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1. Section 6.4 Problem 1
Solve $u_{xx} + u_{yy} = 0$ in the exterior $\{r > a\}$ of a disk, with the boundary condition $u = 1 + 3\sin\theta$ on $r = a$, and the condition at infinity that u be bounded as $r \to \infty$.

Solution: We need to solve $\Delta u = 0$ but we can rewrite this in polar coordinates as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

From the boundary consition we can see that the solution will be of the form $u = 1 + f(r)\sin\theta$. We can plug this into the Laplace equation to get

$$\begin{aligned} f''(r)\sin(\theta) + \frac{1}{r}f'(r)\sin(\theta) - \frac{1}{r^2}f(r)\sin(\theta) &= 0 \\ r^2 f''(r) + rf'(r) - f(r) &= 0 \end{aligned}$$

We can solve this ODE by guessing that the solution is of the form $f(r) = r^m$. Plugging this into the ODE we get

$$\begin{aligned} r^2 m(m-1)r^{m-2} + rmr^{m-1} - r^m &= 0 \\ m(m-1) + m - 1 &= 0 \\ m^2 - 1 &= 0 \\ m &= \pm 1 \end{aligned}$$

Thus $f(r) = C_1 r + C_2 r^{-1}$.
We can determine the constants C_1 and C_2 by plugging in the boundary condition. We get

$$\begin{aligned} f(a) = 3 &\implies C_1 a + C_2 a^{-1} = 3 \\ f(\infty) = \text{bounded} &\implies C_1 = 0 \end{aligned}$$

Thus $f(r) = \frac{3a}{r}$. Then the solution is $u = 1 + \frac{3a}{r}\sin\theta$. Now convert this back to Cartesian coordinates to get

$$\begin{aligned} u(x,y) &= 1 + \frac{3a}{\sqrt{x^2+y^2}}\frac{y}{\sqrt{x^2+y^2}} \\ &= 1 + \frac{3ay}{x^2+y^2} \end{aligned}$$

2. Section 6.4 Problem 2
Solve $u_{xx} + u_{yy} = 0$ in the disk $r < a$ with the boundary condition $\frac{\partial u}{\partial r} - hu = f(\theta)$, where

$f(\theta)$ is an arbitrary function. Write the answer in terms of the Fourier coefficients of $f(\theta)$.

Solution: We can convert the Laplace equation to polar coordinates to get

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

With BC $\frac{\partial u}{\partial r} - hu = f(\theta)$. and $u(0,\theta) =$ bounded. and $u(a,0) = u(a,2\pi), u_\theta(a,0) = u_r(0,2\pi)$.
We can guess that the solution is of the form $u = R(r)\Theta(\theta)$.
The BC (with the exception of the first one) can be written as

$$\Theta(0) = \Theta(2\pi)$$
$$\Theta'(0) = \Theta'(2\pi)$$
$$R(0) = \text{bounded}$$

By plugging back into the Laplace equation we get

$$r^2\frac{R''}{R} = r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \alpha$$

We can first solve the Θ equation for α:
We can clearly see that for $\alpha = -\lambda^2$ we cannot solve the BC.
For $\alpha = 0$ we get $\Theta = C_1$ and $R = C_2 ln(r) + C_3$.
The only way the BC of $R(0) =$ bounded can be satisfied is if $C_2 = 0$. Thus the eigenvalue of $\alpha = 0$ has eigenfunctions of constants.
For $\alpha = \lambda^2$ we get eigenfunctions $\Theta = C_1\cos(\lambda\theta) + C_2\sin(\lambda\theta)$ with eigenvalues of $\lambda = n$. For R we get $R = C_3 r^n + C_4 r^{-n}$.
For the BC to be satisfied we need $C_4 = 0$. Thus the eigenfunctions for $\alpha = \lambda^2$ are $\Theta = C_1\cos(n\theta) + C_2\sin(n\theta)$ and $R = C_3 r^n$. with $\lambda = n$ as eigenvalues

Due to superposition we can write the solution as

$$u(r,\theta) = A_0 + \sum_{n=0}^{\infty}(r^n)(A_n\cos(n\theta) + B_n\sin(n\theta))$$

We can determine the coefficients by applying our last BC. We get

$$\frac{\partial u}{\partial r} - hu = f(\theta)$$

$$u_r = \sum_{n=0}^{\infty} nr^{n-1}(A_n\cos(n\theta) + B_n\sin(n\theta))$$

$$u_r(a,\theta) - hu(a,\theta) = f(\theta)$$

$$\sum_{n=0}^{\infty} na^{n-1}(A_n \cos(n\theta) + B_n \sin(n\theta)) - h[A_0 + \sum_{n=0}^{\infty} a^n(A_n \cos(n\theta) + B_n \sin(n\theta))] = f(\theta)$$

$$-hA_0 + \sum_{n=0}^{\infty}(na^{-1} - h)a^n(A_n \cos(n\theta) + B_n \sin(n\theta)) = f(\theta)$$

$$-hA_0 + \sum_{n=0}^{\infty} \left[(na^{-1} - h)a^n A_n \cos n\theta + (na^{-1} - h)a^n B_n \sin n\right] = f(\theta)$$

$$\int_0^{2\pi} -hA_0 + \sum_{n=0}^{\infty} \left[(na^{-1} - h)a^n A_n \cos n\theta + (na^{-1} - h)a^n B_n \sin n\right] d\theta = \int_0^{2\pi} f(\theta) d\theta$$

$$-hA_0 2\pi = \int_0^{2\pi} f(\theta) d\theta$$

$$A_0 = \frac{-1}{2\pi h} \int_0^{2\pi} f(\theta) d\theta$$

Through a similar process where we first multiply through by $\cos(n\theta)$ and $\sin(n\theta)$ and then integrate we get

$$A_n = \frac{a^{1-n}}{\pi(n - ah)} \int_0^{2\pi} f(\theta)\cos(n\theta) d\theta$$

$$B_n = \frac{a^{1-n}}{\pi(n - ah)} \int_0^{2\pi} f(\theta)\sin(n\theta) d\theta$$

3. Section 6.4 Problem 10
Solve $u_{xx} + u_{yy} = 0$ in the quarter-disk $\{x^2 + y^2 < a^2, x > 0, y > 0\}$ with the following boundary conditions: $u = 0$ on $x = 0$ and on $y = 0$ and $\frac{\partial u}{\partial r} = 1$ on $r = a$. Write the answer as an infinite series and write the first two nonzero terms explicitly.

Solution: Since the domain is on a quarter disk we can convert the Laplace equation to polar coordinates to get

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

With BC $u(0,\theta) =$ bounded, $u(r,0) = 0$, $u_r(a,\theta) = 1$, and $u(r, \frac{\pi}{2}) = 0$.
We can use separation of variables to get $u = R(r)\Theta(\theta)$. Plugging this back into the Laplace equation we get

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \alpha$$

The BC can be written as

$$\Theta(0) = 0$$
$$\Theta(\frac{\pi}{2}) = 0$$
$$R(0) = \text{bounded}$$

We can solve the Θ equation.
Clearly $\alpha = -\lambda^2$ will not satisfy the BC.
For $\alpha = 0$ we get a trivial solution.
For $\alpha = \lambda^2$ we get $\Theta = C_1 \cos(\lambda\theta) + C_2 \sin(\lambda\theta)$.
By applying the BC we get $\Theta = C_2 \sin(\lambda\theta)$ with eigenvalues $\lambda = 2n$.
Thus we get $R = C_3 r^n + C_4 r^{-n}$.
For the BC to be satisfied we need $C_4 = 0$. Thus the eigenfunctions for $\alpha = \lambda^2$ are $\Theta = C_2 \sin(\lambda\theta)$ and $R = C_3 r^\lambda$. with $\lambda = 2n$ as eigenvalues.
Due to superposition we can write the solution as

$$u(r, \theta) = \sum_{n=0}^{\infty} r^{2n}(B_n \sin(2n\theta))$$

We can now apply our BC of $u_r(a, \theta) = 1$ to get

$$u_r = \sum_{n=0}^{\infty} 2na^{2n-1} B_n \sin(2n\theta)$$
$$u_r(a, \theta) = \sum_{n=0}^{\infty} 2na^{2n-1} B_n \sin(2n\theta) = 1$$

By multiplying through by $\sin(2m\theta)$ and integrating we get

$$2na^{2n-1} B_n \int_0^{\frac{\pi}{2}} \sin(2n\theta)\sin(2n\theta)d\theta = \int_0^{\frac{\pi}{2}} \sin(2n\theta)d\theta$$

Thus $B_n = \frac{1}{n\pi a^{2n-1}} \cdot \frac{1-(-1)^n}{n}$.

Since we can see that $B_n = 0$ for even n, The first two nonzero terms are

$$u(r, \theta) = \frac{2}{\pi a} r^2 sin(2\theta) + \frac{2}{9\pi a^5} r^6 \sin(6\theta) + \dots$$

4. Section 5.2 Problem 9 Let $\phi(x)$ be a function of period π. If $\phi(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$ for all x, find the odd coefficients.

Solution: $\phi(x)$ being a function of period π means that $\phi(x) = \phi(x+\pi)$.
We have $\phi(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$.
So

$$\begin{aligned}\phi(x+\pi) &= \sum_{n=1}^{\infty} a_n \sin(n(x+\pi)) \\ &= \sum_{n=1}^{\infty} a_n \sin(nx + n\pi) \\ &= \sum_{n=1}^{\infty} a_n \sin(nx)\cos(n\pi) + a_n \cos(nx)\sin(n\pi)\end{aligned}$$

Since $n \in \mathbb{N}$ we know that $\cos(n\pi) = (-1)^n$ and $\sin(n\pi) = 0$. Thus

$$\phi(x+\pi) = \sum_{n=1}^{\infty} a_n (-1)^n \sin(nx)$$

Thus

$$\phi(x) = \phi(x+\pi) \implies \sum_{n=1}^{\infty} a_n \sin(nx) = \sum_{n=1}^{\infty} a_n (-1)^n \sin(nx)$$

Thus $a_n = (-1)^n a_n$ which implies that $a_n = 0$ for odd n. Thus the odd coefficients are $a_n = 0$ for odd n.

5. Section 5.2 Problem 17
Show that a complex-valued function $f(x)$ is real-valued if and only if its complex Fourier coefficients satisfy $c_n = \overline{c_{-n}}$, where $\overline{c_{-n}}$ denotes the complex conjugate.

Solution: If $f(x)$ is real valued and the interval is $-\pi, \pi$ for simplicity (otherwise we can tranform the function to fit this property), The complex Fourier series of a function $f(x)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

If $f(x)$ is real valued then $f(x) = \overline{f(x)}$. Thus

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$
$$\overline{f(x)} = \sum_{n=-\infty}^{\infty} \overline{c_n} e^{-inx}$$

Now for the series we can substitute $-k$ for n to get

$$\overline{f(x)} = \sum_{k=\infty}^{-\infty} \overline{c_{-k}} e^{ikx}$$

Since $f(x) = \overline{f(x)}$ we can see that $c_n = \overline{c_{-n}}$.

Now if $c_n = \overline{c_{-n}}$ then we can see that

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \sum_{n=-\infty}^{\infty} \overline{c_{-n}} e^{-inx}$$

substituting $-k$ for n we get

$$f(x) = \sum_{k=\infty}^{-\infty} \overline{c_k} e^{ikx}$$
$$= \overline{\sum_{k=\infty}^{-\infty} c_k e^{ikx}}$$

Clearly $\overline{\sum_{k=\infty}^{-\infty} c_k e^{ikx}} = \overline{f(x)}$.
Thus $f(x) = \overline{f(x)}$ and $f(x)$ is real valued.

6. Section 5.3 Problem 2
(a) On the interval $[-1, 1]$, show that the function x is orthogonal to the constant functions.
(b) Find a quadratic polynomial that is orthogonal to both 1 and x.
(c) Find a cubic polynomial that is orthogonal to all quadratics. (These are the first few Legendre polynomials.)

Solution: Note for sake of ease we will use the inner product defined as $\int_{-1}^{1} f(x)g(x)dx$.
(a)

We can show that the function x is orthogonal to the constant functions by taking the inner product of the two functions.

$$\int_{-1}^{1} x \cdot C dx = C \int_{-1}^{1} x dx = C \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$$

Thus x is orthogonal to the constant functions.

(b)

We need a, b, c such that $ax^2 + bx + c$ is orthogonal to both 1 and x.
Thus $< 1, ax^2 + bx + c >= 0$ and $< x, ax^2 + bx + c >= 0$.
We can see that if under the integreal if there is a term with an odd power it will be zero.
Clealry $< 1, ax^2 + bx + c >= \frac{2}{3}a + 2a = 0$
And $< x, ax^2 + bx + c >= \frac{2}{3}b = 0$
Thus $c = \frac{-1}{3}a$ and $b = 0$.
Thus the quadratic polynomial that is orthogonal to both 1 and x is $f(x) = ax^2 - \frac{1}{3}a$ or $f(x) = 3x^2 - 1$.

(c)

We need a, b, c, d such that $ax^3 + bx^2 + cx + d$ is orthogonal to $\alpha x^2 + \beta x + \gamma$.
Thus $< \alpha x^2 + \beta x + \gamma, ax^3 + bx^2 + cx + d >= 0$.

$$\int_{-1}^{1} (\alpha x^2 + \beta x + \gamma)(ax^3 + bx^2 + cx + d)dx = 0$$

$$\int_{-1}^{1} \alpha x^2(ax^3 + bx^2 + cx + d) + \beta x(ax^3 + bx^2 + cx + d) + \gamma(ax^3 + bx^2 + cx + d)dx = 0$$

$$\int_{-1}^{1} (\alpha ax^5 + \alpha bx^4 + \alpha cx^3 + \alpha dx^2 + \beta ax^4 + \beta bx^3 + \beta cx^2 + \beta dx + \gamma ax^3 + \gamma bx^2 + \gamma cx + \gamma d)dx = 0$$

for the terms with an odd power of x we can see that they will be zero.
Thus we can simplify to

$$\int_{-1}^{1} (\alpha bx^4 + \alpha dx^2 + \beta ax^4 + \beta cx^2 + \gamma bx^2 + \gamma d)dx = 0$$

$$\alpha b\frac{2}{5} + \alpha d\frac{2}{3} + \beta a\frac{2}{5} + \beta c\frac{2}{3} + \gamma b\frac{2}{3} + 2\gamma d = 0$$

Now we can group the terms in terms of $\alpha\beta\gamma$ to get

$$(2b/5 + 2d/3)\alpha + (2a/5 + 2c/3)\beta + (2b/3 + 2d)\gamma = 0$$

Since α, β, γ are arbitrary we can see that the coefficients of each term must be zero. Thus through some mantipulation we can see that $b = 0, d = 0$ ad $c = -\frac{3}{5}a$ Thus our cubic polynomial is $f(x) = ax^3 - \frac{3}{5}ax$ or $f(x) = 5x^3 - 3x$.

7. Section 5.3 Problem 6
   Find the complex eigenvalues of the first-derivative operator $\frac{d}{dx}$ subject to the single boundary condition $X(0) = X(1)$. Are the eigenfunctions orthogonal on the interval $(0, 1)$?

Solution: This is an eigenvalue problem where we need to solve

$$\frac{d}{dx}X(x) = \lambda X(x)$$

This cn be solve with seperation of variables and we can see that

$$X(x) = Ce^{x\lambda}$$

$$X(x) = C(\cos(i\lambda) - i\sin(i\lambda))$$

Our BC of $X(0) = X(1)$ implies

$$A = A(\cos(i\lambda) - i\sin(i\lambda))$$

Thus we need to solve $cos(i\lambda) - i\sin(i\lambda) = 1$
We can match the real and imaginary parts to get

$$\cos(i\lambda) = 1$$
$$\sin(i\lambda) = 0$$

Thus $\lambda = 2\pi ni.$ for $n \in \mathbb{Z}$
These have corresponing eigenfucntions of $X(x) = e^{2\pi nix}$
We can check for orthogonality by taking the inner product of two eigenfunctions.

$$\begin{aligned}
\int_0^1 X_n\overline{X_m}dx &= \int_0^1 e^{2\pi nix}e^{-2\pi mix}dx \\
&= \int_0^1 e^{2\pi(n-m)ix}dx \\
&= \frac{1}{2\pi(n-m)i}[e^{2\pi(n-m)ix}]_0^1 \\
&= \frac{1}{2\pi(n-m)i}[e^{2\pi(n-m)i} - 1] \\
&= \frac{1}{2\pi(n-m)i}[1 - 1] \\
&= 0
\end{aligned}$$

Therefore the eigenfunctions are orthogonal on the interval $(0, 1)$.