Chapter 4

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Chapter 4

Markov Property

If the probability of the nest state only depends on the current state, it satisfies the "Markov Property".
Drunkards walk example

$$\mathbb{P}(x_{i+1} = x_i \pm 1) = \frac{1}{2}\mathbb{P}(x_{i+1} \neq x_i \pm 1) = 0$$

$$\mathbb{P}(x_{i+1} = x + 1 | x_i = x) = 1/2$$

$$\mathbb{P}(x_{i+1} = x - 1 | x_i = x) = 1/2$$

Formal Definition

Let $\{X_n, n \in \mathbb{N}\}$ be a stochastic process that takes discrete time values. Suppose $\mathbb{P}(X_{n+1} = j | X_n = i_n ... X_0 = i_0) = P_{i,j}$ Such a stochastic process is called a Markov Chain. P_{ij} is the transition probability from state i to state j.

Transition Probability Matrix

Let $i, j \in \mathbb{N}$ be possible states of the Markov Chain. The matrix $P = [P_{ij}]$ is called the transition probability matrix of the Markov Chain. Where $P_{ij} = \mathbb{P}(x_{n+1} = j | x_n = i)$. **Ex 4.1**

$$\mathbb{P}(\text{rain tomorrow}|\text{rain today}) = \alpha$$

$$\mathbb{P}(\text{rain tomorrow}|\text{no rain today}) = \beta$$

$$\text{Let} \begin{cases} 0 = \text{rain} \\ 1 = \text{no rain} \end{cases}$$

$$P = \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix}$$

Ex 4.4 Suppose whether it rains tomorrow or not depends on both todays and yesterdays weather.

Today's Weather	Yesterdays's Weather	Value
Rain	Rain	0
Rain	No Rain	1
No Rain	Rain	2
No Rain	No Rain	3

Suppose:
$\mathbb{P}$(rain tomorrow|rain today, rain yesterday) = .7
$\mathbb{P}$(rain tomorrow|rain today, no rain yesterday) = .5
$\mathbb{P}$(rain tomorrow|no rain today, rain yesterday) = .4
$\mathbb{P}$(rain tomorrow|no rain today, no rain yesterday) = .2

$$P = \begin{bmatrix} .7 & 0 & .3 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .4 & 0 & .6 \\ 0 & .2 & 0 & .8 \end{bmatrix}$$

4.2 Chapman-Kolmogorov Theorem

P_{ij} = probability of going from state i to state j
$P_{ij}^{(n)}$ = probability of going from state i to state j in n steps.
$P_{ij}^{(n+m)} = \sum_k P_{ik}^{(n)} P_{kj}^{(m)}$ (pg.197)
Look at example 4.10 for next class
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Proof of Chapman-Kolmogorov Theorem

Equation: $P_{ij}^{(n+m)} = \sum_k P_{ik}^{(n)} P_{kj}^{(m)}$
We can visualize this as a graph with n+m steps and we consider all the paths $i \to j$ and sum them with the law of total probability.
Proof:

$$P_{ij}^{(n+m)} = \mathbb{P}(X_{n+m} = j | X_0 = i)$$
$$= \sum_k \mathbb{P}(X_{n+m} = j, X_n = k | X_0 = i)$$
$$= \sum_k \mathbb{P}(X_{n+m} = j | X_n = k, X_0 = i) \mathbb{P}(X_n = k | X_0 = i)$$

Note that this is the probabilty of going from k to j in m steps(which doesn't depend on $x_0 = i$ due to the Markov Property) and from i to k in n steps.
Homogeneity of a Markov Chain.
Example 4.10 An urn always contains 2 balls. Possible ball colors are red and

blue. Each stage of the process we pick a ball and randomly replace it with another ball. Replacement of the same color is .8 and replacement of a different color is .2.
If initially both the first balls are red, what is the probability that the 5th ball is red?

$$P = \begin{bmatrix} .8 & .2 & 0 \\ .1 & .8 & .1 \\ 0 & .2 & .8 \end{bmatrix}$$

Note: for a set up where the probability of changing colors is invariant of the color of the ball, the transition matrix will be visually "radially" symmetric***.

$$\begin{aligned} \mathbb{P}(X_5 = \text{red}) &= P_{22}^{(4)} + \frac{1}{2}P_{21}^{(4)} + 0P_{12}^{(4)} \\ &= 0.7048 \end{aligned}$$

Ask what are other Properties of stochastic matrix
$a_{i,j} = an - i, n - j$
Example 4.11
In a sequence of independent flips of a fair coint, let N denote the number of flips until there is a run of 3 heads.
Find (a) $P(N \leq 8)$ (b) $P(N = 8)$
Consider 4 states: 0,1,2,3. given by n = the number of consecutive heads

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) $= P_{03}^{(8)}$
(b) $= \frac{1}{2}P_{02}^{(7)}$

4.3 Classification of States

Definition: State j of is accessible from state i if $P_{ij}^{(n)} > 0$ for some $n \geq 0$. If the states are accessible from each other, they are said to communicate.
Communication is an equivalence relation.
Reflexive and symmetric are obvious.
Transitive is proven by the Chapman-Kolmogorov Theorem.
This relation divides the states into classes.

Reccurent and Transient States

Definition: A given state i of a MC let f_i denote the probability that the chain will eventually return to state i.

A state is called Reccurent if $f_i = 1$ and Transient if $f_i < 1$.
The expected number of tevisits to a reccuent state is infinite.
for a transient state the probability of being in state i for exactly n times period is $f_i^n(1 - f_i)$: Geometric distribution