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1 Chapter 11: Confidence Intervals

Given a α such that $0 < \alpha < 1$, an Interval Estimation stratgey provides two statistics (r.v) L and R s.t $P(L < \theta < R) = 1 - \alpha$. The interval $[L, R]$ is called a $(1 - \alpha)\%$ confidence interval for θ.
We can say that if you repeat the expirment N times and gte N intervals, then $(1 - \alpha)\%$ of the intervals will contain the true value of θ.

Definition (Confidence Interval)**.** A confidence interval with (1-α) confidence level are two statistics L and R such that $P(L < \theta < R) = 1 - \alpha$.

Remark. If someone says after an expirment that $2 < \lambda < 2.1$ with 90% confidence, it means that on average that 90% of the intervals will contain the true value of λ.

We want CI to be symetric about $\bar{X}$

CI so far: $N(\mu, \sigma^2)$ population

1. σ^2 known: $\bar{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$

2. σ^2 unknown: $\bar{X} \pm t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}$

We can think of $z_{\alpha/2}$ in the normal curve as the shaded area in the tails.

New Context: Two pops $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ and sample from both n_1 and n_2 from each

$$X_{11} \dots X_{1n_1} \sim N(\mu_1, \sigma_1^2)$$

$$X_{21} \dots X_{2n_2} \sim N(\mu_2, \sigma_2^2)$$

These are independent but not necessarily identically distributed.
Let $\bar{X}_1$ and $\bar{X}_2$ be the sample means and S_1^2 and S_2^2 be the sample variances.
Want CI for $\mu_1 - \mu_2$
Case 1: σ_1^2 and σ_2^2 are known.
We can use point estimators:

$$\bar{X}_1 - \bar{X}_2 \sim N(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2})$$

$$\bar{X}_1 - \bar{X}_2 \pm z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Case 2: σ_1^2 and σ_2^2 are unknown. but $\sigma_1^2 = \sigma_2^2$
we can define a pooled sample variance:

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

Also

$$\frac{(n_1 + n_2 - 2)S_p^2}{\sigma^2} = \frac{(n_1 - 1)S_1^2}{\sigma^2} + \frac{(n_2 - 1)S_2^2}{\sigma^2} \sim \chi^2_{n_1+n_2-2}$$

Remark. This is a weighted average of the sample variances. with weights $n_1 - 1$ and $n_2 - 1$

Remark.

$$S_p^2 := \frac{\sum^{n_1}(X_{1i} - \bar{X}_1)^2 + \sum^{n_2}(X_{2i} + \bar{X}_2)^2}{n_1 + n_2 - 2}$$

$$\sim \chi^2_{n_1+n_2-2}$$

Remark. Then the r.v T is

$$T = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2}$$

Consider making a CI for σ^2

Remark. We know that S^2 is a point estimator for σ^2

$$K = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

Consider the graph of the χ^2 distribution. It is definied from $0, \infty$ and we can trap an α area in the tails.
We can call this $\left(\chi^2_{n-1}\right)_{\alpha/2}$ and $\left(\chi^2_{n-1}\right)_{1-(\alpha/2)}$
Notice the non symmetry of the χ^2 distribution.
We can say that $P\left(\left(\chi^2_{n-1}\right)_{\alpha/2} < K < \left(\chi^2_{n-1}\right)_{1-(\alpha/2)}\right) = 1-\alpha$
In a more insightful form we have

$$P\left(\frac{(n-1)S^2}{\chi^2_{n-1,\alpha/2}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{n-1,1-\alpha/2}}\right) = 1-\alpha$$

Definition (F distribution)**.** F distribution with $\nu_1 > 0$ and $\nu_2 > 0$ degrees of freedom is defined by

$$F = \frac{\frac{U}{\nu_1}}{\frac{V}{\nu_2}}$$

where $U \sim \chi^2_{\nu_1}$ and $V \sim \chi^2_{\nu_2}$
ie $F = \frac{U\nu_2}{V\nu_1}$

Remark. Order is important

Example. $\{X_i\}_{i \in range(5)}$ independent standard normal r.v.
Then $\frac{\alpha(Z_1^2+Z_2^2)}{Z_3^2+Z_4^2+Z_5^2} \sim F_{2,3}$ where $\alpha = 3/2$
This is due to the fact that $Z_i^2 \sim \chi^2_1$ and $\sum_i^n Z_i^2 \sim \chi^2_n$

Theorem 1. *PDF of f_{ν_1,ν_2} is*

$$g(x) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} x^{\frac{\nu_1}{2}-1} \left(1 + \frac{\nu_1}{\nu_2}x\right)^{-\frac{\nu_1+\nu_2}{2}}$$

If we consider two normal pops $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ if

$$\frac{(n_1-1)S_1^2}{\sigma_1^2} \sim \chi^2_{n_1-1} and \frac{(n_2-1)S_2^2}{\sigma_2^2} \sim \chi^2_{n_2-1}$$

then

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{n_1-1,n_2-1}$$

This is useful for a CI for ratio of variances.

Hypothesis Testing

Definition (Hypothesis Testing)**.** It is an assertion about the theoretical distribution. for example about parameters.

Example. In a coin toss expiriment C $Ber(p)$ "coin is unfair": $p \neq 0.5$

It is simple if it completely determines the distribution
It is not simple it is composite

Example. Coin is fair (ie $p = 0.5$)
is a simple hypothesis

Definition (Statisical Test)**.** it is a criterion used to ecide whether to reject (or acept) one hypothesis called the null hypothesis H_0 in favor of an alternative hypothesis H_1.

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Example. $N(\mu, \sigma^2 = 4)$ sample to get $\{X_i\}$ want a test to find $\bar{X} < C$
$H_0 : \mu = 4$
$H_1 : \mu = 1$
$\alpha = P(\text{reject } H_0 | H_0 \text{ is true})$
$\beta = P(\text{accept } H_0 | H_1 \text{ is true})$
We found that if α is small, then β is not if n is fixed.
Increasing n will decrease β

Example. Consdier $Exp(\lambda)$ $H_0 : \lambda = 4$
$H_1 : \lambda = 1$
Suppose we get $\{X_i\}$
$\bar{X} < C$
Find α and β for a fixed $C > 0$ Assume sample size is 1
PDF is $f(x) = e^{-x/\lambda}/\lambda$
Thus $\alpha = P(\bar{X} < C | \lambda = 4)$
$\beta = P(\bar{X} > C | \lambda = 1)$
We can use the CDF of the exponential distribution to find these values.

Context:
Given pdf type, parameters unnown:
Two statements: H_0 and H_1
Base on sample decide among these two.
Test spifies criterion on sample under which H_0 is rejected.
Where if Outcomes falls H_0 is rejected
This reigon is called the critical region.

With error $\alpha = P(\text{reject } H_0 | H_0 \text{ is true})$
α is the area of the critical region or level of significance.
Now define $1-\beta$ as the power of the test.

Definition (Power of a Test)**.** The power of a test is the probability of rejecting H_0 when H_1 is true.
ie $1-\beta = P(\text{reject } H_0 | H_1 \text{ is true})$

If we have multiple x_i we use jpdf.

Definition (Neyman-Pearson Lemma)**.** Suppose $H_0 : \theta = \theta_0$ and $H_1 : \theta = \theta_1$
These are both simple
Let $L_0(x_1...x_n) = f(x_1...x_n|\theta_0)$ ie the jpf of $\theta = \theta_0$
Similarly $L_1(x_1...x_n) = f(x_1...x_n|\theta_1)$ ie the jpf of $\theta = \theta_1$
Suppose C is a reigon in the coutocme space where $L_0/L_1 \leq k$
But outside C $L_0/L_1 > k$
Then the test that rejects H_0 if $L_0/L_1 \leq k$ is the most powerful test of size α

Example. $N(\mu, \sigma^2)$
$H_0 : \mu = 1$
$H_1 : \mu = 3$
Find max power crit region
$L_0 = f(x_1...x_n|\mu = 1)$
$L_1 = f(x_1...x_n|\mu = 3)$
$L_0/L_1 = \frac{e^{-\frac{1}{2\sigma^2}\sum_i^n (x_i-1)^2}}{e^{-\frac{1}{2\sigma^2}\sum_i^n (x_i-3)^2}} = e^{-\frac{1}{2}\sum_i^n \left((x_i-1)^2-(x_i-3)^2\right)}$

$$e^{-\sum_i^n (2x_i-4)} \leq k$$

Definition (Likelihood Ratio Test)**.** Last class considered the NPL for simple hypothesis.
$H_0 : \theta = \theta_0$
$H_1 : \theta = \theta_1$
L_0, L_1 are functions of $x_1...x_n$
$C = \{x_1...x_n | L_0/L_1 \leq k\}$
For some k
aka it is smaller up to a scaling factor.
Then C is the Critical region with max power $(1-\beta)$ among all tests of same α

We can consider w, w' as two non-intersection subsets of the parameter space
Consider that $H_0 : \theta \in w$ and $H_1 : \theta \in w'$

Example. $N(\mu, \sigma^2 = 1)$
$H_0 : \mu = 2$

$H_1 : \mu \neq 2$
Clealry this is simple and composite hypothesis respectively.
$w = \{2\}$ and $w' = R \setminus \{2\}$

"Meaningful" tests Reject H_0 iff the Likelihood that Θ is in w is small.
We can see that Likelihood is now

$$L_w(x_1...x_n) = \max_{\theta \in w} f(x_1...x_n|\theta)$$

$L_\Omega = \max_{\theta \in \Omega} f(x_1...x_n|\theta)$
Let $\Lambda = \frac{L_w}{L_\Omega}$

Remark. L_Ω is at least as large as L_w
Thus Λ is in $0, 1$

Definition (Likelihood Ratio Test)**.** Reject H_0 if $\Lambda \leq k$
where k is fixed number between 0 and 1

Example. $N(\mu, \sigma^1 = 1)$ $H_0 : \mu = 2$
$H_1 : \mu \neq 2$
Now when we have $L(x_1...x_n|\mu)$ for this to maximize we take a same method as MLE. We see that $\mu = \mu_{MLE} = \bar{X}$
So for Likelihood ratio statistic

$$\Lambda = \frac{L_w}{L_\Omega}$$
$$= e^{-\frac{1}{2}\sum_i^n (x_i - 2)^2} / e^{-\frac{1}{2}\sum_i^n (x_i - \bar{X})^2}$$

Now we can simplify since the function is monotone inc.
thus

$$ln(\Lambda) = \frac{1}{2}\sum_i^n (x_i - \bar{X})^2 - \frac{1}{2}\sum_i^n (x_i - 2)^2$$

Doing some math we get

$$|\bar{x} - 2| \geq \sqrt{\frac{-2ln(k)}{n}}$$

Rename constants without x_i as $\tilde{k}$

$$|\bar{x} - 2| \geq \tilde{k}$$

Review

Definition (F distribution)**.** F dist with $\nu_1 > 0$ and $\nu_2 > 0$ degrees of freedom is defined by

$$F = \frac{\frac{U}{\nu_1}}{\frac{V}{\nu_2}}$$

where $U \sim \chi^2_{\nu_1}$ and $V \sim \chi^2_{\nu_2}$
If rv $F \sim f_{\nu_1,\nu_2}$ then the rv $\frac{1}{F} = f_{\nu_2,\nu_1}$ ie indices are swapped.
Now consider $f_{\nu_1,\nu_2,\alpha}$ which is $P(F > f_{\nu_1,\nu_2,\alpha}) = \alpha$
Thus $P(F \geq f_{\nu_1,\nu_2,1-\alpha}) = \alpha \leftrightarrow P(\frac{1}{F} \leq \frac{1}{f_{\nu_1,\nu_2,\alpha}}) = \alpha$

$$P(\frac{1}{F} \geq \frac{1}{f_{\nu_1,\nu_2,\alpha}}) = 1 - \alpha$$

Since $\frac{1}{F} \sim f_{\nu_2,\nu_1}$

$$f_{\nu_2,\nu_1,1-\alpha} = \frac{1}{f_{\nu_1,\nu_2,\alpha}}$$

Definition (Test of Significance)**.** $H_0 : \theta = \theta_0$
$H_1 : \theta \neq \theta_0$ or $H_1 : \theta > \theta_0$ or $H_1 : \theta < \theta_0$
Idea of test: given α, β not specified.
Outcome: Reject H_0 or Fail to reject H_0
Test: Compare the value of a test stat with a fixed value that depends on dist $\propto \alpha$