Homework 4: 292H

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Question 1

Compute e^{tA} for the matrix

$$A = \begin{bmatrix} -1 & 2 & 2 \\ -2 & -1 & 1 \\ -2 & -1 & -1 \end{bmatrix}$$

The eigenvalues of the matrix is given by $A - \mu I = \begin{bmatrix} -1-\mu & 2 & 2 \\ -2 & -1-\mu & 1 \\ -2 & -1 & -1-\mu \end{bmatrix}$

The charecteristic polynomial of the matrix is given by

$$(-1-\mu)[(-1-\mu)(-1-\mu)+1] - 2[2(-1-\mu)-2] + 2[2(-1-\mu)+2]$$

$$-\mu^3 - 3\mu^2 - 12\mu - 10$$

$$(-\mu - 1)(\mu^2 + 2\mu + 10)$$

The roots of the charecteristic polynomial are $\mu_1 = -1, \mu_2 = -1-3i, \mu_3 = -1+3i$.

The eigenvector for the eigenvalue $\mu_1 = -1$ is given by solving the equation $(A+I)X = 0$. Here the matrix $[A+I|0]$ is given by

$$\begin{bmatrix} 0 & 2 & 2 & 0 \\ -2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \end{bmatrix}$$

The reduced row echelon form of the matrix is

$$\begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the eigenvector for the eigenvalue $\mu_1 = -1$ is given by

$$\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

The eigenvector for the eigenvalue $\mu_2 = -1-3i$ is given by solving the equation $(A-(-1-3i)I)X = 0$. Here the matrix $[A-(-1-3i)I|0]$ is given by

$$\begin{bmatrix} 3i & 2 & 2 & 0 \\ -2 & 3i & 1 & 0 \\ -2 & -1 & 3i & 0 \end{bmatrix}$$

The reduced row echelon form of the matrix is

$$\begin{bmatrix} 3i & 2 & 2 & 0 \\ 0 & 5i/3 & 1-4i/3 & 0 \\ 0 & -1-4i/3 & 5i/3 & 0 \end{bmatrix} \begin{bmatrix} 3i & 2 & 2 & 0 \\ 0 & 5i/3 & 1-4i/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3i & 2 & 2 & 0 \\ 0 & 1 & -4/5-3i/5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3i & 0 & 18/5+6i/5 & 0 \\ 0 & 1 & -4/5-3i/5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2/5-6i/5 & 0 \\ 0 & 1 & -4/5-3i/5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This leads us to a system of equations

$$\begin{cases} 5x + (2-6i)z = 0 \\ 5y - (4+3i)z = 0 \end{cases}$$

Thus the eigenvector for the eigenvalue $\mu_2 = -1-3i$ is given by

$$\begin{bmatrix} -2+6i \\ 4+3i \\ 5 \end{bmatrix}$$

We can get 2 linearly indpendent general solutions from the real and imaginary parts.

$$e^{tA} = e^{-t}(cos(-3t) + isin(-3t)) \begin{bmatrix} -2+6i \\ 4+3i \\ 5 \end{bmatrix}$$

$$e^{-t} \begin{bmatrix} 2cos(3t) - 6sin(3t) \\ 4cos(3t) - 3sin(3t) \\ 5cos(3t) \end{bmatrix} + e^{-t} \begin{bmatrix} 6cos(3t) - 2sin(3t) \\ 3cos(3t) + 4sin(3t) \\ 5sin(3t) \end{bmatrix}$$

When we make e^{tA}, we can use the real and imaginary solution as different linearly indpendent solutions. Thus the solution to the system of differential equations is given by

$$e^{tA} = e^{-t} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + e^{-t} \begin{bmatrix} 2cos(3t) - 6sin(3t) \\ 4cos(3t) - 3sin(3t) \\ 5cos(3t) \end{bmatrix} + e^{-t} \begin{bmatrix} 6cos(3t) - 2sin(3t) \\ 3cos(3t) + 4sin(3t) \\ 5sin(3t) \end{bmatrix}$$

a

Yes it does converge to zero. It does so at a rate of e^{-t} as it is the term with most "power".

b

The intial points that would lead to a spiral would be any intial points that are in the form

$$\begin{bmatrix} 0 \\ a \\ b \end{bmatrix}$$

for $a, b \in \mathbb{R}$ and $a, b \neq 0$.

c

It would be $\frac{2\pi}{3}$ as the period of the solution since there is a $t3$ in the argument of the cosine and sine functions.

Question 2

$$A = \begin{bmatrix} -4 & 2 & 7 \\ -1 & -1 & -1 \\ -1 & 2 & 4 \end{bmatrix}$$

The eigenvalues of the matrix are

$$\mu = 1, 1, -3$$

The eigenvector for the eigenvalue 1 is

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

The eigenvector for the eigenvalue -3 is

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

To get get e^{tA} we need to use the JC form of the matrix.

$$P = \begin{bmatrix} 1 & ? & 2 \\ -1 & ? & 1 \\ 1 & ? & 0 \end{bmatrix}, U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

To find the 2nd column of P, we can find $(A - 1I)w = v$ where v is the eigenvector for the eigenvalue 1 and w is the generalized eigenvector. Thus we need a augmented matrix $[A - I|v]$.

$$\begin{bmatrix} -5 & 2 & 7 & 1 \\ -1 & -2 & -1 & -1 \\ -1 & 2 & 3 & 1 \end{bmatrix}$$

The reduced row echelon form of the matrix is

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the generalized eigenvector is given by the system

$$\begin{cases} x - z = 0 \\ y + z = 1/2 \end{cases}$$

Thus the generalized eigenvector is given by

$$\begin{bmatrix} 0 \\ 1/2 \\ 0 \end{bmatrix}$$

Thus the matrix P is given by

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1/2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Thus the matrix P^{-1} is given by

$$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 2 & 3 \\ 0.5 & 0 & -0.5 \end{bmatrix}$$

The matrix e^{tU} is given by $e^{tD}e^{tN}$

$$\begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix} (tN + I)$$

$$\begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix}$$

Thus the matrix e^{tA} is given by

$$Pe^{tU}P^{-1}$$

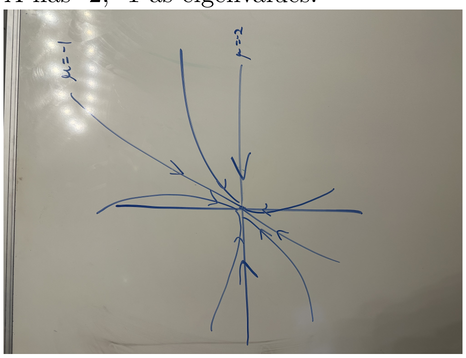
$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1/2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 2 & 3 \\ 0.5 & 0 & -0.5 \end{bmatrix}$$

Question 3

Phase Portrait of the system of differential equations: $v_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

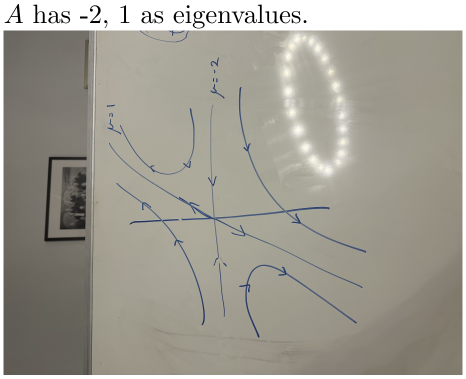
a

A has -2, -1 as eigenvalues.



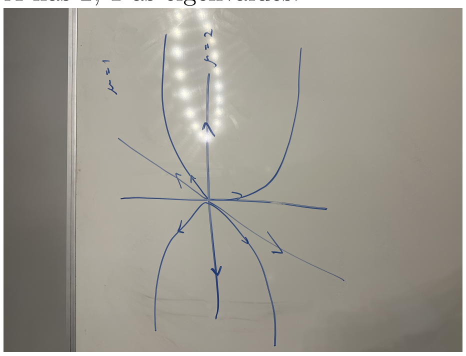
b

A has -2, 1 as eigenvalues.



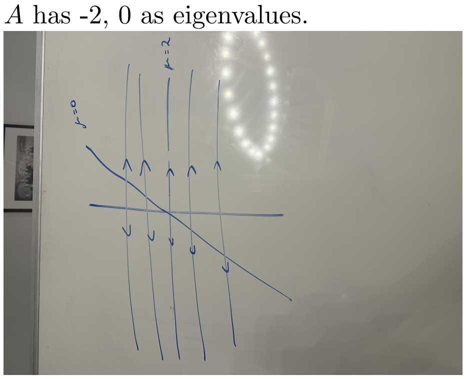
c

A has 2, 1 as eigenvalues.



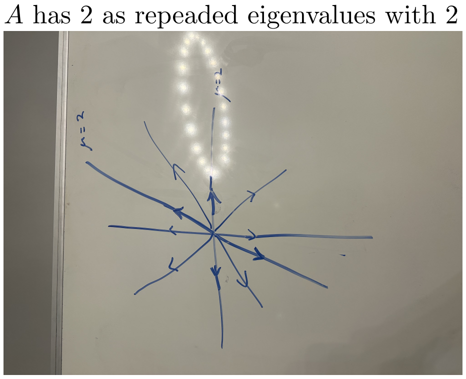
d

A has -2, 0 as eigenvalues.



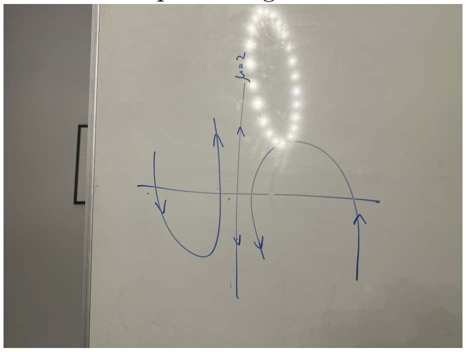
e

A has 2 as repeaded eigenvalues with 2 LI eigenvectors.



f

A has 2 as repeated eigenvalues with 1 LI eigenvector.



Question 4

Solve $X'(t) = \begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix} X(t) + \begin{bmatrix} te^{-2t} \\ e^{-2t} \end{bmatrix}$ With initial condition $X(0) = X_0$ Using Duhamels formula we can see the solution will be in the form of

$$X(t) = e^{(t-t_0)A}X(0) + \int_{t_0}^{t} e^{(t-s)A}g(s)ds$$

To get $e^{(t-t_0)A}$ we need to find the eigenvalues of the matrix $A = \begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}$. The charecteristic polynomial of the matrix is given by

$$\begin{vmatrix} -\lambda & -1 \\ -2 & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = 0$$

Thus the eigenvalues of the matrix are $\mu_1 = -1, \mu_2 = 2$. The eigenvector for the eigenvalue $\mu_1 = -1$ is given by solving the equation $(A + I)X = 0$. Here the matrix $[A + I|0]$ is given by

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$$

The reduced row echelon form of the matrix is

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Thus the eigenvector for the eigenvalue $\mu_1 = -1$ is given by

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenvector for the eigenvalue $\mu_2 = 2$ is given by solving the equation $(A-2I)X = 0$. Here the matrix $[A-2I|0]$ is given by

$$\begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix}$$

The reduced row echelon form of the matrix is

$$\begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$$

Thus the eigenvector for the eigenvalue $\mu_2 = 2$ is given by

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Thus the matrix $e^{(t-t_0)A}$ is given by

$$\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{-(t-t_0)} & 0 \\ 0 & e^{2(t-t_0)} \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{bmatrix}$$

$$\begin{bmatrix} e^{t_0-t} & -e^{2(t-t_0)} \\ e^{t_0-t} & 2e^{2(t-t_0)} \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{bmatrix}$$

$$\begin{bmatrix} \frac{2e^{-(t-t_0)}}{3} + \frac{e^{2(t-t_0)}}{3} & \frac{e^{-(t-t_0)}}{3} - \frac{e^{2(t-t_0)}}{3} \\ \frac{2e^{-(t-t_0)}}{3} - \frac{2e^{2(t-t_0)}}{3} & \frac{e^{-(t-t_0)}}{3} + \frac{2e^{2(t-t_0)}}{3} \end{bmatrix}$$

Now we need to find the integral of $e^{(t-s)A}g(s)ds$.

$$\int_0^t \begin{bmatrix} \frac{2e^{-(t-s)}}{3} + \frac{e^{2(t-s)}}{3} & \frac{e^{-(t-s)}}{3} - \frac{e^{2(t-s)}}{3} \\ \frac{2e^{-(t-s)}}{3} - \frac{2e^{2(t-s)}}{3} & \frac{e^{-(t-s)}}{3} + \frac{2e^{2(t-s)}}{3} \end{bmatrix} \begin{bmatrix} se^{-2s} \\ e^{-2s} \end{bmatrix} ds$$

The multiplication is:

$$\frac{1}{3}\int_0^t \begin{bmatrix} se^{-2s}(\frac{2e^{-(t-s)}}{3} + \frac{e^{2(t-s)}}{3}) + e^{-2s}(\frac{2e^{-(t-s)}}{3} - \frac{2e^{2(t-s)}}{3}) \\ se^{-2s}(\frac{e^{-(t-s)}}{3} - \frac{e^{2(t-s)}}{3}) + e^{-2s}(\frac{e^{-(t-s)}}{3} + \frac{2e^{2(t-s)}}{3}) \end{bmatrix} ds$$

We can split this integral into its components. The first component integrates to:

$$-\frac{1}{144}e^{-2t}[7e^{4t} - 64e^t + 36t + 57]$$

The second component integrates to:

$$\frac{1}{144}e^{-2t}[7e^{4t} - 32e^t + 12t - 39]$$

Thus our final solution is given by

$$X(t) = \begin{bmatrix} \frac{2e^{-(t)}}{3} + \frac{e^{2(t)}}{3} & \frac{e^{-(t)}}{3} - \frac{e^{2(t)}}{3} \\ \frac{2e^{-(t)}}{3} - \frac{2e^{2(t)}}{3} & \frac{e^{-(t)}}{3} + \frac{2e^{2(t)}}{3} \end{bmatrix} X_0 + \begin{bmatrix} -\frac{1}{144}e^{-2t}[7e^{4t} - 64e^t + 36t + 57] \\ \frac{1}{144}e^{-2t}[7e^{4t} - 32e^t + 12t - 39] \end{bmatrix}$$

Question 5

We can write the sum of trig functions a product of trig functions.
Using the two identies:

$$cos(a-b) - cos(a+b) = 2sin(a)sin(b)$$

$$sin(a+b) + sin(a-b) = 2cos(a)sin(b)$$

a

$$cos(5t) - cos(3t) = 2sin(4t)sin(t)$$

b

$$cos(5t) + cos(4t) = 2cos(4.5t)cos(0.5t)$$

c

$$sin(5t) - sin(2t) = 2cos(3.5t)sin(1.5t)$$

d

$$sin(6t) - sin(3t) = 2cos(4.5t)sin(1.5t)$$