PDEs Pranav Tikkawar

Transport Equation: $u_t + cu_x = 0 \implies u(x,t) = f(x-ct)$ Solved via method of characteristics: $\frac{dx}{dt} = c \implies x = at + C$ and function is constant along the characteristic lines.

FOL PDE: $a(x,y)u_x + b(x,y)u_y = 0$ or $\langle \nabla u, (a,b)\rangle = 0$. if a, b constant $u(x,y) = f(bx - ay)$ Solved via method of characteristics: $\frac{dy}{dx} = \frac{b}{a} \implies y = \frac{b}{a}x + C$ and function is constant along the characteristic lines. OR by Change of Var for $x' = ax + by, y' = bx - ay$

If non constant a, b then solve for $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} \implies y =$ something. Solve for constant and that is that the function is constant along.

Well Posed: 1. Existence 2. Uniqueness 3. Stable/Continuous

Waves: $u_{tt} = c^2 u_{xx} \implies u(x,t) = f(x-ct) + g(x+ct)$. Product of 2 transports. **With IC:** $u(x,0) = \phi(x), u_t(x,0) = \psi(x)$. On infinite x: $u(x,t) = \frac{\phi(x+ct) - \phi(x-ct)}{2} - \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$ aka D'Alembert's Formula.

Energy For Wave: $E(t) = \frac{1}{2}\int_{-\infty}^{\infty}(u_t^2 + c^2 u_x^2)dx$ can make $c^2 = T/\rho$. Energy is conserved. Finite speed or propagation.

Diffusion: $u_t = ku_{xx}$

Max Principle Weak: max on all is on boundary. Strong: max of all can only be on boundary.

Uniqueness Heat is unique proven by either max principle with difference of 2 solutions = 0 or by energy method.

Energy for Heat: $E(t) = \int_{-\infty}^{\infty} u^2 dx$

Stablility: $\int |u(x,t) - v(x,t)|^2 dx \leq \int |u(x,0) - v(x,0)|^2 dx$

Invariance on Whole Line: 1. Translation, 2. Derivative, 3. Linear Combination, 4. Integral of sol with anything. 5. Scaling by $u(x,t) \to u(\alpha x, \alpha^2 t)$

Fundamental Solution: $u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\phi(y)dy$. for IC $u(x,0) = \phi(x)$ $S = \frac{1}{\sqrt{4\pi kt}}e^{-\frac{x^2}{4kt}}$ is the fundamental solution. S is a green's function.

Separation of Variables: $u(x,t) = X(x)T(t)$. Create rato of X and T and set equal to constant. Solve for X and T separately with the BC/IC that are homogeonous.

Dirichlet BC: $u(0,t) = u(L,t) = 0$: $X(x) = sin(\frac{n\pi x}{L})$. Waves: $T(t) = A_n cos(\frac{n\pi ct}{L}) + B_n sin(\frac{n\pi ct}{L})$ Diffusion: $T(t) = A_n e^{-\frac{n^2\pi^2 kt}{L^2}}$ Laplace: $Y(y) = A_n cosh(\frac{n\pi y}{L}) + B_n sinh(\frac{n\pi y}{L})$ Connected to a fourier sine series.

Neumann BC: $u_x(0,t) = u_x(L,t) = 0$: $X(x) = cos(\frac{n\pi x}{L})$. Waves: $T(t) = A_n cos(\frac{n\pi ct}{L}) + B_n sin(\frac{n\pi ct}{L})$ Diffusion: $T(t) = A_n e^{-\frac{n^2\pi^2 kt}{L^2}}$ Laplace: $Y(y) = A_n cosh(\frac{n\pi y}{L}) + B_n sinh(\frac{n\pi y}{L})$ Connected to a fourier cosine series.

Robin BC: $u_x(0,t) - \alpha_0 u(0,t) = u_x(L,t) + \alpha_1 u(L,t) = 0$: Waves: $T(t) = A_n cos(\sqrt{\lambda_n}t) + B_n sin(\sqrt{\lambda_n}t)$ Diffusion: $T(t) = A_n e^{-\lambda_n kt}$. They take the form of a fourier series.

Fourier Sine Series: $f(x) = \sum_{n=1}^{\infty} B_n sin(\frac{n\pi x}{L}) \implies B_n = \frac{2}{L}\int_0^L f(x)sin(\frac{nx\pi}{L})dx$ Relates to Odd functions.

Fourier Cosine Series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n cos(\frac{n\pi x}{L}) \implies A_n = \frac{2}{L}\int_0^L f(x)cos(\frac{nx\pi}{L})dx$ Relates to Even functions.

Fourier Series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n cos(\frac{n\pi x}{L}) + B_n sin(\frac{n\pi x}{L}) \implies A_n = \frac{1}{L}\int_{-L}^{L} f(x)cos(\frac{n\pi x}{L})dx, B_n = \frac{1}{L}\int_{-L}^{L} f(x)sin(\frac{n\pi$

Note that our interval is 2L.

Generally $c_n = \frac{1}{2l}\langle f(x), e^{inx}\rangle \quad c_0 = \frac{a_0}{2} \quad c_n = \frac{a_n - ib_n}{2} \quad c_{-n} = \frac{a_n + ib_n}{2} \quad a_n = c_n + c_{-n} \quad b_n = i(c_n - c_{-n})$

Converges: Each point of continuity converges to the average of the left and right limits. All L^2 functions converge to the function. **Bessel's Inequality:** $\sum_{N+} |c_n|^2 \leq ||f||^2 \quad c_n \to 0$ **Plancharel's Theorem:** $||f||^2 = \sum_{-\infty}^{\infty} |c_n|^2$

Laplace's Equation: $\Delta u = 0 \quad u_{xx} + u_{yy} = 0 \quad u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$

Fundamental solution in 2D is $u = \frac{1}{2\pi}\log(r)$ and 3d is $u = \frac{1}{4\pi r}$.

Max Principle: The max of u is found on the boundary.

Uniqueness: Given suffecient BC (on all sides) the solution is unique.

Invariance Invariant under Rigid Motion. This implies that the solution doesnt care about rotation/direction hence the solution is a function of r only.
Rectangle and Cube: Split each inhomogeneous side into its own solution as $u = u_1 + u_2 + u_3 + u_4$ where u_i is the solution to the ith side with inhomogeneous rest homogenous. Not that most solutions will be in form of $(\cos + \sin)(\cosh + \sinh)$ where terms die out based of IC
Poisson's Formula: $\Delta u = 0$ for $x^2 + y^2 < a^2 \quad u = h(\theta)$ for $x^2 + y^2 = a^2$
For $\boldsymbol{x} = (r, \theta) \quad \boldsymbol{x}' = (a, \phi)$ we have $r = |x| \quad a = |x'| \quad |x - x'|^2 = a^2 - 2ar\cos(\theta - \phi) + r^2$
$u(r, \theta) = \frac{(a^2 - r^2)}{2\pi} \int_0^{2\pi} \frac{h(\theta)}{a^2 - 2ar\cos(\theta - \phi) + r^2} d\phi$ and $u(\boldsymbol{x}) = \frac{a^2 - |\boldsymbol{x}|^2}{2\pi} \int_{|x'| = a} \frac{u(x')}{|x - x'|^2}$
Mean Value Property: $u(0) = \frac{a^2}{2\pi a} \int_{|x| = a} \frac{u(x')}{a^2} dS$ in other words the value at the center is the average of the boundary.
Wedges: $\{0 < \theta < \theta_0, 0 < r < a\}$. **Annulus:** $\{a < r < b, 0 < \theta < 2\pi\}$. **Exterior of Circle:** $\{r > a, 0 < \theta < 2\pi\}$. Make sure to consider **ALL** eigenvalues.

Divergence Theorem: $\int_D \nabla \cdot F dx = \int_{\partial D} F \cdot n dS$
Green's First Identity: $\int_{\partial D} v \frac{\partial u}{\partial n} dS = \int_D \nabla v \cdot \nabla u dx + \int_D v \Delta u dx$
Mean Value Principle: $u(0) = \frac{1}{\text{Area of D}} \int_D u dx$. Derived by recognized the value of the integral in the sphere is not dependent on the radius.
Dirichlet Principle Among all w that solve $w = h$ on ∂D, the one with lowest energy is the harmonic solution. Where $E[w] = \frac{1}{2} \int_D |\nabla w|^2 dx$
Green's Second Identity: $\int_D u \Delta v - v \Delta u dx = \int_{\partial D} u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} dS$
Representation Formula: $u(x_0) = \int_{\partial D} [-u(x)] \frac{\partial}{\partial n} (\frac{1}{|x - x_0|}) + \frac{1}{|x - x_0|} \frac{\partial u}{\partial n} \frac{dS}{4\pi}$
Green's Functions: the Green's Function $G(x)$ for the operator $-\Delta$ and the domain D at the point $x_0 \in D$ is a function defined for $x \in D$ st.
(i) $G(x)$ has cont. 2nd Derivatives and $\Delta G = 0 \quad \in D$ except at $x = x_0$ where $\Delta G = -\delta(x - x_0)$
(ii) $G(x) = 0 \quad$ on ∂D
(iii) The fucntions $G(x) + \frac{1}{4\pi |x - x_0|}$ is finite at x_0 and has continuous 2nd Derivatives and is harmonic at x_0
We can also denote it as $G(x, x_0)$
Green's Function for Laplace's Equation with Dirichlet BC: $u(x) = \int_{\partial D} u \frac{\partial G}{\partial n} dS$
Symmetric: Green's function is Symmetric $G(x, x_0) = G(x_0, x)$
Half Space: $G(x, x_0) = -\frac{1}{4\pi |x - x_0|} + \frac{1}{4\pi |x - x_0^*|}$ where $x_0^* = (x_0, y_0, -z_0)$ Note that x_0^* is the reflection of x_0 across the plane $z = 0$ and procides the "opposite" energy field to make the field 0 on the boundary.
Half Space with Dirichlet BC: $u = h \in \partial D \quad u(x_0) = \frac{z_0}{2\pi} \int_{\partial D} \frac{h(x)}{|x - x_0|^3} dS$
Sphere For a sphere of radius a centered at 0, $x_0^* = \frac{a^2 x_0}{|x_0|^2}$ we denote $\rho = |x - x_0| \quad \rho* = |x - x_0^*|$ Then the greens fucntion is $G(x, x_0) = -\frac{1}{4\pi\rho} + \frac{a}{|x_0|} \frac{1}{4\pi\rho*}$ and for $G(x, 0) = -\frac{1}{4\pi |x|} + \frac{1}{4\pi a}$ Note that in application this become the solution using Possion's Formula.

Sum/Difference $sin(A) + sin(B) = 2sin(\frac{A+B}{2})cos(\frac{A-B}{2}) \quad sin(A) - sin(B) = 2cos(\frac{A+B}{2})sin(\frac{A-B}{2})$
$cos(A) + cos(B) = 2cos(\frac{A+B}{2})cos(\frac{A-B}{2}) \quad cos(A) - cos(B) = -2sin(\frac{A+B}{2})sin(\frac{A-B}{2})$
Add/Subtract $sin(A \pm B) = sin(A)cos(B) \pm cos(A)sin(B) \quad cos(A \pm B) = cos(A)cos(B) \mp sin(A)sin(B)$
Product: $sin(A)sin(B) = \frac{1}{2}[cos(A - B) - cos(A + B)] \quad cos(A)cos(B) = \frac{1}{2}[cos(A - B) + cos(A + B)]$
$sin(A)cos(B) = \frac{1}{2}[sin(A + B) + sin(A - B)]$
Squared: $sin^2(A) = \frac{1}{2}[1 - cos(2A)] \quad cos^2(A) = \frac{1}{2}[1 + cos(2A)]$
Double Angle: $sin(2A) = 2sin(A)cos(A) \quad cos(2A) = cos^2(A) - sin^2(A) = 1 - 2\sin^2(A) = 2cos^2(A) - 1$