01:640:481 - Lecture 11

Pranav Tikkawar

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1. Question 1.
   We saw in class that for $U[0, a]$, if we use Y_n (sample max) as an estimator for a, it is biased because $E[Yn] = n/n + 1 \cdot a$. Keeping in mind properties of expectation, modify Y_n so we get an unbiased estimator for a. What is desirable in an unbiased estimator: High variance or low variance?

   Solution: We can use $Y'_n = \frac{n+1}{n} Y_n$ as an unbiased estimator for a. This is because $E[Y'_n] = a$.
   Additionally we would want a low variance for our unbiased estimator as it would lead to more of our values to group towards the mean.

2. Question 2
   Suppose you are dealing with a population with pdf $\alpha x^{\alpha-1}$, $0 < x < 1$ with a parameter $\alpha > 0$ that is unknown. An observation of 3 independent random variables from this population came up as $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.3$. What is the estimated value of α using the maximum-likelihood idea of estimation?

   Solution: To estimate α using the maximum-likelihood estimation (MLE), we first write down the likelihood function. Given the pdf $f(x; \alpha) = \alpha x^{\alpha-1}$, the likelihood function for the observed data x_1, x_2, x_3 is:

$$
\begin{aligned}
L(\alpha) &= \prod_{i=1}^{3} \alpha x_i^{\alpha-1} \\
&= \alpha^3 \cdot x_1^{\alpha-1} \cdot x_2^{\alpha-1} \cdot x_3^{\alpha-1} \\
&= \alpha^3 \cdot 0.2^{\alpha-1} \cdot 0.4^{\alpha-1} \cdot 0.3^{\alpha-1} \\
&= \alpha^3 (.024)^{\alpha-1}
\end{aligned}
$$

Now the log likelihood function is:

$$\begin{aligned}\log L(\alpha) &= \log \alpha^3 + (\alpha - 1)\log(.024) \\ &= 3\log\alpha + (\alpha - 1)\log(.024)\end{aligned}$$

To find the MLE, we differentiate the log likelihood function with respect to α and set it to 0:

$$\frac{d}{d\alpha}\log L(\alpha) = \frac{3}{\alpha} + \log(.024) = 0$$
$$\implies \alpha_{\hat{MLE}} = -\frac{3}{\log(.024)}$$