01:640:350H - Homework 11

Pranav Tikkawar

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1. Section 6.1 Problem 2
   Let $x = (2, 1+i, i)$ and $y = (2-i, 2, 1+2i)$. be vectors in C^3 Compute $< x, y >$ $||x||$, $||y||$, and $||x+y||$. Then verify the Cauchy-Schwarz inequality and the triangle inequality for these vectors.

Solution:

$$\begin{aligned}
< x, y > &= 2(2+i) + (1+i)(2) + i(1-2i) = 4 + 2i + 2 + 2i + i + 2 = 8 + 5i \\
||x|| &= \sqrt{2^2 + (1+i)(1-i) + i(-i)} = \sqrt{4+2+1} = \sqrt{7} \\
||y|| &= \sqrt{(2-i)(2+i) + 2^2 + (1+2i)(1-2i)} = \sqrt{5+4+5} = \sqrt{14} \\
||x+y|| &= \sqrt{(4+i)(4-i) + (3+i)(3-i) + (1+3i)(1-3i)} = \sqrt{17+10+10} = \sqrt{37} \\
|< x, y >| \leq ||x|| \cdot ||y|| &\implies |8+5i| \leq \sqrt{7} \cdot \sqrt{14} \implies \sqrt{64+25} \leq \sqrt{98} \implies \sqrt{89} \leq \sqrt{98} \\
||x+y|| \leq ||x|| + ||y|| &\implies \sqrt{37} \leq \sqrt{7} + \sqrt{14}
\end{aligned}$$

Through minor comuptation we can see that this is true.

2. Section 6.1 Problem 3
   In $C([0,1])$ let $f(t) = t$ and $g(t) = e^t$. Then compute $< f, g >$, $||f||$, $||g||$, and $||f + g||$. Then verify the Cauchy-Schwarz inequality and the triangle inequality for these functions.

Solution:

$$\begin{aligned}
< f, g > &= \int_0^1 t \cdot e^t dt = te^t - e^t \Big|_0^1 = 1 \\
||f|| &= \sqrt{\int_0^1 t^2 dt} = \sqrt{\frac{1}{3}} \\
||g|| &= \sqrt{\int_0^1 e^{2t} dt} = \sqrt{\frac{e^2-1}{2}} \\
||f+g|| &= \sqrt{\int_0^1 (t+e^t)^2 dt} = \sqrt{\int_0^1 t^2 + 2te^t + e^{2t} dt} = \sqrt{\frac{1}{3} + 2 + \frac{e^2-1}{2}} \\
|< f, g >| \leq ||f|| \cdot ||g|| &\implies |1| \leq \sqrt{\frac{1}{3}} \cdot \sqrt{\frac{e^2-1}{2}} \implies 1 \leq \sqrt{\frac{e^2-1}{6}} \\
||f+g|| \leq ||f|| + ||g|| &\implies \sqrt{\frac{1}{3} + 2 + \frac{e^2-1}{2}} \leq \sqrt{\frac{1}{3}} + \sqrt{\frac{e^2-1}{2}}
\end{aligned}$$

Through minor computation we can see that this is true.

3. Section 6.1 Problem 9
   Let β be a basis for a finite dimentional inner product space.
   (a) Prove that if $< x, z >= 0$ for all $z \in V$, then $x = 0$.
   (b) Prove that if $< x, z >=< y, z >$ for all $z \in V$, then $x = y$.

Solution: Part a: If we take $z = x$ then we get $< x, x >= 0$ which implies that $x = 0$.
Part b: If we take $z = x - y$ then we get $< x - y, x - y >= 0$ which implies that $x = y$.

4. Section 6.1 Problem 11
   Prove the parallellogram law on an inner product space V; that is show

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2 \quad \text{for all } x, y \in V$$

Solution: We can start off by rewriting the equation in inner product form:

$$< x + y, x + y > + < x - y, x - y >= 2 < x, x > +2 < y, y >$$

We can rewrite the left hand side by the (almost) linearity of both elemnts of the inner product:

$$\begin{aligned} < x + y, x + y > + < x - y, x - y > &=< x, x + y > + < y, x + y > + < x, x - y > - < y, x - y > \\ &=< x, x > + < x, y > + < y, x > + < y, y > \\ &+ < x, x > - < x, y > - < y, x > + < y, y > \\ &= 2 < x, x > +2 < y, y > \end{aligned}$$

Thus we have shown that the parallelogram law holds.

5. Section 6.1 Problem 12 Let $\{v_1, v_2, ..., v_k\}$ be an orthononal set in V and let $a_1, a_2, ..., a_k$ be scalars. Prove that

$$||\sum_{i=1}^{k} a_i v_i||^2 = \sum_{i=1}^{k} |a_i|^2 ||v_i||^2$$

Solution: We can start off by rewriting the left hand side in inner product form:

$$\begin{aligned} ||\sum_{i=1}^{k} a_i v_i||^2 &= < \sum_{i=1}^{k} a_i v_i, \sum_{j=1}^{k} a_j v_j > \\ &= \sum_{i=1}^{k} < a_i v_i, \sum_{j=1}^{k} a_j v_j > \\ &= \sum_{i=1}^{k} \sum_{j=1}^{k} < a_i v_i, a_j v_j > \\ &= \sum_{i=1}^{k} < a_i v_i, a_i v_i > \quad \text{since the set is orthogonal} \\ &= \sum_{i=1}^{k} |a_i|^2 < v_i, v_i > \\ &= \sum_{i=1}^{k} |a_i|^2 ||v_i||^2 \end{aligned}$$

Thus we have shown that the equation holds.

6. Section 6.1 Problem 16(b) Let $V = C([0,1])$ and define

$$< f, g >= \int_0^{1/2} f(t)g(t)dt$$

Is this a inner product on V? Justify your answer.

Solution: This is not an inner product as if we take a continous function that is zero on the interval $(0, 1/2)$

$f(t) = \begin{cases} 0 & \text{if } t \in [0, 1/2) \\ t - \frac{1}{2} & \text{if } t = 1/2 \end{cases}$.

We can see that $< f, f >= 0$ but $f \neq 0$.