01:640:311H - Homework 9

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1. Define $h(x) = \begin{cases} x^3 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$

   (a) Show that h is differentiable everywhere.

   Solution: To show that h is differentiable everywhere, we need to show that the limit $\lim_{x\to c} \frac{h(x)-h(c)}{x-c}$ exists for all $c \in \mathbb{R}$. We can see that for $c \neq 0$, the function is differentiable since it is a composition of differentiable functions. For $c = 0$, we need to check the limit:

   $$h'(0) = \lim_{x\to 0} \frac{h(x) - h(0)}{x - 0} = \lim_{x\to 0} \frac{x^3 \sin\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

   Thus, h is differentiable at $x = 0$ and hence differentiable everywhere.

   (b) Show that h' is continuous everywhere.

   Solution: Consider $h' = \begin{cases} 3x^2 \sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ To show that h' is continuous everywhere, we need to show that $\lim_{x\to c} h'(x) = h'(c)$ for all $c \in \mathbb{R}$. For $c \neq 0$, h' is continuous since it is a composition of continuous functions. For $c = 0$, we can let $\epsilon > 0$ be given.

   $$\begin{aligned} |h'(x) - h'(0)| &= \left|3x^2 \sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right)\right| \\ &\leq |3x^2| + |x| \\ &\leq 4|x| \end{aligned}$$

   If we take $\delta = \min\left(1, \sqrt{\frac{\epsilon}{4}}\right)$, then for $|x| < \delta$, we have

   $$\begin{aligned} |h'(x) - h'(0)| &\leq 3|x|^2 + |x| \\ &\leq 4|x| < 4\delta \\ &< \epsilon. \end{aligned}$$

Therefore, h' is continuous at $x = 0$ and hence continuous everywhere.

(c) Show that h' is not differentiable at $x = 0$.

Solution: To show that h' is not differentiable at $x = 0$, we need to check the limit

$$h''(0) = \lim_{x\to 0} \frac{h'(x) - h'(0)}{x - 0} = \lim_{x\to 0} \frac{3x^2 \sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right)}{x}.$$

This simplifies to

$$\begin{aligned} h''(0) &= \lim_{x\to 0} \left(3x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right) \\ &= \lim_{x\to 0} 3x \sin\left(\frac{1}{x}\right) - \lim_{x\to 0} \cos\left(\frac{1}{x}\right). \end{aligned}$$

This limit does not exist. Since we can take two sequences approaching 0, $x_n = \frac{1}{n\pi}$ and $y_n = \frac{1}{n\pi + \frac{\pi}{2}}$, we find that they concerge to different values. Thus, h' is not differentiable at $x = 0$.

2. Suppose $f : (a, b) \to \mathbb{R}$ is continuous on (a, b) and differentiable at $c \in (a, b)$. Prove that the function $g : (a, b) \to \mathbb{R}$ defined as

$$g(x) = \begin{cases} \frac{f(x)-f(c)}{x-c} & x \neq c \\ f'(c) & x = c \end{cases}$$

is continuous.

Solution: To show that g is continuous at c, we need to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $|x - c| < \delta$, then $|g(x) - g(c)| < \epsilon$. We know that since f is differentiable at c, we have that there exists a $\delta > 0$ such that

$$|x - c| < \delta \implies \left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| < \epsilon$$

We also know that

$$\begin{aligned} |g(x) - g(c)| &= \left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| \\ &< \epsilon \quad \text{for } |x - c| < \delta. \end{aligned}$$

Thus, we have shown that g is continuous at c. Since c was arbitrary, g is continuous on (a, b).

3. Prove that $f : I \to \mathbb{R}$ is differentiable at $c \in I$ with derivative $f'(c)$ if and only if we can write

$$f(x) = f(c) + f'(c)(x - c) + R_c(x)(x - c)$$

where $R_c(x)$ is continuous at $x = c$ and $R_c(c) = 0$.

Solution: Forward Direction: Assume f is differentiable at c with derivative $f'(c)$.
Then we can construct $R_c(x)$ as follows:

$$R_c(x) = \begin{cases} \frac{f(x)-f(c)-f'(c)(x-c)}{x-c} & x \neq c \\ 0 & x = c \end{cases}$$

Clealry $R_c(c) = 0$ and the equation holds for all $x \in I$. Now, we need to show that $R_c(x)$ is continuous at c. We have

$$\begin{aligned} \lim_{x\to c} R_c(x) &= \lim_{x\to c} \frac{f(x) - f(c) - f'(c)(x - c)}{x - c} \\ &= \lim_{x\to c} \frac{f(x) - f(c)}{x - c} - f'(c) \\ &= f'(c) - f'(c) = 0. \end{aligned}$$

Thus, $R_c(x)$ is continuous at c and the forward direction is proved.
Backward Direction: Assume that we can write

$$f(x) = f(c) + f'(c)(x - c) + R_c(x)(x - c)$$

where $R_c(x)$ is continuous at c and $R_c(c) = 0$.
Then we can rearrange this to get

$$\frac{f(x) - f(c)}{x - c} - f'(c) = R_c(x).$$

Since $R_c(x)$ is continuous at c and $R_c(c) = 0$, we have that for arbitrary $\epsilon > 0$ there exists a $\delta > 0$ such that for all $|x - c| < \delta$, we have

$$|R_c(x)| < \epsilon.$$

Thus we can write

$$\lim_{x\to c} \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| = \lim_{x\to c} |R_c(x)| < \epsilon.$$

And thus we have shown that f is differentiable at c with derivative $f'(c)$.

4. Given a differentiable function $f : A \to \mathbb{R}$, we say that f is uniformly differentiable on A when for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $|x - y| < \delta$, then

$$\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \epsilon.$$

(a) Is $f(x) = x^2$ uniformly differentiable on $\mathbb{R}$?

Solution: Let $\epsilon > 0$ then take $\delta = \epsilon$. Then for any $|x - y| < \delta$, we have

$$\begin{aligned} \left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| &= \left| \frac{x^2 - y^2}{x - y} - 2y \right| \\ &= |(x + y) - 2y| \\ &= |x - y| < \delta = \epsilon. \end{aligned}$$

Thus, $f(x) = x^2$ is uniformly differentiable on $\mathbb{R}$.

(b) Is $g(x) = x^3$ uniformly differentiable on $\mathbb{R}$?

Solution: Note that

$$\begin{aligned} \left| \frac{g(x) - g(y)}{x - y} - g'(y) \right| &= \left| \frac{x^3 - y^3}{x - y} - 3y^2 \right| \\ &= |x^2 + xy - 2y^2| \\ &= |(x - y)(x + y) + y(x - y)| \\ &= |x - y||x + 2y| \end{aligned}$$

Since there is a $|x + 2y|$ term, there are issues with the uniformity of the derivative.
Specifically taking $x_n = n$ and $y_n = n + \frac{1}{n}$, we have

$$\begin{aligned} \lim |x_n - y_n| &= 0 \\ \frac{f(x_n) - f(y_n)}{x_n - y_n} &= 3n^2 + 3 + \frac{1}{n^2} \end{aligned}$$

which converges to 3 as $n \to \infty$. Thus the limit does not converge to $g'(y)$ uniformly. Therefore, $g(x) = x^3$ is not uniformly differentiable on $\mathbb{R}$.

(c) Show that if f is uniformly differentiable on an interval I, then f' must be continuous on I.

Solution: Suppose f is uniformly differentiable on an interval I. By definition, for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $|x-y| < \delta$, then

$$\left| \frac{f(x)-f(y)}{x-y} - f'(y) \right| < \epsilon/2$$

and

$$\left| \frac{f(x)-f(y)}{x-y} - f'(x) \right| < \epsilon/2.$$

This implies that

$$\begin{aligned} |f'(x)-f'(y)| &= \left| \frac{f(x)-f(y)}{x-y} - f'(y) + f'(y) - f'(x) \right| \\ &\leq \left| \frac{f(x)-f(y)}{x-y} - f'(y) \right| + |f'(y) - f'(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

(d) If f is differentiable on a closed, bounded interval $[a,b]$, is f necessarily uniformly differentiable there? Give a proof or a counterexample to support your answer.

Solution: No, We can use the textbooks example of $g(x) = \begin{cases} x^2 sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ on the interval of $[-1,1]$ We know that $g(x)$ is differentiable on $[-1,1]$ but not uniformly differentiable. To see this, we can take the sequences $x_n = \frac{1}{n\pi}$ and $y_n = \frac{1}{n\pi + \frac{\pi}{2}}$ each converging to 0 and thus their difference converges to 0 but the limit of the difference quotient does not converge.

$$\lim \frac{g(x_n) - g(y_n)}{x_n - y_n} \to \infty \text{ as } n \to \infty.$$

and $g'(y_n) = 2y_n \sin(\frac{1}{y_n}) - \cos(\frac{1}{y_n}) \to 0$ Thus the limit does not converge uniformly to $g'(y)$ and thus g is not uniformly differentiable on $[-1,1]$.

5. If f is twice differentiable on an open interval containing c and f'' is continuous at c, prove that

$$f''(c) = \lim_{h \to 0} \frac{f(c+h) + f(c-h) - 2f(c)}{h^2}.$$

Solution: Consider the follow limits and applying L'Hospital's rule twice where needed

$$\begin{aligned}\lim_{h\to 0}\frac{f(c+h)+f(c-h)-2f(c)}{h^2} &= \lim_{h\to 0}\frac{f'(c+h)-f'(c-h)}{2h}\\ &= \lim_{h\to 0}\frac{f''(c+h)+f''(c-h)}{2}\\ &= \frac{f''(c)+f''(c)}{2}\\ &= f''(c).\end{aligned}$$

6. Suppose that $g : A \to \mathbb{R}$ and a is a limit point of A. Also assume that $g(x) \neq 0$ for any $x \in A$. Show that if $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} \frac{1}{g(x)} = 0$.

Solution: Let $\epsilon > 0$. We know that by the archimedian property of the real numbers there exists and $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$.
We also know that since $\lim_{x\to a} g(x) = \infty$, for all $M \in \mathbb{N}$ there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $g(x) > M$.
Thus, we can take $M = N$ then when $0 < |x - a| < \delta$ we have $0 < \frac{1}{g(x)} < \frac{1}{N} < \epsilon$.
Thus we have shown that $\lim_{x\to a} \frac{1}{g(x)} = 0$.