16:960:665 - Homework 3

Pranav Tikkawar

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Problem (13)**.** Assume that $K(\cdot)$ is a complex-valued function defined on $\mathbb{Z}$, and that $K(\cdot)$ is non-negative definite.

1. Prove that $K(\cdot)$ is Hermitian, *i.e.* $K(h) = \overline{K(-h)}$.

Solution: We know that since $K(\cdot)$ is non-negative definite, thus

$$\sum_{j=1}^{n}\sum_{k=1}^{n} a_j \overline{a_k} K(j-k) \geq 0$$

for any complex numbers $a_1, a_2, \ldots, a_n$ and any positive integer n.
Let the matrix Γ be defined as

$$\Gamma_{j,k} = K(j-k)$$

for $1 \leq j, k \leq n$.
Since we know that Γ is non-negative definite, thus

$$a^* \Gamma a \geq 0$$

Then Γ is also Hermitian, which means that

$$\Gamma = \overline{\Gamma}^T$$

Thus by matching the elements of the matrices, we have

$$K(j-k) = \overline{K(k-j)}$$

for all $j, k \in \mathbb{Z}$.
Let $h = j - k$, then we have

$$K(h) = \overline{K(-h)}$$

for all $h \in \mathbb{Z}$ as desired.

2. Let $K_1(\cdot)$ and $K_2(\cdot)$ be the real and imaginary part of $K(\cdot)$, *i.e.* $K(h) = K_1(h) + iK_2(h)$ for all $h \in \mathbb{Z}$. According to Part (a), we know that $K_1(\cdot)$ is even and $K_2(\cdot)$ is odd. For any positive integer n, define the $(2n) \times (2n)$ matrix

$$L^{(n)} = \frac{1}{2}\begin{pmatrix} K_1^{(n)} & -K_2^{(n)} \\ K_2^{(n)} & K_1^{(n)} \end{pmatrix}, \quad \text{where } K_1^{(n)} := [K_1(j-k)]_{j,k=1}^{n} \text{ and } K_2^{(n)} := [K_2(j-k)]_{j,k=1}^{n}.$$

Prove that $L^{(n)}$ is symmetric and non-negative definite. [Hint. Here you need to use the non-negative definiteness of $K(\cdot)$.]

Solution: Let us write $K(\cdot) = K_1(\cdot) + iK_2(\cdot)$ as given and let $x = u + iv$ where $u, v \in \mathbb{R}^n$.
Then let

$$Q = \sum_{j=1}^{n} \sum_{k=1}^{n} x_j \overline{x_k} K(j-k)$$

Expanding $x_j \overline{x_k} = (u_j + iv_j)(u_k - iv_k)$ as $u_j u_k + v_j v_k + i(v_j u_k - u_j v_k)$, we have

$$Q = Q_1 + iQ_2$$

where

$$\begin{aligned} Q_1 &= \sum_{j=1}^{n} \sum_{k=1}^{n} (u_j u_k + v_j v_k) K_1(j-k) + (v_j u_k - u_j v_k) K_2(j-k) \\ Q_2 &= \sum_{j=1}^{n} \sum_{k=1}^{n} (v_j u_k - u_j v_k) K_1(j-k) + (u_j u_k + v_j v_k) K_2(j-k) \end{aligned}$$

This is the same quadratic form as

$$Q = \begin{pmatrix} u' & v' \end{pmatrix} L^{(n)} \begin{pmatrix} u \\ v \end{pmatrix}$$

We can see that $Q \geq 0$ since $K(\cdot)$ is non-negative definite, thus $L^{(n)}$ is also non-negative definite.
Also, since $K_1(\cdot)$ is even and $K_2(\cdot)$ is odd, we have

$$L^{(n)} = (L^{(n)})'$$

Thus $L^{(n)}$ is symmetric and non-negative definite as desired.

3. Let $(Y_1, \ldots, Y_n, Z_1, \ldots, Z_n)'$ be a random vector which has a multivariate normal distribution with mean zero and covariance matrix $L^{(n)}$. Define $W_t = Y_t + iZ_t$ for $1 \leq t \leq n$. Show that the covariance matrix of $(W_1, \ldots, W_n)'$ is given by $K^{(n)} := [K(j-k)]_{j,k=1}^{n}$.

Solution:

4. Apply the Kolmogorov's Existence Theorem to deduce that there exist a bivariate mean zero Gaussian process $(Y_t, Z_t)'$ such that

$$\begin{aligned} \mathbb{E}(Y_{t+h} Y_t) &= \mathbb{E}(Z_{t+h} Z_t) = \tfrac{1}{2} K_1(h) \\ \mathbb{E}(Z_{t+h} Y_t) &= -\mathbb{E}(Y_{t+h} Z_t) = \tfrac{1}{2} K_2(h). \end{aligned}$$

5. Show that $\{X_t = Y_t + iZ_t,\ t \in \mathbb{Z}\}$ is a complex-valued process with autocovariance function $K(\cdot)$.

Problem (14)**.** Consider n frequencies $-\pi < \lambda_1 < \lambda_2 < \cdots < \lambda_n = \pi$.

1. Let $a_1, a_2, \ldots, a_n$ be complex numbers. Prove that if

$$\sum_{j=1}^{n} a_j e^{it\lambda_j} = 0 \quad \text{for all } t \in \mathbb{Z}$$

   then it must hold that $a_1 = a_2 = \cdots = a_n = 0$.

2. Let $A_1, A_2, \ldots A_n$ be complex random variables, and define $X_t = \sum_{j=1}^{n} A_j e^{it\lambda_j}$. Show that the process $\{X_t,\ t \in \mathbb{Z}\}$ is real-valued if and only if $\lambda_j = -\lambda_{n-j}$ and $A_j = \bar{A}_{n-j}$ for $1 \leq j < n$, and A_n is real.

Problem (15)**.** Prove that if $\gamma(\cdot)$ is real, then its spectral distribution $F(\cdot)$ is symmetric in the sense

$$F(\lambda) = F(\pi^-) - F(-\lambda^-), \quad -\pi < \lambda < \pi.$$

Problem (16)**.** Give an expression and a plot for the spectral density of each of the following processes. [Try to plot many more for fun!]

1. MA(1). $X_t = Z_t \pm 0.9 Z_{t-1}$, where $\{Z_t\} \sim \mathrm{WN}(0, 2)$.
2. AR(1). $X_t = \pm 0.9 X_{t-1} + Z_t$, where $\{Z_t\} \sim \mathrm{WN}(0, 3)$.
3. Each of the processes in Problem 7.

Problem (17)**.** Suppose $\gamma(\cdot)$ is a real-valued autocovariance function such that $\gamma(0) > 0$, and the covariance matrix Γ_n is singular for some $n > 1$. Find out the spectral distribution of $\gamma(\cdot)$.