01:640:350H - Midterm 1 Review

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Contents

1 Lecture 1

Basic 250 review and intro to vector spaces
Intro to fields: $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_2$

Definition 1. A vector space V over a field F consists of a set equipped with vector addtion and scalar multiplication so that $\forall x, y \in V, \exists! x + y \in V$ and $\forall a \in F, \forall x \in V, \exists! ax \in V$
The following are the vector space axioms:

1. $\forall x, y \in V, x + y = y + x$
2. $\forall x, y, z \in V, (x + y) + z = x + (y + z)$
3. $\exists \underline{0} \in V$ such that $\forall x \in V, x + \underline{0} = x$
4. $\forall x \in V, \exists - x \in V$ such that $x + (-x) = \underline{0}$
5. $\forall x \in V, 1 \cdot x = x$
6. $\forall a, b \in F, \forall x \in V, (ab)x = a(bx)$
7. $\forall a \in F, \forall x, y \in V, a(x + y) = ax + ay$
8. $\forall a, b \in F, \forall x \in V, (a + b)x = ax + bx$

In words they are:

1. Commutative property of addition
2. Associative property of addition
3. Additive identity
4. additive inverse
5. multiplicative identity
6. Associativity of scalar multiplication
7. distributivity of 1 vector to 2 scalars
8. distributivity of 2 vectors to 1 scalar

2 Lecture 2

Theorem 1 (Theorem 1.1)**.** *Let V be a vector space over F Let $x, y, z \in V$ and assume $x + z = y + z$. Then $x = y$.*
This is cancellation from the right

Proof. Given $x + z = y + z$. Need $x = y$
$x + z = y + z$
$x + z + (-z) = y + z + (-z)$
$x + \underline{0} = y + \underline{0}$
$x = y$ □

Theorem 2 (Theorem 1.1 ')**.** *Let $x, y, z \in V$ If $z + x = z + y$, then $x = y$.*
This is cancellation from the left

Proof. Given $z + x = z + y$. Need $x = y$
$z + x = z + y$
$z + x + (-z) = z + y + (-z)$
$\underline{0} + x = \underline{0} + y$
$x = y$ □

We can also prove this by using Theorem 1.1 and (VS 1)

Corollary 1 (Corollary 1)**.** The vector $\underline{0}$ (VS 3) is unique.

Proof. Suppose $\underline{0}$ and $\underline{0}'$ are both additive identities.
$\underline{0} + \underline{0}' = \underline{0}$
$\underline{0} + \underline{0}' = \underline{0}'$
By Theorem 1.1, $\underline{0} = \underline{0}'$ □

Corollary 2 (Corollary 2)**.** The vector y or $-x$ in (VS 4) is unique.

Proof. Suppose y and y' are both additive inverses of x.
$y + x = \underline{0}$
$y' + x = \underline{0}$
By Theorem 1.1, $y = y'$ □

3 Lecture 3

Theorem 3 (Theorem 1.2(a))**.** $\forall x \in V, 0 \cdot x = \underline{0}$

Proof. Given $x \in V$. Need $0 \cdot x = \underline{0}$
$0 \cdot x = (0 + 0) \cdot x$
$0 \cdot x = 0 \cdot x + 0 \cdot x$
$0 \cdot x + (-0 \cdot x) = 0 \cdot x + 0 \cdot x + (-0 \cdot x)$
$0 = 0 \cdot x$ □

Theorem 4 (Theorem 1.2(b)). $\forall a \in F, \forall x \in V, (-a) \cdot x = -(a \cdot x) = a(-x)$

Proof. We can show that $(-a) \cdot x + (a \cdot x) = \underline{0}$
$(-a) \cdot x + (a \cdot x) = (-a)x + a(x)$
$(-a) \cdot x + (a \cdot x) = (-a + a)x$
$(-a) \cdot x + (a \cdot x) = 0x$
$(-a) \cdot x + (a \cdot x) = \underline{0}$ □

Definition 2. Let V be a vector space over F. A subset W of V is a subspace of V if W is a vector space over F with the same operations of addition and scalar multiplication as V.

Theorem 5 (Theorem 1.3). *Let $W \subset V$. Then W is a subspace of V iff*

- $\underline{0} \in W$
- *W is closed under addition, i.e. $\forall x, y \in W, x + y \in W$*
- *W is closed under scalar multiplication, i.e. $\forall a \in F, x \in W, ax \in W$*

Note that VS 1,2,5,6,7,8 are inherited from V. So we need to prove VS 3,4.

Definition 3. Let V be a vector space over F and S a nonempty subset of V. A vector $v \in V$ is called a linear combination of vectors of S if $\exists$ finitely many vectors $u_1, \ldots, u_n \in S$ and scalars $a_1, \ldots, a_n \in F$ such that $v = a_1u_1 + \cdots + a_nu_n = \sum_{i=1}^{n} a_iu_i$

4 Lecture 4

Definition 4. Let V be a vector space over F and S a nonempty subset of V. Then the span of S is the set of all linear combinations of vectors of S.
The span of $\emptyset$ is defined to be $\{\underline{0}\}$

Theorem 6 (Theorem 1.5). *The span of any subset S of a vector space V is a subspace of V that contains S more over any subspace of V that contains S also contains the span of S.*

Note that Theorem 1.5 asserts that the spans of S is the smallest subspace of V that contains S.

Definition 5. Let $S \subset V$ then S generates (or spans) V if $V = \text{span}(S)$

Definition 6. A subset S of a vector space V is Linearly dependant if $\exists$ finitely many distinct vectors $u_1, \ldots, u_n \in S$ and scalars $a_1, \ldots, a_n \in F$ not all zero such that $a_1u_1 + \cdots + a_nu_n = \underline{0}$

Definition 7. A subset S of a vector space V is linearly independent if it is not linearly dependent. In other words it only has the trivial solution of $a_1u_1 + \cdots + a_nu_n = \underline{0}$ for all $a_i = 0$

Definition 8. A basis β for a vector space V is a Linearly Independent subset of V that spans V. If β is a basis for V, we also say that the vectors of β form a basis for V.

5 Lecture 5

Theorem 7 (Theorem 1.8)**.** *Let V be a vector space and let $u_1, \ldots, u_n$ be distinct vectors in V. Then $\beta = \{u_1, \ldots, u_n\}$ is a basis for V iff every $v \in V$ can be expressed uniquely as a linear combination of the vectors of β.*
This of this as a making V into F^n

Theorem 8 (Theorem 1.9)**.** *If a vector space V is generate by a finite set S then some subset of S is a basis for V hence it has a finite basis.*