01:640:311 - Homework 8

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April 16, 2025

1. Suppose that $f, g : A \to \mathbb{R}$ are both uniformly continuous. Prove that $f + g$ is also uniformly continuous.

Solution: Supose $f, g : A \to \mathbb{R}$ are uniformly continuous. Then for every $\epsilon > 0$, there exists $\delta_1, \delta_2 > 0$ such that for all $x, y \in A$ with $|x - y| < \delta_1$ and $|x - y| < \delta_2$, we have $|f(x) - f(y)| < \frac{\epsilon}{2}$ and $|g(x) - g(y)| < \frac{\epsilon}{2}$.
Thus if we take $\delta = \min(\delta_1, \delta_2)$ for all $x, y \in A$ with $|x - y| < \delta$, we have:

$$\begin{aligned} |f(x) + g(x) - f(y) - g(y)| &= |(f(x) - f(y)) + (g(x) - g(y))| \\ &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

Therefore, $f + g$ is uniformly continuous.

2. (a) Give an example of uniformly continuous functions f and g such that fg is not uniformly continuous.

Solution: We can take $f : \mathbb{R} \to \mathbb{R}, f(x) := x$ and $g : \mathbb{R} \to \mathbb{R}, g(x) := x$.
It is clear that f and g are uniformly continuous since $\forall x, y \in \mathbb{R}$, we have $|f(x) - f(y)| = |x - y|$ and $|g(x) - g(y)| = |x - y|$. Thus we can take $\delta = \epsilon$.
But we have $fg : \mathbb{R} \to \mathbb{R}, fg(x) = x^2$ which is not uniformly continuous.

(b) Prove that if $f, g : A \to \mathbb{R}$ are uniformly continuous and bounded, then fg is uniformly continuous.

Solution: Let $f, g : A \to \mathbb{R}$ be uniformly continuous and bounded. Then there exists $M_1, M_2 > 0$ such that $|f(x)| < M_1$ and $|g(x)| < M_2$ for all $x \in A$.
Since f, g uniformly continuous, for every $\epsilon > 0$, there exists $\delta_1, \delta_2 > 0$ such that for all $x, y \in A$ with $|x - y| < \delta_1$ and $|x - y| < \delta_2$, we have $|f(x) - f(y)| < \frac{\epsilon}{2M_2}$ and $|g(x) - g(y)| < \frac{\epsilon}{2M_1}$.
Thus if we take $\delta = \min(\delta_1, \delta_2)$ for all $x, y \in A$ with $|x - y| < \delta$, we have:

$$\begin{aligned}
|f(x)g(x) - f(y)g(y)| &= |f(x)(g(x) - g(y)) + g(y)(f(x) - f(y))| \\
&\leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \\
&< M_1 \frac{\epsilon}{2M_1} + M_2 \frac{\epsilon}{2M_2} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore, fg is uniformly continuous.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous, and let K be a compact subset of $\mathbb{R}$. Using only the open cover characterization of continuity, prove that $f(K)$ is compact. (Hint: Question 4 from homework 7 is useful here!).

Solution: Suppose f is continuous and K is a compact subset of $\mathbb{R}$.
Remember that the open cover characterization of continuity states that for every open cover $\{U_i\}_{i \in I}$ of a compact set K, there exists a finite subcover $\{U_{i_1}, U_{i_2}, \ldots, U_{i_n}\}$ such that $K \subseteq \bigcup_{j=1}^{n} U_{i_j}$.
We also know from the HW 7 question 4 that if f is continous and U is open then $f^{-1}(U)$ is open.
Let $\{U_i\}_{i \in I}$ be an open cover of $f(K)$.
Then for each $i \in I$, we have $f^{-1}(U_i)$ is open by the statment from the HW.
Then we can say that $\{f^{-1}(U_i)\}_{i \in I}$ is an open cover of K.
Since K is compact, there exists a finite subcover $\{f^{-1}(U_{i_1}), f^{-1}(U_{i_2}), \ldots, f^{-1}(U_{i_n})\}$ such that $K \subseteq \bigcup_{j=1}^{n} f^{-1}(U_{i_j})$.
Then we have $f(K) \subseteq \bigcup_{j=1}^{n} U_{i_j}$, which is a finite subcover of $f(K)$.
Therefore, $f(K)$ is compact.

4. Suppose $f : A \to \mathbb{R}$ is uniformly continuous and $a \notin A$ is a limit point of A.
   (a) Prove that if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in A, then $\{f(x_n)\}_{n=1}^{\infty}$ is Cauchy.

Solution: Let $\epsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that for all $x, y \in A$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

Since $\{x_n\}$ is a Cauchy sequence, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $|x_m - x_n| < \delta$.
By uniform continuity of f, we have:

$$|x_m - x_n| < \delta \implies |f(x_m) - f(x_n)| < \epsilon.$$

Therefore, $\{f(x_n)\}_{n=1}^{\infty}$ is Cauchy.

(b) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in A converging to a. Explain why the limit $\lim_{n\to\infty} f(x_n)$ exists.

Solution: We know that since $\{x_n\}$ is a Cauchy sequence and from the prior part, we have that $\{f(x_n)\}$ is also a Cauchy sequence.
Since every Cauchy sequence converges, we have that $\{f(x_n)\}$ converges to some $L \in \mathbb{R}$.

(c) Suppose $\{y_n\}_{n=1}^{\infty}$ is another sequence in A that converges to a. Prove that $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(x_n)$.

Solution: Let $\epsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ such that for all $x, y \in A$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.
Since $\{x_n\}$ and $\{y_n\}$ both converge to a, there exists $N_1, N_2 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $|x_n - a| < \frac{\delta}{2}$ and for all $n \geq N_2$, we have $|y_n - a| < \frac{\delta}{2}$.
Thus taking $N = \max(N_1, N_2)$, we have for all $n \geq N$:

$$\begin{aligned}|x_n - y_n| &\leq |x_n - a| + |y_n - a| \\ &< \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta. \\ &\implies |f(x_n) - f(y_n)| < \epsilon.\end{aligned}$$

Therefore, we have: $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(x_n)$.

(d) Prove that $\lim_{x\to a} f(x)$ exists.

Solution: Since $\{x_n\}$ and $\{y_n\}$ both converge to a, and we have that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$ from the prior part, we have that $\lim_{x\to a} f(x)$ exists by the contrapositive of the divergence criterion of functional limits.

5. Suppose $f : [0, 1] \to [0, 1]$ is continuous. Prove that there exists an $x \in [0, 1]$ with $f(x) = x$.

Solution: Let $g(x) = f(x) - x$. Then g is continuous on $[0, 1]$.
We have $g(0) = f(0) - 0 \geq 0$ and $g(1) = f(1) - 1 \leq 0$.
Since g is continuous and $g(0) \geq 0$ and $g(1) \leq 0$, by the intermediate value theorem, there exists $c \in [0, 1]$ such that $g(c) = 0$.
More rigorously, we have that $g(0) \geq 0 \geq g(1)$. and since g is continuous, we have that g takes all values between $g(0)$ and $g(1)$.
Thus, there exists $c \in [0, 1]$ such that $g(c) = 0$.
Thus, we have $f(c) = c$.

6. A function $f : A \to \mathbb{R}$ is said to be increasing when for every $x, y \in A$ with $x \leq y$, $f(x) \leq f(y)$. Prove that if f is increasing and has the intermediate value property, then f is continuous on A.

Solution: Suppose f is increasing and has the intermediate value property.
Let $\epsilon > 0$ and $x \in A^\circ$.
Since f has the intermediate value property and is increasing, we have that there exists $x_1, x_2 \in A$ such that $x_1 < x_0 < x_2$ and $f(x) - \epsilon < f(x_1) < f(x) < f(x_2) < f(x) + \epsilon$.
For all $c \in A$ such that $x_1 < c < x_2$, we have $f(c) \in (f(x_1), f(x_2))$.
Take $\delta = \min(x - x_1, x_2 - x)$.
Then for all $x \in A$ such that $|x - c| < \delta \implies x - \delta < c < x - \delta \implies x_1 < c < x_2 \implies f(x_1) < f(c) < f(x_2) \implies f(x) - \epsilon < f(c) < f(x) + \epsilon \implies |f(x) - f(c)| < \epsilon$.
Thus we have that f is continuous at x. Since x was arbitrary, we have that f is continuous on A and the boundary of the points of A are continous as well with the only change of the proof being that we take the single sided limit instead of the two sided limit.