01:640:495 - Lecture 3

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Problem 1

Let P_1, P_2 be the set of polynomials in x with degree at most 1 and 2 respectively, endowed with the usual + and scalar multiplication with real numbers.

(a) Consider x^2 a "vector" in P_2. What are its coordinates with respect to the ordered basis $B : 1, x, x^2$?

Solution: The coordinates are $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(b) What are its coordinates with respect to the basis $B' : (x)(x+1), (x-1)(x+1), (x-1)(x)$?

Solution: The coordinates are $\begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$

This is due to the fact that $x^2 = \frac{1}{2}(x)(x+1) + \frac{1}{2}(x-1)(x)$

(b) Obtain $\Phi : R^3 \to R^3$ the map that converts coordinates in terms of B to in terms of B'.

Solution: The map from B to B' is given by the matrix $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Clealry clear the ith element of B when multiplied Φ is the ith element of B'

Note this can be written as $\begin{bmatrix} \frac{a_1}{2} + \frac{a_2}{2} + \frac{a_3}{2} \\ -a_1 \\ \frac{a_1}{2} - \frac{a_2}{2} + \frac{a_3}{2} \end{bmatrix}$

(c) Obtain $\Psi : R^3 \to R^3$ the map that converts coordinates in terms of B' to in terms of B.

Solution: The map from B' to B is given by the matrix $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$
Clealry clear the ith element of B' when multiplied Ψ is the ith element of B
It is also the inverse of Φ
Note this can be written as $\begin{bmatrix} -a_2 \\ a_1 - a_3 \\ a_1 + a_2 + a_3 \end{bmatrix}$

(d) Show that their compositions are identity maps $\Phi \circ \Psi(\vec{a}) = \vec{a}$ (and $\Phi \circ \Psi(\vec{a}) = \vec{a}$)

Solution: Cleary $\Phi \circ \Psi(\vec{a}) = \vec{a}$ since by definiton they are the inverse of each other
Intuitively, Φ is the map that takes vectors in the basis of B to the coordinates in B' and Ψ is the map that takes vectors in the basis of B' to the coordinates in B
So composing those two maps gives the identity map
This is also true when taking $\Psi \circ \Phi$

(e) Write M_Φ and M_Φ be the matrices (with respect to the standard basis) and check using matrix multiplication that their product is identity matrix.

Solution:

$$M_\Phi = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$
$$M_\Psi = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$
$$M_\Phi M_\Psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$M_\Psi M_\Phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 2

Consider a map $F : P_2 \to P_1$ defined by $F(f) = f'$, that maps a polynomial to its derivative with respect to x.

(a) Say why the map is well defined -ie, why $F(f) \in P_1$.

Solution: This is due to the fact tat any arbitray element of P_2 is a polynomial of degree at most 2 an thus can be represented as $f(x) = a_2x^2 + a_1x + a_0$
Thus $F(f) = f' = \frac{d}{dx}(a_2x^2 + a_1x + a_0) = 2a_2x + a_1$
This is a polynomial of degree at most 1 and thus is in P_1

(b) Show the map is linear.

Solution: We need to show that $F(f + g) = F(f) + F(g)$ and $F(cf) = cF(f)$ for all $f, g \in P_2, c \in R$.

$$\begin{aligned} F(f + g) &= (f + g)' = f' + g' = F(f) + F(g) \\ F(cf) &= (cf)' = cf' = cF(f) \end{aligned}$$

(c) Describe $ker(F), im(F)$ and obtain their dimensions.

Solution: Clearly $ker(F) = \{f \in P_2 : f' = 0\} = \{a_2x^2 + a_1x + a_0 : a_2 = 0, a_1 = 0\} = span\{1\}$
Thus $dim(ker(F)) = 1$
Now $im(F) = \{f \in P_1 : f = F(g) \textbf{ for some } g \in P_2\}$ thus $im(F) = span\{x, 1\}$ Thus $dim(im(F)) = 2$

(d) Write the matrix of F with basis B of P_2 and $1, x$ of P_1.

Solution: The matrix is given by $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(e) Write the matrix of F with basis B' of P_2 and $1, x$ of P_1.

Solution: The matrix is given by $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 2 \end{bmatrix}$

(f) Now, regard $F : P_2 \to P_2$ without changing its defining formula. What is the matrix of F with basis B of P_2? With basis B'?

Solution: The matrix from in basis B is given by $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

The matrix from in basis B' is given by $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

(g) What is the rank of each of the matrices above?

Solution: The rank of the matrix in basis B is 2
The rank of the matrix in basis B' is 2

Problem 3

Let S be the set of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ satisfying $x_1 + 2x_2 - x_3 = 0$. It is a subspace of R^4.

(a) Find a basis of S.

(b) Re-trace the steps in the derivation of the projection formula and obtain matrix (with respect to the standard basis of R^4) of $\pi : R^4 \to S$, the orthogonal project.