01:640:478 - Intense notes

Pranav Tikkawar

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Brownian motion:
(1) Deriving conditional distribution given future values

$$X(s)|X(t), s \leq t \sim N(\frac{s}{t}B, \frac{s}{t}(t-s))$$

where $X(t) = B$
Ex 10.1 part II,
(2) Hitting times T_a

$$P(T_a < t) = \frac{2}{\sqrt{2\pi}} \int_{|a|/\sqrt{t}}^{\infty} e^{-y^2} dy$$

(3) Max of a Brownian Motion in an interval
Today we look at variaions of Brownian motion.
(1) BM with a drift
Defined as

$$\begin{cases} X(0) = 0 \\ \{X(t), t \geq 0\} \text{ has stationary independent increments} \\ X(t) \text{is normally distributed with mean } \mu t \text{ and variance } \sigma^2 t \end{cases}$$

And equivalent definition is

$$X(t) = \mu t + \sigma B(t)$$

where $B(t)$ is a Brownian motion.
It is simlar to a regular BM but slanted upwards.

Example. Let $(\{X(t)\}, t \geq 0)$ be a BM with drift $\mu = .8$ and variance $\sigma^2 = .4$ Find the probability that $2 \leq X(8) \leq 5$

Solution: Look at time t = 8

$$X(8) = .8 \cdot 8 + \sqrt{.4}B(8)$$
$$X(8) = 6.4 + \sqrt{.4}B(8)$$
$$X(8) = 6.4 + \sqrt{.4}Z \quad \text{where } Z \sim N(0,1)$$
$$P(2 \leq X(8) \leq 5) = P(2 \leq 6.4 + \sqrt{.4}Z \leq 5)$$
$$P(2 - 6.4 \leq \sqrt{.4}Z \leq 5 - 6.4) = P(-4.4 \leq \sqrt{.4}Z \leq -1.4)$$
$$P\left(\frac{-4.4}{\sqrt{.4}} \leq Z \leq \frac{-1.4}{\sqrt{.4}}\right)$$

or

$$X(8) \sim N(.8 \cdot 8, .4 \cdot 8)$$

$$P(2 \leq X(8) \leq 5) = P(\frac{2-6.4}{\sqrt{3.2}} < Z < \frac{5-6.4}{\sqrt{3.2}})$$

it is $\approx .2108$

(2) Geometric Brownian Motion

If $\{Y(t), t \geq 0\}$ is a GM then $\{X(t), t \geq 0\}$ is a GBM if

$$X(t) = e^{Y(t)}$$

where $Y(t)$ is a BM with drift μ and variance σ^2

The expected value of a GBM given the history of the process up to a given time is

$$\begin{aligned} E[X(t)|X(u), 0 \leq u \leq s] &= E[e^{Y(t)}|Y(u), 0 \leq u \leq s] \\ &= E[e^{Y(t)-Y(s)+Y(s)}|Y(u), 0 \leq u \leq s] \\ &= e^{Y(s)}E[e^{Y(t)-Y(s)}|Y(u), 0 \leq u \leq s] \\ &= X(t)E[e^{Y(t)-Y(s)}] \\ &= X(t)e^{\mu(t-s)+\frac{\sigma^2}{2}(t-s)} \end{aligned}$$

Therefore,

$$E[X(t)|X(u)0 < u < s] = X(t)e^{(t-s)(\mu+\sigma^2/2)}$$

We can use this to model stock prices over time in general non negative random fluctuations. In general we can consider this as a percentage changes in prices are independent and independently distributed.

Let X_n = price of some stock at time n

Assume $\frac{X_n}{X_{n-1}} < 1$ iid

Define $Y_n = \frac{X_n}{X_{n-1}}$ or $X_n = X_{n-1}Y_n$

We can then iterate to see that $X_n = X_0 \prod_{i=1}^{n} Y_i$

Thus $\log(X_n) = \log(X_0) + \sum_{i=1}^{n} \log(Y_i)$

Since $log(Y_i), i \geq 1$ are iid, $\log(X_n)$ will, when suitably normalized, approximatly be Brownian motion with drift, and thus X_n will be a GBM.