Gradient Computation

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This document contains the gradient formula for the linear and nonlinear lasso problem.

1 Gradient computation

Define two points in spherical coordinates as follows:

$$\boldsymbol{Q} = (1, \theta, \psi),\ \boldsymbol{P_i} = (r_i, \theta_i, \psi_i). \tag{1}$$

The normalized forward solution reads:

$$\hat{\Phi}(\mu)(\boldsymbol{Q}) = \sum_{i=1}^{N} \frac{I_i}{G_{\boldsymbol{P_i}}} \phi_{\boldsymbol{P_i}}(\boldsymbol{Q}),\ \boldsymbol{Q} \in \partial\Omega,\ where, \tag{2}$$

$$G_{\boldsymbol{P_i}} = \sqrt{\int_{\partial\Omega} |F(\boldsymbol{Q}, \boldsymbol{P_i})|^2 d\boldsymbol{Q}} = \sqrt{\int_{\partial\Omega} \phi^2_{\boldsymbol{P_i}}(\boldsymbol{Q}) d\boldsymbol{Q}} \tag{3}$$

where,

$$\begin{aligned}
\phi_{\boldsymbol{P_i}}(\boldsymbol{Q}) &= \frac{2}{||\boldsymbol{Q} - \boldsymbol{P_i}||_2} - \log(1 - \langle \boldsymbol{P}_i,\ \boldsymbol{Q} \rangle + ||\boldsymbol{Q} - \boldsymbol{P_i}||_2) \\
&= \frac{2}{\sqrt{1 + r_i^2 - 2r_i \cos\gamma_i}} - \log(1 - r_i \cos\gamma_i + \sqrt{1 + r_i^2 - 2r_i \cos\gamma_i}),\ where, \\
\cos\gamma_i &= \cos\theta \cos\theta_i + \sin\theta \sin\theta_i \cos(\psi - \psi_i).
\end{aligned} \tag{4}$$

Define the source parameter vector as follows:

$$\boldsymbol{V} = \{\overbrace{I_1, I_2, \cdots}^{N}, \overbrace{r_1, r_2, \cdots}^{N}, \overbrace{\theta_1, \theta_2, \cdots}^{N}, \overbrace{\psi_1, \psi_2, \cdots}^{N}\}, \tag{5}$$

$$\boldsymbol{I_{all}} = \{I_1, I_2, \cdots, I_N\}. \tag{6}$$

And for the following of this report, we use the symbol D^i to represent the partial derivative with respect to the *ith* component of the unknown parameter.

1.1 Linear lasso problem

For the linear lasso problem, locations of the source (i.e., $\boldsymbol{P_i}$) are given, we define the forward estimation as follows:

$$Est(\boldsymbol{I_{all}}, \boldsymbol{Q}) = \sum_{i=1}^{N} \frac{I_i}{G_{\boldsymbol{P_i}}} \phi_{\boldsymbol{P_i}}(\boldsymbol{Q}),\ \boldsymbol{Q} \in \partial\Omega, \tag{7}$$

The objective function reads:

$$Ojb = \frac{1}{2} ||Est(\boldsymbol{I_{all}}, \boldsymbol{Q}) - \phi^d(\boldsymbol{Q})||_2^2 + \lambda ||\boldsymbol{I_{all}}||_1, \tag{8}$$

$$= \int_{\partial\Omega} [Est(\boldsymbol{I_{all}}, \boldsymbol{Q}) - \phi^d(\boldsymbol{Q})]^2 d\boldsymbol{Q} + \lambda \sum_i |I_i|. \tag{9}$$

For convenience, we introduce the following two terms:

$$f_{des}(\boldsymbol{I_{all}}) := \frac{1}{2}||Est(\boldsymbol{I_{all}}, \boldsymbol{Q}) - \phi^d(\boldsymbol{Q})||_2^2 = \frac{1}{2}\int_{\partial\Omega}[Est(\boldsymbol{I_{all}}, \boldsymbol{Q}) - \phi^d(\boldsymbol{Q})]^2 d\boldsymbol{Q}, \tag{10}$$

$$f_{reg} := \lambda||\boldsymbol{I_{all}}||_1. \tag{11}$$

1.2 Subgradient of the regularization term

$$D^i f_{reg} = \lambda\ sign(I_i). \tag{12}$$

FBS

For the FBS solver, only the gradient of the discrepancy term is required, and locations of the source are given. Therefore we only compute:

$$\begin{aligned} Grad_i = D^i f_{des}(\boldsymbol{Q}) &= \int_{\partial\Omega}(Est(\boldsymbol{I_{all}}, \boldsymbol{Q}) - \phi^d(\boldsymbol{Q}))D^i Est(\boldsymbol{I_{all}}, \boldsymbol{Q})d\boldsymbol{Q}, \\ &= \int_{\partial\Omega}(Est(\boldsymbol{I_{all}}, \boldsymbol{Q}) - \phi^d(\boldsymbol{Q}))\frac{1}{G_{\boldsymbol{P_i}}}\phi_{\boldsymbol{P_i}}(\boldsymbol{Q})d\boldsymbol{Q}, \\ i &= 1, 2, \cdots, N. \end{aligned} \tag{13}$$

In the minimization iteration, source location $\boldsymbol{P_i}$ is provided by the solver in each search step.

Quasi Newton

For quasi newton solver, gradient of the whole objective function needs to be computed.

$$PseudoGrad_i = \int_{\partial\Omega}(Est(\boldsymbol{I_{all}}, \boldsymbol{Q}) - \phi^d(\boldsymbol{Q}))\frac{1}{G_{\boldsymbol{P_i}}}\phi_{\boldsymbol{P_i}}(\boldsymbol{Q})d\boldsymbol{Q} + \lambda\ sign(I_i). \tag{14}$$

The Lipschitz constant of the discrepancy gradient

Proposition 1.1. *Let $\boldsymbol{I_{all,1}}, \boldsymbol{I_{all,2}}$ be two intensity solution vectors and $I_{i,1}, I_{i,2}$ be their components, and let $||\boldsymbol{x}|| := \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$ for $\boldsymbol{x} \in R^N$. Define a vector as follows:*

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{Q}) &:= \{D^1 Est(\boldsymbol{I_{all}}, \boldsymbol{Q}), D^2 Est(\boldsymbol{I_{all}}, \boldsymbol{Q}), \cdots, D^N Est(\boldsymbol{I_{all}}, \boldsymbol{Q})\} \\ &= \{\frac{1}{G_{\boldsymbol{P_1}}}\phi_{\boldsymbol{P_1}}(\boldsymbol{Q}), \frac{1}{G_{\boldsymbol{P_2}}}\phi_{\boldsymbol{P_2}}(\boldsymbol{Q}), \cdots, \frac{1}{G_{\boldsymbol{P_N}}}\phi_{\boldsymbol{P_N}}(\boldsymbol{Q})\}. \end{aligned}$$

, then the gradient $\nabla f_{des}(\boldsymbol{I_{all}})$ has a Lipschitz constant as follows:

$$L = \int_{\partial\boldsymbol{Q}}||\boldsymbol{A}(\boldsymbol{Q})\ ||^4 d\boldsymbol{Q} = \sum_i \int_{\partial\Omega}[\frac{1}{G_{\boldsymbol{P_i}}}\phi_{\boldsymbol{P_i}}(\boldsymbol{Q})]^4 d\boldsymbol{Q}\ . \tag{15}$$

Proof. Notice that:

$$\begin{aligned} ||\nabla f_{des}(\boldsymbol{I_{all,1}}) - \nabla f_{des}(\boldsymbol{I_{all,2}})||^2 &= ||\int_{\partial\Omega}[Est(\boldsymbol{I_{all,1}}, \boldsymbol{Q}) - Est(\boldsymbol{I_{all,2}}, \boldsymbol{Q})] \cdot \nabla Est(\boldsymbol{I_{all}}, \boldsymbol{Q})d\boldsymbol{Q}\ ||^2 \\ &= ||\int_{\partial\Omega}\langle \boldsymbol{A}(\boldsymbol{Q}), \boldsymbol{I_{all,1}} - \boldsymbol{I_{all,2}}\rangle \boldsymbol{A}(\boldsymbol{Q})d\boldsymbol{Q}\ ||^2, \end{aligned} \tag{16}$$

Using the Cauchy Schwarz inequality to the vector l2 norm, we conclude that:

$$\int_{\partial\Omega}\langle \boldsymbol{A}(\boldsymbol{Q}), \boldsymbol{I_{all,1}} - \boldsymbol{I_{all,2}}\rangle \boldsymbol{A}(\boldsymbol{Q})d\boldsymbol{Q} \leq \int_{\partial\Omega}||\boldsymbol{A}(\boldsymbol{Q})\ ||\ ||\boldsymbol{I_{all,1}} - \boldsymbol{I_{all,2}}||\boldsymbol{A}(\boldsymbol{Q})d\boldsymbol{Q}. \tag{17}$$

Return to Equation (16), we derive the following:

$$||\nabla f_{des}(\boldsymbol{I_{all,1}}) - \nabla f_{des}(\boldsymbol{I_{all,2}})||^2 = || \int_{\partial\Omega} \langle \boldsymbol{A}(\boldsymbol{Q}), \boldsymbol{I_{all,1}} - \boldsymbol{I_{all,2}} \rangle \boldsymbol{A}(\boldsymbol{Q}) d\boldsymbol{Q} \; ||^2 \tag{18}$$

$$\leq \sum_i [\; ||\boldsymbol{I_{all,1}} - \boldsymbol{I_{all,2}}|| \int_{\partial\Omega} ||\boldsymbol{A}(\boldsymbol{Q}) \; || A_i(\boldsymbol{Q}) d\boldsymbol{Q} \;]^2 \tag{19}$$

$$\leq (\int_{\partial \boldsymbol{Q}} ||\boldsymbol{A}(\boldsymbol{Q}) \; ||^4 d\boldsymbol{Q}) \; ||\boldsymbol{I_{all,1}} - \boldsymbol{I_{all,2}}||^2 \tag{20}$$

therefore, we obtain an estimation of the Lipschitz constant as follows,

$$L = \int_{\partial \boldsymbol{Q}} ||\boldsymbol{A}(\boldsymbol{Q}) \; ||^4 d\boldsymbol{Q} = \sum_i \int_{\partial\Omega} [\frac{1}{G_{\boldsymbol{P_i}}} \phi_{\boldsymbol{P_i}}(\boldsymbol{Q})]^4 d\boldsymbol{Q} \; . \tag{21}$$

□

1.3 Nonlinear lasso problem

For the nonlinear lasso problem, both locations and intensities are unknown, the forward estimation reads:

$$Est(\boldsymbol{V}, \boldsymbol{Q}) = \sum_{i=1}^{N} \frac{I_i}{G_{\boldsymbol{P_i}}} \phi_{\boldsymbol{P_i}}(\boldsymbol{Q}), \;\; \boldsymbol{Q} \in \partial\Omega, \tag{22}$$

The objective function reads:

$$Ojb = \frac{1}{2} ||Est(\boldsymbol{V}, \boldsymbol{Q}) - \phi^d(\boldsymbol{Q})||_2^2 + \lambda ||I_{all}||_1, \tag{23}$$

$$= \int_{\partial\Omega} [Est(\boldsymbol{V}, \boldsymbol{Q}) - \phi^d(\boldsymbol{Q})]^2 d\boldsymbol{Q} + \lambda \sum_i |I_i|. \tag{24}$$

The pseudo gradient of the nonlinear lasso problem consists of two component: the gradient of the discrepancy term and the subgradient of the l1 regularizatoin term.

The discrepancy gradient

Next we compute the following terms:

$$D^i ||\boldsymbol{Q} - \boldsymbol{P_i}||_2, \quad D^i \langle \boldsymbol{Q}, \boldsymbol{P_i} \rangle.$$

for $i = N+1 \to 2N$:

$$D^i \langle \boldsymbol{Q}, \boldsymbol{P_i} \rangle = \cos\gamma_i, \tag{25}$$

$$D^i ||\boldsymbol{Q} - \boldsymbol{P_i}||_2 = \frac{\partial ||\boldsymbol{Q} - \boldsymbol{P_i}||_2}{\partial r_i} = \frac{r_i - \cos\gamma_i}{||\boldsymbol{Q} - \boldsymbol{P_i}||_2}, \tag{26}$$

for $i = 2N+1 \to 3N$:

$$D^i \langle \boldsymbol{Q}, \boldsymbol{P_i} \rangle = r_i(-\cos\theta \sin\theta_i + \sin\theta \cos\theta_i \cos(\psi - \psi_i)), \tag{27}$$

$$D^i ||\boldsymbol{Q} - \boldsymbol{P_i}||_2 = \frac{\partial ||\boldsymbol{Q} - \boldsymbol{P_i}||_2}{\partial \theta_i} = \frac{-r_i(-\cos\theta \sin\theta_i + \sin\theta \cos\theta_i \cos(\psi - \psi_i))}{||\boldsymbol{Q} - \boldsymbol{P_i}||_2}, \tag{28}$$

for $i = 3N+1 \to 4N$:

$$D^i \langle \boldsymbol{Q}, \boldsymbol{P_i} \rangle = r_i(\cos\theta \cos\theta_i - \sin\theta \sin\theta_i \sin(\psi - \psi_i), \tag{29}$$

$$D^i ||\boldsymbol{Q} - \boldsymbol{P_i}||_2 = \frac{\partial ||\boldsymbol{Q} - \boldsymbol{P_i}||_2}{\partial \psi_i} = \frac{-r_i(-\sin\theta \sin\theta_i \sin(\psi - \psi_i))}{||\boldsymbol{Q} - \boldsymbol{P_i}||_2}. \tag{30}$$

$$for \;\; i = 1 \to N,$$

$$Grad_i = D^i f_{des}(\boldsymbol{Q}) = D^i f_{des}(\boldsymbol{Q}) = \int_{\partial\Omega} (Est(\boldsymbol{V}, \boldsymbol{Q}) - \phi^d(\boldsymbol{Q})) \frac{1}{G_{\boldsymbol{P_i}}} \phi_{\boldsymbol{P_i}}(\boldsymbol{Q}) d\boldsymbol{Q}. \tag{31}$$

$$for\ i = N+1 \rightarrow 4N,$$

$$Grad_i = D^i f_{des}(\boldsymbol{Q}) = D^i f_{des}(\boldsymbol{Q}) = \int_{\partial\Omega} (Est(\boldsymbol{V}, \boldsymbol{Q}) - \phi^d(\boldsymbol{Q})) D^i Est(\boldsymbol{V}, \boldsymbol{Q}) d\boldsymbol{Q}, \tag{32}$$

$$D^i Est(\boldsymbol{V}, \boldsymbol{Q}) = I_i \frac{G_{\boldsymbol{P_i}} D^i \phi_{\boldsymbol{P_i}} - \phi_{\boldsymbol{P_i}} D^i G_{\boldsymbol{P_i}}}{G^2_{\boldsymbol{P_i}}}, \tag{33}$$

$$D^i \phi_{\boldsymbol{P_i}} = -\frac{1}{||\boldsymbol{Q} - \boldsymbol{P_i}||_2^3} D^i ||\boldsymbol{Q} - \boldsymbol{P_i}||_2 - \frac{1}{1 - \langle \boldsymbol{P}_i,\ \boldsymbol{Q} \rangle + ||\boldsymbol{Q} - \boldsymbol{P_i}||_2} D^i(-\langle \boldsymbol{P}_i,\ \boldsymbol{Q} \rangle + ||\boldsymbol{Q} - \boldsymbol{P_i}||_2). \tag{34}$$

Recall that,

$$G_{\boldsymbol{P_i}} = \sqrt{\int_{\partial\Omega} |F(\boldsymbol{Q}, \boldsymbol{P_i})|^2 d\boldsymbol{Q}} = \sqrt{\int_{\partial\Omega} \phi^2_{\boldsymbol{P_i}}(\boldsymbol{Q}) d\boldsymbol{Q}},$$

thus,

$$D^i G_{\boldsymbol{P_i}} = \frac{\int_{\partial\Omega} \phi_{\boldsymbol{P_i}}(\boldsymbol{Q}) D^i \phi_{\boldsymbol{P_i}}(\boldsymbol{Q}) d\boldsymbol{Q}}{G_{\boldsymbol{P_i}}}. \tag{35}$$

Pseudo Gradient

Whenever the gradient of the whole objective function is required, we compute the discrepancy gradient and add the subgradient of the regularization term to it, i.e.,

$$PsudoGrad_i = \begin{cases} Grad_i + \lambda * sign(I_i),\ i = 1 \rightarrow N, \\ Grad_i,\ i > N. \end{cases} \tag{36}$$