01:XXX:XXX - Homework n

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October 25, 2024

Missed notes: Counting proesses
$\{N(t), t >= 0\}$ They follow 3 properties:

1. $N(t) \geq 0$
2. $N(t)$ is integer valued
3. $N(t)$ is monotone increasing

$$N(t) : R \to N$$

Monotone increasing function of t

$$N(t) - N(s) = \text{ Number of events in } (t, s]$$

Little o notation
A function f is said to be little o $o(h)$ if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0$$

eg: $f(h) = h^2$ is little $o(h)$
If u add two function in little $o(h)$ then it is still little $o(h)$
Definition: A counting process $\{N(t), t \geq 0\}$ is a Poisson process if:

1. $N(0) = 0$
2. The number of events in disjoint intervals are independent.
3. $P(N(t + h) - N(t) = 1) = \lambda h + o(h)$ where λ is the rate of the Poisson process. (this mean it is dependant on the length of the interval)
4. $P(N(t + h) - N(t) \geq 2) = o(h)$

Lemma 5.1:
Let $\{N(t), t \geq 0\}$ be a Poisson process. Define $\{N_s(t), t \geq 0\}$ by $N_s(t) = N(s + t) - N(s)$
Then $\{N_s(t), t \geq 0\}$ is a Poisson process with rate λ
Proof:

$$\begin{aligned}
N_s(0) = N(s + 0) - N(s) = 0 \\
(a, b) \cap (c, d) = \emptyset \\
P(N_s(b) - N_s(a) = x, N_s(d) - N_s(c) = y) \\
P(N(b - s) - N(a - s) = x, N(d - s) - N(c - s) = y) \\
P(N(b - s) - N(a - s) = x)P(N(d - s) - N(c - s) = y) \\
P(N_s(b) - N_s(a) = x)P(N_s(d) - N_s(c) = y)
\end{aligned}$$

Thus disjoint intervals are independent.

$$P(N_s(t+h) - N_s(t) = 1) = P(N(s+t+h) - N(s+t) = 1)$$

We assume N has stationary increments.

$$P(N(s+t+h) - N(s+t) = 1) = P(N(t+h) - N(t) = 1) = \lambda h + o(h)$$

Lmma 5.2:
Let $T_1 = min(t > 0 : N(t) = 1)$
it is time of arrival
T_1 is exponentially distributed with rate λ
Proof:

$$P_0(t) = P(N(t) = 0)$$
$$P_0(t+h) = P(N(t) = 0, N(t+h) - N(t) = 0)$$
$$P_0(t+h) = P(N(t) = 0)P(N(t+h) - N(t) = 0)$$
$$P_0(t+h) = P_0(t)(1 - \lambda h - 2o(h))$$

note that $-2o(h) = o(h)$ cuz it basically 0

$$P_0(t+h) = P_0(t) - \lambda h P_0(t) + o(h)P_0(t)$$

$$\frac{dP_0(t)}{t} = -\lambda P_0(t) + 0$$

This solves to with IC $P_0(0) = 1$
$$P_0(t) = e^{-\lambda t}$$

Define:
$T_n forn \geq 1$ is the time between the $(n-1)^t h$ and $n^t h$ arrival.
Proposition 5.4:
$T_1, T_2, \dots$ are independent and exponentially distributed with rate λ
Proof:
Rea book.
Remark:
Define $S_n = \sum_{i=1}^{n} T_i$
From last time, S_n has a gamma distribution with parameters n and λ

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$$

Theoremm 5.1
If $\{N(t), t \geq 0\}$ is a Poisson process with parameter λ then $N(t)$ is a poisson random variable with parameter λt

Proof:

$$\begin{aligned} P(N(t) = n) &= \int_0^\infty P(N(t) = n | S_n = t) \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} dt \\ &= P(T_{n+1} = t - s | T_1 + T_2 + \cdots + T_n = s) \\ &= P(T_{n+1} = t - s) \\ &= \frac{(\lambda t)^n e^{-\lambda t}}{n!} \end{aligned}$$

Example
Let $\{N(t), t \geq 0\}$ be a Poisson process with rate $\lambda = \frac{1}{3}$
Find:
a) $P(N(5) > N(3))$
This means there are > 0 events in $(3, 5]$

$$\begin{aligned} P(N(5) > N(3)) &= 1 - P(N(5) - N(3) = 0) \\ &= 1 - P(N(2) = 0) \\ &= 1 - e^{-\frac{2}{3}} \end{aligned}$$

b) $P(\{N(4) = 1\}, \{N(5) = 3\})$

c) $E(N(5)|N(3) = 2)$
d) $E(T_b|N(3) = 4)$
Last time we finished 5.3.2 + examples
5.3.3 Further thinning of a poisson process.
Suppose $\{N(t), t \geq 0\}$ is a Poisson process with rate λ
There are events of 2 types: 1 w/ probability p and 2 w/ probability $1 - p$
Write $N_1(t)$ for the number of type 1 events in $(0, t]$
$N_2(t)$ for the number of type 2 events in $(0, t]$
textbfProposition 5.5
$\{N_1(t), t \geq 0\}$ is a Poisson process with rate $p\lambda$ and $\{N_2(t), t \geq 0\}$ Poisson process with rate $(1 - p)\lambda$
Compound Poisson process
Suppose random variables are iid with disstribution F with mean μ and variance σ^2
The non-negative integer valued random varianle $S = \sum_{i=1}^{N} X_i$ is called a compound Poisson random variable.
Conditional Variance formula

$$Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$$

If N is a poison random variable with parameter λ then:

$$Var(S) = \lambda\sigma^2 + \mu^2\lambda$$

Read example 5.27

1 add Missed info

yes

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$$P_j = \lim_{t\to\infty} P_{ij}(t)$$
$$0 = \sum_{k\neq j} q_{kj}P_k - v_jP_j$$

Read remarks of page 395

Example. Limiting probability: P_j for the Birth-Death process with birth rate λ_j and death rate μ_j
Write the balance equations for P_j

$$\lambda_0 P_0 = \mu_1 P_1$$
$$(\lambda_1 + \mu_1)P_1 = \lambda_0 P_0 + \mu_2 P_2$$
$$(\lambda_n + \mu_n)P_n = \lambda_{n-1}P_{n-1} + \mu_{n+1}P_{n+1}$$

We can now go into canceling.

$$\lambda_1 P_1 = \mu_2 P_2$$
$$P_2 = \frac{\lambda_1}{\mu_2}P_1 = \frac{\lambda_1 \cdot \lambda_0}{\mu_2 \cdot \mu_1}$$
$$P_n = \frac{\lambda_{n-1}\cdot\lambda_{n-2}\dots\lambda_0}{\mu_{n+1}\cdot\mu_n\dots\mu_1}P_0$$

Use $\sum_{j=0}^{\infty} P_j = 1$ to find limiting probability

$$1 = P_0 + P_0\sum_{n=1}^{\infty}\frac{\lambda_{n-1}\cdot\lambda_{n-2}\dots\lambda_0}{\mu_{n+1}\cdot\mu_n\dots\mu_1}$$
$$P_0 = \frac{1}{1+\sum_{n=1}^{\infty}\frac{\lambda_{n-1}\cdot\lambda_{n-2}\dots\lambda_0}{\mu_{n+1}\cdot\mu_n\dots\mu_1}}$$

We need the infinite sume to be finite for it to be a valid probability.

Read Examples 6.13, 6.14, 6.15 and skip 6.16

Chapter 6.6 Time reversibility

If limiting probabilities exist, a CTMC is called ergodic.
Consider an ergotic CTMC that has been running a long time.
First look at the embedded discrete time markov chain.(forget the time spent in each state. Just look at the transitions)
Let π_i be the limiting probability of being in state i of the embedded chain.
Recall that $\pi_1 = \sum_j \pi_j P_{ji}$ and $\sum_i \pi_1 = 1$
Note that π_i is the proportion of transitions
Recall tha $\frac{1}{v_i}$ mean time spent in state i
Claim P_i is the proportion of time the CTMC is in state i more precisely $P_i = \frac{\pi_i/v_i}{\sum_j \pi_j/v_j}$
Verification:
Know $v_i P_i = \sum_{j\neq i} P_j q_{ji} = \sum_{j \neq i} P_j v_j P_{ji} = \sum_j P_j v_j P_{ji}$ Through some more manpulation we can see that this can only be satisfied by π_i
Time reversibility:
Reversing a CTMC Assume the process has been running a long time. Observe it backwards.
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