CHAPTER

1 Systems of Linear Equations and Matrices

CHAPTER CONTENTS

INTRODUCTION

Information in science, business, and mathematics is often organized into rows and columns to form rectangular arrays called “matrices” (plural of “matrix”). Matrices often appear as tables of numerical data that arise from physical observations, but they occur in various mathematical contexts as well. For example, we will see in this chapter that all of the information required to solve a system of equations such as

$$\begin{aligned} 5x + y &= 3 \\ 2x - y &= 4 \end{aligned}$$

is embodied in the matrix

$$\begin{bmatrix} 5 & 1 & 3 \\ 2 & -1 & 4 \end{bmatrix}$$

and that the solution of the system can be obtained by performing appropriate operations on this matrix. This is particularly important in developing computer programs for solving systems of equations because computers are well suited for manipulating arrays of numerical information. However, matrices are not simply a notational tool for solving systems of equations; they can be viewed as mathematical objects in their own right, and there is a rich and important theory associated with them that has a multitude of practical applications. It is the study of matrices and related topics that forms the mathematical field that we call “linear algebra.” In this chapter we will begin our study of matrices.

1.1 Introduction to Systems of Linear Equations

Systems of linear equations and their solutions constitute one of the major topics that we will study in this course. In this first section we will introduce some basic terminology and discuss a method for solving such systems.

Linear Equations

Recall that in two dimensions a line in a rectangular *xy*-coordinate system can be represented by an equation of the form

$$ax + by = c \quad (a,\ b \text{ not both } 0)$$

and in three dimensions a plane in a rectangular *xyz*-coordinate system can be represented by an equation of the form

$$ax + by + cz = d \quad (a,\ b,\ c \text{ not all } 0)$$

These are examples of "linear equations," the first being a linear equation in the variables *x* and *y* and the second a linear equation in the variables *x*, *y*, and *z*. More generally, we define a ***linear equation*** in the *n* variables $x_1, x_2, \ldots, x_n$ to be one that can be expressed in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b \tag{1}$$

where $a_1,\ a_2, \ldots,\ a_n$ and *b* are constants, and the *a*'s are not all zero. In the special cases where $n = 2$ or $n = 3$, we will often use variables without subscripts and write linear equations as

$$a_1x + a_2y = b \quad (a_1, a_2 \text{ not both } 0) \tag{2}$$

$$a_1x + a_2y + a_3z = b \quad (a_1,\ a_2,\ a_3 \text{ not all } 0) \tag{3}$$

In the special case where $b = 0$, Equation 1 has the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0 \tag{4}$$

which is called a ***homogeneous linear equation*** in the variables $x_1,\ x_2, \ldots, x_n$.

EXAMPLE 1 Linear Equations

Observe that a linear equation does not involve any products or roots of variables. All variables occur only to the first power and do not appear, for example, as arguments of trigonometric, logarithmic, or exponential functions. The following are linear equations:

$$x + 3y = 7 \qquad x_1 - 2x_2 - 3x_3 + x_4 = 0$$

$$\frac{1}{2}x - y + 3z = -1 \quad x_1 + x_2 + \ldots + x_n = 1$$

The following are not linear equations:

$$x+3y^2=4 \quad 3x+2y-xy=5$$
$$\sin x+y=0 \quad \sqrt{x_1}+2x_2+x_3=1$$

A finite set of linear equations is called a ***system of linear equations*** or, more briefly, a ***linear system***. The variables are called ***unknowns.*** For example, system 5 that follows has unknowns *x* and *y*, and system 6 has unknowns x_1,x_2, and x_3.

$$5x+y=3 \quad 4x_1-x_2+3x_3=-1 \tag{5}$$

$$2x-y=4 \quad 3x_1+x_2+9x_3=-4 \tag{6}$$

The double subscripting on the coefficients a_{ij} of the unknowns gives their location in the system—the first subscript indicates the equation in which the coefficient occurs, and the second indicates which unknown it multplies. Thus, a_{12} is in the first equation and multiplies x_2.

A general linear system of *m* equations in the *n* unknowns $x_1,\ x_2, \ldots, x_n$ can be written as

$$\begin{array}{c} a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=b_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n=b_m \end{array} \tag{7}$$

A ***solution*** of a linear system in *n* unknowns $x_1,\ x_2, \ldots, x_n$ is a sequence of *n* numbers $s_1,\ s_2, \ldots, x_n$ for which the substitution

$$x_1=s_1,\ x_2=s_2, \ldots,\ x_n=s_n$$

makes each equation a true statement. For example, the system in 5 has the solution

$$x=1,\ y=-2$$

and the system in 6 has the solution

$$x_1=1,\ x_2=2,\ x_3=-1$$

These solutions can be written more succinctly as

$$(1,\ -2) \text{ and } (1,2,\ -1)$$

in which the names of the variables are omitted. This notation allows us to interpret these solutions geometrically as points in two-dimensional and three-dimensional space. More generally, a solution

$$x_1=s_1,\ x_2=s_2, \ldots,\ x_n=s_n$$

of a linear system in *n* unknowns can be written as

$$(s_1,\ s_2, \ldots, s_n)$$

which is called an ***ordered n-tuple***. With this notation it is understood that all variables appear in the same order in each equation. If $n = 2$, then the n-tuple is called an ***ordered pair***, and if $n = 3$, then it is called an ***ordered triple***.

Linear Systems with Two and Three Unknowns

Linear systems in two unknowns arise in connection with intersections of lines. For example, consider the linear system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

in which the graphs of the equations are lines in the xy-plane. Each solution (x, y) of this system corresponds to a point of intersection of the lines, so there are three possibilities (Figure 1.1.1):

1. The lines may be parallel and distinct, in which case there is no intersection and consequently no solution.
2. The lines may intersect at only one point, in which case the system has exactly one solution.
3. The lines may coincide, in which case there are infinitely many points of intersection (the points on the common line) and consequently infinitely many solutions.

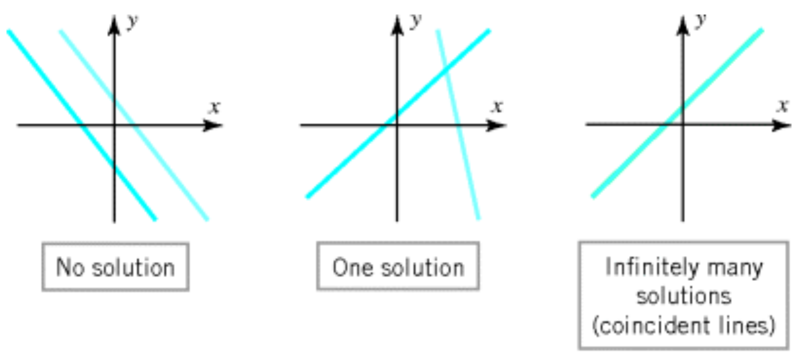


Figure 1.1.1

In general, we say that a linear system is ***consistent*** if it has at least one solution and ***inconsistent*** if it has no solutions. Thus, a *consistent* linear system of two equations in two unknowns has either one solution or infinitely many solutions—there are no other possibilities. The same is true for a linear system of three equations in three unknowns

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

in which the graphs of the equations are planes. The solutions of the system, if any, correspond to points where all three planes intersect, so again we see that there are only three possibilities—no solutions, one solution, or infinitely many solutions (Figure 1.1.2).

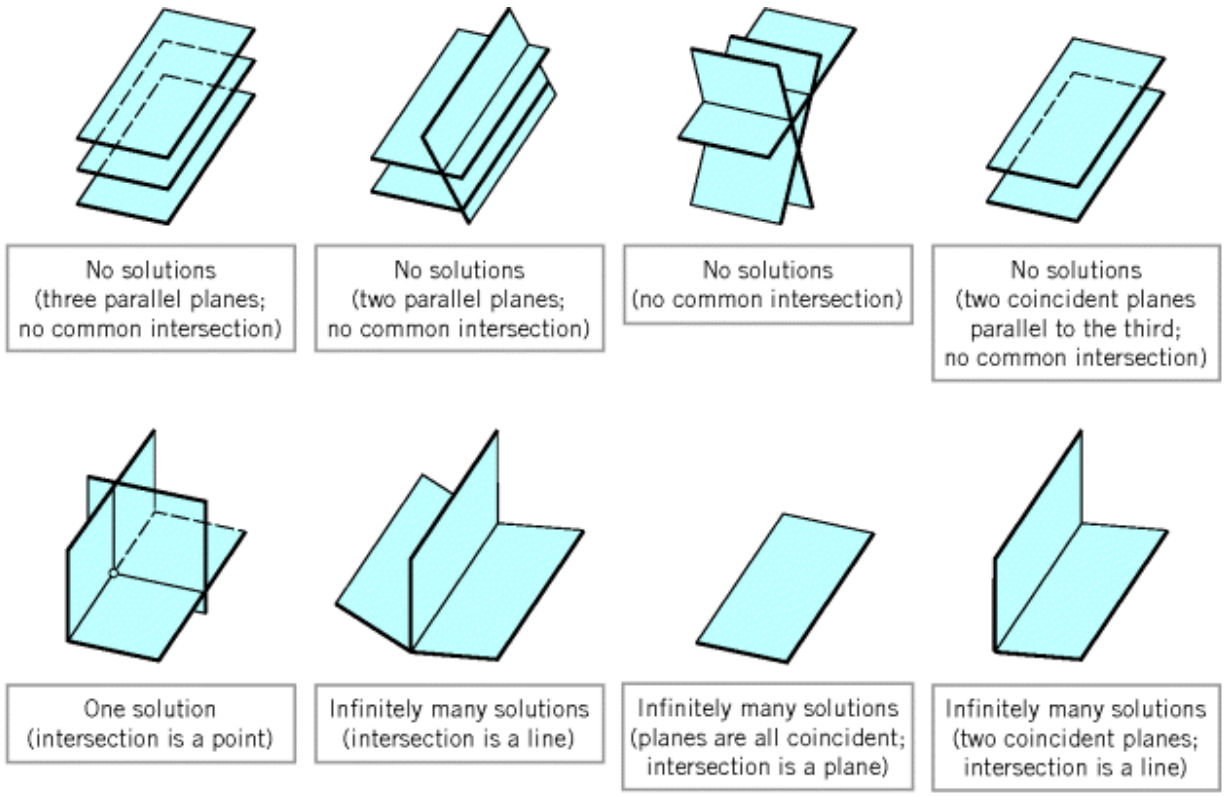


Figure 1.1.2

We will prove later that our observations about the number of solutions of linear systems of two equations in two unknowns and linear systems of three equations in three unknowns actually hold for *all* linear systems. That is:

Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

EXAMPLE 2 A Linear System with One Solution

Solve the linear system

$$\begin{aligned} x - y &= 1 \\ 2x + y &= 6 \end{aligned}$$

Solution We can eliminate x from the second equation by adding -2 times the first equation to the second. This yields the simplified system

$$\begin{aligned} x - y &= 1 \\ 3y &= 4 \end{aligned}$$

From the second equation we obtain $y = \frac{4}{3}$, and on substituting this value in the first equation we obtain $x = 1 + y = \frac{7}{3}$. Thus, the system has the unique solution

$$x = \frac{7}{3},\ y = \frac{4}{3}$$

Geometrically, this means that the lines represented by the equations in the system intersect at the single point $\left(\frac{7}{3},\ \frac{4}{3}\right)$. We leave it for you to check this by graphing the lines.

EXAMPLE 3 A Linear System with No Solutions

Solve the linear system

$$\begin{aligned} x + y &= 4 \\ 3x + 3y &= 6 \end{aligned}$$

Solution We can eliminate x from the second equation by adding −3 times the first equation to the second equation. This yields the simplified system

$$\begin{aligned} x + y &= 4 \\ 0 &= -6 \end{aligned}$$

The second equation is contradictory, so the given system has no solution. Geometrically, this means that the lines corresponding to the equations in the original system are parallel and distinct. We leave it for you to check this by graphing the lines or by showing that they have the same slope but different y-intercepts.

EXAMPLE 4 A Linear System with Infinitely Many Solutions

Solve the linear system

$$\begin{aligned} 4x - 2y &= 1 \\ 16x - 8y &= 4 \end{aligned}$$

In Example 4 we could have also obtained parametric equations for the solutions by solving 8 for y in terms of x, and letting $x = t$ be the parameter. The resulting parametric equations would look different but would define the same solution set.

Solution We can eliminate x from the second equation by adding −4 times the first equation to the second. This yields the simplified system

$$\begin{aligned} 4x - 2y &= 1 \\ 0 &= 0 \end{aligned}$$

The second equation does not impose any restrictions on x and y and hence can be omitted. Thus, the solutions of the system are those values of x and y that satisfy the single equation

$$4x - 2y = 1 \tag{8}$$

Geometrically, this means the lines corresponding to the two equations in the original system coincide. One way to describe the solution set is to solve this equation for *x* in terms of *y* to obtain $x = \frac{1}{4} + \frac{1}{2}y$ and then assign an arbitrary value *t* (called a ***parameter***) to *y*. This allows us to express the solution by the pair of equations (called ***parametric equations***)

$$x = \frac{1}{4} + \frac{1}{2}t, \quad y = t$$

We can obtain specific numerical solutions from these equations by substituting numerical values for the parameter. For example, $t = 0$ yields the solution $\left(\frac{1}{4}, 0\right)$, $t = 1$ yields the solution $\left(\frac{3}{4}, 1\right)$, and $t = -1$ yields the solution $\left(-\frac{1}{4}, -1\right)$. You can confirm that these are solutions by substituting the coordinates into the given equations.

EXAMPLE 5 A Linear System with Infinitely Many Solutions

Solve the linear system

$$\begin{aligned} x - y + 2z &= 5 \\ 2x - 2y + 4z &= 10 \\ 3x - 3y + 6z &= 15 \end{aligned}$$

Solution This system can be solved by inspection, since the second and third equations are multiples of the first. Geometrically, this means that the three planes coincide and that those values of *x*, *y*, and *z* that satisfy the equation

$$x - y + 2z = 5 \tag{9}$$

automatically satisfy all three equations. Thus, it suffices to find the solutions of 9. We can do this by first solving 9 for *x* in terms of *y* and *z*, then assigning arbitrary values *r* and *s* (parameters) to these two variables, and then expressing the solution by the three parametric equations

$$x = 5 + r - 2s, \ y = r, \ z = s$$

Specific solutions can be obtained by choosing numerical values for the parameters *r* and *s*. For example, taking $r = 1$ and $s = 0$ yields the solution (6, 1,0).

Augmented Matrices and Elementary Row Operations

As the number of equations and unknowns in a linear system increases, so does the complexity of the algebra involved in finding solutions. The required computations can be made more manageable by simplifying notation and standardizing procedures. For example, by mentally keeping track of the location of the +'s, the *x*'s, and the ='s in the linear system

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}$$

we can abbreviate the system by writing only the rectangular array of numbers

$$\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{bmatrix}$$

As noted in the introduction to this chapter, the term “matrix” is used in mathematics to denote a rectangular array of numbers. In a later section we will study matrices in detail, but for now we will only be concerned with augmented matrices for linear systems.

This is called the ***augmented matrix*** for the system. For example, the augmented matrix for the system of equations

$$\begin{aligned}
x_1 + x_2 + 2x_3 &= 9 \\
2x_1 + 4x_2 - 3x_3 &= 1 \\
3x_1 + 6x_2 - 5x_3 &= 0
\end{aligned} \quad \text{is} \quad \begin{bmatrix}
1 & 1 & 2 & 9 \\
2 & 4 & -3 & 1 \\
3 & 6 & -5 & 0
\end{bmatrix}$$

The basic method for solving a linear system is to perform appropriate algebraic operations on the system that do not alter the solution set and that produce a succession of increasingly simpler systems, until a point is reached where it can be ascertained whether the system is consistent, and if so, what its solutions are. Typically, the algebraic operations are as follows:

1. Multiply an equation through by a nonzero constant.
2. Interchange two equations.
3. Add a constant times one equation to another.

Since the rows (horizontal lines) of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operations on the rows of the augmented matrix:

1. Multiply a row through by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

These are called ***elementary row operations*** on a matrix.

In the following example we will illustrate how to use elementary row operations and an augmented matrix to solve a linear system in three unknowns. Since a systematic procedure for solving linear systems will be developed in the next section, do not worry about how the steps in the example were chosen. Your objective here should be simply to understand the computations.

EXAMPLE 6 Using Elementary Row Operations

In the left column we solve a system of linear equations by operating on the equations in the system, and in the right column we solve the same system by operating on the rows of the augmented matrix.

$$\begin{aligned} x+y+2z &= 9 \\ 2x+4y-3z &= 1 \\ 3x+6y-5z &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -2 times the first equation to the second to obtain

$$\begin{aligned} x+y+2z &= 9 \\ 2y-7z &= -17 \\ 3x+6y-5z &= 0 \end{aligned}$$

Add -2 times the first row to the second to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -3 times the first equation to the third to obtain

$$\begin{aligned} x+y+2z &= 9 \\ 2y-7z &= -17 \\ 3y-11z &= -27 \end{aligned}$$

Add -3 times the first row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Multiply the second equation by $\frac{1}{2}$ to obtain

$$\begin{aligned} x+y+2z &= 9 \\ y-\tfrac{7}{2}z &= -\tfrac{17}{2} \\ 3y-11z &= -27 \end{aligned}$$

Multiply the second row by $\frac{1}{2}$ to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add -3 times the second equation to the third to obtain

$$\begin{aligned} x+y+2z &= 9 \\ y-\tfrac{7}{2}z &= -\tfrac{17}{2} \\ -\tfrac{1}{2}z &= -\tfrac{3}{2} \end{aligned}$$

Add -3 times the second row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Multiply the third equation by -2 to obtain

$$\begin{aligned} x+y+2z &= 9 \\ y-\tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Multiply the third row by -2 to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add -1 times the second equation to the first to obtain

Add -1 times the second row to the first to obtain

$$\begin{aligned} x \quad + \frac{11}{2}z &= \frac{35}{2} \\ y - \frac{7}{2}z &= -\frac{17}{2} \\ z &= 3 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add $-\frac{11}{2}$ times the third equation to the first and $\frac{7}{2}$ times the third equation to the second to obtain

Add $-\frac{11}{2}$ times the third row to the first and $\frac{7}{2}$ times the third row to the second to obtain

$$\begin{aligned} x \qquad &= 1 \\ y \quad &= 2 \\ z &= 3 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The solution $x = 1,\ y = 2,\ z = 3$ is now evident.

Maxime Bôcher (1867–1918)

Historical Note The first known use of augmented matrices appeared between 200 B.C. and 100 B.C. in a Chinese manuscript entitled *Nine Chapters of Mathematical Art*. The coefficients were arranged in columns rather than in rows, as today, but remarkably the system was solved by performing a succession of operations on the columns. The actual use of the term *augmented matrix* appears to have been introduced by the American mathematician Maxime Bôcher in his book *Introduction to Higher Algebra,* published in 1907. In addition to being an outstanding research mathematician and an expert in Latin, chemistry, philosophy, zoology, geography, meteorology, art, and music, Bôcher was an outstanding expositor of mathematics whose elementary textbooks were greatly appreciated by students and are still in demand today.
[*Image: Courtesy of the American Mathematical Society*]

Concept Review

- Linear equation
- Homogeneous linear equation
- System of linear equations
- Solution of a linear system
- Ordered n-tuple
- Consistent linear system
- Inconsistent linear system
- Parameter
- Parametric equations
- Augmented matrix
- Elemenetary row operations

Skills

- Determine whether a given equation is linear.
- Determine whether a given n-tuple is a solution of a linear system.
- Find the augmented matrix of a linear system.
- Find the linear system corresponding to a given augmented matrix.
- Perform elementary row operations on a linear system and on its corresponding augmented matrix.
- Determine whether a linear system is consistent or inconsistent.
- Find the set of solutions to a consistent linear system.

Exercise Set 1.1

1. In each part, determine whether the equation is linear in x_1, x_2, and x_3.

(a) $x_1 + 5x_2 - \sqrt{2}x_3 = 1$

(b) $x_1 + 3x_2 + x_1x_3 = 2$

(c) $x_1 = -7x_2 + 3x_3$

(d) $x_1^{-2} + x_2 + 8x_3 = 5$

(e) $x_1^{3/5} - 2x_2 + x_3 = 4$

(f) $\pi x_1 - \sqrt{2}x_2 + \frac{1}{3}x_3 = 7^{1/3}$

Answer:

(a), (c), and (f) are linear equations; (b), (d) and (e) are not linear equations

2. In each part, determine whether the equations form a linear system.

(a) $$\begin{aligned} -2x+4y+z&=2 \\ 3x-\frac{2}{y}&=0 \end{aligned}$$

(b) $$\begin{aligned} x&=4 \\ 2x&=8 \end{aligned}$$

(c) $$\begin{aligned} 4x-y+2z&=-1 \\ -x+(\ln 2)y-3z&=0 \end{aligned}$$

(d) $$\begin{aligned} 3z+x&=-4 \\ y+5z&=1 \\ 6x+2z&=3 \\ -x-y-z&=4 \end{aligned}$$

3. In each part, determine whether the equations form a linear system.

(a) $$\begin{aligned} 2x_1 \qquad\qquad\quad - x_4 &= 5 \\ -x_1+5x_2+3x_3-2x_4&=-1 \end{aligned}$$

(b) $$\begin{aligned} \sin(2x_1+x_3)&=\sqrt{5} \\ e^{2x_2-2x_4}&=\frac{1}{x_2} \\ 4x_4&=4 \end{aligned}$$

(c) $$\begin{aligned} 7x_1-x_2+2x_3&=0 \\ 2x_1+x_2-x_3x_4&=3 \\ -x_1+5x_2-x_4&=-1 \end{aligned}$$

(d) $x_1+x_2=x_3+x_4$

Answer:

(a) and (d) are linear systems; (b) and (c) are not linear systems

4. For each system in Exercise 2 that is linear, determine whether it is consistent.

5. For each system in Exercise 3 that is linear, determine whether it is consistent.

Answer:

(a) and (d) are both consistent

6. Write a system of linear equations consisting of three equations in three unknowns with

(a) no solutions.

(b) exactly one solution.

(c) infinitely many solutions.

7. In each part, determine whether the given vector is a solution of the linear system

$$\begin{aligned} 2x_1-4x_2-x_3&=1 \\ x_1-3x_2+x_3&=1 \\ 3x_1-5x_2-3x_3&=1 \end{aligned}$$

(a) (3, 1, 1)

(b) (3, −1, 1)

(c) (13, 5, 2)

(d) $\left(\frac{13}{2}, \frac{5}{2}, 2\right)$

(e) (17, 7, 5)

Answer:

(a), (d), and (e) are solutions; (b) and (c) are not solutions

8. In each part, determine whether the given vector is a solution of the linear system

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 3 \\ 3x_1 - x_2 + x_3 &= 1 \\ -x_1 + 5x_2 - 5x_3 &= 5 \end{aligned}$$

(a) $\left(\frac{5}{7}, \frac{8}{7}, 1\right)$

(b) $\left(\frac{5}{7}, \frac{8}{7}, 0\right)$

(c) (5, 8, 1)

(d) $\left(\frac{5}{7}, \frac{10}{7}, \frac{2}{7}\right)$

(e) $\left(\frac{5}{7}, \frac{22}{7}, 2\right)$

9. In each part, find the solution set of the linear equation by using parameters as necessary.

(a) $7x - 5y = 3$

(b) $-8x_1 + 2x_2 - 5x_3 + 6x_4 = 1$

Answer:

(a) $\begin{aligned} x &= \frac{5}{7}t + \frac{3}{7} \\ y &= t \end{aligned}$

(b) $\begin{aligned} x_1 &= \frac{1}{4}r - \frac{5}{8}s + \frac{3}{4}t - \frac{1}{8} \\ x_2 &= r \\ x_3 &= s \\ x_4 &= t \end{aligned}$

10. In each part, find the solution set of the linear equation by using parameters as necessary.

(a) $3x_1 - 5x_2 + 4x_3 = 7$

(b) $3v - 8w + 2x - y + 4z = 0$

11. In each part, find a system of linear equations corresponding to the given augmented matrix

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 0 & -2 & 5 \\ 7 & 1 & 4 & -3 \\ 0 & -2 & 1 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} 7 & 2 & 1 & -3 & 5 \\ 1 & 2 & 4 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

Answer:

(a) $$\begin{aligned} 2x_1 \phantom{{}- 4x_2} &= 0 \\ 3x_1 - 4x_2 &= 0 \\ x_2 &= 1 \end{aligned}$$

(b) $$\begin{aligned} 3x_1 \phantom{{}+ x_2} - 2x_3 &= 5 \\ 7x_1 + x_2 + 4x_3 &= -3 \\ -2x_2 + x_3 &= 7 \end{aligned}$$

(c) $$\begin{aligned} 7x_1 + 2x_2 + x_3 - 3x_4 &= 5 \\ x_1 + 2x_2 + 4x_3 &= 1 \end{aligned}$$

(d) $$\begin{aligned} x_1 &= 7 \\ x_2 &= -2 \\ x_3 &= 3 \\ x_4 &= 4 \end{aligned}$$

12. In each part, find a system of linear equations corresponding to the given augmented matrix.

(a) $\begin{bmatrix} 2 & -1 \\ -4 & -6 \\ 1 & -1 \\ 3 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \\ 5 & -6 & 1 & 1 \\ -8 & 0 & 0 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 0 & 1 & -4 & 3 \\ -4 & 0 & 4 & 1 & -3 \\ -1 & 3 & 0 & -2 & -9 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}$

13. In each part, find the augmented matrix for the given system of linear equations.

(a) $$\begin{aligned} -2x_1 &= 6 \\ 3x_1 &= 8 \\ 9x_1 &= -3 \end{aligned}$$

(b) $$\begin{aligned} 6x_1 - x_2 + 3x_3 &= 4 \\ 5x_2 - x_3 &= 1 \end{aligned}$$

(c) $$\begin{aligned} 2x_2 - 3x_4 + x_5 &= 0 \\ -3x_1 - x_2 + x_3 &= -1 \\ 6x_1 + 2x_2 - x_3 + 2x_4 - 3x_5 &= 6 \end{aligned}$$

(d) $x_1 - x_5 = 7$

Answer:

(a) $$\begin{bmatrix} -2 & 6 \\ 3 & 8 \\ 9 & -3 \end{bmatrix}$$

(b) $$\begin{bmatrix} 6 & -1 & 3 & 4 \\ 0 & 5 & -1 & 1 \end{bmatrix}$$

(c) $$\begin{bmatrix} 0 & 2 & 0 & -3 & 1 & 0 \\ -3 & -1 & 1 & 0 & 0 & -1 \\ 6 & 2 & -1 & 2 & -3 & 6 \end{bmatrix}$$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 7 \end{bmatrix}$

14. In each part, find the augmented matrix for the given system of linear equations.

(a) $$\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 4x_1 + 5x_2 &= 3 \\ 7x_1 + 3x_2 &= 2 \end{aligned}$$

(b) $$\begin{aligned} 2x_1 + 2x_3 &= 1 \\ 3x_1 - x_2 + 4x_3 &= 7 \\ 6x_1 + x_2 - x_3 &= 0 \end{aligned}$$

(c) $$\begin{aligned} x_1 + 2x_2 - x_4 + x_5 &= 1 \\ 3x_2 + x_3 - x_5 &= 2 \\ x_3 + 7x_4 &= 1 \end{aligned}$$

(d) $$\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \end{aligned}$$

15. The curve $y = ax^2 + bx + c$ shown in the accompanying figure passes through the points (x_1, y_1), (x_2, y_2), and (x_3, y_3). Show that the coefficients a, b, and c are a solution of the system of linear equations whose augmented matrix is

$$\begin{bmatrix} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{bmatrix}$$

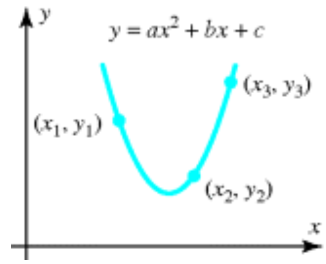


Figure Ex-15

16. Explain why each of the three elementary row operations does not affect the solution set of a linear system.

17. Show that if the linear equations

$$x_1 + kx_2 = c \text{ and } x_1 + lx_2 = d$$

have the same solution set, then the two equations are identical (i.e., $k = 1$ and $c = d$).

True-False Exercises

In parts (a)–(h) determine whether the statement is true or false, and justify your answer.

(a) A linear system whose equations are all homogeneous must be consistent.

Answer:

True

(b) Multiplying a linear equation through by zero is an acceptable elementary row operation.

Answer:

False

(c) The linear system

$$\begin{aligned} x - y &= 3 \\ 2x - 2y &= k \end{aligned}$$

cannot have a unique solution, regardless of the value of k.

Answer:

True

(d) A single linear equation with two or more unknowns must always have infinitely many solutions.

Answer:

True

(e) If the number of equations in a linear system exceeds the number of unknowns, then the system must be inconsistent.

Answer:

False

(f) If each equation in a consistent linear system is multiplied through by a constant c, then all solutions to the new system can be obtained by multiplying solutions from the original system by c.

Answer:

False

(g) Elementary row operations permit one equation in a linear system to be subtracted from another.

Answer:

True

(h) The linear system with corresponding augmented matrix

$$\begin{bmatrix} 2 & -1 & 4 \\ 0 & 0 & -1 \end{bmatrix}$$

is consistent.

Answer:

False

1.2 Gaussian Elimination

In this section we will develop a systematic procedure for solving systems of linear equations. The procedure is based on the idea of performing certain operations on the rows of the augmented matrix for the system that simplifies it to a form from which the solution of the system can be ascertained by inspection.

Considerations in Solving Linear Systems

When considering methods for solving systems of linear equations, it is important to distinguish between large systems that must be solved by computer and small systems that can be solved by hand. For example, there are many applications that lead to linear systems in thousands or even millions of unknowns. Large systems require special techniques to deal with issues of memory size, roundoff errors, solution time, and so forth. Such techniques are studied in the field of *numerical analysis* and will only be touched on in this text. However, almost all of the methods that are used for large systems are based on the ideas that we will develop in this section.

Echelon Forms

In Example 6 of the last section, we solved a linear system in the unknowns x, y, and z by reducing the augmented matrix to the form

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

from which the solution $x = 1, y = 2, z = 3$ became evident. This is an example of a matrix that is in ***reduced row echelon form***. To be of this form, a matrix must have the following properties:

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a ***leading 1***.
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else in that column.

A matrix that has the first three properties is said to be in ***row echelon form***. (Thus, a matrix in reduced row echelon form is of necessity in row echelon form, but not conversely.)

EXAMPLE 1 Row Echelon and Reduced Row Echelon Form

The following matrices are in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The following matrices are in row echelon form but not reduced row echelon form.

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

EXAMPLE 2 More on Row Echelon and Reduced Row Echelon Form

As Example 1 illustrates, a matrix in row echelon form has zeros below each leading 1, whereas a matrix in reduced row echelon form has zeros below *and above* each leading 1. Thus, with any real numbers substituted for the *'s, all matrices of the following types are in row echelon form:

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

All matrices of the following types are in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

If, by a sequence of elementary row operations, the augmented matrix for a system of linear equations is put in *reduced* row echelon form, then the solution set can be obtained either by inspection or by converting certain linear equations to parametric form. Here are some examples.

In Example 3 we could, if desired, express the solution more succinctly as the 4-tuple $(3, -1, 0, 5)$.

EXAMPLE 3 Unique Solution

Suppose that the augmented matrix for a linear system in the unknowns x_1, x_2, x_3, and x_4 has been reduced by elementary row operations to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

This matrix is in reduced row echelon form and corresponds to the equations

$$\begin{aligned} x_1 \qquad\qquad\qquad &= 3 \\ x_2 \qquad\qquad &= -1 \\ x_3 \qquad &= 0 \\ x_4 &= 5 \end{aligned}$$

Thus, the system has a unique solution, namely, $x_1 = 3, x_2 = -1, x_3 = 0, x_4 = 5$.

EXAMPLE 4 Linear Systems in Three Unknowns

In each part, suppose that the augmented matrix for a linear system in the unknowns x, y, and z has been reduced by elementary row operations to the given reduced row echelon form. Solve the system.

$$\text{(a)}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 2 & 0\\0 & 0 & 0 & 1\end{bmatrix}\quad\text{(b)}\begin{bmatrix}1 & 0 & 3 & -1\\0 & 1 & -4 & 2\\0 & 0 & 0 & 0\end{bmatrix}\quad\text{(c)}\begin{bmatrix}1 & -5 & 1 & 4\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

Solution

(a) The equation that corresponds to the last row of the augmented matrix is

$$0x+0y+0z=1$$

Since this equation is not satisfied by any values of x, y, and z, the system is inconsistent.

(b) The equation that corresponds to the last row of the augmented matrix is

$$0x+0y+0z=0$$

This equation can be omitted since it imposes no restrictions on x, y, and z; hence, the linear system corresponding to the augmented matrix is

$$\begin{aligned}x \quad +3z &= -1\\ y-4z &= 2\end{aligned}$$

Since x and y correspond to the leading 1's in the augmented matrix, we call these the ***leading variables***. The remaining variables (in this case z) are called ***free variables***. Solving for the leading variables in terms of the free variables gives

$$\begin{aligned}x &= -1-3z\\ y &= 2+4z\end{aligned}$$

From these equations we see that the free variable z can be treated as a parameter and assigned an arbitrary value, t, which then determines values for x and y. Thus, the solution set can be represented by the parametric equations

$$x=-1-3t,\quad y=2+4t,\quad z=t$$

By substituting various values for t in these equations we can obtain various solutions of the system. For example, setting $t=0$ yields the solution

$$x=-1,\quad y=2,\quad z=0$$

and setting $t=1$ yields the solution

$$x=-4,\quad y=6,\quad z=1$$

(c) As explained in part (b), we can omit the equations corresponding to the zero rows, in which case the linear system associated with the augmented matrix consists of the single equation

$$x-5y+z=4 \tag{1}$$

from which we see that the solution set is a plane in three-dimensional space. Although 1 is a valid form of the solution set, there are many applications in which it is preferable to express the solution set in parametric form. We can convert 1 to parametric form by solving for the leading variable x in terms of the free variables y and z to obtain

$$x=4+5y-z$$

From this equation we see that the free variables can be assigned arbitrary values, say $y=s$ and $z=t$, which then determine the value of x. Thus, the solution set can be expressed parametrically as

$$x = 4 + 5s - t, \quad y = s, \quad z = t \tag{2}$$

We will usually denote parameters in a general solution by the letters $r, s, t, \ldots$, but any letters that do not conflict with the names of the unknowns can be used. For systems with more than three unknowns, subscripted letters such as $t_1, t_2, t_3, \ldots$ are convenient.

Formulas, such as 2, that express the solution set of a linear system parametrically have some associated terminology.

DEFINITION 1

If a linear system has infinitely many solutions, then a set of parametric equations from which all solutions can be obtained by assigning numerial values to the parameters is called a ***general solution*** of the system.

Elimination Methods

We have just seen how easy it is to solve a system of linear equations once its augmented matrix is in reduced row echelon form. Now we will give a step-by-step ***elimination procedure*** that can be used to reduce any matrix to reduced row echelon form. As we state each step in the procedure, we illustrate the idea by reducing the following matrix to reduced row echelon form.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 1. Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Leftmost nonzero column

Step 2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \leftarrow \text{The first and second rows in the preceding matrix were interchanged.}$$

Step 3. If the entry that is now at the top of the column found in Step 1 is a, multiply the first row by $1/a$ in order to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \leftarrow \text{The first row of the preceding matrix was multiplied by } \frac{1}{2}.$$

Step 4. Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \leftarrow -2 \text{ times the first row of the preceding matrix was added to the third row.}$$

Step 5. Now cover the top row in the matrix and begin again with Step 1 applied to the submatrix that remains. Continue in this way until the *entire* matrix is in row echelon form.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

Leftmost nonzero column in the submatrix

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

← The first row in the submatrix was multiplied by $-\frac{1}{2}$ to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

← -5 times the first row of the submatrix was added to the second row of the submatrix to introduce a zero below the leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

← The top row in the submatrix was covered, and we returned again to Step 1.

Leftmost nonzero column in the new submatrix

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

← The first (and only) row in the new submatrix was multiplied by 2 to introduce a leading 1.

The *entire* matrix is now in row echelon form. To find the reduced row echelon form we need the following additional step.

Step 6. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \leftarrow \frac{7}{2} \text{ times the third row of the preceding matrix was added to the second row.}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \leftarrow -6 \text{ times the third row was added to the first row.}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \leftarrow 5 \text{ times the second row was added to the first row.}$$

The last matrix is in reduced row echelon form.

The procedure (or algorithm) we have just described for reducing a matrix to reduced row echelon form is called ***Gauss-Jordan elimination***. This algorithm consists of two parts, a ***forward phase*** in which zeros are introduced below the leading 1's and then a ***backward phase*** in which zeros are introduced above the leading 1's. If only the forward phase is used, then the procedure produces a row echelon form only and is called ***Gaussian elimination***. For example, in the preceding computations a row echelon form was obtained at the end of Step 5.

Carl Friedrich Gauss (1777–1855)

Wilhelm Jordan (1842–1899)

Historical Note Although versions of Gaussian elimination were known much earlier, the power of the method was not recognized until the great German mathematician Carl Friedrich Gauss used it to compute the orbit of the asteroid Ceres from limited data. What happened was this: On January 1, 1801 the Sicilian astronomer Giuseppe Piazzi (1746–1826) noticed a dim celestial object that he believed might be a "missing planet." He named the object Ceres and made a limited number of positional observations but then lost the object as it neared the Sun. Gauss undertook the problem of computing the orbit from the limited data using least squares and the procedure that we now call Gaussian elimination. The work of Gauss caused a sensation when Ceres reappeared

a year later in the constellation Virgo at almost the precise position that Gauss predicted! The method was further popularized by the German engineer Wilhelm Jordan in his handbook on geodesy (the science of measuring Earth shapes) entitled *Handbuch der Vermessungskunde* and published in 1888.
[*Images: Granger Collection (Gauss); wikipedia (Jordan)*]

EXAMPLE 5 Gauss-Jordan Elimination

Solve by Gauss-Jordan elimination.

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \phantom{{}-2x_4} + 2x_5 \phantom{{}-3x_6} &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\ 5x_3 + 10x_4 \phantom{{}+4x_5} + 15x_6 &= 5 \\ 2x_1 + 6x_2 \phantom{{}-5x_3} + 8x_4 + 4x_5 + 18x_6 &= 6 \end{aligned}$$

Solution The augmented matrix for the system is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

Adding —2 times the first row to the second and fourth rows gives

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

Multiplying the second row by -1 and then adding -5 times the new second row to the third row and -4 times the new second row to the fourth row gives

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

Interchanging the third and fourth rows and then multiplying the third row of the resulting matrix by $\frac{1}{6}$ gives the row echelon form

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{This completes the forward phase since there are zeros below the leading 1's .}$$

Adding -3 times the third row to the second row and then adding 2 times the second row of the resulting matrix to the first row yields the reduced row echelon form

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{This completes the backward phase since there are zeros above the leading 1's .}$$

The corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 \quad + 4x_4 + 2x_5 \quad &= 0 \\ x_3 + \ 2x_4 \qquad &= 0 \\ x_6 &= \frac{1}{3} \end{aligned} \tag{3}$$

Note that in constructing the linear system in 3 we ignored the row of zeros in the corresponding augmented matrix. Why is this justified?

Solving for the leading variables we obtain

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \frac{1}{3} \end{aligned}$$

Finally, we express the general solution of the system parametrically by assigning the free variables x_2, x_4, and x_5 arbitrary values r, s, and t, respectively. This yields

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = \frac{1}{3}$$

Homogeneous Linear Systems

A system of linear equations is said to be ***homogeneous*** if the constant terms are all zero; that is, the system has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}$$

Every homogeneous system of linear equations is consistent because all such systems have $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ as a solution. This solution is called the ***trivial solution***; if there are other solutions, they are called ***nontrivial solutions***.

Because a homogeneous linear system always has the trivial solution, there are only two possibilities for its solutions:

- The system has only the trivial solution.
- The system has infinitely many solutions in addition to the trivial solution.

In the special case of a homogeneous linear system of two equations in two unknowns, say

$$\begin{aligned} a_1x + b_1y = 0 \quad (\boldsymbol{a_1}, \boldsymbol{b_1} \ \textbf{not both zero}) \\ a_2x + b_2y = 0 \quad (\boldsymbol{a_2}, \boldsymbol{b_2} \ \textbf{not both zero}) \end{aligned}$$

the graphs of the equations are lines through the origin, and the trivial solution corresponds to the point of intersection at the origin (Figure 1.2.1).

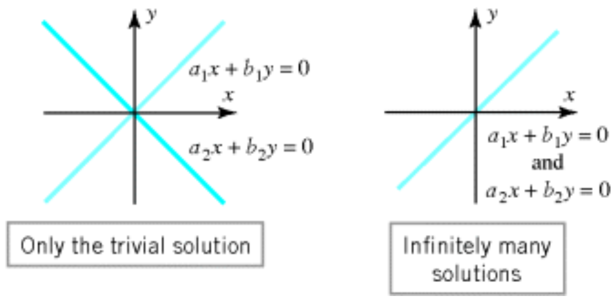


Figure 1.2.1

There is one case in which a homogeneous system is assured of having nontrivial solutions—namely, whenever the system involves more unknowns than equations. To see why, consider the following example of four equations in six unknowns.

EXAMPLE 6 A Homogeneous System

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\quad &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 \qquad\quad + 15x_6 &= 0 \\ 2x_1 + 6x_2 \qquad\quad + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned} \tag{4}$$

Solution Observe first that the coefficients of the unknowns in this system are the same as those in Example 5; that is, the two systems differ only in the constants on the right side. The augmented matrix for the given homogeneous system is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix} \tag{5}$$

which is the same as the augmented matrix for the system in Example 5, except for zeros in the last column. Thus, the reduced row echelon form of this matrix will be the same as that of the augmented matrix in Example 5, except for the last column. However, a moment's reflection will make it evident that a column of zeros is not changed by an elementary row operation, so the reduced row echelon form of 5 is

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

The corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 \quad + 4x_4 + 2x_5 \quad &= 0 \\ x_3 + 2x_4 \qquad\quad &= 0 \\ x_6 &= 0 \end{aligned}$$

Solving for the leading variables we obtain

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= 0 \end{aligned} \tag{7}$$

If we now assign the free variables x_2, x_4, and x_5 arbitrary values r, s, and t, respectively, then we can express the solution set parametrically as

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$$

Note that the trivial solution results when $r = s = t = 0$.

Free Variable in Homogeneous Linear Systems

Example 6 illustrates two important points about solving homogeneous linear systems:

1. Elementary row operations do not alter columns of zeros in a matrix, so the reduced row echelon form of the augmented matrix for a homogeneous linear system has a final column of zeros. This implies that the linear system corresponding to the reduced row echelon form is homogeneous, just like the original system.
2. When we constructed the homogeneous linear system corresponding to augmented matrix 6, we ignored the row of zeros because the corresponding equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 = 0$$

does not impose any conditions on the unknowns. Thus, depending on whether or not the reduced row echelon form of the augmented matrix for a homogeneous linear system has any rows of zero, the linear system corresponding to that reduced row echelon form will either have the same number of equations as the original system or it will have fewer.

Now consider a general homogeneous linear system with *n* unknowns, and suppose that the reduced row echelon form of the augmented matrix has *r* nonzero rows. Since each nonzero row has a leading 1, and since each leading 1 corresponds to a leading variable, the homogeneous system corresponding to the reduced row echelon form of the augmented matrix must have *r* leading variables and $n - r$ free variables. Thus, this system is of the form

$$\begin{array}{cccccc} x_{k_1} & & & + & \sum() = 0 \\ & x_{k_2} & & + & \sum() = 0 \\ & & \ddots & & \vdots \\ & & x_{k_r} & + & \sum() = 0 \end{array} \tag{8}$$

where in each equation the expression $\sum()$ denotes a sum that involves the free variables, if any [see 7, for example]. In summary, we have the following result.

THEOREM 1.2.1 Free Variable Theorem for Homogeneous Systems

If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has n - r free variables.

Note that Theorem 1.2.2 applies only to homogeneous systems—a *nonhomogeneous* system with more unknowns than equations need not be consistent. However, we will prove later that if a nonhomogeneous system with more unknowns then equations is consistent, then it has in infinitely many solutions.

Theorem 1.2.1 has an important implication for homogeneous linear systems with more unknowns than equations. Specifically, if a homogeneous linear system has m equations in n unknowns, and if $m < n$, then it must also be true that $r < n$ (why?). This being the case, the theorem implies that there is at least one free variable, and this implies in turn that the system has infinitely many solutions. Thus, we have the following result.

THEOREM 1.2.2

A homogeneous linear system with more unknowns than equations has infinitely many solutions.

In retrospect, we could have anticipated that the homogeneous system in Example 6 would have infinitely many solutions since it has four equations in six unknowns.

Gaussian Elimination and Back-Substitution

For small linear systems that are solved by hand (such as most of those in this text), Gauss-Jordan elimination (reduction to reduced row echelon form) is a good procedure to use. However, for large linear systems that require a computer solution, it is generally more efficient to use Gaussian elimination (reduction to row echelon form) followed by a technique known as ***back-substitution*** to complete the process of solving the system. The next example illustrates this technique.

EXAMPLE 7 Example 5 Solved by Back-Substitution

From the computations in Example 5, a row echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To solve the corresponding system of equations

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \qquad + 2x_5 \qquad &= 0 \\ x_3 + 2x_4 \qquad + 3x_6 &= 1 \\ x_6 &= \frac{1}{3} \end{aligned}$$

we proceed as follows:

Step 1. Solve the equations for the leading variables.

$$\begin{aligned} x_1 &= -3x_2 + 2x_3 - 2x_5 \\ x_3 &= 1 - 2x_4 - 3x_6 \\ x_6 &= \frac{1}{3} \end{aligned}$$

Step 2. Beginning with the bottom equation and working upward, successively substitute each equation into all the equations above it.

Substituting $x_6 = \frac{1}{3}$ into the second equation yields

$$x_1 = -3x_2 + 2x_3 - 2x_5$$
$$x_3 = -2x_4$$
$$x_6 = \frac{1}{3}$$

Substituting $x_3 = -2x_4$ into the first equation yields

$$x_1 = -3x_2 - 4x_4 - 2x_5$$
$$x_3 = -2x_4$$
$$x_6 = \frac{1}{3}$$

Step 3. Assign arbitrary values to the free variables, if any.

If we now assign x_2, x_4, and x_5 the arbitrary values r, s, and t, respectively, the general solution is given by the formulas

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = \frac{1}{3}$$

This agrees with the solution obtained in Example 5.

EXAMPLE 8

Suppose that the matrices below are augmented matrices for linear systems in the unknowns x_1, x_2, x_3, and x_4. These matrices are all in row echelon form but not reduced row echelon form. Discuss the existence and uniqueness of solutions to the corresponding linear systems

$$\text{(a)}\begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(b)}\begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(c)}\begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Solution

(a) The last row corresponds to the equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$$

from which it is evident that the system is inconsistent.

(b) The last row corresponds to the equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 0$$

which has no effect on the solution set. In the remaining three equations the variables x_1, x_2, and x_3 correspond to leading 1's and hence are leading variables. The variable x_4 is a free variable. With a little algebra, the leading variables can be expressed in terms of the free variable, and the free variable can be assigned an arbitrary value. Thus, the system must have infinitely many solutions.

(c) The last row corresponds to the equation

$$x_4 = 0$$

which gives us a numerical value for x_4. If we substitute this value into the third equation, namely,

$$x_3 + 6x_4 = 9$$

we obtain $x_3 = 9$. You should now be able to see that if we continue this process and substitute the known values of x_3 and x_4 into the equation corresponding to the second row, we will obtain a unique numerical value for x_2; and if, finally, we substitute the known values of x_4, x_3, and x_2 into the equation corresponding to the first row, we will produce a unique numerical value for x_1. Thus, the system has a unique solution.

Some Facts About Echelon Forms

There are three facts about row echelon forms and reduced row echelon forms that are important to know but we will not prove:

1. Every matrix has a unique reduced row echelon form; that is, regardless of whether you use Gauss-Jordan elimination or some other sequence of elementary row operations, the same reduced row echelon form will result in the end.*
2. Row echelon forms are not unique; that is, different sequences of elementary row operations can result in different row echelon forms.
3. Although row echelon forms are not unique, all row echelon forms of a matrix A have the same number of zero rows, and the leading 1's always occur in the same positions in the row echelon forms of A. Those are callled the ***pivot positions*** of A. A column that contains a pivot position is called a ***pivot column*** of A.

EXAMPLE 9 Pivot Positions and Columns

Earlier in this section (immediately after Definition 1) we found a row echelon form of

$$A=\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

to be

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The leading 1's occur in positions (row 1, column 1), (row 2, column 3), and (row 3, column 5). These are the pivot positions. The pivot columns are columns 1,3, and 5.

Roundoff Error and Instability

There is often a gap between mathematical theory and its practical implementation—Gauss-Jordan elimination and Gaussian elimination being good examples. The problem is that computers generally approximate numbers, thereby introducing ***roundoff*** errors, so unless precautions are taken, successive calculations may degrade an answer to a degree that makes it useless. Algorithms (procedures) in which this happens are called ***unstable***. There are various techniques for minimizing roundoff error and instability. For example, it can be shown that for large linear systems Gauss-Jordan elimination involves roughly 50% more operations than Gaussian elimination, so most computer algorithms are based on the latter method. Some of these matters will be considered in Chapter 9.

Concept Review

- Reduced row echelon form
- Row echelon form
- Leading 1
- Leading variables
- Free variables
- General solution to a linear system
- Gaussian elimination
- Gauss-Jordan elimination
- Forward phase
- Backward phase
- Homogeneous linear system
- Trivial solution
- Nontrivial solution
- Dimension Theorem for Homogeneous Systems
- Back-substitution

Skills

- Recognize whether a given matrix is in row echelon form, reduced row echelon form, or neither.
- Construct solutions to linear systems whose corresponding augmented matrices that are in row echelon form or reduced row echelon form.
- Use Gaussian elimination to find the general solution of a linear system.
- Use Gauss-Jordan elimination in order to find the general solution of a linear system.
- Analyze homogeneous linear systems using the Free Variable Theorem for Homogeneous Systems.

Exercise Set 1.2

1. In each part, determine whether the matrix is in row echelon form, reduced row echelon form, both, or neither.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

Answer:

(a) Both
(b) Both
(c) Both
(d) Both
(e) Both
(f) Both
(g) Row echelon

2. In each part, determine whether the matrix is in row echelon form, reduced row echelon form, both, or neither.

(a) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 7 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

3. In each part, suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given reduced row echelon form. Solve the system.

(a) $\begin{bmatrix} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 7 & -2 & 0 & -8 & -3 \\ 0 & 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Answer:

(a) $x_1 = -37,\ x_2 = -8,\ x_3 = 5$

(b) $x_1 = 13t - 10,\ x_2 = 13t - 5,\ x_3 = -t + 2,\ x_4 = t$

(c) $x_1 = -7s + 2t - 11,\ x_2 = s,\ x_3 = -3t - 4,\ x_4 = -3t + 9,\ x_5 = t$

(d) Inconsistent

4. In each part, suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given reduced row echelon form. Solve the system.

(a) $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -6 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

In Exercises 5–8, solve the linear system by Gauss-Jordan elimination.

5. $$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

Answer:

$x_1 = 3,\ x_2 = 1,\ x_3 = 2$

6.
$$\begin{aligned} 2x_1+2x_2+2x_3 &= 0 \\ -2x_1+5x_2+2x_3 &= 1 \\ 8x_1+x_2+4x_3 &= -1 \end{aligned}$$

7.
$$\begin{aligned} x-\ \ y+2z-\ \ w &= -1 \\ 2x+\ \ y-2z-2w &= -2 \\ -x+2y-4z+\ \ w &= 1 \\ 3x\qquad\qquad -3w &= -3 \end{aligned}$$

Answer:

$x=t-1,\ y=2s,\ z=s,\ w=t$

8.
$$\begin{aligned} -2b+3c &= 1 \\ 3a+6b-3c &= -2 \\ 6a+6b+3c &= 5 \end{aligned}$$

In Exercises 9–12, solve the linear system by Gaussian elimination.

9. Exercise 5

Answer:

$x_1=3,\ x_2=1,\ x_3=2$

10. Exercise 6

11. Exercise 7

Answer:

$x=t-1,\ y=2s,\ z=s,\ w=t$

12. Exercise 8

In Exercises 13–16, determine whether the homogeneous system has nontrivial solutions by inspection (without pencil and paper).

13.
$$\begin{aligned} 2x_1-3x_2+4x_3-\ \ x_4 &= 0 \\ 7x_1+\ \ x_2-8x_3+9x_4 &= 0 \\ 2x_1+8x_2+\ \ x_3-\ \ x_4 &= 0 \end{aligned}$$

Answer:

Has nontrivial solutions

14.
$$\begin{aligned} x_1+3x_2-x_3 &= 0 \\ x_2-8x_3 &= 0 \\ 4x_3 &= 0 \end{aligned}$$

15.
$$\begin{aligned} a_{11}x_1+a_{12}x_2+a_{13}x_3 &= 0 \\ a_{21}x_1+a_{22}x_2+a_{23}x_3 &= 0 \end{aligned}$$

Answer:

Has nontrivial solutions

16.
$$\begin{aligned} 3x_1-2x_2 &= 0 \\ 6x_1-4x_2 &= 0 \end{aligned}$$

In Exercises 17–24, solve the given homogeneous linear system by any method.

17.
$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

Answer:

$x_1 = 0,\ x_2 = 0,\ x_3 = 0$

18.
$$\begin{aligned} 2x - y - 3z &= 0 \\ -x + 2y - 3z &= 0 \\ x + y + 4z &= 0 \end{aligned}$$

19.
$$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

Answer:

$x_1 = -s,\ x_2 = -t - s,\ x_3 = 4s,\ x_4 = t$

20.
$$\begin{aligned} v + 3w - 2x &= 0 \\ 2u + v - 4w + 3x &= 0 \\ 2u + 3v + 2w - x &= 0 \\ -4u - 3v + 5w - 4x &= 0 \end{aligned}$$

21.
$$\begin{aligned} 2x + 2y + 4z &= 0 \\ w - y - 3z &= 0 \\ 2w + 3x + y + z &= 0 \\ -2w + x + 3y - 2z &= 0 \end{aligned}$$

Answer:

$w = t,\ x = -t,\ y = t,\ z = 0$

22.
$$\begin{aligned} x_1 + 3x_2 + x_4 &= 0 \\ x_1 + 4x_2 + 2x_3 &= 0 \\ -2x_2 - 2x_3 - x_4 &= 0 \\ 2x_1 - 4x_2 + x_3 + x_4 &= 0 \\ x_1 - 2x_2 - x_3 + x_4 &= 0 \end{aligned}$$

23.
$$\begin{aligned} 2I_1 - I_2 + 3I_3 + 4I_4 &= 9 \\ I_1 - 2I_3 + 7I_4 &= 11 \\ 3I_1 - 3I_2 + I_3 + 5I_4 &= 8 \\ 2I_1 + I_2 + 4I_3 + 4I_4 &= 10 \end{aligned}$$

Answer:

$I_1 = -1,\ I_2 = 0,\ I_3 = 1,\ I_4 = 2$

24.
$$\begin{aligned} Z_3 + Z_4 + Z_5 &= 0 \\ -Z_1 - Z_2 + 2Z_3 - 3Z_4 + Z_5 &= 0 \\ Z_1 + Z_2 - 2Z_3 - Z_5 &= 0 \\ 2Z_1 + 2Z_2 - Z_3 + Z_5 &= 0 \end{aligned}$$

In Exercises 25–28, determine the values of a for which the system has no solutions, exactly one solution, or infinitely many solutions.

25. $$\begin{aligned} x+2y- \quad 3z &= 4 \\ 3x-y+ \quad 5z &= 2 \\ 4x+y+\left(a^2-14\right)z &= a+2 \end{aligned}$$

Answer:

If $a=4$, there are infinitely many solutions; if $a=-4$, there are no solutions; if $a\neq\pm 4$, there is exactly one solution.

26. $$\begin{aligned} x+2y+ \quad z &= 2 \\ 2x-2y+ \quad 3z &= 1 \\ x+2y-\left(a^2-3\right)z &= a \end{aligned}$$

27. $$\begin{aligned} x+ \quad 2y &= 1 \\ 2x+\left(a^2-5\right)y &= a-1 \end{aligned}$$

Answer:

If $a=3$, there are infinitely many solutions; if $a=-3$, there are no solutions; if $a\neq\pm 3$, there is exactly one solution.

28. $$\begin{aligned} x+y+ \quad 7z &= -7 \\ 2x+3y+ \quad 17z &= -16 \\ x+2y+\left(a^2+1\right)z &= 3a \end{aligned}$$

In Exercises 29–30, solve the following systems, where *a*, *b*, and *c* are constants.

29. $$\begin{aligned} 2x+y &= a \\ 3x+6y &= b \end{aligned}$$

Answer:

$x=\frac{2a}{3}-\frac{b}{9},\ y=-\frac{a}{3}+\frac{2b}{9}$

30. $$\begin{aligned} x_1+x_2+x_3 &= a \\ 2x_1 \quad +2x_3 &= b \\ 3x_2+3x_3 &= c \end{aligned}$$

31. Find two different row echelon forms of

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

This exercise shows that a matrix can have multiple row echelon forms.

Answer:

$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are possible answers.

32. Reduce

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix}$$

to reduced row echelon form without introducing fractions at any intermediate stage.

33. Show that the following nonlinear system has 18 solutions if $0 \le \alpha \le 2\pi$, $0 \le \gamma \le 2\pi$, and $0 \le \gamma < 2\pi$.

$$\begin{aligned} \sin\alpha + 2\cos\beta + 3\tan\gamma &= 0 \\ 2\sin\alpha + 5\cos\beta + 3\tan\gamma &= 0 \\ -\sin\alpha - 5\cos\beta + 5\tan\gamma &= 0 \end{aligned}$$

[*Hint:* Begin by making the substitutions $x = \sin\alpha$, $y = \cos\beta$, and $z = \tan\gamma$.]

34. Solve the following system of nonlinear equations for the unknown angles α, β, and γ, where $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$, and $0 \le \gamma < \pi$.

$$\begin{aligned} 2\sin\alpha - \cos\beta + 3\tan\gamma &= 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma &= 2 \\ 6\sin\alpha - 3\cos\beta + \tan\gamma &= 9 \end{aligned}$$

35. Solve the following system of nonlinear equations for *x*, *y*, and *z*.

$$\begin{aligned} x^2 + y^2 + z^2 &= 6 \\ x^2 - y^2 + 2z^2 &= 2 \\ 2x^2 + y^2 - z^2 &= 3 \end{aligned}$$

[*Hint:* Begin by making the substitutions $X = x^2$, $Y = y^2$, $Z = z^2$.]

Answer:

$x = \pm 1,\ y = \pm\sqrt{3},\ z = \pm\sqrt{2}$

36. Solve the following system for *x*, *y*, and *z*.

$$\begin{aligned} \frac{1}{x} + \frac{2}{y} - \frac{4}{z} &= 1 \\ \frac{2}{x} + \frac{3}{y} + \frac{8}{z} &= 0 \\ -\frac{1}{x} + \frac{9}{y} + \frac{10}{z} &= 5 \end{aligned}$$

37. Find the coefficients *a*, *b*, *c*, and *d* so that the curve shown in the accompanying figure is the graph of the equation $y = ax^3 + bx^2 + cx + d$.

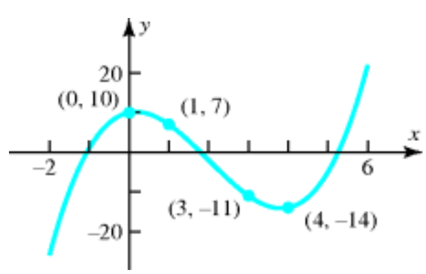


Figure Ex-37

Answer:

$a = 1,\ b = -6,\ c - 2,\ d = 10$

38. Find the coefficients a, b, c, and d so that the curve shown in the accompanying figure is given by the equation $ax^2 + ay^2 + bx + cy + d = 0$.

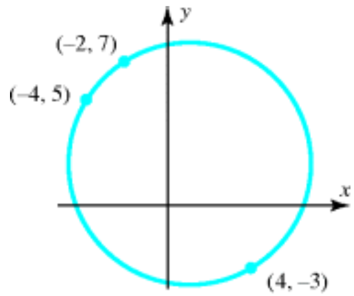


Figure Ex-38

39. If the linear system

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x - b_2y + c_2z &= 0 \\ a_3x + b_3y - c_3z &= 0 \end{aligned}$$

has only the trivial solution, what can be said about the solutions of the following system?

$$\begin{aligned} a_1x + b_1y + c_1z &= 3 \\ a_2x - b_2y + c_2z &= 7 \\ a_3x + b_3y - c_3z &= 11 \end{aligned}$$

Answer:

The nonhomogeneous system will have exactly one solution.

40. (a) If A is a 3×5 matrix, then what is the maximum possible number of leading 1's in its reduced row echelon form?

(b) If B is a 3×6 matrix whose last column has all zeros, then what is the maximum possible number of parameters in the general solution of the linear system with augmented matrix B?

(c) If C is a 5×3 matrix, then what is the minimum possible number of rows of zeros in any row echelon form of C?

41. (a) Prove that if $ad - bc \neq 0$, then the reduced row echelon form of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Use the result in part (a) to prove that if $ad - bc \neq 0$, then the linear system

$$\begin{aligned} ax + by &= k \\ cx + dy &= l \end{aligned}$$

has exactly one solution.

42. Consider the system of equations

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \\ ex + fy &= 0 \end{aligned}$$

Discuss the relative positions of the lines $ax + by = 0$, $cx + dy = 0$, and $ex + fy = 0$ when (a) the system has only the trivial solution, and (b) the system has nontrivial solutions.

43. Describe all possible reduced row echelon forms of

(a) $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

(b) $\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{bmatrix}$

True-False Exercises

In parts (a)–(i) determine whether the statement is true or false, and justify your answer.

(a) If a matrix is in reduced row echelon form, then it is also in row echelon form.

Answer:

True

(b) If an elementary row operation is applied to a matrix that is in row echelon form, the resulting matrix will still be in row echelon form.

Answer:

False

(c) Every matrix has a unique row echelon form.

Answer:

False

(d) A homogeneous linear system in n unknowns whose corresponding augmented matrix has a reduced row echelon form with r leading 1's has $n - r$ free variables.

Answer:

True

(e) All leading 1's in a matrix in row echelon form must occur in different columns.

Answer:

True

(f) If every column of a matrix in row echelon form has a leading 1 then all entries that are not leading 1's are zero.

Answer:

False

(g) If a homogeneous linear system of n equations in n unknowns has a corresponding augmented matrix with a reduced row echelon form containing n leading 1's, then the linear system has only the trivial solution.

Answer:

True

(h) If the reduced row echelon form of the augmented matrix for a linear system has a row of zeros, then the system must have infinitely many solutions.

Answer:

False

(i) If a linear system has more unknowns than equations, then it must have infinitely many solutions.

Answer:

False

1.3 Matrices and Matrix Operations

Rectangular arrays of real numbers arise in contexts other than as augmented matrices for linear systems. In this section we will begin to study matrices as objects in their own right by defining operations of addition, subtraction, and multiplication on them.

Matrix Notation and Terminology

In Section 1.2 we used rectangular arrays of numbers, called *augmented matrices*, to abbreviate systems of linear equations. However, rectangular arrays of numbers occur in other contexts as well. For example, the following rectangular array with three rows and seven columns might describe the number of hours that a student spent studying three subjects during a certain week:

	Mon.	**Tues.**	**Wed.**	**Thurs.**	**Fri.**	**Sat.**	**Sun.**
Math	2	3	2	4	1	4	2
History	0	3	1	4	3	2	2
Language	4	1	3	1	0	0	2

If we suppress the headings, then we are left with the following rectangular array of numbers with three rows and seven columns, called a “matrix”:

$$\begin{bmatrix} 2 & 3 & 2 & 4 & 1 & 4 & 2 \\ 0 & 3 & 1 & 4 & 3 & 2 & 2 \\ 4 & 1 & 3 & 1 & 0 & 0 & 2 \end{bmatrix}$$

More generally, we make the following definition.

DEFINITION 1

A ***matrix*** is a rectangular array of numbers. The numbers in the array are called the ***entries*** in the matrix.

A matrix with only one column is called a ***column vector*** or a ***column matrix***, and a matrix with only one row is called a ***row vector*** or a ***row matrix***. In Example 1, the 2×1 matrix is a column vector, the 1×4 matrix is a row vector, and the 1×1 matrix is both a row vector and a column vector.

EXAMPLE 1 Examples of Matrices

Some examples of matrices are

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad [2 \;\; 1 \;\; 0 \;\; -3], \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad [4]$$

The ***size*** of a matrix is described in terms of the number of rows (horizontal lines) and columns (vertical lines) it contains. For example, the first matrix in Example 1 has three rows and two columns, so its size is 3 by 2 (written 3×2). In a size description, the first number always denotes the number of rows, and the second denotes the number of columns. The remaining matrices in Example 1 have sizes 1×4, 3×3, 2×1, and 1×1, respectively.

We will use capital letters to denote matrices and lowercase letters to denote numerical quantities; thus we might write

$$A = \begin{bmatrix} 2 & 1 & 7 \\ 3 & 4 & 2 \end{bmatrix} \text{ or } C = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

When discussing matrices, it is common to refer to numerical quantities as ***scalars***. Unless stated otherwise, *scalars will be real numbers*; complex scalars will be considered later in the text.

Matrix brackets are often omitted from 1×1 matrices, making it impossible to tell, for example, whether the symbol 4 denotes the number “four” or the matrix [4]. This rarely causes problems because it is usually possible to tell which is meant from the context.

The entry that occurs in row *i* and column *j* of a matrix *A* will be denoted by a_{ij}. Thus a general 3×4 matrix might be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{1}$$

When a compact notation is desired, the preceding matrix can be written as

$$[a_{ij}]_{m \times n} \text{ or } [a_{ij}]$$

the first notation being used when it is important in the discussion to know the size, and the second being used when the size need not be emphasized. Usually, we will match the letter denoting a matrix with the letter denoting its entries; thus, for a matrix *B* we would generally use b_{ij} for the entry in row *i* and column *j*, and for a matrix *C* we would use the notation c_{ij}.

The entry in row *i* and column *j* of a matrix *A* is also commonly denoted by the symbol $(A)_{ij}$. Thus, for matrix 1 above, we have

$$(A)_{ij} = a_{ij}$$

and for the matrix

$$A = \begin{bmatrix} 2 & -3 \\ 7 & 0 \end{bmatrix}$$

we have $(A)_{11} = 2,\ (A)_{12} = -3,\ (A)_{21} = 7$, and $(A)_{22} = 0$.

Row and column vectors are of special importance, and it is common practice to denote them by boldface lowercase letters rather than capital letters. For such matrices, double subscripting of the entries is unnecessary. Thus a general $1 \times n$ row vector a and a general $m \times 1$ column vector **b** would be written as

$$\mathbf{a} = [a_1\ a_2\ \cdots\ a_n] \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A matrix *A* with *n* rows and *n* columns is called a ***square matrix of order n***, and the shaded entries $a_{11},\ a_{22},\ \ldots,\ a_{nn}$ in 2 are said to be on the ***main diagonal*** of *A*.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{2}$$

Operations on Matrices

So far, we have used matrices to abbreviate the work in solving systems of linear equations. For other applications, however, it is desirable to develop an "arithmetic of matrices" in which matrices can be added, subtracted, and multiplied in a useful way. The remainder of this section will be devoted to developing this arithmetic.

DEFINITION 2

Two matrices are defined to be ***equal*** if they have the same size and their corresponding entries are equal.

The equality of two matrices

$$A = [a_{ij}] \text{ and } B = [b_{ij}]$$

of the same size can be expressed either by writing

$$(A)_{ij} = (B)_{ij}$$

or by writing

$$a_{ij} = b_{ij}$$

where it is understood that the equalities hold for all values of *i* and *j*.

EXAMPLE 2 Equality of Matrices

Consider the matrices

$$A=\begin{bmatrix}2 & 1\\ 3 & x\end{bmatrix},\quad B=\begin{bmatrix}2 & 1\\ 3 & 5\end{bmatrix},\quad C=\begin{bmatrix}2 & 1 & 0\\ 3 & 4 & 0\end{bmatrix}$$

If $x=5$, then $A=B$, but for all other values of x the matrices A and B are not equal, since not all of their corresponding entries are equal. There is no value of x for which $A=C$ since A and C have different sizes.

DEFINITION 3

If A and B are matrices of the same size, then the ***sum*** $A+B$ is the matrix obtained by adding the entries of B to the corresponding entries of A, and the ***difference*** $A-B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A. Matrices of different sizes cannot be added or subtracted.

In matrix notation, if $A=[a_{ij}]$ and $B=[b_{ij}]$ have the same size, then

$$(A+B)_{ij}=(A)_{ij}+(B)_{ij}=a_{ij}+b_{ij} \text{ and } (A-B)_{ij}=(A)_{ij}-(B)_{ij}=a_{ij}-b_{ij}$$

EXAMPLE 3 Addition and Subtraction

Consider the matrices

$$A=\begin{bmatrix}2 & 1 & 0 & 3\\ -1 & 0 & 2 & 4\\ 4 & -2 & 7 & 0\end{bmatrix},\quad B=\begin{bmatrix}-4 & 3 & 5 & 1\\ 2 & 2 & 0 & -1\\ 3 & 2 & -4 & 5\end{bmatrix},\quad C=\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$$

Then

$$A+B=\begin{bmatrix}-2 & 4 & 5 & 4\\ 1 & 2 & 2 & 3\\ 7 & 0 & 3 & 5\end{bmatrix} \text{ and } A-B=\begin{bmatrix}6 & -2 & -5 & 2\\ -3 & -2 & 2 & 5\\ 1 & -4 & 11 & -5\end{bmatrix}$$

The expressions $A+C$, $B+C$, $A-C$, and $B-C$ are undefined.

DEFINITION 4

If A is any matrix and c is any scalar, then the ***product*** cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is said to be a ***scalar multiple*** of A.

In matrix notation, if $A=[a_{ij}]$, then

$$(cA)_{ij}=c(A)_{ij}=ca_{ij}$$

EXAMPLE 4 Scalar Multiples

For the matrices

$$A=\begin{bmatrix}2 & 3 & 4\\ 1 & 3 & 1\end{bmatrix},\quad B=\begin{bmatrix}0 & 2 & 7\\ -1 & 3 & -5\end{bmatrix},\quad C=\begin{bmatrix}9 & -6 & 3\\ 3 & 0 & 12\end{bmatrix}$$

we have

$$2A=\begin{bmatrix}4 & 6 & 8\\ 2 & 6 & 2\end{bmatrix},\quad (-1)B=\begin{bmatrix}0 & -2 & -7\\ 1 & -3 & 5\end{bmatrix},\quad \tfrac{1}{3}C=\begin{bmatrix}3 & -2 & 1\\ 1 & 0 & 4\end{bmatrix}$$

It is common practice to denote $(-1)B$ by $-B$.

Thus far we have defined multiplication of a matrix by a scalar but not the multiplication of two matrices. Since matrices are added by adding corresponding entries and subtracted by subtracting corresponding entries, it would seem natural to define multiplication of matrices by multiplying corresponding entries. However, it turns out that such a definition would not be very useful for most problems. Experience has led mathematicians to the following more useful definition of matrix multiplication.

DEFINITION 5

If A is an $m\times r$ matrix and B is an $r\times n$ matrix, then the ***product*** AB is the $m\times n$ matrix whose entries are determined as follows: To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together, and then add up the resulting products.

EXAMPLE 5 Multiplying Matrices

Consider the matrices

$$A=\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix},\quad B=\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}$$

Since A is a 2×3 matrix and B is a 3×4 matrix, the product AB is a 2×4 matrix. To determine, for example, the entry in row 2 and column 3 of AB, we single out row 2 from A and column 3 from B. Then, as illustrated below, we multiply corresponding entries together and add up these products.

$$\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}=\begin{bmatrix}\square & \square & \square & \square\\ \square & \square & \boxed{26} & \square\end{bmatrix}$$

$$(2\cdot 4)+(6\cdot 3)+(0\cdot 5)=26$$

The entry in row 1 and column 4 of AB is computed as follows:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}\begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \boxed{13} \\ \square & \square & \square & \square \end{bmatrix}$$

$$(1 \cdot 3) + (2 \cdot 1) + (4 \cdot 2) = 13$$

The computations for the remaining entries are

$$\begin{aligned} (1.4) + (2.0) + (4.2) &= 12 \\ (1.1) - (2.1) + (4.7) &= 27 \\ (1.4) + (2.3) + (4.5) &= 30 \\ (2.4) + (6.0) + (0.2) &= 8 \\ (2.1) - (6.1) + (0.7) &= -4 \\ (2.3) + (6.1) + (0.2) &= 12 \end{aligned} \qquad AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

The definition of matrix multiplication requires that the number of columns of the first factor *A* be the same as the number of rows of the second factor *B* in order to form the product *AB*. If this condition is not satisfied, the product is undefined. A convenient way to determine whether a product of two matrices is defined is to write down the size of the first factor and, to the right of it, write down the size of the second factor. If, as in 3, the inside numbers are the same, then the product is defined. The outside numbers then give the size of the product.

$$\overset{A}{m \times r} \qquad \overset{B}{r \times n} = \overset{AB}{m \times n} \tag{3}$$

Inside

Outside

Gotthold Eisenstein (1823–1852)

Historical Note The concept of matrix multiplication is due to the German mathematician Gotthold Eisenstein, who introduced the idea around 1844 to simplify the process of making substitutions in linear systems. The idea was then expanded on and formalized by Cayley in his *Memoir on the Theory of Matrices* that was published in 1858. Eisenstein was a pupil of Gauss, who ranked him as the equal of Isaac Newton and Archimedes. However, Eisenstein, suffering from bad health his entire life, died at age 30, so his potential was never realized.
[*Image: wikipedia*]

EXAMPLE 6 Determining Whether a Product Is Defined

Suppose that *A, B*, and *C* are matrices with the following sizes:

$$\begin{matrix} A & B & C \\ 3\times 4 & 4\times 7 & 7\times 3 \end{matrix}$$

Then by 3, *AB* is defined and is a 3×7 matrix; *BC* is defined and is a 4×3 matrix; and *CA* is defined and is a 7×4 matrix. The products *AC, CB*, and *BA* are all undefined.

In general, if $A=[a_{ij}]$ is an $m\times r$ matrix and $B=[b_{ij}]$ is an $r\times n$ matrix, then, as illustrated by the shading in 4,

$$AB=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix} \tag{4}$$

the entry $(AB)_{ij}$ in row *i* and column *j* of *AB* is given by

$$(AB)_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+a_{i3}b_{3j}+\cdots+a_{ir}b_{rj} \tag{5}$$

Partitioned Matrices

A matrix can be subdivided or ***partitioned*** into smaller matrices by inserting horizontal and vertical rules between selected rows and columns. For example, the following are three possible partitions of a general 3×4 matrix *A*—the first is a partition of *A* into four ***submatrices*** A_{11}, A_{12}, A_{21}, and A_{22}; the second is a partition of *A* into its row vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$; and the third is a partition of *A* into its column vectors $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, and $\mathbf{c}_4$:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}=\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}$$

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}=[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3 \quad \mathbf{c}_4]$$

Matrix Multiplication by Columns and by Rows

Partitioning has many uses, one of which is for finding particular rows or columns of a matrix product *AB* without computing the entire product. Specifically, the following formulas, whose proofs are left as exercises, show how individual column vectors of *AB* can be obtained by partitioning *B* into column vectors and how individual row vectors of *AB* can be obtained by partitioning *A* into row vectors.

$$AB = A[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n] = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_n] \tag{6}$$

$$(AB \text{ computed column by column})$$

$$AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B \end{bmatrix} \tag{7}$$

$$(AB \text{ computed row by row})$$

In words, these formulas state that

$$j\text{ th column vector of } AB = A[j\text{ th column vector of } B] \tag{8}$$

$$i\text{ th row vector of } AB = [i\text{ th row vector of } A]B \tag{9}$$

EXAMPLE 7 Example 5 Revisited

If A and B are the matrices in Example 5, then from 8 the second column vector of AB can be obtained by the computation

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \underset{\substack{\uparrow \\ \text{Second column of } B}}{\begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}} = \underset{\substack{\uparrow \\ \text{Second column of } AB}}{\begin{bmatrix} 27 \\ -4 \end{bmatrix}}$$

and from 9 the first row vector of AB can be obtained by the computation

$$\underset{\text{First row of } A}{[1 \quad 2 \quad 4]} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \underset{\text{First row of } AB}{[12 \quad 27 \quad 30 \quad 13]}$$

Matrix Products as Linear Combinations

We have discussed three methods for computing a matrix product AB—entry by entry, column by column, and row by row. The following definition provides yet another way of thinking about matrix multiplication.

DEFINITION 6

If $A_1,\ A_2,\ \ldots,\ A_r$ are matrices of the same size, and if $c_1,\ c_2,\ \ldots,\ c_r$ are scalars, then an expression of the form

$$c_1A_1 + c_2A_2 + \cdots + c_rA_r$$

is called a ***linear combination*** of $A_1, A_2, \ldots, A_r$ with ***coefficients*** $c_1, c_2, \ldots, c_r$.

To see how matrix products can be viewed as linear combinations, let A be an $m \times n$ matrix and **x** an $n \times 1$ column vector, say

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then

$$A = \begin{bmatrix} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n \\ \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \tag{10}$$

This proves the following theorem.

THEOREM 1.3.1

If A is an $m \times n$ matrix, and if **x** *is an $n \times 1$ column vector, then the product A***x** *can be expressed as a linear combination of the column vectors of A in which the coefficients are the entries of* **x**.

EXAMPLE 8 Matrix Products as Linear Combinations

The matrix product

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

can be written as the following linear combination of column vectors

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

EXAMPLE 9 Columns of a Product *AB* as Linear Combinations

We showed in Example 5 that

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

It follows from Formula 6 and Theorem 1.3.1 that the j th column vector of AB can be expressed as a linear combination of the column vectors of A in which the coefficients in the linear combination are the entries from the j th column of B. The computations are as follows:

$$\begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 27 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ 26 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 5\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 13 \\ 12 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Matrix Form of a Linear System

Matrix multiplication has an important application to systems of linear equations. Consider a system of m linear equations in n unknowns:

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = b_2 \\ \vdots & & \vdots & & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = b_m \end{array}$$

Since two matrices are equal if and only if their corresponding entries are equal, we can replace the m equations in this system by the single matrix equation

$$\begin{bmatrix} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n \\ \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The $m \times 1$ matrix on the left side of this equation can be written as a product to give

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If we designate these matrices by A, $\mathbf{x}$, and $\mathbf{b}$, respectively, then we can replace the original system of m equations in n unknowns has been replaced by the single matrix equation

$$A\mathbf{x} = \mathbf{b}$$

The matrix A in this equation is called the ***coefficient matrix*** of the system. The ***augmented matrix*** for the system is obtained by adjoining $\mathbf{b}$ to A as the last column; thus the augmented matrix is

$$[A|\mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

The vertical bar in $[A|\mathbf{b}]$ is a convenient way to separate A from $\mathbf{b}$ visually; it has no mathematical significance.

Transpose of a Matrix

We conclude this section by defining two matrix operations that have no analogs in the arithmetic of real numbers.

DEFINITION 7

If A is any $m \times n$ matrix, then the ***transpose of A***, denoted by A^T, is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A; that is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth.

EXAMPLE 10 Some Transposes

The following are some examples of matrices and their transposes.

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B=\begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C=\begin{bmatrix} 1 & 3 & 5 \end{bmatrix}, \quad D=[4]$$

$$A^T=\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}, \quad B^T=\begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}, \quad C^T=\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad D^T=[4]$$

Observe that not only are the columns of A^T the rows of A, but the rows of A^T are the columns of A. Thus the entry in row i and column j of A^T is the entry in row j and column i of A; that is,

$$\left(A^T\right)_{ij} = (A)_{ji} \tag{11}$$

Note the reversal of the subscripts.

In the special case where A is a square matrix, the transpose of A can be obtained by interchanging entries that are symmetrically positioned about the main diagonal. In 12 we see that A^T can also be obtained by "reflecting" A about its main diagonal.

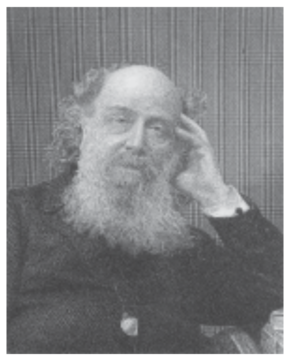
(12)

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 3 & -5 \\ -2 & 7 & 8 \\ 4 & 0 & 6 \end{bmatrix}$$

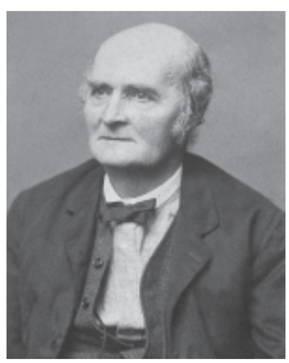
Interchange entries that are symmetrically positioned about the main diagonal.

DEFINITION 8

If A is a square matrix, then the ***trace of A***, denoted by $\mathrm{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.



James Sylvester (1814–1897)



Arthur Cayley (1821–1895)

Historical Note The term *matrix* was first used by the English mathematician (and lawyer) James Sylvester, who defined the term in 1850 to be an “oblong arrangement of terms.” Sylvester communicated his work on matrices to a fellow English mathematician and lawyer named Arthur Cayley, who then introduced some of the basic operations on matrices in a book entitled *Memoir on the Theory of Matrices* that was published in 1858. As a matter of interest, Sylvester, who was Jewish, did not get his college degree because he refused to sign a required oath to the Church of England. He was appointed to a chair at the University of Virginia in the United States but resigned after swatting a student with a stick because he was reading a newspaper in class.

Sylvester, thinking he had killed the student, fled back to England on the first available ship. Fortunately, the student was not dead, just in shock!
[*Images: The Granger Collection, New York*]

EXAMPLE 11 Trace of a Matrix

The following are examples of matrices and their traces.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\operatorname{tr}(A) = a_{11} + a_{22} + a_{33} \quad \operatorname{tr}(B) = -1 + 5 + 7 + 0 = 11$$

In the exercises you will have some practice working with the transpose and trace operations.

Concept Review

- Matrix
- Entries
- Column vector (or column matrix)
- Row vector (or row matrix)
- Square matrix
- Main diagonal
- Equal matrices
- Matrix operations: sum, difference, scalar multiplication
- Linear combination of matrices
- Product of matrices (matrix multiplication)
- Partitioned matrices
- Submatrices
- Row-column method
- Column method
- Row method
- Coefficient matrix of a linear system
- Transpose
- Trace

Skills

- Determine the size of a given matrix.
- Identify the row vectors and column vectors of a given matrix.
- Perform the arithmetic operations of matrix addition, subtraction, scalar multiplication, and multiplication.
- Determine whether the product of two given matrices is defined.
- Compute matrix products using the row-column method, the column method, and the row method.
- Express the product of a matrix and a column vector as a linear combination of the columns of the matrix.
- Express a linear system as a matrix equation, and identify the coefficient matrix.
- Compute the transpose of a matrix.
- Compute the trace of a square matrix.

Exercise Set 1.3

1. Suppose that A, B, C, D, and E are matrices with the following sizes:

A	B	C	D	E
(4×5)	(4×5)	(5×2)	(4×2)	(5×4)

In each part, determine whether the given matrix expression is defined. For those that are defined, give the size of the resulting matrix.

(a) BA

(b) $AC + D$

(c) $AE + B$

(d) $AB + B$

(e) $E(A + B)$

(f) $E(AC)$

(g) E^TA

(h) $(A^T + E)D$

Answer:

(a) Undefined

(b) 4×2

(c) Undefined

(d) Undefined

(e) 5×5

(f) 5×2

(g) Undefined

(h) 5×2

2. Suppose that A, B, C, D, and E are matrices with the following sizes:

$$\begin{array}{ccccc} A & B & C & D & E \\ (3\times 1) & (3\times 6) & (6\times 2) & (2\times 6) & (1\times 3) \end{array}$$

In each part, determine whether the given matrix expression is defined. For those that are defined, give the size of the resulting matrix.

(a) EA

(b) AB^T

(c) $B^T\left(A+E^T\right)$

(d) $2A+C$

(e) $\left(C^T+D\right)B^T$

(f) $CD+B^TE^T$

(g) $\left(BD^T\right)C^T$

(h) $DC+EA$

3. Consider the matrices

$$A=\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix},\quad B=\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix},\quad C=\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix},\quad D=\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix},\quad E=\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

In each part, compute the given expression (where possible).

(a) $D+E$

(b) $D-E$

(c) $5A$

(d) $-7C$

(e) $2B-C$

(f) $4E-2D$

(g) $-3(D+2E)$

(h) $A-A$

(i) $\mathrm{tr}(D)$

(j) $\mathrm{tr}(D-3E)$

(k) $4\ \mathrm{tr}(7B)$

(l) $\mathrm{tr}(A)$

Answer:

(a) $\begin{bmatrix} 7 & 6 & 5 \\ -2 & 1 & 3 \\ 7 & 3 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 15 & 0 \\ -5 & 10 \\ 5 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} -7 & -28 & -14 \\ -21 & -7 & -35 \end{bmatrix}$

(e) Undefined

(f) $\begin{bmatrix} 22 & -6 & 8 \\ -2 & 4 & 6 \\ 10 & 0 & 4 \end{bmatrix}$

(g) $\begin{bmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{bmatrix}$

(h) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

(i) 5

(j) -25

(k) 168

(l) Undefined

4. Using the matrices in Exercise 3, in each part compute the given expression (where possible).

(a) $2A^T + C$

(b) $D^T - E^T$

(c) $(D - E)^T$

(d) $B^T + 5C^T$

(e) $\frac{1}{2}C^T - \frac{1}{4}A$

(f) $B - B^T$

(g) $2E^T - 3D^T$

(h) $\left(2E^T - 3D^T\right)^T$

(i) $(CD)E$

(j) $C(BA)$

(k) $\mathrm{tr}(DE^T)$

(l) $\mathrm{tr}(BC)$

5. Using the matrices in Exercise 3, in each part compute the given expression (where possible).

(a) AB

(b) BA

(c) $(3E)D$

(d) $(AB)C$

(e) $A(BC)$

(f) CC^T

(g) $(DA)^T$

(h) $\left(C^TB\right)A^T$

(i) $\text{tr}(DD^T)$

(j) $\text{tr}\left(4E^T - D\right)$

(k) $\text{tr}\left(C^TA^T + 2E^T\right)$

(l) $\text{tr}\left(\left(EC^T\right)^TA\right)$

Answer:

(a) $\begin{bmatrix} 12 & -3 \\ -4 & 5 \\ 4 & 1 \end{bmatrix}$

(b) Undefined

(c) $\begin{bmatrix} 42 & 108 & 75 \\ 12 & -3 & 21 \\ 36 & 78 & 63 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & 45 & 9 \\ 11 & -11 & 17 \\ 7 & 17 & 13 \end{bmatrix}$

(f) $\begin{bmatrix} 21 & 17 \\ 17 & 35 \end{bmatrix}$

(g) $\begin{bmatrix} 0 & -2 & 11 \\ 12 & 1 & 8 \end{bmatrix}$

(h) $\begin{bmatrix} 12 & 6 & 9 \\ 48 & -20 & 14 \\ 24 & 8 & 16 \end{bmatrix}$

(i) 61

(j) 35

(k) 28

(l) 99

6. Using the matrices in Exercise 3, in each part compute the given expression (where possible).

(a) $\left(2D^T - E\right)A$

(b) $(4B)C + 2B$

(c) $(-AC)^T + 5D^T$

(d) $\left(BA^T - 2C\right)^T$

(e) $B^T\left(CC^T - A^TA\right)$

(f) $D^TE^T - (ED)^T$

7. Let

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$$

Use the row method or column method (as appropriate) to find

(a) the first row of *AB*.

(b) the third row of *AB*.

(c) the second column of *AB*.

(d) the first column of *BA*.

(e) the third row of *AA*.

(f) the third column of *AA*.

Answer:

(a) $[67\ 41\ 41]$

(b) $[63\ 67\ 57]$

(c) $\begin{bmatrix} 41 \\ 21 \\ 67 \end{bmatrix}$

(d) $\begin{bmatrix} 6 \\ 6 \\ 63 \end{bmatrix}$

(e) $[24\ 56\ 97]$

(f) $\begin{bmatrix} 76 \\ 98 \\ 97 \end{bmatrix}$

8. Referring to the matrices in Exercise 7, use the row method or column method (as appropriate) to find

(a) the first column of *AB*.

(b) the third column of *BB*.

(c) the second row of *BB*.

(d) the first column of *AA*.

(e) the third column of *AB*.

(f) the first row of *BA*.

9. Referring to the matrices A and B in Exercise 7, and Example 9,

(a) express each column vectorof *AA* as a linear combination of the column vectors of *A*.

(b) express each column vector of *BB* as a linear combination of the column vectors of *B*.

Answer:

(a) $\begin{bmatrix}-3\\48\\24\end{bmatrix}=3\begin{bmatrix}3\\6\\0\end{bmatrix}+6\begin{bmatrix}-2\\5\\4\end{bmatrix}$; $\begin{bmatrix}12\\29\\56\end{bmatrix}=-2\begin{bmatrix}3\\6\\0\end{bmatrix}+5\begin{bmatrix}-2\\5\\4\end{bmatrix}+4\begin{bmatrix}7\\4\\9\end{bmatrix}$; $\begin{bmatrix}76\\98\\97\end{bmatrix}=7\begin{bmatrix}3\\6\\0\end{bmatrix}+4\begin{bmatrix}-2\\5\\4\end{bmatrix}+9\begin{bmatrix}7\\4\\9\end{bmatrix}$

(b) $\begin{bmatrix}64\\21\\77\end{bmatrix}=6\begin{bmatrix}6\\0\\7\end{bmatrix}+7\begin{bmatrix}4\\3\\5\end{bmatrix}$; $\begin{bmatrix}14\\22\\28\end{bmatrix}=-2\begin{bmatrix}6\\0\\7\end{bmatrix}+\begin{bmatrix}-2\\1\\7\end{bmatrix}+7\begin{bmatrix}4\\3\\5\end{bmatrix}$; $\begin{bmatrix}38\\18\\74\end{bmatrix}=4\begin{bmatrix}6\\0\\7\end{bmatrix}+3\begin{bmatrix}-2\\1\\7\end{bmatrix}+5\begin{bmatrix}4\\3\\5\end{bmatrix}$

10. Referring to the matrices A and B in Exercise 7, and Example 9,

(a) express each column vector of *AB* as a linear combination of the column vectors of *A*.

(b) express each column vector of *BA* as a linear combination of the column vectors of *B*.

11. In each part, find matrices *A*, **x**, and **b** that express the given system of linear equations as a single matrix equation $A\mathbf{x}=\mathbf{b}$, and write out this matrix equation.

(a)
$$\begin{aligned}2x_1-3x_2+5x_3&=7\\9x_1-x_2+x_3&=-1\\x_1+5x_2+4x_3&=0\end{aligned}$$

(b)
$$\begin{aligned}4x_1\qquad-3x_3+x_4&=1\\5x_1+x_2\qquad-8x_4&=3\\2x_1-5x_2+9x_3-x_4&=0\\3x_2-x_3+7x_4&=2\end{aligned}$$

Answer:

(a) $\begin{bmatrix}2&-3&5\\9&-1&1\\1&5&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}7\\-1\\0\end{bmatrix}$

(b) $\begin{bmatrix}4&0&-3&1\\5&1&0&-8\\2&-5&9&-1\\0&3&-1&7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}1\\3\\0\\2\end{bmatrix}$

12. In each part, find matrices *A*, **x**, and **b** that express the given system of linear equations as a single matrix equation $A\mathbf{x}=\mathbf{b}$, and write out this matrix equation.

(a)
$$\begin{aligned}x_1-2x_2+3x_3&=-3\\2x_1+x_2\qquad&=0\\-3x_2+4x_3&=1\\x_1\qquad+x_3&=5\end{aligned}$$

(b)
$$\begin{aligned}3x_1+3x_2+3x_3&=-3\\-x_1-5x_2-2x_3&=3\\-4x_2+x_3&=0\end{aligned}$$

13. In each part, express the matrix equation as a system of linear equations.

(a) $\begin{bmatrix}5&6&-7\\-1&-2&3\\0&4&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\0\\3\end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 5 & -3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -9 \end{bmatrix}$

Answer:

(a) $\begin{array}{rcrcrcr} 5x_1 & + & 6x_2 & - & 7x_3 & = & 2 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & 0 \\ & & 4x_2 & - & x_3 & = & 3 \end{array}$

(b) $\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 3x_2 & & & = & 2 \\ 5x_1 & - & 3x_2 & - & 6x_3 & = & -9 \end{array}$

14. In each part, express the matrix equation as a system of linear equations.

(a) $\begin{bmatrix} 3 & -1 & 2 \\ 4 & 3 & 7 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -2 & 0 & 1 \\ 5 & 0 & 2 & -2 \\ 3 & 1 & 4 & 7 \\ -2 & 5 & 1 & 6 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

In Exercises 15–16, find all values of *k*, if any, that satisfy the equation.

15. $\begin{bmatrix} k & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix}\begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = 0$

Answer:

-1

16. $\begin{bmatrix} 2 & 2 & k \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ k \end{bmatrix} = 0$

In Exercises 17–18, solve the matrix equation for *a*, *b*, *c*, and *d*.

17. $\begin{bmatrix} a & 3 \\ -1 & a+b \end{bmatrix} = \begin{bmatrix} 4 & d-2c \\ d+2c & -2 \end{bmatrix}$

Answer:

$a = 4,\ b = -6,\ c = -1,\ d = 1$

18. $\begin{bmatrix} a-b & b+a \\ 3d+c & 2d-c \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 7 & 6 \end{bmatrix}$

19. Let *A* be any $m \times n$ matrix and let *0* be the $m \times n$ matrix each of whose entries is zero. Show that if $kA = 0$, then $k = 0$ or $A = 0$.

20. (a) Show that if *AB* and *BA* are both defined, then *AB* and *BA* are square matrices.

(b) Show that if *A* is an $m \times n$ matrix and *A*(*BA*) is defined, then *B* is an $n \times m$ matrix.

21. Prove: If A and B are $n \times n$ matrices, then $\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B)$.

22. (a) Show that if A has a row of zeros and B is any matrix for which AB is defined, then AB also has a row of zeros.

(b) Find a similar result involving a column of zeros.

23. In each part, find a 6×6 matrix $[a_{ij}]$ that satisfies the stated condition. Make your answers as general as possible by using letters rather than specific numbers for the nonzero entries.

(a) $a_{ij} = 0$ if $i \neq j$

(b) $a_{ij} = 0$ if $i > j$

(c) $a_{ij} = 0$ if $i < j$

(d) $a_{ij} = 0$ if $|i - j| > 1$

Answer:

(a) $\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix}$

(b) $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ 0 & 0 & a_{33} & a_{34} & a_{35} & a_{36} \\ 0 & 0 & 0 & a_{44} & a_{45} & a_{46} \\ 0 & 0 & 0 & 0 & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix}$

(c) $\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$

(d) $\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} & 0 \\ 0 & 0 & 0 & a_{54} & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} \end{bmatrix}$

24. Find the 4×4 matrix $A = [a_{ij}]$ whose entries satisfy the stated condition.

(a) $a_{ij} = i + j$

(b) $a_{ij} = i^{j-1}$

(c) $a_{ij} = \begin{cases} 1 & \text{if } |i-j| > 1 \\ -1 & \text{if } |i-j| \le 1 \end{cases}$

25. Consider the function $y = f(x)$ defined for 2×1 matrices x by $y = Ax$, where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Plot $f(x)$ together with x in each case below. How would you describe the action of f?

(a) $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(b) $x = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

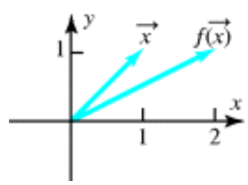
(c) $x = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

(d) $x = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

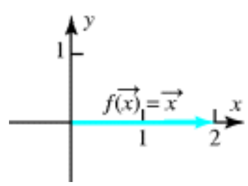
Answer:

$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 \end{pmatrix}$

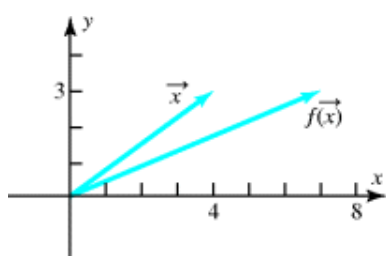
(a) $f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



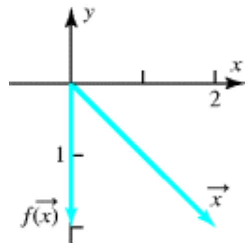
(b) $f\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$



(c) $f\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$



(d) $f\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$



26. Let I be the $n \times n$ matrix whose entry in row i and column j is

$$\begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Show that $AI = IA = A$ for every $n \times n$ matrix A.

27. How many 3×3 matrices A can you find such that

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \\ 0 \end{bmatrix}$$

for all choices of x, y, and z?

Answer:

One; namely, $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

28. How many 3×3 matrices A can you find such that

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} xy \\ 0 \\ 0 \end{bmatrix}$$

for all choices of x, y, and z?

29. A matrix B is said to be a ***square root*** of a matrix A if $BB = A$.

(a) Find two square roots of $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

(b) How many different square roots can you find of $A = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}$?

(c) Do you think that every 2×2 matrix has at least one square root? Explain your reasoning.

Answer:

(a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$

(b) Four; $\begin{bmatrix} \sqrt{5} & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} -\sqrt{5} & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} \sqrt{5} & 0 \\ 0 & -3 \end{bmatrix}$, $\begin{bmatrix} -\sqrt{5} & 0 \\ 0 & -3 \end{bmatrix}$

30. Let *0* denote a 2×2 matrix, each of whose entries is zero.

(a) Is there a 2×2 matrix A such that $A \neq 0$ and $AA = 0$? Justify your answer.

(b) Is there a 2×2 matrix A such that $A \neq 0$ and $AA = A$? Justify your answer.

True-False Exercises

In parts (a)–(o) determine whether the statement is true or false, and justify your answer.

(a) The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ has no main diagonal.

Answer:

True

(b) An $m \times n$ matrix has *m* column vectors and *n* row vectors.

Answer:

False

(c) If *A* and *B* are 2×2 matrices, then $AB = BA$.

Answer:

False

(d) The *i* th row vector of a matrix product *AB* can be computed by multiplying *A* by the *i*th row vector of *B*.

Answer:

False

(e) For every matrix *A*, it is true that $\left(A^T\right)^T = A$.

Answer:

True

(f) If *A* and *B* are square matrices of the same order, then $\text{tr}(AB) = \text{tr}(A)\text{tr}(B)$.

Answer:

False

(g) If *A* and *B* are square matrices of the same order, then $(AB)^T = A^T B^T$.

Answer:

False

(h) For every square matrix *A*, it is true that $\text{tr}\left(A^T\right) = \text{tr}(A)$.

Answer:

True

(i) If *A* is a 6×4 matrix and *B* is an $m \times n$ matrix such that $B^T A^T$ is a 2×6 matrix, then $m = 4$ and $n = 2$.

Answer:

True

(j) If *A* is an $n \times n$ matrix and *c* is a scalar, then $\operatorname{tr}(cA) = c \operatorname{tr}(A)$.

Answer:

True

(k) If *A*, *B*, and *C* are matrices of the same size such that $A - C = B - C$, then $A = B$.

Answer:

True

(l) If *A*, *B*, and *C* are square matrices of the same order such that $AC = BC$, then $A = B$.

Answer:

False

(m) If $AB + BA$ is defined, then *A* and *B* are square matrices of the same size.

Answer:

True

(n) If *B* has a column of zeros, then so does *AB* if this product is defined.

Answer:

True

(o) If *B* has a column of zeros, then so does *BA* if this product is defined.

Answer:

False