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Pranav Tikkawar

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1. A set is called clopen when it is both closed and open. Prove that the only clopen subsets of $\mathbb{R}$ are $\mathbb{R}$ and $\emptyset$. (Hint: $\mathbb{R}$ is connected.)

Solution: We will first prove that $\emptyset$ is clopen. Clearly it is vacously true that $\emptyset$ is both open and closed ie clopen. We know that the compliment of an open set is closed and the compliment of a closed set is open. Thus, the compliment of $\emptyset$ is $\mathbb{R}$ thus $\mathbb{R}$ is clopen.

Now we will prove that the only clopen subsets of $\mathbb{R}$ are $\mathbb{R}$ and $\emptyset$. Let A be a clopen subset of $\mathbb{R}$. Then we know that $A^c \cup A = \mathbb{R}$ since A is both open and closed, but this is only possible if A^c is empty or A is empty since $\mathbb{R}$ is connected.
Thus $A^c = \emptyset$ or $A = \emptyset$. Hence $A = \mathbb{R}$ or $A = \emptyset$. Thus the only clopen subsets of $\mathbb{R}$ are $\mathbb{R}$ and $\emptyset$.

2. Using the concept of open covers and compactness (and explicitly avoiding the Bolzano–Weierstrass Theorem for sequences or Problem 2 from Homework 5), prove that every bounded infinite set has a limit point.

Solution: Suppose A is a infinite bounded set bounded by $[-M, M]$ and suppose A has no limit points. Then there exists an $\epsilon > 0$ such that $V_\epsilon(x) \cap A = \{x\}$ for all $x \in A$. This means we can find an open cover of A which is finite since we can take the open cover $\{V_\epsilon(x)\}_{x \in A}$. This contradicts the fact that A is infinite. Thus every bounded infinite set has a limit point.

3. Prove that every closed connected set containing at least two points is perfect.

> **Solution:** Suppose A is a closed connected set containing at least two points. We need show that A is perfect, that A has no isolated points, or every point in A is a limit point of A.
> Suppose $x \in A$ is an isolated point. Then there exists an $\epsilon > 0$ such that $V_\epsilon(x) \cap A = \{x\}$. This means we can find an open cover of A which is finite since we can take the open cover $\{V_\epsilon(x)\}_{x \in A}$. This contradicts the fact that A is connected. Thus every point in A is a limit point of A. Hence A is perfect.

4. Let $Q = \{q_1, q_2, \cdots, q_n, \cdots\}$. Also define $\epsilon_n = 2^{-n}$ and let

$$U = \bigcup_{n=1}^{\infty} V_{\epsilon_n}(q_n), \quad F = U^c$$

(a) Prove that F is a closed, nonempty set containing only irrational numbers.

> **Solution:** We can see that F is closed since U is the union of open sets, and thus open. We know that the compliment of an open set is closed. Thus F is closed.
> To show that F is nonempty, we can by contradiction take $F = \emptyset$ and thus $U = \mathbb{R}$. Now we can consider the idea that The "length" of the interval of $V_{\epsilon_n}(q_n)$ is $2\epsilon_n = 2^{-n+1}$. Thus we can see that the total length of the union of all the intervals is $\sum_{n=1}^{\infty} 2^{-n+1} = 2$. Thus we can see that the union of all the intervals is bounded by 2 and thus U cannot be $\mathbb{R}$. Thus F is nonempty.
> To show that F contains only irrational numbers, we can see that by definition $\mathbb{Q} \subset U$. Thus since $F = U^c$, we can see that F cannot contain any rational numbers. Thus F contains only irrational numbers.

(b) Does F contain any nonempty open intervals?

> **Solution:** Let us assume by contradiction that there is a nonempty open interval O contained in F. We know that by the density of $\mathbb{Q}$ in $\mathbb{R}$, there will exist a rational number $q \in O$. This means that q is in the open interval O and thus $q \in F$. But we know that F contains only irrational numbers. Thus we can see that there cannot be a nonempty open interval contained in F.
> Thus we can see that F does not contain any nonempty open intervals.

5. Let C be the Cantor set, and define

$$C + C = \{x + y : x, y \in C\}$$

Since $C \subset [0, 1]$, $C + C \subseteq [0, 2]$. Surprisingly, we also have that $C + C \supseteq [0, 2]$. Let $s \in [0, 2]$ be arbitrary.

(a) If $C_1 = [0, 1/3] \cup [2/3, 1]$ is the first level Cantor set, prove that there exist points $x_1, y_1 \in C$ such that $x_1 + y_1 = s$.

Solution: Let $s \in [0, 2]$. Then either $s \in [0, 2/3], s \in [2/3, 4/3]$ or $s \in [4/3, 2]$. Let us label the closed intervals A_1, A_2, A_3 respectively. Also let us define $B_1 = [0, 1/3]$ and $B_2 = [2/3, 1]$. Then we can see that $C_1 = B_1 \cup B_2$.
If $s \in [0, 2/3]$, then we can see for $x_1, y_1 \in B_1$, $x_1 + y_1 \in A_1$ and thus $x_1 + y_1 = s$.
If $s \in [4/3, 2]$, then we can see for $x_1, y_1 \in B_2$, $x_1 + y_1 \in A_3$ and thus $x_1 + y_1 = s$.
If $s \in [2/3, 4/3]$, then we can see WLOG $x_1 \in B_1$ and $y_1 \in B_2$, $x_1 + y_1 \in A_2$ and thus $x_1 + y_1 = s$.
Thus we can see that for any $s \in [0, 2]$, we can find $x_1, y_1 \in C_1$ such that $x_1 + y_1 = s$.

(b) Now, let C_n denote the nth level Cantor set. Prove that there exist $x_n, y_n \in C_n$ such that $x_n + y_n = s$.

Solution: We can do this by induction on the number of levels of the Cantor set, n. We can clealy see the base case is true by part a.
Our induction hypothesis is that for all $n \leq k$, there exist $x_n, y_n \in C_n$ such that $x_n + y_n = s$.
Now we will show that for $n = k + 1$, there exist $x_{k+1}, y_{k+1} \in C_{k+1}$ such that $x_{k+1} + y_{k+1} = s$.

$$\begin{aligned}
C_{k+1} + C_{k+1} &= \left(\left(\frac{1}{3}C_k\right) \cup \left(\frac{2}{3} + \frac{C_k}{3}\right)\right) + \left(\left(\frac{1}{3}C_k\right) \cup \left(\frac{2}{3} + \frac{C_k}{3}\right)\right) \\
&= \left(\frac{2}{3}(C_k + C_k)\right) \cup \left(\frac{1}{3}(C_k + C_k) + \frac{2}{3}\right) \cup \left(\frac{4}{3} + \frac{1}{3}(C_k + C_k)\right) \\
&= \left(\frac{2}{3}[0, 2]\right) \cup \left(\frac{1}{3}[0, 2] + \frac{2}{3}\right) \cup \left(\frac{4}{3} + \frac{1}{3}[0, 2]\right) \\
&= [0, \frac{4}{3}] \cup [\frac{2}{3}, \frac{4}{3}] \cup [\frac{4}{3}, 2] \\
&= [0, 2].
\end{aligned}$$

Thus we can see that for $n = k + 1$, there exist $x_{k+1}, y_{k+1} \in C_{k+1}$ such that $x_{k+1} + y_{k+1} = s$.
Thus by induction we can see that for all $n \in \mathbb{N}$, there exist $x_n, y_n \in C_n$ such that $x_n + y_n = s$.

(c) The sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ you constructed may not converge. Prove, nevertheless, that there exist $x, y \in C$ with $x + y = s$.

Solution: The sequences sequences $\{x_n\}$ and $\{\}$ are bounded by $[0, 2]$ and thus by Bolzano-Weierstrass, they have convergent subsequences. Let us call the

convergent subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$. and suppose they converge to x, y respectively. Since the Cantor set is compact we can also say that these limits are also in C. Thus we can see that $x + y = s$.
Thus we can see that there exist $x, y \in C$ with $x + y = s$.

6. (The Lebesgue covering lemma) Suppose K is compact and $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ is an open cover of K. Prove that there exists a $\delta > 0$ such that for every $x \in K$, there exists an α such that
$$V_\delta(x) \subset U_\alpha.$$

Solution: Suppose by K is a compact set with an open cover $\mathscr{U}$. Since K is compact then there is a finite subcover from $\mathscr{U}$ let us call it $F = \{U_1, U_2, \cdots, U_n\}$ for $n \in \mathbb{N}$ such that $K \subseteq \bigcup_{i=1}^{n} U_i$. Let $x \in K$ be arbitrary. Then

$$\begin{aligned} x \in K &\implies x \in \bigcup_{i=1}^{n} U_i \\ &\implies \exists i \in \{1, 2, \cdots, n\} \text{ such that } x \in U_i \\ &\implies V_{\delta_x}(x) \subset U_i \text{ for some } \delta_x > 0. \end{aligned}$$

We can say the the collection of these $V_{\delta_x}(x)$ is an open cover of K. Since K is compact, we can find a finite subcover of K from the collection of $V_{\delta_x}(x)$. Let us call this finite subcover $F = \{V_{\delta_{x_1}}(x_1), V_{\delta_{x_2}}(x_2), \cdots, V_{\delta_{x_n}}(x_n)\}$. Let $\delta = \min\{\delta_{x_1}, \delta_{x_2}, \cdots, \delta_{x_n}\}$. Then we can see that for every $x \in K$, there exists an α such that $V_\delta(x) \subset U_\alpha$.
Thus we can see that there exists a $\delta > 0$ such that for every $x \in K$, there exists an α such that $V_\delta(x) \subset U_\alpha$.