01:640:311H - Chapter 1

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What are the Real Numbers?

The real numbers are a **complete ordered field**.
This uniquely determines the real numbers.
No what do these words mean: complete, ordered, field.

0.1 field

A field is a set of numbers with two operations, addition and multiplication, that satisfy the following properties $\forall x, y, z \in \mathbb{R}$:

$$
\begin{aligned}
(x+y)+z &= x+(y+z) \\
x+y &= y+x \\
(x*y)*z &= x*(y*z) \\
x*y &= y*x \\
x*(y+z) &= x*y+x*z \\
\exists 0 \text{ s.t. } x+0 &= x \\
\forall x \exists -x \text{ s.t. } x+(-x) &= 0 \\
\exists 1 \text{ s.t. } x*1 &= x \\
0 &\neq 1 \\
\forall x \neq 0 \exists x^{-1} \text{ s.t. } x*x^{-1} &= 1
\end{aligned}
$$

Theorem 1. *For all real numbers x: $0x = 0$.*

Proof.

$$
\begin{aligned}
0*x+0*x &= (0+0)*x \\
0*x+0*x &= 0*x \\
0*x+0*x &= 0*x+0 \\
0*x &= 0
\end{aligned}
$$

□

0.2 ordered

For all $x, y, z \in \mathbb{R}$:

$$
\begin{aligned}
x<y &\implies x+z<y+z \\
x<y \text{ and } y<z &\implies x<z \\
\textbf{Trichotomy Law:} & x<y \text{ or } x=y \text{ or } x>y
\end{aligned}
$$

Theorem 2.

$$0 < 1$$

Proof. We do this by the Trichotomy Law.
We know that $0 \neq 1$
we can do this by contradiction: Suppose $1 < 0$

$$\begin{aligned} 1 + (-1) &< 0 + (-1) \\ 0 &< -11 * (-1) \qquad\qquad\qquad\qquad < 0 * (-1) \\ 1 * (-1) &< 01 * (-1) + (1 * 1) < 0 + (1 * 1) \\ 0 &< 1 \end{aligned}$$

□

Definition. If S is a set of real then we say b is an upper bound of S if $\forall x \in S : x \leq b$.

Definition. Given a set of S of reals. we say b is least uper bound or supremem of S when

1. b is an upper bound of S
2. If c is an upper bound of S then $b \leq c$

we denote this as $b = \sup S$

0.3 complete

Every non empty set of real numbers that is bounded above has a least upper bound.

Theorem 3. *$x = \sup S$ if and only if x is an upper bound of S for all $\epsilon > 0$ there exists $s \in S$ such that $x - \epsilon < s$*

Proof. $\Longrightarrow$ Suppose $x = \sup S$
Then x is an upper bound of S
We only need to show that for all $\epsilon > 0$ there exists $s \in S$ such that $x - \epsilon < s$
Let $\epsilon > 0$
Since $x = \sup S$ every other uppw bound of S is greater than x
So $x - \epsilon$ is not an upper bound of S
So there exists $s \in S$ such that $x - \epsilon < s$
Suppose for all $\epsilon > 0$ there exists $s \in S$ such that $x - \epsilon < s$
We need to show that $x = \sup S$
We know that x is an upper bound of S
And we know that is $b < x$ then b is not an upper bound of S
So $x - \epsilon$ is not an upper bound of S
so there exists $s \in S$ such that $x - \epsilon < s$
$\Longleftarrow$ Now suppose x is an ub $\forall \epsilon > 0$ there exists $s \in S$ such that $x - \epsilon < s$
Since we know x is an upper bound of S we only need to show that if b is an upper bound of S then $b \geq x$
By contrapoitive, this is equivalent to showing that if $b < x$ then b is not an upper bound of

S
Let $\epsilon = x - b$
Then there exists an $s \in S$ such that $x - \epsilon < s$
and $x - \epsilon = b$
and b is not an upper bound of S
Thus $x = \sup S$ □

Note that we get that every non empty set of real numbers that is bounded below has a greatest lower bound for free from the completeness of the real numbers.
This is due to the fact multiplication by -1 is a reflection across the origin which maps upper bounds to lower bounds.

Theorem 4. *Define* $-S = \{-s \text{ s.t. } s \in S\}$
Then if b *is an upper bound of* S *then* $-b$ *is a lower bound of* $-S$

Proof. □

Theorem 5. *If* $b = \sup S$ *then* $-b = \inf -S$

Proof. HW □

Theorem 6 (Nested Interval)**.**

Theorem 7 (Archimedan property)**.**

0.4 Existence of $\sqrt{2}$

Lemma 1. *If* $a > 0$ *and* $b \in \mathbb{R}$ *then* $a^2 > b^2 \implies a > b$

Proof. By contrapositive, suppose $a \leq b$
Then $a^2 \leq ab < b^2$
So $a^2 < b^2$ □

Theorem 8. *There is an* $x > 0$ *such that* $x^2 = 2$

Proof. Let $S = \{s \in \mathbb{R} \text{ s.t. } s > 0 \text{ and } s^2 < 2\}$
We can see that $0 \in S$ so S is non empty
More over $2^2 = 4 > 2$ so $2 \notin S$ so S is bounded above
Let $x = \sup S$ then we WTS $x^2 = 2$
Suppose $x^2 > 2$
Let $\epsilon =$ very small Let us consider $(x - \frac{1}{n})^2$

$$\begin{aligned}(x - \frac{1}{n})^2 &= x^2 - 2x\frac{1}{n} + \frac{1}{n}^2 \\ &\geq x^2 - 2x\frac{1}{n}\end{aligned}$$

We want $x^2 - 2x\frac{1}{n} > 2$
Know that $x^2 > 2$
Thus $x^2 - 2 > \frac{2x}{n}$ and $\frac{1}{n} < \frac{x^2-2}{2x}$
So by the Archimedan property there exists, We can take an n such that $\frac{1}{n} < \frac{x^2-2}{2x}$
Thus $(x - \frac{1}{n})^2 > 2$, so $x - \frac{1}{n}$ is an upper bound of S resulting in a contradiction.
Now consider $x^2 < 2$
Let $\epsilon = 2 - x^2$
Then there exists $s \in S$ such that $x < s$
Then $s^2 < 2$
Then $s^2 < x^2$
Then $s < x$

$\square$

Definition. We say a set is countable if $A \sim \mathbb{N}$

Lemma 2. *Any infinite subset of a countable set is countable*

Proof. Let $A \subset \mathbb{N}$ be infinite and we define $f : \mathbb{N} \to A$

$$\begin{aligned}
f(1) &= \min A \\
f(2) &= \min(A \setminus \{f(1)\}) \\
f(3) &= \min(A \setminus \{f(1), f(2)\}) \\
&\vdots \\
f(n) &= \min(A \setminus \{f(1), f(2), \dots, f(n-1)\})
\end{aligned}$$

This is a bijection between $\mathbb{N}$ and A $\square$

Corollary. If there exists an inject form $A to \mathbb{N}$ then either A is finite or $A \sim \mathbb{N}$(countable)

Proof. If A is finite then we are doen
If A is infite the $f : A \to Im(f)$ stays injective and becmes surjective so $A \sim Im(f)$ Since $Im(f) \subset \mathbb{N}$ is infinite then $A \sim Im(f) \sim \mathbb{N}$ $\square$

Proposition 1. $\mathbb{N} \times \mathbb{N}$ *is countable*

Proof. $\mathbb{N} \times \mathbb{N}$ is inifite, so if we could contracit and inject $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ then $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$
Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
$f(a, b) = 2^a 3^b$
By unique prime facotization if $(a, b) \neq (c, d)$ then $f(a, b) \neq f(c, d)$
So f is injective and the corollary gives us that $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$ $\square$

Corollary. N^n is countable for all n.

Theorem 9. *If $S_1, S_2, \dots$ is a sequence of sets each finite or countable then $\cup_{n=1}^{\infty} S_n$ is finite or countable*

Proof. By defining $\tilde{S}_i = \{s \in S : s \notin S_j \forall j < i\}$
We can assume WLOG that the S_i are disjoint
Foreach S_i we can enumrate the elements is $S_i = \{S_{i,1}, S_{i,2}, ...\}$ and $S = \bigcup_{i=1}^{\infty} \{S_{1,1}...S_{1,n_1}, S_{2,1}...S_{2,n_2}, ...\}$
Each element of S has a unique index, so the function is $f \to \mathbb{N} \times \mathbb{N}$ by $f(S_{i,j}) = (i, j)$ is well definied and injective.
Since $N \times \mathbb{N}$ is countable there is a bijeections g and there is $h = g \circ f$ that is injectibve and so by the lemma S is either finite or countable $\square$

Theorem 10. $\mathbb{Q}$ *is countable*

Proof. Write $\mathbb{Q} - \bigcup_{i=1}^{\infty} A_i$
where $A_i = \left\{\pm\frac{a}{n} : a \in \mathbb{N} \cup \{0\}, b \in \mathbb{N}, \text{ and } a + b = i\right\}$
Now each A_i is finite so by the previous theorem $\mathbb{Q}$ is countable $\square$

Theorem 11. $\mathbb{R}$ *is not countable*

Proof. Suppose $f : \mathbb{N} \to \mathbb{R}$ we will prove that f is not surjective
so $N \not\sim \mathbb{R}$
First for each n write $x_n = f(n)$ bw fore $n = 1$ we can dind an inreral I_1 not contaieding x_1 now by splitting I_1 into 3 pieces we can always fine a piece excldieing x_2 Call this closed bounded interval I_2 and so on.
Iterating we get anested set of closed bounded intervals $I_1 \supset I_2 \supset I_3 \supset \dots$ with $x_n \notin I_n$
Thus $x_n \notin \bigcap_{m=1}^{\infty} I_m$
Property $\exists x \in \bigcap_{m=1}^{\infty} I_m$ by NIP Since $x \neq x_n \forall n$ f is not surhective. and thus $\mathbb{R}$ is not countable

$\square$

Theorem 12.

Theorem 13. *for any set A,* $\{0, 1\}^A \sim P(A)$

Theorem 14. *For any set A,* $A \not\sim \{0, 1\}^A$