01:640:350H - Homework 8

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1. (10 points) Sec. 5.1 Question 4(c) For each of the following matrices:

   (a) Determine the eigenvalues of the matrix.

   (b) For each eigenvalue, determine the eigenvectors.

   (c) If possible find a basis for F^n consisting of eigenvectors of the matrix.

   (d) If successful in finding such a basis determine and invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

$$A = \begin{bmatrix} i & 1 \\ 2 & -i \end{bmatrix}, \qquad \text{for } F = C$$

Solution: Part (a)
The characteristic polynomial of A is given by

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} i-\lambda & 1 \\ 2 & -i-\lambda \end{vmatrix} \\ &= (i-\lambda)(-i-\lambda) - 2 \\ &= \lambda^2 + 1 - 2 \\ &= \lambda^2 - 1 \\ &= (\lambda - 1)(\lambda + 1) \end{aligned}$$

So the eigenvalues of A are $\lambda = 1, -1$.
Part (b)
For $\lambda = 1$, we have

$$A - I = \begin{bmatrix} i-1 & 1 \\ 2 & -i-1 \end{bmatrix}$$

$$\begin{bmatrix} i-1 & 1 \\ 2 & -i-1 \end{bmatrix} \xrightarrow{(-i-1)r_1 + r_2 \to r_2} \begin{bmatrix} i-1 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{i-1} r_1 \to r_1} \begin{bmatrix} 1 & \frac{-i+1}{2} \\ 0 & 0 \end{bmatrix}$$

We can see that vectors of the form of $\begin{bmatrix} i-1 \\ 2 \end{bmatrix}$ are eigenvectors of A corresponding to $\lambda = 1$.
We can call the eigenspace of $\lambda = 1$ as E_1.
For $\lambda = -1$, we have

$$A + I = \begin{bmatrix} i+1 & 1 \\ 2 & -i+1 \end{bmatrix}$$

$$\begin{bmatrix} i+1 & 1 \\ 2 & -i+1 \end{bmatrix} \xrightarrow{(-i+1)r_1 + r_2 \to r_2} \begin{bmatrix} i+1 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{i+1} r_1 \to r_1} \begin{bmatrix} 1 & \frac{-i-1}{2} \\ 0 & 0 \end{bmatrix}$$

$\begin{bmatrix} i+1 \\ 2 \end{bmatrix}$ are eigenvectors of A corresponding to $\lambda = -1$.
We can call the eigenspace of $\lambda = -1$ as E_{-1} as.
Part (c)
By the theorems in chapter 5 we can see that the sets E_1 and E_{-1} are lineanrly independent sets, thus $E_1 \cup E_{-1}$ is also linearly independent.
Since the largest linearly independent combination of vectors in the set $E_1 \cup E_{-1}$ is of size 2, and C^2 has dimension 2, we can see that the eigenvectors of A are a basis for C^2.
Thus the basis is $\left\{ \begin{bmatrix} i-1 \\ 2 \end{bmatrix}, \begin{bmatrix} i+1 \\ 2 \end{bmatrix} \right\}$
Part (d)
Let $Q = \begin{bmatrix} i-1 & i+1 \\ 2 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
We can see that $Q^{-1}AQ = D$. Obviosuly Q is invertible since $\det(Q) = 2i - 2 - 2i - 2 = -4 \neq 0$. Thus we have found the required Q and D.

2. (10 points) Sec. 5.1 Question 4(d)

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{bmatrix}, \qquad \text{for } F = R$$

Solution: The characteristic polynomial of A is given by

$$\begin{aligned} det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 0 & -1 \\ 4 & 1-\lambda & -4 \\ 2 & 0 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda) \begin{vmatrix} 2-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda)((2-\lambda)(-1-\lambda)+2) \\ &= -\lambda(1-\lambda)^2 \end{aligned}$$

Thus the eigenvalues of A are $\lambda = 0, 1$. Where 1 has a multiplicity of 2
Part (b)

For $\lambda = 0$, we have

$$A - 0I = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{r_2 - 2r_1 \to r_2, r_3 - r_1 \to r_3} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

We can see that the eigenvectors of A corresponding to $\lambda = 0$ are of the form $\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$.

We can call the eigenspace of $\lambda = 0$ as E_0.
For $\lambda = 1$, we have

$$A - I = \begin{bmatrix} 1 & 0 & -1 \\ 4 & 0 & -4 \\ 2 & 0 & -2 \end{bmatrix}$$

$$\xrightarrow{r_2 - 4r_1 \to r_2, r_3 - 2r_1 \to r_3} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We can see that the eigenvectors of A corresponding to $\lambda = 1$ are of the form $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

We can call the eigenspace of $\lambda = 1$ as E_1.

Part (c)

By the theorems in chapter 5 we can see that the sets E_0 and E_1 are lineanrly independent sets, thus $E_0 \cup E_1$ is also linearly independent.
Since the largest linearly independent combination of vectors in the set $E_0 \cup E_1$ is of size 3, and R^3 has dimension 3, we can see that the eigenvectors of A are a basis for R^3.

Thus the basis is $\left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Part (d)

Let $Q = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We can see that $Q^{-1}AQ = D$. Obviosuly Q is invertible since $\det(Q) = 1(0-1) - 1(4-0) + 0 = -4 \neq 0$. Thus we have found the required Q and D.

3. (10 points) Sec. 5.1 Question 5(f) For each linear operator T on V fin the eigenvalues of T and an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

$$V = P_3(R), \quad T(f(x)) = f(x) + f(2)x$$

Solution: For T we can say that $T = L_A$ where A is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 4 & 8 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The characteristic polynomial of A is given by

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 1 & 3-\lambda & 4 & 8 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix}$$
$$= (3-\lambda) \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} \qquad = (3-\lambda)(1-\lambda)^3$$

Thus the eigenvalues of T are $\lambda = 3, 1$, with 1 having a multiplicity of 3.
For $\lambda = 1$ we have

$$A - I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We can see that the eigenvectors of A corresponding to $\lambda = 1$ are of the form $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -8 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.
We can call this eigenspace as E_1.
For $\lambda = 3$ we have

$$A - 3I = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & 0 & 4 & 8 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$\xrightarrow{r_2+\frac{1}{2}r_1+2r_3+4r_4\to r_2}\begin{bmatrix}-2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -2 & 0\\ 0 & 0 & 0 & -2\end{bmatrix}$$

We can see that the eigenvectors of A corresponding to $\lambda = 3$ are of the form $\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$

We can call this eigenspace as E_3.
By the theorems in chapter 5 we can see that the sets E_1 and E_3 are lineanrly independent sets, thus $E_1 \cup E_3$ is also linearly independent.

Solution: Continued:
Since the largest linearly independent combination of vectors in the set $E_1 \cup E_3$ is of size 4, and $P_3(R)$ has dimension 4, we can see that eigenvectors of T is a basis for $P_3(R)$.

Thus we can let $\beta = \left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-4\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-8\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right\}$

We can see that $[T]_\beta$ is a diagonal matrix.

$$[T]_\beta = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 3\end{bmatrix}$$

4. (10 points) Sec. 5.1 Question 9(a) Prove that a linear operator T on a finite dimensional vector space is invertable iff 0 is not an eigenvalue of T.

Solution: Assume T is a linear operator on a finite dimensional vector space. Let us call T a function from V to V.
Proof of $\implies$:
Suppose T is invertible.
Need to show that 0 is not an eigenvalue of T.
Note that since T is a linear operator on a finite dimensional vector space, we can take $T = L_A$ for some matrix A.

Suppose w is an eigenvector of T corresponding to the eigenvalue λ.
Then we have $T(w) = \lambda w$.
Notice that we have v be non zero and we have that for all $w \in V$ we have $T(w) = 0$ iff $w = 0$.
Thus must have that $\lambda \neq 0$. as desired.
Proof of $\Longleftarrow$:
Suppose 0 is not an eigenvalue of T.
Need to show that T is invertible.
Since we know that 0 is not an eigenvalue of T, we know that there is no non zero vector v such that $T(v) = 0$.
This means that T is injective.
This also implies that T is surjective since T is a linear operator on a finite dimensional vector space.
Thus T is bijective and thus invertible.

5. (10 points) Sec. 5.1 Question 9(b) Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T iff λ^{-1} is an eigenvalue of T^{-1}.

Solution: Since T is the inverse of T^{-1}, we only need to consider the forward direction.
Suppose λ is an eigenvalue of T.
Then we have $T(v) = \lambda v$ for some non zero vector v.
We can apply T^{-1} to both sides to get $T^{-1}(T(v)) = T^{-1}(\lambda v)$.
This implies that $v = \lambda T^{-1}(v)$.
Thus we have that $\lambda^{-1}v = T^{-1}(v)$ Thus λ^{-1} is an eigenvalue of T^{-1}.

6. (10 points) Sec. 5.1 Question 10 Prove that the eigenvalue of an upper triangular matrix M are the diagonal entries of M.

Solution: We can prove this by induction for any $n \times n$.
For a base case of 1×1 matrix, we have that the only eigenvalue is the diagonal entry.
Suppose that the statement is true for all $n \times n$ matrices.
Let M be an $(n + 1) \times (n + 1)$ upper triangular matrix.
Let λ be an eigenvalue of M.
Then we have that $det(M - \lambda I) = 0$.
We can expand this determinant along the last row to get

$$(m_{n+1,n+1} - \lambda)det(M_{n\times n} - \lambda I) = 0$$

Since $M_{n\times n} - \lambda I$ is an upper triangular matrix, we can see that the eigenvalues of $M_{n\times n}$ are the diagonal entries of $M_{n\times n}$.
Thus the eigenvalues of M are the diagonal entries of M.

7. (10 points) Sec. 5.2 Question 2(d) For each of the following matrices $A \in M_{n\times n}(R)$ test A for diagonalizability and if A is diagonalizable find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

$$A = \begin{bmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{bmatrix}$$

Solution: First we can find the characteristic polynomial of A.

$$\begin{aligned} det(A - \lambda I) &= \begin{vmatrix} 7-\lambda & -4 & 0 \\ 8 & -5-\lambda & 0 \\ 6 & -6 & 3-\lambda \end{vmatrix} \\ &= (3-\lambda)\begin{vmatrix} 7-\lambda & -4 \\ 8 & -5-\lambda \end{vmatrix} \\ &= (3-\lambda)((7-\lambda)(-5-\lambda)+32) \\ &= (3-\lambda)(\lambda^2 - 2\lambda - 3) \\ &= (3-\lambda)^2(-\lambda - 1) \end{aligned}$$

Thus the eigenvalues of A are $\lambda = 3, -1$ with 3 having a multiplicity of 2.
For $\lambda = 3$ we have

$$\begin{aligned} A - 3I &= \begin{bmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{bmatrix} \\ &\xrightarrow{r_2 - 2r_1 \to r_2, r_3 - \frac{3}{2}r_1 \to r_3} \begin{bmatrix} 4 & -4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

We can see that the eigenvectors of A corresponding to $\lambda = 3$ are of the form $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

We can call this eigenspace as E_3.
For $\lambda = -1$ we have

$$A + I = \begin{bmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -6 & 4 \end{bmatrix}$$
$$\xrightarrow{r_2 - r_1 \to r_2, r_3 - \frac{3}{2}r_1 \to r_3} \begin{bmatrix} 8 & -4 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & 4 \end{bmatrix}$$

We can clearly see that the eigenvectors of A corresponding to $\lambda = -1$ are of the form $\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$.

Solution: Continued:
We can call this eigenspace as E_{-1}.
We can see that the sets E_3 and E_{-1} are lineanrly independent sets, thus $E_3 \cup E_{-1}$ is also linearly independent.
Since the largest linearly independent combination of vectors in the set $E_3 \cup E_{-1}$ is of size 3, and R^3 has dimension 3, we can see that the eigenvectors of A are a basis for R^3.
This must mean that A is diagonalizable.
Let $Q = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.
Thus we have found the required Q and D.

8. (10 points) Sec. 5.2 Question 2(f)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution: Since the matrix is already upper triangular, we can see that the eigenvalues of A are the diagonal entries of A.
Thus the eigenvalues of A are $\lambda = 1, 3$ with 1 having a multiplicity of 2.

For $\lambda = 1$ we have

$$A - I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

We can see that the eigenvectors of A corresponding to $\lambda = 1$ are of the form $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

We can see that this eigenspace is of dimension 1. but the algebraic multiplicity of the eigenvalue is 2.
Thus A is not diagonalizable.

9. (10 points) Sec. 5.2 Question 3(d) For each of the following linear operators T on a vector space V, test T and if T is diagonalizable find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

$$V = P_2(R), \quad T(f(x)) = f(0) + f(1)(x + x^2)$$

Solution: Let us assert that $T = L_A$ for some matrix A.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The characteristic polynomial of A is given by

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} \qquad = (1-\lambda)(\lambda^2 - 2\lambda)$$

$$= -\lambda(1-\lambda)(2-\lambda)$$

Thus the eigenvalues of A are $\lambda = 0, 1, 2$.
Since each eigenvalue has a multiplicity of 1, we can see that A is diagonalizable.
For $\lambda = 0$ we have

$$A - 0I = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{r_3 - r_2 \to r_3, r_2 - r_1 \to r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We can see that the eigenvectors of A corresponding to $\lambda = 0$ are of the form $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

We can call this eigenspace as E_0.
For $\lambda = 1$ we have

$$A - I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

We can see that the eigenvectors of A corresponding to $\lambda = 1$ are of the form $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

We can call this eigenspace as E_1.
For $\lambda = 2$ we have

$$\begin{aligned} A - 2I &= \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \\ &\xrightarrow{r_2+r_1 \to r_2, r_3+r_1 \to r_3} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \\ &\xrightarrow{r_3+r_2 \to r_3} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

We can see that the eigenvectors of A corresponding to $\lambda = 2$ are of the form $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

We can call this eigenspace as E_2.
Thus we can see that the sets E_0, E_1, and E_2 are lineanrly independent sets, thus $E_0 \cup E_1 \cup E_2$ is also linearly independent.
Since the largest linearly independent combination of vectors in the set $E_0 \cup E_1 \cup E_2$ is of size 3, and $P_2(R)$ has dimension 3, we can see that the eigenvectors of A are a basis for $P_2(R)$.
Thus our basis β is $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

10. (10 points) Sec. 5.2 Question 13 Let T be an invertible linear operator on a finite dimensional vector space V

(a) Recall that for any eigenvalue λ of T, λ^-1 is an eigenvalue of T^-1. Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^-1 corresponding to λ^-1.

(b) Prove that if T is diagonalizable, then T^-1 is also diagonalizable.

Solution: Part (a)
Suppose λ is an eigenvalue of T and v is an eigenvector of T corresponding to λ.
Then we have $T(v) = \lambda v$.
We can apply T^{-1} to both sides to get $T^{-1}(T(v)) = T^{-1}(\lambda v)$.
This implies that $v = \lambda T^{-1}(v)$.
Thus we have that $\lambda^{-1}v = T^{-1}(v)$.
Thus v is an eigenvector of T^{-1} corresponding to λ^{-1}.
Thus the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.
Part (b)
Suppose T is diagonalizable.
Then we have that $T = L_A$ for some matrix A.
Let Q be the matrix such that $Q^{-1}AQ = D$ where D is a diagonal matrix.
We can see that $T^{-1} = L_{A^{-1}}$.
We can see that $Q^{-1}A^{-1}Q = D^{-1}$.
Thus T^{-1} is diagonalizable.

11. (10 points) Sec. 5.2 Question 21 Let $W_1, W_2, W_3, \ldots, W_k$ be subspaces of finite-dimensional vector space V such that

$$\sum_{i=1}^{k} W_i = V$$

Prove that V is that direct sum of $W_1, W_2, W_3, \ldots, W_k$ if and only

$$dim(V) = \sum_{i=1}^{k} dim(W_i)$$

Solution: Proof of $\implies$:
Suppose V is the direct sum of $W_1, W_2, W_3, \ldots, W_k$.
Need to show that $dim(V) = \sum_{i=1}^{k} dim(W_i)$.
Let γ_i be a basis for W_i.
Since all γ_i are linearly independent and $\gamma_i \cap \gamma_j = \emptyset$ for $i \neq j$, we can see that $\gamma = \gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_k$ is a basis for V.
Thus $dim(V) = |(\gamma)| = \sum_{i=1}^{k} dim(W_i)$.

Note that I use $|(\gamma)|$ to denote the number of elements in the basis γ. I frankly don't know if this is allowed but know I am denoting it as such.

Proof of $\Longleftarrow$:
Suppose $dim(V) = \sum_{i=1}^{k} dim(W_i)$.
Need to show that V is the direct sum of $W_1, W_2, W_3, \ldots, W_k$.
Let γ_i be a basis for W_i.
Since $V = \sum_{i=1}^{k} W_i$, we can see that $\forall v \in V$, $v = \sum_{i=1}^{k} w_i$ for some $w_i \in W_i$.
Thus $\gamma = \gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_k$ is a generating set for V.
Since $dim(V) = \sum_{i=1}^{k} dim(W_i)$ Then $\sum_{i=1}^{k} dim(W_i) = \sum_{i=1}^{k} dim(span(\gamma_i)) = dim(span(\gamma)) = |(\gamma)|$.
Since $|(\gamma)| = dim(V)$, so we can see that the size of γ is the same as the dimension of V. and thus by theorem 5.9 we can see that V is the direct sum of $W_1, W_2, W_3, \ldots, W_k$.

12. (10 points) Sec. 5.2 Question 22 Let V be a finite dimensional vector space with a basis β and let $\beta_1, \ldots, \beta_k$ be a partition of β (ie $\beta_1, \ldots, \beta_k$ are subsets of β such that $\beta = \beta_1 \cup \cdots \cup \beta_k$ and $\beta_i \cap \beta_j = \emptyset$ for $i \neq j$). Prove that $V = span\beta_1 \oplus \cdots \oplus span\beta_k$

Solution: We need to prove that $V = span\beta_1 + \cdots + span\beta_k$ and $span(\beta_i)span(\beta_j) = 0$ for $i \neq j$
Clearly the $span\beta_i \cap span(\beta_j) = 0$ for $i \neq j$ since $\beta_i \cap \beta_j = \emptyset$.
We can see that $V = span\beta_1 + \cdots + span\beta_k$ since $\beta = \beta_1 \cup \cdots \cup \beta_k$.
Thus $V = span\beta_1 \oplus \cdots \oplus span\beta_k$.