Gradient Computation

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This document contains the gradient formula for the linear and nonlinear lasso problem.

1 Gradient computation

Define two points in spherical coordinates as follows:

$$\boldsymbol{Q} = (1, \theta, \psi),\ \boldsymbol{P_i} = (r_i, \theta_i, \psi_i). \tag{1}$$

The normalized forward solution reads:

$$Est(\boldsymbol{Q}) = \hat{\Phi}(\mu)(\boldsymbol{Q}) = \sum_{i=1}^{N} \frac{I_i}{G(\boldsymbol{P_i})}\phi_{\boldsymbol{P_i}}(\boldsymbol{Q}),\ \boldsymbol{Q} \in \partial\Omega,\ where, \tag{2}$$

$$G(\boldsymbol{P_i}) = \sqrt{\int_{\partial\Omega} |F(\boldsymbol{Q}, \boldsymbol{P_i})|^2 d\boldsymbol{Q}} = \sqrt{\int_{\partial\Omega} \phi^2_{\boldsymbol{P_i}}(\boldsymbol{Q}) d\boldsymbol{Q}} \tag{3}$$

where,

$$\begin{aligned}
\phi_{\boldsymbol{P_i}}(\boldsymbol{Q}) &= \frac{2}{||\boldsymbol{Q} - \boldsymbol{P_i}||_2} - \log(1 - \langle \boldsymbol{P}_i,\ \boldsymbol{Q}\rangle + ||\boldsymbol{Q} - \boldsymbol{P_i}||_2) \\
&= \frac{2}{\sqrt{1 + r_i^2 - 2r_i\cos\gamma_i}} - \log(1 - r_i\cos\gamma_i + \sqrt{1 + r_i^2 - 2r_i\cos\gamma_i}),\ where, \\
\cos\gamma_i &= \cos\theta\cos\theta_i - \sin\theta\sin\theta_i(\psi - \psi_i).
\end{aligned} \tag{4}$$

Define the source parameter vector as follows:

$$\boldsymbol{V} = \{\overbrace{I_1, I_2, \cdots}^{N}, \overbrace{r_1, r_2, \cdots}^{N}, \overbrace{\theta_1, \theta_2, \cdots}^{N}, \overbrace{\psi_1, \psi_2, \cdots}^{N}\}, \tag{5}$$

$$I_{tot} = \{I_1, I_2, \cdots, I_N\} \tag{6}$$

For the linear lasso problem, we compute the pseudo gradient of the term:

$$Ojb = \frac{1}{2}||Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})||_2^2 + \lambda||I_{tot}||_1, \tag{7}$$

$$= \int_{\partial\Omega} [Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})]^2 d\boldsymbol{Q} + \lambda\sum_i |I_i|. \tag{8}$$

For convenience, we introduce the following two terms:

$$Descrepency(\boldsymbol{Q}) := \frac{1}{2}||Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})||_2^2 = \frac{1}{2}\int_{\partial\Omega} [Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})]^2 d\boldsymbol{Q}, \tag{9}$$

$$Regularization := \lambda||I_{tot}||_1. \tag{10}$$

1.1 Subgradient of the regularization term

$$\frac{\partial Regularization}{\partial I_i} = \lambda\ sign(I_i). \tag{11}$$

1.2 Linear lasso problem

FBS

For the FBS solver, only the gradient of the descrepency term is required, and locations of the source are given. Therefore we only compute:

$$\begin{aligned}
Grad_i = \frac{\partial Descrepency(\boldsymbol{Q})}{\partial I_i} &= \int_{\partial\Omega} (Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})) \frac{\partial Est(\boldsymbol{Q})}{\partial I_i} d\boldsymbol{Q}, \\
&= \int_{\partial\Omega} (Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})) \frac{1}{G(\boldsymbol{P_i})} \phi_{\boldsymbol{P_i}}(\boldsymbol{Q}) d\boldsymbol{Q}, \\
i &= 1, 2, \cdots, N.
\end{aligned} \tag{12}$$

In the minimization iteration, source location $\boldsymbol{P_i}$ is provided by the solver in each search step.

Quasi Newton

For quasi newton solver, gradient of the whole objective function needs to be computed.

$$PseudoGrad_i = \int_{\partial\Omega} (Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})) \frac{1}{G(\boldsymbol{P_i})} \phi_{\boldsymbol{P_i}}(\boldsymbol{Q}) d\boldsymbol{Q} + \lambda \; sign(I_i). \tag{13}$$

1.3 Nonlinear lasso problem

In the nonlinear lasso problem, the locations are also unknown. The pseudo gradient of the nonlinear lasso problem consists of two component: the gradient of the descrepency term and the subgradient of the l1 regularizatoin term.

The descrepency gradient

$$\begin{aligned}
&for \; i = 1 \to N, \\
&Grad_i = \frac{\partial Descrepency(\boldsymbol{Q})}{\partial \boldsymbol{V}_i} = \frac{\partial Descrepency(\boldsymbol{Q})}{\partial I_i} = \int_{\partial\Omega} (Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})) \frac{1}{G(\boldsymbol{P_i})} \phi_{\boldsymbol{P_i}}(\boldsymbol{Q}) d\boldsymbol{Q}.
\end{aligned} \tag{14}$$

$$\begin{aligned}
&for \; i = N+1 \to 2N, \\
&Grad_i = \frac{\partial Descrepency(\boldsymbol{Q})}{\partial \boldsymbol{V}_i} = \frac{\partial Descrepency(\boldsymbol{Q})}{\partial r_i} = \int_{\partial\Omega} (Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})) \frac{\partial Est(\boldsymbol{Q})}{\partial r_i} d\boldsymbol{Q},
\end{aligned} \tag{15}$$

$$\frac{\partial Est(\boldsymbol{Q})}{\partial r_i} = I_i \frac{\frac{\partial \phi_{\boldsymbol{P_i}}}{\partial r_i} G(\boldsymbol{P_i}) - \phi_{\boldsymbol{P_i}} \frac{\partial G(\boldsymbol{P_i})}{\partial r_i}}{G^2(\boldsymbol{P_i})}, \tag{16}$$

where,

$$\frac{\partial \phi_{\boldsymbol{P_i}}}{\partial r_i} = -\frac{2}{||\boldsymbol{Q} - \boldsymbol{P_i}||_2^2} \frac{\partial ||\boldsymbol{Q} - \boldsymbol{P_i}||_2}{\partial r_i} - \frac{1}{1 - \langle \boldsymbol{P}_i, \; \boldsymbol{Q} \rangle + ||\boldsymbol{Q} - \boldsymbol{P_i}||_2} \frac{\partial(-\langle \boldsymbol{P}_i, \; \boldsymbol{Q} \rangle + ||\boldsymbol{Q} - \boldsymbol{P_i}||_2)}{\partial r_i} \tag{17}$$

$$= -2 \frac{r_i - \cos\gamma_i}{||\boldsymbol{Q} - \boldsymbol{P_i}||_2^3} - \frac{-\cos\gamma_i + \frac{r_i - \cos\gamma_i}{||\boldsymbol{Q} - \boldsymbol{P_i}||_2}}{1 - \langle \boldsymbol{P}_i, \; \boldsymbol{Q} \rangle + ||\boldsymbol{Q} - \boldsymbol{P_i}||_2} \tag{18}$$

$$\begin{aligned}
&for \; i = 2N+1 \to 3N, \\
&Grad_i = \frac{\partial Descrepency(\boldsymbol{Q})}{\partial \boldsymbol{V}_i} = \frac{\partial Descrepency(\boldsymbol{Q})}{\partial \theta_i} = \int_{\partial\Omega} (Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})) \frac{\partial Est(\boldsymbol{Q})}{\partial \theta_i} d\boldsymbol{Q},
\end{aligned} \tag{19}$$

$$\frac{\partial Est(\boldsymbol{Q})}{\partial \theta_i} = I_i \frac{\frac{\partial \phi_{\boldsymbol{P_i}}}{\partial \theta_i} G(\boldsymbol{P_i}) - \phi_{\boldsymbol{P_i}} \frac{\partial G(\boldsymbol{P_i})}{\partial \theta_i}}{G^2(\boldsymbol{P_i})}. \tag{20}$$

$$for\ i = 3N+1 \rightarrow 4N,$$

$$Grad_i = \frac{\partial Descrepency(\boldsymbol{Q})}{\partial \boldsymbol{V}_i} = \frac{\partial Descrepency(\boldsymbol{Q})}{\partial \psi_i} = \int_{\partial\Omega} (Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})) \frac{\partial Est(\boldsymbol{Q})}{\partial \psi_i} d\boldsymbol{Q}, \tag{21}$$

$$\frac{\partial Est(\boldsymbol{Q})}{\partial \psi_i} = I_i \frac{\frac{\partial \phi_{\boldsymbol{P_i}}}{\partial \psi_i} G(\boldsymbol{P_i}) - \phi_{\boldsymbol{P_i}} \frac{\partial G(\boldsymbol{P_i})}{\partial \psi_i}}{G^2(\boldsymbol{P_i})}. \tag{22}$$