01:640:481 - Homework 5

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1. Question 1
   If x is a value of a random variable having an exponential distribution, find k so that the interval from 0 to kx is a $(1-\alpha) \times 100\%$ confidence interval for the parameter θ.

Solution: We need to solve for k where

$$P(0 < \theta < kx) = 1 - \alpha$$

$$\begin{aligned} P(0 < \theta < kx) &= 1 - \alpha \\ &= P(x > \theta/k) \\ \int_{\theta/k}^{\infty} \frac{1}{\theta} e^{-x/\theta} dx &= e^{-1/k} = 1 - \alpha \\ -\frac{1}{k} &= \ln(1-\alpha) \\ k &= -\frac{1}{\ln(1-\alpha)} \end{aligned}$$

2. Question 2
   Making use of the method of section 8.7. It can be shown that for a random sample of size $n = 2$ from the population of excersize 11.2, the distribition of the sample range is given by

$$f(R) = \begin{cases} \frac{2}{\theta^2}(\theta - R) & 0 \leq R \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

use this to find c such that $R < \theta < cR$ is a $(1-\alpha) \times 100\%$ confidence interval for θ.

Solution: We can see that for the sample range R, the PDF is given by $f(R) = \frac{2}{\theta^2}(\theta - R)$ for $0 \leq R \leq \theta$. We need to find c such that $P(R < \theta < cR) = 1 - \alpha$.

$$\begin{aligned}
P(R < \theta < cR) &= 1 - \alpha \\
P(R < \theta/c \cap \theta < R) &= P(\theta/c < R < \theta) \\
P(\theta/c < R < \theta) &= \int_{\theta/c}^{\theta} \frac{2}{\theta^2}(\theta - R)dR \\
&= \frac{2}{\theta^2}\left[\theta R - \frac{R^2}{2}\right]_{\theta/c}^{\theta} \\
\frac{2}{\theta^2}\left[\theta^2 - \frac{\theta^2}{2} - \frac{\theta^2}{c} + \frac{\theta^2}{2c}\right] &= 1 - \alpha \\
1 - \frac{2}{c} + \frac{1}{2c^2} &= 1 - \alpha \\
\alpha c^2 - 2c + 1 &= 0 \\
c &= \frac{1 \pm \sqrt{1-\alpha}}{\alpha}
\end{aligned}$$

3. Question 3
   Show that for $\nu > 2$ the variance of the t-distribution with ν degrees of freedom is $\frac{\nu}{\nu - 2}$. (Hint: Make the substitution $1 + \frac{t^2}{\nu} = \frac{1}{u}$.)
   Hint: Note that the t-distribution has mean 0. Thus the variance is the expected value of t^2. (use other hints in question page)

Solution: We can see that the t-distribution has mean 0. Thus the variance is the expected value of t^2 ie $\int_{-\infty}^{\infty} t^2 f(t)dt$. We can use the fact that the PDF of the t-distribution is given by

$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

By the hint we can see that $dt = -\frac{\sqrt{\nu}}{2\sqrt{1-\nu}}du$ and the limits of integreation become 0 to 1 as t goes from $-\infty$ to ∞ and u goes from 0 to 1. For sake of ease we can let c bt the constant at the begining of the equation $\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)}$. We can now substitute these values into the integral to get

$$E[t^2] = 2\nu^{3/2}c\int_0^1 (\frac{1}{u} - 1)u^{\frac{\nu+1}{2}}\frac{1}{\sqrt{1-u}}du$$

Because type setting is hard and I am lazy, I will skip the rest of the computation and reach the conclusion that I can convert this to ther form of a beta distribution and use the properties of the beta distribution to get

$$\frac{\nu\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)} \cdot \frac{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)}{\nu\Gamma\left(\frac{\nu+1}{2}\right)} \cdot \frac{2}{\nu-2} = \frac{\nu}{\nu-2}$$

Clealry $E[t^2] = \frac{\nu}{\nu-2}$.

4. Question 4
We are dealing with a normal population with known standard deviation $\sigma = 0.3$. After a sampling, we sample values x_1, x_2, x_3 which are 1.3, 1.5, and 1.7. Use the formula we derived in class to obtain a 95% confidence interval for the population mean μ. Use the formula that gives a CI that is a symmetric interval around the sample mean. (NOTE: $1-\alpha$ would be 0.95) Do the same thing as in previous question but now considering sigma as unknown. (o72)

Solution: We essentially need to find the confidence interval for the population mean μ when the standard deviation is unknown.
The formula is

$$\bar{x} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}$$

where $t_{\alpha/2,n-1}$ is the value of the t-distribution with $n-1$ degrees of freedom. and s is the sample standard deviation.
Clearly

$$\begin{aligned}
\bar{x} &= \frac{1.3+1.5+1.7}{3} = 1.5 \\
s &= \sqrt{\frac{(1.3-1.5)^2+(1.5-1.7)^2+(1.7-1.5)^2}{2}} = 0.2 \\
n &= 3 \\
t_{\alpha/2,n-1} &= t_{0.025,2} = 4.303
\end{aligned}$$

Thus the confidence interval is

$$\begin{aligned}
1.5 - 4.303\frac{0.2}{\sqrt{3}} < \mu < 1.5 + 4.303\frac{0.2}{\sqrt{3}} \\
1.003 < \mu < 1.997
\end{aligned}$$

5. Question 5
Use the PDF of t distribution with an appropriate value of the parameter ν to obtain

the the value of the number given by the definite integral $\int_0^\infty \frac{1}{(1+m^2)^5} dm$.

Solution: We can first consider the PDF of the t-distribution. We know that the PDF of the t-distribution is given by

$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

We can see that for $\frac{\nu+1}{2} = 5$, $\nu = 9$. Thus the PDF of the t-distribution is given by

$$f(t) = \frac{\Gamma(5)}{\sqrt{9\pi}\Gamma\left(\frac{9}{2}\right)}\left(1+\frac{t^2}{9}\right)^{-5}$$

We can see that our integral with a subsitution of $m = t/3$ becomes

$$\int_0^\infty \frac{1}{(1+m^2)^5} dm = \int_0^\infty \frac{1}{(1+t^2/9)^5}\frac{1}{3}dt$$

Now since we know that the t-distribution is symmetric about zero, the integral from 0 to ∞ is .5. Thus the integral is

$$\frac{1}{3}\int \frac{1}{(1+t^2/9)^5}dt = \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{\sqrt{9\pi}\Gamma(9/2)}{\Gamma(5)}$$

The left hand side simplifies to

$$\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{\sqrt{9\pi}\Gamma(9/2)}{\Gamma(5)} = \frac{35\pi}{256}$$

(note. I do not want to type set all the algebra so I hope this is acceptable)
Therefore the value of the integral is $\frac{35\pi}{256}$.

6. Question 6
Consider two random variables X and Y with the joint probability density

$$f(x,y) = \begin{cases} 12xy(1-y) & \text{for } 0<x<1, 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the probability density of $Z = XY^2$ by using Theorem 1 to determine the joint probability density of Y and Z and then integrating out y.

Solution: We know that $f(x, y) = 12xy(1 - y)$ for $0 < x < 1$
We can convert this to a function of y and z by using the transformation $z = xy^2$ more fittingly $x = z/y^2$.
By theorem 1 we know that $g(y) = f(w(y))|w'(y)|$. Thus applying it to the problem we see that $|w'(y)| = \frac{dx}{dz} = \frac{1}{y^2}$. Thus we can see that

$$g(z, y) = 12\frac{z}{y}(1 - y) \cdot \frac{1}{y^2}$$

$$g(z, y) = 12z(y^{-3} - y^{-2})$$

We can see that our function is bounded on $0 < z < y^2$ and $0 < y < 1$. Thus we can integrate out y along the bounds of $\sqrt{z} < y < 1$ to get

$$\begin{aligned} h(z) &= 12z \int_{\sqrt{z}}^{1} (y^{-3} - y^{-2}) dy \\ &= 12z \left[-\frac{1}{2} y^{-2} + y^{-1} \right]_{\sqrt{z}}^{1} \\ &= 12z \left[-\frac{1}{2} + 1 + \frac{1}{2\sqrt{z}} - \sqrt{z} \right] \\ &= 6z + 6 - 12\sqrt{z} \end{aligned}$$

Thus the probability density of $Z = XY^2$ is given by

$$h(z) = \begin{cases} 6z + 6 - 12\sqrt{z} & 0 < z < 1 \\ 0 & \text{elsewhere} \end{cases}$$