

16:960:665 - Syllabus

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Contents

1	Notes	2
1.1	9/2/2025 Lecture 1	2
1.2	9/9/2025 Lecture 2	4
1.3	9/11/2025 Lecture 3	6
1.4	9/16/2025 Lecture 4	8
1.5	9/18/2025 Lecture 5	11
1.6	9/23/2025 Lecture 6	13
1.7	9/25/2025 Lecture 7	15
1.8	9/30/2025 Lecture 8	17
1.9	10/2/2025 Lecture 9	19
1.10	10/7/2025 Lecture 10	21

Syllabus

Time Series: Theory and Methods. Brockwell and Davis
Asymptotic Theory of Weakly dependent Random Process
Martingale Limit Theory

Durrett - Probability Theory and Examples

Questions

Ask what I need to get and review before classes start
Measure Theory: not hardcore
look into textbooks and ergodic theory
Ask the professors of the classes to audit
What is the $X(\omega)$ notation

Acronyms

R.V. - Random Variable
S.P. - Stochastic Process
fn - Function
dist - Distribution
G.P. - Gaussian Process
iid - independent and identically distributed
a.s. - Almost Surely
w.p 1 - with probability 1

1 Notes

1.1 9/2/2025 Lecture 1

We use Stochastic Process to model time series data

Definition (Stochastic Process). A stochastic process is a family of random variables $\{X_t : t \in \mathcal{T}\}$ defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

$\mathcal{T} = \mathbb{N}, \mathbb{Z}$ Discrete Time

$\mathcal{T} = \mathbb{R}$ Continuous Time (not focusing on this)

$\mathcal{T} \subseteq \mathbb{R}^n$ Geospatial, with location and time, (not focusing on this)

$\mathcal{T} \subseteq \mathbb{S}^3$ Unit Sphere w/Geophysics.

Definition (Realization of a S.P.). The functions $\{X(\omega), \omega \in \Omega\}(\mathcal{T} \rightarrow \mathbb{R})$ are realizations or sample paths of the process.

- Fix t , X_t is a fn of Ω
- Fix an outcome $\omega \in \Omega$, $X_t(\omega)$ is a fn on \mathcal{T}
- The time series we observe is a realization of the S.P.
- Conventionally the observed time series is indexed by $\{1, 2, \dots, n\}$ ie $\{X_1, X_2, \dots, X_n\}$ (known as the lens/sample size)

Example (1.2.1 from book). Suppose $A \geq 0$ is a R.V and given by $\Theta \sim Uniform(0, 2\pi)$. and they are independent. and $v > 0$ is a known constant

Then $X_t = A \cos(vt + \Theta), t \in \mathbb{Z}$

For every $\omega \in \Omega$, $A(\omega), \Theta(\omega)$ are fixed

$X_t(\omega) = A(\omega) \cos(vt + \Theta(\omega))$

A determines the amplitude and Θ determines the phase.

What we do is we take a model, and have the data as a realization, and solve the inverse problem of determining the parameters of the model.

Example (1.2.2 from the book). Consider X_1, X_2, X_3, \dots are IID and take value $1, -1$ with probability $1/2$

Im considering to use some binomial theorem thing...

Example (1.2.3 from the book). Suppose X_t coming from prior question.

$S_t = \sum_{i=1}^t X_i = X_1 + X_2 + \dots + X_t$ $S_t : t \in \mathbb{N}$ is a S.P. called a simple symmetric random walk

Consider a man in 1D who starts at 0, and takes a random draw to walk left or right. The path of this miserable guys is S_t

The realization is a plot of $S_t(\omega)$ against t .

Definition (The Distribution of a Stochastic Process). Let \mathcal{I} be the collection of all tuples $\{\mathbf{t} = (t_1, t_2, \dots, t_n)' \in \mathcal{T}, t_1 < t_2 < \dots < t_n\}$ The finite dimensional dist. fns of $\{X_t, t \in \mathcal{T}\}$ are the collection of fns $\{F_t(\cdot) : \mathbf{t} \in \mathcal{I}\}$ where

$$F_{\mathbf{t}}(\mathbf{x}) = P(X_{t_1} = x_1, X_{t_2} = x_2, \dots, X_{t_n} = x_n)$$

$$\mathbf{x} = (x_1, x_2, \dots, x_n)' \in \mathbb{R}^n$$

ie $F_{\mathbf{t}}(\mathbf{x})$ is the joint distribution of the process of the R.V. \mathbf{x} .

Theorem 1 (Kolmogorov (consistency) Theorem). *The prob. distribution fns $\{F_{\mathbf{t}}(\cdot) : \mathbf{t} \in \mathcal{I}\}$ are the distribution functions of some S.P. \iff for any $n \in \mathbb{N}$, $\mathbf{t} = (t_1, t_2, \dots, t_n)' \in \mathcal{T}$ and $1 \leq i \leq n$*

$$\lim_{x_i \rightarrow \infty} F_{\mathbf{t}}(\mathbf{x}) = F_{\mathbf{t}_i}(\mathbf{x}_i)$$

Where $\mathbf{x} = (x_1, x_2, \dots, x_n)'$,

$\mathbf{t}_i = (t_1, t_2, \dots, t_{i-1}, t_{i+1}, \dots, t_n)'$ and $\mathbf{x}_i = (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n)'$

essentially the i are the missing ones

$$F(x_1, x_2) = \mathbb{P}(X_1 \leq x_1, X_2 \leq x_2)$$

$$\lim_{x_2 \rightarrow \infty} F(x_1, x_2) = \mathbb{P}(X_1 \leq x_1)$$

"https://en.wikipedia.org/wiki/Kolmogorov_extension_theorem"

We essentially only need to specify the consistency of the finite dimensional distributions to define a S.P.

1.2 9/9/2025 Lecture 2

Definition (Autocovariance function). If $X_t, t \in \mathcal{T}$ is a S.P. s.t $E(X_t^2) < \infty$, then for every $t \in \mathcal{T}$ the the autocovariance function is defined as

$$\gamma_x(r, s) = Cov(X_r, X_s), r, s \in \mathcal{T}$$

Definition (Autocorrelation function). If $X_t, t \in \mathcal{T}$ is a S.P. s.t $E(X_t^2) < \infty$, then for every $t \in \mathcal{T}$ the the autocorrelation function is defined as

$$\rho_x(r, s) = Corr(X_r, X_s) = \frac{\gamma_x(r, s)}{\sqrt{\gamma_x(r, r)\gamma_x(s, s)}}, r, s \in \mathcal{T}$$

Definition (Stationary S.P). A stochastic process $X_t, t \in \mathcal{T}$ is said to be stationary

- $E(X_t^2) < \infty$ for all $t \in \mathcal{T}$
- $E(X_t) = \mu$ for all $t \in \mathcal{T}$
- $\gamma_x(r, s) = \gamma_x(r + h, s + h)$ for all $r, s, h \in \mathcal{T}$

Weakly Stationary/Covariance Stationary/Wide Sense Stationary/Second Order Stationary

ASK: If our \mathcal{T} is a non convex set, does this still hold?

Also if X_t is stationary, then $\gamma_x(r, s) = \gamma_x(0, s - r) = \gamma_x(s - r)$ ie we can define the autocovariance as a fn of the one variable: the lag $h = s - r$

Similarly $\rho_x(h) = \frac{\gamma_x(h)}{\gamma_x(0)}$

Definition (Strict Stationarity). A stochastic process $X_t, t \in \mathcal{T}$ is said to be strictly stationary if for every $n \in \mathbb{N}$, $t_1, t_2, \dots, t_n \in \mathcal{T}$ and $h \in \mathcal{T}$ the random vectors $(X_{t_1}, X_{t_2}, \dots, X_{t_n})'$ and $(X_{t_1+h}, X_{t_2+h}, \dots, X_{t_n+h})'$ have the same distribution.

ie the finite dimensional distributions are shift invariant.

If Strict Stationarity with finite second moments \implies Weak Stationarity.

Definition (Gaussian Time Series (S.P)). A Gaussian S.P. is a S.P. $X_t, t \in \mathcal{T}$ if all the finite dimensional distributions fns of $\{X_t\}$ are multivariate normal.

ie for every $n \in \mathbb{N}$ and $t_1, t_2, \dots, t_n \in \mathcal{T}$ the random vector $(X_{t_1}, X_{t_2}, \dots, X_{t_n})'$ has a multivariate normal distribution. - IF a G.P. is stationary, then it is strictly stationary.

Definition (Stationarity of IID). IID variables are strictly stationary.

Definition (White Noise). A S.P. X_t is said to be white noise if can also be written as $WN(0, \sigma^2)$

- $E(X_t) = 0$ for all t
- $Var(X_t) = \sigma^2 < \infty$ for all t
- $Cov(X_t, X_s) = 0$ for all $t \neq s$

It is a weakly stationary S.P.

Example (Example of White Noise not Strictly Stationary). Let X_t with $t = \text{even}$ be $N(0, 1)$ and X_t with $t = \text{odd}$ be *Rademacher*(0, 1)
Then X_t is white noise but not strictly stationary.

Example (1.3.1). $X_t = A \cos(\Theta t) + B \sin(\Theta t)$ where $E(A) = E(B) = 0$, $Var(A) = Var(B) = 1$, $Cov(A, B) = 0$

- $E(X_t) = 0$
- $Var(X_t) = E(A^2 \cos^2(\Theta t) + B^2 \sin^2(\Theta t)) = \cos^2(\Theta t) + \sin^2(\Theta t) = 1$
- $Cov(X_t, X_s) = E(X_t X_s) = E[(A \cos(\Theta t) + B \sin(\Theta t))(A \cos(\Theta s) + B \sin(\Theta s))] = E[A^2] \cos(\Theta t) \cos(\Theta s) + E[B^2] \sin(\Theta t) \sin(\Theta s) = \cos(\Theta t) \cos(\Theta s) + \sin(\Theta t) \sin(\Theta s) = \cos(\Theta(t - s))$

Note that the $Cov(X_t, X_s)$ is only a fn of $t - s$

Thus X_t is weakly stationary.

Example (1.3.2). Let $Z_t, t \in \mathbb{Z}$ be IID(0, σ^2)

$$X_t = Z_t + \Theta Z_{t-1}$$

- $E(X_t) = 0$
- $Var(X_t) = Var(Z_t) + \Theta^2 Var(Z_{t-1}) = (1 + \Theta^2) \sigma^2$
- $Cov(X_t, X_s) = E(X_t X_s) = E[(Z_t + \Theta Z_{t-1})(Z_s + \Theta Z_{s-1})] = \Theta \sigma^2$ if $|t - s| = 1$, $(1 + \Theta^2) \sigma^2$ if $t = s$, 0 otherwise

Thus X_t is weakly stationary.

Example (1.3.4). Assume X_t is IID(0, σ^2)

$$S_t = X_1 + X_2 + \dots + X_t \quad t \geq 1$$

- $E(S_t) = 0$
- $Var(S_t) = t \sigma^2$ Not constant

- $Cov(S_r, S_t) = E(S_r S_t) = r\sigma^2$ WLOG $r \leq t$
- $Cov(S_r, S_t) = (r \wedge t)\sigma^2$

Proposition 1 (1.5.1). Suppose X_t is weakly stationary with $\gamma_x(h), \rho_x(h)$ as the autocovariance and autocorrelation fns. Then

- $\gamma_x(0) \geq 0$
- $|\gamma_x(h)| \leq \gamma_x(0)$ for all $h \in \mathcal{T}$
- $\gamma_x(h) = \gamma_x(-h)$ for all $h \in \mathcal{T}$

Remark (Some Statistics...). Observe $\{X_t\}, t = 1, 2, \dots, n$ Want to estimate $\mu, \gamma(0), \gamma(1), \dots, \gamma(n-1)$

$$\begin{aligned}\hat{\mu} &= \frac{1}{n} \sum_{i=1}^n X_i := \bar{X} \\ \hat{\gamma}(0) &= \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2 \\ \hat{\gamma}(1) &= \frac{1}{n} \sum_{i=1}^{n-1} (X_i - \bar{X})(X_{i+1} - \bar{X}) \\ \hat{\gamma}(h) &= \frac{1}{n} \sum_{i=1}^{n-h} (X_i - \bar{X})(X_{i+h} - \bar{X})\end{aligned}$$

The reason why we divide by n we want to shrink it. intuition is that we want to make autocorrelation smaller as n increases.

1.3 9/11/2025 Lecture 3

Remark (Matrix Form of Autocovariance). Observe X_1, X_2, \dots, X_n
 $\hat{\gamma}(h) = \frac{1}{n} \sum_{i=1}^{n-h} (X_i - \bar{X})(X_{i+h} - \bar{X})$.

$$\Gamma_n = \text{Cov} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} \gamma_x(0) & \gamma_x(1) & \gamma_x(2) & \dots & \gamma_x(n-1) \\ \gamma_x(1) & \gamma_x(0) & \gamma_x(1) & \dots & \gamma_x(n-2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_x(n-1) & \gamma_x(n-2) & \gamma_x(n-3) & \dots & \gamma_x(0) \end{bmatrix}$$

This is a Toeplitz matrix. ie constant along the diagonals. It is also positive semidefinite. ie $a' \Gamma_n a \geq 0$ for all $a \in \mathbb{R}^n$.

For the Sample version, we have

$$\hat{\Gamma}_n = \begin{bmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) & \hat{\gamma}(2) & \dots & \hat{\gamma}(n-1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) & \hat{\gamma}(1) & \dots & \hat{\gamma}(n-2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\gamma}(n-1) & \hat{\gamma}(n-2) & \hat{\gamma}(n-3) & \dots & \hat{\gamma}(0) \end{bmatrix}$$

We use n as a common denominator to ensure that $\hat{\Gamma}_n$ is positive semidefinite.

Γ_n is called the order- n autocovariance matrix of the process.

$\hat{\Gamma}_n$ is called the order- n sample autocovariance

Theorem 2. *A real valued fn defined on the integers is the autocovariance fn of a weakly stationary Time Series iff*

- It is even. ie $\gamma(h) = \gamma(-h)$ for all $h \in \mathcal{T}$
- It is non-negative definite. ie for every $n \in \mathbb{N}$ and $a_1, a_2, \dots, a_n \in \mathbb{R}$

IE $\sum_{i,j}^n a_i k(t_i - t_j) a_j \geq 0$ for all $n \geq 1, \mathbf{t} = (t_1, t_2, \dots, t_n)' \in \mathbb{N}^n$, and $\mathbf{a} = (a_1, a_2, \dots, a_n)' \in \mathbb{R}^n$

Proof. **LOOK MORE INTO THIS THEOREM**

\implies

It is straightforward to see that $\gamma_x(h)$ is even.

Let $n \in \mathbb{N}$, $\mathbf{t} = (t_1, t_2, \dots, t_n)' \in \mathbb{N}^n$, and $\mathbf{a} = (a_1, a_2, \dots, a_n)' \in \mathbb{R}^n$

$$\sum_{i,j}^n a_i \gamma_x(t_i - t_j) a_j = \sum_{i,j}^n a_i \text{Cov}(X_{t_i}, X_{t_j}) a_j = \text{Cov}\left(\sum_{i=1}^n a_i X_{t_i}, \sum_{j=1}^n a_j X_{t_j}\right) = \text{Var}\left(\sum_{i=1}^n a_i X_{t_i}\right) \geq 0$$

\Longleftarrow

Let $k(h)$ be a real valued fn defined on the integers which is even and non-negative definite.

Let $n \in \mathbb{N}$, $\mathbf{t} = (t_1, t_2, \dots, t_n)' \in \mathbb{N}^n$, and $\mathbf{a} = (a_1, a_2, \dots, a_n)' \in \mathbb{R}^n$

Define $\Gamma_n = [k(t_i - t_j)]_{i,j=1}^n$

Then Γ_n is a non-negative definite matrix. ie $\mathbf{a}' \Gamma_n \mathbf{a} \geq 0$ for all $\mathbf{a} \in \mathbb{R}^n$.

Thus by the spectral theorem, there exists a random vector $\mathbf{X} = (X_{t_1}, X_{t_2}, \dots, X_{t_n})'$ with mean 0 and covariance matrix Γ_n . ie $E(\mathbf{X}) = 0$ and $\text{Cov}(\mathbf{X}) = \Gamma_n$.

ie $\text{Cov}(X_{t_i}, X_{t_j}) = k(t_i - t_j)$ for all $1 \leq i, j \leq n$

By Kolmogorov's theorem, there exists a S.P. $X_t, t \in \mathbb{Z}$ with autocovariance fn $k(h)$.

□

Example. Suppose $k(h) = \begin{cases} 1 & h = 0 \\ \rho & h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$

When is k an autocovariance fn of a weakly stationary S.P.?

- $|\rho| \leq .5$ then

Remember Z_t is IID(0, σ^2), $X_t = Z_t + \Theta Z_{t-1}$ with acovf $\gamma_x(h) = \begin{cases} (1 + \Theta^2)\sigma^2 & h = 0 \\ \Theta\sigma^2 & h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$

$\rho(1) = \frac{\Theta}{1+\Theta^2}$ then $1 + \Theta^2 \leq 2\theta$ ie $|\rho| \leq .5$

- If $.5 < \rho \leq 1$ then $k(h)$ is not an acovf.

Then you can find a n s.t.

$$\sum_{i,j}^{2n} a_i a_j k(i-j) = 2n - 2(n-1)\rho < 0$$

Where does this formula on the RHS come from?

- If $-1 \leq \rho < -.5$ then $k(h)$ is not an acovf.

Definition (Mixing Conditions). Suppose \mathcal{G} and \mathcal{H} are two sub σ -fields on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $\mathcal{G} \subset \mathcal{F}$ and $\mathcal{H} \subset \mathcal{F}$.

Definition (α -mixing:). α -mixing: $\alpha(\mathcal{G}, \mathcal{H}) = \sup_{G \in \mathcal{G}, H \in \mathcal{H}} |\mathbb{P}(G \cap H) - \mathbb{P}(G)\mathbb{P}(H)|$
 X_1 and X_2 are independent $\mathcal{G} = \sigma(X_1) = \sigma([X_1 \leq c], c \in \mathbb{R})$ and $\mathcal{H} = \sigma(X_2)$

- $\alpha(\mathcal{G}, \mathcal{H}) = 0$ iff \mathcal{G} and \mathcal{H} are independent
- $0 \leq \alpha(\mathcal{G}, \mathcal{H}) \leq .25$ for all \mathcal{G}, \mathcal{H} sub σ -fields of \mathcal{F}
- $\mathbb{P}(G \cap H) - \mathbb{P}(G)\mathbb{P}(H) = \mathbb{E}[I_G I_H] - \mathbb{E}[I_G]\mathbb{E}[I_H] = \text{Cov}(I_G, I_H)$

$$|\text{Cov}(c_1 I_G, c_2 I_H)| \leq c_1 c_2 \alpha(\mathcal{G}, \mathcal{H})$$

Definition (ϕ -mixing:). ϕ -mixing: $\phi(\mathcal{G}, \mathcal{H}) = \sup_{G \in \mathcal{G}, H \in \mathcal{H}, \mathbb{P}(G) > 0} |\mathbb{P}(H|G) - \mathbb{P}(H)|$

- $\phi(\mathcal{G}, \mathcal{H}) = 0$ iff \mathcal{G} and \mathcal{H} are independent
- $0 \leq \phi(\mathcal{G}, \mathcal{H}) \leq 1$ for all \mathcal{G}, \mathcal{H} sub σ -fields of \mathcal{F}
- $\alpha(\mathcal{G}, \mathcal{H}) \leq \frac{1}{2} \phi(\mathcal{G}, \mathcal{H})$

Example. X is G -measureable and Y is H -measureable, $|X| \leq C_1$ and $|Y| \leq C_2$ a.s.
Then $|\text{Cov}(X, Y)| \leq 4C_1 C_2 \alpha(\mathcal{G}, \mathcal{H})$

1.4 9/16/2025 Lecture 4

Remark (Last Class Review). Mixing Conditions:

Suppose \mathcal{G} and \mathcal{H} are two sub σ -fields on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $\mathcal{G} \subset \mathcal{F}$ and $\mathcal{H} \subset \mathcal{F}$. LOOK INTO TEXTBOOK ASSIGNMENTS

- α -mixing: $\alpha(\mathcal{G}, \mathcal{H}) = \sup_{G \in \mathcal{G}, H \in \mathcal{H}} |\mathbb{P}(G \cap H) - \mathbb{P}(G)\mathbb{P}(H)|$
- ϕ -mixing: $\phi(\mathcal{G}, \mathcal{H}) = \sup_{G \in \mathcal{G}, H \in \mathcal{H}, \mathbb{P}(G) > 0} |\mathbb{P}(H|G) - \mathbb{P}(H)|$

1. $\alpha(\mathcal{G}, \mathcal{H}) = 0$ iff \mathcal{G} and \mathcal{H} are independent

2. $0 \leq \alpha(\mathcal{G}, \mathcal{H}) \leq .25$ for all \mathcal{G}, \mathcal{H} sub σ -fields of \mathcal{F} , $0 \leq \phi(\mathcal{G}, \mathcal{H}) \leq 1$ for all \mathcal{G}, \mathcal{H} sub σ -fields of \mathcal{F}
3. $\alpha(\mathcal{G}, \mathcal{H}) \leq \frac{1}{2}\phi(\mathcal{G}, \mathcal{H})$

Equal definition: $\alpha(X, Y) = \sup_{c_1, c_2 \in \mathbb{R}} |\mathbb{P}(X \leq c_1, Y \leq c_2) - \mathbb{P}(X \leq c_1)\mathbb{P}(Y \leq c_2)|$
 Equal definition: $\phi(X, Y) = \sup_{c_1, c_2 \in \mathbb{R}, \mathbb{P}(X \leq c_1) > 0} |\mathbb{P}(Y \leq c_2 | X \leq c_1) - \mathbb{P}(Y \leq c_2)|$

Theorem 3 (Ibragimov 1962). $\mathbb{P}(G \cap H) - \mathbb{P}(G)\mathbb{P}(H) = \text{Cov}(I_G, I_H)$

$|\text{Cov}(c_1 I_G, c_2 I_H)| \leq c_1 c_2 \alpha(\mathcal{G}, \mathcal{H})$

Sup. $|X| \leq C_1$ and $|Y| \leq C_2$ a.s.

Then $|E(XY) - E(X)E(Y)| \leq 4c_1 c_2 \alpha(\mathcal{G}, \mathcal{H})$

Proof.

$$\begin{aligned} E(XY) - E(X)E(Y) &= E[X(Y - E(Y))] \\ &= E[X(E(Y|X) - E(Y))] \\ &= E[E(XY|X) - E(Y)] \\ |E(XY) - E(X)E(Y)| &= |E[X(E(Y|X) - E(Y))]| \\ &\leq c_1 E|E(Y|X) - E(Y)| \end{aligned}$$

Define $\eta = \text{sign}(E(Y|X) - E(Y))$

$$\begin{aligned} &= c_1 E[\eta(E(Y|X) - E(Y))] \\ \eta E(Y|X) &= E(\eta Y|X) \\ c_1 E[E(\eta Y|X) - \eta E(Y)] &= c_1 [E(\eta Y) - E(\eta)E(Y)] \\ E(\eta Y) - E(\eta)E(Y) &\leq E[Y[E(\eta|Y) - E(\eta)]] \end{aligned}$$

Let $\xi = \text{sign}(E(\eta|Y) - E(\eta))$

$$\begin{aligned} E(\eta Y) - E(\eta)E(Y) &\leq c_2 (E[\xi \eta] - E(\xi)E(\eta)) \\ E(XY) - E(X)E(Y) &\leq c_1 c_2 (E[\xi \eta] - E(\xi)E(\eta)) \end{aligned}$$

$$\eta = I_{\eta=1} - I_{\eta=-1}, \xi = I_{\xi=1} - I_{\xi=-1}$$

$$\begin{aligned} \text{Cov}(\xi, \eta) &= \text{Cov}(I_{\xi=1} - I_{\xi=-1}, I_{\eta=1} - I_{\eta=-1}) \\ &= \text{Cov}(I_{\xi=1}, I_{\eta=1}) + \text{Cov}(I_{\xi=-1}, I_{\eta=-1}) \\ &\quad - \text{Cov}(I_{\xi=1}, I_{\eta=-1}) - \text{Cov}(I_{\xi=-1}, I_{\eta=1}) \end{aligned}$$

$$\implies |\text{Cov}(\xi, \eta)| \leq 4\alpha(\mathcal{G}, \mathcal{H})$$

$$|E(XY) - E(X)E(Y)| \leq 4c_1 c_2 \alpha(\mathcal{G}, \mathcal{H})$$

□

Why are we doing this?

Consider $X_1, X_2, \dots \text{IID}(0, \sigma^2)$

$$\frac{X_1 + X_2 + \dots + X_n}{n} \xrightarrow{d} N(0, \sigma^2)$$

Now how do we get CLT?

Consider X_1, X_2, \dots is a weakly stationary S.P, with $E(X_t) = 0$

$$\frac{X_1 + X_2 + \dots + X_n}{n} \xrightarrow{d} N(0, \sigma^2)$$

We can see this is the variance $S_n = X_1 + X_2 + \dots + X_n$

$$\text{Var}(S_n) = \sum_{i=1}^n \text{Var}(X_i) + 2 \sum_{1 \leq i < j \leq n} \text{Cov}(X_i, X_j)$$

$$= n\gamma_x(0) + 2 \sum_{1 \leq i < j \leq n} (\gamma_x(j-i))$$

$$= n\gamma_x(0) + 2 \sum_{h=1}^{n-1} (n-h)\gamma_x(h)$$

$$\text{Var}\left(\frac{S_n}{\sqrt{n}}\right) = \gamma_x(0) + 2 \sum_{h=1}^{n-1} \left(1 - \frac{h}{n}\right) \gamma_x(h)$$

$$\lim_{n \rightarrow \infty} \text{Var}\left(\frac{S_n}{\sqrt{n}}\right) = \gamma_x(0) + 2 \sum_{h=1}^{\infty} \gamma_x(h)$$

We want this infinite series to converge. ie $\sum_{h=1}^{\infty} |\gamma_x(h)| < \infty$.

Consider X_1, X_2, \dots is a strictly stationary S.P.

Define $\alpha_0 = \frac{1}{2}$, and $\alpha_n = \alpha(X_0, X_n)$ for $n \geq 1$

Assume $E|X_0|^p < \infty$ for some $p > 2$

$$\text{Then } |\gamma_x(k)| = |\text{Cov}(X_0, X_k)| \leq 8 \|X_0\|_p^2 \alpha_k^{1-\frac{2}{p}}$$

Corollary (Only Y is bounded). Suppose $E[X^2] < \infty$ for some $p > 1$ and $|Y| \leq C$ a.s.

Then $E(XY) - E(X)E(Y) \leq 6C \|X\|_p [\alpha(X, Y)]^{1-\frac{1}{p}}$ where $\|X\|_p = (E|X|^p)^{\frac{1}{p}}$

Proof. Through Truncation:

$$X_1 = XI_{|X| \leq C_1} \text{ and } X_2 = X - X_1$$

$$\begin{aligned} |E(XY) - E(X)E(Y)| &\leq |E(X_1Y) - E(X_1)E(Y)| + |E(X_2Y) - E(X_2)E(Y)| \\ &\leq 4CC_1\alpha(X, Y) + 2CE|X_2| \end{aligned}$$

$$E|X_2| = E|XI_{|X| > C_1}| \leq \frac{E|X|^p}{C_1^{p-1}}$$

$$\begin{aligned} I_{|X| > C_1} &< \frac{|X|^p}{C_1^{p-1}} \\ &= \frac{\|X\|_p^p}{C_1^{p-1}} \end{aligned}$$

$$\text{Thus } |E(XY) - E(X)E(Y)| \leq 4CC_1\alpha(X, Y) + \frac{\|X\|_p^p}{C_1^{p-1}}.$$

Take $C_1 = \alpha^{-\frac{1}{p}} \|X\|_p$ to get best bound.

Then the corollary follows.

Look into bernstein inequality

□

Corollary (No bounded (Davydov 1968)). Suppose $E|X|^p < \infty$ and $E|Y|^q < \infty$ for some $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} < 1$ then

$$|E(XY) - E(X)E(Y)| \leq 8 \|X\|_p \|Y\|_q [\alpha(X, Y)]^{1-\frac{1}{p}-\frac{1}{q}}$$

Review of Hilbert Spaces

Definition (Inner Product Space). A vector space \mathcal{V} over the field \mathbb{F} is called an inner product space if there exists a fn $\langle \cdot, \cdot \rangle$

- $\langle u, v \rangle = \overline{\langle v, u \rangle}$ for all $u, v \in \mathcal{V}$
- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in \mathcal{V}$
- $\langle cu, v \rangle = c\langle u, v \rangle$ for all $u, v \in \mathcal{V}$ and $c \in \mathbb{F}$
- $\langle u, u \rangle \geq 0$ for all $u \in \mathcal{V}$
- $\langle u, u \rangle = 0$ iff $u = 0$

We will see that for the prob space $\langle X, Y \rangle = E[XY]$ but this only holds a.s.

1.5 9/18/2025 Lecture 5

Definition (Inner Product Space). \mathcal{H} is an inner product space with inner product $\langle \cdot, \cdot \rangle$

Example (2.2.2). $L^2(\Omega, \mathcal{F}, \mathbb{P}) = \{X : \Omega \rightarrow \mathbb{R} | X \text{ is measurable and } E(X^2) < \infty\}$

$$\langle X, Y \rangle = E(XY) = \int_{\Omega} X(\omega)Y(\omega)d\mathbb{P}(\omega)$$

$$\langle X, X \rangle = E(X^2) = 0 \implies X = 0 \text{ a.s.}$$

Define an equivalence relation $X \sim Y$ if $X = Y$ a.s.

- The elements of L^2 are equivalence classes
- $\|X\| = \sqrt{\langle X, X \rangle} = \sqrt{E(X^2)}$ is a norm on L^2

Remark. IP Properties:

- $|\langle x, y \rangle| \leq \|x\| \|y\|$ (Cauchy-Schwarz Inequality)
- $\|x + y\| \leq \|x\| + \|y\|$ (Triangle Inequality)
- If $\|x_n - x\| \rightarrow 0$ and $\|y_n - y\| \rightarrow 0$ then $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$ (Continuity of Inner Product)

Definition (Limit of a Sequence in Hilbert Space). Let \mathcal{H} be a Hilbert Space and $\{x_n\}$ be a sequence in \mathcal{H}

We say that $x_n \rightarrow x$ if $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$.

Suppose $\{X_n\}$ is a sequence of random variables in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ which converges to X . Then consider the RV 1 (constant)

$$\text{Consider } \langle X_n, 1 \rangle \rightarrow \langle X, 1 \rangle$$

$$\text{ie } E(X_n) \rightarrow E(X)$$

$$X_n \rightarrow X$$

$$\langle X_n, X_n \rangle \rightarrow \langle X, X \rangle \text{ ie } E(X_n^2) \rightarrow E(X^2)$$

$$\begin{aligned} X_n &\rightarrow X, Y_n \rightarrow Y \\ \langle X_n, Y_n \rangle &\rightarrow \langle X, Y \rangle \\ \text{ie } E(X_n Y_n) &\rightarrow E(XY) \end{aligned}$$

Definition (Cauchy Sequence). A sequence of elements $\{x_n\}$ in an inner product space \mathcal{H} is called a Cauchy sequence if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x_n - x_m\| < \epsilon$ for all $n, m \geq N$.

Definition (Hilbert Space). An inner product space \mathcal{H} is called a Hilbert Space if every Cauchy sequence in \mathcal{H} converges to an element in \mathcal{H} .

Example. Consider $\mathcal{M}(\Omega, \mathcal{F}, \mathbb{P}) = \{X : |X| \leq C, C > 0\}$

$$\langle X, Y \rangle = E(XY)$$

$$X \sim N(0, 1)$$

$$X_n = XI_{|X| \leq n}$$

$$E|X - X_n|^2 = E[X^2 I_{|X| > n}] \rightarrow 0 \text{ by DCT (Dominated Convergence Theorem)}$$

So $X_n \rightarrow X$ in L^2 but $X \notin \mathcal{M}$

Thus \mathcal{M} is not a Hilbert Space.

Definition (Complex Random Variable). A complex random variable is a fn $Z : \Omega \rightarrow \mathbb{C}$ such that $Z = X + iY$ where X, Y are real random variables.

Definition (Closed Subspace). A linear subspace of a Hilbert Space \mathcal{H} is called a closed subspace if \mathcal{M} contains its limit points. ie if $\{x_n\} \subset \mathcal{M}$ and $x_n \rightarrow x$ in \mathcal{H} then $x \in \mathcal{M}$.

Proposition 2 (2.3.1). Review the definition If \mathcal{M} is a closed subset of a H.S \mathcal{H} then the orthogonal complement $\mathcal{M}^\perp = \{x \in \mathcal{H} : x \perp y, \forall y \in \mathcal{M}\}$ closed linear subspace of \mathcal{H} .

Theorem 4 (2.3.1 Projection Theorem). If \mathcal{M} is a closed linear subspace of a H.S \mathcal{H} and $x \in \mathcal{H}$ then

(i) there is a unique element $\hat{x} \in \mathcal{M}$ such that $\|x - \hat{x}\| = \inf_{y \in \mathcal{M}} \|x - y\|$

(ii) $\hat{x} \in \mathcal{M}$ and $\|x - \hat{x}\| = \inf_{y \in \mathcal{M}} \|x - y\|$ iff $\hat{x} \in \mathcal{M}$ and $x - \hat{x} \in \mathcal{M}^\perp$

Definition (2.4.1 Closed Span). The closed span $\overline{\text{sp}}\{X_t, t \in \mathcal{T}\}$ of any subset $\{X_t, t \in \mathcal{T}\}$ of a H.S \mathcal{H} is the smallest closed linear subspace of \mathcal{H} containing $\{X_t, t \in \mathcal{T}\}$.

Definition (Orthonormal Set). A set $\{e_t : t \in \mathcal{T}\}$ of element of an IP space is said to be

$$\text{orthonormal if } \langle e_s, e_t \rangle = \begin{cases} 1 & s = t \\ 0 & s \neq t \end{cases} \text{ for all } s, t \in \mathcal{T}$$

Definition (Complete Orthonormal Set). An orthonormal set $\{e_t : t \in \mathcal{T}\}$ in a H.S \mathcal{H} is said to be complete if $\overline{\text{sp}}\{e_t, t \in \mathcal{T}\} = \mathcal{H}$

Definition (Seperability). The HS is separable if it has a finite or countable infinite complete orthonormal set.

Example (Separable HS). 1. \mathbb{R}^d
 2. $L^2(\Omega, \mathcal{F}, \mathbb{P})$

Theorem 5 (2.4.2). If \mathcal{H} is a separable H.S and $\mathcal{H} = \overline{\text{sp}}\{e_t : t \in \mathcal{T}\}$ where $\{e_t : t \in \mathcal{T}\}$ is an orthonormal set then

- The set of all finite linear combinations of $\{e_t : t \in \mathcal{T}\}$ is dense in \mathcal{H} . ie for every $x \in \mathcal{H}$ and $\epsilon > 0$ there exists $y = \sum_{j=1}^n a_j e_{t_j}$ such that $\|x - y\| < \epsilon$
- $x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i$ for each $x \in \mathcal{H}$ ie $\|x - \sum_{i=1}^n \langle x, e_i \rangle e_i\| \rightarrow 0$ as $n \rightarrow \infty$
- $\|x\|^2 = \sum_{i=1}^{\infty} |\langle x, e_i \rangle|^2$ for each $x \in \mathcal{H}$ (Parseval's Identity)
- $\langle x, y \rangle = \sum_{i=1}^{\infty} \langle x, e_i \rangle \langle y, e_i \rangle$ for all $x, y \in \mathcal{H}$
- $x = 0 \iff \langle x, e_i \rangle = 0$ for all $i \geq 1$

1.6 9/23/2025 Lecture 6

Definition (ARMA models: $\text{ARMA}(p, q)$). Let $\{Z_t\} \sim WN(0, \sigma^2)$. The process $\{X_t, t \in \mathbb{Z}\}$ is said to be an $\text{ARMA}(p, q)$ process if

- $\{X_t\}$ is stationary for all $t \in \mathbb{Z}$
- $X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}$ for all $t \in \mathbb{Z}$ where $\phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$ are real constants with $\phi_p, \theta_q \neq 0$.

Remark. There are a few special cases of the $\text{ARMA}(p, q)$ model:

- When $q = 0$ we can write the model as $X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p} = Z_t$ and call it an $\text{AR}(p)$ model.
- When $p = 0$ we can write the model as $X_t = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}$ and call it a $\text{MA}(q)$ model.
- When $p = 0$ and $q = 0$ we have $X_t = Z_t$ and call it a white noise model.
- $\{X_t\}$ is defined relative to the white noise process $\{Z_t\}$.
- Stationarity is a critical requirement for the $\text{ARMA}(p, q)$ model.
- AR polynomial: $\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p$
- MA polynomial: $\theta(z) = 1 + \theta_1 z + \dots + \theta_q z^q$
- Backshift operator: $BX_t = X_{t-1}$, $B^2 X_t = X_{t-2}$, \dots , $B^k X_t = X_{t-k}$
- $\text{AR}(p)$ model: $\phi(B)X_t = Z_t$
- $\text{MA}(q)$ model: $X_t = \theta(B)Z_t$

- ARMA(p, q) model: $\phi(B)X_t = \theta(B)Z_t$
- More general model with a mean: $\{X_t + \mu : t \in \mathbb{Z}\}$
- Can also be characterized by $X_t = \phi_0 + \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}$ where $\phi_0 = \mu(1 - \phi_1 - \dots - \phi_p)$

Example (Stationary solution to AR(1)).

$$\begin{aligned}
X_t &= \phi X_{t-1} + Z_t \\
&= Z_t + \phi(Z_{t-1} + \phi X_{t-2}) = Z_t + \phi Z_{t-1} + \phi^2 X_{t-2} \\
&= Z_t + \phi Z_{t-1} + \phi^2(Z_{t-2} + \phi X_{t-3}) = Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2} + \phi^3 X_{t-3} \\
&\vdots \\
&= Z_t + \phi Z_{t-1} + \phi^2 Z_{t-2} + \dots + \phi^k Z_{t-k} + \phi^{k+1} X_{t-(k+1)}
\end{aligned}$$

If $|\phi| < 1$ then $\phi^{k+1} X_{t-(k+1)} \rightarrow 0$ as $k \rightarrow \infty$

Thus the stationary solution is $X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$

If $|\phi| \geq 1$ then there is no stationary solution since we can see that $X_{t+1} = \phi X_t + Z_{t+1} \iff X_t = -\frac{1}{\phi} Z_{t+1} + \frac{1}{\phi} X_{t+1}$

$$\begin{aligned}
X_t &= \phi^{-1} X_{t+1} - \phi^{-1} Z_{t+1} \\
&= \phi^{-1}(\phi^{-1} X_{t+2} - \phi^{-1} Z_{t+2}) - \phi^{-1} Z_{t+1} \\
&= \phi^{-2} X_{t+2} - \phi^{-1} Z_{t+1} - \phi^{-2} Z_{t+2} \\
&\vdots \\
&= \phi^{-k} X_{t+k} - \sum_{j=1}^k \phi^{-j} Z_{t+j} \\
&= -\sum_{j=1}^{\infty} \phi^{-j} Z_{t+j}
\end{aligned}$$

We will see later that why this is the unique stationary solution when $|\phi| < 1$

Remark. Uniqueness of stationary solution to AR(1):

- If $X_t = \phi X_{t-1} + Z_t$, where $|\phi| > 1$ then we can rewrite this as $X_t = \phi^* X_{t-1} + Z_t^*$ with $\phi^* < 1$ and $Z_t^* \sim WN(0, \sigma^2)$ Homework problem

Definition (3.1.3: Causality). An ARMA(p, q) process $\phi(B)X_t = \theta(B)Z_t$ is said to be causal if there exists a sequence of constants $\{\psi_j\}$ such that $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ for all $t \in \mathbb{Z}$.

Proposition 3 (3.1.1). *If $\{X_t, t \in \mathbb{Z}\}$ is a sequence of rv st. $\sup_t E|X_t| < \infty$ and if $\{\psi_j\}_{j \geq 0}$ is a sequence of numbers s.t $\sum_{j=0}^{\infty} |\psi_j| < \infty$ then the series $\psi(B)X_t = \left(\sum_{j=0}^{\infty} \psi_j B^j\right) X_t = \sum_{j=0}^{\infty} \psi_j X_{t-j}$ converges absolutely w.p 1
If in addition $\sup_t E(X_t^2) < \infty$ then the series converges in L^2 to the same limit.*

Proof. • Consider $\sum_{j=0}^{\infty} |\psi_j| |X_{t-j}|$, which always exists (may be infinite)

- Monotone Convergence Theorem implies $E\left(\sum_{j=0}^{\infty} |\psi_j| |X_{t-j}|\right) = \sum_{j=0}^{\infty} |\psi_j| E|X_{t-j}| \leq (\sup_t E|X_t|) \sum_{j=0}^{\infty} |\psi_j| < \infty \implies \sum_{j=0}^{\infty} |\psi_j| |X_{t-j}| < \infty$ w.p 1
- $\implies \sum_{j=0}^{\infty} \psi_j X_{t-j}$ converges absolutely w.p 1, call the limit W_t .
- Verify $\sum_{j=0}^n \psi_j X_{t-j}$ is a Cauchy sequence in L^2 : We do this by showing $\|\sum_{j=n}^m \psi_j X_{t-j}\|_2 \rightarrow 0$ as $n, m \rightarrow \infty$.
- So it converges in L^2 to some limit S_t .
- $E(S_t - W_t)^2 = E[\liminf_n (S - \sum_{j=0}^n \psi_j X_{t-j})^2]$ by Fatou's Lemma
 $\leq \liminf_n E(S - \sum_{j=0}^n \psi_j X_{t-j})^2 = 0$
 $\implies S_t = W_t$ a.s. since the second moment is 0.

□

1.7 9/25/2025 Lecture 7

Remark (Review). Review of last week:

- ARMA(p, q) process: $\phi(B)X_t = \theta(B)Z_t$ where $\{Z_t\} \sim WN(0, \sigma^2)$
- MA(q) process: $X_t = \theta(B)Z_t$

Proposition 4 (3.1.2). *If $\{X_t\}$ is a stationary process with autocovariance function $\gamma_x(\cdot)$ and if $\{\psi_j\}_{j \geq 0}$, $\sum_{j=0}^{\infty} |\psi_j| < \infty$, define $Y_t = \sum_{j=0}^{\infty} \psi_j X_{t-j}$ (converges absolutely, w.p 1). Then Y_t is also stationary with autocovariance function $\gamma_y(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_x(h + j - k)$ where $\psi_j = 0$ for $j < 0$.*

Proof. We need to show that $E(Y_t)$ is constant and $\text{Cov}(Y_{t+h}, Y_t)$ depends only on h .

$$\begin{aligned}
E(Y_t) &= E\left(\sum_{j=0}^{\infty} \psi_j X_{t-j}\right) = \sum_{j=0}^{\infty} \psi_j E(X_{t-j}) = \mu_x \sum_{j=0}^{\infty} \psi_j \text{ (constant)} \\
\text{Cov}(Y_{t+h}, Y_t) &= E[(Y_{t+h} - E(Y_{t+h}))(Y_t - E(Y_t))] \\
&= E\left[\left(\sum_{j=0}^{\infty} \psi_j (X_{t+h-j} - \mu_x)\right) \left(\sum_{k=0}^{\infty} \psi_k (X_{t-k} - \mu_x)\right)\right] \\
&= E\left[\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \psi_j \psi_k (X_{t+h-j} - \mu_x)(X_{t-k} - \mu_x)\right] \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \psi_j \psi_k E[(X_{t+h-j} - \mu_x)(X_{t-k} - \mu_x)] \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \psi_j \psi_k \gamma_x(h+j-k) \\
&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_x(h+j-k) \text{ where } \psi_j = 0 \text{ for } j < 0
\end{aligned}$$

□

Remark. Let $\alpha(B) = \sum_{j=0}^{\infty} \alpha_j B^j$ and $\beta(B) = \sum_{j=0}^{\infty} \beta_j B^j$ $\sum_{j=0}^{\infty} |\alpha_j| < \infty$ $\sum_{j=0}^{\infty} |\beta_j| < \infty$. Then the product $\psi(B) = \alpha(B)\beta(B) = \sum_{j=0}^{\infty} \psi_j B^j$ then $\sum_{j=0}^{\infty} |\psi_j| < \infty$

Theorem 6 (3.1.1.a). *If $\phi(z)$ and $\theta(z)$ have no common zeros, if $\phi(z) \neq 0$ for $|z| = 1$ and if $\{Z_t\} \sim WN(0, \sigma^2)$ then exists a unique stationary solution given by*

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \text{ where } \psi(z) = \frac{\theta(z)}{\phi(z)} = \sum_{j=0}^{\infty} \psi_j z^j$$

and $\sum_{j=0}^{\infty} |\psi_j| < \infty$. so that X_t is well-defined and causal.

Proof. (i) Find Solution

If $\phi(z) \neq 0$ for $|z| = 1$ then $\exists \epsilon > 0$ such that

$$\begin{aligned}
\frac{1}{\phi(z)} &:= \sum_{j=0}^{\infty} \zeta_j z^j =: \zeta(z), |z| \leq 1 + \epsilon \\
&\implies |\zeta_j| \leq (1 + \epsilon/2)^{-j} \text{ for some } K > 0
\end{aligned}$$

Consider $\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j$ for $|z| < 1$

Consider $\frac{1}{1-0.5z} = \sum_{j=0}^{\infty} (0.5z)^j$ for $|z| < 2$

$\phi(z) = \prod_{j=1}^p (1 - w_j z)$, ie each of the roots are $\frac{1}{w_j}$.

Then $\frac{1}{\phi(z)} = \prod_{j=1}^p \frac{1}{1-w_j z}$

$$\implies \frac{1}{\phi(z)} = \sum_{j=0}^{\infty} \zeta_j z^j \text{ for } |z| < \min_{1 \leq j \leq p} |w_j|^{-1}$$

We know that $\forall j, |w_j| < 1$ and then if we take $\epsilon = \min_{1 \leq j \leq p} |w_j|^{-1} - 1 > 0$ then we are done.

(ii) Find Stationary Solution

Define $X_t = \frac{\theta(B)}{\phi(B)} Z_t$ which is stationary

$$\phi(B)X_t = \theta(B)Z_t$$

(iii) Uniqueness of Stationary Solution

Suppose $\{W_t\}$ is another stationary solution to $\phi(B)W_t = \theta(B)Z_t$

$$\begin{aligned} \phi(B)W_t &= \theta(B)Z_t \\ \zeta(B[\phi(B)W_t]) &= \zeta(B[\theta(B)Z_t]) \\ \implies W_t &= \zeta(B)[\theta(B)Z_t] = \frac{\theta(B)}{\phi(B)} Z_t = X_t \end{aligned}$$

□

Theorem 7 (3.1.1.b). Assume $\phi(z)$ and $\theta(z)$ have no common zeros. If there exists a stationary solution which is also causal then $\phi(z) \neq 0$ for $|z| \leq 1$.

1.8 9/30/2025 Lecture 8

Remark (Review). Prior class review:

- ARMA(p, q) process: $\phi(B)X_t = \theta(B)Z_t$ where $\{Z_t\} \sim WN(0, \sigma^2)$

$\phi(z)$ and $\theta(z)$ have no common zeros.

Theorem 8 (3.1.1.a & .b). (a) If $\phi(z) \neq 0$ for all $|z| \leq 1$ then there exists a unique stationary solution given by

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \text{ where } \psi(z) = \frac{\theta(z)}{\phi(z)} = \sum_{j=0}^{\infty} \psi_j z^j$$

and they satisfy $\sum_{j=0}^{\infty} |\psi_j| < \infty$.

(b) If there exists a stationary solution which is also causal then $\phi(z) \neq 0$ for all $|z| \leq 1$.

Remark. Not proving

- If $\phi(z) \neq 0$ for all $|z| = 1$ then there a unique stationary solution.
- If $\phi(z) = 0$ for some $|z| = 1$ then there is no stationary solution.
- If $\phi(z) \neq 0$ for all $|z| = 1$ and $\{X_t\}$ is the unique stationary solution then one can find $\hat{\phi}(z)$ and $WN\{Z_t^*\}$ st $\hat{\phi}(z)X_t = \phi(B)Z_t^*$ and $\hat{\phi}(z) \neq 0$ for all $|z| \leq 1$.
- Only Focus on Causal and Invertable ARMA models

Definition (3.1.4). Suppose $\{X_t\}$ is a stationary solution of $\phi(B)X_t = \theta(B)Z_t$, it is said to be invertible if $\exists \pi_j$ such that $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and $Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$ for all $t \in \mathbb{Z}$.

Theorem 9 (3.1.2). Suppose X_t is the unique stationary solution of $\phi(B)X_t = \theta(B)Z_t$, then it is invertible iff $\theta(z) \neq 0$ for all $|z| \leq 1$.

When the condition holds $\{\pi_j\}$ are determined by $\pi(z) = \frac{\phi(z)}{\theta(z)} = \sum_{j=0}^{\infty} \pi_j z^j$.

Remark. IF the definition of invertability is relaxed to:

$$Z_t \in \overline{\text{sp}}\{X_t, X_{t-1}, \dots\}$$

then the condition relaxed to $\theta(z) \neq 0$ for all $|z| < 1$

Definition (3.2.1). Suppose $\{Z_t\} \sim WN(0, \sigma^2)$, we say $\{X_t\}$ is an infinite order moving average denoted by $MA(\infty)$ if

$$\exists \{\psi_j\} \text{ such that } \sum_{j=0}^{\infty} |\psi_j| < \infty \text{ and } X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

May relax condition to $\sum_{j=0}^{\infty} \psi_j^2 < \infty$ then take X_t as the L^2 limit.
Sometimes $MA(\infty)$ is called the linear process.

This is related to the Wold Decomposition Theorem.

Proposition 5 (3.2.1). If $\{X_t\}$ is a zero-mean stationary process with autocovariance function $\gamma_x(\cdot)$ such that $\gamma_x(h) = 0$ for $|h| > q$ and $\gamma_x(q) \neq 0$ then $\{X_t\}$ is an $MA(q)$ process.

IE: $\exists WN\{Z_t\}$ s.t. $X_t = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}$ where $\theta_q \neq 0$.

Proof. • Find the WN $\{Z_t\}$

- Show that $X_t = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}$ for some $\theta_1, \dots, \theta_q$ with $\theta_q \neq 0$

□

Definition (Linear Predictor). Suppose $Y \in \mathbb{R}$, $E[Y] = 0$, $\mathbf{X} \in \mathbb{R}^d$, $E[\mathbf{X}] = \mathbf{0}$.

$$\text{Cov}\left(\begin{bmatrix} Y \\ \mathbf{X} \end{bmatrix}\right) = \begin{bmatrix} \sigma_Y^2 & \sigma'_{\mathbf{YX}} \\ \sigma_{\mathbf{YX}} & \Sigma_X \end{bmatrix}$$

A linear predictor takes the form $C^T \mathbf{X}$ where $C \in \mathbb{R}^d$.

The best linear predictor (BLP) of Y based on \mathbf{X} is the linear predictor $\hat{Y} = C^T \mathbf{X}$ that minimizes the mean squared error $\min_{C \in \mathbb{R}^d} E[(Y - C^T \mathbf{X})^2]$.

$$\begin{aligned} E[(Y - C^T \mathbf{X})^2] &= E[Y^2] - 2C^T E[Y\mathbf{X}] + C^T E[\mathbf{X}\mathbf{X}^T]C \\ &= \sigma_Y^2 - 2C^T \sigma_{\mathbf{YX}} + C^T \Sigma_X C \end{aligned}$$

The best solution is given taking the partial derivative and setting it to 0:

$$\begin{aligned} \frac{\partial}{\partial C} E[(Y - C^T \mathbf{X})^2] &= -2\sigma_{\mathbf{YX}} + 2\Sigma_X C = 0 \\ \implies \hat{C} &= \Sigma_X^{-1} \sigma_{\mathbf{YX}} \\ \implies \hat{Y} &= \hat{C}^T \mathbf{X} = \sigma'_{\mathbf{YX}} \Sigma_X^{-1} \mathbf{X} \\ E[(Y - \hat{Y})^2] &= \sigma_Y^2 - \sigma'_{\mathbf{YX}} \Sigma_X^{-1} \sigma_{\mathbf{YX}} \end{aligned}$$

Remark. $\{X_t\}$ is a mean-zero stationary process.
 Want to predict X_{k+1} based on $\{X_1, \dots, X_k\}$.

$$\min_{\phi_1, \dots, \phi_k} E[(X_{k+1} - \hat{X}_{k+1})^2]$$

Where $\hat{X}_{k+1} = \sum_{j=1}^k \phi_j X_{k+1-j}$

$$\text{Gamma}_{k+1} = \text{Cov}\left(\begin{bmatrix} X_{k+1} \\ X_k \\ \vdots \\ X_1 \end{bmatrix}\right) = \begin{bmatrix} \gamma(0) & \gamma(\mathbf{k})' \\ \gamma(\mathbf{k}) & \Gamma_k \end{bmatrix}$$

Where $\mathbf{gamma}(\mathbf{k}) = [\gamma(1), \dots, \gamma(k)]'$ and $\Gamma_k = [\gamma(i-j)]_{i,j=1}^k$

$$\begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \vdots \\ \hat{\phi}_k \end{bmatrix} = \Gamma_k^{-1} \gamma(\mathbf{k})$$

1.9 10/2/2025 Lecture 9

Proposition 6 (3.2.1). *IF $\{X_t\}$ is a zero-mean stationary process with autocovariance function $\gamma_x(\cdot)$ such that $\gamma_x(h) = 0$ for $|h| > q$ and $\gamma_x(q) \neq 0$ then $\{X_t\}$ is an $MA(q)$ process.*

Proof. Need to show

- Find $WN\{Z_t\}$
- Show that $X_t = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}$ for some $\theta_1, \dots, \theta_q$ with $\theta_q \neq 0$

Linear prediction problem: Predict X_{k+1} using X_1, \dots, X_k

$$\underline{\hat{\phi}_k} = \arg \min_{\phi_1, \dots, \phi_k} E[(X_{k+1} - \hat{X}_{k+1})^2]$$

$$\hat{X}_{k+1} = \sum_{j=1}^k \phi_j X_{k+1-j}$$

$$\underline{\hat{\phi}_k} = \Gamma_k^{-1} \underline{\gamma(k)} \text{ where } \underline{\gamma(k)} = [\gamma(1), \dots, \gamma(k)]'$$

$$E[(X_{k+1} - \underline{\hat{\phi}_k}' X_k)^2] = \gamma(0) - \underline{\gamma(k)}' \Gamma_k^{-1} \underline{\gamma(k)} = \nu_k$$

Note that if we want

$$X_t = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}$$

$$Z_t = \sum_{j=0}^{\infty} \psi_j X_{t-j}$$

$$Z_{t-1}, \dots, Z_{t-q} \in \overline{sp}\{X_{t-1}, X_{t-2}, \dots\}$$

Proof:

Define $\mathcal{M}_t = \overline{\text{sp}}\{X_t, X_{t-1}, \dots\}$ and $Z_t = X_t - \mathcal{P}_{\mathcal{M}_{t-1}}(X_t)$ Notice that $Z_t \in \mathcal{M}_t$ and $Z_t \perp \mathcal{M}_{t-1}$

- $E[Z_s Z_t] = 0$ if $s \neq t$ $s > t$.

$$\begin{aligned} Z_t &\in \mathcal{M}_t \subset \mathcal{M}_{s-1} \\ Z_s &:= X_s - \mathcal{P}_{\mathcal{M}_{s-1}}(X_s) \\ Z_s \perp \mathcal{M}_{s-1} &\implies Z_s \perp Z_t \\ X_t &= Z_t + \phi_1 Z_{t-1} + \dots + \phi_q Z_{t-q} \text{ for some } \phi_1, \dots, \phi_q \\ Z_t &= \sum_{j=0}^{\infty} \psi_j X_{t-j} \text{ for some } \psi_j \\ Z_{t-1}, \dots, Z_{t-q} &\in \overline{\text{sp}}\{X_{t-1}, X_{t-2}, \dots\} \end{aligned}$$

and $\mathcal{P}_{\overline{\text{sp}}\{X_{t-1}, X_{t-2}, \dots\}}(X_t) \rightarrow \mathcal{P}_{\mathcal{M}_t} X_t$ in L^2 as $k \rightarrow \infty$.

- $\|Z_t\| = \lim_{n \rightarrow \infty} \|X_t - \mathcal{P}_{\overline{\text{sp}}\{X_{t-1}, \dots, X_{t-n}\}}(X_t)\|$
 $= \lim_{n \rightarrow \infty} E[(X_{t-1} - \mathcal{P}_{\overline{\text{sp}}\{X_{t-2}, \dots, X_{t-n-1}\}}(X_{t-1}))^2] = \|Z_{t-1}\|$
Denote $\sigma^2 = \|Z_t\|^2$ then $\{Z_t\} \sim WN(0, \sigma^2)$
- $\mathcal{M}_{t-1} = \overline{\text{sp}}\{X_{t-1}, X_{t-2}, \dots\} = \overline{\text{sp}}\{Z_{t-1}, \dots, Z_{t-q}, \dots, X_{t-q-1}, X_{t-q-2}, \dots\}$
Also $\overline{\text{sp}}\{Z_{t-1}, \dots, Z_{t-q}\} \perp \overline{\text{sp}}\{X_{t-q-1}, X_{t-q-2}, \dots\}$
So $\mathcal{M}_{t-1} = \overline{\text{sp}}\{Z_{t-1}, \dots, Z_{t-q}\} \oplus \mathcal{M}_{t-q-1}$

$$\begin{aligned} \mathcal{P}_{\mathcal{M}_{t-1}}(X_t) &= \mathcal{P}_{\overline{\text{sp}}\{Z_{t-1}, \dots, Z_{t-q}\}}(X_t) + \mathcal{P}_{\mathcal{M}_{t-q-1}}(X_t) \\ &= \frac{\langle X_t, Z_{t-1} \rangle}{\langle Z_{t-1}, Z_{t-1} \rangle} Z_{t-1} + \dots + \frac{\langle X_t, Z_{t-q} \rangle}{\langle Z_{t-q}, Z_{t-q} \rangle} Z_{t-q} + \mathcal{P}_{\mathcal{M}_{t-q-1}}(X_t) \end{aligned}$$

Note that $\gamma_x(h) = 0$ for $|h| > q \implies \mathcal{P}_{\mathcal{M}_{t-q-1}}(X_t) = 0$

$$\implies X_t = Z_t + \sum_{j=1}^q \frac{\langle X_t, Z_{t-j} \rangle}{\sigma^2} Z_{t-j}$$

□

Proposition 7 (5.2.1 Durbin Levinson Algorithm). *Suppose $\{X_t\}$ is a zero-mean stationary process with autocovariance function $\gamma(\cdot)$. Such that $\gamma(0) > 0$ and $\gamma(h) \rightarrow 0$ as $h \rightarrow \infty$.*

*Then $\phi_{11} = \rho(1)$ and $\nu_1 = \gamma(0)(1 - \rho(1)^2)$
 $\phi_{nn} = [\gamma(n) - \sum_{j=1}^{n-1} \phi_{n-1,j} \gamma(n-j)] \nu_{n-1}^{-1}$ for $n \geq 2$*

$$\begin{bmatrix} \phi_{n1} \\ \phi_{n2} \\ \vdots \\ \phi_{n,n-1} \end{bmatrix} = \begin{bmatrix} \phi_{n-1,1} \\ \phi_{n-1,2} \\ \vdots \\ \phi_{n-1,n-1} \end{bmatrix} - \phi_{nn} \begin{bmatrix} \phi_{n-1,n-1} \\ \phi_{n-1,n-2} \\ \vdots \\ \phi_{n-1,1} \end{bmatrix} \text{ for } n \geq 2$$

$$\nu_n = \nu_{n-1}(1 - \phi_{nn}^2)$$

Given ϕ_{n-1} and ν_{n-1} How do we get ϕ_n and ν_n ?

The complexity is $O(n^2)$ instead of $O(n^3)$.

Proof. (1) $\phi_{11} = \rho(1)$ and $\nu_1 = \gamma(0)[1 - \rho(1)^2]$

Suppose we have $\underline{\phi}_{n-1}$ and ν_{n-1} , we want to find $\underline{\phi}_n$ and ν_n .

Want: $\mathcal{P}_{\overline{sp}\{X_1, \dots, X_n\}}(X_{n+1})$

Let $w_1 = X_1 - \mathcal{P}_{\overline{sp}\{X_2, \dots, X_n\}}(X_1)$ then $\begin{cases} w_1 \perp \overline{sp}\{X_n, \dots, X_{n-2}\} \\ \overline{sp}\{X_1, \dots, X_n\} = \overline{sp}\{w_1\} \oplus \overline{sp}\{X_2, \dots, X_n\} \end{cases}$

$$\mathcal{P}_{\overline{sp}\{X_n, \dots, X_2\}}(X_{n+1}) = \underline{\phi}_{n-1}' \begin{bmatrix} X_n \\ X_{n-1} \\ \vdots \\ X_2 \end{bmatrix} \text{ Then want } \frac{\langle X_{n+1}, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$\mathcal{P}_{\overline{sp}\{X_n, \dots, X_{n-2}\}}(X_1) = \underline{\phi}_{n-1}' \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

$$w_1 = X_1 - \underline{\phi}_{n-1}' \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \quad ||w_1||^2 = \nu_{n-1} \quad \langle X_{n+1}, w_1 \rangle = \gamma(n) - \underline{\phi}_{n-1}' \begin{bmatrix} \gamma(n-1) \\ \gamma(n-2) \\ \vdots \\ \gamma(1) \end{bmatrix}$$

$$\mathcal{P}_{\overline{sp}_{w_1}}(X_{n+1}) = \frac{\langle X_{n+1}, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = \phi_{nn} w_1 \underline{\phi}_n' \begin{bmatrix} X_n \\ X_{n-1} \\ \vdots \\ X_1 \end{bmatrix} - \phi_{nn} \begin{bmatrix} \phi_{n-1,n-1} \\ \phi_{n-1,n-2} \\ \vdots \\ \phi_{n-1,1} \end{bmatrix} \quad \nu_n = \nu_{n-1} - \phi_{nn}^2 \nu_{n-1} =$$

$$(1 - \phi_{nn}^2) \nu_{n-1}$$

□

PLEASE REVIEW THIS WHAT THE HELL IS THIS

1.10 10/7/2025 Lecture 10

Remark. Review of last class:

- Linear Predictor: Suppose $Y \in \mathbb{R}$, $E[Y] = 0$, $\mathbf{X} \in \mathbb{R}^d$, $E[\mathbf{X}] = \mathbf{0}$.

$$\text{Cov} \begin{pmatrix} Y \\ \mathbf{X} \end{pmatrix} = \begin{bmatrix} \sigma_Y^2 & \Sigma_{YX} \\ \Sigma_{XY} & \Sigma_X \end{bmatrix}$$

The BLP is given by $\hat{X} = \phi_n' X_n$ where $\phi_n = \begin{bmatrix} \phi_{n1} \\ \phi_{n2} \\ \vdots \\ \phi_{nn} \end{bmatrix}$ and $X_n = \begin{bmatrix} X_n \\ X_{n-1} \\ \vdots \\ X_1 \end{bmatrix}$

Suppose $\{X_t\}$ follows causal AR(p) process: $\phi(B)X_t = Z_t$ where $\{Z_t\} \sim WN(0, \sigma^2)$
 Predict X_t based on X_{t-1}, \dots, X_{t-p}