01:640:350H - Homework 4

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1. Question 1.4 15 pg(35)
   Let S_1 and S_2 be subsets of a vector space V. Prove that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$ Give an example in which $\text{span}(S_1 \cap S_2)$ and $\text{span}(S_1) \cap \text{span}(S_2)$ are equal and one in which they are not equal.
   Proof:
   Assume that S_1 and S_2 are subsets of a vector space V.
   Let $v \in \text{span}(S_1 \cap S_2)$.
   Then, v can be written as a linear combination of elements in $S_1 \cap S_2$.
   $v = \sum_{i=0}^{n} a_i v_i$ where $v_i \in S_1 \cap S_2$ and $a_i \in \mathbb{R}$.
   Since $v_i \in S_1 \cap S_2$, $v_i \in S_1$ and $v_i \in S_2$.
   Due to the closure of addtion and scalar multiplication properties of a vector subspace, any linear combination of elements in S_1 will be in $\text{span}(S_1)$ and any linear combination of elements in S_2 will be in $\text{span}(S_2)$.
   Since we can clealry see that v is a linear combination of elements in S_1 and S_2, $v \in \text{span}(S_1)$ and $v \in \text{span}(S_2)$.
   Therefore, $v \in \text{span}(S_1) \cap \text{span}(S_2)$. $\square$

   Example of span$(S_1 \cap S_2) =$ span$(S_1) \cap$ span(S_2):
   We can consider V to be R^2.

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$S_1 \cap S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{span}(S_1) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = R^2$$

$$\text{span}(S_2) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{span}(S_1) \cap \text{span}(S_2) = R^2 \cap R^2 = R^2$$

$$\text{span}(S_1 \cap S_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = R^2$$

Clealry this is an example where $\text{span}(S_1 \cap S_2) = \text{span}(S_1) \cap \text{span}(S_2)$

Example of span($S_1 \cap S_2) \neq$ span($S_1) \cap$ span(S_2):
We can consider V to be R^2.

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$S_1 \cap S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

$$\text{span}(S_1) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = R^2$$

$$\text{span}(S_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = R^2$$

$$\text{span}(S_1) \cap \text{span}(S_2) = R^2 \cap R^2 = R^2$$

$$\text{span}(S_1 \cap S_2) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \neq R^2$$

Clealry this is an example where $\text{span}(S_1 \cap S_2) \neq \text{span}(S_1) \cap \text{span}(S_2)$

2. Question 1.5 15 pg(43)
Let $S = \{u_1, u_2, \ldots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent iff $u_1 = 0$ or $u_{k+1} \in \text{span}(u_1, u_2, \ldots, u_k)$ for some $k(1 \leq k < n)$.
Also: Can k be allowed to be 0 here (rather than equal to or greater than 1, as the problem says)? For sake of ease I will refer to S is linearly dependent as Q and $u_1 = 0$ or $u_k + 1 \in \text{span}(u_1, u_2, \ldots, u_k)$ for some $k(1 \leq k < n)$ as P
Proof:

Proof of Q $\to$ P
We can do this by contradiction Assume that $S = \{u_1, u_2, \ldots, u_n\}$ is a finite set of vectors.
Assume that S is linearly dependent.
Assume that $u_1 \neq 0$ and $u_k + 1 \notin \text{span}(u_1, u_2, \ldots, u_k)$ for all $k(1 \leq k < n)$.
Since S is Linearly dependant, then there exists $a_i \in \mathbb{R}$ such that $\sum_{i=1}^{n} a_i u_i = 0$ where not all a_i are 0.

Also $\frac{1}{-a_n}\sum_{i=1}^{n-1} a_i u_i = u_n$.
This is a contraction of $u \notin \text{span}(u_1, u_2, \ldots, u_{n-1})$.
Also note that if $u_1 = 0$ then we can take a_1 to be any non-zero element of the field to generate the zero vector which is contradictory to the fact that the zero vector is a trivial linear combination.

Proof of P $\rightarrow$ Q
Assume that $S = \{u_1, u_2, \ldots, u_n\}$ is a finite set of vectors.
Assume that $u_1 = 0$ or $u_k + 1 \in \text{span}(u_1, u_2, \ldots, u_k)$ for some $k(1 \leq k < n)$.
Need to show that S is linearly dependent.
If $u_1 = 0$, then $\sum_{i=2}^{n} 0u_i = u_1$ is a non-trivial linear combination of elements in S that equals 0.
If $u_k + 1 \in \text{span}(u_1, u_2, \ldots, u_k)$ for some $k(1 \leq k < n)$, then $u_{k+1} = \sum_{i=1}^{k} a_i u_i$ for some $a_i \in \mathbb{R}$.
Thus we can consider $\sum_{i=1}^{k} a_i u_i - u_{k+1} = 0$ as a non-trivial linear combination of elements in S that equals 0.
Thus S is linearly dependent.

Extra Question: Can k be allowed to be 0 here (rather than equal to or greater than 1, as the problem says)?
Yes, as if $k = 0$, it would imply that $u_1 \in \text{span}(\emptyset)$.
Since the span of the empty set is $\{0\}$, $u_1 = 0$.
Thus, $u_1 = 0$ is a valid condition for S to be linearly dependent.

3. Question 2.1 16
Let $T : P(R) \rightarrow P(R)$ be defined by $T(f) = f'$. Recall that T is linear. Prove that T is onto but not one-to-one.
Proof:

Onto:
Let $g \in P(R)$.
Need to show that there exists an $f \in P(R)$ such that $T(f) = f' = g$.
Take $g = \sum_{i=0}^{n} a_i x^i$.
Then $f = \sum_{i=0}^{n} \frac{a_i}{i+1} x^{i+1}$
Then $T(f) = f' = \sum_{i=0}^{n} a_i x^i = g$.
Therefore, T is onto.

Not One-to-One:
Need to show that there exists $T(f_1) = T(f_2)$ but $f_1 \neq f_2$.
Let $f_1 = 1$ and $f_2 = 0$.
Then $T(f_1) = T(1) = 0$ and $T(f_2) = T(0) = 0$.

Therefore, T is not one-to-one.

4. Question 2.3 3a
Let $g(x) = 3 + x$ Let $T : P_2(R) \to P_2(R)$ and $U : P_2(R) \to R^3$ be the linear transformations defined by

$$T(f(x)) = f'(x)g(x) + 2f(x) \text{ and } U(a + bx + cx^2) = (a + b, c, a - b)$$

Compute $[U]_\beta^\gamma, [T]_\beta$ and $[UT]_\beta^\gamma$ directly, then use theorem 2.11 to verify your result.
To compute $[U]_\beta^\gamma$, we need to find $U(e_1), U(e_2), U(e_3)$ where $e_1 = 1, e_2 = x, e_3 = x^2$.
$U(e_1) = (1, 0, 1)$
$U(e_2) = (1, 0, -1)$
$U(e_3) = (0, 1, 0)$.
Therefore, $[U]_\beta^\gamma = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$.
To compute $[T]_\beta$, we need to find $T(e_1), T(e_2), T(e_3)$ where $e_1 = 1, e_2 = x, e_3 = x^2$
$T(e_1) = 2$
$T(e_2) = (3 + x) + 2x = 3 + 3x$
$T(e_3) = 2x(3 + x) + 2x^2 = 6x + 4x^2$
Thus $[T]_\beta = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}$
To compute $[UT]_\beta^\gamma$, we need to find $UT(e_1), UT(e_2), UT(e_3)$ where $e_1 = 1, e_2 = x, e_3 = x^2$
$UT(e_1) = U(2) = (2, 0, 2)$
$UT(e_2) = U(3 + 3x) = (6, 0, 0)$
$UT(e_3) = U(6x + 4x^2) = (6, 4, -6)$
Thus $[UT]_\beta^\gamma = \begin{bmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{bmatrix}$
To verify our result, we can use theorem 2.11.
$[UT]_\beta^\gamma = [U]_\beta^\gamma [T]_\beta$
$\begin{bmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}$
After multiplying the matrices, we can clearly see that the result is the same as $[UT]_\beta^\gamma$

5. Question 2.3 3b
Let $g(x) = 3 + x$ Let $T : P_2(R) \to P_2(R)$ and $U : P_2(R) \to R^3$ be the linear transformations defined by

$$T(f(x)) = f'(x)g(x) + 2f(x) \text{ and } U(a + bx + cx^2) = (a + b, c, a - b)$$

Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[Uh(x)]_\gamma$. Then use $[U]_\beta^\gamma$ from (a) and Theorem 2.14 to verify your result.
To compute $[h(x)]_\beta$ we can simply define it as:

$$[h(x)]_\beta = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

To compute $[Uh(x)]_\gamma$, we need to find $U(h(x))$.
$U(h(x)) = U(3 - 2x + x^2) = (3 - 2, 1, 3 + 2) = (1, 1, 5)$
Therefore, $[Uh(x)]_\gamma = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$
To verify our result, we can use theorem 2.14.
$[Uh(x)]_\gamma = [U]_\beta^\gamma [h(x)]_\beta$
$\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$
After multiplying the matrices, we can clearly see that the result is the same as $[Uh(x)]_\gamma$

6. Question 2.3 4d based on Sec. 2.2 5(d)
For each of the following parts let T be the linear transformation define in the corresponding part of Excerise 5 of section 2.2. Use Theorem 2.14 to compute the following vectors:

$$[T(f(x))]_\gamma, \text{ where } f(x) = 6 - x + 2x^2$$

$$\text{Define } T : P_2(R) \to R \text{ by } T(f(x)) = f(2)$$

$$[T(f(x))]_\gamma = [T]_\beta^\gamma [f(x)]_\beta$$

$[f(x)]_\beta = \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix}$
We can use $\gamma = \{1\}$ and $\beta = \{1, x, x^2\}$

$$T(1) = 1$$

$$T(x) = 2$$

$$T(x^2) = 4$$

Therefore, $[T]_\beta^\gamma = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$ Using theorem 2.14, we can compute $[T(f(x))]_\gamma$

$$[T(f(x))]_\gamma = [T]_\beta^\gamma [f(x)]_\beta$$

$$[T(f(x))]_\gamma = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix}$$

After multiplying the matrices, we can clearly see that the result is $[T(f(x))]_\gamma = 12$

7. Question 2.4 2c
   For each of the following linear transformation T, determine whether T is invertible and justify your answer.

$$T : R^3 \to R^3 \text{ defined by } T(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$$

   We can first assert that $T = L_A$ where $A = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$
   We can then utilize theorem 2.18 to determine if T is invertible.
   The contents of theorem 2.18 imply that T is invertible if and only $[T]_\beta^\gamma$ is invertible.
   Since $[T]_\beta^\gamma = A$(as we have determined from the prior HW), we can see that it would be sufficient to determine if A is invertible.
   Utilizing RREF theroy if we can put in A in RREF form and get the identity matrix, then A is invertible.

$$\begin{bmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix} \xrightarrow{R_3 - R_1 \to R_3} \begin{bmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 4 & 2 \end{bmatrix} \xrightarrow{R_3 - 4R_2 \to R_3} \begin{bmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{1/2R_3 \to R_3} \begin{bmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{1/3R_1 \to R_1} \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 + 2/3R_3 \to R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   Since we have obtained the identity matrix, A is invertible.
   Therefore, T is invertible.

8. Question 2.4 2d
   For each of the following linear transformation T, determine whether T is invertible and justify your answer.

$$T : R^3 \to R^2 \text{ defined by } T(p(x)) = p'(x)$$

   We can first assert that $T = L_A$ where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
   We can then utilize theorem 2.18 to determine if T is invertible.
   The contents of theorem 2.18 imply that T is invertible if and only $[T]_\beta^\gamma$ is invertible.
   Since $[T]_\beta^\gamma = A$(as we have determined from the prior HW), we can see that it would be sufficient to determine if A is invertible.
   Since A is a 2x3 matrix, we can already see that A is not invertible.
   Therefore, T is not invertible.
   Additionally, we have seen from a prior HW question that T is not one-to-one.
   Thus it is not bijection, and thus not invertible.

Finally, we can see that the domain and codomain of T are not the same dimension, and thus T is not invertible by Theorem 2.17 Corollary

9. Question 2.4 7a
   Let A be an $n \times n$ matrix.
   Suppose $A^2 = \underline{O}$. Prove that A is not invertible.
   Proof:
   Assume that A is an $n \times n$ matrix.
   Assume that $A^2 = \underline{O}$.
   Need to show that A is not invertible.
   Assume that A is invertible.
   Then there exists a matrix B such that $AB = I$.
   Then $A^2B^2 = A(AB)B = AIB = AB = I$.
   Since $A^2 = \underline{O}$, $A^2B^2 = \underline{O}B^2 = \underline{O}$.
   Clearly $I \neq \underline{O}$.
   This is a contradiction. Thus, A is not invertible.

10. Question 2.4 7b Let A be an $n \times n$ matrix.
    Suppose $AB = \underline{O}$ for some non-zero $n \times n$ matrix B. Could A be invertible?
    Proof:
    Assume that A is an $n \times n$ matrix.
    Assume that $AB = \underline{O}$ for some non-zero $n \times n$ matrix B.
    Need to show that A is not invertible.
    Assume that A is invertible.
    Then there exists a matrix C such that $CA = I$.
    Then $CAB = IB = B$.
    Since $AB = \underline{O}$, $CAB = C\underline{O}$
    Thus $B = \underline{O}$.
    This goes against the assumption that B is non-zero.
    Therefore, A is not invertible.