Intro to Quantum Computating - Homework 1

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1. In the class, we showed that the eigenvectors of the Pauli Z operator are given by $|0\rangle$, $|1\rangle$ with eigenvalues 1, -1 respectively:

$$Z|0\rangle = |0\rangle, \quad Z|1\rangle = -|1\rangle,$$

where

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Find the eigenvectors of the Pauli X operator and the corresponding eigenvalues. Find the 2×2 matrix that relates the eigenvectors of the X operator to the eigenvectors of the Z operator.

Solution: We can do this by solving the eigenvalue equation

$$X|v\rangle = \lambda|v\rangle,$$

where X is the Pauli X operator given by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of the Pauli X operator are $\lambda = 1$ and $\lambda = -1$. The eigenvectors corresponding to these eigenvalues are

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The 2x2 matrix that relates the eigenvectors of the X operator to the eigenvectors of the Z operator is given by

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

2. Recall that the commutator of two operators is given by $[A, B] = AB - BA$. Show that (below the symbol † indicates Hermitian conjugation)
   (a) $[A, B]^\dagger = [B^\dagger, A^\dagger]$
   (b) $[A, B] = -[B, A]$
   (c) If A, B are Hermitian, then $i[A, B]$ is also Hermitian

Solution: a

$$\begin{aligned}
[A, B]^\dagger &= (AB - BA)^\dagger \\
&= (AB)^\dagger - (BA)^\dagger \\
&= B^\dagger A^\dagger - A^\dagger B^\dagger \\
&= [B^\dagger, A^\dagger]
\end{aligned}$$

b

$$\begin{aligned}
[A, B] &= AB - BA \\
&= -BA + AB \\
&= -[B, A]
\end{aligned}$$

c Suppose A, B are Hermitian, then $A = A^\dagger$ and $B = B^\dagger$. Then we need that $i[A, B] = i[A, B]^\dagger$. We have

$$\begin{aligned}
(i[A, B])^\dagger &= -i[B^\dagger, A^\dagger] \\
&= i[A^\dagger, B^\dagger] \\
&= i[A, B] \\
&= i[A, B]
\end{aligned}$$

Thus $i[A, B]$ is Hermitian.

3. An operator A is called anti-Hermitian if $A^\dagger = -A$.
   (a) Show that eigenvalues of anti-Hermitian operators are purely imaginary.
   (b) Show that expectation values of anti-Hermitian operators are purely imaginary for any given state $|\psi\rangle$.

Solution: **a** Suppose A is an anti-Hermitian operator with eigenvalue a and eigenvector $|a\rangle$. Then we have

$$\begin{aligned}
A|a\rangle &= a|a\rangle \\
A^\dagger|a\rangle &= a^*|a\rangle \\
-A|a\rangle &= a^*|a\rangle \\
-a|a\rangle &= a^*|a\rangle \\
-a &= a^*
\end{aligned}$$

Thus a is purely imaginary. **b** Suppose A is an anti-Hermitian operator and $|\psi\rangle$ is any state. We want to show that $\langle\psi|A|\psi\rangle^* = -\langle\psi|A|\psi\rangle$. We have

$$\begin{aligned}
\langle\psi|A|\psi\rangle^* &= \langle\psi|A^\dagger|\psi\rangle \\
&= -\langle\psi|A|\psi\rangle
\end{aligned}$$

Thus $\langle\psi|A|\psi\rangle$ is purely imaginary.