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1 Multiple Time Series

1.1 Introductory Theory

1.1.1 Introduction

Considering a Probability space triplet: $\Omega, \mathscr{A}, \mathbb{P}$. We can consider Ω to be our sample space. The borel algebra $\mathscr{A}$ to be the collection of historic events. And the Probability Measure $\mathbb{P}$. We are mainly concerned with the finite second moments of the Stochastic.

$$\mathbb{E}(x_j(s), x_k(t)) = \gamma_{j,k}(s,t) < \infty$$

We can arrange this into a Symmetric Matrix $\Gamma(s,t)$

Theorem 1. *In order that $\gamma_{j,k}(s,t)$ may be continuous, $j,k = 1,\dots,p$, it is necessary and sufficient that $\gamma_j(s,t)$ be continuous at $s = t$ for $j = 1,\dots,p$*

Proof. pg 5 □

This equivalent to $\lim_{u\to 0} \mathbb{E}[(x_j(s+u) - x_j(s))^2] = 0$, $j = 1,\dots,p$. This is called mean-square continuity.

1.1.2 Differentiation and Integration of Stochastic Processes

Definition (Convergence in the Mean-Square Sense)**.** Let $x_{n}{}_{n=1}^{\infty}$, be a sequence of random variables, $\mathbb{E}|x_n|^2 < \infty$. Then the seqeuences converges to R.V x if

$$\lim_n \mathbb{E}[|x - x_n|^2] = ||x - x_n||^2 = 0$$

To denote this we write $x_n \to x$

Definition (Cauchy Condition)**.**

$$\lim_{n,m} ||x_n - x_m|| = 0$$

Note if $\mathbb{E}[x\bar{y}]$ is a continuous function of x and y, so that $||x_n||$ and $||y_n||$ are finite and $x_n \to x$ and $y_n \to y$ then $\mathbb{E}(x_n\bar{y}_n) \to \mathbb{E}(x\bar{y})$

Definition (Mean-Square Differentiable)**.** We say a the scalar process $x(t)$ is MS Differentiable at t if $\delta^{-1}[x(t+\delta) - x(t)]$ has a unique limit as $\delta \to 0$
The cauchy criterion for MS Diff. is as follows:

$$\lim_{\delta_1,\delta_2\to 0} ||\delta_1^{-1}[x(t+\delta_1) - x(t)] - \delta_2^{-1}[x(t+\delta_2) - x(t)]|| = 0$$

Then a nessesary and sufficient condition for the cauchy criterion definition is as follows:

$$\lim_{\delta_1,\delta_2\to 0} \mathbb{E}[\delta_1^{-1}[x(t+\delta_1) - x(t)] - \delta_2^{-1}[x(t+\delta_2) - x(t)]] =$$
$$\lim_{\delta_1,\delta_2\to 0} \frac{\gamma(t+\delta_1, t+\delta_2) - \gamma(t+\delta_1, t) - \gamma(t, t+\delta_2) + \gamma(t,t)}{\delta_1\delta_2}$$

In turn it is sufficient that $\frac{\partial^2\gamma(s,t)}{\partial s \partial t}$ exists and be continuous.

If $\dot{x}(t)$ is the MS derivative, this equation above has covariance function $\frac{\partial^2\gamma(s,t)}{\partial s\partial t}$ and $\mathbb{E}[x(s)\dot{x}(t)] = \frac{\partial\gamma(s,t)}{\partial t}$
We want to represent this in the form of $\int_{-\infty}^{\infty} x(t)m(dt)$ where m is a σ-finite measure adjusted* so that the corresponding distribution function is continuous from the right.
Similarly we only consider mean-square continuous functions and functions of the form $\iint_{-\infty}^{\infty} \gamma(s,t)m(ds)m(dt) < \infty$
We can consider the integral $\int_a^b x(t)m(dt)$ by approximating sums $\sum_{i=1}^{n} x(t_i)m((s_{j-1}, s_j))$ wherein the points s_j divide the interval $[a,b]$ into intervals less than ϵ and $t_i \in (s_{j-1}, s_j)$. We call this the integral of $x(t)$ wrt $m(t)$ over $[a,b]$ This is called the Riemann-Stieltjes integral.

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