01:640:495 - Lecture 1

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Lecture 1

1. Given 3 (non colinear) points A, B, C in the plane, Find a quadratic polynomial $f(x)$ passes through all three points.
   Define $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$

Solution: Given these three points, we can write the following equations:

$$\begin{aligned} f(x_1) &= y_1 \\ f(x_2) &= y_2 \\ f(x_3) &= y_3 \end{aligned}$$

where $f(x) = ax^2 + bx + c$. Substituting the values of x_1, x_2, x_3 in the above equations, we get:

$$\begin{aligned} ax_1^2 + bx_1 + c &= y_1 \\ ax_2^2 + bx_2 + c &= y_2 \\ ax_3^2 + bx_3 + c &= y_3 \end{aligned}$$

we solve this system by solving the following matrix equation:

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Clearly the matrix is invertible since the points are non-colinear. Thus, we can solve for a, b, c and get the quadratic polynomial $f(x)$.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Replacing the values of a, b, c in $f(x)$, we get the required quadratic polynomial.

We can similarly motivate this by choosing the expression in a way that is aligned w/ the data :

$$f(x) = \theta_1(x - x_2)(x - x_3) + \theta_2(x - x_1)(x - x_3) + \theta_3(x - x_1)(x - x_2)$$

Thus our goal is to find $\theta_1, \theta_2, \theta_3$ such that $f(x_1) = y_1, f(x_2) = y_2, f(x_3) = y_3$. Substituting the values of x_1, x_2, x_3 in the above equation, we get:

$$\begin{aligned} \theta_1(x_1 - x_2)(x_1 - x_3) &= y_1 \\ \theta_2(x_2 - x_1)(x_2 - x_3) &= y_2 \\ \theta_3(x_3 - x_1)(x_3 - x_2) &= y_3 \end{aligned}$$

We can solve this system by solving the following matrix equation:

$$\begin{bmatrix} (x_1 - x_2)(x_1 - x_3) & 0 & 0 \\ 0 & (x_2 - x_1)(x_2 - x_3) & 0 \\ 0 & 0 & (x_3 - x_1)(x_3 - x_2) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Clearly the matrix is invertible and the solution given by:

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} (x_1 - x_2)(x_1 - x_3) & 0 & 0 \\ 0 & (x_2 - x_1)(x_2 - x_3) & 0 \\ 0 & 0 & (x_3 - x_1)(x_3 - x_2) \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{aligned} \theta_1 &= \frac{y_1}{(x_1 - x_2)(x_1 - x_3)} \\ \theta_2 &= \frac{y_2}{(x_2 - x_1)(x_2 - x_3)} \\ \theta_3 &= \frac{y_3}{(x_3 - x_1)(x_3 - x_2)} \end{aligned}$$

Lecture 2

Set is the most important mathematical object.
We then use **functions** to map between sets.
Injections are functions that map distinct elements to distinct elements. AKA one-to-one functions.
Surjections are functions that map to every element in the codomain. AKA onto functions.
Composition of functions is a function that is the result of applying two functions. We must make sure that you can apply the functions in the correct order and the sets match up.

Example. Show that $(x-1)(x-2), x(x-2), x(x-1)$ are linearly independent.

Solution: We can see that they are olinearly independiant if the only solution to the equation $a(x-1)(x-2) + b(x)(x-2) + c(x)(x-1) = 0$ is $a = b = c = 0$.
Expanding the equation, we get:

$$\begin{aligned} a(x^2 - 3x + 2) + b(x^2 - 2x) + c(x^2 - x) &= 0 \\ (a + b + c)x^2 + (-3a - 2b - c)x + 2a &= 0 \end{aligned}$$

We can make this a system of equations/ matrix equation:

$$\begin{bmatrix} 1 & 1 & 1 \\ -3 & -2 & -1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We can see that the matrix is invertible and thus the only solution is $a = b = c = 0$. Thus the functions are linearly independent.

Solution: Similarly we can plug in the values of $x = 0, 1, 2$ to get the following equations:

$$a(-1)(-2) = 0$$
$$b(0)(-2) = 0$$
$$c(0)(-1) = 0$$

We can see that the only solution to this system is $a = b = c = 0$. Thus the functions are linearly independent.

Lecture 4

Inner Product:

$$\langle x, y \rangle = x^T y$$

When taking the inner product of a vector with itself, we get the **norm** of the vector: Positive definite, symmetric, bilinear.
0 only if $x = 0$.
If we are looking for a

$$\langle v - s^*, s \rangle = 0$$

Then we can write this as:

$$\langle b_i, s* \rangle = \langle b_i, v \rangle$$

for all $b_i \in B$ the basis and $s^* \in S$ the solution.
Since s^* in the span of B, we can write:

$$s^* = \sum_{i=1}^{n} \alpha_i b_i$$

Thus we can make a matrix equation:

$$\begin{bmatrix} < b_1, b_1 > & < b_1, b_2 > & ... \\ < b_2, b_1 > & < b_2, b_2 > & ... \\ ... & ... & ... \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ ... \end{bmatrix} = \begin{bmatrix} < b_1, v > \\ < b_2, v > \\ ... \end{bmatrix}$$

Lecture 7

Teqnique: Find Orthogonal projection
Found a matrix P $n \times n$ such that $\pi(v) = Pv$ The idea is $\pi(v) = \sum_{i=1}^{n} \lambda_i b_i$ where b_i are the basis vectors.
we can take P as the matrix of basis inner products. Notice that with choice of distance $\sum(y_i - (\theta_0 + \theta_1 x_i))^2$ we get distance function of

$$d(\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \begin{bmatrix} \theta_0 + \theta_1 x_1 \\ \theta_0 + \theta_1 x_2 \\ \vdots \\ \theta_0 + \theta_1 x_n \end{bmatrix})$$

As θ_0, θ_1 vary, we get a plane in $\mathbb{R}^n$ Solving this is equivalent to finding the orthogonal projection of y onto the span of b_1, b_2. If we have weights we can redifine the inner product as:

$$\langle x, y \rangle = \sum w_i x_i y_i$$

And then we can then define our P as:

$$P = B(B^T B)^{-1} B^T$$

Where $B^T B$ is the matrix of inner products with ordinary dot product as the inner product.