01:640:481 - Homework 4

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1. Question 10.53 Given a random sample of size n from a Poisson population, use the method of moments to obtain an estimator for the parameter λ.

Solution: We need to solve the following equation for λ:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \lambda$$

Thus, $\hat{\lambda} = \bar{X}$ is the method of moments estimator for λ.

2. Question 10.56 If $X_1, X_2, \ldots, X_n$ is a random sample from a population given by

$$g(x;\theta,\delta) = \begin{cases} 1/\theta e^{-(x-\delta)/\theta} & \text{if } x > \delta \\ 0 & \text{otherwise} \end{cases}$$

find estimators for δ and θ by the method of moments. This distribution is sometimes referred to as the two-parameter exponential distribution, and for $\theta = 1$ it is the distribution of Example 3.

Solution: We can solve the following equations for δ and θ:

$$\begin{aligned} m_1' = \mu_1' &= \bar{X} = \delta + \theta \\ m_2' = \mu_2' &= \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \delta^2 + 2\delta\theta + \theta^2 \\ \delta &= \bar{X} - \theta \\ \theta &= \sqrt{\mu_2' - \mu_1'^2} \\ \delta &= \mu_1' - \sqrt{\mu_2' - \mu_1'^2} \end{aligned}$$

Thus we have a method of moments estimator for δ and θ.

3. Question 10.59 Use the method of maximum likelihood to rework Exercise 53.

Solution: We want to max the likelihood function $L(\lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$ We can take the log of the likelihood function and solve for λ:

$$ln(L(\lambda)) = \sum_{i=1}^{n} x_i ln(\lambda) - n\lambda - \sum_{i=1}^{n} ln(x_i!)$$
$$\frac{\partial ln(L(\lambda))}{\partial \lambda} = \frac{1}{\lambda}\sum_{i=1}^{n} x_i - n = 0$$
$$\lambda = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{X}$$

Thus $\hat{\lambda} = \bar{X}$ is the maximum likelihood estimator for λ.

4. Question 10.66 Use the method of maximum likelihood to rework Exercise 56

Solution: We want to max the likelihood function $L(\delta, \theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-(x_i - \delta)/\theta}$ We can take the log of the likelihood function and solve for δ and θ:

$$ln(L(\delta, \theta)) = -\sum_{i=1}^{n} \frac{x_i - \delta}{\theta} - nln(\theta)$$
$$\frac{\partial ln(L(\delta, \theta))}{\partial \delta} = \frac{n}{\theta} = 0$$
$$\frac{\partial ln(L(\delta, \theta))}{\partial \theta} = \sum_{i=1}^{n} \frac{x_i - \delta}{\theta^2} - \frac{n}{\theta} = 0$$

We can solve the above equations to get the maximum likelihood estimators for δ and θ. We can see that $\hat{\delta} = min(X_i)$ and $\hat{\theta} = \bar{x} - min(X_i)$.

5. Question 10.3 Use the formula for the sampling distribution of $\tilde{X}$ on page 253 to show that for random samples of size $n = 3$ the median is an unbiased estimator of the parameter θ of a uniform population with $\alpha = \theta - \frac{1}{2}$ and $\beta = \theta + \frac{1}{2}$.

Solution: We can notice that the sample median for this population is $h(x) = \frac{(2n-1)!}{m!m!} \cdot \int_{-\infty}^{x} f(x)dx \cdot \int_{x}^{\infty} f(x)dx f(x)$.

$$h(x) = 6\left(x - \theta + \frac{1}{2}\right)\left(\theta + \frac{1}{2} - x\right)$$

$$E[x] = 6\int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} x\left(x - \theta + \frac{1}{2}\right)\left(\theta + \frac{1}{2} - x\right)$$

After a bunch of algebra, we can see that $E[x] = \theta$. Thus, the median is an unbiased estimator of the parameter θ of a uniform population with $\alpha = \theta - \frac{1}{2}$ and $\beta = \theta + \frac{1}{2}$.

6. Question 10.15 Show that the mean of a random sample of size n is a minimum variance unbiased estimator of the parameter λ of a Poisson population.

Solution: Consider the possion distribution $f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$. The mean of a random sample of size n is $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. We know that the mean of a Poisson distribution is λ. Thus, $\bar{X}$ is an unbiased estimator of λ. We also know that the variance of a Poisson distribution is λ. We can caluclate the CRLB of λ by solving the following equation:

$$var(\bar{X}) = \frac{1}{nE\left[\frac{\partial ln(f(X))}{\partial \lambda}^2\right]}$$

We can see that the CRLB is

$$\begin{aligned} ln(f(X)) &= -\lambda + xln(\lambda) - ln(x!) \\ \frac{\partial ln(f(X))}{\partial \lambda} &= \frac{x}{\lambda} - 1 \\ E\left[\frac{\partial ln(f(X))}{\partial \lambda}^2\right] &= E\left[\left(\frac{x}{\lambda} - 1\right)^2\right] = \frac{1}{\lambda} \\ var(\bar{X}) &= \frac{1}{n \cdot \frac{1}{\lambda}} = \frac{\lambda}{n} \end{aligned}$$

Since the variance of $\bar{X}$ is $\frac{\lambda}{n}$, we can see that the mean of a random sample of size n is a minimum variance unbiased estimator of the parameter λ of a Poisson population.

7. Question 10.18 Show that for the unbiased estimator of Example 4, $\frac{n+1}{n} \cdot Y_n$, the Cramer-Rao inequality is not satisfied.

Solution: We kno the sample distribution of Y_n is

$$\frac{n}{\beta^n} \cdot y_n^{n-1}$$

We know that the CRLB is given by

$$var(\hat{\theta}) = \frac{1}{nE\left[\frac{\partial ln(f(X))}{\partial \theta}^2\right]}$$

We can calculate the CRLB for the unbiased estimator $\frac{n+1}{n} \cdot Y_n$ by solving the following equation:

$$\begin{aligned} ln(f(X)) &= \ln(n) - n\ln(\beta) + (n-1)\ln(y_n) \\ \frac{\partial ln(f(X))}{\partial \beta} &= -\frac{n}{\beta} \\ E\left[\frac{\partial ln(f(X))}{\partial \beta}^2\right] &= E\left[\left(-\frac{n}{\beta}\right)^2\right] = \frac{n^2}{\beta^2} \\ var(\hat{\theta}) &= \frac{1}{n \cdot \frac{n^2}{\beta^2}} = \frac{\beta^2}{n^3} \end{aligned}$$

We can see that the CRLB is $\frac{\beta^2}{n^3}$. We can calculate the variance of the unbiased estimator $\frac{n+1}{n} \cdot Y_n$ by solving the following equation:

$$var\left(\frac{n+1}{n} \cdot Y_n\right) = \left(\frac{n+1}{n}\right)^2 \cdot var(Y_n) = \left(\frac{n+1}{n}\right)^2 \cdot \frac{\beta^2}{n^3} = \frac{\beta^2(n+1)^2}{n^4}$$

We can see that the variance of the unbiased estimator $\frac{n+1}{n} \cdot Y_n$ is $\frac{\beta^2(n+1)^2}{n^4}$ and that it is greater than the CRLB $\frac{\beta^2}{n^3}$. Thus, the Cramer-Rao inequality is not satisfied.