

PDEs

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Introduction

What is a PDE?

Start with ODE: $u = u(x)$, equation involving independent variable x and dependent variable u as well as its derivatives.

Example: $u'' - xu = 0, x \in I$ (Airy Functions). Second order Linear ODE. $Lu = u'' - xu$ Where L is an operator.

Linearity means 2 things: $L(u_1 + u_2) = Lu_1 + Lu_2$ and $L(cu) = cLu$

$\forall u_1, u_2 \in \mathcal{F}, \forall c \in \mathbb{F}$

PDE: $u = u(x, y, \dots)$ equation involving independent variables x, y, \dots and Function u as well as its partial derivatives $u_x, u_y, u_{xx}, u_{yy}, u_{xy}$

Example: $x^2u - \sin(xy)u_{xy} + 3u_x = 0$ 4th order linear PDE of 2 vars.

Remark: Importance of linearity: say u_1, u_2 are solutions of a Linear PDE:

$Lu_1 = 0, Lu_2 = 0$ then $c_1u_1 + c_2u_2, (\forall c_1, c_2 \in \mathbb{R})$ is also a solution of $Lu = 0$

More generally, if u_1, \dots, u_n are solutions, then $\sum_{j=1}^n c_j u_j$ is also a solution.

Example: $u = u(x, y)$, solve $u_{xx} = 0$. $u_x = f(y), u = f(y)x + g(y)$ where f, g are arbitrary functions. ($\forall f, g \in \mathcal{F}$)

$Lu = 0$ is homogenous, $Lu = f$ is non-homogenous.

1.2 First Order PDE of x,y

$$x, y, u_x, u_y, u$$

Generally: $a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0$

Example 1: $u_x = 0$: $u = f(y)$ No change in the x direction, hence the function stays constant on all horizontal lines.

Example 2: Geometric Method $au_x + bu_y = 0, (a, b \in \mathbb{R})$

$\vec{v} = (a, b)$; $\nabla u = \langle u_x, u_y \rangle$; $\nabla u \cdot \vec{v} = 0 \rightarrow D_{\vec{v}}u = 0$

No change in the " \vec{v} " direction (say $|\vec{v}| = 1$)

$x = ta, y = tb \rightarrow ay - bx = 0$ On the lines $ay - bx = c$ where c is a constant, the function u is constant. Let's call its value $f(c)$

$u(x, y) = f(ay - bx)$ where f is a function of a single variable

The lines where these are solutions/constant are called **characteristic lines**

Check: $u_x = -bf'(ay - bx)$, $u_y = af'(ay - bx)$

Change of variable Change our plane such that \vec{v} is our "x" axis.

View $(x, y) = x + iy$, $(x', y') = x' + iy'$. Multiplying by $e^{i\alpha}$ rotates the plane ccw by α

$x' + iy' = (x + iy)e^{i\alpha}$ where $\vec{v} = (a, b) = (\cos(\alpha), \sin(\alpha))$

$x' = x\cos(\alpha) + y\sin(\alpha)$, $y' = -x\sin(\alpha) + y\cos(\alpha)$

Rewrite PDE in our new system: $u = u(x', y') = u(x'(x, y), y'(x, y))$

$$u_x = u_{x'} \frac{\partial x'}{\partial x} + u_{y'} \frac{\partial y'}{\partial x}$$

$$u_x = au_{x'} - bu_{y'}$$

$$u_y = u_{x'} \frac{\partial x'}{\partial y} + u_{y'} \frac{\partial y'}{\partial y}$$

$$u_y = au_{y'} + bu_{x'}$$

$$au_x + bu_y = 0$$

$$a^2 u_{x'} + b^2 u_{y'} = 0$$

$$u_{x'} = 0$$

$$u = f(y') = f(ay - bx)$$

Example 3: $u_x + yu_y = 0$

u doesn't change in the direction of $\vec{v} = (1, y)$ at the point (x, y)

Lets call C the characteristic curve: $\begin{cases} x = x(t), y = y(t) \end{cases}$ tangent to \vec{v} at any (x, y)

$$\frac{d}{dt} u(x(t), y(t)) = 0$$

$$\frac{dy}{dx} = y \rightarrow y(x) = ce^x, (\forall c \in \mathbb{R})$$

$$u(x, ce^x) = f(c)$$

$$u(x, y) = f(ye^{-x})$$

Remark: More generally $a(x, y)u_x + b(x, y)u_y = 0$

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}: \text{ODE for characteristic curves}$$

1.3.1 Mass flow/Transport Equation/Continuity Equation

Substance that flows in space. (eg. fluid)

$\rho = \rho(x, y, z, t)$ density of the substance at point (x, y, z) at time t

$\vec{v} = \vec{v}(x, y, z, t)$ velocity of the substance at point (x, y, z) at time t

Consider R as an arbitrary region in space.

Conservation of mass: $m(t) = \int_R \rho(x, y, z, t) dV$ mass in R at time t

Consider $[t, t + \Delta t]$, $m(t + \Delta t) = \int_R \rho(x, y, z, t + \Delta t) dV$

Substance leaves/enters in R through the boundary ∂R

Consider a small part of the boundary call it ∂S and see how much mass has left through this boundary patch over time period $[t, t + \Delta t]$

We want to introduce the "normal" \vec{n} over the boundary ∂S

$$\text{height} = \vec{v} \Delta t \cdot \vec{n} \text{ and area of base} = dS \rightarrow \text{volume} = \Delta t \vec{v} \cdot \vec{n} dS$$

$$\rho = \text{mass/vol} \rightarrow \Delta t \rho \vec{v} \cdot \vec{n} dS$$

$$\Delta m = \Delta t \int_{\partial R} \rho \vec{v} \cdot \vec{n} dS$$

Mass Conservation: $m(t + \Delta t) = m(t) - \Delta m$

$$\begin{aligned} \frac{1}{\Delta t} \int_R \rho(x, y, z, t + \Delta t) - \rho(x, y, z, t) dV &= \int_{\partial R} \rho \vec{v} \cdot \vec{n} dS \\ &= \int_R \text{div}(\rho \vec{v}) dV \end{aligned}$$

Where $\text{div}(\vec{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

Let $\Delta t \rightarrow 0$

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{v}) = 0$$

This is the Transport Equation.

Example: $\vec{v} = c(1, 0)$ and $\rho = \rho(x, t)$

$$\frac{\partial \rho}{\partial t} + c \frac{\partial \rho}{\partial x} = 0$$

$$\rho(t, x) = f(x - ct)$$

$\rho_t + c\rho_x = 0, t > 0, x \in \mathbb{R}, \rho(0, x) = \rho_0(x), x \in \mathbb{R}$ Initial condition

$$\rho(t, x) = \rho_0(x - ct)$$

1.3.2 Heat Equation/Diffusion/Energy Flux

Flow of energy: $\vec{q}(x, y, z, t)$ energy flux at point (x, y, z) at time t

During the time interval $[t, t + \Delta t]$ the energy $\Delta E = \Delta t \int_{\partial R} \vec{q} \cdot \vec{n} dS$ has left the test volume R through the boundary ∂R

Consider the patch ∂S of the boundary ∂R and the normal \vec{n}

$$\Delta t \vec{q} \cdot \vec{n} dS \rightarrow e(\vec{n}) \Delta t dS$$

Cauchy tensor deformation.

To measure the energy inside R we need the specific heat $c(x, y, z)$ and it measures the energy containing in 1 degree of temperature in 1 unit mass.

$$c = \frac{e}{T \cdot \text{mass}} = \frac{e}{T \cdot \rho \text{vol}}$$

$$Tc\rho = \frac{e}{\text{vol}}$$

$$E(t) = \int_R T(\vec{x}, t) \rho(\vec{x}) c(\vec{x}) dV$$

This is the energy inside R at time t .

$$E(t + \Delta t) = E(t) - \Delta E$$

$$\int_R T_t(\vec{x}, t) \rho(\vec{x}) c(\vec{x}) + \text{div} \vec{q} dV = 0$$

$$T_t c \rho + \text{div} \vec{q} = 0$$

Incomplete: we need to know how \vec{q} depends on T

Fourier's law of heat conduction: $\vec{q} = -k \nabla T$

$k(\vec{x})$ = heat conductivity of the material

Heat flows from hot to cold.

$$\text{grad}(T)$$

is the direction of the greatest increase of T

$$c\rho T_t - \text{div}(k \nabla T) = 0$$

Specific case: Assume c, ρ, k are constants.

$$\nabla T = \begin{pmatrix} T_x \\ T_y \\ T_z \end{pmatrix} \cdot (d_x, d_y, d_z)$$

$$\text{div} \nabla T = \nabla \cdot \nabla T = \nabla^2 T = T_{xx} + T_{yy} + T_{zz}$$

This is the laplacian of T

$$T_t = \mathbf{D} \nabla^2 T, \mathbf{D} = \frac{k}{c\rho}$$

Fick's law of Diffusion.
High density to low density.

Wave Equation

Consider a string
We have x and $u(x, t)$
Consider a small part of the string $[x, x + \Delta x]$ called $d\ell$
There is a tangent force $T(x + \Delta x, t)$
The mass of the string is $m(x)$ from origin to x
Newton's law: $F = ma$

$$T(x + \Delta x, t) - T(x, t) = (m(x + \Delta x) - m(x))\vec{r}_t(x, t)$$

Divide by Δx and let $\Delta x \rightarrow 0$

$$\vec{T}_x(x, t) = \rho(x)\vec{r}_{tt}(x, t)$$

Where ρ is the linear density of the string
 T is tangent to the string: T is parallel to \vec{r}_x
introduce $\vec{\tau} = \frac{\vec{r}_x}{|\vec{r}_x|}$

$$T = T(x, t)\vec{\tau}$$

Where T is constant along the string.
Assume small vibration so that $|u_x|$ is small.

$$\begin{aligned}\vec{r} &= (1, u_x) = (1, 0) \\ u_t + \nabla \cdot (u\vec{v}) &= 0\end{aligned}$$

1D wave equation: $u_{tt} = c^2 u_{xx}$
In general $u_{tt} - c^2 \Delta u_{xx} =$

Laplaces Equation: $\Delta u = 0$

Remark describes equilibrium

Functions that solves Laplace's equation are called harmonic functions.

Example: 1D: $u_{xx} = 0$: Linear

Example: 2D: $u_{xx} + u_{yy} = 0$:

$$f(z) = u(x, y) + iv(x, y)$$

holomorphic /complex analytic

then $\Delta u = 0$ and $\Delta v = 0$

Then taking the real and imaginary parts of these equations we get the harmonic functions in 2D.

Imp Example: $u = \ln(x^2 + y^2)$

Remark: Characterization of steady(no time); irrotational(zero curl); incompressible (zero divergence) flow fields \vec{F}

$$\nabla \times \vec{F} = 0 \rightarrow \vec{F} = \nabla u$$

where u is the potential of the flow field.

$$\nabla \cdot \vec{F} = 0 \rightarrow \operatorname{div} \nabla u = \Delta u = 0$$

1.4 Initial and Boundary Conditions

PDE + BCs = Boundary Value Problem (BVP)

PDEs describing equilibrium phenomena are paired with boundary conditions (BCs).

Dirichlet BC Suppose a space D and a boundary ∂D

$$\begin{cases} \Delta u = 0 \text{ in } D \\ u = \phi \text{ on } \partial D \end{cases}$$

Body D in thermal equilibrium knowing the boundary temperature find the temperature inside the body.

Prescribing the function on the boundary is called Dirichlet BC.

Neumann BC

$$\begin{cases} \Delta u = 0 \text{ in } D \\ \partial_n u = \psi \text{ on } \partial D \end{cases} \quad \text{Neumann BC.}$$

$\partial_n u = \Delta u \cdot n$ This is heat flux

EG: Insulated object: $\partial_n u = 0$ on ∂D

if u is a solution then so is $u + c$

$$\text{Mixed Boundary Conditions} \begin{cases} \Delta u = 0 \text{ in } D \\ u = \phi \text{ on } \partial D_1 \\ \partial_n u = \psi \text{ on } \partial D_2 \end{cases}$$

$$\text{Robin BC} \begin{cases} \Delta u = 0 \text{ in } D \\ \alpha u + \beta \partial_n u = \gamma \text{ on } \partial D \end{cases}$$

Example:

$$k \partial_n u + c(u - u_\infty) = 0 \text{ on } \partial D$$

where k is thermal conductivity, c is convective heat transfer coefficient, u_∞ is the ambient temperature.

Example

$$\begin{cases} u'' = 0 \text{ on } (0, 1) \\ u(0) = 0 \\ u(1) = 1 \end{cases} \rightarrow u(x) = x$$

PDEs describing dynamic processes We have time variable! Usually are paired with initial conditions (IC) and BC.

- Thermodynamics: u_t

IC: $u(t_0, x) = \phi(x), \forall x \in D$

- Newtonian Mechanics: u_{tt}

$$\text{IC: } \begin{cases} u(t_0, x) = \phi(x) \\ u_t(t_0, x) = \psi(x) \end{cases}, \forall x \in D$$

Example

$$u_t - u_{xx} = f(t, x), t > 0, x \in (0, 1)$$

$$u(0, x) = u_0(x) \text{ IC}$$

$$u(t, 0) = \phi(t) \text{ BC}$$

$$u(t, 1) = \psi(t) \text{ BC}$$

Remark: If D is unbounded, we'll need conditions at infinity.

1.5 Well Posed problems

Well pose problems has 3 criteria:

- Existence: There exists a solution
- Uniqueness: The solution is unique
- Stability: The solution depends continuously on the data. (IC, BC, source terms)

$$Ax = b$$

Where A is $m \times n$ matrix, x is $n \times 1$ vector, b is $m \times 1$ vector.

- $m > n$: Existence may fail too many variables
- $m < n$: Uniqueness may fail too many equations
- $m = n$ and A is invertible: Existence, uniqueness and stability!

$$x = A^{-1}b$$

$$\begin{cases} Ax = b \\ A(x_\epsilon) = b + \epsilon \end{cases}$$

$$\|x - x_\epsilon\| \leq \|A^{-1}\epsilon\| \leq \|A^{-1}\| \cdot \|\epsilon\|$$

Remark If A has a very small eigenvalue, then $\|A^{-1}\|$ is very large. Ill conditioned problems

1.6 Types of 2nd Order PDEs

In the case of 2 variables x, y

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0$$

$$Q(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$$

$$H = b^2 - 4ac \text{ the Discriminant}$$

Definition

$$\begin{cases} H < 0 \text{ Elliptic: By linear change of variable it can be reduced to the normal form: } u_{xx} + u_{yy} + \dots = 0 \\ H = 0 \text{ Parabolic: By linear change of variable it can be reduced to the normal form: } u_{xx} + \dots = 0 \\ H > 0 \text{ Hyperbolic By linear change of variable it can be reduced to the normal form: } u_{xx} - u_{yy} = 0 \end{cases}$$

$$au_{xx} + bu_{xy} + cu_{yy} + \dots = 0$$

Can be converted to quadratic form Proof of cases: $u_{xx} + bu_{xy} + cu_{yy} \dots = 0$ This is (1)

$$\begin{aligned} x^2 + bxy + cy^2 &= x^2 + 2xby/2 + (by)^2/4 + cy^2 - (by)^2/4 \\ &= (x + by/2)^2 - Hy^2/4 \end{aligned}$$

Let $x = \xi, y = b\xi/2 + \sqrt{H}\eta/2$

Extended to a, b, c functions of x, y.

$$H = H(x, y)$$

Same definition categorizes the type of equation at (x, y)

Example Euler-Tricomi Equation

$$u_{xx} - xu_{yy} = 0$$

$$H = 4x$$

This corresponds to transonic flow.

Hyperbolic become subsonic

Elliptic become supersonic

Parabolic is sonic boom

Matrix Perspective Rewrite (1) as $\text{div}(A\nabla u) + \dots = 0$

$$A = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

$$\partial_x(au_x + b/2u_y) + \partial_y(b/2u_x + cu_y)$$

$$H < 0 \Leftrightarrow \det(A) > 0$$

Evals have the same sign (Elliptic)

$$H = 0 \Leftrightarrow \det(A) = 0$$

One eval is zero (Parabolic)

$$H > 0 \Leftrightarrow \det(A) < 0$$

Evals have different signs (Hyperbolic)

For more variables: $u(x_1, x_2, \dots, x_n)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

A is symmetric

$$\operatorname{div}(A\nabla u) + \dots = 0. \text{ This is (2)}$$

$$\sum a_{ij} u_{x_i x_j} + \dots = 0$$

Def. (2) is

- Elliptic if all eigenvalues of A have same sign Δ
- Parabolic if one eval is 0 but all others have same sign $\partial_t - \Delta$
- Hyperbolic if evals one eval is of one sign and all the others are of the opposite sign. $\partial_t^2 - \Delta$

Consider the equation $Ax \cdot x$

Where A is symmetric

This is a quadratic form

Goal reduce to normal form.

$$A = P \Lambda P^T$$

Where Λ is the eigenvalue decomposition

And $P^T = P^{-1}$ (orthogonal)

$$\begin{aligned} Ax \cdot x &= P \Lambda P^T x \cdot x \\ &= \Lambda P^T x \cdot P^T x \end{aligned}$$

Let $y = P^T x$

$$= \wedge y \cdot y$$

$$= \sum \lambda_i y_i^2$$

Where λ_i are the eigenvalues of A

If all $\lambda_i > 0$ let $z_i = \sqrt{\lambda_i} y_i$

$$= \sum z_i^2$$

If all $\lambda_i < 0$ let $z_i = \sqrt{-\lambda_i} y_i$

$$= -\sum z_i^2$$

In z -variables

$$Ax \cdot x = z_1^2 - z_2^2 + \dots$$

This is called Sylvester's law of inertia.

$$\text{div}(A \nabla u) + \dots = 0$$

$$A = P \wedge P^T$$

$$y = P^T x$$

$$\nabla_x u = P \nabla_y u$$

Try and figure this out

Note that $\nabla_x = (u_x 1, u_x 2, u_x 3, \dots)$

$$\text{div}_x(F) = \text{div}_y(P^T F)$$

$$\text{div}_x(P \wedge P^T \nabla_x u) = \text{div}_x(P \wedge \nabla_y u)$$

$$= \text{div}_y(P P^T \wedge \nabla_y u) = \text{div}_y(\wedge \nabla_y u)$$

$$= \sum \lambda_i u_{y_i y_i}$$