1.4 Inverses; Algebraic Properties of Matrices

In this section we will discuss some of the algebraic properties of matrix operations. We will see that many of the basic rules of arithmetic for real numbers hold for matrices, but we will also see that some do not.

Properties of Matrix Addition and Scalar Multiplication

The following theorem lists the basic algebraic properties of the matrix operations.

THEOREM 1.4.1 Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

(a) $A+B=B+A$ (**Commutative law for addition**)

(b) $A+(B+C)=(A+B)+C$ (**Associative law for addition**)

(c) $A(BC)=(AB)C$ (**Associative law for multiplication**)

(d) $A(B+C)=AB+AC$ (**Left distributive law**)

(e) $(B+C)A=BA+CA$ (**Right distributive law**)

(f) $A(B-C)=AB-AC$

(g) $(B-C)A=BA-CA$

(h) $a(B+C)=aB+aC$

(i) $a(B-C)=aB-aC$

(j) $(a+b)C=aC+bC$

(k) $(a-b)C=aC-bC$

(l) $a(bC)=(ab)C$

(m) $a(BC)=(aB)C=B(aC)$

To prove any of the equalities in this theorem we must show that the matrix on the left side has the same size as that on the right and that the corresponding entries on the two sides are the same. Most of the proofs follow the same pattern, so we will prove part (*d*) as a sample. The proof of the associative law for multiplication is more complicated than the rest and is outlined in the exercises.

There are three basic ways to prove that two matrices of the same size are equal—prove that corresponding entries are the same, prove that corresponding row vectors are the same, or prove that corresponding column vectors are the same.

Proof (d) We must show that $A(B+C)$ and $AB+AC$ have the same size and that corresponding entries are equal. To form $A(B+C)$, the matrices *B* and *C* must have the same size, say $m \times n$, and the matrix *A* must then have *m* columns, so its size must be of the form $r \times m$. This makes $A(B+C)$ an $r \times n$ matrix. It follows that $AB+AC$ is also an $r \times n$ matrix and, consequently, $A(B+C)$ and $AB+AC$ have the same size.

Suppose that $A=[a_{ij}]$, $B=[b_{ij}]$,and $C=[c_{ij}]$. We want to show that corresponding entries of $A(B+C)$ and $AB+AC$ are equal; that is,

$$[A(B+C)]_{ij}=[AB+AC]_{ij}$$

for all values of *i* and *j*. But from the definitions of matrix addition and matrix multiplication, we have

$$\begin{aligned}[A(B+C)]_{ij} &= a_{i1}(b_{1j}+c_{1j})+a_{i2}(b_{2j}+c_{2j})+\cdots+a_{im}(b_{mj}+c_{mj}) \\ &= (a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{im}b_{mj})+(a_{i1}c_{1j}+a_{i2}c_{2j}+\cdots+a_{im}c_{mj}) \\ &= [AB]_{ij}+[AC]_{ij}=[AB+AC]_{ij}\end{aligned}$$

Remark Although the operations of matrix addition and matrix multiplication were defined for pairs of matrices, associative laws (*b*) and (*c*) enable us to denote sums and products of three matrices as $A+B+C$ and *ABC* without inserting any parentheses. This is justified by the fact that no matter how parentheses are inserted, the associative laws guarantee that the same end result will be obtained. In general, *given any sum or any product of matrices, pairs of parentheses can be inserted or deleted anywhere within the expression without affecting the end result*.

EXAMPLE 1 Associativity of Matrix Multiplication

As an illustration of the associative law for matrix multiplication, consider

$$A=\begin{bmatrix}1 & 2\\ 3 & 4\\ 0 & 1\end{bmatrix},\quad B=\begin{bmatrix}4 & 3\\ 2 & 1\end{bmatrix},\quad C=\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}$$

Then

$$AB=\begin{bmatrix}1 & 2\\ 3 & 4\\ 0 & 1\end{bmatrix}\begin{bmatrix}4 & 3\\ 2 & 1\end{bmatrix}=\begin{bmatrix}8 & 5\\ 20 & 13\\ 2 & 1\end{bmatrix} \text{ and } BC=\begin{bmatrix}4 & 3\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}=\begin{bmatrix}10 & 9\\ 4 & 3\end{bmatrix}$$

Thus

$$(AB)C=\begin{bmatrix}8 & 5\\ 20 & 13\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}=\begin{bmatrix}18 & 15\\ 46 & 39\\ 4 & 3\end{bmatrix}$$

and

$$A(BC)=\begin{bmatrix}1 & 2\\ 3 & 4\\ 0 & 1\end{bmatrix}\begin{bmatrix}10 & 9\\ 4 & 3\end{bmatrix}=\begin{bmatrix}18 & 15\\ 46 & 39\\ 4 & 3\end{bmatrix}$$

so $(AB)C=A(BC)$, as guaranteed by Theorem 1.4.1(*c*).

Properties of Matrix Multiplication

Do not let Theorem 1.4.1 lull you into believing that *all* laws of real arithmetic carry over to matrix arithmetic. For example, you know that in real arithmetic it is always true that $ab = ba$, which is called the *commutative law for multiplication*. In matrix arithmetic, however, the equality of *AB* and *BA* can fail for three possible reasons:

1. *AB* may be defined and *BA* may not (for example, if *A* is 2×3 and *B* is 3×4).
2. *AB* and *BA* may both be defined, but they may have different sizes (for example, if *A* is 2×3 and *B* is 3×2).
3. *AB* and *BA* may both be defined and have the same size, but the two matrices may be different (as illustrated in the next example).

Do not read too much into Example 2—it does not rule out the possibility that *AB* and *BA* may be equal in *certain* cases, just that they are not equal in *all* cases. If it so happens that $AB = BA$, then we say that *AB* and *BA* ***commute***.

EXAMPLE 2 Order Matters in Matrix Multiplication

Consider the matrices

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

Multiplying gives

$$AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

Thus, $AB \neq BA$.

Zero Matrices

A matrix whose entries are all zero is called a ***zero matrix***. Some examples are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [0]$$

We will denote a zero matrix by *0* unless it is important to specify its size, in which case we will denote the $m \times n$ zero matrix by $0_{m \times n}$.

It should be evident that if A and 0 are matrices with the same size, then

$$A + 0 = 0 + A = A$$

Thus, 0 play s the same role in this matrix equation that the number 0 plays in the numerical equation $a + 0 = 0 + a = a$.

The following theorem lists the basic properties of zero matrices. Since the results should be self-evident, we will omit the formal proofs.

THEOREM 1.4.2 Properties of Zero Matrices

If c is a scalar, and if the sizes of the matrices are such that the operations can be perfomed, then:

(a) $A + 0 = 0 + A = A$

(b) $A - 0 = A$

(c) $A - A = A + (-A) = 0$

(d) $0A = 0$

(e) If $cA = 0$, then $c = 0$ or $A = 0$.

Since we know that the commutative law of real arithmetic is not valid in matrix arithmetic, it should not be surprising that there are other rules that fail as well. For example, consider the following two laws of real arithmetic:

- If $ab = bc$ and $a \neq 0$, then $b = c$. **[The cancellation law]**
- If $ab = 0$, then at least one of the factors on the left is 0.

The next two examples show that these laws are not universally true in matrix arithmetic.

EXAMPLE 3 Failure of the Cancellation Law

Consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$

We leave it for you to confirm that

$$AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

Although $A \neq 0$, canceling A from both sides of the equation $AB = AC$ would lead to the incorrect conclusion that $B = C$. Thus, the cancellation law does not hold, in general, for matrix multiplication.

EXAMPLE 4 A Zero Product with Nonzero Factors

Here are two matrices for which $AB = 0$, but $A \neq 0$ and $B \neq 0$:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$$

Identity Matrices

A square matrix with 1's on the main diagonal and zeros elsewhere is called an ***identity matrix***. Some examples are

$$[1], \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

An ***identity matrix*** is denoted by the letter I. If it is important to emphasize the size, we will write I_n for the $n \times n$ identity matrix.

To explain the role of identity matrices in matrix arithmetic, let us consider the effect of multiplying a general 2×3 matrix A on each side by an identity matrix. Multiplying on the right by the 3×3 identity matrix yields

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

and multiplying on the left by the 2×2 identity matrix yields

$$I_2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

The same result holds in general; that is, if A is any $m \times n$ matrix, then

$$AI_n = A \quad \text{and} \quad I_mA = A$$

Thus, the identity matrices play the same role in these matrix equations that the number 1 plays in the numerical equation $a \cdot 1 = 1 \cdot a = a$.

As the next theorem shows, identity matrices arise naturally in studying reduced row echelon forms of *square* matrices.

THEOREM 1.4.3

If R is the reduced row echelon form of an $n \times n$ matrix A, then either R has a row of zeros or R is the identity matrix I_n.

Proof Suppose that the reduced row echelon form of A is

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix}$$

Either the last row in this matrix consists entirely of zeros or it does not. If not, the matrix contains no zero rows, and consequently each of the n rows has a leading entry of 1. Since these leading 1's occur progressively farther to the right as we move down the matrix, each of these 1's must occur on the main diagonal. Since the other entries in the same column as one of these 1's are zero, R must be I_n. Thus, either R has a row of zeros or $R = I_n$.

Inverse of a Matrix

In real arithmetic every nonzero number a has a reciprocal $a^{-1} (= 1/a)$ with the property

$$a \cdot a^{-1} = a^{-1} \cdot a = 1$$

The number a^{-1} is sometimes called the *multiplicative inverse* of a. Our next objective is to develop an analog of this result for matrix arithmetic. For this purpose we make the following definition.

DEFINITION 1

If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be ***invertible*** (or ***nonsingular***) and B is called an ***inverse*** of A. If no such matrix B can be found, then A is said to be ***singular***.

Remark The relationship $AB = BA = I$ is not changed by interchanging A and B, so if A is invertible and B is an inverse of A, then it is also true that B is invertible, and A is an inverse of B. Thus, when

$$AB = BA = I$$

we say that A and B are *inverses of one another*.

EXAMPLE 5 An Invertible Matrix

Let

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

Then

$$\begin{aligned} AB &= \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \\ BA &= \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \end{aligned}$$

Thus, A and B are invertible and each is an inverse of the other.

EXAMPLE 6 Class of Singular Matrices ◄

In general, a square matrix with a row or column of zeros is singular. To help understand why this is so, consider the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

To prove that A is singular we must show that there is no 3×3 matrix B such that $AB = BA = I$. For this purpose let $\mathbf{c}_1, \ \mathbf{c}_2, \ \mathbf{0}$ be the column vectors of A. Thus, for any 3×3 matrix B we can express the product BA as

$$BA = B[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{0}] = [B\mathbf{c}_1 \quad B\mathbf{c}_2 \quad \mathbf{0}] \quad [\textbf{Formula (6) of Section 1.3}]$$

The column of zeros shows that $BA \neq I$ and hence that A is singular.

Properties of Inverses

It is reasonable to ask whether an invertible matrix can have more than one inverse. The next theorem shows that the answer is no—*an invertible matrix has exactly one inverse*.

THEOREM 1.4.4

If B and C are both inverses of the matrix A, then $B = C$.

Proof Since B is an inverse of A, we have $BA = I$. Multiplying both sides on the right by C gives $(BA)C = IC = C$. But it is also true that $(BA)C = B(AC) = BI = B$, so $C = B$.

As a consequence of this important result, we can now speak of “the” inverse of an invertible matrix. If A is invertible, then its inverse will be denoted by the symbol A^{-1}. Thus,

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I \tag{1}$$

The inverse of A plays much the same role in matrix arithmetic that the reciprocal a^{-1} plays in the numerical relationships $aa^{-1} = 1$ and $a^{-1}a = 1$.

In the next section we will develop a method for computing the inverse of an invertible matrix of any size. For now we give the following theorem that specifies conditions under which a 2×2 matrix is invertible and provides a simple formula for its inverse.

THEOREM 1.4.5

The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{2}$$

We will omit the proof, because we will study a more general version of this theorem later. For now, you should at least confirm the validity of Formula 2 by showing that $AA^{-1} = A^{-1}A = I$.

Historical Note The formula for A^{-1} given in Theorem 1.4.5 first appeared (in a more general form) in Arthur Cayley's 1858 *Memoir on the Theory of Matrices*. The more general result that Cayley discovered will be studied later.

The quantity $ad - bc$ in Theorem 1.4.5 is called the ***determinant*** of the 2×2 matrix A and is denoted by

$$\det(A) = ad - bc$$

or alternatively by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Remark Figure 1.4.1 illustrates that the determinant of a 2×2 matrix A is the product of the entries on its main diagonal minus the product of the entries *off* its main diagonal. In words, Theorem 1.4.5 states that a 2×2 matrix A is invertible if and only if its determinant is nonzero, and if invertible, then its inverse can be obtained by interchanging its diagonal entries, reversing the signs of its off-diagonal entries, and multiplying the entries by the reciprocal of the determinant of A.

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Figure 1.4.1

EXAMPLE 7 Calculating the Inverse of a 2 × 2 Matrix

In each part, determine whether the matrix is invertible. If so, find its inverse.

(a) $A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

Solution

(a) The determinant of A is $\det(A) = (6)(2) - (1)(5) = 7$, which is nonzero. Thus, A is invertible, and its inverse is

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

We leave it for you to confirm that $AA^{-1} = A^{-1}A = I$.

(b) The matrix is not invertible since $\det(A) = (-1)(-6) - (2)(3) = 0$.

EXAMPLE 8 Solution of a Linear System by Matrix Inversion

A problem that arises in many applications is to solve a pair of equations of the form

$$u = ax + by$$
$$v = cx + dy$$

for x and y in terms of u and v. One approach is to treat this as a linear system of two equations in the unknowns x and y and use Gauss–Jordan elimination to solve for x and y. However, because the coefficients of the unknowns are *literal* rather than *numerical*, this procedure is a little clumsy. As an alternative approach, let us replace the two equations by the single matrix equation

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

which we can rewrite as

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

If we assume that the 2×2 matrix is invertible (i.e., $ad - bc \neq 0$), then we can multiply through on the left by the inverse and rewrite the equation as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix}$$

Using Theorem 1.4.5, we can rewrite this equation as

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix}$$

from which we obtain

$$x=\frac{du-bv}{ad-bc}, \quad y=\frac{av-cu}{ad-bc}$$

The next theorem is concerned with inverses of matrix products.

THEOREM 1.4.6

If A and B are invertible matrices with the same size, then AB is invertible and

$$(AB)^{-1}=B^{-1}A^{-1}$$

Proof We can establish the invertibility and obtain the stated formula at the same time by showing that

$$(AB)\left(B^{-1}A^{-1}\right)=\left(B^{-1}A^{-1}\right)(AB)=I$$

But

$$(AB)\left(B^{-1}A^{-1}\right)=A\left(BB^{-1}\right)A^{-1}=AIA^{-1}=AA^{-1}=I$$

and similarly, $\left(B^{-1}A^{-1}\right)(AB)=I$.

Although we will not prove it, this result can be extended to three or more factors:

A product of any number ofinvertible matrices is invertible, and the inverse of the product is the product of the inverses in the reverse order.

EXAMPLE 9 The Inverse of a Product

Consider the matrices

$$A=\begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix},\quad B=\begin{bmatrix}3 & 2\\ 2 & 2\end{bmatrix}$$

We leave it for you to show that

$$AB=\begin{bmatrix}7 & 6\\ 9 & 8\end{bmatrix},\quad (AB)^{-1}=\begin{bmatrix}4 & -3\\ -\frac{9}{2} & \frac{7}{2}\end{bmatrix}$$

and also that

$$A^{-1}=\begin{bmatrix}3 & -2\\ -1 & 1\end{bmatrix},\quad B^{-1}=\begin{bmatrix}1 & -1\\ -1 & \frac{3}{2}\end{bmatrix},\quad B^{-1}A^{-1}=\begin{bmatrix}1 & -1\\ -1 & \frac{3}{2}\end{bmatrix}\begin{bmatrix}3 & -2\\ -1 & 1\end{bmatrix}=\begin{bmatrix}4 & -3\\ -\frac{9}{2} & \frac{7}{2}\end{bmatrix}$$

Thus, $(AB)^{-1}=B^{-1}A^{-1}$ as guaranteed by Theorem 1.4.6.

Powers of a Matrix

If A is a *square* matrix, then we define the nonnegative integer powers of A to be

$$A^0=I \quad \text{and} \quad A^n=AA\cdots A \quad [\boldsymbol{n}\ \textbf{factors}]$$

and if A is invertible, then we define the negative integer powers of A to be

$$A^{-n}=\left(A^{-1}\right)^n=A^{-1}A^{-1}\cdots A^{-1} \quad [\boldsymbol{n}\ \textbf{factors}]$$

Because these definitions parallel those for real numbers, the usual laws of nonnegative exponents hold; for example,

$$A^rA^s=A^{r+s} \quad \text{and} \quad \left(A^r\right)^s=A^{rs}$$

If a product of matrices is singular, then at least one of the factors must be singular. Why?

In addition, we have the following properties of negative exponents.

THEOREM 1.4.7

If A is invertible and n is a nonnegative integer, then:

(a) A^{-1} is invertible and $\left(A^{-1}\right)^{-1}=A$.

(b) A^n is invertible and $\left(A^n\right)^{-1} = A^{-n} = \left(A^{-1}\right)^n$.

(c) kA is invertible for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$.

We will prove part (c) and leave the proofs of parts (a) and (b) as exercises.

Proof (c) Properties (c) and (m) in Theorem 1.4.1 imply that

$$(kA)\left(k^{-1}A^{-1}\right) = k^{-1}(kA)A^{-1} = \left(k^{-1}k\right)AA^{-1} = (1)I = I$$

and similarly, $\left(k^{-1}A^{-1}\right) = (kA) = I$. Thus, kA is invertible and $(kA)^{-1} = k^{-1}A^{-1}$.

EXAMPLE 10 Properties of Exponents

Let A and A^{-1} be the matrices in Example 9; that is,

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Then

$$A^{-3} = \left(A^{-1}\right)^3 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

Also,

$$A^3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

so, as expected from Theorem 1.4.7(b),

$$\left(A^3\right)^{-1} = \frac{1}{(11)(41) - (30)(15)}\begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = \left(A^{-1}\right)^3$$

EXAMPLE 11 The Square of a Matrix Sum

In real arithmetic, where we have a commutative law for multiplication, we can write

$$(a+b)^2 = a^2 + ab + ba + b^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$$

However, in matrix arithmetic, where we have no commutative law for multiplication, the best we can do is to write

$$(A+B)^2 = A^2 + AB + BA + B^2$$

It is only in the special case where A and B *commute* (i.e., $AB = BA$) that we can go a step further and write

$$(A+B)^2 = A^2 + 2AB + B^2$$

Matrix Polynomials

If A is a square matrix, say $n \times n$, and if

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$$

is any polynomial, then we define the $n \times n$ matrix $p(A)$ to be

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_m A^m \tag{3}$$

where I is the $n \times n$ identity matrix; that is, $p(A)$ is obtained by substituting A for x and replacing the constant term a_0 by the matrix $a_0 I$. An expression of form 3 is called a ***matrix polynomial in A***.

EXAMPLE 12 A Matrix Polynomial

Find $p(A)$ for

$$p(x) = x^2 - 2x - 3 \quad \text{and} \quad A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$$

Solution

$$\begin{aligned} p(A) &= A^2 - 2A - 3I \\ &= \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

or more briefly, $p(A) = 0$.

Remark It follows from the fact that $A^r A^s = A^{r+s} = A^{s+r} = A^s A^r$ that powers of a square matrix commute, and since a matrix polynomial in A is built up from powers of A, any two matrix polynomials in A also commute; that is, for any polynomials p_1 and p_2 we have

$$p_1(A) p_2(A) = p_2(A) p_1(A) \tag{4}$$

Properties of the Transpose

The following theorem lists the main properties of the transpose.

THEOREM 1.4.8

If the sizes of the matrices are such that the stated operations can be performed, then:

(a) $\left(A^T\right)^T = A$

(b) $(A+B)^T = A^T + B^T$

(c) $(A-B)^T = A^T - B^T$

(d) $(kA)^T = kA^T$

(e) $(AB)^T = B^T A^T$

If you keep in mind that transposing a matrix interchanges its rows and columns, then you should have little trouble visualizing the results in parts (*a*)–(*d*). For example, part (*a*) states the obvious fact that interchanging rows and columns twice leaves a matrix unchanged; and part (*b*) states that adding two matrices and then interchanging the rows and columns produces the same result as interchanging the rows and columns before adding. We will omit the formal proofs. Part (*e*) is a less obvious, but for brevity we will omit its proof as well. The result in that part can be extended to three or more factors and restated as:

The transpose of a product of any number of matrices is the product of the transposes in the reverse order.

The following theorem establishes a relationship between the inverse of a matrix and the inverse of its transpose.

THEOREM 1.4.9

If A is an invertible matrix, then A^T is also invertible and

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

Proof We can establish the invertibility and obtain the formula at the same time by showing that

$$A^T\left(A^{-1}\right)^T = \left(A^{-1}\right)^T A^T = I$$

But from part (*e*) of Theorem 1.4.8 and the fact that $I^T = I$, we have

$$
\begin{aligned}
A^T\left(A^{-1}\right)^T &= \left(A^{-1}A\right)^T = I^T = I \\
\left(A^{-1}\right)^T A^T &= \left(AA^{-1}\right)^T = I^T = I
\end{aligned}
$$

which completes the proof.

EXAMPLE 13 Inverse of a Transpose

Consider a general 2×2 invertible matrix and its transpose:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Since A is invertible, its determinant $ad - bc$ is nonzero. But the determinant of A^T is also $ad - bc$ (verify), so A^T is also invertible. It follows from Theorem 1.4.5 that

$$\left(A^T\right)^{-1} = \begin{bmatrix} \dfrac{d}{ad-bc} & -\dfrac{c}{ad-bc} \\ -\dfrac{b}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

which is the same matrix that results if A^{-1} is transposed (verify). Thus,

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

as guaranteed by Theorem 1.4.9.

Concept Review

- Commutative law for matrix addition
- Associative law for matrix addition
- Associative law for matrix multiplication
- Left and right distributive laws
- Zero matrix
- Identity matrix
- Inverse of a matrix
- Invertible matrix
- Nonsingular matrix
- Singular matrix
- Determinant
- Power of a matrix

- Matrix polynomial

Skills

- Know the arithmetic properties of matrix operations.
- Be able to prove arithmetic properties of matrices.
- Know the properties of zero matrices.
- Know the properties of identity matrices.
- Be able to recognize when two square matrices are inverses of each other.
- Be able to determine whether a 2×2 matrix is invertible.
- Be able to solve a linear system of two equations in two unknowns whose coefficient matrix is invertible.
- Be able to prove basic properties involving invertible matrices.
- Know the properties of the matrix transpose and its relationship with invertible matrices.

Exercise Set 1.4

1. Let

$$A=\begin{bmatrix}2&-1&3\\0&4&5\\-2&1&4\end{bmatrix},\quad B=\begin{bmatrix}8&-3&-5\\0&1&2\\4&-7&6\end{bmatrix},\quad C=\begin{bmatrix}0&-2&3\\1&7&4\\3&5&9\end{bmatrix},\quad a=4,\quad b=-7$$

Show that

(a) $A+(B+C)=(A+B)+C$

(b) $(AB)C=A(BC)$

(c) $(a+b)C=aC+bC$

(d) $a(B-C)=aB-aC$

2. Using the matrices and scalars in Exercise 1, verify that

(a) $a(BC)=(aB)C=B(aC)$

(b) $A(B-C)=AB-AC$

(c) $(B+C)A=BA+CA$

(d) $a(bC)=(ab)C$

3. Using the matrices and scalars in Exercise 1, verify that

(a) $\left(A^T\right)^T=A$

(b) $(A+B)^T=A^T+B^T$

(c) $(aC)^T=aC^T$

(d) $(AB)^T=B^TA^T$

In Exercises 4–7 use Theorem 1.4.5 to compute the inverses of the following matrices.

4. $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

5. $B = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$

Answer:

$$B^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix}$$

6. $C = \begin{bmatrix} 6 & 4 \\ -2 & -1 \end{bmatrix}$

7. $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Answer:

$$D^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

8. Find the inverse of

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

9. Find the inverse of

$$\begin{bmatrix} \frac{1}{2}(e^x + e^{-x}) & \frac{1}{2}(e^x - e^{-x}) \\ \frac{1}{2}(e^x - e^{-x}) & \frac{1}{2}(e^x + e^{-x}) \end{bmatrix}$$

Answer:

$$\begin{bmatrix} \frac{1}{2}(e^x + e^{-x}) & -\frac{1}{2}(e^x - e^{-x}) \\ -\frac{1}{2}(e^x - e^{-x}) & \frac{1}{2}(e^x + e^{-x}) \end{bmatrix}$$

10. Use the matrix A in Exercise 4 to verify that $(A^T)^{-1} = (A^{-1})^T$.

11. Use the matrix B in Exercise 5 to verify that $(B^T)^{-1} = (B^{-1})^T$.

12. Use the matrices A and B in 4 and 5 to verify that $(AB)^{-1} = B^{-1}A^{-1}$.

13. Use the matrices A, B, and C in Exercises 4–6 to verify that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

In Exercises 14–17, use the given information to find A.

14. $A^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$

15. $(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$

Answer:

$$A = \begin{bmatrix} \frac{2}{7} & 1 \\ \frac{1}{7} & \frac{3}{7} \end{bmatrix}$$

16. $\left(5A^T\right)^{-1} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$

17. $(I + 2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$

Answer:

$$\begin{bmatrix} -\frac{9}{13} & \frac{1}{13} \\ \frac{2}{13} & -\frac{6}{13} \end{bmatrix}$$

18. Let A be the matrix

$$\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$$

In each part, compute the given quantity.

(a) A^3

(b) A^{-3}

(c) $A^2 - 2A + I$

(d) $p(A)$, where $p(x) = x - 2$

(e) $p(A)$, where $p(x) = 2x^2 - x + 1$

(f) $p(A)$, where $p(x) = x^3 - 2x + 4$

19. Repeat Exercise 18 for the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

Answer:

(a) $\begin{bmatrix} 41 & 15 \\ 30 & 11 \end{bmatrix}$

(b) $\begin{bmatrix} 11 & -15 \\ -30 & 41 \end{bmatrix}$

(c) $\begin{bmatrix} 6 & 2 \\ 4 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

(e) $\begin{bmatrix} 20 & 7 \\ 14 & 6 \end{bmatrix}$

(f) $\begin{bmatrix} 39 & 13 \\ 26 & 13 \end{bmatrix}$

20. Repeat Exercise 18 for the matrix

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & -2 & 0 \\ 5 & 0 & 2 \end{bmatrix}$$

21. Repeat Exercise 18 for the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & -3 & -1 \end{bmatrix}$$

Answer:

(a) $\begin{bmatrix} 27 & 0 & 0 \\ 0 & 26 & -18 \\ 0 & 18 & 26 \end{bmatrix}$

(b) $\begin{bmatrix} \frac{1}{27} & 0 & 0 \\ 0 & 0.026 & 0.018 \\ 0 & -0.018 & 0.026 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & -5 & -12 \\ 0 & 12 & -5 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 3 \\ 0 & -3 & -3 \end{bmatrix}$

(e) $\begin{bmatrix} 16 & 0 & 0 \\ 0 & -14 & -15 \\ 0 & 15 & -14 \end{bmatrix}$

(f) $\begin{bmatrix} 25 & 0 & 0 \\ 0 & 32 & -24 \\ 0 & 24 & 32 \end{bmatrix}$

In Exercises 22–24, let $p_1(x) = x^2 - 9$, $p_2(x) = x + 3$, and $p_3(x) = x - 3$. Show that $p_1(A) = p_2(A)p_3(A)$ for the given matrix.

22. The matrix *A* in Exercise 18.

23. The matrix *A* in Exercise 21.

24. An arbitrary square matrix *A*.

25. Show that if $p(x) = x^2 - (a+d)x + (ad - bc)$ and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then $p(A) = 0$.

26. Show that if $p(x) = x^3 - (a+b+c)x^2 + (ab + ae + be - cd)x - a(be - cd)$ and

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{bmatrix}$$

then $p(A) = 0$.

27. Consider the matrix

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

where $a_{11}a_{22} \cdots a_{nn} \neq 0$. Show that *A* is invertible and find its inverse.

Answer:

$$\begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix}$$

28. Show that if a square matrix *A* satisfies $A^2 - 3A + I = 0$, then $A^{-1} = 3I - A$.

29. (a) Show that a matrix with a row of zeros cannot have an inverse.

(b) Show that a matrix with a column of zeros cannot have an inverse.

30. Assuming that all matrices are $n \times n$ and invertible, solve for *D*.

$$ABC^TDBA^TC = AB^T$$

31. Assuming that all matrices are $n \times n$ and invertible, solve for D.

$$C^T B^{-1} A^2 BAC^{-1} DA^{-2} B^T C^{-2} = C^T$$

Answer:

$D = CA^{-1}B^{-1}A^{-2}BC^2\left(B^T\right)^{-1}A^2$

32. If A is a square matrix and n is a positive integer, is it true that $\left(A^n\right)^T = \left(A^T\right)^n$? Justify your answer.

33. Simplify:

$$(AB)^{-1}\left(AC^{-1}\right)\left(D^{-1}C^{-1}\right)^{-1}D^{-1}$$

Answer:

B^{-1}

34. Simplify:

$$\left(AC^{-1}\right)^{-1}\left(AC^{-1}\right)\left(AC^{-1}\right)^{-1}AD^{-1}$$

In Exercises 35–37, determine whether A is invertible, and if so, find the inverse. [*Hint:* Solve $AX = I$ for X by equating corresponding entries on the two sides.]

35. $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Answer:

$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

36. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

37. $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

Answer:

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

38. Prove Theorem 1.4.2.

In Exercises 39–42, use the method of Example 8 to find the unique solution of the given linear system.

39. $$\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 4x_1 + 5x_2 &= 3 \end{aligned}$$

Answer:

$$x_1 = \frac{1}{23},\ x_2 = \frac{13}{23}$$

40. $$\begin{aligned} -x_1 + 5x_2 &= 4 \\ -x_1 - 3x_2 &= 1 \end{aligned}$$

41. $$\begin{aligned} 6x_1 + x_2 &= 0 \\ 4x_1 - 3x_2 &= -2 \end{aligned}$$

Answer:

$$x_1 = -\frac{1}{11},\ x_2 = \frac{6}{11}$$

42. $$\begin{aligned} 2x_1 - 2x_2 &= 4 \\ x_1 + 4x_2 &= 4 \end{aligned}$$

43. Prove part *(a)* of Theorem 1.4.1.

44. Prove part *(c)* of Theorem 1.4.1.

45. Prove part *(f)* of Theorem 1.4.1.

46. Prove part *(b)* of Theorem 1.4.2.

47. Prove part *(c)* of Theorem 1.4.2.

48. Verify Formula 4 in the text by a direct calculation.

49. Prove part *(d)* of Theorem 1.4.8.

50. Prove part *(e)* of Theorem 1.4.8.

51. (a) Show that if A is invertible and $AB = AC$, then $B = C$.

(b) Explain why part (a) and Example 3 do not contradict one another.

52. Show that if A is invertible and k is any nonzero scalar, then $(kA)^n = k^n A^n$ for all integer values of n.

53. (a) Show that if A, B, and $A + B$ are invertible matrices with the same size, then

$$A\left(A^{-1} + B^{-1}\right)B(A + B)^{-1} = I$$

(b) What does the result in part (a) tell you about the matrix $A^{-1} + B^{-1}$?

54. A square matrix A is said to be ***idempotent*** if $A^2 = A$.

(a) Show that if A is idempotent, then so is $I - A$.

(b) Show that if A is idempotent, then $2A - I$ is invertible and is its own inverse.

55. Show that if A is a square matrix such that $A^k = 0$ for some positive integer k, then the matrix A is invertible and

$$(1 - A)^{-1} = I + A + A^2 + \cdot \cdot \cdot + A^{k-1}$$

True-False Exercises

In parts (a)–(k) determine whether the statement is true or false, and justify your answer.

(a) Two $n \times n$ matrices, A and B, are inverses of one another if and only if $AB = BA = 0$.

Answer:

False

(b) For all square matrices A and B of the same size, it is true that $(A + B)^2 = A^2 + 2AB + B^2$.

Answer:

False

(c) For all square matrices A and B of the same size, it is true that $A^2 - B^2 = (A - B)(A + B)$.

Answer:

False

(d) If A and B are invertible matrices of the same size, then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

Answer:

False

(e) If A and B are matrices such that AB is defined, then it is true that $(AB)^T = A^T B^T$.

Answer:

False

(f) The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$.

Answer:

True

(g) If *A* and *B* are matrices of the same size and *k* is a constant, then $(kA+B)^T = kA^T + B^T$.

Answer:

True

(h) If *A* is an invertible matrix, then so is A^T.

Answer:

True

(i) If $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$ and *I* is an identity matrix, then $p(I) = a_0 + a_1 + a_2 + \cdots + a_m$.

Answer:

False

(j) A square matrix containing a row or column of zeros cannot be invertible.

Answer:

True

(k) The sum of two invertible matrices of the same size must be invertible.

Answer:

False

1.5 Elementary Matrices and a Method for Finding A^{-1}

In this section we will develop an algorithm for finding the inverse of a matrix, and we will discuss some of the basic properties of invertible matrices.

In Section 1.1 we defined three elementary row operations on a matrix A:

1. Multiply a row by a nonzero constant c.
2. Interchange two rows.
3. Add a constant c times one row to another.

It should be evident that if we let B be the matrix that results from A by performing one of the operations in this list, then the matrix A can be recovered from B by performing the corresponding operation in the following list:

1. Multiply the same row by $1/c$.
2. Interchange the same two rows.
3. If B resulted by adding c times row r_1 of A to row r_2, then add $-c$ times r_1 to r_2.

It follows that if B is obtained from A by performing a sequence of elementary row operations, then there is a second sequence of elementary row operations, which when applied to B recovers A (Exercise 43). Accordingly, we make the following definition.

DEFINITION 1

Matrices A and B are said to be ***row equivalent*** if either (hence each) can be obtained from the other by a sequence of elementary row operations.

Our next goal is to show how matrix multiplication can be used to carry out an elementary row operation.

DEFINITION 2

An $n \times n$ matrix is called an ***elementary matrix*** if it can be obtained from the $n \times n$ identity matrix I_n by performing a *single* elementary row operation.

EXAMPLE 1 Elementary Matrices and Row Operations

Listed below are four elementary matrices and the operations that produce them.

$$\begin{bmatrix}1 & 0\\0 & -3\end{bmatrix} \qquad \begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\end{bmatrix} \qquad \begin{bmatrix}1 & 0 & 3\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \qquad \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
Multiply the second row of I_2 by -3.	Interchange the second and fourth rows of I_4.	Add 3 times the third row of I_3 to the first row.	Multiply the first row of I_3 by 1.

The following theorem, whose proof is left as an exercises, shows that when a matrix A is multiplied on the *left* by an elementary matrix E, the effect is to perform an elementary row operation on A.

THEOREM 1.5.1 Row Operations by Matrix Multiplication

If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product EA is the matrix that results when this same row operation is performed on A.

EXAMPLE 2 Using Elementary Matrices

Consider the matrix

$$A = \begin{bmatrix}1 & 0 & 2 & 3\\2 & -1 & 3 & 6\\1 & 4 & 4 & 0\end{bmatrix}$$

and consider the elementary matrix

$$E = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\3 & 0 & 1\end{bmatrix}$$

which results from adding 3 times the first row of I_3 to the third row. The product EA is

$$EA = \begin{bmatrix}1 & 0 & 2 & 3\\2 & -1 & 3 & 6\\4 & 4 & 10 & 9\end{bmatrix}$$

which is precisely the same matrix that results when we add 3 times the first row of A to the third row.

Theorem 1.5.1 will be a useful tool for developing new results about matrices, but as a practical matter it is usually preferable to perform row operations directly.

We know from the discussion at the beginning of this section that if E is an elementary matrix that results from performing an elementary row operation on an identity matrix I, then there is a second elementary row operation, which when applied to E, produces I back again. Table 1 lists these operations. The operations on the right side of the table are called the ***inverse operations*** of the corresponding operations on the left.

Table 1

Row Operation on *I* That Produces *E*	**Row Operation on *E* That Reproduces *I***
Multiply row i by $c \neq 0$	Multiply row i by $1/c$
Interchange rows i and j	Interchange rows i and j
Add c times row i to row j	Add $-c$ times row i to row j

EXAMPLE 3 Row Operations and Inverse Row Operations

In each of the following, an elementary row operation is applied to the 2×2 identity matrix to obtain an elementary matrix E, then E is restored to the identity matrix by applying the inverse row operation.

$$\begin{bmatrix}1&0\\0&1\end{bmatrix} \xrightarrow[\text{Multiply the second row by 7.}]{} \begin{bmatrix}1&0\\0&7\end{bmatrix} \xrightarrow[\text{Multiply the second row by } \frac{1}{7}.]{} \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0\\0&1\end{bmatrix} \xrightarrow[\text{Interchange the first and second rows.}]{} \begin{bmatrix}0&1\\1&0\end{bmatrix} \xrightarrow[\text{Interchange the first and second rows.}]{} \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0\\0&1\end{bmatrix} \xrightarrow[\text{Add 5 times the second row to the first.}]{} \begin{bmatrix}1&5\\0&1\end{bmatrix} \xrightarrow[\text{Add } -5 \text{ times the second row to the first.}]{} \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

The next theorem is a key result about invertibility of elementary matrices. It will be a building block for many results that follow.

THEOREM 1.5.2

Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Proof If E is an elementary matrix, then E results by performing some row operation on I. Let E_0 be the matrix that results when the inverse of this operation is performed on I. Applying Theorem 1.5.1 and using the fact that inverse row operations cancel the effect of each other, it follows that

$$E_0E = I \text{ and } EE_0 = I$$

Thus, the elementary matrix E_0 is the inverse of E.

Equivalence Theorem

One of our objectives as we progress through this text is to show how seemingly diverse ideas in linear algebra are related. The following theorem, which relates results we have obtained about invertibility of matrices, homogeneous linear systems, reduced row echelon forms, and elementary matrices, is our first step in that direction. As we study new topics, more statements will be added to this theorem.

THEOREM 1.5.3 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false.

(a) A is invertible.

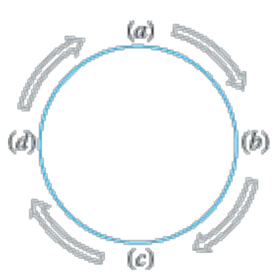
(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) The reduced row echelon form of A is I_n.

(d) A is expressible as a product of elementary matrices.

It may make the logic of our proof of Theorem 1.5.3 more apparent by writing the implications

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$$



This makes it evident visually that the validity of any one statement implies the validity of all the others, and hence that the falsity of any one implies the falsity of the others.

Proof We will prove the equivalence by establishing the chain of implications:

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$$

$(a) \Rightarrow (b)$ Assume A *is invertible and let* $\mathbf{x}_0$ *be any solution of*. Multiplying both sides of this equation by the matrix A^{-1} gives $A^{-1}(A\mathbf{x}_0) = A^{-1}\mathbf{0}$, or $\left(A^{-1}A\right)\mathbf{x}_0 = \mathbf{0}$, or $I\mathbf{x}_0 = \mathbf{0}$, or $\mathbf{x}_0 = \mathbf{0}$. Thus, $A\mathbf{x} = 0$ has only the trivial solution.

$(b) \Rightarrow (c)$ Let $A\mathbf{x} = \mathbf{0}$ be the matrix form of the system

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = 0 \end{array} \tag{1}$$

and assume that the system has only the trivial solution. If we solve by Gauss-Jordan elimination, then the system of equations corresponding to the reduced row echelon form of the augmented matrix will be

$$\begin{array}{llll} x_1 & & & = 0 \\ & x_2 & & = 0 \\ & & \ddots & \\ & & & x_n = 0 \end{array} \tag{2}$$

Thus the augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 \end{bmatrix}$$

for 1 can be reduced to the augmented matrix

$$\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}$$

for 2 by a sequence of elementary row operations. If we disregard the last column (all zeros) in each of these matrices, we can conclude that the reduced row echelon form of A is I_n.

$(c) \Rightarrow (d)$ Assume that the reduced row echelon form of A is I_n, so that A can be reduced to I_n by a finite sequence of elementary row operations. By Theorem 1.5.1, each of these operations can be accomplished by multiplying on the left by an appropriate elementary matrix. Thus we can find elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \cdots E_2 E_1 A = I_n \tag{3}$$

By Theorem 1.5.2, $E_1, E_2, \ldots, E_k$ are invertible. Multiplying both sides of Equation 3 on the left successively by $E_k^{-1}, \ldots, E_2^{-1}, E_1^{-1}$ we obtain

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n = E_1^{-1} E_2^{-1} \cdots E_k^{-1} \tag{4}$$

By Theorem 1.5.2, this equation expresses A as a product of elementary matrices.

$(d) \Rightarrow (a)$ If A is a product of elementary matrices, then from Theorem 1.4.7 and Theorem 1.5.2, the matrix A is a product of invertible matrices and hence is invertible.

A Method for Inverting Matrices

As a first application of Theorem 1.5.3, we will develop a procedure (or algorithm) that can be used to tell whether a given matrix is invertible, and if so, produce its inverse. To derive this algorithm, assume for the moment, that A is an invertible $n \times n$ matrix. In Equation 3, the elementary matrices execute a sequence of row operations that reduce A to I_n. If we multiply both sides of this equation on the right by A^{-1} and simplify, we obtain

$$A^{-1} = E_k \cdots E_2 E_1 I_n$$

But this equation tells us that *the same sequence of row operations that reduces A to I_n will transform I_n to A^{-1}*. Thus, we have established the following result.

Inversion Algorithm

To find the inverse of an invertible matrix A, find a sequence of elementary row operations that reduces A to the identity and then perform that same sequence of operations on I_n to obtain A^{-1}.

A simple method for carrying out this procedure is given in the following example.

EXAMPLE 4 Using Row Operations to Find A^{-1}

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Solution We want to reduce A to the identity matrix by row operations and simultaneously apply these operations to I to produce A^{-1}. To accomplish this we will adjoin the identity matrix to the right side of A, thereby producing a partitioned matrix of the form

$$[A \mid I]$$

Then we will apply row operations to this matrix until the left side is reduced to I; these operations will convert the right side to A^{-1}, so the final matrix will have the form

$$\left[I \mid A^{-1}\right]$$

The computations are as follows:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \leftarrow \text{We added } -2 \text{ times the first row to the second and } -1 \text{ times the first row to the third.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \leftarrow \text{We added 2 times the second row to the third.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \leftarrow \text{We multiplied the third row by} -1.$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \leftarrow \text{We added 3 times the third row to the second and } -3 \text{ times the third row to the first.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \leftarrow \text{We added } -2 \text{ times the second row to the first.}$$

Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Often it will not be known in advance if a given $n \times n$ matrix A is invertible. However, if it is not, then by parts (*a*) and (*c*) of Theorem 1.5.3 it will be impossible to reduce A to I_n by elementary row operations. This will be signaled by a row of zeros appearing on the *left side* of the partition at some stage of the inversion algorithm. If this occurs, then you can stop the computations and conclude that A is not invertible.

EXAMPLE 5 Showing That a Matrix Is Not Invertible

Consider the matrix

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

Applying the procedure of Example 4 yields

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right] \leftarrow \text{We added } -2 \text{ times the first row to the second and added the first row to the third.}$$

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \leftarrow \text{We added the second row to the third.}$$

Since we have obtained a row of zeros on the left side, A is not invertible.

EXAMPLE 6 Analyzing Homogeneous Systems

Use Theorem 1.5.3 to determine whether the given homogeneous system has nontrivial solutions.

(a)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 5x_2 + 3x_3 &= 0 \\ x_1 \qquad\quad + 8x_3 &= 0 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 6x_2 + 4x_3 &= 0 \\ 2x_1 + 4x_2 - x_3 &= 0 \\ -x_1 + 2x_2 + 5x_3 &= 0 \end{aligned}$$

Solution From parts (*a*) and (*b*) of Theorem 1.5.3 a homogeneous linear system has only the trivial solution if and only if its coefficient matrix is invertible. From Example 4 and Example 5 the coefficient matrix of system (a) is invertible and that of system (b) is not. Thus, system (a) has only the trivial solution whereas system (b) has nontrivial solutions.

Concept Review

- Row equivalent matrices
- Elementary matrix
- Inverse operations
- Inversion algorithm

Skills

- Determine whether a given square matrix is an elementary.
- Determine whether two square matrices are row equivalent.
- Apply the inverse of a given elementary rwo operation to a matrix.
- Apply elementary row operations to reduce a given square matrix to the identity matrix.

- Understand the relationships between statements that are equivalent to the invertibility of a square matrix (Theorem 1.5.3).
- Use the inversion algorithm to find the inverse of an invertible matrix.
- Express an invertible matrix as a product of elementary matrices.

Exercise Set 1.5

1. Decide whether each matrix below is an elementary matrix.

(a) $\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} -5 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Answer:

(a) Elementary

(b) Not elementary

(c) Not elementary

(d) Not elementary

2. Decide whether each matrix below is an elementary matrix.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

3. Find a row operation and the corresponding elementry matrix that will restore the given elementary matrix to the identity matrix.

(a) $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} -7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Answer:

(a) Add 3 times row 2 to row 1: $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

(b) Multiply row 1 by $-\frac{1}{7}$: $\begin{bmatrix} -\frac{1}{7} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) Add 5 times row 1 to row 3: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$

(d) Swap rows 1 and 3: $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4. Find a row operation and the corresponding elementry matrix that will restore the given elementary matrix to the identity matrix.

(a) $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & -\frac{1}{7} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

5. In each part, an elementary matrix *E* and a matrix *A* are given. Write down the row operation corresponding to *E* and show that the product *EA* results from applying the row operation to *A*.

(a) $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix}$

(b) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$

(c) $E = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Answer:

(a) Swap rows 1 and 2: $EA = \begin{bmatrix} 3 & -6 & -6 & -6 \\ -1 & -2 & 5 & -1 \end{bmatrix}$

(b) Add -3 times row 2 to row 3: $EA = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ -1 & 9 & 4 & -12 & -10 \end{bmatrix}$

(c) Add 4 times row 3 to row 1: $EA = \begin{bmatrix} 13 & 28 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

6. In each part, an elementary matrix *E* and a matrix *A* are given. Write down the row operation corresponding to *E* and show that the product *EA* results from applying the row operation to *A*.

(a) $E = \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix}$

(b) $E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$

(c) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

In Exercises 7–8, use the following matrices.

$$A=\begin{bmatrix}3 & 4 & 1\\ 2 & -7 & -1\\ 8 & 1 & 5\end{bmatrix},\ B=\begin{bmatrix}8 & 1 & 5\\ 2 & -7 & -1\\ 3 & 4 & 1\end{bmatrix}$$
$$C=\begin{bmatrix}3 & 4 & 1\\ 2 & -7 & -1\\ 2 & -7 & 3\end{bmatrix},\ D=\begin{bmatrix}8 & 1 & 5\\ -6 & 21 & 3\\ 3 & 4 & 1\end{bmatrix}$$
$$F=\begin{bmatrix}8 & 1 & 5\\ 8 & 1 & 1\\ 3 & 4 & 1\end{bmatrix}$$

7. Find an elementary matrix *E* that satisfies the equation.

(a) $EA=B$

(b) $EB=A$

(c) $EA=C$

(d) $EC=A$

Answer:

(a) $\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$

(b) $\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$

(c) $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -2 & 0 & 1\end{bmatrix}$

(d) $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}$

8. Find an elementary matrix *E* that satisfies the equation.

(a) $EB=D$

(b) $ED=B$

(c) $EB=F$

(d) $EF=B$

In Exercises 9–24, use the inversion algorithm to find the inverse of the given matrix, if the inverse exists.

9. $\begin{bmatrix}1 & 4\\ 2 & 7\end{bmatrix}$

Answer:

$$\begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$$

10. $\begin{bmatrix} -3 & 6 \\ 4 & 5 \end{bmatrix}$

11. $\begin{bmatrix} -1 & 3 \\ 3 & -2 \end{bmatrix}$

Answer:

$$\begin{bmatrix} \frac{2}{7} & \frac{3}{7} \\ \frac{3}{7} & \frac{1}{7} \end{bmatrix}$$

12. $\begin{bmatrix} 6 & -4 \\ -3 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$

Answer:

$$\begin{bmatrix} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}$$

14. $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

15. $\begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$

Answer:

No inverse

16. $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$

17. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Answer:

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

18. $\begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$

Answer:

$$\begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

20. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{bmatrix}$

21. $\begin{bmatrix} 2 & -4 & 0 & 0 \\ 1 & 2 & 12 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -4 & -5 \end{bmatrix}$

Answer:

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & -3 & 0 \\ -\frac{1}{8} & \frac{1}{4} & -\frac{3}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{bmatrix}$$

22. $\begin{bmatrix} -8 & 17 & 2 & \frac{1}{3} \\ 4 & 0 & \frac{2}{5} & -9 \\ 0 & 0 & 0 & 0 \\ -1 & 13 & 4 & 2 \end{bmatrix}$

23. $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & 3 & -2 & 6 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

Answer:

$$\begin{bmatrix} -\frac{7}{12} & \frac{5}{24} & \frac{5}{8} & -\frac{1}{4} \\ \frac{5}{6} & \frac{5}{12} & \frac{1}{4} & -\frac{1}{2} \\ \frac{5}{12} & \frac{5}{24} & \frac{5}{8} & -\frac{1}{4} \\ -\frac{1}{12} & -\frac{1}{24} & -\frac{1}{8} & \frac{1}{4} \end{bmatrix}$$

24. $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 5 & -3 \end{bmatrix}$

In Exercises 25–26, find the inverse of each of the following 4×4 matrices, where k_1, k_2, k_3, k_4, and k are all nonzero.

25. (a) $\begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix}$

(b) $\begin{bmatrix} k & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Answer:

(a) $\begin{bmatrix} \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 0 & \frac{1}{k_4} \end{bmatrix}$

(b) $\begin{bmatrix} \frac{1}{k} & -\frac{1}{k} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{k} & -\frac{1}{k} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

26. (a) $\begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix}$

In Exercise 27–Exercise 28, find all values of c, if any, for which the given matrix is invertible.

27. $\begin{bmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{bmatrix}$

Answer:

$c \neq 0, 1$

28. $\begin{bmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{bmatrix}$

In Exercises 29–32, write the given matrix as a product of elementary matrices.

29. $\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}$

Answer:

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

30. $\begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

Answer:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

32. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

In Exercises 33–36, write the *inverse* of the given matrix as a product of elementary matrices.

33. The matrix in Exercise 29.

Answer:

$$\begin{bmatrix} -\frac{1}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

34. The matrix in Exercise 30.

35. The matrix in Exercise 31.

Answer:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

36. The matrix in Exercise 32.

In Exercises 37–38, show that the given matrices A and B are row equivalent, and find a sequence of elementary row operations that produces B from A.

37.
$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{bmatrix}$

Answer:

Add -1 times the first row to the second row. Add -1 times the first row to the third row. Add -1 times the second row to the first row. Add the second row to the third row.

38.
$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 9 & 4 \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$

39. Show that if

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix}$$

is an elementary matrix, then at least one entry in the third row must be a zero.

40. Show that

$$A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

is not invertible for any values of the entries.

41. Prove that if A and B are $m \times n$ matrices, then A and B are row equivalent if and only if A and B have the same reduced row echelon form.

42. Prove that if A is an invertible matrix and B is row equivalent to A, then B is also invertible.

43. Show that if B is obtained from A by performing a sequence of elementary row operations, then there is a second sequence of elementary row operations, which when applied to B recovers A.

True-False Exercises

In parts (a)–(g) determine whether the statement is true or false, and justify your answer.

(a) The product of two elementary matrices of the same size must be an elementary matrix.

Answer:

False

(b) Every elementary matrix is invertible.

Answer:

True

(c) If A and B are row equivalent, and if B and C are row equivalent, then A and C are row equivalent.

Answer:

True

(d) If A is an $n \times n$ matrix that is not invertible, then the linear system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

Answer:

True

(e) If A is an $n \times n$ matrix that is not invertible, then the matrix obtained by interchanging two rows of A cannot be invertible.

Answer:

True

(f) If A is invertible and a multiple of the first row of A is added to the second row, then the resulting matrix is invertible.

Answer:

True

(g) An expression of the invertible matrix A as a product of elementary matrices is unique.

Answer:

False

1.6 More on Linear Systems and Invertible Matrics

In this section we will show how the inverse of a matrix can be used to solve a linear system and we will develop some more results about invertible matrices.

Number of Solutions of a Linear System

In Section 1.1 we made the statement (based on Figures 1.1.1 and 1.1.2) that every linear system has either no solutions, has exactly one solution, or has infinitely many solutions. We are now in a position to prove this fundamental result.

THEOREM 1.6.1

A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

Proof If $A\mathbf{x} = \mathbf{b}$ is a system of linear equations, exactly one of the following is true: (a) the system has no solutions, (b) the system has exactly one solution, or (c) the system has more than one solution. The proof will be complete if we can show that the system has infinitely many solutions in case (c).

Assume that $A\mathbf{x} = \mathbf{b}$ has more than one solution, and let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are any two distinct solutions. Because $\mathbf{x}_1$ and $\mathbf{x}_2$ are distinct, the matrix $\mathbf{x}_0$ is nonzero; moreover,

$$A\mathbf{x}_0 = A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = 0$$

If we now let k be any scalar, then

$$\begin{aligned} A(\mathbf{x}_1 + k\mathbf{x}_0) &= A\mathbf{x}_1 + A(k\mathbf{x}_0) = A\mathbf{x}_1 + k(A\mathbf{x}_0) \\ &= \mathbf{b} + k\mathbf{0} = \mathbf{b} + \mathbf{0} = \mathbf{b} \end{aligned}$$

But this says that $\mathbf{x}_1 + k\mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{b}$. Since $\mathbf{x}_0$ is nonzero and there are infinitely many choices for k, the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

Solving Linear Systems by Matrix Inversion

Thus far we have studied two *procedures* for solving linear systems–Gauss–Jordan elimination and Gaussian elimination. The following theorem provides an actual *formula* for the solution of a linear system of n equations in n unknowns in the case where the coefficient matrix is invertible.

THEOREM 1.6.2

If A is an invertible $n \times n$ matrix, then for each $n \times 1$ *matrix* $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof Since $A\left(A^{-1}\mathbf{b}\right) = \mathbf{b}$, it follows that $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution of $A\mathbf{x} = \mathbf{b}$. To show that this is the only solution, we will assume that $\mathbf{x}_0$ is an arbitrary solution and then show that $\mathbf{x}_0$ must be the solution $A^{-1}\mathbf{b}$.

If $\mathbf{x}_0$ is any solution of $A\mathbf{x} = \mathbf{b}$, then $A\mathbf{x}_0 = \mathbf{b}$. Multiplying both sides of this equation by A^{-1}, we obtain $\mathbf{x}_0 = A^{-1}\mathbf{b}$.

EXAMPLE 1 Solution of a Linear System Using A^{-1}

Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad + 8x_3 &= 17 \end{aligned}$$

In matrix form this system can be written as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

In Example 4 of the preceding section, we showed that A is invertible and

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

By Theorem 1.6.2, the solution of the system is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

or $x_1 = 1, \ x_2 = -1, \ x_3 = 2$.

Keep in mind that the method of Example 1 only applies when the system has as many equations as unknowns and the coefficient matrix is invertible.

Linear Systems with a Common Coefficient Matrix

Frequently, one is concerned with solving a sequence of systems

$$A\mathbf{x} = \mathbf{b}_1, \quad A\mathbf{x} = \mathbf{b}_2, \quad A\mathbf{x} = \mathbf{b}_3, \ldots, \quad A\mathbf{x} = \mathbf{b}_k$$

each of which has the same square coefficient matrix A. If A is invertible, then the solutions

$$\mathbf{x}_1 = A^{-1}\mathbf{b}_1, \quad \mathbf{x}_2 = A^{-1}\mathbf{b}_2, \quad \mathbf{x}_3 = A^{-1}\mathbf{b}_3, \ldots, \quad \mathbf{x}_k = A^{-1}\mathbf{b}_k$$

can be obtained with one matrix inversion and k matrix multiplications. An efficient way to do this is to form the partitioned matrix

$$[A|\mathbf{b}_1|\mathbf{b}_2|\cdot\cdot\cdot|\mathbf{b}_k|] \tag{1}$$

in which the coefficient matrix A is “augmented” by all k of the matrices $\mathbf{b}_1$, $\mathbf{b}_2, \ldots, \mathbf{b}_k$, and then reduce 1 to reduced row echelon form by Gauss-Jordan elimination. In this way we can solve all k systems at once. This method has the added advantage that it applies even when A is not invertible.

EXAMPLE 2 Solving Two Linear Systems at Once

Solve the systems

(a)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 5x_2 + 3x_3 &= 5 \\ x_1 \qquad + 8x_3 &= 9 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 3x_3 &= 6 \\ x_1 \qquad + 8x_3 &= -6 \end{aligned}$$

Solution The two systems have the same coefficient matrix. If we augment this coefficient matrix with the columns of constants on the right sides of these systems, we obtain

$$\left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 8 & 9 & -6 \end{array}\right]$$

Reducing this matrix to reduced row echelon form yields (verify)

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

It follows from the last two columns that the solution of system (a) is $x_1 = 1, x_2 = 0, x_3 = 1$ and the solution of system (b) is $x_1 = 2, x_2 = 1, x_3 = -1$.

Properties of Invertible Matrices

Up to now, to show that an $n \times n$ matrix A is invertible, it has been necessary to find an $n \times n$ matrix B such that

$$AB = I \quad \text{and} \quad BA = I$$

The next theorem shows that if we produce an $n \times n$ matrix B satisfying *either* condition, then the other condition holds automatically.

THEOREM 1.6.3

Let A be a square matrix.

(a) If B is a square matrix satisfying $BA = I$, then $B = A^{-1}$.

(b) If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$.

We will prove part (*a*) and leave part (*b*) as an exercise.

Proof (a) Assume that $BA = I$. If we can show that A is invertible, the proof can be completed by multiplying $BA = I$ on both sides by A^{-1} to obtain

$$BAA^{-1} = IA^{-1} \quad \text{or} \quad BI = IA^{-1} \quad \text{or} \quad B = A^{-1}$$

To show that A is invertible, it suffices to show that the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution (see Theorem 1.5.3). Let $\mathbf{x}_0$ be any solution of this system. If we multiply both sides of $A\mathbf{x}_0 = \mathbf{0}$ on the left by B, we obtain $BA\mathbf{x}_0 = B\mathbf{0}$ or $I\mathbf{x}_0 = \mathbf{0}$ or $\mathbf{x}_0 = \mathbf{0}$. Thus, the system of equations $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Equivalence Theorem

We are now in a position to add two more statements to the four given in Theorem 1.5.3.

THEOREM 1.6.4 Equivalent Statements

If A is an $n \times n$ matrix, then the following are equivalent.

(a) A is invertible.

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) The reduced row echelon form of A is I_n.

(d) A is expressible as a product of elementary matrices.

(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

It follows from the equivalency of parts (*e*) and (*f*) that if you can show that $A\mathbf{x} = \mathbf{b}$ has at *least one* solution for every $n \times 1$ matrix $\mathbf{b}$, then you can conclude that it has *exactly one* solution for every $n \times 1$ matrix $\mathbf{b}$.

Proof Since we proved in Theorem 1.5.3 that (*a*), (*b*), (*c*), and (*d*) are equivalent, it will be sufficient to prove that $(a) \Rightarrow (f) \Rightarrow (e) \Rightarrow (a)$.

$(a) \Rightarrow (f)$ This was already proved in Theorem 1.6.2.

$(f) \Rightarrow (e)$ This is self-evident, for if $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$, then $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

$(e) \Rightarrow (a)$ If the system $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$, then, in particular, this is so for the systems

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be solutions of the respective systems, and let us form an $n \times n$ matrix C having these solutions as columns. Thus C has the form

$$C = [\mathbf{x}_1 | \mathbf{x}_2 | \cdots | \mathbf{x}_n]$$

As discussed in Section 1.3, the successive columns of the product AC will be

$$A\mathbf{x}_1, \ A\mathbf{x}_2, \ \ldots, \ A\mathbf{x}_n$$

[see Formula 8 of Section 1.3]. Thus,

$$AC = [A\mathbf{x}_1 | A\mathbf{x}_2 | \cdots | A\mathbf{x}_n] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

By part (*b*) of Theorem 1.6.3, it follows that $C = A^{-1}$. Thus, A is invertible.

We know from earlier work that invertible matrix factors produce an invertible product. Conversely, the following theorem It shows that if the product of square matrices is invertible, then the factors themselves must be invertible.

THEOREM 1.6.5

Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.

In our later work the following fundamental problem will occur frequently in various contexts.

A Fundamental Problem

Let A be a fixed $m \times n$ matrix. Find all $m \times 1$ matrices $\mathbf{b}$ such that the system of equations $A\mathbf{x} = \mathbf{b}$ is consistent.

If A is an invertible matrix, Theorem 1.6.2 completely solves this problem by asserting that for *every* $m \times 1$ matrix $\mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$. If A is not square, or if A is square but not invertible, then Theorem 1.6.2 does not apply. In these cases the matrix $\mathbf{b}$ must usually satisfy certain conditions in order for $A\mathbf{x} = \mathbf{b}$ to be consistent. The following example illustrates how the methods of Section 1.2 can be used to determine such conditions.

EXAMPLE 3 Determining Consistency by Elimination

What conditions must b_1, b_2, and b_3 satisfy in order for the system of equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= b_1 \\ x_1 \phantom{{}+x_2} + x_3 &= b_2 \\ 2x_1 + x_2 + 3x_3 &= b_3 \end{aligned}$$

to be consistent?

Solution The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{bmatrix}$$

which can be reduced to row echelon form as follows:

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix} \quad \leftarrow \quad -1 \text{ times the first row was added to the second and } -2 \text{ times the first row was added to the third.}$$

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix} \quad \leftarrow \quad \text{The second row was multiplied by} -1.$$

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix} \quad \leftarrow \quad \text{The second row was added to the third.}$$

It is now evident from the third row in the matrix that the system has a solution if and only if b_1, b_2, and b_3 satisfy the condition

$$b_3 - b_2 - b_1 = 0 \quad \text{or} \quad b_3 = b_1 + b_2$$

To express this condition another way, $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is a matrix of the form

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_1 + b_2 \end{bmatrix}$$

where b_1 and b_2 are arbitrary.

EXAMPLE 4 Determining Consistency by Elimination

What conditions must b_1, b_2, and b_3 satisfy in order for the system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 3x_3 &= b_2 \\ x_1 \qquad\; + 8x_3 &= b_3 \end{aligned}$$

to be consistent?

Solution The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 5 & 3 & b_2 \\ 1 & 0 & 8 & b_3 \end{bmatrix}$$

Reducing this to reduced row echelon form yields (verify)

$$\begin{bmatrix} 1 & 0 & 0 & -40b_1 + 16b_2 + 9b_3 \\ 0 & 1 & 0 & 13b_1 - 5b_2 - 3b_3 \\ 0 & 0 & 1 & 5b_1 - 2b_2 - b_3 \end{bmatrix} \tag{2}$$

In this case there are no restrictions on b_1, b_2, and b_3, so the system has the unique solution

$$x_1 = -40b_1 + 16b_2 + 9b_3, \quad x_2 = 13b_1 - 5b_2 - 3b_3, \quad x_3 = 5b_1 - 2b_2 - b_3 \tag{3}$$

for all values of b_1, b_2, and b_3.

What does the result in Example 4 tell you about the coefficient matrix of the system?

Skills

- Determine whether a linear system of equations has no solutions, exactly one solution, or infinitely many solutions.
- Solve linear systems by inverting its coefficient matrix.
- Solve multiple linear systems with the same coefficient matrix simultaneously.

- Be familiar with the additional conditions of invertibility stated in the Equivalence Theorem.

Exercise Set 1.6

In Exercises 1–8, solve the system by inverting the coefficient matrix and using Theorem 1.6.2.

1. $$\begin{aligned} x_1 + x_2 &= 2 \\ 5x_1 + 6x_2 &= 9 \end{aligned}$$

Answer:

$x_1 = 3,\ x_2 = -1$

2. $$\begin{aligned} 4x_1 - 3x_2 &= -3 \\ 2x_1 - 5x_2 &= 9 \end{aligned}$$

3. $$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 3 \end{aligned}$$

Answer:

$x_1 = -1,\ x_2 = 4,\ x_3 = -7$

4. $$\begin{aligned} 5x_1 + 3x_2 + 2x_3 &= 4 \\ 3x_1 + 3x_2 + 2x_3 &= 2 \\ x_2 + x_3 &= 5 \end{aligned}$$

5. $$\begin{aligned} x + y + z &= 5 \\ x + y - 4z &= 10 \\ -4x + y + z &= 0 \end{aligned}$$

Answer:

$x = 1,\ x = 5,\ x = -1$

6. $$\begin{aligned} - x - 2y - 3z &= 0 \\ w + x + 4y + 4z &= 7 \\ w + 3x + 7y + 9z &= 4 \\ -w - 2x - 4y - 6z &= 6 \end{aligned}$$

7. $$\begin{aligned} 3x_1 + 5x_2 &= b_1 \\ x_1 + 2x_2 &= b_2 \end{aligned}$$

Answer:

$x_1 = 2b_1 - 5b_2,\ x_2 = -b_1 + 3b_2$

8. $$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 5x_3 &= b_2 \\ 3x_1 + 5x_2 + 8x_3 &= b_3 \end{aligned}$$

In Exercises 9–12, solve the linear systems together by reducing the appropriate augmented matrix.

9. $$\begin{aligned} x_1 - 5x_2 &= b_1 \\ 3x_1 + 2x_2 &= b_2 \end{aligned}$$

(i) $b_1 = 1,\ \ b_2 = 4$

(ii) $b_1 = -2,\ \ b_2 = 5$

Answer:

(i) $x_1 = \frac{22}{17},\ x_2 = \frac{1}{17}$

(ii) $x_1 = \frac{21}{17},\ x_2 = \frac{11}{17}$

10.
$$\begin{aligned} -x_1+4x_2+\ x_3 &= b_1 \\ x_1+9x_2-\ 2x_3 &= b_2 \\ 6x_1+4x_2-\ 8x_3 &= b_3 \end{aligned}$$

(i) $b_1=0,\ \ b_2=1,\ \ b_3=0$

(ii) $b_1=-3,\ \ b_2=4,\ \ b_3=-5$

11.
$$\begin{aligned} 4x_1-7x_2&=b_1 \\ x_1+2x_2&=b_2 \end{aligned}$$

(i) $b_1=0,\ \ b_2=1$

(ii) $b_1=-4,\ \ b_2=6$

(iii) $b_1=-1,\ \ b_2=3$

(iv) $b_1=-5,\ \ b_2=1$

Answer:

(i) $x_1=\frac{7}{15},\ x_2=\frac{4}{15}$

(ii) $x_1=\frac{34}{15},\ x_2=\frac{28}{15}$

(iii) $x_1=\frac{19}{15},\ x_2=\frac{13}{15}$

(iv) $x_1=-\frac{1}{5},\ x_2=\frac{3}{5}$

12.
$$\begin{aligned} x_1+3x_2+5x_3&=b_1 \\ -x_1-\ 2x_2\qquad\ &=b_2 \\ 2x_1+5x_2+4x_3&=b_3 \end{aligned}$$

(i) $b_1=1,\ \ b_2=0,\ \ b_3=-1$

(ii) $b_1=0,\ \ b_2=1,\ \ b_3=1$

(iii) $b_1=-1,\ \ b_2=-1,\ \ b_3=0$

In Exercises 13–17, determine conditions on the b_i's, if any, in order to guarantee that the linear system is consistent.

13.
$$\begin{aligned} x_1+3x_2&=b_1 \\ -2x_1+\ x_2&=b_2 \end{aligned}$$

Answer:

No conditions on b_1 and b_2

14.
$$\begin{aligned} 6x_1-4x_2&=b_1 \\ 3x_1-2x_2&=b_2 \end{aligned}$$

15.
$$\begin{aligned} x_1-2x_2+5x_3 &= b_1 \\ 4x_1-5x_2+8x_3 &= b_2 \\ -3x_1+3x_2-3x_3 &= b_3 \end{aligned}$$

Answer:

$b_3=b_1-b_2$

16.
$$\begin{aligned} x_1-2x_2-\ x_3 &= b_1 \\ -4x_1+5x_2+2x_3 &= b_2 \\ -4x_1+7x_2+4x_3 &= b_3 \end{aligned}$$

17.
$$\begin{aligned} x_1-\ x_2+3x_3+2x_4 &= b_1 \\ -2x_1+\ x_2+5x_3+\ x_4 &= b_2 \\ -3x_1+2x_2+2x_3-\ x_4 &= b_3 \\ 4x_1-3x_2+\ x_3+3x_4 &= b_4 \end{aligned}$$

Answer:

$b_1=b_3+b_4,\ b_2=2b_3+b_4$

18. Consider the matrices

$$A=\begin{bmatrix}2 & 1 & 2\\ 2 & 2 & -2\\ 3 & 1 & 1\end{bmatrix} \text{ and } \mathrm{x}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

(a) Show that the equation $A\mathbf{x}=\mathbf{x}$ can be rewritten as $(A-I)\mathbf{x}=\mathbf{0}$ and use this result to solve $A\mathbf{x}=\mathbf{x}$ for $\mathbf{x}$.

(b) Solve $A\mathbf{x}=4\mathbf{x}$.

In Exercises 19–20, solve the given matrix equation for X.

19. $\begin{bmatrix}1 & -1 & 1\\ 2 & 3 & 0\\ 0 & 2 & -1\end{bmatrix}X=\begin{bmatrix}2 & -1 & 5 & 7 & 8\\ 4 & 0 & -3 & 0 & 1\\ 3 & 5 & -7 & 2 & 1\end{bmatrix}$

Answer:

$$X=\begin{bmatrix}11 & 12 & -3 & 27 & 26\\ -6 & -8 & 1 & -18 & -17\\ -15 & -21 & 9 & -38 & -35\end{bmatrix}$$

20. $\begin{bmatrix}-2 & 0 & 1\\ 0 & -1 & -1\\ 1 & 1 & -4\end{bmatrix}X=\begin{bmatrix}4 & 3 & 2 & 1\\ 6 & 7 & 8 & 9\\ 1 & 3 & 7 & 9\end{bmatrix}$

21. Let $A\mathbf{x}=\mathbf{0}$ be a homogeneous system of n linear equations in n unknowns that has only the trivial solution. Show that if k is any positive integer, then the system $A^k\mathbf{x}=\mathbf{0}$ also has only the trivial solution.

22. Let $A\mathbf{x}=\mathbf{0}$ be a homogeneous system of n linear equations in n unknowns, and let Q be an invertible $n\times n$ matrix. Show that $A\mathbf{x}=\mathbf{0}$ has just the trivial solution if and only if $(QA)\mathbf{x}=\mathbf{0}$ has just the trivial solution.

23. Let $A\mathbf{x}=\mathbf{b}$ be any consistent system of linear equations, and let $\mathbf{x}_1$ be a fixed solution. Show that every solution to the system can be written in the form $\mathbf{x}=\mathbf{x}_1+\mathbf{x}_0$, where $\mathbf{x}_0$ is a solution to $A\mathbf{x}=\mathbf{0}$. Show also that every matrix of this form is a solution.

24. Use part *(a)* of Theorem 1.6.3 to prove part *(b)*.

True-False Exercises

In parts (a)–(g) determine whether the statement is true or false, and justify your answer.

(a) It is impossible for a linear system of linear equations to have exactly two solutions.

Answer:

True

(b) If the linear system $A\mathbf{x}=\mathbf{b}$ has a unique solution, then the linear system $A\mathbf{x}=\mathbf{c}$ also must have a unique solution.

Answer:

True

(c) If A and B are $n\times n$ matrices such that $AB=I_n$, then $BA=I_n$.

Answer:

True

(d) If A and B are row equivalent matrices, then the linear systems $A\mathbf{x}=\mathbf{0}$ and $B\mathbf{x}=\mathbf{0}$ have the same solution set.

Answer:

True

(e) If A is an $n\times n$ matrix and S is an $n\times n$ invertible matrix, then if $\mathbf{x}$ is a solution to the linear system $(S^{-1}AS)\mathbf{x}=\mathbf{b}$, then $S\mathbf{x}$ is a solution to the linear system $A\mathbf{y}=S\mathbf{b}$.

Answer:

True

(f) Let *A* be an $n \times n$ matrix. The linear system $A\mathbf{x} = 4\mathbf{x}$ has a unique solution if and only if $A - 4I$ is an invertible matrix.

Answer:

True

(g) Let *A* and *B* be $n \times n$ matrices. If *A* or *B* (or both) are not invertible, then neither is *AB*.

Answer:

True

1.7 Diagonal, Triangular, and Symmetric Matrices

In this section we will discuss matrices that have various special forms. These matrices arise in a wide variety of applications and will also play an important role in our subsequent work.

Diagonal Matrices

A square matrix in which all the entries off the main diagonal are zero is called a *diagonal matrix*. Here are some examples:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

A general $n \times n$ diagonal matrix D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & d_n \end{bmatrix} \tag{1}$$

A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero; in this case the inverse of 1 is

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \ldots & 0 \\ 0 & 1/d_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1/d_n \end{bmatrix} \tag{2}$$

Confirm Formula 2 by showing that

$$DD^{-1} = D^{-1}D = I$$

Powers of diagonal matrices are easy to compute; we leave it for you to verify that if D is the diagonal matrix 1 and k is a positive integer, then

$$D^k = \begin{bmatrix} d_1^k & 0 & \ldots & 0 \\ 0 & d_2^k & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & d_n^k \end{bmatrix} \tag{3}$$

EXAMPLE 1 Inverses and Powers of Diagonal Matrices

If

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

then

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -243 & 0 \\ 0 & 0 & 32 \end{bmatrix}, \quad A^{-5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{243} & 0 \\ 0 & 0 & \frac{1}{32} \end{bmatrix}$$

Matrix products that involve diagonal factors are especially easy to compute. For example,

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & d_1a_{13} & d_1a_{14} \\ d_2a_{21} & d_2a_{22} & d_2a_{23} & d_2a_{24} \\ d_3a_{31} & d_3a_{32} & d_3a_{33} & d_3a_{34} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_2a_{12} & d_3a_{13} \\ d_1a_{21} & d_2a_{22} & d_3a_{23} \\ d_1a_{31} & d_2a_{32} & d_3a_{33} \\ d_1a_{41} & d_2a_{42} & d_3a_{43} \end{bmatrix}$$

In words, to multiply a matrix A on the left by a diagonal matrix D, one can multiply successive rows of A by the successive diagonal entries of D, and to multiply A on the right by D, one can multiply successive columns of A by the successive diagonal entries of D.

Triangular Matrices

A square matrix in which all the entries above the main diagonal are zero is called ***lower triangular***, and a square matrix in which all the entries below the main diagonal are zero is called ***upper triangular***. A matrix that is either upper triangular or lower triangular is called ***triangular***.

EXAMPLE 2 Upper and Lower Triangular Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

A general 4 × 4 upper triangular matrix

A general 4 × 4 lower triangular matrix

Remark Observe that diagonal matrices are both upper triangular and lower triangular since they have zeros below and above the main diagonal. Observe also that a *square* matrix in row echelon form is upper triangular since it has zeros below the main diagonal.

Properties of Triangular Matrices

Example 2 illustrates the following four facts about triangular matrices that we will state without formal proof.

- A square matrix $A = [a_{ij}]$ is upper triangular if and only if all entries to the left of the main diagonal are zero; that is, $a_{ij} = 0$ if $i > j$ (Figure 1.7.1).
- A square matrix $A = [a_{ij}]$ is lower triangular if and only if all entries to the right of the main diagonal are zero; that is, $a_{ij} = 0$ if $i < j$ (Figure 1.7.1).
- A square matrix $A = [a_{ij}]$ is upper triangular if and only if the *i*th row starts with at least $i - 1$ zeros for every *i*.
- A square matrix $A = [a_{ij}]$ is lower triangular if and only if the *jth* column starts with at least $j - 1$ zeros for every *j*.

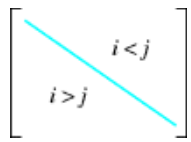


Figure 1.7.1

The following theorem lists some of the basic properties of triangular matrices.

THEOREM 1.7.1

(a) The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.

(b) The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.

(c) A triangular matrix is invertible if and only if its diagonal entries are all nonzero.

(d) The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

Part *(a)* is evident from the fact that transposing a square matrix can be accomplished by reflecting the entries about the main diagonal; we omit the formal proof. We will prove *(b)*, but we will defer the proofs of *(c)* and *(d)* to the next chapter, where we will have the tools to prove those results more efficiently.

Proof (b) We will prove the result for lower triangular matrices; the proof for upper triangular matrices is similar. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be lower triangular $n \times n$ matrices, and let $C = [c_{ij}]$ be the product $C = AB$. We can prove that C is lower triangular by showing that $c_{ij} = 0$ for $i < j$. But from the definition of matrix multiplication,

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

If we assume that $i < j$, then the terms in this expression can be grouped as follows:

$$c_{ij} = \underbrace{a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{i(j-1)}b_{(j-1)j}}_{\substack{\textbf{Terms in which the row} \\ \textbf{number of } \boldsymbol{b} \textbf{ is less than the} \\ \textbf{column number of } \boldsymbol{b}}} + \underbrace{a_{ij}b_{jj} + \cdots + a_{in}b_{nj}}_{\substack{\textbf{Terms in which the row} \\ \textbf{number of } \boldsymbol{a} \textbf{ is less than} \\ \textbf{the column number of } \boldsymbol{a}}}$$

In the first grouping all of the *b* factors are zero since *B* is lower triangular, and in the second grouping all of the *a* factors are zero since *A* is lower triangular. Thus, $c_{ij} = 0$, which is what we wanted to prove.

EXAMPLE 3 Computations with Triangular Matrices

Consider the upper triangular matrices

$$A=\begin{bmatrix}1 & 3 & -1\\ 0 & 2 & 4\\ 0 & 0 & 5\end{bmatrix},\quad B=\begin{bmatrix}3 & -2 & 2\\ 0 & 0 & -1\\ 0 & 0 & 1\end{bmatrix}$$

It follows from part (*c*) of Theorem 1.7.1 that the matrix A is invertible but the matrix B is not. Moreover, the theorem also tells us that A^{-1}, AB, and BA must be upper triangular. We leave it for you to confirm these three statements by showing that

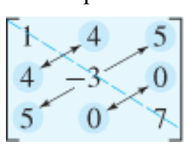
$$A^{-1}=\begin{bmatrix}1 & -\frac{3}{2} & \frac{7}{5}\\ 0 & \frac{1}{2} & -\frac{2}{5}\\ 0 & 0 & \frac{1}{5}\end{bmatrix},\quad AB=\begin{bmatrix}3 & -2 & -2\\ 0 & 0 & 2\\ 0 & 0 & 5\end{bmatrix},\quad BA=\begin{bmatrix}3 & 5 & -1\\ 0 & 0 & -5\\ 0 & 0 & 5\end{bmatrix}$$

Symmetric Matrices

DEFINITION 1

A square matrix A is said to be ***symmetric*** if $A=A^T$.

It is easy to recognize a symmetric matrix by inspection: The entries on the main diagonal have no restrictions, but mirror images of entries *across* the main diagonal must be equal. Here is a picture using the second matrix in Example 4:



All diagonal matrices, such as the third matrix in Example 4, obviously have this property.

EXAMPLE 4 Symmetric Matrices

The following matrices are symmetric, since each is equal to its own transpose (verify).

$$\begin{bmatrix}7 & -3\\ -3 & 5\end{bmatrix},\quad \begin{bmatrix}1 & 4 & 5\\ 4 & -3 & 0\\ 5 & 0 & 7\end{bmatrix},\quad \begin{bmatrix}d_1 & 0 & 0 & 0\\ 0 & d_2 & 0 & 0\\ 0 & 0 & d_3 & 0\\ 0 & 0 & 0 & d_4\end{bmatrix}$$

Remark It follows from Formula 11 of Section 1.3 that a square matrix $A = [a_{ij}]$ is symmetric if and only if

$$(A)_{ij} = (A)_{ji} \tag{4}$$

for all values of *i* and *j*.

The following theorem lists the main algebraic properties of symmetric matrices. The proofs are direct consequences of Theorem 1.4.8 and are omitted.

THEOREM 1.7.2

If A and B are symmetric matrices with the same size, and if k is any scalar, then:

(a) A^T is symmetric.

(b) $A + B$ and $A - B$ are symmetric.

(c) kA is symmetric.

It is not true, in general, that the product of symmetric matrices is symmetric. To see why this is so, let A and B be symmetric matrices with the same size. Then it follows from part (*e*) of Theorem 1.4.8 and the symmetry of A and B that

$$(AB)^T = B^T A^T = BA$$

Thus, $(AB)^T = AB$ if and only if $AB = BA$, that is, if and only if A and B commute. In summary, we have the following result.

THEOREM 1.7.3

The product of two symmetric matrices is symmetric if and only if the matrices commute.

EXAMPLE 5 Products of Symmetric Matrices

The first of the following equations shows a product of symmetric matrices that *is not* symmetric, and the second shows a product of symmetric matrices that *is* symmetric. We conclude that the factors in the first equation do not commute, but those in the second equation do. We leave it for you to verify that this is so.

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

Invertibility of Symmetric Matrices

In general, a symmetric matrix need not be invertible. For example, a diagonal matrix with a zero on the main diagonal is symmetric but not invertible. However, the following theorem shows that if a symmetric matrix happens to be invertible, then its inverse must also be symmetric.

THEOREM 1.7.4

If A is an invertible symmetric matrix, then A^{-1} is symmetric.

Proof Assume that A is symmetric and invertible. From Theorem 1.4.9 and the fact that $A = A^T$, we have

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1} = A^{-1}$$

which proves that A^{-1} is symmetric.

Products AA^T and A^TA

Matrix products of the form AA^T and A^TA arise in a variety of applications. If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix, so the products AA^T and A^TA are both square matrices—the matrix AA^T has size $m \times m$, and the matrix A^TA has size $n \times n$. Such products are always symmetric since

$$\left(AA^T\right)^T = \left(A^T\right)^T A^T = AA^T \quad \text{and} \quad \left(A^TA\right)^T = A^T\left(A^T\right)^T = A^TA$$

EXAMPLE 6 The Product of a Matrix and Its Transpose Is Symmetric

Let A be the 2×3 matrix

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$$

Then

$$A^TA = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}$$

Observe that A^TA and AA^T are symmetric as expected.

Later in this text, we will obtain general conditions on A under which AA^T and A^TA are invertible. However, in the special case where A is *square*, we have the following result.

THEOREM 1.7.5

If A is an invertible matrix, then AA^T and A^TA are also invertible.

Proof Since A is invertible, so is A^T by Theorem 1.4.9. Thus AA^T and A^TA are invertible, since they are the products of invertible matrices.

Concept Review

- Diagonal matrix
- Lower triangular matrix
- Upper triangular matrix
- Triangular matrix
- Symmetric matrix

Skills

- Determine whether a diagonal matrix is invertible with no computations.
- Compute matrix products involving diagonal matrices by inspection.
- Determine whether a matrix is triangular.
- Understand how the transpose operation affects diagonal and triangular matrices.
- Understand how inversion affects diagonal and triangular matrices.
- Determine whether a matrix is a symmetric matrix.

Exercise Set 1.7

In Exercises 1–4, determine whether the given matrix is invertible.

1. $\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$

Answer:

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{5} \end{bmatrix}$$

2. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

3. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

Answer:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

4. $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

In Exercises 5–8, determine the product by inspection.

5. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -4 & 1 \\ 2 & 5 \end{bmatrix}$

Answer:

$$\begin{bmatrix} 6 & 3 \\ 4 & -1 \\ 4 & 10 \end{bmatrix}$$

6. $\begin{bmatrix} 1 & 2 & -5 \\ -3 & -1 & 0 \end{bmatrix}\begin{bmatrix} -4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

7. $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} -3 & 2 & 0 & 4 & -4 \\ 1 & -5 & 3 & 0 & 3 \\ -6 & 2 & 2 & 2 & 2 \end{bmatrix}$

Answer:

$$\begin{bmatrix} -15 & 10 & 0 & 20 & -20 \\ 2 & -10 & 6 & 0 & 6 \\ 18 & -6 & -6 & -6 & -6 \end{bmatrix}$$

8. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 4 & -1 & 3 \\ 1 & 2 & 0 \\ -5 & 1 & -2 \end{bmatrix}\begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

In Exercises 9–12, find A^2, A^{-2}, and A^{-k} (where k is any integer) by inspection.

9. $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

Answer:

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix},\ A^{-2} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix},\ A^{-k} = \begin{bmatrix} 1 & 0 \\ 0 & 1/(-2)^k \end{bmatrix}$$

10. $A = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

11. $A=\begin{bmatrix}\frac{1}{2} & 0 & 0\\ 0 & \frac{1}{3} & 0\\ 0 & 0 & \frac{1}{4}\end{bmatrix}$

Answer:

$$A^2=\begin{bmatrix}\frac{1}{4} & 0 & 0\\ 0 & \frac{1}{9} & 0\\ 0 & 0 & \frac{1}{16}\end{bmatrix},\ A^{-2}=\begin{bmatrix}4 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 16\end{bmatrix},\ A^{-k}=\begin{bmatrix}2^k & 0 & 0\\ 0 & 3^k & 0\\ 0 & 0 & 4^k\end{bmatrix}$$

12. $A=\begin{bmatrix}-2 & 0 & 0 & 0\\ 0 & -4 & 0 & 0\\ 0 & 0 & -3 & 0\\ 0 & 0 & 0 & 2\end{bmatrix}$

In Exercises 13–19, decide whether the given matrix is symmetric.

13. $\begin{bmatrix}-8 & -8\\ 0 & 0\end{bmatrix}$

Answer:

Not symmetric

14. $\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}$

15. $\begin{bmatrix}0 & -7\\ -7 & 7\end{bmatrix}$

Answer:

Symmetric

16. $\begin{bmatrix}3 & 4\\ 4 & 0\end{bmatrix}$

17. $\begin{bmatrix}0 & 1 & 2\\ 1 & 5 & -6\\ 2 & 6 & 6\end{bmatrix}$

Answer:

Not symmetric

18. $\begin{bmatrix}2 & -1 & 3\\ -1 & 5 & 1\\ 3 & 1 & 7\end{bmatrix}$

19. $\begin{bmatrix}0 & 0 & 1\\ 0 & 2 & 0\\ 3 & 0 & 0\end{bmatrix}$

Answer:

Not symmetric

In Exercises 20–22, decide by inspection whether the given matrix is invertible.

20. $\begin{bmatrix} -1 & 2 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

21. $\begin{bmatrix} 0 & 1 & -2 & 5 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Answer:

Not invertible

22. $\begin{bmatrix} 2 & 0 & 0 & 0 \\ -3 & -1 & 0 & 0 \\ -4 & -6 & 0 & 0 \\ 0 & 3 & 8 & -5 \end{bmatrix}$

In Exercises 23–24, find all values of the unknown constant(s) in order for A to be symmetric.

23. $A = \begin{bmatrix} 4 & -3 \\ a+5 & -1 \end{bmatrix}$

Answer:

$a = -8$

24. $A = \begin{bmatrix} 2 & a-2b+2c & 2a+b+c \\ 3 & 5 & a+c \\ 0 & -2 & 7 \end{bmatrix}$

In Exercises 25–26, find all values of x in order for A to be invertible.

25. $A = \begin{bmatrix} x-1 & x^2 & x^4 \\ 0 & x+2 & x^3 \\ 0 & 0 & x-4 \end{bmatrix}$

Answer:

$x \neq 1, \ -2, 4$

26. $A = \begin{bmatrix} x-\frac{1}{2} & 0 & 0 \\ x & x-\frac{1}{3} & 0 \\ x^2 & x^3 & x-\frac{1}{4} \end{bmatrix}$

In Exercises 27–28, find a diagonal matrix A that satisfies the given condition.

27. $A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Answer:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

28. $$A^{-2} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

29. Verify Theorem 1.7.1(*b*) for the product *AB*, where

$$A = \begin{bmatrix} -1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -8 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

30. Verify Theorem 1.7.1(*d*) for the matrices *A* and *B* in Exercise 29.

31. Verify Theorem 1.7.4 for the given matrix *A*.

(a) $A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$,

(b) $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 1 & -7 \\ 3 & -7 & 4 \end{bmatrix}$

32. Let *A* be an $n \times n$ symmetric matrix.

(a) Show that A^2 is symmetric.

(b) Show that $2A^2 - 3A + I$ is symmetric.

33. Prove: If $A^T A = A$, then *A* is symmetric and $A = A^2$.

34. Find all 3×3 diagonal matrices *A* that satisfy $A^2 - 3A - 4I = 0$.

35. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Determine whether *A* is symmetric.

(a) $a_{ij} = i^2 + j^2$

(b) $a_{ij} = i^2 - j^2$

(c) $a_{ij} = 2i + 2j$

(d) $a_{ij} = 2i^2 + 2j^3$

Answer:

(a) Yes

(b) No (unless $n = 1$)

(c) Yes

(d) No (unless $n = 1$)

36. On the basis of your experience with Exercise 35, devise a general test that can be applied to a formula for a_{ij} to determine whether $A = [a_{ij}]$ is symmetric.

37. A square matrix *A* is called ***skew-symmetric*** if $A^T = -A$.

Prove:

(a) If *A* is an invertible skew-symmetric matrix, then A^{-1} is skew-symmetric.

(b) If *A* and *B* are skew-symmetric matrices, then so are A^T, $A + B$, $A - B$, and kA for any scalar *k*.

(c) Every square matrix A can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix. [*Hint:* Note the identity $A=\frac{1}{2}\left(A+A^{T}\right)+\frac{1}{2}\left(A-A^{T}\right)$.]

In Exercises 38–39, fill in the missing entries (marked with ×) to produce a skew-symmetric matrix.

38. $A=\begin{bmatrix} \times & \times & 4 \\ 0 & \times & \times \\ \times & -1 & \times \end{bmatrix}$

39. $A=\begin{bmatrix} \times & 0 & \times \\ \times & \times & -4 \\ 8 & \times & \times \end{bmatrix}$

Answer:

$$\begin{bmatrix} 0 & 0 & -8 \\ 0 & 0 & -4 \\ 8 & 4 & 0 \end{bmatrix}$$

40. Find all values of a, b, c, and d for which A is skew-symmetric.

$$A=\begin{bmatrix} 0 & 2a-3b+c & 3a-5b+5c \\ -2 & 0 & 5a-8b+6c \\ -3 & -5 & d \end{bmatrix}$$

41. We showed in the text that the product of symmetric matrices is symmetric if and only if the matrices commute. Is the product of commuting skew-symmetric matrices skew- symmetric? Explain. [*Note:* See Exercise 37 for the deffinition of *skew-symmetric*.]

42. If the $n\times n$ matrix A can be expressed as $A=LU$, where L is a lower triangular matrix and U is an upper triangular matrix, then the linear system $A\mathbf{x}=\mathbf{b}$ can be expressed as $LU\mathbf{x}=\mathbf{b}$ and can be solved in two steps:

Step 1. Let $U\mathbf{x}=\mathbf{y}$, so that $LU\mathbf{x}=\mathbf{b}$ can be expressed as $L\mathbf{y}=\mathbf{b}$. Solve this system.

Step 2. Solve the system $U\mathbf{x}=\mathbf{y}$ for $\mathbf{x}$.

In each part, use this two-step method to solve the given system.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ -3 & -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -5 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}$

43. Find an upper triangular matrix that satisfies

$$A^3=\begin{bmatrix} 1 & 30 \\ 0 & -8 \end{bmatrix}$$

Answer:

$$A=\begin{bmatrix} 1 & 10 \\ 0 & -2 \end{bmatrix}$$

True-False Exercises

In parts (a)–(m) determine whether the statement is true or false, and justify your answer.

(a) The transpose of a diagonal matrix is a diagonal matrix.

Answer:

True

(b) The transpose of an upper triangular matrix is an upper triangular matrix.

Answer:

False

(c) The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix.

Answer:

False

(d) All entries of a symmetric matrix are determined by the entries occurring on and above the main diagonal.

Answer:

True

(e) All entries of an upper triangular matrix are determined by the entries occurring on and above the main diagonal.

Answer:

True

(f) The inverse of an invertible lower triangular matrix is an upper triangular matrix.

Answer:

False

(g) A diagonal matrix is invertible if and only if all of its diagonal entries are positive.

Answer:

False

(h) The sum of a diagonal matrix and a lower triangular matrix is a lower triangular matrix.

Answer:

True

(i) A matrix that is both symmetric and upper triangular must be a diagonal matrix.

Answer:

True

(j) If A and B are $n \times n$ matrices such that $A + B$ is symmetric, then A and B are symmetric.

Answer:

False

(k) If A and B are $n \times n$ matrices such that $A + B$ is upper triangular, then A and B are upper triangular.

Answer:

False

(l) If A^2 is a symmetric matrix, then A is a symmetric matrix.

Answer:

False

(m) If kA is a symmetric matrix for some $k \neq 0$, then A is a symmetric matrix.

Answer:

True