PDEs

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Introduction

What is a PDE?
Start with ODE: $u = u(x)$, equation involing indepednant variable x and dependent variable u as well as its derivatives.
Example: $u'' - xu = 0, x \in I$ (Airy Functions). Second order Linear ODE.
$Lu = u'' - xu$ Where L is an operator.
Linearity means 2 things: $L(u_1 + u_2) = Lu_1 + Lu_2$ and $L(cu) = cLu$
$\forall u_1, u_2 \in \mathcal{F}, \forall c \in \mathbb{F}$
PDE: $u = u(x, y, ...)$ equation involving independent variables $x, y, ...$ and Function u as well as its partial derivatives $u_x, u_y, u_{xx}, u_{yy}, u_{xy}$
Example: $x^2u - sin(xy)u_{xxyy} + 3u_x = 0$ 4th order linear PDE of 2 vars.
Remark: Importance of linearity: say u_1, u_2 are solutions of a Linear PDE: $Lu_1 = 0, Lu_2 = 0$ then $c_1u_1 + c_2u_2, (\forall c_1, c_2 \in \mathbb{R})$ is also a solution of $Lu = 0$
More generally, if $u_1, ..., u_n$ are solutions, then $\sum_{j=1}^{n} c_j u_j$ is also a solution.
Example: $u = u(x, y)$, solve $u_{xx} = 0$. $u_x = f(y)$, $u = f(y)x + g(y)$ where f, g are arbitrary functions. $(\forall f, g \in \mathcal{F})$
$Lu = 0$ is homogenous, $Lu = f$ is non-homogenous.

1.2 First Order PDE of x,y

$$x, y, u_x, u_y, u$$

Generally: $a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0$
Example 1: $u_x = 0$: $u = f(y)$ No change in the x direction, hence the function stays constant on all horizontal lines.
Example 2: Geometric Method $au_x + bu_y = 0, (a, b \in \mathbb{R})$
$\vec{v} = (a, b)$; $\nabla u =< u_x, u_y >$; $\nabla u \cdot \vec{v} = 0 \to D_{\vec{v}}u = 0$
No change in the "v" direction (say $|\vec{v}| = 1$)
$x = ta, y = tb \to ay - bx = 0$ On the lines $ay - bx = c$ where c is a constant, the function u is constant. Lets call its value $f(c)$
$u(x, y) = f(ay - bx)$ where f is a function of a single variable

The lines where these are solutions/constant are called **characteristic lines**
Check: $u_x = -bf'(ay - bx)$, $u_y = af'(ay - bx)$
Change of variable Change our plane such that $\vec{v}$ is our "x" axis.
View $(x, y) = x + iy, (x', y') = x' + iy'$. Multiplying by $e^{i\alpha}$ rotates the plane ccw by α
$x' + iy' = (x + iy)e^{i\alpha}$ where $\vec{v} = (a, b) = (cos(\alpha), sin(\alpha))$
$x' = xcos(\alpha) + ysin(\alpha)$, $y' = -xsin(\alpha) + ycos(\alpha)$
Rewrite PDE in our new system: $u = u(x', y') = u(x'(x, y), y'(x, y))$

$$u_x = u_{x'}\frac{\partial x'}{\partial x} + u_{y'}\frac{\partial y'}{\partial x}$$
$$u_x = au_{x'} - bu_{y'}$$
$$u_y = u_{x'}\frac{\partial x'}{\partial y} + u_{y'}\frac{\partial y'}{\partial y}$$
$$u_y = au_{y'} + bu_{x'}$$
$$au_x + bu_y = 0$$
$$a^2u_{x'} + b^2u_{y'} = 0$$
$$u_{x'} = 0$$
$$u = f(y') = f(ay - bx)$$

Example 3: $u_x + yu_y = 0$
u doesnt change in the direction of $\vec{v} = (1, y)$ at the point (x, y)
Lets call C the characteristic curve: $\begin{cases} x = x(t), y = y(t) \end{cases}$ tangent to $\vec{v}$ at any (x, y)

$$\frac{d}{dt}u(x(t), y(t)) = 0$$

$$\frac{dy}{dx} = y \to y(x) = ce^x, (\forall c \in \mathbb{R})$$
$$u(x, ce^x) = f(c)$$
$$u(x, y) = f(ye^{-x})$$

Remark: More generally $a(x, y)u_x + b(x, y)u_y = 0$

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}\text{: ODE for characteristic curves}$$

1.3.1 Mass flow/Transport Equation/Continuity Equation

Substance that flows in space. (eg. fluid)
$\rho = \rho(x, y, z, t)$ density of the substance at point (x, y, z) at time t
$\vec{v} = \vec{v}(x, y, z, t)$ velocity of the substance at point (x, y, z) at time t
Consider R as an arbitrary region in space.
Conservation of mass: $m(t) = \int_R \rho(x, y, z, t)dV$ mass in R at time t
Consider $[t, t + \Delta t]$, $m(t + \Delta t) = \int_R \rho(x, y, z, t + \Delta t)dV$
Substance leaves/enters in R through the boundary ∂R
Consider a small part of the boundary call it ∂S and see how much mass has left through this boundary patch over time period $[t, t + \Delta t]$
We want to introdouce the "normal" $\vec{n}$ over the boundary ∂S

$$\text{height} = \vec{v}\Delta t \cdot \vec{n} \text{ and area of base} = dS \rightarrow \text{volume} = \Delta t\vec{v} \cdot \vec{n}dS$$

$$\rho = \text{mass/vol} \rightarrow \Delta t\rho\vec{v} \cdot \vec{n}dS$$

$$\Delta m = \Delta t \int_{\partial R} \rho\vec{v} \cdot \vec{n}dS$$

Mass Conservation: $m(t + \Delta t) = m(t) - \Delta m$

$$\frac{1}{\Delta t} \int_R \rho(x, y, z, t + \Delta t) - \rho(x, y, z, t)dV = \int_{\partial R} \rho\vec{v} \cdot \vec{n}dS$$

$$= \int_R div(\rho\vec{v})dV$$

Where $div(\vec{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$
Let $\Delta t \rightarrow 0$

$$\frac{\partial \rho}{\partial t} + div(\rho\vec{v}) = 0$$

This is the Transport Equation.
Example: $\vec{v} = c(1, 0)$ and $\rho = \rho(x, t)$

$$\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = 0$$

$$\rho(t, x) = f(x - ct)$$

$\rho_t + c\rho_x = 0, t > 0, x \in \mathbb{R}, \rho(0, x) = \rho_0(x), x \in \mathbb{R}$ Initial condition

$$\rho(t, x) = \rho_0(x - ct)$$

1.3.2 Heat Equation/Diffusion/Energy Flux

Flow of energy: $\vec{q}(x, y, z, t)$ energy flux at point (x, y, z) at time t
During the time interval $[t, t + \Delta t]$ the energy $\Delta E = \Delta t \int_{\partial R} \vec{q} \cdot \vec{n} dS$ has left the test volume R through the boundary ∂R
Consider the patch ∂S of the boundary ∂R and the normal $\vec{n}$

$$\Delta t \vec{q} \cdot \vec{n} dS \to e(\vec{n}) \Delta t dS$$

Cauchy tensor deformation.
To measure the energy inside R we need the specific heat $c(x, y, z)$ and it measure the energy containing in 1 degree of temperature in 1 unit mass.

$$c = \frac{e}{T \cdot \text{mass}} = \frac{e}{T \cdot \rho \text{vol}}$$

$$T c \rho = \frac{e}{\text{vol}}$$

$$E(t) = \int_R T(\vec{x}, t) \rho(\vec{x}) c(\vec{x}) dV$$

This is the energy inside R at time t.

$$E(t + \Delta t) = E(t) - \Delta E$$

$$\int_R T_t(\vec{x}, t) \rho(\vec{x}) c(\vec{x}) + div \vec{q} dV = 0$$

$$T_t c \rho + div \vec{q} = 0$$

Incomplete: we need to know how $\vec{q}$ depends on T
Forier's law of heat conduction: $\vec{q} = -k \nabla T$
$k(\vec{x})$ = heat conductivity of the material
Heat flows from hot to cold.

$$grad(t)$$

is the direction of the greatest increase of T

$$c \rho T_t - div(k \nabla T) = 0$$

Specific case: Assume c, ρ, k are constants.

$$\nabla T = \begin{pmatrix} T_x \\ T_y \\ T_z \end{pmatrix} \cdot \left(d_x, d_y, d_z\right)$$

$$div \nabla T = \nabla \cdot \nabla T = \nabla^2 T = T_{xx} + T_{yy} + T_{zz}$$

This is the laplacian of T

$$T_t = \mathbf{D} \nabla^2 T, \mathbf{D} = \frac{k}{c \rho}$$

Fick's law of Diffusion.
High density to low density.

Wave Equation

Consider a string
We have x and $u(x,t)$
Consider a small part of the string $[x, x+\Delta x]$ called $d\ell$
There is a tangent force $T(x+\Delta x, t)$
The mass of the string is $m(x)$ from origin to x
Newton's law: $F = ma$

$$T(x+\Delta x, t) - T(x,t) = (m(x+\Delta x) - m(x))\vec{r}_t t(x,t)$$

Divide by Δx and let $\Delta x \to 0$

$$\vec{T}_x(x,t) = \rho(x)\vec{r}_t t(x,t)$$

Where ρ is the linear density of the string
T is tangent to the string: T is parallel to $\vec{r}_x$
introduce $\vec{\tau} = \frac{\vec{r}_x}{|\vec{r}_x|}$

$$T = T(x,t)\vec{\tau}$$

Where T is constant along the string.
Assume small vibration so that $|u_x|$ is small.

$$\vec{r} = (1, u_x) = (1, 0)$$

$$u_t + \nabla \cdot (u\vec{v}) = 0$$

1D wave equation: $u_{tt} = c^2 u_{xx}$
In general $u_{tt} - c^2 \Delta u_{xx} =$
Laplaces Equation: $\Delta u = 0$
Remark describes equilibrium
Functions that solves Laplace's equation are called harmonic functions.
Example: 1D: $u_{xx} = 0$: Linear
Example: 2D: $u_{xx} + u_{yy} = 0$:
$f(z) = u(x,y) + iv(x,y)$
holomorphic /complex analytic
then $\Delta u = 0$ and $\Delta v = 0$
Then taking the real and imaginary parts of these equations we get the harmonic functions in 2D.
Imp Example: $u = ln(x^2 + y^2)$
Remark: Characterization of steady(no time); irrotational(zero curl); incompressible (zero divergence) flow fiels $\vec{F}$

$$\nabla \times \vec{F} = 0 \to \vec{F} = \nabla u$$

where u is the potential of the flow field.

$$\nabla \cdot \vec{F} = 0 \to \text{div}\nabla u = \Delta u = 0$$

1.4 Initial and Boundary Conditions

PDE + BCs = Boundary Value Problem (BVP)
PDEs describing equilibrium phenomena are paired wiht boundary conditions (BCs).
Dirichlet BC Suppose a space D and a boundary ∂D
$\begin{cases} \Delta u = 0 \text{ in } D \\ u = \phi \text{ on } \partial D \end{cases}$
Body D in thermal equilibrium knowing the boundary temperature find the temperature inside the body.
Prescribing the function on the boundary is called Dirichlet BC.
Neumann BC
$\begin{cases} \Delta u = 0 \text{ in } D \\ \partial n \vec{u} = \psi \text{ on } \partial D \end{cases}$ Neumann BC.
$\partial_n u = \Delta u \cdot n$ This is heat flux
EG: Insulated object: $\partial_n u = 0$ on ∂D
if u is a solution then so is $u + c$
Mixed Boundary Conditions $\begin{cases} \Delta u = 0 \text{ in } D \\ u = \phi \text{ on } \partial D_1 \\ \partial_n u = \psi \text{ on } \partial D_2 \end{cases}$
Robin BC $\begin{cases} \Delta u = 0 \text{ in } D \\ \alpha u + \beta \partial_n u = \gamma \text{ on } \partial D \end{cases}$
Example:

$$k\partial_n u + c(u - u_\infty) = 0 \text{ on}$$

where k is thermal conductivity, c is convective heat transfer coefficient, u_∞ is the ambient temperature.
Example

$$\begin{cases} u'' = 0 \text{ on } (0,1) \\ u(0) = 0 \\ u(1) = 1 \end{cases} \to u(x) = x$$

PDEs describing dynamic processes We have time variable! Usually are paired with initial conditions (IC) and BC.

- Thermodynamics: u_t

IC: $u(t_0, x) = \phi(x), \forall x \in D$

- Newtonian Mechanics: u_{tt}

IC: $\begin{cases} u(t_0,x) = \phi(x) \\ u_t(t_0,x) = \psi(x) \end{cases}, \forall x \in D$

Example

$$u_t - u_{xx} = f(t,x), t > 0, x \in (0,1)$$
$$u(0,x) = u_0(x) \text{ IC}$$
$$u(t,0) = \phi(t) \text{ BC}$$
$$u(t,1) = \psi(t) \text{ BC}$$

Remark: If D is unbounded, we'll need conditions at infinity.

1.5 Well Posed problems

Well pose problems has 3 criteria:

- Existence: There exists a solution
- Uniqueness: The solution is unique
- Stability: The solution depends continuously on the data. (IC,BC,source terms)

$$Ax = b$$

Where A is $m \times n$ matrix, x is $n \times 1$ vector, b is $m \times 1$ vector.

- $m > n$: Existence may fail too many variables
- $m < n$: Uniqueness may fail too many equations
- $m = n$ and A is invertible: Existence, uniqueness and stability!

$x = A^{-1}b$

$$\begin{cases} Ax = b \\ A(x_\epsilon) = b + \epsilon \end{cases}$$

$$||x - x_\epsilon|| \leq |A^{-1}\epsilon| \leq ||A^{-1}|| \cdot ||\epsilon||$$

Remark If A has a very small eigenvalue, then $||A^{-1}||$ is very large. Ill conditioned problems

1.6 Types of 2nd Order PDEs

In the case of 2 variables x, y

$$\begin{gathered} au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0 \\ Q(x,y) = ax^2 + bxy + cy^2 + dx + ey + f \\ H = b^2 - 4ac\text{the Discriminant} \end{gathered}$$

Definition

$$\begin{cases} H < 0 \text{ Elliptic: By linear change of varibale it can be reduced to the normal form: } u_{xx} + u_{yy} + ... = 0 \\ \\ H = 0 \text{ Parabolic: By linear change of variable it can be reduced to the normal form: } u_{xx} + .. = 0 \\ \\ H > 0 \text{ Hyperbolic By linear change of variable it can be reduced to the normal form: } u_{xx} - u_{yy} = 0 \end{cases}$$

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$$au_{xx} + bu_{xy} + cu_{yy} + ... = 0$$

Can be converted to quaratic form Proof of cases: $u_{xx} + bu_{xy} + cu_{yy} ... = 0$ This is **(1)**

$$\begin{aligned} x^2 + bxy + cy^2 &= x^2 + 2xby/2 + (by)^2/4 + cy^2 - (by)^2/4 \\ &= (x + by/2)^2 + -Hy^2/4 \end{aligned}$$

Let $x = \xi, y = b\xi/2 + \sqrt{H}\eta/2$
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Extended to a,b,c functions of x,y.
$H = H(x, y)$
Same definition categorizes the type of equation at (x,y)
Example Euler-Tricomi Equation

$$u_{xx} - xu_{yy} = 0$$

$$H = 4x$$

This corresponds to transonic flow.
Hyperbolic become subsonic
Elliptic become supersonic
Parabolic is sonic boom
Matrix Perspective Rewrite (1) as $div(A\nabla u) + ... = 0$

$$A = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

$$\partial_x(au_x + b/2u_y) + \partial_y(b/2u_x + cu_y)$$

$$H < 0 \Leftrightarrow det(A) > 0$$

Evals have the same sign (Elliptic)

$$H = 0 \Leftrightarrow det(A) = 0$$

One eval is zero (Parabolic)

$$H > 0 \Leftrightarrow det(A) < 0$$

Evals have different signs (Hyperbolic)
For more variables: $u(x_1, x_2, ..., x_n)$

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix}$$

A is symmetric

$$div(A\nabla u) + ... = 0. \text{ This is (2)}$$

$$\sum a_i j u_{x_i x_j} + ... = 0$$

Def. (2) is

- Elliptic if all eigenvalues of A have same sign Δ
- Parabolic if one eval is 0 but all others have same sign $\partial_t - \Delta$
- Hyperbolic if evals one eval is of one sign and all the others are of the opposite sign. $\partial_t^2 - \Delta$

Consider the equation $Ax \cdot x$
Where A is symmetric
This is a quaratic form
Goal reduce to normal form.

$$A = P \wedge P^T$$

Where $\wedge$ is the eigenvalue decomposition
And $P^T = P^{-1}$ (orthogonal)

$$Ax \cdot x = P \wedge P^T x \cdot x$$
$$= \wedge P^T x \cdot P^T x$$

Let $y = P^T x$

$$= \wedge y \cdot y$$
$$= \sum \lambda_i y_i^2$$

Where λ_i are the eigenvalues of A
If all $\lambda_i > 0$ let $z_i = \sqrt{\lambda_i} y_i$

$$= \sum z_i^2$$

If all $\lambda_i < 0$ let $z_i = \sqrt{-\lambda_i} y_i$

$$= -\sum z_i^2$$

In z-variables

$$Ax \cdot x = z_1^2 - z_2^2 + ...$$

This is called Sylvester's law of inertia.

$$div(A\nabla u) + ... = 0$$
$$A = P \wedge P^T$$
$$y = P^T x$$
$$\nabla_x u = P \nabla_y u$$

Try and figure this out
Note that $\nabla_x = (u_x 1, u_x 2, u_x 3, ...)$

$$div_x(F) = div_y(P^T F)$$
$$div_x(P \wedge P^T \nabla_x u) = div_x(P \wedge \nabla_y u)$$
$$= div_y(PP^T \wedge \nabla_y u) = div_y(\wedge \nabla_y u)$$
$$= \sum \lambda_i u_{y_i y_i}$$