01:640:350H - Homework 7

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October 28, 2024

1. Question 4.1 9 Prove that $det(AB) = det(A)det(B)$ for any $A, B \in M_{2x2}(\mathbb{R})$.

Solution: We can consider the matrices A and B in the following cases: (WLOG)

(a) A is not invertible

(b) A is invertible

Case 1: A is not invertible If A is not invertible, then $det(A) = 0$. Thus $det(A)det(B) = 0$ and by A being non-invertible, AB is also non-invertible due to the fact that A has a rank less than 2 therefore regardless of the rank of B the matrix AB will have a rank less than 2. Thus $det(AB) = 0$.
therefore $det(AB) = det(A)det(B)$.
Case 2: A is invertible If A is invertible, then $det(A) \neq 0$. We know that A^{-1} exists. Thus $ABy = Ay$ for some $x, y \in F$ is equivalent to $By = x$.
We can consider the augemented matrix $[A|I]$ and row reduce by a series of elementary row operation $E_1...E_n$ to $[I|A^{-1}]$.
We can consider the augemented matrix $[AB|I]$ and row reduce by the same series of elementary row operation $E_1...E_n$ to $[B|A^{-1}]$
We also know that for for any row additions it will not change the determinant, for row swaps it will change the determinant by a factor of -1, and for row scaling it will change the determinant by a factor of the scalar.
So for k row swaps and l row scaling we can consider $1 = det(I) = (-1)^k * c_1 * c_2 * ... * c_l * det(A)$
Then we can see that $det(B) = \frac{1}{(-1)^k c_1 * c_2 * ... * c_l} det(AB)$
Which implies that $det(AB) = det(A)det(B)$

Alternatively: We can consider the matrices A and B as arbitrary matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. We can see that

$$\begin{aligned}
det(AB) &= det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) \\
&= det\left(\begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}\right) \\
&= (ae+bg)(cf+dh) - (af+bh)(ce+dg) \\
&= aecf + aedh + bgcf + bhdh - afce - afdg - bghc - bhgd \\
&= aecf + aedh + bgcf + bhdh - afce - afdg - bghc - bhgd \\
&= a(ecf - fdg) + b(gcf - hgd) + c(aed - bhc) + d(bh - af) \\
&= det(A)det(B)
\end{aligned}$$

2. Question 4.1 11 Let $\delta : M_{2x2}(F) \to F$ be a function with the following three properties:

1. δ is a linear function of each row of the matrix when the other row is fixed.

2. if the two rows of A are identical, then $\delta(A) = 0$.
3. $\delta(I) = 1$.

(a) Prove that $\delta(E) = det(E)$ for any elementary matrix E.

(b) Prove that $\delta(EA) = \delta(E)\delta(A)$ for any elementary matrix E and any $A \in M_{2x2}(F)$.

Solution: Part 1: We can consider the elementary matrices E in the following cases:

1. E is a row swap matrix
2. E is a row scaling matrix
3. E is a row addition matrix

Case 1: E is a row swap matrix We can consider the matrix E as $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We can see that

$$\begin{aligned}
\delta(E) &= \delta\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \\
&= \delta\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) + \delta\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right)_{\text{Goes to 0}} \\
&= \delta\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)_{\text{Goes to 0}} + \delta\left(\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\right) \\
&= \delta\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) + \delta\left(\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\right) \\
&= \delta\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) - \delta\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right)_{\text{Goes to 0}} \\
&= -1
\end{aligned}$$

Case 2: E is a row scaling matrix We can consider the matrix E as $E = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$.
We can see that by property 1, $\delta(E) = k\delta\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = k$.
Additionally if we consider the matrix E as $E = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$, we can see that $\delta(E) = k\delta\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = k$.

Case 3: E is a row addition matrix We can consider the matrix E as $E = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$.

We can see that

$$\begin{aligned}\delta(E) &= \delta\left(\begin{bmatrix}1 & 0\\ k & 1\end{bmatrix}\right)\\ &= \delta\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) + \delta\left(\begin{bmatrix}0 & 0\\ k & 1\end{bmatrix}\right)\\ &= 1 + k\delta\left(\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\right)\\ &= 1\end{aligned}$$

Additionally if we consider the matrix E as $E = \begin{bmatrix}1 & k\\ 0 & 1\end{bmatrix}$, we can see that

$$\begin{aligned}\delta(E) &= \delta\left(\begin{bmatrix}1 & k\\ 0 & 1\end{bmatrix}\right)\\ &= \delta\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) + \delta\left(\begin{bmatrix}0 & k\\ 0 & 1\end{bmatrix}\right)\\ &= 1 + k\delta\left(\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\right)\\ &= 1\end{aligned}$$

Thus we can see that $\delta(E) = det(E)$ for any elementary matrix E.

Part 2: We can take an aribtrary matrix $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$ and notice that $\delta(A) = ad - bc = det(A)$.

Since E is an elementary matrix and we know that $det(EA) = det(E)det(A)$, we can see that $\delta(EA) = \delta(E)\delta(A)$. by the fact that $\delta(E) = det(E)$ and $\delta(A) = det(A)$.

3. Question 4.1 12 Let $\delta : M_{2x2}(F) \to F$ be a function with properties from the prior question. Prove that $\delta(A) = det(A)$ for any $A \in M_{2x2}(F)$.

Solution: We can consider an arbitrary matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{aligned}
\delta(A) &= \delta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\
&= \delta\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right) + \delta\left(\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}\right) \\
&= \delta\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) + \delta\left(\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}\right) + \delta\left(\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}\right) + \delta\left(\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}\right) \\
&= ad - bc
\end{aligned}$$

Clealy $\delta(A) = det(A)$ for any $A \in M_{2x2}(F)$.

4. Question 4.2 7 Cofactor Expansion:

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}$$

along the second row.

Solution:

$$det(A) = (-1)^{(3)}(-1)det\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} + (-1)^{(4)}(0)det\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} + (-1)^{(5)}(-3)det\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

$$det(A) = -(-1(-6)) + 0 - (-3(-2)) = -12$$

5. Question 4.2 8 Cofactor Expansion:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{bmatrix}$$

along the third row.

Solution:

$$det(A) = (-1)^{(6)}(-1)det\begin{bmatrix} 0 & 2 \\ 1 & 5 \end{bmatrix} + (-1)^{(7)}(3)det\begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} + (-1)^{(8)}(0)det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$det(A) = (-1(-2)) - 3(5) + 0 = -13$$

6. Question 4.2 14

$$det\left(\begin{bmatrix}2 & 3 & 4\\ 5 & 6 & 0\\ 7 & 0 & 0\end{bmatrix}\right)$$

Solution: We will cofactor expand along the third row.

$$det(A) = (-1)^{(6)}(7)det\begin{bmatrix}3 & 4\\ 6 & 0\end{bmatrix} + (-1)^{(7)}(0)det\begin{bmatrix}2 & 4\\ 5 & 0\end{bmatrix} + (-1)^{(8)}(0)det\begin{bmatrix}2 & 3\\ 5 & 6\end{bmatrix}$$

$$det(A) = -7(-24) - 0 + 0 = 168$$

7. Question 4.2 18

$$det\left(\begin{bmatrix}1 & -2 & 3\\ -1 & 2 & -5\\ 3 & -1 & 2\end{bmatrix}\right)$$

Solution: We will cofactor expand along the first row.

$$det(A) = (-1)^0(1)det\begin{bmatrix}2 & -5\\ -1 & 2\end{bmatrix} + (-1)^1(-2)det\begin{bmatrix}-1 & -5\\ 3 & 2\end{bmatrix} + (-1)^2(3)det\begin{bmatrix}-1 & 2\\ 3 & -1\end{bmatrix}$$

$$det(A) = 1(4-5) + 2(-2+15) + 3(1-6) = -1 + 26 - 15 = 10$$

8. Question 4.2 23 Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.

Solution: Let the upper triangular matrix be

$$\begin{bmatrix}a_{11} & a_{12} & a_{13} & ... & a_{1n}\\ 0 & a_{22} & a_{23} & ... & a_{2n}\\ 0 & 0 & a_{33} & ... & a_{3n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & ... & a_{nn}\end{bmatrix}$$

We can see that by cofactor expansion along the last row, we can see that

$$det(A) = (-1)^{(n^2-n)}0 + (-1)^{(n^2-n+1)}0 + ... + (-1)^{(n^2-1)}a_{nn}det\begin{bmatrix}a_{11} & a_{12} & ... & a_{1n-1}\\ 0 & a_{22} & ... & a_{2n-1}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & ... & a_{n-1n-1}\end{bmatrix}$$

and by continuously cofactor expanding along the last row, We can continue only requiring one non-trivial term. Thus the solution will be

$$\Pi_{i=1}^{n}(-1)^{n^2-1}a_{ii} = a_{11}a_{22}...a_{nn}$$

9. Question 4.3 12 A matrix $Q \in M_{nxn}(\mathbb{C})$ is called orthogonal if $QQ^t = I$. Prove that if Q is orthogonal, then $det(Q) = \pm 1$.

Solution: We can see that $det(QQ^t) = det(I) = 1$. We can also see that $det(QQ^t) = det(Q)det(Q^t) = det(Q)det(Q) = det(Q)^2$. Thus $det(Q)^2 = 1$ and $det(Q) = \pm 1$.

10. Question 4.3 15 Prove that if $A, B \in M_{nxn}(F)$ are similar, then $det(A) = det(B)$.

Solution: We can see that if A and B are similar, then there exists an invertible matrix Q such that $B = Q^{-1}AQ$. We can see that $det(B) = det(Q^{-1}AQ) = det(Q^{-1})det(A)det(Q) = det(A)$. Thus $det(A) = det(B)$.

11. Question 4.3 24 Let $A \in M_{nxn}(F)$ have the form

$$A = \begin{bmatrix} 0 & 0 & ... & 0 & a_1 \\ -1 & 0 & ... & 0 & a_2 \\ 0 & -1 & ... & 0 & a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & ... & -1 & a_n \end{bmatrix}$$

Compute $det(A - tI)$.

Solution: We can do a cofactor expansion along the first row.

$$det(A-tI) = (t)det\left(\begin{bmatrix} t & 0 & ... & a_1 \\ -1 & t & ... & a_2 \\ 0 & -1 & ... & a_3 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & t + a_n \end{bmatrix}\right) + (-1)^{n-1}a_0 det\left(\begin{bmatrix} -1 & t & ... & 0 \\ 0 & -1 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & -1 \end{bmatrix}\right)$$

Clealry this will continue to be a series of t's and a_i's. Thus we can see that

$$det(A - tI) = t(t(t...(t + a_n)... + a_2) + a_1) + a_0$$

$$det(A - tI) = t^n + a_{n-1}t^{n-1} + ... + a_1 t + a_0$$

Additionally notice that this is the matrix for an nth order linear recurrence relation: where the ith element of the vector x when multiplied by A will give the $i+1$th element of the vector x and the "base element" be given by a_0. Thus we can see that the characteristic polynomial of the matrix A is $det(A - tI)$.

12. Question 4.4 2(c) Evaluate the determinant of the matrix

$$A = \begin{bmatrix} 2+i & -1+3i \\ 1-2i & 3-i \end{bmatrix}$$

Solution:

$$det(A) = (2+i)(3-i) - (-1+3i)(1-2i) = 6 - 2i + 3i - i^2 + 1 - 2i - 3i + 6i^2 = 2 - 4i$$

13. Question 4.4 3(c) Evaluate the determinant of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}$$

Along the second column.

Solution:

$$det(A) = (-1)^1(1)det\begin{bmatrix} -1 & -3 \\ 2 & 0 \end{bmatrix} + (-1)^2(0)det\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} + (-1)^3(3)det\begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}$$

$$det(A) = -(6) - 0 - (3(2)) = -12$$

14. Question 4.4 3(e) Evaluate the determinant of the matrix

$$A = \begin{bmatrix} 0 & 1+i & 2 \\ -2i & 0 & 1-i \\ 3 & 4i & 0 \end{bmatrix}$$

Along the third row.

Solution:

$$det(A) = (-1)^6(3)det\begin{bmatrix}1+i & 2 \\ 0 & 1-i\end{bmatrix} + (-1)^7(4i)det\begin{bmatrix}0 & 2 \\ -2i & 1-i\end{bmatrix} + (-1)^8(0)det\begin{bmatrix}0 & 1+i \\ -2i & 0\end{bmatrix}$$

$$det(A) = 3((1-i)(1+i)-0) - 4i(0-2(-2i)) + 0 = 6 + 16 = 22$$