16:960:665 - Time Series Analysis - Homework 2

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Problem (6). (a) Suppose $\mathscr{H}$ is a separable Hilbert space and $\mathscr{H} = \overline{\text{sp}}\{x_i,\ i = 1, 2, \infty\}$.Let x be an element of $\mathscr{H}$. Show that

$$\mathcal{P}_{\overline{\text{sp}}\{x_1, x_2, \dots, x_n\}}(x) \to x \quad \text{as } n \to \infty.$$

Solution: Let $V_n = \overline{\text{sp}}\{x_1, x_2, \dots, x_n\}$. Since $V_n \subseteq V_{n+1}$, we have a nested sequence of closed subspaces. Since $\mathscr{H}$ is separable, then $\bigcup_{n=1}^{\infty} V_n$ is dense in $\mathscr{H}$. Therefore, for any $x \in \mathscr{H}$ and any $\epsilon > 0$, there exists an N such that for all $n \geq N$, there exists a $y_n \in V_n$ with $\|x - y_n\| < \epsilon$.

Since $\mathcal{P}_{V_n}(x)$ is the orthogonal projection of x onto V_n, it minimizes the distance from x to any point in V_n. Thus, we have:

$$\|x - \mathcal{P}_{V_n}(x)\| \leq \|x - y_n\| < \epsilon \quad \text{for all } n \geq N.$$

This shows that $\|x - \mathcal{P}_{V_n}(x)\| \to 0$ as $n \to \infty$, which implies that $\mathcal{P}_{V_n}(x) \to x$ in the norm of $\mathscr{H}$. Hence, we conclude that:

$$\mathcal{P}_{\overline{\text{sp}}\{x_1, x_2, \dots, x_n\}}(x) \to x \quad \text{as } n \to \infty.$$

(b) Suppose $\{X_t,\ t \in \mathbb{Z}\}$ is a stationary process. Show that

$$\mathcal{P}_{\overline{\text{sp}}\{X_{n-j},\ 1 \leq j \leq \infty\}}(X_n) = \lim_{r \to \infty} \mathcal{P}_{\overline{\text{sp}}\{X_{n-1}, X_{n-2}, \dots, X_{n-r}\}}(X_n).$$

Solution: Let $V_r = \overline{\text{sp}}\{X_{n-1}, X_{n-2}, \ldots, X_{n-r}\}$. Since $V_r \subseteq V_{r+1}$, we have a nested sequence of closed subspaces. The union $\bigcup_{r=1}^{\infty} V_r$ is dense in $V_\infty := \overline{\text{sp}}\{X_{n-j},\ 1 \leq j \leq \infty\}$ because it includes all finite linear combinations of the X_{n-j}'s.
For any $X_n \in \mathscr{H}$, and any $\epsilon > 0$, there exists an R such that for all $r \geq R$, there exists a $Y_r \in V_r$ with $\|X_n - Y_r\| < \epsilon$. Since $\mathcal{P}_{V_r}(X_n)$ is the orthogonal projection of X_n onto V_r, it minimizes the distance from X_n to any point in V_r. Thus, we have:

$$\|X_n - \mathcal{P}_{V_r}(X_n)\| \leq \|X_n - Y_r\| < \epsilon \quad \text{for all } r \geq R.$$

This shows that $\|X_n - \mathcal{P}_{V_r}(X_n)\| \to 0$ as $r \to \infty$, which implies that $\mathcal{P}_{V_r}(X_n) \to X_n$ in the norm of $\mathscr{H}$. Hence, we conclude that:

$$\mathcal{P}_{\overline{\text{sp}}\{X_{n-j},\, 1\leq j\leq\infty\}}(X_n) = \lim_{r\to\infty} \mathcal{P}_{\overline{\text{sp}}\{X_{n-1},X_{n-2},\ldots,X_{n-r}\}}(X_n).$$

Problem (7)**.** Consider the following ARMA processes.

(i) AR(3): $r_t = 0.3 + 0.8r_{t-1} - .5r_{t-2} - .2r_{t-3} + a_t$.

(ii) MA(3): $r_t = 0.3 + a_t + 0.8a_{t-1} - .5a_{t-2} - .2a_{t-3}$.

(iii) ARMA(3,2): $r_t = 0.3 + 0.8r_{t-1} - .5r_{t-2} - .2r_{t-3} + a_t + 0.5a_{t-1} + 0.3a_{t-2}$.

Assume all a_t are i.i.d $N(0, 4)$. For each of the three preceding process, do the following:

(a) Calculate the ACF up to lag 12. [Hint. You may need to read Section 3.3 before trying (iii).]

Solution: We can approach this by using the

(i) AR(3): $r_t = 0.3 + 0.8r_{t-1} - .5r_{t-2} - .2r_{t-3} + a_t$.
Write it in the form of: $\phi(B)r_t = a_t$ where $\phi(B) = 1 - 0.8B + 0.5B^2 + 0.2B^3$.
We can then write the system of equations for the ACF $\rho(h)$ as follows:

$$\begin{aligned}
\rho(0) &= 1\\
\rho(1) &= 0.8 - 0.5\rho(1) - 0.2\rho(2)\\
\rho(2) &= 0.8\rho(1) - 0.5 - 0.2\rho(1)\\
\rho(3) &= 0.8\rho(2) - 0.5\rho(1) - 0.2
\end{aligned}$$

We can see that by solving this we get $\rho(1) = .556$, $\rho(2) = -.167$, $\rho(3) = -.611$. For $h > 3$, we can use the recursive relation:

$$\rho(h) = 0.8\rho(h-1) - 0.5\rho(h-2) - 0.2\rho(h-3)$$

Thus we get the values:

$$\begin{aligned}
\rho(4) &= -.517 \\
\rho(5) &= -.074 \\
\rho(6) &= -.321 \\
\rho(7) &= .397 \\
\rho(8) &= .172 \\
\rho(9) &= -.125 \\
\rho(10) &= -.266 \\
\rho(11) &= -.184 \\
\rho(12) &= -.010
\end{aligned}$$

(ii) MA(3): $r_t = 0.3 + a_t + 0.8a_{t-1} - .5a_{t-2} - .2a_{t-3}$.
We have $\theta(B) = 1 + 0.8B - 0.5B^2 - 0.2B^3$. The ACF for an MA(q) process is given by:

$$\begin{aligned}
\gamma(h) &= \sigma^2 \sum_{j=0}^{q-h} \theta_j \theta_{j+h} \quad \text{for } h = 0, 1, \ldots, q \\
\gamma(h) &= 0 \quad \text{for } h > q
\end{aligned}$$

Thuswe can calculate:

$$\begin{aligned}
&\gamma(0) = 4(1^2 + 0.8^2 + (-0.5)^2 + (-0.2)^2) = 4(1 + 0.64 + 0.25 + 0.04) = 4(1.93) = 7.72 \\
&\gamma(1) = 4(1 * 0.8 + 0.8 * (-0.5) + (-0.5) * (-0.2)) = 4(0.8 - 0.4 + 0.1) = 4(0.5) = 2 \\
&\gamma(2) = 4(1 * (-0.5) + 0.8 * (-0.2)) = 4(-0.5 - 0.16) = 4(-0.66) = -2.64 \\
&\gamma(3) = 4(1 * (-0.2)) = -0.8 \\
&\gamma(h) = 0 \quad \text{for } h > 3
\end{aligned}$$

Now we can divide by $\gamma(0)$ to get the ACF:

$$\begin{aligned}
\rho(0) &= 1 \\
\rho(1) &= \frac{2}{7.72} \approx 0.259 \\
\rho(2) &= \frac{-2.64}{7.72} \approx -0.342 \\
\rho(3) &= \frac{-0.8}{7.72} \approx -0.104 \\
\rho(h) &= 0 \quad \text{for } h > 3
\end{aligned}$$

(iii) ARMA(3,2): $r_t = 0.3 + 0.8r_{t-1} - .5r_{t-2} - .2r_{t-3} + a_t + 0.5a_{t-1} + 0.3a_{t-2}$.
We can write it in the form of: $\phi(B)r_t = c + \theta(B)a_t$ where $\phi(B) = 1 - 0.8B +$

$0.5B^2 + 0.2B^3$ and $\theta(B) = 1 + 0.5B + 0.3B^2$.
To find the ACF, we can use the formula for ARMA processes:

$$\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}$$

where $\psi(z) = \frac{\theta(z)}{\phi(z)}$.
The solution for the ACF is given by

$$\psi_0 = \theta_0 = 1$$
$$\psi_1 = \theta_1 + \phi_1\psi_0 = 0.5 + 0.8 * 1 = 1.3$$
$$\psi_2 = \theta_2 + \phi_1\psi_1 + \phi_2\psi_0 = \theta_2 + \phi_2 + \theta_1\phi_1 + \phi_1^2 = .3 + .5 + .4 + .64 = 1.84$$
$$\psi_n = ***$$

(b) Simulate a series of length $T = 250$, give the time series plot.

Solution:

(c) Compare the true ACF plot (plot what you obtained in Part (a)) with the sample ACF plot (use the `R` function `acf()`).

Solution:

Problem (8)**.** Consider the AR(1) process $X_t = 2X_{t-1} + Z_t$, where $Z_t \sim \text{WN}(0, \sigma^2)$. Define

$$Z_t^* := .25Z_t - \frac{3}{4}\sum_{j=1}^{\infty} 2^{-j} Z_{t+j}$$

(a) Express the unique stationary solution X_t in terms of Z_t.

Solution: We can write the AR(1) process as:

$$(1 - 2B)X_t = Z_t$$

The unique stationary solution is given by:

$$\begin{aligned} X_t &= \frac{1}{1-2B} Z_t \\ &= -\frac{1}{2B} \frac{1}{1-\frac{1}{2B}} Z_t \\ &= -\sum_{j=1}^{\infty} 2^{-j} Z_{t+j} \end{aligned}$$

This is the unique stationary solution for X_t in terms of Z_t. Note this is not causal.

(b) Prove that $\{Z_t^*\}$ is a white noise. What is its variance?

Solution: Mean:

$$\begin{aligned} E[Z_t^*] &= .25E[Z_t] - \frac{3}{4} \sum_{j=1}^{\infty} 2^{-j} E[Z_{t+j}] \\ &= 0 - 0 = 0 \end{aligned}$$

Variance:

$$\begin{aligned}
\mathrm{Var}(Z_t^*) &= E[(Z_t^*)^2] \\
&= E\left[\left(.25Z_t - \frac{3}{4}\sum_{j=1}^{\infty} 2^{-j} Z_{t+j}\right)^2\right] \\
&= E\left[\frac{1}{16}Z_t^2 - \frac{3}{8}Z_t\sum_{j=1}^{\infty} 2^{-j} Z_{t+j} + \frac{9}{16}\left(\sum_{j=1}^{\infty} 2^{-j} Z_{t+j}\right)^2\right] \\
&= \frac{1}{16}E[Z_t^2] + \frac{3}{8}E\left[Z_t\sum_{j=1}^{\infty} 2^{-j} Z_{t+j}\right] + \frac{9}{16}E\left[\left(\sum_{j=1}^{\infty} 2^{-j} Z_{t+j}\right)^2\right] \\
&= \frac{1}{16}\sigma^2 + 0 + \frac{9}{16}E\left[\sum_{j=1}^{\infty} 4^{-j} Z_{t+j}^2\right] \\
&= \frac{1}{16}\sigma^2 + \frac{9}{16}\sum_{j=1}^{\infty} 4^{-j} E[Z_{t+j}^2] \\
&= \frac{1}{16}\sigma^2 + \frac{9}{16}\sum_{j=1}^{\infty} 4^{-j}\sigma^2 \\
&= \frac{1}{16}\sigma^2 + \frac{3}{16}\sigma^2 \\
&= \frac{1}{4}\sigma^2
\end{aligned}$$

(c) Prove that $X_t = .5X_{t-1} + Z_t^*$.

Solution: Note the noncausal solution for X_t from part (a):

$$X_t = -\sum_{j=1}^{\infty} 2^{-j} Z_{t+j}$$

Computing $.5 * X_{t-1}$:

$$
\begin{aligned}
X_{t-1} &= -\sum_{j=1}^{\infty} 2^{-j} Z_{t-1+j} \\
&= -\frac{1}{2} Z_t - \sum_{j=1}^{\infty} 2^{-j+1} Z_{t+j} \\
.5X_{t-1} &= -\frac{1}{4} Z_t - \frac{1}{4} \sum_{j=1}^{\infty} 2^{-j} Z_{t+j}
\end{aligned}
$$

Then $X_t - .5X_{t-1}$:

$$
\begin{aligned}
X_t - .5X_{t-1} &= -\sum_{j=1}^{\infty} 2^{-j} Z_{t+j} + \frac{1}{4} Z_t + \frac{1}{4} \sum_{j=1}^{\infty} 2^{-j} Z_{t+j} \\
X_t - .5X_{t-1} &= .25 Z_t - \frac{3}{4} \sum_{j=1}^{\infty} 2^{-j} Z_{t+j} \\
`' X_t - .5X_{t-1} &= Z_t^*
\end{aligned}
$$

Problem (9)**.** Suppose that $\{X_t\}$ and $\{Y_t\}$ are two zero-mean stationary processes with the same autocovariance function, and that Y_t is an ARMA(p, q) process.

(a) If $\phi_1, \dots, \phi_p$ are the AR coefficients for Y_t, define $W_t := X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p}$. Show that $\{W_t\}$ has an autocovariance function which is zero for lags $|h| > q$.

Solution: Note W_t is a linear combination of X_t's. Since $\{X_t\}$ is stationary, W_t is also stationary. The autocovariance function of W_t is given by:

$$
\begin{aligned}
\gamma_W(h) &= \mathrm{Cov}(W_t, W_{t+h}) \\
&= \mathrm{Cov}\left(X_t - \sum_{i=1}^{p} \phi_i X_{t-i}, X_{t+h} - \sum_{j=1}^{p} \phi_j X_{t+h-j} \right) \\
\text{If } |h| > p, \text{WLOG } h = p+1 &= \mathrm{Cov}\left(X_t - \sum_{i=1}^{p} \phi_i X_{t-i}, X_{t+p+1} - \sum_{j=1}^{p} \phi_j X_{t+p+1-j} \right)
\end{aligned}
$$

Note that There are no overlapping terms between $X_t - \sum_{i=1}^{p} \phi_i X_{t-i}$ and $X_{t+p+1} - \sum_{j=1}^{p} \phi_j X_{t+p+1-j}$ since the maximum lag in the first term is p and the minimum lag in the second term is $p+1$. And for any other choice of h the difference in lags will also be bigger. Therefore, all covariance terms will be zero. Thus, we have:

$$\gamma_W(h) = 0 \quad \text{for } |h| > p$$

(b) Apply Proposition 3.2.1 to $\{W_t\}$ to conclude that $\{X_t\}$ is also an ARMA(p, q) process.

Solution: Note that Proposition 3.2.1 states: If $\{X_t\}$ is a zero-mean stationary process with an autocovariance function $\gamma(\cdot)$ such that $\gamma(h) = 0$ for $|h| > q$, then $\{X_t\}$ is an MA(q) process.
From part (a), we have shown that $\{W_t\}$ has an autocovariance function $\gamma_W(h)$ such that $\gamma_W(h) = 0$ for $|h| > q$. Thus by Proposition 3.2.1, $\{W_t\}$ is an MA(q) process. IE it can be written as:

$$\begin{aligned} W_t &= Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + ... + \theta_q Z_{t-q} \\ \text{where } Z_t &\sim WN(0, \sigma^2) \end{aligned}$$

Now, recall the definition of W_t:

$$W_t = X_t - \phi_1 X_{t-1} - ... - \phi_p X_{t-p}$$

Equating the two expressions for W_t, we have:

$$\begin{aligned} X_t - \phi_1 X_{t-1} - ... - \phi_p X_{t-p} &= Z_t + \theta_1 Z_{t-1} + ... + \theta_q Z_{t-q} \\ \Rightarrow X_t &= \phi_1 X_{t-1} + ... + \phi_p X_{t-p} + Z_t + \theta_1 Z_{t-1} + ... + \theta_q Z_{t-q} \end{aligned}$$

Thus we have expressed X_t as an ARMA(p, q) process. Hence, we conclude that $\{X_t\}$ is also an ARMA(p, q) process.

Problem (10)**.** Read Proposition 5.1.1 and its proof (a very nice one!) before you work on this problem. Suppose there are n observations $X_1, X_2, \ldots, X_n$ of a stationary time series. Define

$$\hat{\gamma}(h) = \begin{cases} n^{-1} \sum_{t=1}^{n-|h|} (X_{t+h} - \bar{X})(X_t - \bar{X}) & \text{if } |h| < n, \\ 0 & \text{if } |h| > n. \end{cases}$$

Note that although the sample autocovariannces are usually only defined for lags $|h| < n$, here $\hat{\gamma}(\cdot)$ is defined as a function on all integers, where it takes value 0 when $|h| \geq n$.

Proposition 1 (5.1.1)**.** *If $\gamma(0) > 0$ and $\gamma(h) \to 0$ as $|h| \to \infty$, then the Covariance Matrix Γ_n is non-singular for all n.*

(a) Show that the function $\hat{\gamma}(\cdot)$ is non-negative definite.

Solution: To show that $\hat{\gamma}(\cdot)$ is non-negative definite, we need $\sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j \hat{\gamma}(i-j) \geq 0$ for any finite set of real numbers $a_1, a_2, \ldots, a_m$.
Consider:

$$\begin{aligned} Q &= \sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j \hat{\gamma}(i-j) \\ \text{By definition} &= \sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j \left(n^{-1} \sum_{t=1}^{n-|i-j|} (X_{t+i-j} - \bar{X})(X_t - \bar{X}) \right) \\ \text{rearranging the sums} &= n^{-1} \sum_{t=1}^{n} \left(\sum_{i=1}^{m} a_i (X_t - \bar{X}) \right)^2 \end{aligned}$$

This is a sum of squares, and thus is always non-negative. Therefore, we conclude that $\hat{\gamma}(\cdot)$ is non-negative definite.

(b) There is nothing you need to do for this part. But observe that (i) by Theorem 1.5.1, there exists some stationary process $\{Y_t\}$ of which $\hat{\gamma}(\cdot)$ is the autocovariance function; and (ii) from Proposition 3.2.1 it then follows that $\{Y_t\}$ is an MA$(n-1)$ process.

Solution: Nice!

(c) Prove that if $\hat{\gamma}(0) > 0$, then $\hat{\Gamma}_n$ is non-singular. (In the last Homework, you showed that $\hat{\Gamma}_n$ is non-negative definite, and now you know that it is also strictly positive-definite unless the n observations are all equal.)

Solution: from part (a), we know that

$$a^T \hat{\Gamma}_n a = n^{-1} \sum_{t=1}^{n} \left(\sum_{i=1}^{n} a_i (X_t - \bar{X}) \right)^2$$

We know that since $\gamma(0) > 0$, not all X_t are equal. Therefore, there exists at least one t such that $X_t - \bar{X} \neq 0$. Thus, for any non-zero vector a, the term $\left(\sum_{i=1}^{n} a_i (X_t - \bar{X})\right)^2$ will be positive for at least one t. Hence, we have:

$$a^T \hat{\Gamma}_n a > 0 \quad \text{for all non-zero } a$$

This implies that $\hat{\Gamma}_n$ is strictly positive-definite, and therefore non singular.

Problem (11)**.**

(a) Consider a MA(∞) process $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$, where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$, and $\sum_{j=0}^{\infty} |\psi_j| < \infty$. Show that the autocovariance function $\gamma(\cdot)$ of $\{X_t\}$ satisfies $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$.

Solution: We know that the autocovariance function for an MA(∞) process is given by:

$$\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}$$

Now we can compute the sum of absolute values of the autocovariances:

$$\begin{aligned}
|\gamma(h)| &= \sigma^2 \left| \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|} \right| \\
&\leq \sigma^2 \sum_{j=0}^{\infty} |\psi_j||\psi_{j+|h|}| \quad \text{(by triangle inequality)} \\
\sum_{h=-\infty}^{\infty} |\gamma(h)| &= |\gamma(0)| + 2 \sum_{h=1}^{\infty} |\gamma(h)| \\
\sum_{h=0}^{\infty} |\gamma(h)| &\leq \sigma^2 \sum_{h=0}^{\infty} \sum_{j=0}^{\infty} |\psi_j||\psi_{j+h}| \\
&= \sigma^2 \left(\sum_{j=0}^{\infty} |\psi_j| \right) \left(\sum_{k=0}^{\infty} |\psi_k| \right) \quad \text{(by changing index)} \\
&= \sigma^2 \left(\sum_{j=0}^{\infty} |\psi_j| \right)^2 < \infty \quad \text{(by our assumption)}
\end{aligned}$$

(b) Let $\{X_t\}$ be a causal ARMA process with autocovariance function $\gamma(\cdot)$. Show that there exist a constant $C > 0$ and another constant $s \in (0, 1)$ such that $|\gamma(h)| \leq C s^{|h|}$ for all $h \in \mathbb{Z}$, and hence $\sum_h |\gamma(h)| < \infty$.

Solution: We know that for a causal ARMA process, the autocovariance function $\gamma(h)$ it can be expressed as an MA(∞) process:

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

where ψ_j are the coefficients of the MA(∞) representation.
THe acf of this process is given by:

$$\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}$$

We know that for $h > \max(p, q)$ the acf satisfies the recursive relation:

$$\gamma(h) = \phi_1 \gamma(h-1) + \phi_2 \gamma(h-2) + ... + \phi_p \gamma(h-p)$$

And the general solution to this is of the form:

$$\gamma(h) = \sum_{i=1}^{k} C_i r_i^{|h|}$$

Thus we can bound $|\gamma(h)|$ as follows:

$$\begin{aligned} |\gamma(h)| &\leq \sum_{i=1}^{k} |C_i||r_i|^{|h|} \\ &\leq C s^{|h|} \quad \text{where } C = \sum_{i=1}^{k} |C_i| \text{ and } s = \max_i |r_i| < 1 \end{aligned}$$

Since $s \in (0, 1)$, we have:

$$\begin{aligned} \sum_{h=-\infty}^{\infty} |\gamma(h)| &\leq \sum_{h=-\infty}^{\infty} C s^{|h|} \\ &= C \left(1 + 2 \sum_{h=1}^{\infty} s^h \right) \\ &= C \left(1 + 2 \frac{s}{1-s} \right) < \infty \end{aligned}$$

Problem (12)**.** The process $X_t = Z_t - Z_{t-1}$, where $\{Z_t\} \sim \text{WN}(0, \sigma^2)$, is not invertible according to Definition 3.1.4. Show however that $Z_t \in \overline{\text{sp}}\{X_j, -\infty < j \leq t\}$ by considering the mean square limit of the sequence $\sum_{j=0}^{n}(1 - j/n)X_{t-j}$ as $n \to \infty$.

Solution:

Definition (3.1.4)**.** Suppose $\{X_t\}$ is a staitionary solution of $\phi(B)X_t = \theta(B)Z_t$, it is said to be invertible if $\exists \pi_j$ such that $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and $Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$ for all $t \in \mathbb{Z}$.

We have $X_t = Z_t - Z_{t-1}$. Rearranging, we get $Z_t = X_t + Z_{t-1}$. Iterating this, we have:

$$\begin{aligned} Z_t &= X_t + X_{t-1} + Z_{t-2} \\ &= X_t + X_{t-1} + X_{t-2} + Z_{t-3} \\ &\vdots \\ &= \sum_{j=0}^{n} X_{t-j} + Z_{t-n-1} \end{aligned}$$

Now, consider the sequence $\sum_{j=0}^{n}(1 - j/n)X_{t-j}$:

$$\begin{aligned} S_n &= \sum_{j=0}^{n}(1 - j/n)X_{t-j} \\ &= \sum_{j=0}^{n}(1 - j/n)(Z_{t-j} - Z_{t-j-1}) \\ &= \sum_{j=0}^{n}(1 - j/n)Z_{t-j} - \sum_{j=0}^{n}(1 - j/n)Z_{t-j-1} \\ &= Z_t - \frac{1}{n}\sum_{j=1}^{n} Z_{t-j} - (1 - \frac{n+1}{n})Z_{t-n-1} + \frac{1}{n}\sum_{j=0}^{n-1} Z_{t-j-1} \\ &= Z_t - \frac{1}{n}Z_{t-n-1} \end{aligned}$$

As $n \to \infty$, the term $\frac{1}{n}Z_{t-n-1} \to 0$ in mean square since Z_t is white noise with finite variance. Therefore, we have:

$$\lim_{n\to\infty} S_n = Z_t$$

This shows that Z_t can be expressed as the mean square limit of a sequence of linear combinations of X_j's for $j \leq t$. Hence, we conclude that:

$$Z_t \in \overline{\text{sp}}\{X_j, -\infty < j \leq t\}$$