Utilizing Finite State Stochastic Graphs for Reliability Analysis in Aircraft Systems

Pranav Tikkawar

July 21, 2025

Abstract

This paper presents a novel approach for reliability analysis in aircraft systems by utilizing finite state stochastic graphs. Traditional methods often rely on the Weibull distribution to model time-to-failure, which requires extensive failure and repair data that may not always be available. Our method leverages finite state stochastic graphs to approximate system reliability, capturing component states and their transitions without the need for detailed historical data. Through simulations, we demonstrate how this framework can effectively analyze and predict the reliability of complex aircraft systems, offering a practical alternative for scenarios with limited data.

1 Introduction

1.1 Background

1.2 Components and States

Consider the system to be a collection of components denoted by $C = \{C_1, C_2, \ldots, C_n\}$. Each component C_i, there exists a finite number of states it could be in, denoted by $S_i = \{s_{i1}, s_{i2}, \ldots, s_{im}\}$. The states represent the operational conditions of the component, such as new, functional, degraded, or failed. The transition between these states is governed by a set of probabilities, which can be represented as a transition matrix P_i for each component C_i. The transition matrix captures the likelihood of moving from one state to another over time. With the assumption that the system does not require the specific time a component is in a state, we can consider the distribution of changing states over time to follow an exponential distribution with a parameter $\lambda_{i,j}$ representing the rate of transition from state $s_{i,j}$ to state s_{ik}. The mean time to transition is given by $\frac{1}{\lambda_{ij}}$.

1.3 Graph Representation

We can represent the system as a directed graph where each component C_i is a vertex and the edges represent the dependencies between components. The edges can be weighted to indicate the strength of the dependency, which can affect the transition probabilities of connected components. This graph representation allows us to visualize and analyze the relationships between components and their states.

2 Stochastic Graph Model for Component Transitions

Consider a directed graph $G(V, E)$ representing a system of components over discrete time $t \in \{1, \dots, T\}$. Each vertex $C_i \in V$ represents a component with state space $S^{(i)} = \{s_1^{(i)}, s_2^{(i)}, \dots, s_{n_i}^{(i)}\}$. The transition matrix $P^{(i)}$ is time-homogeneous with entries:

$$P_{ij}^{(A)} = \mathbb{P}(A(t) = j \mid A(t-1) = i)$$

$$P^{(A)} = \begin{bmatrix} P_{11}^{(A)} & P_{12}^{(A)} & \cdots & P_{1n}^{(A)} \\ P_{21}^{(A)} & P_{22}^{(A)} & \cdots & P_{2n}^{(A)} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1}^{(A)} & P_{n2}^{(A)} & \cdots & P_{nn}^{(A)} \end{bmatrix}$$

The edge set $E = \{(C_a, C_b) \mid C_a C_b \in V\}$ has a influence/weight matrix W where the weight w_{ij} represents the adjustment in the jth row's probability in the target C_b when the component C_a transitions to the state i when it was priorly not in state i. Note that the matrix W must be of size $n \times m$ where n is the size

$$W = \begin{bmatrix} w_{11} & w_{12} & w_{13} & \cdots & w_{1m} \\ w_{21} & w_{22} & w_{23} & \cdots & w_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & w_{n3} & \cdots & w_{nm} \end{bmatrix}$$

The adjusted transition matrix $\widetilde{P}^{(j)}$ is defined per row k as:

$$\widetilde{P}_{kk}^{(j)} = \sigma\left(\text{logit}\left(P_{kk}^{(j)}\right) + W_j\right)$$

$$\widetilde{P}_{kl}^{(j)} = P_{kl}^{(j)} \cdot \frac{1 - \widetilde{P}_{kk}^{(j)}}{1 - P_{kk}^{(j)}}, \quad l \neq k$$

where $\sigma(z) = (1 + e^{-z})^{-1}$ is the sigmoid function and $\text{logit}(x) = \log(x/(1-x))$.

$$\tilde{P}_{kk}^{(b)} = \sigma\left(\ell(P_{kk}^{(b)}) + \sum_{n \in \mathcal{M}} W^{(n,b)}[i]\right)$$

$$\tilde{P}_{kl}^{(b)} = P_{kl}^{(b)} \cdot \frac{1 - \tilde{P}_{kk}^{(b)}}{1 - P_{kk}^{(b)}}, \quad l \neq k$$

$$\sigma(z) = \frac{1}{1+e^{-z}}$$
$$\ell(z) = \log\left(\frac{z}{1-z}\right)$$

Model Properties

(i) **Probability Preservation**: $\widetilde{P}^{(j)}_{kl} \in [0,1]$ for all k, l

(ii) **Idempotence at Zero Influence**: $W_j = 0 \implies \widetilde{P}^{(j)} = P^{(j)}$

(iii) **Row-Stochasticity**: $\sum_l \widetilde{P}^{(j)}_{kl} = 1$ for all k

(iv) **Surjective Adjustment**: $\forall y \in (0,1), \exists W_j$ such that $\widetilde{P}^{(j)}_{kk} = y$

(v) **Interpretability**: W_j represents additive log-odds adjustment:

$$\log\left(\frac{\widetilde{P}^{(j)}_{kk}}{1-\widetilde{P}^{(j)}_{kk}}\right) = \log\left(\frac{P^{(j)}_{kk}}{1-P^{(j)}_{kk}}\right) + W_j$$

Advantages Over Initial Formulation

- **Aggregated Influence**: Handles multiple dependencies via $W_j = \sum_i w_{ij}$, eliminating edge-order ambiguity
- **Row-Consistent Adjustment**: Off-diagonal terms explicitly reference same-row diagonal probability
- **Stochasticity Guarantee**: Row sums remain unity by construction
- **Probabilistic Coherence**: Maintains interpretable log-odds relationship

Limitations and Extensions

- **State-Independent Weights**: Future work may incorporate $w_{ij}(s_i, s_j)$
- **Temporal Dynamics**: Extension to time-varying $W_j(t)$ possible
- **Higher-Order Dependencies**: Non-Markovian edges could be considered

3 Parameter Estimation Framework

This paper presents a rigorous parameter estimation framework for stochastic graph models with logit-adjusted transition probabilities. Given a directed graph $G(V, E)$ with components $\{C_i\}_{i=1}^n$ and state spaces $\{S^{(i)}\}_{i=1}^n$, we estimate:

(i) Base transition probabilities $P^{(i)} = [P_{jk}^{(i)}]$

(ii) Edge weights $\mathbf{w} = \{w_{ij}\}_{(i,j)\in E}$

from observed state transition data.

4 Data Requirements

The estimation framework requires the following data:

- **State trajectories**: Time-series $\mathcal{D} = \{\mathbf{s}(t)\}_{t=0}^T$ where $\mathbf{s}(t) = (s_1(t), \ldots, s_n(t))$ and $s_j(t) \in S^{(j)}$
- **Graph topology**: Known directed graph $G(V, E)$
- **Transition records**: Documented state transitions for all components

5 Estimation of Base Transition Probabilities

The base transition matrices $P^{(i)}$ are estimated independently per component.

5.1 Maximum Likelihood Estimation

For each component C_i, the transition probability from state j to k is:

$$\hat{P}_{jk}^{(i)} = \frac{N_{jk}^{(i)}}{\sum_{l=1}^{|S^{(i)}|} N_{jl}^{(i)}} \tag{1}$$

where $N_{jk}^{(i)}$ counts observed transitions $j \to k$.

5.2 Regularization

For sparse data, apply Dirichlet smoothing:

$$\tilde{P}_{jk}^{(i)} = \frac{N_{jk}^{(i)} + \alpha}{\sum_{l=1}^{|S^{(i)}|} N_{jl}^{(i)} + |S^{(i)}|\alpha}, \quad \alpha > 0 \tag{2}$$

6 Estimation of Edge Weights

The edge weights $\mathbf{w} = \{w_{ij}\}_{(i,j)\in E}$ are estimated jointly via maximum likelihood.

6.1 Likelihood Formulation

EM formula:

$$\begin{aligned} L(\theta : X) &= p(X \mid \theta) \\ Q(\theta \mid \theta^{(n)}) &= \mathbb{E}_{Z|X,\theta^{(n)}}\left[\ln p(X, Z \mid \theta)\right] \\ \theta^{(n+1)} &= \underset{\theta}{\operatorname{argmax}}\; Q(\theta \mid \theta^{(n)}) \end{aligned}$$

The complete-data log-likelihood is:

$$\mathcal{L}(\mathbf{w}) = \sum_{t=1}^{T-1} \sum_{j=1}^{|V|} \ln \widetilde{P}^{(j)}_{s_j(t), s_j(t+1)}(\mathbf{w}) \tag{3}$$

where $\widetilde{P}^{(j)}$ is the adjusted transition matrix:

$$\begin{aligned} \widetilde{P}^{(j)}_{kk}(\mathbf{w}) &= \sigma\left(\operatorname{logit}(P^{(j)}_{kk}) + W_j\right) \\ \widetilde{P}^{(j)}_{kl}(\mathbf{w}) &= P^{(j)}_{kl} \cdot \frac{1 - \widetilde{P}^{(j)}_{kk}(\mathbf{w})}{1 - P^{(j)}_{kk}}, \quad l \neq k \end{aligned}$$

with $W_j = \sum_{i:(i,j)\in E} w_{ij}$.

6.2 Optimization

Maximize the log-likelihood:

$$\mathbf{w}^* = \underset{\mathbf{w}}{\operatorname{argmax}} \mathcal{L}(\mathbf{w}) \tag{4}$$

using gradient-based methods. The gradient components are:

$$\frac{\partial \mathcal{L}}{\partial w_{ij}} = \sum_{t=1}^{T-1} \frac{1}{\widetilde{P}^{(j)}_{k_t l_t}} \frac{\partial \widetilde{P}^{(j)}_{k_t l_t}}{\partial w_{ij}} \tag{5}$$

where $k_t = s_j(t)$, $l_t = s_j(t+1)$.

7 Estimation Algorithm

The joint estimation procedure is:

1. **Initialize**:

$$\hat{P}_{jk}^{(i)} \leftarrow \frac{N_{jk}^{(i)} + \alpha}{\sum_l N_{jl}^{(i)} + |S^{(i)}|\alpha}$$
$$\mathbf{w}^{(0)} \leftarrow \mathbf{0}$$

2. **Iterate until convergence**:

$$\mathbf{w}^{(n+1)} = \mathbf{w}^{(n)} + \gamma_n \nabla_{\mathbf{w}} \mathcal{L}(\mathbf{w}^{(n)})$$

with adaptive step size γ_n.

3. **Convergence criterion**:

$$\|\nabla_{\mathbf{w}} \mathcal{L}(\mathbf{w}^{(n)})\|_2 < \epsilon \tag{6}$$

8 Regularization

To prevent overfitting, use Tikhonov regularization:

$$\mathcal{L}_{\text{reg}}(\mathbf{w}) = \mathcal{L}(\mathbf{w}) - \lambda \|\mathbf{w}\|_2^2, \quad \lambda > 0 \tag{7}$$

9 Asymptotic Properties

Under standard regularity conditions:

Theorem 1 (Consistency)**.** *As $T \to \infty$, the MLE satisfies:*

$$\hat{\mathbf{w}}_T \xrightarrow{p} \mathbf{w}_0 \tag{8}$$

where $\mathbf{w}_0$ is the true parameter vector.

Theorem 2 (Asymptotic Normality)**.**

$$\sqrt{T}(\hat{\mathbf{w}}_T - \mathbf{w}_0) \xrightarrow{d} \mathcal{N}(0, \mathcal{I}^{-1}(\mathbf{w}_0)) \tag{9}$$

where $\mathcal{I}(\mathbf{w})$ is the Fisher information matrix.

10 Identifiability Conditions

For unique identifiability, we require:

1. **Observational equivalence**: $\widetilde{P}^{(j)}(\mathbf{w}_1) = \widetilde{P}^{(j)}(\mathbf{w}_2)\, \forall j \implies \mathbf{w}_1 = \mathbf{w}_2$
2. **Connectivity**: Each component has $\deg_{\text{in}}(C_j) \geq 1$
3. **Acyclicity**: $G(V, E)$ is a directed acyclic graph

11 Practical Considerations

- **Data requirements**: Minimum trajectory length $T \gg \max_j |S^{(j)}|^2$
- **Initialization**: Warm-start weights using correlation analysis
- **Validation**: Use k-fold cross-validation for hyperparameter tuning

12 Conclusion

This framework provides statistically rigorous methods for estimating parameters in stochastic graph transition models. The maximum likelihood approach with gradient-based optimization ensures consistent estimates while preserving model structure and interpretability.

13 Results

13.1 Simulation Setup

13.2 Simulation Results

14 Discussion

15 Conclusion

16 Future Work

17 References