

Gradient Computation

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This document contains the gradient formula for the linear and nonlinear lasso problem.

1 Gradient computation

Define two points in spherical coordinates as follows:

$$\mathbf{Q} = (1, \theta, \psi), \quad \mathbf{P}_i = (r_i, \theta_i, \psi_i). \quad (1)$$

The normalized forward solution reads:

$$Est(\mathbf{Q}) = \hat{\Phi}(\mu)(\mathbf{Q}) = \sum_{i=1}^N \frac{I_i}{G(\mathbf{P}_i)} \phi_{\mathbf{P}_i}(\mathbf{Q}), \quad \mathbf{Q} \in \partial\Omega. \quad (2)$$

where,

$$\begin{aligned} \phi_{\mathbf{P}_i}(\mathbf{Q}) &= \frac{2}{\|\mathbf{Q} - \mathbf{P}_i\|_2} + \log\left(\frac{1}{1 - \langle \mathbf{P}_i, \mathbf{Q} \rangle + \|\mathbf{Q} - \mathbf{P}_i\|_2}\right) \\ &= \frac{2}{\sqrt{r^2 + r_i^2 - 2rr_i \cos \gamma_i}} - \log(1 - rr_i \cos \gamma_i + \sqrt{r^2 + r_i^2 - 2rr_i \cos \gamma_i}), \quad \text{where,} \\ \cos \gamma_i &= \cos \theta \cos \theta_i - \sin \theta \sin \theta_i (\psi - \psi_i). \end{aligned} \quad (3)$$

Define the source parameter vector as follows:

$$\mathbf{V} = \{\overbrace{I_1, I_2, \dots}^N, \overbrace{r_1, r_2, \dots}^N, \overbrace{\theta_1, \theta_2, \dots}^N, \overbrace{\psi_1, \psi_2, \dots}^N\}, \quad (4)$$

$$I_{tot} = \{I_1, I_2, \dots, I_N\} \quad (5)$$

For the linear lasso problem, we compute the pseudo gradient of the term:

$$Obj = \frac{1}{2} \|Est(\mathbf{Q}) - \phi^d(\mathbf{Q})\|_2^2 + \lambda \|I_{tot}\|_1, \quad (6)$$

$$= \int_{\partial\Omega} [Est(\mathbf{Q}) - \phi^d(\mathbf{Q})]^2 d\mathbf{Q} + \lambda \sum_i |I_i|. \quad (7)$$

For convenience, we introduce the following two terms:

$$Discrepancy(\mathbf{Q}) := \frac{1}{2} \|Est(\mathbf{Q}) - \phi^d(\mathbf{Q})\|_2^2 = \frac{1}{2} \int_{\partial\Omega} [Est(\mathbf{Q}) - \phi^d(\mathbf{Q})]^2 d\mathbf{Q}, \quad (8)$$

$$Regularization := \lambda \|I_{tot}\|_1. \quad (9)$$

1.1 Subgradient of the regularization term

$$\frac{\partial Regularization}{\partial I_i} = \lambda \text{sign}(I_i). \quad (10)$$

1.2 Linear lasso problem

FBS

For the FBS solver, only the gradient of the discrepancy term is required, and locations of the source are given. Therefore we only compute:

$$\begin{aligned} Grad_i &= \frac{\partial Discrepancy(\mathbf{Q})}{\partial I_i} = \int_{\partial\Omega} (Est(\mathbf{Q}) - \phi^d(\mathbf{Q})) \frac{\partial Est(\mathbf{Q})}{\partial I_i} d\mathbf{Q}, \\ &= \int_{\partial\Omega} (Est(\mathbf{Q}) - \phi^d(\mathbf{Q})) \frac{1}{G(\mathbf{P}_i)} \phi_{\mathbf{P}_i}(\mathbf{Q}) d\mathbf{Q}, \\ i &= 1, 2, \dots, N. \end{aligned} \quad (11)$$

In the minimization iteration, source location \mathbf{P}_i is provided by the solver in each search step.

Quasi Newton

For quasi newton solver, gradient of the whole objective function needs to be computed.

$$PseudoGrad_i = \int_{\partial\Omega} (Est(\mathbf{Q}) - \phi^d(\mathbf{Q})) \frac{1}{G(\mathbf{P}_i)} \phi_{\mathbf{P}_i}(\mathbf{Q}) d\mathbf{Q} + \lambda \text{sign}(I_i). \quad (12)$$

1.3 Nonlinear lasso problem

In the nonlinear lasso problem, the locations are also unknown. The pseudo gradient of the nonlinear lasso problem consists of two component: the gradient of the discrepancy term and the subgradient of the l1 regularization term.

The discrepancy gradient

$$\begin{aligned} &\text{for } i = 1 \rightarrow N, \\ Grad_i &= \frac{\partial Discrepancy(\mathbf{Q})}{\partial \mathbf{V}_i} = \frac{\partial Discrepancy(\mathbf{Q})}{\partial I_i} = \int_{\partial\Omega} (Est(\mathbf{Q}) - \phi^d(\mathbf{Q})) \frac{1}{G(\mathbf{P}_i)} \phi_{\mathbf{P}_i}(\mathbf{Q}) d\mathbf{Q}. \end{aligned} \quad (13)$$

$$\begin{aligned} &\text{for } i = N + 1 \rightarrow 2N, \\ Grad_i &= \frac{\partial Discrepancy(\mathbf{Q})}{\partial \mathbf{V}_i} = \frac{\partial Discrepancy(\mathbf{Q})}{\partial r_i} = \int_{\partial\Omega} (Est(\mathbf{Q}) - \phi^d(\mathbf{Q})) \frac{\partial Est(\mathbf{Q})}{\partial r_i} d\mathbf{Q}. \end{aligned} \quad (14)$$

$$\begin{aligned} &\text{for } i = 2N + 1 \rightarrow 3N, \\ Grad_i &= \frac{\partial Discrepancy(\mathbf{Q})}{\partial \mathbf{V}_i} = \frac{\partial Discrepancy(\mathbf{Q})}{\partial \theta_i} = \int_{\partial\Omega} (Est(\mathbf{Q}) - \phi^d(\mathbf{Q})) \frac{\partial Est(\mathbf{Q})}{\partial \theta_i} d\mathbf{Q}. \end{aligned} \quad (15)$$

$$\begin{aligned} &\text{for } i = 3N + 1 \rightarrow 4N, \\ Grad_i &= \frac{\partial Discrepancy(\mathbf{Q})}{\partial \mathbf{V}_i} = \frac{\partial Discrepancy(\mathbf{Q})}{\partial \psi_i} = \int_{\partial\Omega} (Est(\mathbf{Q}) - \phi^d(\mathbf{Q})) \frac{\partial Est(\mathbf{Q})}{\partial \psi_i} d\mathbf{Q}. \end{aligned} \quad (16)$$