. So aRa.

Antisymmetric

Suppose $a, b \in A$ Assume aRb and bRa. Need: $a = b$
Since aRb and bRa, $(a, b) \in P \cap Q$ and $(b, a) \in P \cap Q$. This means that $(a, b) \in P$ and $(a, b) \in Q$ and $(b, a) \in P$ and $(b, a) \in Q$. Since P and Q are partial orders and antisymmetric, aPb and aQb and bPa and bQa implies $a = b$. So R is antisymmetric.

Transitive

Suppose $a, b, c \in A$. Assume aRb and bRc. Need aRc
Since aRb and bRc, $(a, b) \in P \cap Q$ and $(b, c) \in P \cap Q$. This means that $(a, b) \in P$ and $(a, b) \in Q$ and $(b, c) \in P$ and $(b, c) \in Q$. Thus aPb and aQb and bPc and bQc. Since P and Q are partial order and transitive, aPc and aQc. Since aPc and aQc, $a \in P \cap Q$. So aRc. So R is transitive.

b

Give an example of two partial orders P and Q on $A\{1, 2, 3\}$ such that $P \cup Q$ is not a partial order on A.

Solution

Let $A = \{1, 2, 3\}$
Let $P = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$
Let $Q = \{(1, 1), (2, 2), (3, 3), (3, 1)\}$
Then $P \cup Q = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$
Which is not antisymmetric, thus not a partial order.

Problem 20

a

Let $\leq_1$ *and* $\leq_2$ be total orders on the same nonempty set A. Let P be the relation on A defined by aPb iff $a \leq_1 b$ and $a \leq_2 b$. Prove that P is a partial order on A.

Reflexive

Suppose $a \in A$. Need: aPa
Since $\leq_1$ and $\leq_2$ are total orders, $a \leq_1 a$ and $a \leq_2 a$. So aPa.

Antisymmetric

Suppose $a, b \in A$. Assume aPb and bPa. Need: $a = b$
Since aPb and bPa, $a \leq_1 b$ and $a \leq_2 b$ and $b \leq_1 a$ and $b \leq_2 a$. Since $\leq_1$ and $\leq_2$ are total orders and antisymmetric, $a = b$ as desired

Transitive

Suppose $a, b, c \in A$. Assume aPb and bPc. Need aPc
Since aPb and bPc, $a \leq_1 b$ and $a \leq_2 b$ and $b \leq_1 c$ and $b \leq_2 c$. Since $\leq_1$ and $\leq_2$ are total orders and transitive, $a \leq_1 c$ and $a \leq_2 c$. So aPc. So P is transitive.

b

Give an example of 2 total orders on the same set A such that the relation P is not a total order on A.
aPb iff $a \leq_3 b$ and $a \leq_4 b$

Solution

Let $A = \{1, 2, 3\}$
Let $\leq_3 := \leq$ Let $\leq_4 := \geq$
$\leq_3 = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$
$\leq_4 = \{(1,1), (2,1), (3,1), (2,2), (3,2), (3,3)\}$
$P = \{(1,1), (2,2), (3,3)\}$
Counter to Total order: $1 \not{P} 2$ and $2 \not{P} 1$
$I_A \subseteq P$

Problem 21

Let P be a partial order on a set A, and let $B \subseteq A$. Prove that if B contains one of its upper bounds s then s is the least upper bound of B.

Proof

Suppose A, B are sets and P is a partial order on A where $B \subseteq A$
Assume $s \in B$ and s is an upper bound of B.
Need: s is the least upper bound of B. In other words $\forall t$ that are upper bounds sPt
Let t be an upper bound of B. Since s is an upper bound of B that is also in B, there does not exist another upperbound in B that is less than s since the definition of an upper bound is $\{s : \forall a \in B, aPs\}$. So every upperbound of B other than s must related to s by P. In other words sPt thus s is the least upper bound of B.

Problem 22

Let $S \subseteq \mathbb{R}$ be a bounded set and let T be an non-empty subset of S. Prove that

$$inf(s) \leq inf(T) \leq sup(T) \leq sup(S)$$

Proof

Proof of existence Suppose S is a bounded set and T is a non-empty subset of S.
Since S is bounded, $inf(S)$ and $sup(S)$ exist. Since T is a non-empty subset of S, T is also bounded. So $inf(T)$ and $sup(T)$ exist.
Proof of $inf(s) \leq inf(T)$
Let $i_s = inf(S)$ and $i_t = inf(T)$.
Since T is a subset of S, $inf(S)$ is a lower bound of T. So by the definition of infimum $inf(S) \leq inf(T)$.
Proof of $inf(T) \leq sup(T)$
Let $i_t = inf(T)$ and $s_t = sup(T)$.
Since T is a bounded set and is non empty $inf(T) \leq sup(T)$.
Proof of $sup(T) \leq sup(S)$
Let $s_t = sup(T)$ and $s_s = sup(S)$.
Since T is a subset of S, $sup(S)$ is an upperbound of T. So by the definition of supremum $sup(T) \leq sup(S)$.

Problem 23

Let B and C be non-empty sets of real numbers such that $b \leq c$ for all $b \in B$ and $c \in C$. Prove that $sup(B) \leq inf(C)$. **Proof** Suppose B and C are non-empty sets of real numbers such that $b \leq c$ for all $b \in B$ and $c \in C$.
Need: $sup(B) \leq inf(C)$.
Since B is a non-empty set of real numbers, $sup(B)$ exists. Since C is a non-empty set of real numbers, $inf(C)$ exists due to the axiom of completeness.
Let $c \in C$. Since $(\forall b \in B) b \leq c$, c is an upper bound of B. So by definition of supremum $sup(B) \leq c$. Since c is an arbitrary element of C and $sup(B) \leq c$, $sup(B)$ is a lower bound of C. So by definition of infimum $sup(B) \leq inf(C)$.

Problem 24

Let A be an arbitrary nonempty set and let

Problem 25

Let A be the set of all closed subintervals of $[0, 1]$ with positive length. A is a partially ordered by set inclusion. A set $B \subseteq A$ is a collection of intervals, also note $[0, 1] \in A$.

i

Does every nonempty subset B of A have an upper bound?
Yes as for every set B in A, $[0, 1]$ will be an element such that $1 \in A$ and $\forall x \in B, x \subseteq [0, 1]$. So $[0, 1]$ is an upper bound for B.

ii

Does every nonempty subset B of A have a least upper bound?
Yes, as since there exists an upper bound for every nonempty subset B of A, the least upper bound will be the interval s such that all of the upperbounds of B contain s. In other words $sup(B) := s \in A$ such that $\forall x$ that are upper bounds of $B, s \subseteq x$.
Since B is bounded then it has a maximum which is in B. This maximum is an upper bound of B and is a subset of all other upper bounds of B. So the maximum is the least upper bound of B.
Review for Later

iii

Does every nonempty subset B of A have a maximum?

iv

v

vi

Problem 26

Problem 27

Definitions

Power Set

Let A be a set. The power set of A is the set of all subsets of A.
$\mathscr{P}(A) := \{X : X \subseteq A\}$

Cartesian Product

Let A and B be sets. The Cartesian product of A and B is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.
$A \times B := \{(a, b) : a \in A, b \in B\}$

Set Partition

Let A be a set. A set partition of A is a collection of nonempty subsets of A such that every element of A is in exactly one of the subsets.
$P = \{A_1, A_2, \ldots, A_n\}$ is a partition of A if:
$\forall x \in A : \exists i \in \{1, 2, \ldots, n\} : x \in A_i$
$\forall i, j \in \{1, 2, \ldots, n\} : i \neq j \Rightarrow A_i \cap A_j = \emptyset$

Identity Relation

Let A be a set. The identity relation on A is the relation that relates every element of A to itself.
$I_A := \{(a, a) : a \in A\}$

Composition

Let P and Q be relations. $P : A \to B$ and $Q : B \to C$. The composition of P and Q is the relation that relates a to c if there exists b such that aPb and bQc.
$P \circ Q := \{(a, c) : \exists b \in B : aPb \land bQc\}$

Inverse

Let P be a relation from A to B. The inverse of P is the relation that relates b to a if aPb.
$\overleftarrow{P} := \{(b, a) : a \in A, b \in B, aPb\}$

Dom

Let P be a relation from A to B. The domain of P is the set of all elements of A that are related to some element of B.
$dom(P) := \{a \in A : \exists b \in B : aPb\}$

Range

Let P be a relation from A to B. The range of P is the set of all elements of B that are related to some element of A.
$ran(P) := \{b \in B : \exists a \in A : aPb\}$

title