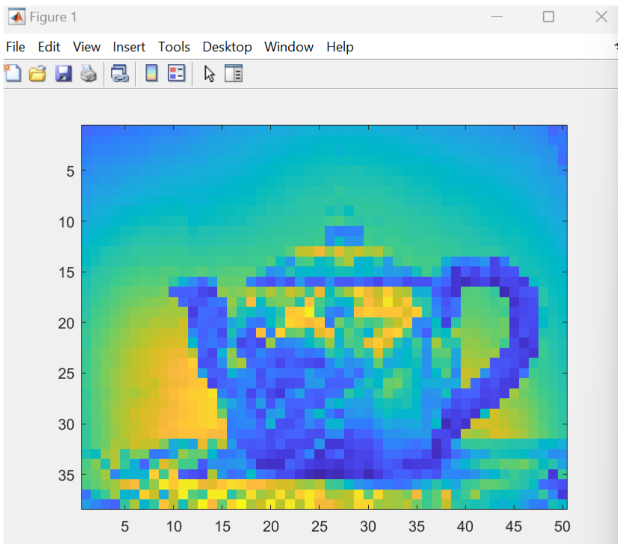
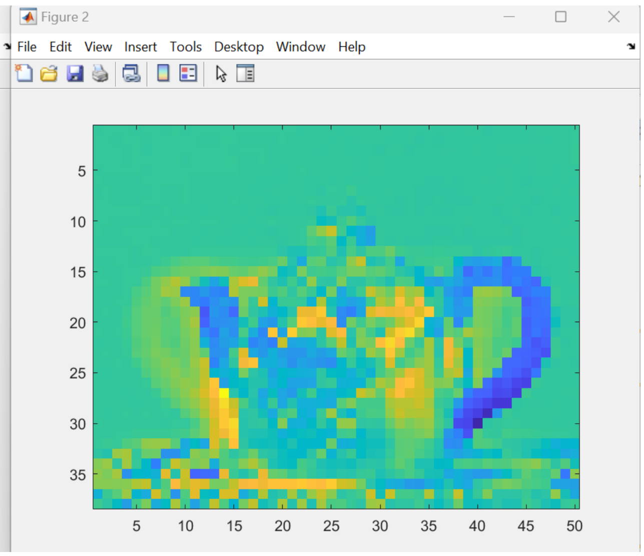
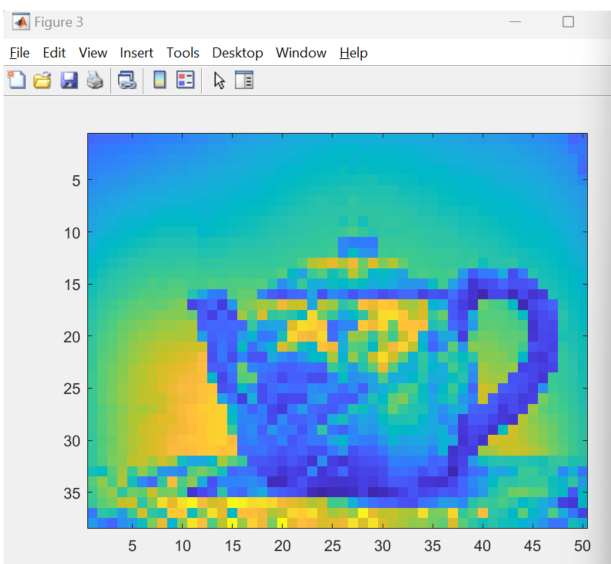
# HW4

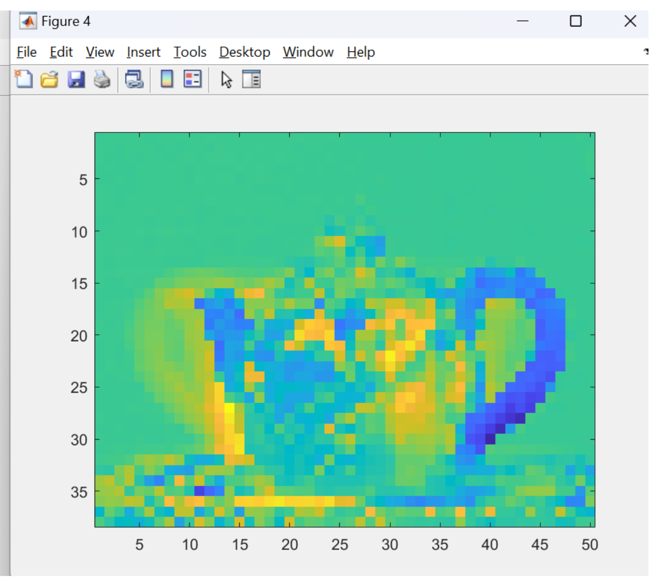
## Q1

After observing the following 10 sets of pictures (the original one on the left, the generated one on the right), it can be concluded that PCA reduced the dimensions of the original data by keeping the most relevant and effective features.



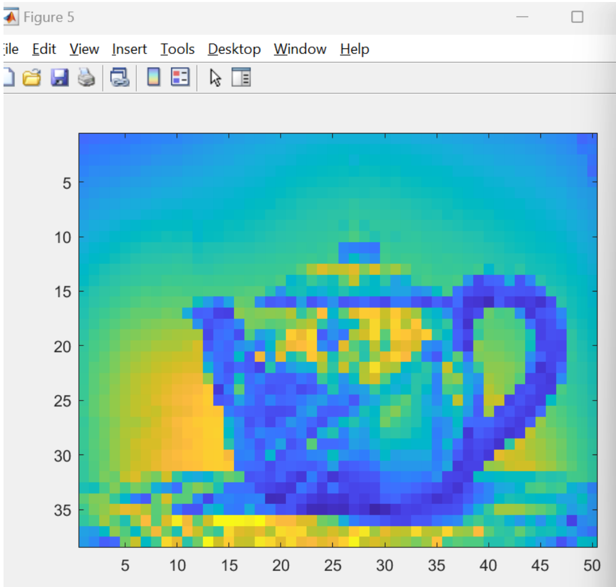


Figure 5
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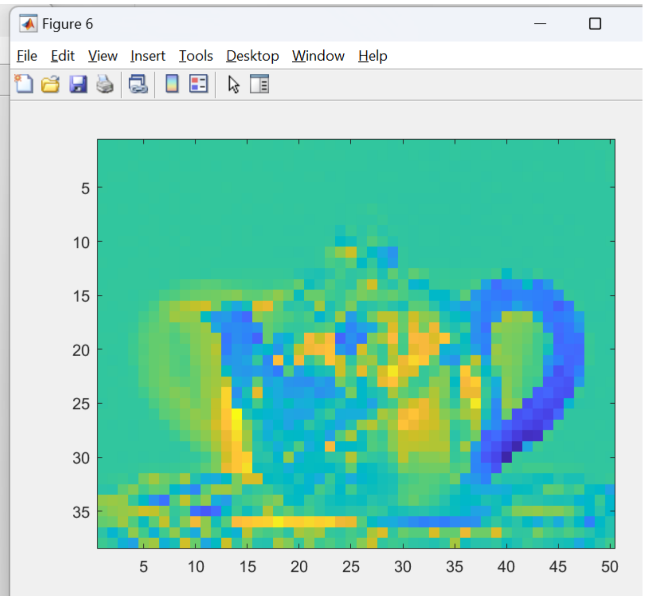
5 10 15 20 25 30 35
5 10 15 20 25 30 35 40 45 50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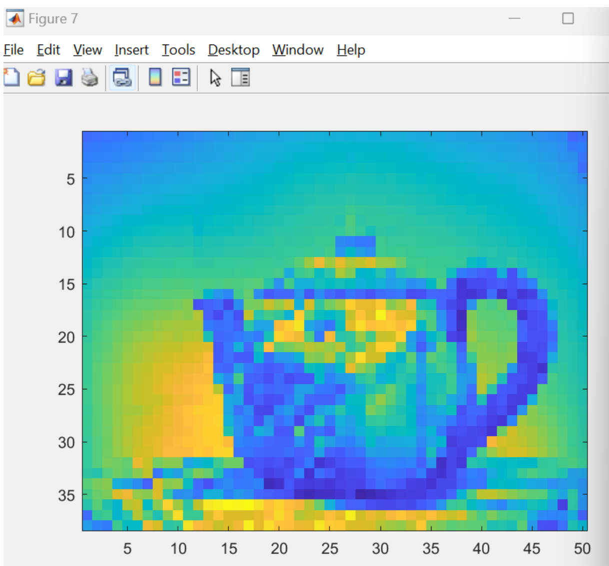
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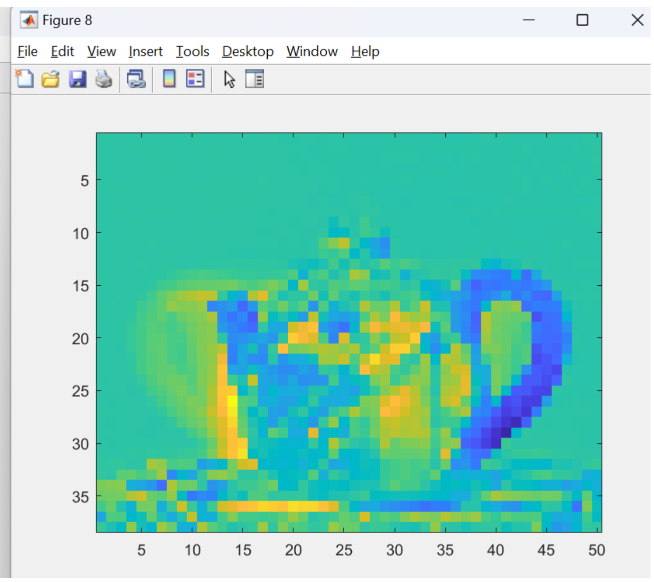
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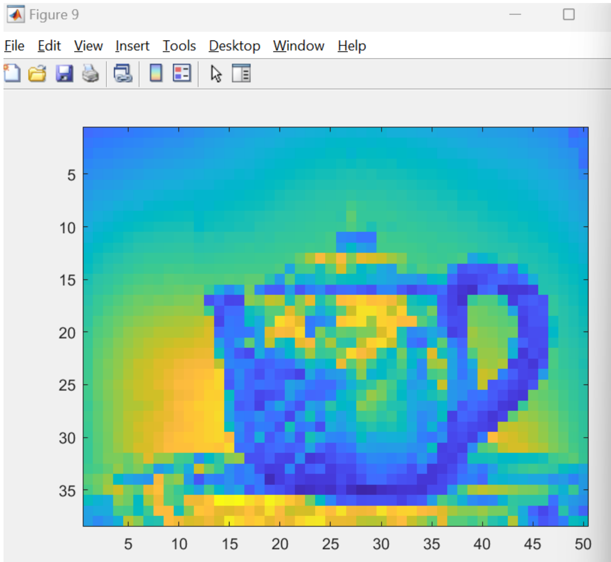
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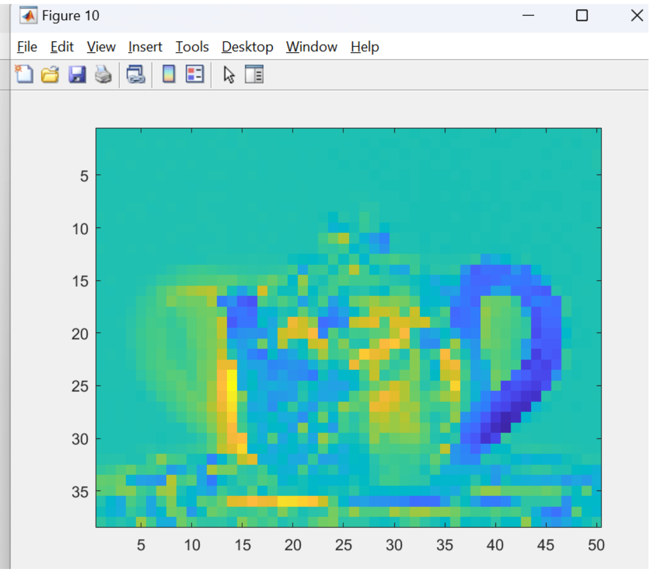
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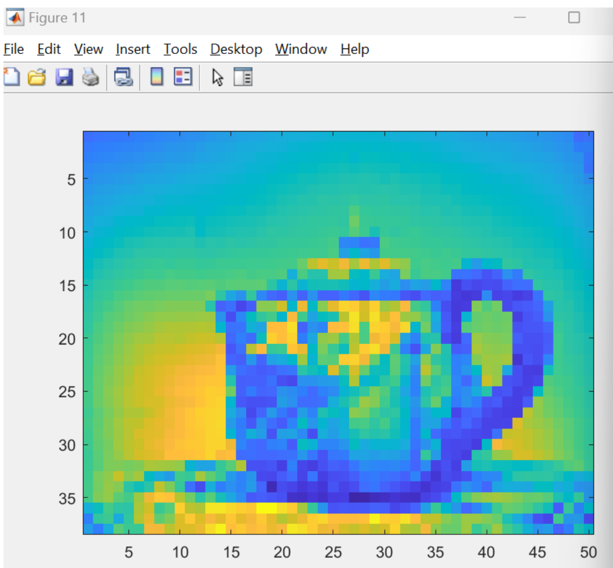
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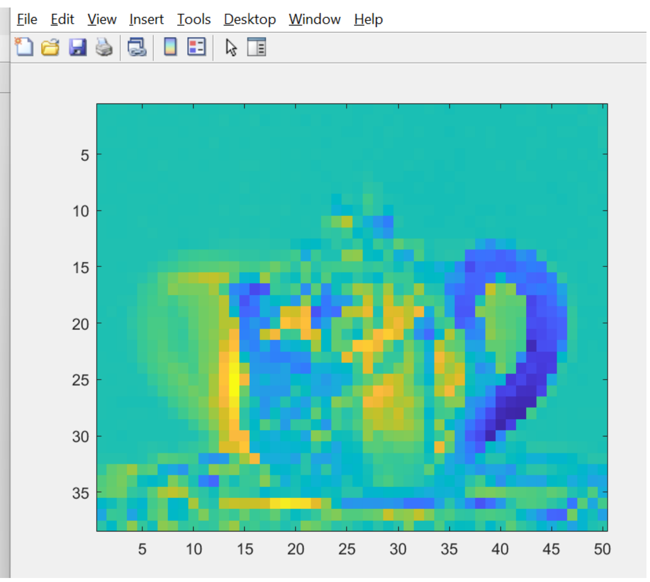
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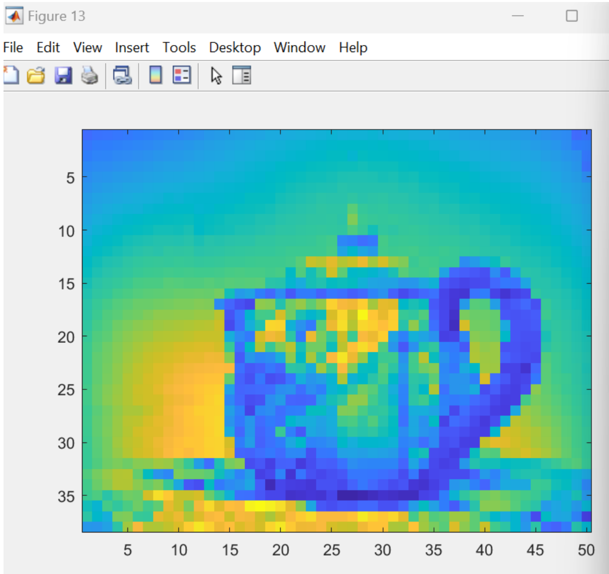
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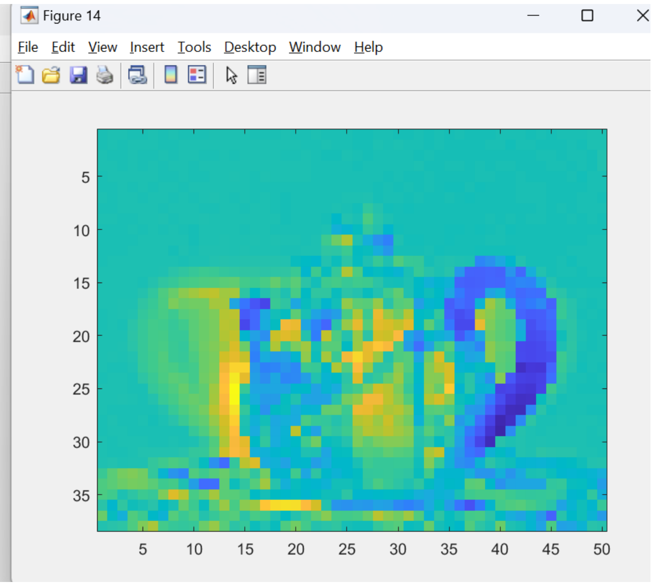
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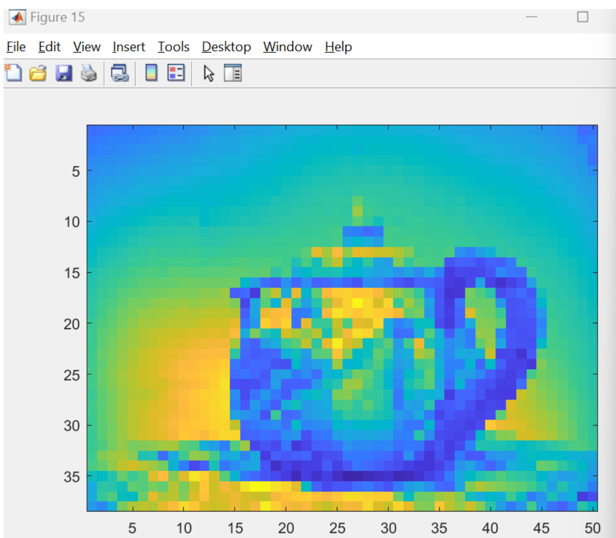
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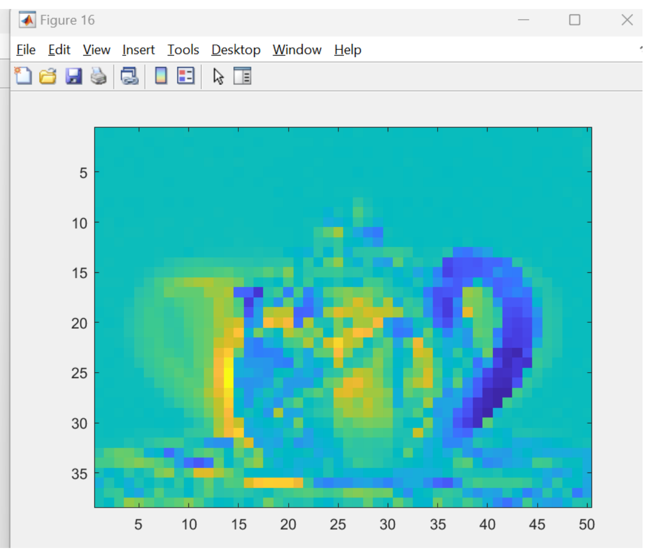
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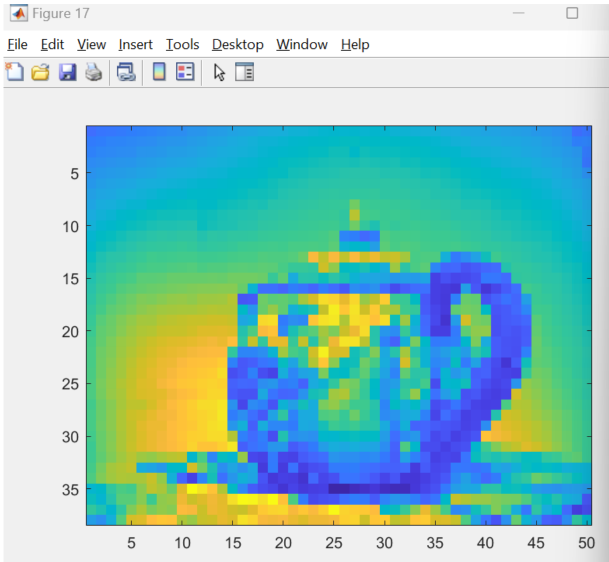
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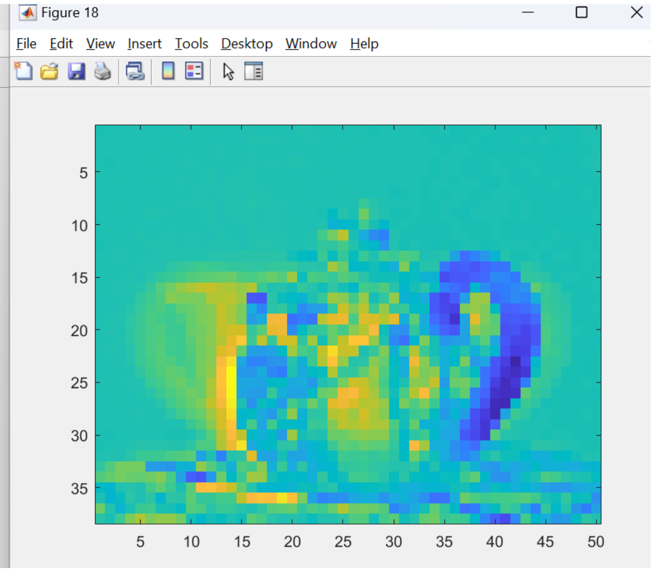
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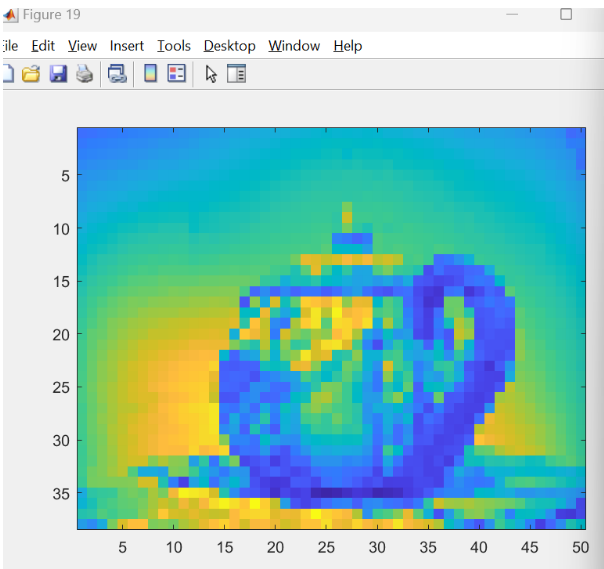
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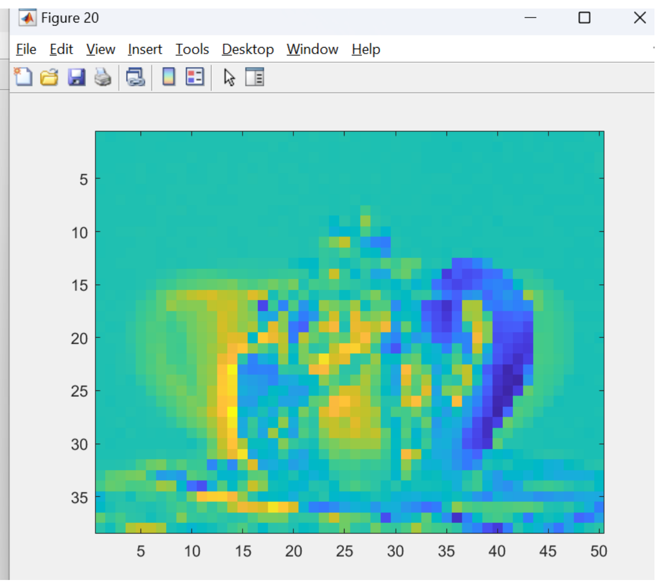
Figure 20
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**Q2**

$X$: draw Apple $\quad E_1$: Box 1 $\quad E_2$: Box 2

$$P(E_1|X) = \frac{P(X|E_1)\,P(E_1)}{P(X)} = \frac{P(X|E_1)P(E_1)}{P(X|E_1)P(E_1) + P(X|E_2)P(E_2)}$$

$$= \frac{\frac{8}{12} \cdot \frac{1}{2}}{\frac{8}{12} \cdot \frac{1}{2} + \frac{10}{12} \cdot \frac{1}{2}} = \frac{4}{9}$$

**Q3**

$$P(y|\theta) = \alpha^{y}(1-\alpha)^{1-y}$$

$$\sum_{i=1}^{N} \log P(y_i|\alpha) = \sum_{i=1}^{N} \log \alpha^{y_i}(1-\alpha)^{1-y_i}$$

$$\Rightarrow \quad \frac{\partial}{\partial\alpha}\sum_{i=1}^{N} \log \alpha^{y_i}(1-\alpha)^{1-y_i} = 0$$

$$\frac{\partial}{\partial\alpha}\sum_{i=1}^{N} y_i \log\alpha + (1-y_i)\log(1-\alpha) = 0$$

$$\frac{\partial}{\partial\alpha}\sum_{i\in \text{Class } 1} \log\alpha + \sum_{i\in\text{Class } 0} \log(1-\alpha) = 0$$

$$\sum_{i\in\text{Class }1} \frac{1}{\alpha} + \sum_{\dots} \frac{-1}{1-\alpha} = 0$$

$$N_1 \frac{1}{\alpha} - N_0 \frac{1}{1-\alpha} = 0$$

$$N_1(1-\alpha) - N_0\alpha = 0$$

$$N_1 - \alpha(N_1+N_0) = 0$$

$$\boxed{\alpha = \frac{N_1}{N_1+N_0}}$$

$$\sum_{i=1}^{N} \log P(\vec{x}_i|\vec{\mu},\Sigma) = \sum_{i=1}^{N} \log \frac{1}{(2\pi)^{D/2}\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2}(\vec{x}_i-\vec{\mu})^T \Sigma^{-1}(\vec{x}_i-\vec{\mu})\right)$$

$\rightarrow$ Max over $\mu$

$$\frac{\partial}{\partial\mu}\left(\sum_{i=1}^{N} \log \frac{1}{(2\pi)^{D/2}\sqrt{|\Sigma|}} \exp(-\frac{1}{2}(\vec{x}_i-\vec{\mu})\Sigma^{-1}(\vec{x}_i-\vec{\mu})\right) = 0$$

$$\sum_{i=1}^{N} (\vec{x}_i - \vec{\mu})\Sigma^{-1} = \vec{0}$$

$$\boxed{\vec{\mu} = \frac{1}{N}\sum_{i=1}^{N} \vec{x}_i}$$

$$l = \sum_{i=1}^{N} -\frac{D}{2}\log 2\pi - \frac{1}{2}\log|\Sigma| - \frac{1}{2}(\vec{x}_i - \vec{\mu})^T \Sigma^{-1} (\vec{x}_i - \vec{\mu})$$

$$= -\frac{ND}{2}\log 2\pi + \frac{N}{2}\log|\Sigma^{-1}| - \frac{1}{2}\sum_{i=1}^{N} tr\left[(\vec{x}_i - \vec{\mu})^T \Sigma^{-1} (\vec{x}_i - \vec{\mu})\right]$$

$$= -\frac{ND}{2}\log 2\pi + \frac{N}{2}\log A - \frac{1}{2}\sum_{i=1}^{N} tr\left[(\vec{x}_i - \vec{\mu})(\vec{x}_i - \vec{\mu})^T A\right]$$

$$\frac{\partial l}{\partial A} = -0 + \frac{N}{2}(A^{-1})^T - \frac{1}{2}\sum_{i=1}^{N}\left[(\vec{x}_i - \vec{\mu})(\vec{x}_i - \vec{\mu})^T\right]^T$$

$$= \frac{N}{2}\Sigma - \frac{1}{2}\sum_{i=1}^{N}(\vec{x}_i - \vec{\mu})(\vec{x}_i - \vec{\mu})^T \Rightarrow 0$$

$$\Sigma = \frac{1}{N}\sum_{i=1}^{N}(\vec{x}_i - \vec{\mu})(\vec{x}_i - \vec{\mu})^T$$

$Y = \arg\max P(\hat{Y}|X) \quad \hat{Y} \in \{0,1\}$

$\Rightarrow Y = \underset{\hat{Y} \in \{0,1\}}{\arg\max}\, P(X|\hat{Y}) P(\hat{Y})$

$$P(X|\hat{Y}) = \frac{1}{(2\pi)^{d/2} |\Sigma_Y|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(X-\mu_Y)^T \Sigma_Y^{-1} (X-\mu_Y)\right)$$

$$\Rightarrow Y = \underset{\hat{Y} \in \{0,1\}}{\arg\max} \left(-\frac{1}{2}(X-\mu_Y)^T \Sigma_Y^{-1} (X-\mu_Y) + \log P(Y) - \frac{d}{2}\log(2\pi) - \frac{1}{2}\log(\Sigma_Y)\right)$$

assume $\Sigma_0 = \Sigma_1 = \Sigma$

~~$\Rightarrow Y = \underset{\hat{Y} \in \{0,1\}}{\arg\max}$~~

$$\Rightarrow Y = \underset{\hat{Y} \in \{0,1\}}{\arg\min} \left((X-\mu_Y)^T \Sigma^{-1} (X-\mu_Y) - 2\log P(Y)\right)$$

$$Y = \underset{\hat{Y} \in \{0,1\}}{\arg\min} \left(X^T \Sigma^{-1} X - 2X^T \Sigma^{-1} \mu_Y + \mu_Y^T \Sigma^{-1} \mu_Y - 2\log P(Y)\right)$$

$$\Rightarrow Y = \arg\max \left(X^T \Sigma^{-1} \mu_Y - \frac{1}{2}\mu_Y^T \Sigma^{-1} \mu_Y + \log P(Y)\right)$$

Define $W_0 = \Sigma^{-1}\mu_0 - \frac{1}{2}\mu_0^T \Sigma^{-1} \mu_0 + \log P(0)$

$W_1 = \Sigma^{-1}\mu_1 - \frac{1}{2}\mu_1^T \Sigma^{-1} \mu_1 + \log P(1)$

$\Rightarrow Y = \arg\max X^T W_Y$

$(W_0 - W_1)^T X = 0$ decision boundary, it is linear.

$$\log P(x \mid Y=0) = -\frac{d}{2}\log(2\pi) - \frac{1}{2}\log|\Sigma_y| - \frac{1}{2}(x-\mu_y)^T \Sigma_y (x-\mu_y)$$

$$\Rightarrow -\frac{d}{2}\log(2\pi) - \frac{1}{2}\log\Sigma_0 - \frac{1}{2}(x-\mu_0)^T\Sigma_0^{-1}(x-\mu_0) + \log P(Y=0) =$$
$$-\frac{d}{2}\log(2\pi) - \frac{1}{2}\log\Sigma_1 - \frac{1}{2}(x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1) + \log P(Y=1)$$

$$\Rightarrow \frac{1}{2}\log(\Sigma_0/\Sigma_1) - \frac{1}{2}(x^T\Sigma_0^{-1}x - x^T\Sigma_1^{-1}x - 2x^T\Sigma_0^{-1}\mu_0 + 2x^T\Sigma_1^{-1}\mu_1 + \mu_0^T\Sigma_0^{-1}\mu_0$$
$$-\mu_1^T\Sigma_1^{-1}\mu_1) + \log(P(Y=0)/P(Y=1)) = 0$$

This is a quadratic equation in $x$, therefore the decision boundary is quadratic.