Problem 1 — Merton's Jump–Diffusion Process

Write a program to implement Merton's jump–diffusion process. Suggested parameters:

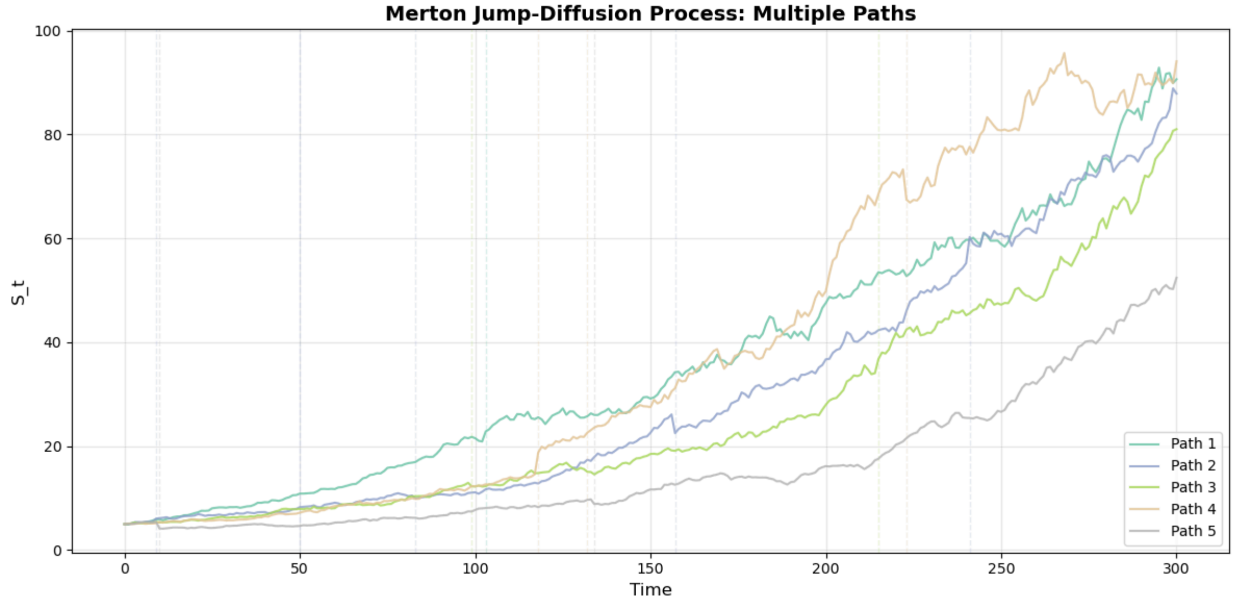
- $T = 300$, $dt = 1$, $\lambda = 0.01$, $S0 = 5$, $\mu = 0.01$, $\sigma = 0.02$

Y may adjust parameters if the plot isn't informative. Report:

- Plots of several generated paths
- The times at which jumps occur (per path)

In [24]:
```
import numpy as np
import matplotlib.pyplot as plt

# Parameters
T = 300          # total time
dt = 1.0         # time step
lam = 0.01       # Poisson jump intensity λ
S0 = 5.0
mu = 0.01
sigma = 0.02

# jump size distribution parameters
mu_J = -0.1      # mean of log(1 + J)
sigma_J = 0.2    # std of log(1 + J)

N_steps = int(T / dt)
t_grid = np.arange(N_steps + 1)

# Number of paths to generate
n_paths = 5

# Store all paths and their jump times
all_paths = []
all_jump_times = []

rng = np.random.default_rng(123)

# Generate multiple paths
for path_num in range(n_paths):
    S = np.zeros(N_steps + 1)
    S[0] = S0
    jump_times = []

    for k in range(N_steps):
        # Brownian increment
        dW = np.sqrt(dt) * rng.normal()

        # Poisson jumps: number of jumps in [t, t+dt]
        N_t = rng.poisson(lam * dt)

        # total jump factor over this step
        if N_t > 0:
            # log(1+J) ~ N(mu_J, sigma_J^2)
            Y = rng.normal(mu_J, sigma_J, size=N_t)
```

```
            J_factor = np.exp(Y).prod()
            jump_times.append(t_grid[k+1])
        else:
            J_factor = 1.0

        # Merton SDE step (Euler)
        S[k+1] = S[k] * np.exp((mu - 0.5 * sigma**2) * dt + sigma * dW) * J_factor

    all_paths.append(S)
    all_jump_times.append(jump_times)

# Plot all paths
fig, ax = plt.subplots(figsize=(12, 6))

colors = plt.cm.Set2(np.linspace(0, 1, n_paths))

for path_num in range(n_paths):
    ax.plot(t_grid, all_paths[path_num], label=f"Path {path_num + 1}",
            color=colors[path_num], alpha=0.8, linewidth=1.5)

    # Mark jump times for this path with subtle vertical lines
    for jt in all_jump_times[path_num]:
        ax.axvline(jt, color=colors[path_num], alpha=0.2, linestyle='--', linewidth=1)

ax.set_xlabel("Time", fontsize=12)
ax.set_ylabel("S_t", fontsize=12)
ax.set_title("Merton Jump-Diffusion Process: Multiple Paths", fontsize=14, fontweight=
ax.legend(loc='best', fontsize=10)
ax.grid(alpha=0.3)
plt.tight_layout()
plt.show()

# Print jump times for each path
print("=" * 60)
print("Jump Times by Path")
print("=" * 60)
for path_num in range(n_paths):
    n_jumps = len(all_jump_times[path_num])
    print(f"Path {path_num + 1}: {n_jumps} jump(s) at times {all_jump_times[path_num]}

print("=" * 60)
print(f"Expected number of jumps: λ × T = {lam} × {T} = {lam * T:.1f}")
print(f"Actual average across {n_paths} paths: {np.mean([len(jt) for jt in all_jump_ti
print("=" * 60)
```




```
============================================================
Jump Times by Path
============================================================
Path 1: 1 jump(s) at times [103]
Path 2: 5 jump(s) at times [9, 50, 83, 157, 241]
Path 3: 2 jump(s) at times [99, 215]
Path 4: 3 jump(s) at times [118, 132, 223]
Path 5: 2 jump(s) at times [10, 134]
============================================================
Expected number of jumps: λ × T = 0.01 × 300 = 3.0
Actual average across 5 paths: 2.60
============================================================
```

Problem 2 — MLE for Uniform[0, θ], Bootstrap, and Probabilities

Suppose X_1, ..., X_50 are i.i.d. Uniform[0, θ].

(a) Find the MLE for θ and compute the distribution of the MLE.

(a) MLE for θ and its distribution

Assume $X_1, \ldots, X_n \overset{iid}{\sim} \mathrm{Uniform}[0, \theta]$. The joint density of the sample is

$$f(x_1, \ldots, x_n; \theta) = \prod_{i=1}^{n} \frac{1}{\theta} \mathbf{1}_{[0,\theta]}(x_i) = \theta^{-n} \mathbf{1}\{0 \le x_{(1)}, x_{(n)} \le \theta\},$$

where $x_{(1)} = \min_i x_i$ and $x_{(n)} = \max_i x_i$.

For fixed observed data, this likelihood is

- zero if $\theta < x_{(n)}$, because at least one observation would lie outside $[0, \theta]$, and
- equal to θ^{-n} for all $\theta \ge x_{(n)}$.

On the interval $[x_{(n)}, \infty)$, the function θ^{-n} is strictly decreasing in θ. Therefore, the likelihood is maximized by choosing the smallest admissible value of θ, namely $\theta = x_{(n)}$. Hence the maximum likelihood estimator is

$$\hat{\theta} = X_{(n)} = \max\{X_1, \dots, X_n\}.$$

To find the distribution of $\hat{\theta}$, note that

$$P(\hat{\theta} \le x) = P(X_{(n)} \le x) = P(X_1 \le x, \dots, X_n \le x) = \prod_{i=1}^{n} P(X_i \le x) = \left(\frac{x}{\theta}\right)^n, \quad 0 \le x \le$$

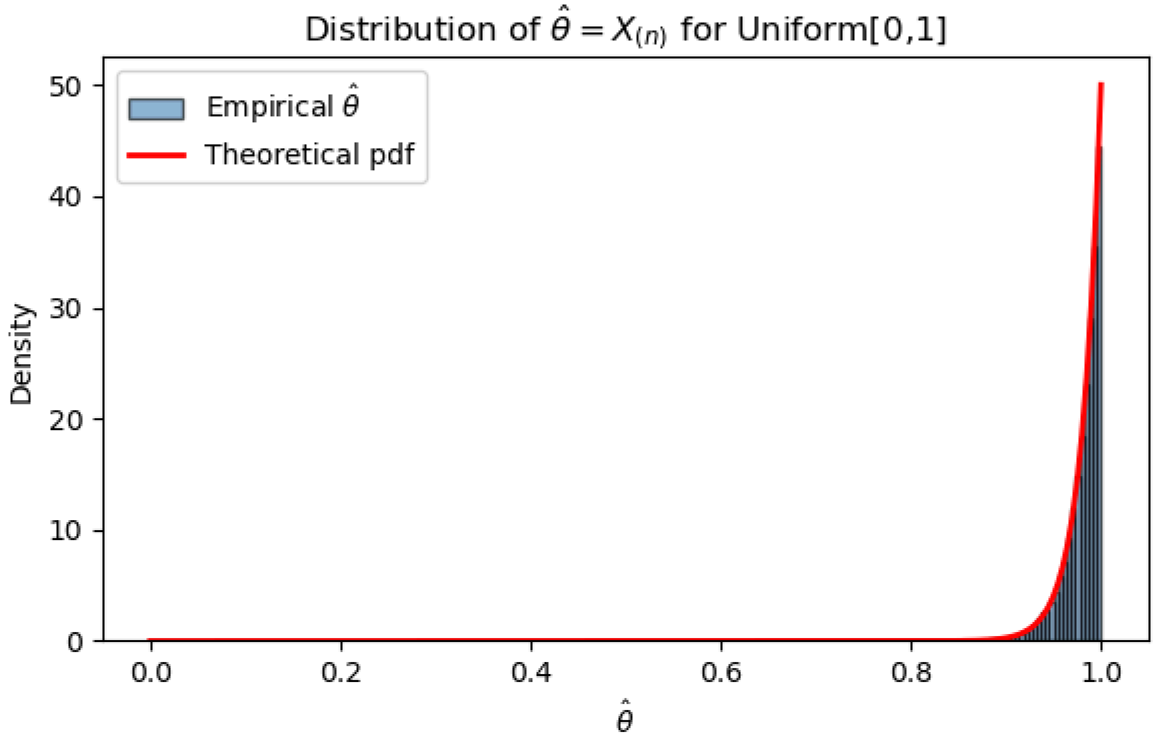
since each X_i is Uniform $[0, \theta]$ and $P(X_i \le x) = x/\theta$ for $x \in [0, \theta]$.

Differentiating this cdf with respect to x gives the pdf of $\hat{\theta}$:

$$f_{\hat{\theta}}(x) = \frac{d}{dx}\left(\frac{x}{\theta}\right)^n = \frac{nx^{n-1}}{\theta^n}, \quad 0 \le x \le \theta.$$

Equivalently, the scaled estimator $\hat{\theta}/\theta$ has a $\mathrm{Beta}(n, 1)$ distribution.

In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import beta, uniform

# Settings
np.random.seed(42)
theta_true = 1.0
n = 50
n_rep = 10000  # number of Monte Carlo replications

# Simulate MLEs
theta_hats = np.empty(n_rep)
for r in range(n_rep):
    sample = np.random.uniform(0, theta_true, size=n)
    theta_hats[r] = np.max(sample)

plt.figure(figsize=(6, 4))
plt.hist(theta_hats, bins=40, density=True, alpha=0.6, color='steelblue', edgecolor='k

x_grid = np.linspace(0, theta_true, 200)
f_theta_hat = n * x_grid**(n - 1) / theta_true**n
plt.plot(x_grid, f_theta_hat, 'r-', lw=2, label='Theoretical pdf')

plt.xlabel(r'$\hat\theta$')
plt.ylabel('Density')
plt.title(r'Distribution of $\hat\theta = X_{(n)}$ for Uniform[0,1]')
plt.legend()
plt.tight_layout()
plt.show()
```

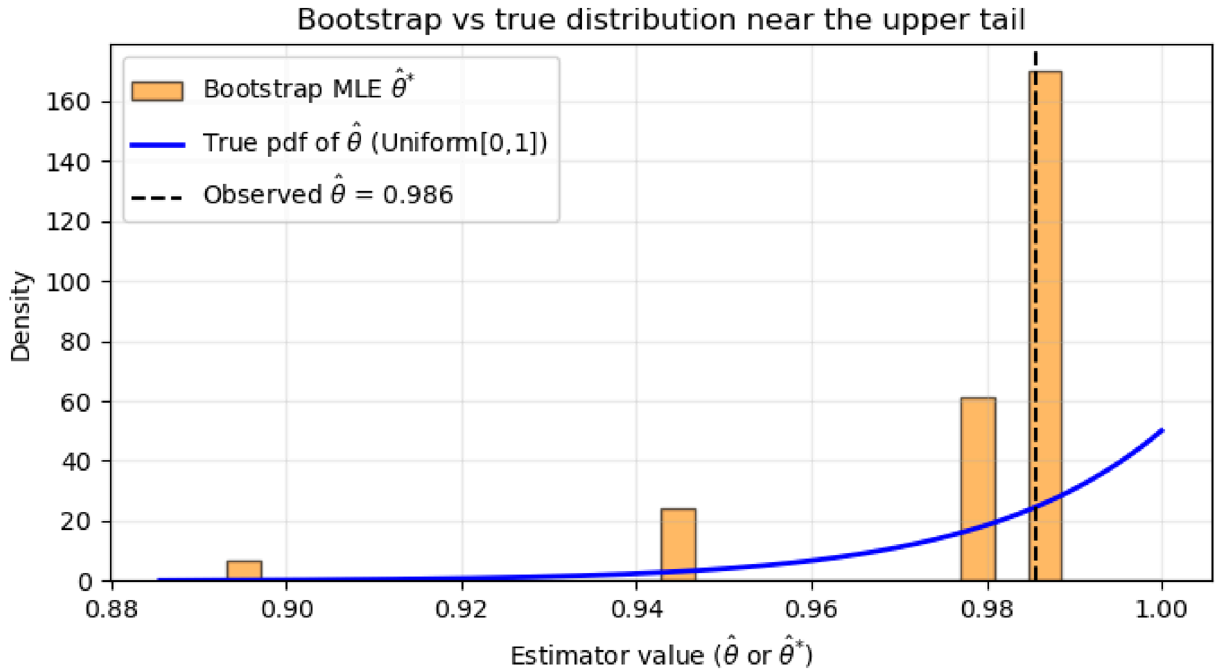


(b) Generate your own observations $x_1, \ldots, x_n$ (with only one maximum and $\theta = 1$), and based on them plot a histogram of the bootstrap samples (N = 1000). Compare the distribution of the MLE and this histogram.

In [20]:
```
import numpy as np
import matplotlib.pyplot as plt

np.random.seed(123)

theta_true = 1.0
n = 50
N_boot = 1000

x = np.random.uniform(0, theta_true, size=n)
theta_hat = np.max(x)

theta_boot = np.empty(N_boot)
for b in range(N_boot):
    idx = np.random.randint(0, n, size=n)
    x_star = x[idx]
    theta_boot[b] = np.max(x_star)

p_equal_empirical = np.mean(np.isclose(theta_boot, theta_hat))
p_equal_theoretical = 1 - (1 - 1/n)**n

print(f"Observed MLE (theta_hat): {theta_hat:.4f}")
print(f"Empirical P(theta* = theta_hat | data): {p_equal_empirical:.4f}")
print(f"Theoretical P(theta* = theta_hat | data): {p_equal_theoretical:.4f}")

fig, ax = plt.subplots(figsize=(7, 4))

x_min_zoom = max(0, theta_hat - 0.1)
x_max_zoom = 1.0
```

```
ax.hist(
    theta_boot,
    bins=30,
    range=(x_min_zoom, x_max_zoom),
    density=True,
    alpha=0.6,
    color='darkorange',
    edgecolor='k',
    label=r'Bootstrap MLE $\hat\theta^{*}$'
)

x_grid_zoom = np.linspace(x_min_zoom, x_max_zoom, 400)
f_true_zoom = n * x_grid_zoom**(n - 1)
ax.plot(
    x_grid_zoom,
    f_true_zoom,
    'b-',
    lw=2,
    label=r'True pdf of $\hat\theta$ (Uniform[0,1])'
)

ax.axvline(
    theta_hat,
    color='k',
    linestyle='--',
    lw=1.5,
    label=rf'Observed $\hat\theta$ = {theta_hat:.3f}'
)

ax.set_xlabel(r'Estimator value ($\hat\theta$ or $\hat\theta^{*}$)')
ax.set_ylabel('Density')
ax.set_title('Bootstrap vs true distribution near the upper tail')
ax.legend(loc='upper left')
ax.grid(alpha=0.25)
fig.tight_layout()
plt.show()
```

```
Observed MLE (theta_hat): 0.9856
Empirical P(theta* = theta_hat | data): 0.6360
Theoretical P(theta* = theta_hat | data): 0.6358
```



(c) This example shows that bootstrap does poorly. In fact, P(θ̂ = 1 | θ = 1) = 0 but P(θ* = θ | data) ≈ 0.636. Prove these two statements.

Hint for (c): P(θ* = θ | data) = 1 − (1 − 1/n)^n. For illustration, use n = 50.

Under the true model $X_1, \ldots, X_n \overset{iid}{\sim} \text{Uniform}[0,1]$, the distribution is continuous, so for any fixed point a we have $P(X_i = a) = 0$.
In particular, the MLE is $\hat{\theta} = \max_i X_i$, and

$$P(\hat{\theta} = 1) = P(\max_i X_i = 1) = 0,$$

because this would require at least one X_i to be exactly 1, an event of probability 0.

Now condition on an observed sample $\{x_1, \ldots, x_n\}$ with a unique maximum $x_{(n)}$.
In a nonparametric bootstrap sample of size n, each resampled observation is drawn from $\{x_1, \ldots, x_n\}$ with probability $1/n$ for each point, so the probability that a single draw is *not* the maximum $x_{(n)}$ is $1 - 1/n$.
The probability that none of the n bootstrap draws equals $x_{(n)}$ is therefore $(1 - 1/n)^n$, hence

$$P(\hat{\theta}^* = \hat{\theta} \mid \text{data}) = P(\text{bootstrap sample contains } x_{(n)} \text{ at least once}) = 1 - (1 - 1/n)^n.$$

For $n = 50$, this gives

$$P(\hat{\theta}^* = \hat{\theta} \mid \text{data}) = 1 - (49/50)^{50} \approx 0.636.$$

Thus, under the true continuous model, $P(\hat{\theta} = 1) = 0$, but under the bootstrap (conditional on the data) the probability that the bootstrap MLE equals the observed MLE is about 0.636.
This mismatch near the boundary shows that the nonparametric bootstrap provides a poor approximation to the sampling distribution of the MLE for the Uniform$[0, \theta]$ endpoint.