

## 0.1 Solution of the monopole potential

In spherical coordinate system, consider this equation on  $B^3$ :

$$\Delta\phi(\mathbf{Q}) = \sum_{i=1}^N I_i \delta_{P_i}(\mathbf{Q}), \quad \mathbf{Q}, \mathbf{P}_i \in \Omega \quad (1)$$

$$\frac{\partial\phi(\mathbf{Q})}{\partial\mathbf{n}} = 0, \quad \mathbf{Q} \in \partial\Omega,$$

where  $\mathbf{Q} = (r, \theta, \psi)$ ,  $\mathbf{P}_i = (r_i, \theta_i, \psi_i)$ ,  $\Omega = B^3(0, 1)$ .

The dirac delta function is defined as follows:

$$\delta_{\mathbf{x}}(\mathbf{y}) := \begin{cases} \infty, & \text{if } \mathbf{x} = \mathbf{y}, \\ 0, & \text{else,} \end{cases} \quad (2)$$

$$\text{s.t., } \int_{R^3} \delta_{\mathbf{x}}(\mathbf{y}) d\mathbf{y} = 1, \text{ for } \mathbf{x}, \mathbf{y} \in R^3.$$

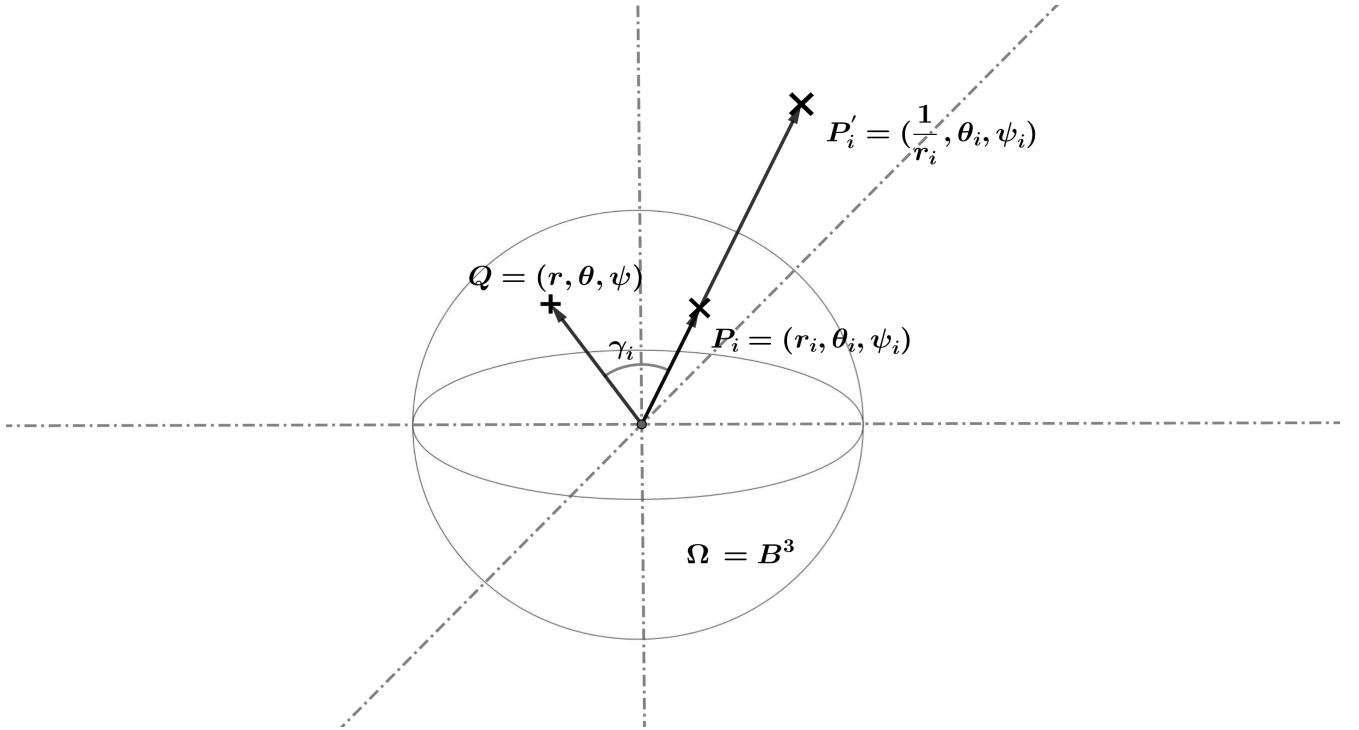


Figure 1: Spherical Domain

**Theorem 0.1.1.** Equation (1) has a solution:

$$\phi(\mathbf{Q}) = \sum_{i=1}^N \left[ \frac{I_i}{l(\mathbf{Q}, \mathbf{P}_i)} + \frac{I_i}{r_i l(\mathbf{Q}, \mathbf{P}'_i)} + I_i \log\left(\frac{1}{1 - rr_i \cos \gamma_i + r_i l(\mathbf{Q}, \mathbf{P}'_i)}\right) \right] + \text{constant}, \quad (3)$$

where,

$$l(\mathbf{Q}, \mathbf{P}_i) := \|\mathbf{Q} - \mathbf{P}_i\|_2 = \sqrt{r^2 + r_i^2 - 2rr_i \cos \gamma_i},$$

$$l(\mathbf{Q}, \mathbf{P}'_i) := \|\mathbf{Q} - \mathbf{P}'_i\|_2 = \sqrt{r^2 + \frac{1}{r_i^2} - 2\frac{r}{r_i} \cos \gamma_i},$$

$$\cos \gamma_i := \cos \theta_i \cos \theta + \sin \theta_i \sin \theta \cos(\psi - \psi_i),$$

$$\mathbf{Q} = (r, \theta, \psi) \in \Omega \cup \partial\Omega, \quad \mathbf{P}_i = (r_i, \theta_i, \psi_i) \in \Omega, \quad \mathbf{P}'_i = \left(\frac{1}{r_i}, \theta_i, \psi_i\right) \notin \Omega \cup \partial\Omega.$$

In Cartesian coordinates, we denote the coordinates of these points as follows:

$$\mathbf{Q} = (x, y, z), \mathbf{P}_i = (x_i, y_i, z_i), \mathbf{P}'_i = (x'_i, y'_i, z'_i).$$

The coordinate transformation reads:

$$x = r \sin \theta \sin \psi, \quad y = r \sin \theta \cos \psi, \quad z = r \cos \theta.$$

Using these identities, we derive that:

$$rr_i \cos \gamma_i = (x, y, z) \cdot (x_i, y_i, z_i)^T = xx_i + yy_i + zz_i. \quad (4)$$

Meanwhile, the Euclidean distance in Cartesian coordinates reads:

$$\begin{aligned} l(\mathbf{Q}, \mathbf{P}_i) &= \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}, \\ l(\mathbf{Q}, \mathbf{P}'_i) &= \sqrt{(x - x'_i)^2 + (y - y'_i)^2 + (z - z'_i)^2}. \end{aligned} \quad (5)$$

Substitute 4 and 5 back to Solution 3, we get:

$$\begin{aligned} \phi(x, y, z) &= \sum_{i=1}^N \left[ \frac{I_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}} + \frac{I_i}{\sqrt{x_i^2 + y_i^2 + z_i^2} \sqrt{(x - x'_i)^2 + (y - y'_i)^2 + (z - z'_i)^2}} + \right. \\ &\quad \left. I_i \log \left( \frac{1}{1 - xx_i + yy_i + zz_i + \sqrt{x_i^2 + y_i^2 + z_i^2} \sqrt{(x - x'_i)^2 + (y - y'_i)^2 + (z - z'_i)^2}} \right) \right] + \text{constant}, \end{aligned} \quad (6)$$

where,

$$(x'_i, y'_i, z'_i) = \frac{1}{\sqrt{x_i^2 + y_i^2 + z_i^2}} (x_i, y_i, z_i).$$

## 0.2 Forward problem solution

For the forward problem, we measure the potential at the boundary of the domain. Therefore we restrict the reference point to be  $\mathbf{Q} \in \partial\Omega$ , thus,

$$\mathbf{Q} = (1, \theta, \psi),$$

and then we derive to:

$$l(\mathbf{Q}, \mathbf{P}_i) = \|\mathbf{Q} - \mathbf{P}_i\|_2 = r_i \|\mathbf{Q} - \mathbf{P}'_i\|_2 = r_i l(\mathbf{Q}, \mathbf{P}'_i), \quad \text{for } \mathbf{Q} \in \partial\Omega.$$

Thus, returning to Solution 3, when  $\mathbf{Q} \in \partial\Omega$ , the forward solution becomes:

$$\begin{aligned} \phi(\mathbf{Q}) &= \sum_{i=1}^N \left[ 2 \frac{I_i}{l(\mathbf{Q}, \mathbf{P}_i)} - I_i \log(1 - r_i \cos \gamma_i + l(\mathbf{Q}, \mathbf{P}_i)) \right], \quad \mathbf{Q} \in \partial\Omega, \\ l(\mathbf{Q}, \mathbf{P}_i) &= \sqrt{1 + r_i^2 - 2r_i \cos \gamma_i}, \\ \cos \gamma_i &= \cos \theta_i \cos \theta + \sin \theta_i \sin \theta \cos(\psi - \psi_i). \end{aligned} \quad (7)$$

In Cartesian coordinates, Solution 7 becomes:

$$\begin{aligned} \phi(x, y, z) &= \sum_{i=1}^N \left[ 2 \frac{I_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}} - \right. \\ &\quad \left. I_i \log(1 - xx_i + yy_i + zz_i + \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}) \right], \end{aligned} \quad (8)$$

where,

$$(x, y, z) \in \partial\Omega, \quad \text{i.e., } x^2 + y^2 + z^2 = 1. \quad (9)$$

### 0.3 Formulate the forward solution using the Dirac delta measure in spherical coordinates

Dirac delta measure has the following property:

$$\int f(y) d\delta_x(y) = f(x). \quad (10)$$

Let,

$$\mathbf{S} = (r, \theta, \psi), \quad \mathbf{Q} = (\tilde{r}, \tilde{\theta}, \tilde{\psi}), \quad \mathbf{P}_i = (r_i, \theta_i, \psi_i).$$

Define the source term in Equation 1 as a measure, as follows:

$$\mu(\mathbf{Q}) := \sum_{i=1}^N I_i \delta_{\mathbf{P}_i}(\mathbf{Q}), \quad (11)$$

where the dirac delta function is defined in Definition 2. Using Property 33 to Solution 3, we derive to:

$$\phi(\mathbf{Q}) = \int_{\Omega} F(\mathbf{Q}, \mathbf{S}) d\mu(\mathbf{S}) = \Phi(\mu), \quad \mathbf{Q}, \mathbf{S} \in \Omega, \quad (12)$$

where,

$$F(\mathbf{Q}, \mathbf{S}) := \frac{1}{\sqrt{r^2 + \tilde{r}^2 - 2r\tilde{r} \cos \tilde{\gamma}}} + \frac{1}{r\sqrt{\frac{1}{r^2} + \tilde{r}^2 - 2\frac{\tilde{r}}{r} \cos \tilde{\gamma}}} - \log(1 - r\tilde{r} \cos \tilde{\gamma} + r\sqrt{\frac{1}{r^2} + \tilde{r}^2 - 2\frac{\tilde{r}}{r} \cos \tilde{\gamma}}), \quad (13)$$

$$\cos \tilde{\gamma} := \cos \tilde{\theta} \cos \theta + \sin \tilde{\theta} \sin \theta \cos(\psi - \tilde{\psi}).$$

Apply this to the boundary solution, i.e., Solution 7, we have the following:

$$\phi(\mathbf{Q})|_{\mathbf{Q} \in \partial\Omega} = \int_{\Omega} F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega} d\mu(\mathbf{S}) = \Phi(\mu), \quad \mathbf{S} \in \Omega, \quad (14)$$

where,

$$\Phi : \mathcal{X}(\Omega) \rightarrow L^2(\partial\Omega), \text{ s.t.}, \quad (15)$$

$$\Phi(\mu) = \langle F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega}, \mu \rangle_{\Omega} = \int_{\Omega} F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega} d\mu(\mathbf{S}), \quad \mu \in \mathcal{X}, \quad \mathbf{S} \in \Omega. \quad (16)$$

where  $\mathcal{X}$  is a space of measure. For the operator  $\Phi$ , its adjoint is given as follows:

$$\Phi^* : L^2(\partial\Omega) \rightarrow C^1(\Omega), \text{ s.t.}, \quad (17)$$

$$\Phi^*(p) = \langle F(\mathbf{Q}, \mathbf{S}), p \rangle_{\partial\Omega} = \int_{\partial\Omega} F(\mathbf{Q}, \mathbf{S}) p(\mathbf{Q}) d\mathbf{Q}, \quad \mathbf{Q} \in \partial\Omega, \quad \mathbf{S} \in \Omega, \quad p \in L^2(\partial\Omega). \quad (18)$$

Using the explicit form of  $F(\mathbf{Q}, \mathbf{S})$ , we expand 41 as follows:

$$\Phi^*(p) = \int_0^{2\pi} \int_0^{\pi} \left[ \frac{1}{\sqrt{r^2 + 1 - 2r \cos \tilde{\gamma}}} + \frac{1}{r\sqrt{\frac{1}{r^2} + 1 - 2\frac{1}{r} \cos \tilde{\gamma}}} - \log(1 - r \cos \tilde{\gamma} + r\sqrt{\frac{1}{r^2} + 1 - 2\frac{1}{r} \cos \tilde{\gamma}}) \right] p(\tilde{\theta}, \tilde{\psi}) \sin \tilde{\theta} d\tilde{\theta} d\tilde{\psi},$$

$$\text{thus, } \Phi^*(p) = \int_0^{2\pi} \int_0^{\pi} \left[ \frac{2}{\sqrt{r^2 + 1 - 2r \cos \tilde{\gamma}}} - \log(1 - r \cos \tilde{\gamma} + \sqrt{r^2 + 1 - 2r \cos \tilde{\gamma}}) \right] p(\tilde{\theta}, \tilde{\psi}) \sin \tilde{\theta} d\tilde{\theta} d\tilde{\psi}, \quad (19)$$

$$\text{where, } \cos \tilde{\gamma} := \cos \tilde{\theta} \cos \theta + \sin \tilde{\theta} \sin \theta \cos(\psi - \tilde{\psi}) \quad (20)$$

### 0.3.1 Integral 21 is bounded on the boundary

Let

$$\mathbf{Q} = (1, \tilde{\theta}, \tilde{\psi}), \quad \mathbf{S} = (r, \theta, \psi).$$

In this section, we investigate the integral:

$$\begin{aligned} I &= \int_{\partial\Omega} F(\mathbf{Q}, \mathbf{S}) p(\mathbf{Q}) d\mathbf{Q} \\ &= \int_0^{2\pi} \int_0^\pi \left[ \frac{2}{\sqrt{r^2 + 1 - 2r \cos \tilde{\gamma}}} - \log(1 - r \cos \tilde{\gamma} + \sqrt{r^2 + 1 - 2r \cos \tilde{\gamma}}) \right] p(\tilde{\theta}, \tilde{\psi}) \sin \tilde{\theta} d\tilde{\theta} d\tilde{\psi}. \end{aligned} \quad (21)$$

**Proposition 0.3.1.** *Assume that  $p : \mathbf{Q} \rightarrow p(\mathbf{Q})$  is square integrable, then Integral 21 is bounded if and only if  $\mathbf{S} \notin \partial\Omega$ .*

*Proof.* Cauchy Schwarz inequality yields:

$$\int_{\partial\Omega} F(\mathbf{Q}, \mathbf{S}) p(\mathbf{Q}) d\mathbf{Q} \leq \sqrt{\int_{\partial\Omega} F^2(\mathbf{Q}, \mathbf{S}) d\mathbf{Q} \int_{\partial\Omega} p^2(\mathbf{Q}) d\mathbf{Q}}. \quad (22)$$

Since  $p \in L^2(\partial\Omega)$ ,

$$\int_{\partial\Omega} p^2(\mathbf{Q}) d\mathbf{Q} < \infty. \quad (23)$$

When  $\mathbf{S} \notin \partial\Omega$ ,  $F(\mathbf{Q}, \mathbf{S})$  is bounded, therefore,

$$\int_{\partial\Omega} F^2(\mathbf{Q}, \mathbf{S}) d\mathbf{Q} < \infty. \quad (24)$$

Now we consider the case when  $\mathbf{S} \in \partial\Omega$ . Notice that:

$$I_1(\mathbf{S}) := \int_{\partial\Omega} \left( \frac{1}{\sqrt{r^2 + 1 - 2r \cos \tilde{\gamma}}} \right)^2 d\mathbf{Q} = \int_{\partial\Omega} \frac{1}{\|\mathbf{Q} - \mathbf{S}\|_2^2} d\mathbf{Q}, \quad (25)$$

Since  $\mathbf{S} \in \partial\Omega = \partial B(0, 1)$ ,  $I_1(\mathbf{S})$  is rotational invariant. Convert the problem to Cartesian coordinates, such that:

$$\mathbf{Q} = (\cos \tilde{\theta} \sin \tilde{\psi}, \sin \tilde{\theta} \sin \tilde{\psi}, \cos \tilde{\theta}). \quad (26)$$

Using the rotational invariant of  $I_1(\mathbf{S})$ , we can choose  $\mathbf{S}$  as follows:

$$\mathbf{S} = (0, 0, 1), \quad (27)$$

then,

$$I_1(\mathbf{S}) = \int_{\partial\Omega} \frac{1}{\|\mathbf{Q} - \mathbf{S}\|_2^2} d\mathbf{Q} = \int_0^{2\pi} \int_0^\pi \frac{1}{2 - 2 \cos \tilde{\theta}} \sin \tilde{\theta} d\tilde{\theta} d\tilde{\psi} = \frac{1}{2} \int_0^{2\pi} \int_{-1}^1 \frac{1}{u - 1} du d\tilde{\psi} = \infty. \quad (28)$$

□

**Corollary 0.3.1.** *The integral kernel  $F(\mathbf{Q}, \mathbf{S})$  is square integrable if and only if  $\mathbf{S} \notin \partial\Omega$ .*

## 0.4 Normalization

Following the convention, we normalize the integral kernel  $F(\mathbf{Q}, \mathbf{S})$  as follows:

$$\hat{F}(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega} = \frac{F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega}}{\sqrt{\int_{\partial\Omega} |F(\mathbf{Q}, \mathbf{S})|^2 d\mathbf{Q}}}. \quad (29)$$

Let  $G(\mathbf{S})$  be such that:

$$G(\mathbf{S}) = \sqrt{\int_{\partial\Omega} |F(\mathbf{Q}, \mathbf{S})|^2 d\mathbf{Q}} = \sqrt{\int_0^{2\pi} \int_0^\pi \left[ \frac{2}{\sqrt{r^2+1-2r\cos\tilde{\gamma}}} - \log(1-r\cos\tilde{\gamma} + \sqrt{r^2+1-2r\cos\tilde{\gamma}}) \right]^2 \sin\tilde{\theta} d\tilde{\theta} d\tilde{\psi}},$$

where,  $\cos\tilde{\gamma} := \cos\tilde{\theta}\cos\theta + \sin\tilde{\theta}\sin\theta\cos(\psi - \tilde{\psi})$ .

Then the normalized forward solution is derived to:

$$\hat{\Phi}(\mu) = \int_{\Omega} \hat{F}(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega} d\mu(\mathbf{S}) = \int_{\Omega} \frac{F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega}}{G(\mathbf{S})} d\mu(\mathbf{S}), \quad \mu \in \mathcal{X}, \quad \mathbf{S} \in \Omega. \quad (30)$$

Therefore,

$$\hat{\Phi}^*(p) = \int_{\partial\Omega} \hat{F}(\mathbf{Q}, \mathbf{S}) p(\mathbf{S}) d\mathbf{S}, \quad \mathbf{Q} \in \Omega, \quad \mathbf{S} \in \partial\Omega, \quad p \in L^2(\partial\Omega), \quad (31)$$

$$\text{or, } \hat{\Phi}^*(p) = \int_0^{2\pi} \int_0^\pi \frac{\left[ \frac{2}{\sqrt{r^2+1-2r\cos\tilde{\gamma}}} - \log(1-r\cos\tilde{\gamma} + \sqrt{r^2+1-2r\cos\tilde{\gamma}}) \right]}{\sqrt{\int_0^{2\pi} \int_0^\pi \left[ \frac{2}{\sqrt{r^2+1-2r\cos\tilde{\gamma}}} - \log(1-r\cos\tilde{\gamma} + \sqrt{r^2+1-2r\cos\tilde{\gamma}}) \right]^2 \sin\tilde{\theta} d\tilde{\theta} d\tilde{\psi}}} p(\tilde{\theta}, \tilde{\psi}) \sin\tilde{\theta} d\tilde{\theta} d\tilde{\psi}, \quad (32)$$

where,  $\cos\tilde{\gamma} := \cos\tilde{\theta}\cos\theta + \sin\tilde{\theta}\sin\theta\cos(\psi - \tilde{\psi})$ .

## 0.5 Formulate the forward solution using the Dirac delta measure in Cartesian coordinates

Dirac delta measure has the following property:

$$\int f(y) d\delta_x(y) = f(x). \quad (33)$$

Let,

$$\mathbf{S} = (\tilde{x}, \tilde{y}, \tilde{z}), \quad \mathbf{Q} = (x, y, z), \quad \mathbf{P}_i = (x_i, y_i, z_i).$$

Define the source term in Equation 1 as a measure, as follows:

$$\mu(\mathbf{Q}) := \sum_{i=1}^N I_i \delta_{\mathbf{P}_i}(\mathbf{Q}), \quad (34)$$

where the dirac delta function is defined in Definition 2. Using Property 33 to Solution 3, we derive to:

$$\phi(\mathbf{Q}) = \int_{\Omega} F(\mathbf{Q}, \mathbf{S}) d\mu(\mathbf{S}) = \Phi(\mu), \quad \mathbf{Q}, \mathbf{S} \in \Omega, \quad (35)$$

where,

$$F(\mathbf{Q}, \mathbf{S}) := \frac{1}{\sqrt{(x-\tilde{x})^2 + (y-\tilde{y})^2 + (z-\tilde{z})^2}} + \frac{1}{\sqrt{x^2 + y^2 + z^2} \sqrt{(\tilde{x}-x')^2 + (\tilde{y}-y')^2 + (\tilde{z}-z')^2}} - \log(1 - x\tilde{x} + y\tilde{y} + z\tilde{z} + \sqrt{x^2 + y^2 + z^2} \sqrt{(x'-\tilde{x})^2 + (y'-\tilde{y})^2 + (z'-\tilde{z})^2}), \quad (36)$$

where,

$$(x', y', z') = \frac{1}{\sqrt{x^2 + y^2 + z^2}} (x, y, z).$$

Apply this to the boundary solution, i.e., Solution 7, we have the following:

$$\phi(\mathbf{Q})|_{\mathbf{Q} \in \partial\Omega} = \int_{\Omega} F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega} d\mu(\mathbf{S}) = \Phi(\mu), \quad \mathbf{S} \in \Omega, \quad (37)$$

where,

$$\Phi : \mathcal{X}(\Omega) \rightarrow L^2(\partial\Omega), \text{ s.t.,} \quad (38)$$

$$\Phi(\mu) = \langle F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega}, \mu \rangle_{\Omega} = \int_{\Omega} F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega} d\mu(\mathbf{S}), \quad \mu \in \mathcal{X}, \quad \mathbf{S} \in \Omega. \quad (39)$$

where  $\mathcal{X}$  is a space of measure. For the operator  $\Phi$ , its adjoint is given as follows:

$$\Phi^* : L^2(\partial\Omega) \rightarrow C^1(\Omega), \text{ s.t.,} \quad (40)$$

$$\Phi^*(p) = \langle F(\mathbf{Q}, \mathbf{S}), p \rangle_{\partial\Omega} = \int_{\partial\Omega} F(\mathbf{Q}, \mathbf{S}) p(\mathbf{S}) d\mathbf{S}, \quad \mathbf{Q} \in \Omega, \quad \mathbf{S} \in \partial\Omega, \quad p \in L^2(\partial\Omega). \quad (41)$$

The restriction of  $\mathbf{S} \in \partial\Omega$  yields the following identity:

$$\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 = 1, \quad (42)$$

it defines an implicit function, reads:

$$\tilde{z} = \tilde{z}(\tilde{x}, \tilde{y}).$$

Using the explicit form of  $F(\mathbf{Q}, \mathbf{S})$  and the definition of a surface integral, we expand Equation 41 as follows:

$$\begin{aligned} \Phi^*(p) = & \int_{-1}^1 \int_{-\sqrt{1-\tilde{x}^2}}^{\sqrt{1-\tilde{x}^2}} \left[ \frac{1}{\sqrt{(x-\tilde{x})^2 + (y-\tilde{y})^2 + (z-\tilde{z})^2}} + \frac{1}{\sqrt{x^2 + y^2 + z^2} \sqrt{(\tilde{x}-x')^2 + (\tilde{y}-y')^2 + (\tilde{z}-z')^2}} \right. \\ & \left. - \log(1 - x\tilde{x} + y\tilde{y} + z\tilde{z} + \sqrt{x^2 + y^2 + z^2} \sqrt{(x'-\tilde{x})^2 + (y'-\tilde{y})^2 + (z'-\tilde{z})^2}) \right] p(\tilde{x}, \tilde{y}, \tilde{z}) \sqrt{\frac{\partial \tilde{z}^2}{\partial \tilde{x}} + \frac{\partial \tilde{z}^2}{\partial \tilde{y}} + 1} d\tilde{x} d\tilde{y}, \end{aligned} \quad (43)$$

using the restriction given in Equation 42, Equation 43 simplifies to:

$$\begin{aligned} \Phi^*(p) = & \int_{-1}^1 \int_{-\sqrt{1-\tilde{x}^2}}^{\sqrt{1-\tilde{x}^2}} \left[ \frac{2}{\sqrt{(x-\tilde{x})^2 + (y-\tilde{y})^2 + (z-\tilde{z}(\tilde{x}, \tilde{y}))^2}} - \right. \\ & \left. \log(1 - x\tilde{x} + y\tilde{y} + z\tilde{z}(\tilde{x}, \tilde{y}) + \sqrt{(x-\tilde{x})^2 + (y-\tilde{y})^2 + (z-\tilde{z}(\tilde{x}, \tilde{y}))^2}) \right] \\ & p(\tilde{x}, \tilde{y}, \tilde{z}(\tilde{x}, \tilde{y})) \frac{1}{|\tilde{z}(\tilde{x}, \tilde{y})|} d\tilde{x} d\tilde{y}, \end{aligned} \quad (44)$$

### 0.5.1 Normalization in Cartesian coordinates

Following the convention, we normalize the integral kernel  $F(\mathbf{Q}, \mathbf{S})$  as follows:

$$\hat{F}(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega} = \frac{F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega}}{\sqrt{\int_{\partial\Omega} |F(\mathbf{Q}, \mathbf{S})|^2 d\mathbf{Q}}}, \quad (45)$$

where,

$$F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega} := \frac{2}{\sqrt{(x-\tilde{x})^2 + (y-\tilde{y})^2 + (z-\tilde{z})^2}} - \log(1 - x\tilde{x} + y\tilde{y} + z\tilde{z} + \sqrt{(x-\tilde{x})^2 + (y-\tilde{y})^2 + (z-\tilde{z})^2}) \quad (46)$$

with a restriction:  $x^2 + y^2 + z^2 = 1$ .

Notice that the restriction in Definition 46 defines an implicit function that reads:

$$z = z(x, y). \quad (47)$$

The surface integral in Normalization 45 is defined as follows:

$$\int_{\partial\Omega} |F(\mathbf{Q}, \mathbf{S})|^2 d\mathbf{Q} = \int_{-1}^1 \int_{-\sqrt{1-\tilde{x}^2}}^{\sqrt{1-\tilde{x}^2}} F^2(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega} \sqrt{\frac{\partial z^2}{\partial x} + \frac{\partial z^2}{\partial y} + 1} dx dy. \quad (48)$$

where  $F(\mathbf{Q}, \mathbf{S})|_{\mathbf{Q} \in \partial\Omega}$  is defined in Definition 46.

## 0.6 BLASSO scheme for the inverse problem

**Definition 0.6.1.** Source tracing problem in the space of measure.

Given the measurement data  $\phi^d \in L^2(\partial\Omega)$ , we reconstruct a measure  $\mu^*$  from:

$$\begin{aligned} \mu^* &= \operatorname{argmin}_{\mu \in \mathcal{X}} \left\{ \frac{1}{2} \|\Phi\mu - \phi^d\|_2^2 + \lambda \|\mu\| \right\}, \quad \|\mu\| = \|\mathbf{I}\|_1, \\ \mathbf{I} &:= \{I_1, I_2, \dots, I_N\}, \end{aligned} \quad (49)$$

where  $I_i$  is the intensity of the  $i$ th source.

The optimality condition for Problem 49 is given as follows:

$$0 \in \Phi^*(\Phi\mu - \phi^d) + \lambda \partial \|\mu\|, \quad \text{where } \partial \|\mu\| := \{g \mid \|g\|_\infty \leq 1 \text{ \& } \int g d\mu = \|\mu\|\}. \quad (50)$$

## 0.7 Sliding Frank Wolfe algorithm

To solve Problem 49, we implement the Sliding Frank Wolfe algorithm. We present the algorithm in this section.

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**Result:** Solve  $\mathbf{I}_* = \{I_1, I_2, \dots\}$ ,  $\mathbf{P}_* = \{\mathbf{P}_1, \mathbf{P}_2, \dots\}$

Initialize:  $\mathbf{I}_*^0 = \emptyset$ ,  $\mathbf{P}_*^0 = \emptyset$ ,  $k = 0$  ;

**while**  $\|\eta_k\|_\infty > 1$ , **do**

$\mu_k = \sum_{i=0}^k I_i \delta_{\mathbf{P}_i}(\mathcal{S})$  ;

Solve  $\mathbf{P}_{k+\frac{1}{2}}$  from:

$$\mathbf{P}_{k+\frac{1}{2}} = \operatorname{argmax}_{\mathbf{P} \in R^3} \|\eta_k\|_\infty, \quad \text{where } \eta_k = \frac{\Phi^*(p_k)}{\lambda}, \quad p_k = \Phi\mu_k - \phi^d. \quad (51)$$

$\mathbf{P}_*^{k+\frac{1}{2}} = [\mathbf{P}_*^k; \mathbf{P}_{k+\frac{1}{2}}]$ , updates  $\mu$  to  $\mu_{k+\frac{1}{2}} = \sum_{i=0}^{k+\frac{1}{2}} I_i \delta_{\mathbf{P}_i}(\mathcal{S})$  .

Solve  $\mathbf{I}_*^{k+\frac{1}{2}} = \{I_1, I_2, \dots, I_{k+\frac{1}{2}}\}$  from:

$$\mathbf{I}_*^{k+\frac{1}{2}} = \operatorname{argmin}_{\mathbf{I} \in R^{k+1}} \left\{ \frac{1}{2} \|\Phi\mu_k|_{\mathbf{P}=\mathbf{P}_*^{k+\frac{1}{2}}} - \phi^d\|_2^2 + \lambda \|\mu_k\|_1 \right\}. \quad (52)$$

Initialize with  $\mathbf{I}_*^{k+\frac{1}{2}}, \mathbf{P}_*^{k+\frac{1}{2}}$ , s.t.,  $\mu_{k+1} = \sum_{i=0}^{k+\frac{1}{2}} I_i \delta_{\mathbf{P}_i}(\mathcal{Q})$ , and solve:

$$\mathbf{I}_*^{k+1}, \mathbf{P}_*^{k+1} = \operatorname{argmin}_{\mathbf{I} \in R^{k+1}, \mathbf{P} \in R^{3(k+1)}} \left\{ \frac{1}{2} \|\Phi\mu_{k+1} - \phi^d\|_2^2 + \lambda \|\mu_{k+1}\|_1 \right\}. \quad (53)$$

$k = k + 1$ .

**end**

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## 0.8 Some results