01:XXX:XXX - Homework n

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1. Sec 7.1 2(a) For each matrix A, find a basis for each generalized eigenspaces of L_A consisting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J for A.
$$A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

Solution: We must first find the eigenvalues of A. The characteristic polynomial of A is given by

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} \\ &= (1-\lambda)(3-\lambda) + 1 \\ &= \lambda^2 - 4\lambda + 4 \\ &= (\lambda - 2)^2 \end{aligned}$$

Thus, the eigenvalue of A is $\lambda = 2$ with multiplicity 2. We can now find the eigenvectors of A by solving the system $(A - 2I)\vec{x} = \vec{0}$.

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus our eigenvectors are of the form $\begin{bmatrix} a \\ a \end{bmatrix}$. We can choose $a = 1$ to get the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We can now find the generalized eigenvectors of A by solving the system $(A-2I)\vec{x} = \vec{v}$ where $\vec{v}$ is the eigenvector we found earlier.

$$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Thus our generalized eigenvectors solve the equation $x_1 + x_2 = 1$. We can choose $x_1 = 1$ to get the generalized eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Thus, the basis for the generalized eigenspace of A is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

We can now find the Jordan canonical form of A by constructing the matrix P whose columns are the basis vectors of the generalized eigenspaces of A.

$$J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

2. Sec 7.1 2(b)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

Solution: We must first find the eigenvalues of A. The characteristic polynomial of A is given by

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)(2-\lambda) - 6 \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda - 4)(\lambda + 1) \end{aligned}$$

Thus, the eigenvalues of A are $\lambda = 4, -1$. We can now find the eigenvectors of A by solving the system $(A - 4I)\vec{x} = \vec{0}$.

$$\left[\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array}\right]$$

Thus our eigenvectors are of the form $\begin{bmatrix} 2a \\ 3a \end{bmatrix}$. We can choose $a = 1$ to get the eigenvector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

We can now find the generalized eigenvectors of A for $\lambda = -4$ by solving the system $(A + 4I)\vec{x} = \vec{v}$ where $\vec{v}$ is the eigenvector we found earlier.

3. Sec 7.1 3(a) For each linear operator T find a basis for each generalized eigenspace of T consiting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J for T.

$$\text{Define } T \text{ on } P_2(R) \text{ by } T(f(x)) = 2f(x) - f'(x)$$

Solution:

4. Sec 7.1 3(b)

V is the real vector space of functions spanned by the set of real valued functions $\left\{1, t, t^2, e^t, te^t\right\}$, ar

Solution:

5. Sec 7.2 4(a) For each of the matrices A that follow, find a Jordan canonical form J and an invertible matrix Q such that $J = Q^{-1}AQ$.

$$A = \begin{bmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{bmatrix}$$

Solution: