

1 Solution of the monopole potential

In spherical coordinate system, consider this equation on B^3 :

$$\begin{aligned} \Delta\phi(\mathbf{Q}) &= \sum_{i=1}^N I_i \delta_{\mathbf{P}_i}(\mathbf{Q}), \quad \mathbf{Q}, \mathbf{P}_i \in \Omega \\ \frac{\partial\phi(\mathbf{Q})}{\partial\mathbf{n}} &= 0, \quad \mathbf{Q} \in \partial\Omega, \\ \text{where } \mathbf{Q} &= (r, \theta, \psi), \quad \mathbf{P}_i = (r_i, \theta_i, \psi_i), \quad \Omega = B^3(0, 1). \end{aligned} \tag{1}$$

The dirac delta function is defined as follows:

$$\begin{aligned} \delta_{\mathbf{x}}(\mathbf{y}) &:= \begin{cases} \infty, & \text{if } \mathbf{x} = \mathbf{y}, \\ 0, & \text{else,} \end{cases} \\ \text{s.t., } \int_{R^3} \delta_{\mathbf{x}}(\mathbf{y}) d\mathbf{y} &= 1, \text{ for } \mathbf{x}, \mathbf{y} \in R^3. \end{aligned} \tag{2}$$

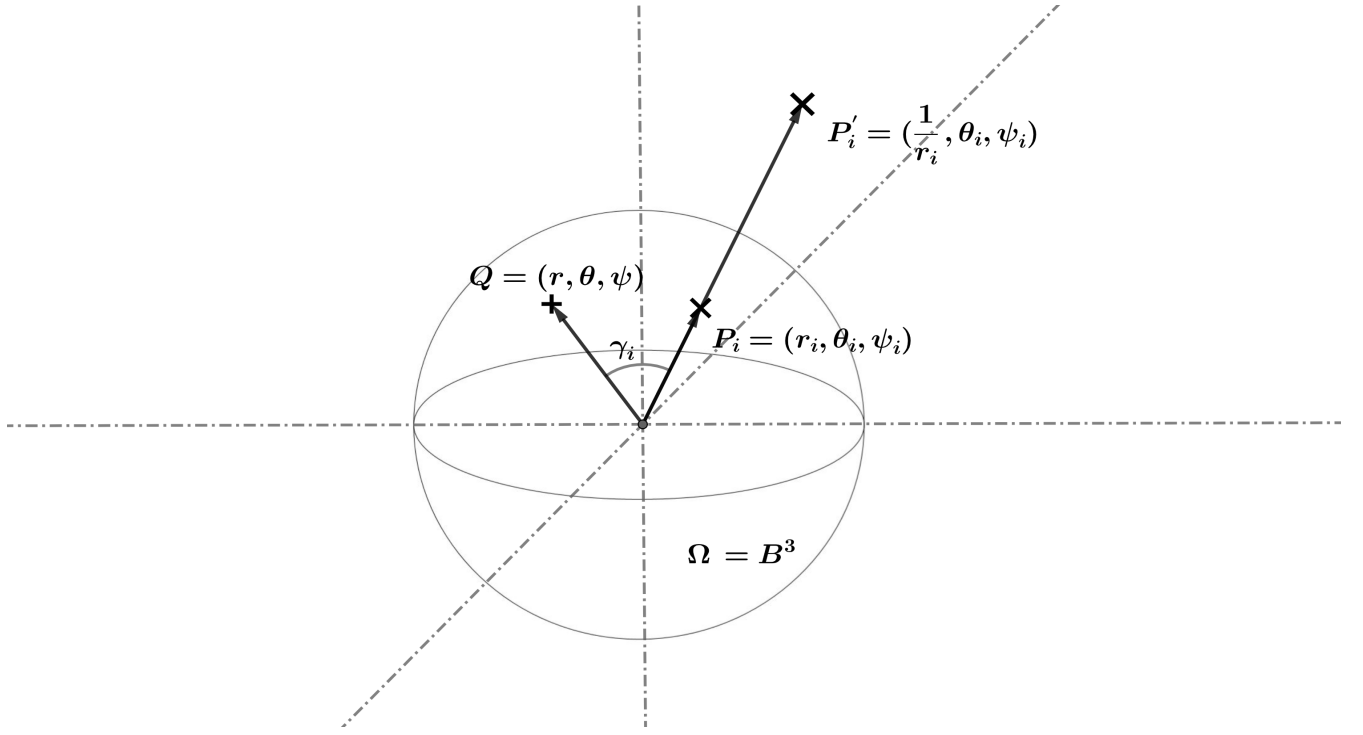


Figure 1: Spherical Domain

Theorem 1.1. Equation (1) has a solution:

$$\phi(\mathbf{Q}) = \sum_{i=1}^N \left[\frac{I_i}{\|\mathbf{Q} - \mathbf{P}_i\|_2} + \frac{I_i}{r_i \|\mathbf{Q} - \mathbf{P}'_i\|_2} + I_i \log\left(\frac{1}{1 - rr_i \cos \gamma_i + r_i \|\mathbf{Q} - \mathbf{P}'_i\|_2}\right) \right] + \text{constant}, \tag{3}$$

where,

$$\|\mathbf{Q} - \mathbf{P}_i\|_2 := \sqrt{r^2 + r_i^2 - 2rr_i \cos \gamma_i},$$

$$\|\mathbf{Q} - \mathbf{P}'_i\|_2 := \sqrt{r^2 + \frac{1}{r_i^2} - 2\frac{r}{r_i} \cos \gamma_i},$$

$$\cos \gamma_i := \cos \theta_i \cos \theta + \sin \theta_i \sin \theta \cos(\psi - \psi_i),$$

$$\mathbf{Q} = (r, \theta, \psi) \in \Omega \cup \partial\Omega, \quad \mathbf{P}_i = (r_i, \theta_i, \psi_i) \in \Omega, \quad \mathbf{P}'_i = \left(\frac{1}{r_i}, \theta_i, \psi_i\right) \notin \Omega \cup \partial\Omega.$$

To simplify the expression of the solution, we define a function as follows:

$$\phi_{\mathbf{P}_i}(\mathbf{Q}) := \frac{1}{\|\mathbf{Q} - \mathbf{P}_i\|_2} + \frac{1}{\|\mathbf{P}_i\|_2 \|\mathbf{Q} - \mathbf{P}'_i\|_2} + \log\left(\frac{1}{1 - \langle \mathbf{P}_i, \mathbf{Q} \rangle + \|\mathbf{P}_i\|_2 \|\mathbf{Q} - \mathbf{P}'_i\|_2}\right), \text{ where,} \quad (4)$$

$$r_i = \|\mathbf{P}_i\|_2, \quad r = \|\mathbf{Q}\|_2, \quad \langle \mathbf{P}_i, \mathbf{Q} \rangle = rr_i \cos \gamma_i.$$

then the solution becomes:

$$\phi(\mathbf{Q}) = \sum_{i=1}^N I_i \phi_{\mathbf{P}_i}(\mathbf{Q}). \quad (5)$$

In Cartesian coordinates, we denote the coordinates of these points as follows:

$$\mathbf{Q} = (x, y, z), \quad \mathbf{P}_i = (x_i, y_i, z_i), \quad \mathbf{P}'_i = (x'_i, y'_i, z'_i).$$

The coordinate transformation reads:

$$x = r \sin \theta \sin \psi, \quad y = r \sin \theta \cos \psi, \quad z = r \cos \theta.$$

Using these identities, we derive that:

$$rr_i \cos \gamma_i = (x, y, z) \cdot (x_i, y_i, z_i)^T = xx_i + yy_i + zz_i. \quad (6)$$

Meanwhile, the Euclidean distance in Cartesian coordinates reads:

$$\begin{aligned} \|\mathbf{Q} - \mathbf{P}_i\|_2 &= \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}, \\ \|\mathbf{Q} - \mathbf{P}'_i\|_2 &= \sqrt{(x - x'_i)^2 + (y - y'_i)^2 + (z - z'_i)^2}. \end{aligned} \quad (7)$$

Substitute (6) and (7) back to Solution (3), we get:

$$\begin{aligned} \phi(x, y, z) &= \sum_{i=1}^N \left[\frac{I_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}} + \frac{I_i}{\sqrt{x_i^2 + y_i^2 + z_i^2} \sqrt{(x - x'_i)^2 + (y - y'_i)^2 + (z - z'_i)^2}} + \right. \\ &\quad \left. I_i \log\left(\frac{1}{1 - xx_i + yy_i + zz_i + \sqrt{x_i^2 + y_i^2 + z_i^2} \sqrt{(x - x'_i)^2 + (y - y'_i)^2 + (z - z'_i)^2}}\right) \right] + \text{constant}, \end{aligned} \quad (8)$$

where,

$$(x'_i, y'_i, z'_i) = \frac{1}{\sqrt{x_i^2 + y_i^2 + z_i^2}} (x_i, y_i, z_i).$$

Same as the solution in spherical coordinates, we can simplify the expression of the solution as follows:

$$\phi(\mathbf{Q}) = \sum_{i=1}^N I_i \phi_{\mathbf{P}_i}(\mathbf{Q}).$$

2 Forward problem solution

For the forward problem, we measure the potential at the boundary of the domain. Therefore we restrict the reference point to be $\mathbf{Q} \in \partial\Omega$, thus,

$$\mathbf{Q} = (1, \theta, \psi),$$

and then we derive to:

$$\|\mathbf{Q} - \mathbf{P}_i\|_2 = \|\mathbf{Q} - \mathbf{P}_i\|_2 = r_i \|\mathbf{Q} - \mathbf{P}'_i\|_2 = r_i \|\mathbf{Q} - \mathbf{P}'_i\|_2, \text{ for } \mathbf{Q} \in \partial\Omega.$$

Thus, returning to Solution (3), when $\mathbf{Q} \in \partial\Omega$, the forward solution becomes:

$$\begin{aligned}\phi(\mathbf{Q}) &= \sum_{i=1}^N [2 \frac{I_i}{\|\mathbf{Q} - \mathbf{P}_i\|_2} - I_i \log(1 - r_i \cos \gamma_i + \|\mathbf{Q} - \mathbf{P}_i\|_2)], \quad \mathbf{Q} \in \partial\Omega, \\ \|\mathbf{Q} - \mathbf{P}_i\|_2 &= \sqrt{1 + r_i^2 - 2r_i \cos \gamma_i}, \\ \cos \gamma_i &= \cos \theta_i \cos \theta + \sin \theta_i \sin \theta \cos(\psi - \psi_i).\end{aligned}\tag{9}$$

In Cartesian coordinates, Solution (9) becomes:

$$\begin{aligned}\phi(x, y, z) &= \sum_{i=1}^N [2 \frac{I_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}} - \\ &\quad I_i \log(1 - xx_i + yy_i + zz_i + \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2})],\end{aligned}\tag{10}$$

where,

$$(x, y, z) \in \partial\Omega, \text{ i.e., } x^2 + y^2 + z^2 = 1.\tag{11}$$

3 Formulate the forward solution using the Dirac delta measure in spherical coordinates

Dirac delta measure has the following property:

$$\int f(y) d\delta_x(y) = f(x).\tag{12}$$

Let,

$$\mathbf{S} = (r, \theta, \psi), \quad \mathbf{Q} = (\tilde{r}, \tilde{\theta}, \tilde{\psi}), \quad \mathbf{P}_i = (r_i, \theta_i, \psi_i).$$

Define the source term in Equation (1) as a measure, as follows:

$$\mu(\mathbf{Q}) := \sum_{i=1}^N I_i \delta_{\mathbf{P}_i}(\mathbf{Q}),\tag{13}$$

where the dirac delta function is defined in Definition (2). Using Property (40) to Solution (3), we derive to:

$$\phi(\mathbf{Q}) = \int_{\Omega} F(\mathbf{Q}, \mathbf{S}) d\mu(\mathbf{S}) = \Phi(\mu), \quad \mathbf{Q}, \mathbf{S} \in \Omega,\tag{14}$$

where,

$$\begin{aligned}F(\mathbf{Q}, \mathbf{S}) &:= \frac{1}{\sqrt{r^2 + \tilde{r}^2 - 2r\tilde{r} \cos \tilde{\gamma}}} + \frac{1}{r\sqrt{\frac{1}{r^2} + \tilde{r}^2 - 2\frac{\tilde{r}}{r} \cos \tilde{\gamma}}} - \log(1 - r\tilde{r} \cos \tilde{\gamma} + r\sqrt{\frac{1}{r^2} + \tilde{r}^2 - 2\frac{\tilde{r}}{r} \cos \tilde{\gamma}}), \\ \cos \tilde{\gamma} &:= \cos \tilde{\theta} \cos \theta + \sin \tilde{\theta} \sin \theta \cos(\psi - \tilde{\psi}).\end{aligned}\tag{15}$$

Apply this to the boundary solution, i.e., Solution (9), we have the following:

$$\phi(\mathbf{Q})|_{\mathbf{Q} \in \partial\Omega} = \int_{\Omega} F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega} d\mu(\mathbf{S}) = \Phi(\mu), \quad \mathbf{S} \in \Omega,\tag{16}$$

where,

$$\Phi : \mathcal{X}(\Omega) \rightarrow L^2(\partial\Omega), \text{ s.t.,}\tag{17}$$

$$\Phi(\mu) = \langle F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega}, \mu \rangle_{\Omega} = \int_{\Omega} F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega} d\mu(\mathbf{S}), \quad \mu \in \mathcal{X}, \quad \mathbf{S} \in \Omega.\tag{18}$$

where \mathcal{X} is a space of measure. For the operator Φ , its adjoint is given as follows:

$$\Phi^* : L^2(\partial\Omega) \rightarrow C^1(\Omega), \text{ s.t.,} \quad (19)$$

$$\Phi^*(p) = \langle F(\mathbf{Q}, \mathbf{S}), p \rangle_{\partial\Omega} = \int_{\partial\Omega} F(\mathbf{Q}, \mathbf{S}) p(\mathbf{Q}) d\mathbf{Q}, \quad \mathbf{Q} \in \partial\Omega, \quad \mathbf{S} \in \Omega, \quad p \in L^2(\partial\Omega). \quad (20)$$

Using the explicit form of $F(\mathbf{Q}, \mathbf{S})$, we expand 48 as follows:

$$\begin{aligned} \Phi^*(p) &= \int_0^{2\pi} \int_0^\pi \left[\frac{1}{\sqrt{r^2 + 1 - 2r \cos \tilde{\gamma}}} + \frac{1}{r \sqrt{\frac{1}{r^2} + 1 - 2\frac{1}{r} \cos \tilde{\gamma}}} - \right. \\ &\quad \left. \log(1 - r \cos \tilde{\gamma} + r \sqrt{\frac{1}{r^2} + 1 - 2\frac{1}{r} \cos \tilde{\gamma}}) \right] p(\tilde{\theta}, \tilde{\psi}) \sin \tilde{\theta} d\tilde{\theta} d\tilde{\psi}, \\ \text{thus, } \Phi^*(p) &= \int_0^{2\pi} \int_0^\pi \left[\frac{2}{\sqrt{r^2 + 1 - 2r \cos \tilde{\gamma}}} - \log(1 - r \cos \tilde{\gamma} + \sqrt{r^2 + 1 - 2r \cos \tilde{\gamma}}) \right] p(\tilde{\theta}, \tilde{\psi}) \sin \tilde{\theta} d\tilde{\theta} d\tilde{\psi}, \end{aligned} \quad (21)$$

$$\text{where, } \cos \tilde{\gamma} := \cos \tilde{\theta} \cos \theta + \sin \tilde{\theta} \sin \theta \cos(\psi - \tilde{\psi}) \quad (22)$$

3.1 Existence of Integral (23)

Let

$$\mathbf{Q} = (1, \tilde{\theta}, \tilde{\psi}), \quad \mathbf{S} = (r, \theta, \psi).$$

In this section, we investigate the integral:

$$\begin{aligned} I &= \int_{\partial\Omega} F(\mathbf{Q}, \mathbf{S}) p(\mathbf{Q}) d\mathbf{Q} \\ &= \int_0^{2\pi} \int_0^\pi \left[\frac{2}{\sqrt{r^2 + 1 - 2r \cos \tilde{\gamma}}} - \log(1 - r \cos \tilde{\gamma} + \sqrt{r^2 + 1 - 2r \cos \tilde{\gamma}}) \right] p(\tilde{\theta}, \tilde{\psi}) \sin \tilde{\theta} d\tilde{\theta} d\tilde{\psi}. \end{aligned} \quad (23)$$

Proposition 3.1. Assume that $p : \mathbf{Q} \rightarrow p(\mathbf{Q})$ is square integrable, then Integral (23) is bounded if and only if $\mathbf{S} \notin \partial\Omega$.

Proof. Cauchy Schwarz inequality yields:

$$\int_{\partial\Omega} F(\mathbf{Q}, \mathbf{S}) p(\mathbf{Q}) d\mathbf{Q} \leq \sqrt{\int_{\partial\Omega} F^2(\mathbf{Q}, \mathbf{S}) d\mathbf{Q} \int_{\partial\Omega} p^2(\mathbf{Q}) d\mathbf{Q}}. \quad (24)$$

Since $p \in L^2(\partial\Omega)$,

$$\int_{\partial\Omega} p^2(\mathbf{Q}) d\mathbf{Q} < \infty. \quad (25)$$

When $\mathbf{S} \notin \partial\Omega$, $F(\mathbf{Q}, \mathbf{S})$ is bounded, therefore,

$$\int_{\partial\Omega} F^2(\mathbf{Q}, \mathbf{S}) d\mathbf{Q} < \infty. \quad (26)$$

Now we consider the case when $\mathbf{S} \in \partial\Omega$. Notice that:

$$I_1(\mathbf{S}) := \int_{\partial\Omega} \left(\frac{1}{\sqrt{r^2 + 1 - 2r \cos \tilde{\gamma}}} \right)^2 d\mathbf{Q} = \int_{\partial\Omega} \frac{1}{\|\mathbf{Q} - \mathbf{S}\|_2^2} d\mathbf{Q}, \quad (27)$$

Since $\mathbf{S} \in \partial\Omega = \partial B(0, 1)$, $I_1(\mathbf{S})$ is rotational invariant. Convert the problem to Cartesian coordinates, such that:

$$\mathbf{Q} = (\cos \tilde{\theta} \sin \tilde{\psi}, \sin \tilde{\theta} \sin \tilde{\psi}, \cos \tilde{\theta}). \quad (28)$$

Using the rotational invariant of $I_1(\mathbf{S})$, we can choose \mathbf{S} as follows:

$$\mathbf{S} = (0, 0, 1), \quad (29)$$

then,

$$I_1(\mathbf{S}) = \int_{\partial\Omega} \frac{1}{\|\mathbf{Q} - \mathbf{S}\|_2^2} d\mathbf{Q} = \int_0^{2\pi} \int_0^\pi \frac{1}{2 - 2\cos\tilde{\theta}} \sin\tilde{\theta} d\tilde{\theta} d\tilde{\psi} = \frac{1}{2} \int_0^{2\pi} \int_{-1}^1 \frac{1}{u - 1} du d\tilde{\psi} = \infty. \quad (30)$$

□

Corollary 3.1. *The integral kernel $F(\mathbf{Q}, \mathbf{S})$ is square integrable if and only if $\mathbf{S} \notin \partial\Omega$.*

4 Normalization

Following the convention, we normalize the integral kernel $F(\mathbf{Q}, \mathbf{S})$ as follows:

$$\hat{F}(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega} := \frac{F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega}}{\sqrt{\int_{\partial\Omega} |F(\mathbf{Q}, \mathbf{S})|^2 d\mathbf{Q}}}. \quad (31)$$

Let $G(\mathbf{S})$ be such that:

$$G(\mathbf{S}) = \sqrt{\int_{\partial\Omega} |F(\mathbf{Q}, \mathbf{S})|^2 d\mathbf{Q}} = \sqrt{\int_0^{2\pi} \int_0^\pi \left[\frac{2}{\sqrt{r^2 + 1 - 2r\cos\tilde{\gamma}}} - \log(1 - r\cos\tilde{\gamma} + \sqrt{r^2 + 1 - 2r\cos\tilde{\gamma}}) \right]^2 \sin\tilde{\theta} d\tilde{\theta} d\tilde{\psi}},$$

where, $\cos\tilde{\gamma} := \cos\tilde{\theta}\cos\theta + \sin\tilde{\theta}\sin\theta\cos(\psi - \tilde{\psi})$.

4.1 Normalized forward solution

Using the normalized forward kernel given in (31), the forward solution is derived to:

$$\hat{\Phi}(\mu) = \int_{\Omega} \hat{F}(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega} d\mu(\mathbf{S}) = \int_{\Omega} \frac{F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega}}{G(\mathbf{S})} d\mu(\mathbf{S}), \quad \mu \in \mathcal{X}, \quad \mathbf{S} \in \Omega. \quad (32)$$

We recall that:

$$\mu(\mathbf{S}) = \sum_i I_i \delta_{\mathbf{P}_i}(\mathbf{S}), \quad (33)$$

and by the sifting property given in (40), we derive the following solution:

$$\hat{\Phi}(\mu)(\mathbf{Q}) = \sum_{i=1}^N I_i \frac{F(\mathbf{Q}, \mathbf{P}_i)}{G(\mathbf{P}_i)}, \quad \mathbf{Q} \in \partial\Omega. \quad (34)$$

Notice that:

$$F(\mathbf{Q}, \mathbf{P}_i) = \phi_{\mathbf{P}_i}(\mathbf{Q}), \quad \mathbf{Q} \in \partial\Omega, \quad \text{and then,} \quad (35)$$

$$G(\mathbf{P}_i) = \sqrt{\int_{\partial\Omega} |F(\mathbf{Q}, \mathbf{P}_i)|^2 d\mathbf{Q}} = \sqrt{\int_{\partial\Omega} \phi_{\mathbf{P}_i}^2(\mathbf{Q}) d\mathbf{Q}} \quad (36)$$

where $\phi_{\mathbf{P}_i}(\mathbf{Q})$ is defined in (4). We now derive to the normalized forward solution, as follows:

$$\hat{\Phi}(\mu)(\mathbf{Q}) = \sum_{i=1}^N \frac{I_i}{\sqrt{\int_{\partial\Omega} \phi_{\mathbf{P}_i}^2(\mathbf{Q}) d\mathbf{Q}}} \phi_{\mathbf{P}_i}(\mathbf{Q}), \quad \mathbf{Q} \in \partial\Omega. \quad (37)$$

4.2 Normalized adjoint problem

Using the normalized integral kernel, the adjoint problem becomes:

$$\hat{\Phi}^*(p) = \int_{\partial\Omega} \hat{F}(\mathbf{Q}, \mathbf{S}) p(\mathbf{Q}) d\mathbf{Q}, \quad \mathbf{S} \in \Omega, \quad \mathbf{Q} \in \partial\Omega, \quad p \in L^2(\partial\Omega), \quad (38)$$

$$\text{or, } \hat{\Phi}^*(p) = \int_0^{2\pi} \int_0^\pi \frac{\left[\frac{2}{\sqrt{r^2 + 1 - 2r\cos\tilde{\gamma}}} - \log(1 - r\cos\tilde{\gamma} + \sqrt{r^2 + 1 - 2r\cos\tilde{\gamma}}) \right]}{\sqrt{\int_0^{2\pi} \int_0^\pi \left[\frac{2}{\sqrt{r^2 + 1 - 2r\cos\tilde{\gamma}}} - \log(1 - r\cos\tilde{\gamma} + \sqrt{r^2 + 1 - 2r\cos\tilde{\gamma}}) \right]^2 \sin\tilde{\theta} d\tilde{\theta} d\tilde{\psi}}} p(\tilde{\theta}, \tilde{\psi}) \sin\tilde{\theta} d\tilde{\theta} d\tilde{\psi}, \quad (39)$$

where, $\cos\tilde{\gamma} := \cos\tilde{\theta}\cos\theta + \sin\tilde{\theta}\sin\theta\cos(\psi - \tilde{\psi})$.

5 Formulate the forward solution using the Dirac delta measure in Cartesian coordinates

Dirac delta measure has the following property:

$$\int f(y) d\delta_x(y) = f(x). \quad (40)$$

Let,

$$\mathbf{S} = (\tilde{x}, \tilde{y}, \tilde{z}), \quad \mathbf{Q} = (x, y, z), \quad \mathbf{P}_i = (x_i, y_i, z_i).$$

Define the source term in Equation (1) as a measure, as follows:

$$\mu(\mathbf{Q}) := \sum_{i=1}^N I_i \delta_{\mathbf{P}_i}(\mathbf{Q}), \quad (41)$$

where the dirac delta function is defined in Definition (2). Using Property (40) to Solution (3), we derive to:

$$\phi(\mathbf{Q}) = \int_{\Omega} F(\mathbf{Q}, \mathbf{S}) d\mu(\mathbf{S}) = \Phi(\mu), \quad \mathbf{Q}, \mathbf{S} \in \Omega, \quad (42)$$

where,

$$F(\mathbf{Q}, \mathbf{S}) := \frac{1}{\sqrt{(x - \tilde{x})^2 + (y - \tilde{y})^2 + (z - \tilde{z})^2}} + \frac{1}{\sqrt{x^2 + y^2 + z^2} \sqrt{(\tilde{x} - x')^2 + (\tilde{y} - y')^2 + (\tilde{z} - z')^2}} - \log(1 - x\tilde{x} + y\tilde{y} + z\tilde{z} + \sqrt{x^2 + y^2 + z^2} \sqrt{(x' - \tilde{x})^2 + (y' - \tilde{y})^2 + (z' - \tilde{z})^2}), \quad (43)$$

where,

$$(x', y', z') = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x, y, z).$$

Apply this to the boundary solution, i.e., Solution (9), we have the following:

$$\phi(\mathbf{Q})|_{\mathbf{Q} \in \partial\Omega} = \int_{\Omega} F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega} d\mu(\mathbf{S}) = \Phi(\mu), \quad \mathbf{S} \in \Omega, \quad (44)$$

where,

$$\Phi : \mathcal{X}(\Omega) \rightarrow L^2(\partial\Omega), \text{ s.t.}, \quad (45)$$

$$\Phi(\mu) = \langle F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega}, \mu \rangle_{\Omega} = \int_{\Omega} F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega} d\mu(\mathbf{S}), \quad \mu \in \mathcal{X}, \quad \mathbf{S} \in \Omega. \quad (46)$$

where \mathcal{X} is a space of measure. For the operator Φ , its adjoint is given as follows:

$$\Phi^* : L^2(\partial\Omega) \rightarrow C^1(\Omega), \text{ s.t.}, \quad (47)$$

$$\Phi^*(p) = \langle F(\mathbf{Q}, \mathbf{S}), p \rangle_{\partial\Omega} = \int_{\partial\Omega} F(\mathbf{Q}, \mathbf{S}) p(\mathbf{S}) d\mathbf{S}, \quad \mathbf{Q} \in \Omega, \quad \mathbf{S} \in \partial\Omega, \quad p \in L^2(\partial\Omega). \quad (48)$$

The restriction of $\mathbf{S} \in \partial\Omega$ yields the following identity:

$$\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 = 1, \quad (49)$$

it defines an implicit function, reads:

$$\tilde{z} = \tilde{z}(\tilde{x}, \tilde{y}).$$

Using the explicit form of $F(\mathbf{Q}, \mathbf{S})$ and the definition of a surface integral, we expand Equation (48) as follows:

$$\begin{aligned} \Phi^*(p) = & \int_{-1}^1 \int_{-\sqrt{1-\tilde{x}^2}}^{\sqrt{1-\tilde{x}^2}} \left[\frac{1}{\sqrt{(x-\tilde{x})^2 + (y-\tilde{y})^2 + (z-\tilde{z})^2}} + \frac{1}{\sqrt{x^2 + y^2 + z^2} \sqrt{(\tilde{x}-x')^2 + (\tilde{y}-y')^2 + (\tilde{z}-z')^2}} \right. \\ & \left. - \log(1 - x\tilde{x} + y\tilde{y} + z\tilde{z} + \sqrt{x^2 + y^2 + z^2} \sqrt{(x'-\tilde{x})^2 + (y'-\tilde{y})^2 + (z'-\tilde{z})^2}) \right] p(\tilde{x}, \tilde{y}, \tilde{z}) \sqrt{\frac{\partial \tilde{z}^2}{\partial \tilde{x}} + \frac{\partial \tilde{z}^2}{\partial \tilde{y}} + 1} d\tilde{x} d\tilde{y}, \end{aligned} \quad (50)$$

using the restriction given in Equation (49), Equation (50) simplifies to:

$$\begin{aligned} \Phi^*(p) = & \int_{-1}^1 \int_{-\sqrt{1-\tilde{x}^2}}^{\sqrt{1-\tilde{x}^2}} \left[\frac{2}{\sqrt{(x-\tilde{x})^2 + (y-\tilde{y})^2 + (z-\tilde{z}(\tilde{x}, \tilde{y}))^2}} - \right. \\ & \left. \log(1 - x\tilde{x} + y\tilde{y} + z\tilde{z}(\tilde{x}, \tilde{y}) + \sqrt{(x-\tilde{x})^2 + (y-\tilde{y})^2 + (z-\tilde{z}(\tilde{x}, \tilde{y}))^2}) \right] \\ & p(\tilde{x}, \tilde{y}, \tilde{z}(\tilde{x}, \tilde{y})) \frac{1}{|\tilde{z}(\tilde{x}, \tilde{y})|} d\tilde{x} d\tilde{y}, \end{aligned} \quad (51)$$

5.1 Normalization in Cartesian coordinates

Following the convention, we normalize the integral kernel $F(\mathbf{Q}, \mathbf{S})$ as follows:

$$\hat{F}(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega} = \frac{F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega}}{\sqrt{\int_{\partial\Omega} |F(\mathbf{Q}, \mathbf{S})|^2 d\mathbf{Q}}}, \quad (52)$$

where,

$$F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega} := \frac{2}{\sqrt{(x-\tilde{x})^2 + (y-\tilde{y})^2 + (z-\tilde{z})^2}} - \log(1 - x\tilde{x} + y\tilde{y} + z\tilde{z} + \sqrt{(x-\tilde{x})^2 + (y-\tilde{y})^2 + (z-\tilde{z})^2}) \quad (53)$$

with a restriction: $x^2 + y^2 + z^2 = 1$.

Notice that the restriction in Definition (53) defines an implicit function that reads:

$$z = z(x, y). \quad (54)$$

The surface integral in Normalization (52) is defined as follows:

$$\int_{\partial\Omega} |F(\mathbf{Q}, \mathbf{S})|^2 d\mathbf{Q} = \int_{-1}^1 \int_{-\sqrt{1-\tilde{x}^2}}^{\sqrt{1-\tilde{x}^2}} F^2(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega} \sqrt{\frac{\partial z^2}{\partial x} + \frac{\partial z^2}{\partial y} + 1} dx dy. \quad (55)$$

where $F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega}$ is defined in Definition 53.

6 BLASSO scheme for the inverse problem

Definition 6.1. Source tracing problem in the space of measure.

Given the measurement data $\phi^d \in L^2(\partial\Omega)$, we reconstruct a measure μ^* from:

$$\begin{aligned} \mu^* = & \operatorname{argmin}_{\mu \in \mathcal{X}} \left\{ \frac{1}{2} \|\Phi\mu - \phi^d\|_2^2 + \lambda \|\mu\| \right\}, \quad \|\mu\| = \|\mathbf{I}\|_1, \\ \mathbf{I} := & \{I_1, I_2, \dots, I_N\}, \end{aligned} \quad (56)$$

where I_i is the intensity of the i th source.

The optimality condition for Problem (56) is given as follows:

$$0 \in \Phi^*(\Phi\mu - \phi^d) + \lambda \partial\|\mu\|, \quad \text{where } \partial\|\mu\| := \{g \mid \|g\|_\infty \leq 1 \text{ \& } \int g d\mu = \|\mu\|\}. \quad (57)$$

7 Sliding Frank Wolfe algorithm

To solve Problem (56), we implement the Sliding Frank Wolfe algorithm. We present the algorithm in this section.

Result: Solve $\mathbf{I}_* = \{I_1, I_2, \dots\}$, $\mathbf{P}_* = \{\mathbf{P}_1, \mathbf{P}_2, \dots\}$

Initialize: $\mathbf{I}_*^0 = \emptyset$, $\mathbf{P}_*^0 = \emptyset$, $k = 0$;

while $\|\eta_k\|_\infty > 1$, **do**

$\mu_k = \sum_{i=0}^k I_i \delta_{\mathbf{P}_i}(\mathbf{S})$;

Solve $\mathbf{P}_{k+\frac{1}{2}}$ from:

$$\mathbf{P}_{k+\frac{1}{2}} = \operatorname{argmax}_{\mathbf{P} \in R^3} \|\eta_k\|_\infty, \text{ where } \eta_k = \frac{\Phi^*(p_k)}{\lambda}, p_k = \Phi\mu_k - \phi^d. \quad (58)$$

$\mathbf{P}_*^{k+\frac{1}{2}} = [\mathbf{P}_*^k; \mathbf{P}_{k+\frac{1}{2}}]$, updates μ to $\mu_{k+\frac{1}{2}} = \sum_{i=0}^{k+\frac{1}{2}} I_i \delta_{\mathbf{P}_i}(\mathbf{S})$.

Solve $\mathbf{I}_*^{k+\frac{1}{2}} = \{I_1, I_2, \dots, I_{k+\frac{1}{2}}\}$ from:

$$\mathbf{I}_*^{k+\frac{1}{2}} = \operatorname{argmin}_{\mathbf{I} \in R^{k+1}} \left\{ \frac{1}{2} \|\Phi\mu_k|_{\mathbf{P}=\mathbf{P}_*^{k+\frac{1}{2}}} - \phi^d\|_2^2 + \lambda \|\mu_k\|_1 \right\}. \quad (59)$$

Initialize with $\mathbf{I}_*^{k+\frac{1}{2}}, \mathbf{P}_*^{k+\frac{1}{2}}$, s.t., $\mu_{k+1} = \sum_{i=0}^{k+\frac{1}{2}} I_i \delta_{\mathbf{P}_i}(\mathbf{Q})$, and solve:

$$\mathbf{I}_*^{k+1}, \mathbf{P}_*^{k+1} = \operatorname{argmin}_{\mathbf{I} \in R^{k+1}, \mathbf{P} \in R^{3(k+1)}} \left\{ \frac{1}{2} \|\Phi\mu_{k+1} - \phi^d\|_2^2 + \lambda \|\mu_{k+1}\|_1 \right\}. \quad (60)$$

$k = k + 1$.

end

8 Some results