Math 292 Homework 5

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Problem 1

Solve $x''(t) + 4x(t) = 3cos(2t)$ with $x(0) = x_0$ and $x'(0) = y_0$
Creating a driven first order System:
We can rewrite the equation as a first order system by letting $y(t) = x'(t)$
Then we have $y'(t) + 4x(t) = 3cos(2t)$
Thus we get the matrix A in the equation $\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 3cos(2t) \end{bmatrix}$
Where $A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$
Solving the Homogeneous System:
The characteristic equation of A is $det(A - \mu I) = 0$
This gives us $\mu^2 + 4 = 0$
Thus we have $\mu = 2i$ and $\mu = -2i$
The eigenvectors of A are $v_1 = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -2i \end{bmatrix}$
We can then split one of the eigenvectors and imaginary exponential into real and imaginary parts to get $e^{2it} = cos(2t) + isin(2t)$

$$cos(2t) + isin(2t)\begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

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Where $A = \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix}$
Solving the Homogeneous System:
The characteristic equation of A is $det(A - \mu I) = 0$
This gives us $\mu^2 + 4\mu = 0$
Thus we have $\mu = 0$ and $\mu = -4$
The eigenvectors of A are $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

Thus the matrix exponential of A is $e^{At} = \begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} e^{0t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix}^{-1}$

$$e^{At} = \begin{bmatrix} e & \frac{e}{4} - \frac{e^{-4t}}{4} \\ 0 & e^{-4t} \end{bmatrix}$$

Solving the Inhomogeneous System:
Given the matrix exponential of A, we can solve the inhomogeneous system by using the formula $x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}g(s)ds$

Where $g(s) = \begin{bmatrix} 0 \\ 3cos(2s) \end{bmatrix}$

Thus we have $x(t) = \begin{bmatrix} e & \frac{e}{4} - \frac{e^{-4t}}{4} \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \int_0^t \begin{bmatrix} e & \frac{e}{4} - \frac{e^{-4(t-s)}}{4} \\ 0 & e^{-4(t-s)} \end{bmatrix} \begin{bmatrix} 0 \\ 3cos(2s) \end{bmatrix} ds$

The integral can be simplified to $\int_0^t \begin{bmatrix} 3cos(2s)(\frac{e}{4} - \frac{e^{-4(t-s)}}{4}) \\ 3cos(2s)(e^{-4(t-s)}) \end{bmatrix} ds$

Thus the integral evalutes to

$$\begin{bmatrix} \frac{3}{40}((5e-1)sin(2t) - 2cos(2t)) + \frac{3}{20}e^{-4t} \\ \frac{3}{10}(sin(2t) + 2cos(2t)) - \frac{3}{5}e^{-4t} \end{bmatrix}$$

Thus the solution to the system's x component is

$$x(t) = ex_0 + y_0(\frac{e}{4} - \frac{e^{-4t}}{4}) + \frac{3}{40}((5e-1)sin(2t) - 2cos(2t)) + \frac{3}{20}e^{-4t}$$

Problem 2

Consider the vector field $v(x,t) = \begin{bmatrix} -(2+y)(x+y) \\ -y(1-x) \end{bmatrix}$

a

Finding the Equilibrium Points:
$-(2+y)(x+y) = 0, -y(1-x) = 0$
Thus we have $(x,y) = (0,0), (1,-1), (1,-2)$ **Linearizing the System:**
The Jacobian of the system is

$$J = \begin{bmatrix} -2-y & -x-2y-2 \\ y & x-1 \end{bmatrix}$$

Evaluating the Jacobian at the equilibrium points gives us

$$J(0,0) = \begin{bmatrix} -2 & -2 \\ 0 & -1 \end{bmatrix}$$

$$J(1,-1) = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}$$

$$J(1,-2)=\begin{bmatrix}0 & 1\\ -2 & 0\end{bmatrix}$$

Stability of the Equilibrium Points:
The Trace and Determinant of the Jacobian at $(0,0)$ are -3 and 2 respectively. Thus near the equilibrium point $(0,0)$ the system is a sink, and therefore is a stable equilibrium point.
The Trace and Determinant of the Jacobian at $(1,-1)$ are -1 and -1 respectively.
Thus near the equilibrium point $(1,-1)$ the system is a saddle, and therefore is an unstable equilibrium point.
The Trace and Determinant of the Jacobian at $(1,-2)$ are 0 and 2 respectively. Thus near the equilibrium point $(1,-2)$ the system is a periodic orbit, and therefore is a not a stable equilibrium point, but is Lyanupov Stable.

b

Consider $v(x,t)=\begin{bmatrix}(2+y)(x+y)\\ -y(1-x)\end{bmatrix}$ **Finding the Equilibrium Points:**
$(2+y)(x+y)=0, -y(1-x)=0$
Thus we have $(x,y)=(0,0),(1,-1),(1,-2)$ **Linearizing the System:**
The Jacobian of the system is

$$J=\begin{bmatrix}2+y & x+2y+2\\ y & x-1\end{bmatrix}$$

Evaluating the Jacobian at the equilibrium points gives us

$$J(0,0)=\begin{bmatrix}2 & 2\\ 0 & -1\end{bmatrix}$$

$$J(1,-1)=\begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}$$

$$J(1,-2)=\begin{bmatrix}0 & -1\\ -2 & 0\end{bmatrix}$$

Stability of the Equilibrium Points:
The Trace and Determinant of the Jacobian at $(0,0)$ are 1 and -2 respectively. Thus near the equilibrium point $(0,0)$ the system is a saddle, and therefore is an unstable equilibrium point.
The Trace and Determinant of the Jacobian at $(1,-1)$ are 1 and 1 respectively. Thus near the equilibrium point $(1,-1)$ the system is a source, and therefore is an unstable equilibrium point.
The Trace and Determinant of the Jacobian at $(1,-2)$ are 0 and 2 respectively. Thus near the equilibrium point $(1,-2)$ the system is a periodic orbit, and therefore is a not a stable equilibrium point, but is Lyanupov Stable.

Problem 3

Find exact solution of $x'(t) = v(x(t), t)$ and $x(0) = 0$ for $v(x, t) = 2t(1 + x)$
Starting from $X_0 = 0$, then compute X_1, X_2, X_3, X_4
Picard Iteration:
We can solve the equation by using Picard Iteration.

$$X_0 = 0$$
$$X_1 = \int_0^t 2sds = \int_0^t 2sds = t^2$$

$$X_2 = \int_0^t 2s(1 + X_1)ds = \int_0^t 2s(1 + s^2)ds = t^2 + \frac{t^4}{2}$$

$$X_3 = \int_0^t 2s(1 + X_2)ds = \int_0^t 2s(1 + s^2 + \frac{s^4}{2})ds = t^2 + \frac{t^4}{2} + \frac{t^6}{6}$$

$$X_4 = \int_0^t 2s(1 + X_3)ds = \int_0^t 2s(1 + s^2 + \frac{s^4}{2} + \frac{s^6}{6})ds = t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \frac{t^8}{24}$$

Thus the exact solution is $X(t) = \sum_{n=1}^{\infty} \frac{t^{2n}}{n!}$

$$X(t) = e^{t^2} - 1$$