0.1 Solution of the monopole potential

In spherical coordinate system, consider this equation on B^3:

$$\begin{aligned}\Delta\phi(\boldsymbol{Q}) &= \sum_{i=1}^{N} I_i \delta_{\boldsymbol{P}_i}(\boldsymbol{Q}), \quad \boldsymbol{Q}, \boldsymbol{P_i} \in \Omega \\ \frac{\partial \phi(\boldsymbol{Q})}{\partial \boldsymbol{n}} &= 0, \quad \boldsymbol{Q} \in \partial\Omega, \\ where \;\; \boldsymbol{Q} &= (r,\theta,\psi),\ \boldsymbol{P}_i = (r_i,\theta_i,\psi_i),\ \Omega = B^3(0,1).\end{aligned} \tag{1}$$

The dirac delta function is defined as follows:

$$\delta_{\boldsymbol{x}}(\boldsymbol{y}) := \begin{cases} \infty, & if\ \boldsymbol{x} = \boldsymbol{y}, \\ 0, & else, \end{cases} \tag{2}$$
$$\text{s.t.}, \int_{R^3} \delta_{\boldsymbol{x}}(\boldsymbol{y}) d\boldsymbol{y} = 1, \ \text{for } \boldsymbol{x}, \boldsymbol{y} \in R^3.$$

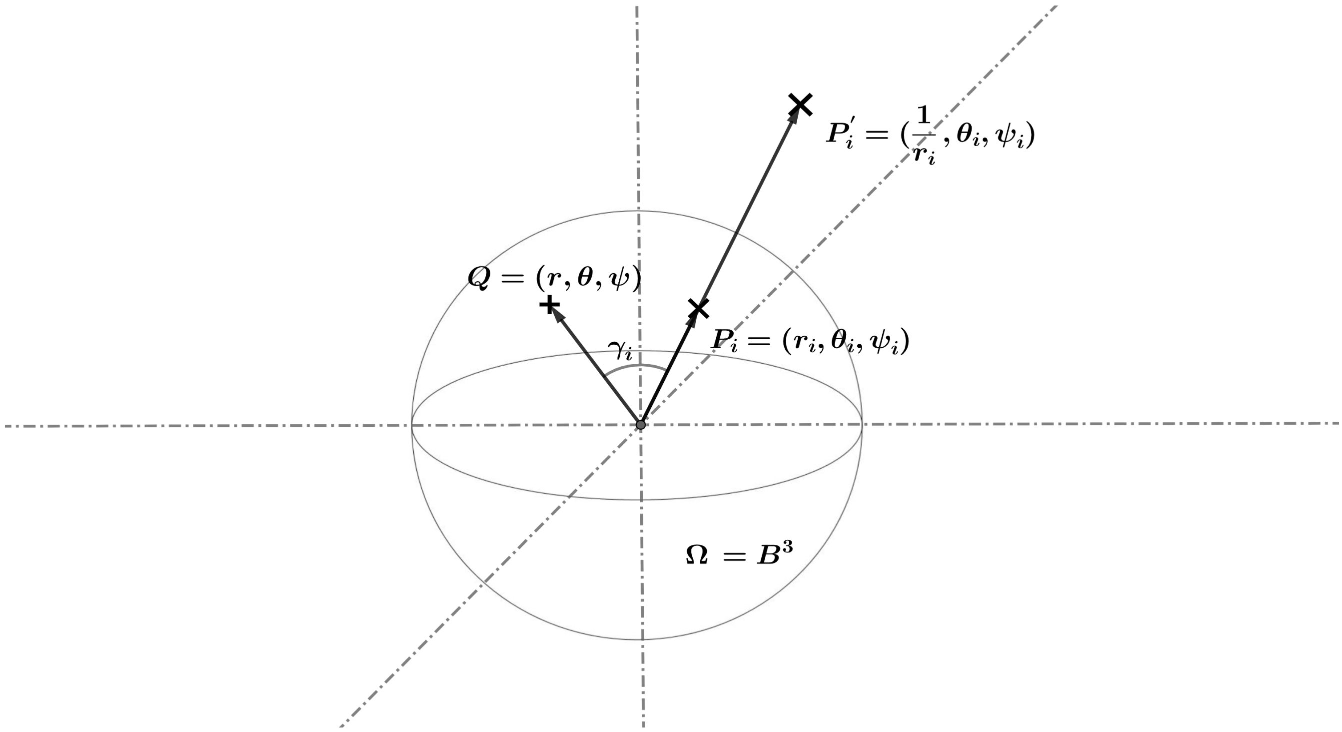

Figure 1: Spherical Domain

Theorem 0.1.1. *Equation (1) has a solution:*

$$\phi(\boldsymbol{Q}) = \sum_{i=1}^{N} [\frac{I_i}{l(\boldsymbol{Q}, \boldsymbol{P}_i)} + \frac{I_i}{r_i l(\boldsymbol{Q}, \boldsymbol{P}_i')} + I_i \log(\frac{1}{1 - r r_i \cos\gamma_i + r_i l(\boldsymbol{Q}, \boldsymbol{P}_i')})] + constant, \tag{3}$$

where,

$$\begin{aligned} l(\boldsymbol{Q}, \boldsymbol{P}_i) &:= ||\boldsymbol{Q} - \boldsymbol{P_i}||_2 = \sqrt{r^2 + r_i^2 - 2 r r_i \cos\gamma_i}, \\ l(\boldsymbol{Q}, \boldsymbol{P}_i') &:= ||\boldsymbol{Q} - \boldsymbol{P_i'}||_2 = \sqrt{r^2 + \frac{1}{r_i^2} - 2\frac{r}{r_i}\cos\gamma_i}, \\ \cos\gamma_i &:= \cos\theta_i \cos\theta + \sin\theta_i \sin\theta \cos(\psi - \psi_i), \\ \boldsymbol{Q} = (r,\theta,\psi) \in \Omega \cup \partial\Omega, \ & \boldsymbol{P_i} = (r_i,\theta_i,\psi_i) \in \Omega, \ \boldsymbol{P_i'} = (\frac{1}{r_i}, \theta_i, \psi_i) \notin \Omega \cup \partial\Omega. \end{aligned}$$

To simplify the expression of the solution, we define a function as follows:

$$\phi_{\boldsymbol{P_i}}(\boldsymbol{Q}) := \frac{1}{l(\boldsymbol{Q}, \boldsymbol{P}_i)} + \frac{1}{||\boldsymbol{P}_i||_2 l(\boldsymbol{Q}, \boldsymbol{P}_i^{'})} + \log(\frac{1}{1 - \langle \boldsymbol{P}_i,\ \boldsymbol{Q}\rangle + ||\boldsymbol{P}_i||_2 l(\boldsymbol{Q}, \boldsymbol{P}_i^{'})}),\ where, \tag{4}$$
$$r_i = ||\boldsymbol{P}_i||_2,\ r = ||\boldsymbol{Q}||_2,\ \langle \boldsymbol{P}_i,\ \boldsymbol{Q}\rangle = r_i \cos\gamma_i.$$

then the solution becomes:

$$\phi(\boldsymbol{Q}) = \sum_{i=1}^{N} I_i \phi_{\boldsymbol{P_i}}(\boldsymbol{Q}). \tag{5}$$

In Cartesian coordinates, we denote the coordinates of these points as follows:

$$\boldsymbol{Q} = (x, y, z),\ \boldsymbol{P_i} = (x_i, y_i, z_i),\ \boldsymbol{P_i^{'}} = (x_i', y_i', z_i').$$

The coordinate transformation reads:

$$x = r\sin\theta\sin\psi,\ y = r\sin\theta\cos\psi,\ z = r\cos\theta.$$

Using these identities, we derive that:

$$rr_i \cos\gamma_i = (x, y, z) \cdot (x_i, y_i, z_i)^T = xx_i + yy_i + zz_i. \tag{6}$$

Meanwhile, the Euclidean distance in Cartesian coordinates reads:

$$\begin{aligned} l(\boldsymbol{Q}, \boldsymbol{P_i}) &= \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}, \\ l(\boldsymbol{Q}, \boldsymbol{P_i'}) &= \sqrt{(x - x_i')^2 + (y - y_i')^2 + (z - z_i')^2}. \end{aligned} \tag{7}$$

Substitute (6) and (7) back to Solution (3), we get:

$$\begin{aligned} \phi(x, y, z) = \sum_{i=1}^{N} [&\frac{I_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}} + \frac{I_i}{\sqrt{x_i^2 + y_i^2 + z_i^2}\sqrt{(x - x_i')^2 + (y - y_i')^2 + (z - z_i')^2}} + \\ &I_i \log(\frac{1}{1 - xx_i + yy_i + zz_i + \sqrt{x_i^2 + y_i^2 + z_i^2}\sqrt{(x - x_i')^2 + (y - y_i')^2 + (z - z_i')^2}}] + constant, \end{aligned} \tag{8}$$

where,

$$(x_i', y_i', z_i') = \frac{1}{\sqrt{x_i^2 + y_i^2 + z_i^2}}(x_i, y_i, z_i).$$

Same as the solution in spherical coordinates, we can simplify the expression of the solution as follows:

$$\phi(\boldsymbol{Q}) = \sum_{i=1}^{N} I_i \phi_{\boldsymbol{P_i}}(\boldsymbol{Q}).$$

0.2 Forward problem solution

For the forward problem, we measure the potential at the boundary of the domain. Therefore we restrict the reference point to be $\boldsymbol{Q} \in \partial\Omega$, thus,

$$\boldsymbol{Q} = (1, \theta, \psi),$$

and then we derive to:

$$l(\boldsymbol{Q}, \boldsymbol{P}_i) = ||\boldsymbol{Q} - \boldsymbol{P}_i||_2 = r_i||\boldsymbol{Q} - \boldsymbol{P}_i^{'}||_2 = r_i l(\boldsymbol{Q}, \boldsymbol{P}_i^{'}),\ \text{for } \boldsymbol{Q} \in \partial\Omega.$$

Thus, returning to Solution (3), when $\boldsymbol{Q} \in \partial\Omega$, the forward solution becomes:

$$\begin{aligned}
\phi(\boldsymbol{Q}) &= \sum_{i=1}^{N}[2\frac{I_i}{l(\boldsymbol{Q},\boldsymbol{P}_i)} - I_i \log(1 - r_i\cos\gamma_i + l(\boldsymbol{Q},\boldsymbol{P}_i))],\ \boldsymbol{Q} \in \partial\Omega, \\
l(\boldsymbol{Q},\boldsymbol{P}_i) &= \sqrt{1 + r_i^2 - 2r_i\cos\gamma_i}, \\
\cos\gamma_i &= \cos\theta_i\cos\theta + \sin\theta_i\sin\theta\cos(\psi - \psi_i).
\end{aligned} \tag{9}$$

In Cartesian coordinates, Solution (9) becomes:

$$\begin{aligned}
\phi(x,y,z) = \sum_{i=1}^{N}[2\frac{I_i}{\sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}} - \\
I_i\log(1 - xx_i + yy_i + zz_i + \sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2})],
\end{aligned} \tag{10}$$

where,

$$(x,y,z) \in \partial\Omega,\ \text{i.e., } x^2 + y^2 + z^2 = 1. \tag{11}$$

0.3 Formulate the forward solution using the Dirac delta measure in spherical coordinates

Dirac delta measure has the following property:

$$\int f(y)d\delta_x(y) = f(x). \tag{12}$$

Let,

$$\boldsymbol{S} = (r,\theta,\psi),\ \boldsymbol{Q} = (\tilde{r},\tilde{\theta},\tilde{\psi}),\ \boldsymbol{P_i} = (r_i,\theta_i,\psi_i).$$

Define the source term in Equation (1) as a measure, as follows:

$$\mu(\boldsymbol{Q}) := \sum_{i=1}^{N} I_i\delta_{\boldsymbol{P}_i}(\boldsymbol{Q}), \tag{13}$$

where the dirac delta function is defined in Definition (2). Using Property (39) to Solution (3), we derive to:

$$\phi(\boldsymbol{Q}) = \int_{\Omega} F(\boldsymbol{Q},\boldsymbol{S})d\mu(\boldsymbol{S}) = \Phi(\mu),\ \boldsymbol{Q},\boldsymbol{S} \in \Omega, \tag{14}$$

where,

$$\begin{aligned}
F(\boldsymbol{Q},\boldsymbol{S}) &:= \frac{1}{\sqrt{r^2 + \tilde{r}^2 - 2r\tilde{r}\cos\tilde{\gamma}}} + \frac{1}{r\sqrt{\frac{1}{r^2} + \tilde{r}^2 - 2\frac{\tilde{r}}{r}\cos\tilde{\gamma}}} - \log(1 - r\tilde{r}\cos\tilde{\gamma} + r\sqrt{\frac{1}{r^2} + \tilde{r}^2 - 2\frac{\tilde{r}}{r}\cos\tilde{\gamma}}), \\
\cos\tilde{\gamma} &:= \cos\tilde{\theta}\cos\theta + \sin\tilde{\theta}\sin\theta\cos(\psi - \tilde{\psi}).
\end{aligned} \tag{15}$$

Apply this to the boundary solution, i.e., Solution (9), we have the following:

$$\phi(\boldsymbol{Q})|_{\boldsymbol{Q}\in\partial\Omega} = \int_{\Omega} F(\boldsymbol{Q},\boldsymbol{S})|_{\boldsymbol{Q}\in\partial\Omega}d\mu(\boldsymbol{S}) = \Phi(\mu),\ \boldsymbol{S} \in \Omega, \tag{16}$$

where,

$$\Phi : \mathcal{X}(\Omega) \to L^2(\partial\Omega),\ \text{s.t.,} \tag{17}$$

$$\Phi(\mu) = \langle F(\boldsymbol{Q},\boldsymbol{S})|_{\boldsymbol{Q}\in\partial\Omega}, \mu\rangle_{\Omega} = \int_{\Omega} F(\boldsymbol{Q},\boldsymbol{S})|_{\boldsymbol{Q}\in\partial\Omega}d\mu(\boldsymbol{S}),\ \mu \in \mathcal{X},\ \boldsymbol{S} \in \Omega. \tag{18}$$

where $\mathcal{X}$ is a space of measure. For the operator Φ, its adjoint is given as follows:

$$\Phi^* : L^2(\partial\Omega) \to C^1(\Omega), \text{ s.t.,} \tag{19}$$

$$\Phi^*(p) = \langle F(\boldsymbol{Q},\boldsymbol{S}), p\rangle_{\partial\Omega} = \int_{\partial\Omega} F(\boldsymbol{Q},\boldsymbol{S})p(\boldsymbol{Q})d\boldsymbol{Q},\ \boldsymbol{Q}\in\partial\Omega,\ \boldsymbol{S}\in\Omega,\ p\in L^2(\partial\Omega). \tag{20}$$

Using the explicit form of $F(\boldsymbol{Q},\boldsymbol{S})$, we expand 47 as follows:

$$\Phi^*(p) = \int_0^{2\pi}\int_0^{\pi}[\frac{1}{\sqrt{r^2+1-2r\cos\tilde{\gamma}}} + \frac{1}{r\sqrt{\frac{1}{r^2}+1-2\frac{1}{r}\cos\tilde{\gamma}}} -$$

$$\log(1-r\cos\tilde{\gamma}+r\sqrt{\frac{1}{r^2}+1-2\frac{1}{r}\cos\tilde{\gamma}})]p(\tilde{\theta},\tilde{\psi})\sin\tilde{\theta}d\tilde{\theta}d\tilde{\psi},$$

$$\text{thus, } \Phi^*(p) = \int_0^{2\pi}\int_0^{\pi}[\frac{2}{\sqrt{r^2+1-2r\cos\tilde{\gamma}}} - \log(1-r\cos\tilde{\gamma}+\sqrt{r^2+1-2r\cos\tilde{\gamma}})]p(\tilde{\theta},\tilde{\psi})\sin\tilde{\theta}d\tilde{\theta}d\tilde{\psi}, \tag{21}$$

$$\text{where, } \cos\tilde{\gamma} := \cos\tilde{\theta}\cos\theta + \sin\tilde{\theta}\sin\theta\cos(\psi-\tilde{\psi}) \tag{22}$$

0.3.1 Existence of Integral (23)

Let

$$\boldsymbol{Q} = (1,\tilde{\theta},\tilde{\psi}),\ \boldsymbol{S} = (r,\theta,\psi).$$

In this section, we investigate the integral:

$$I = \int_{\partial\Omega} F(\boldsymbol{Q},\boldsymbol{S})p(\boldsymbol{Q})d\boldsymbol{Q} \tag{23}$$

$$= \int_0^{2\pi}\int_0^{\pi}[\frac{2}{\sqrt{r^2+1-2r\cos\tilde{\gamma}}} - \log(1-r\cos\tilde{\gamma}+\sqrt{r^2+1-2r\cos\tilde{\gamma}})]p(\tilde{\theta},\tilde{\psi})\sin\tilde{\theta}d\tilde{\theta}d\tilde{\psi}.$$

Proposition 0.3.1. *Assume that $p:\ \boldsymbol{Q}\to p(\boldsymbol{Q})$ is square integrable, then Integral (23) is bounded if and only if $\boldsymbol{S}\notin\partial\Omega$.*

Proof. Cauchy Schwarz inequality yields:

$$\int_{\partial\Omega} F(\boldsymbol{Q},\boldsymbol{S})p(\boldsymbol{Q})d\boldsymbol{Q} \le \sqrt{\int_{\partial\Omega} F^2(\boldsymbol{Q},\boldsymbol{S})d\boldsymbol{Q}\int_{\partial\Omega} p^2(\boldsymbol{Q})d\boldsymbol{Q}}. \tag{24}$$

Since $p\in L^2(\partial\Omega)$,

$$\int_{\partial\Omega} p^2(\boldsymbol{Q})d\boldsymbol{Q} < \infty. \tag{25}$$

When $\boldsymbol{S}\notin\partial\Omega$, $F(\boldsymbol{Q},\boldsymbol{S})$ is bounded, therefore,

$$\int_{\partial\Omega} F^2(\boldsymbol{Q},\boldsymbol{S})d\boldsymbol{Q} < \infty. \tag{26}$$

Now we consider the case when $\boldsymbol{S}\in\partial\Omega$. Notice that:

$$I_1(\boldsymbol{S}) := \int_{\partial\Omega}(\frac{1}{\sqrt{r^2+1-2r\cos\tilde{\gamma}}})^2 d\boldsymbol{Q} = \int_{\partial\Omega}\frac{1}{||\boldsymbol{Q}-\boldsymbol{S}||_2^2}d\boldsymbol{Q}, \tag{27}$$

Since $\boldsymbol{S}\in\partial\Omega=\partial B(0,1)$, $I_1(\boldsymbol{S})$ is rotational invariant. Convert the problem to Cartesian coordinates, such that:

$$\boldsymbol{Q} = (\cos\tilde{\theta}\sin\tilde{\psi}, \sin\tilde{\theta}\sin\tilde{\psi}, \cos\tilde{\theta}). \tag{28}$$

Using the rotational invariant of $I_1(\boldsymbol{S})$, we can choose $\boldsymbol{S}$ as follows:

$$\boldsymbol{S} = (0,0,1), \tag{29}$$

then,

$$I_1(\boldsymbol{S}) = \int_{\partial\Omega} \frac{1}{||\boldsymbol{Q}-\boldsymbol{S}||_2^2} d\boldsymbol{Q} = \int_0^{2\pi}\int_0^{\pi} \frac{1}{2-2\cos\tilde{\theta}} \sin\tilde{\theta} d\tilde{\theta} d\tilde{\psi} = \frac{1}{2}\int_0^{2\pi}\int_{-1}^{1} \frac{1}{u-1} du d\tilde{\psi} = \infty. \tag{30}$$

□

Corollary 0.3.1. *The integral kernel $F(\boldsymbol{Q},\boldsymbol{S})$ is square integralble if and only if $\boldsymbol{S} \notin \partial\Omega$.*

0.4 Normalization

Following the convention, we normalize the integral kernel $F(\boldsymbol{Q},\boldsymbol{S})$ as follows:

$$\hat{F}(\boldsymbol{Q},\boldsymbol{S})|_{\boldsymbol{Q}\in\partial\Omega} := \frac{F(\boldsymbol{Q},\boldsymbol{S})|_{\boldsymbol{Q}\in\partial\Omega}}{\sqrt{\int_{\partial\Omega} |F(\boldsymbol{Q},\boldsymbol{S})|^2 d\boldsymbol{Q}}}. \tag{31}$$

Let $G(\boldsymbol{S})$ be such that:

$$G(\boldsymbol{S}) = \sqrt{\int_{\partial\Omega} |F(\boldsymbol{Q},\boldsymbol{S})|^2 d\boldsymbol{Q}} = \sqrt{\int_0^{2\pi}\int_0^{\pi} [\frac{2}{\sqrt{r^2+1-2r\cos\tilde{\gamma}}} - \log(1 - r\cos\tilde{\gamma} + \sqrt{r^2+1-2r\cos\tilde{\gamma}})]^2 \sin\tilde{\theta} d\tilde{\theta} d\tilde{\psi}},$$

where, $\cos\tilde{\gamma} := \cos\tilde{\theta}\cos\theta + \sin\tilde{\theta}\sin\theta\cos(\psi-\tilde{\psi})$.

0.4.1 Normalized forward solution

Using the normalized forward kernel given in (31), the forward solution is derived to:

$$\hat{\Phi}(\mu) = \int_{\Omega} \hat{F}(\boldsymbol{Q},\boldsymbol{S})|_{\boldsymbol{Q}\in\partial\Omega} d\mu(\boldsymbol{S}) = \int_{\Omega} \frac{F(\boldsymbol{Q},\boldsymbol{S})|_{\boldsymbol{Q}\in\partial\Omega}}{G(\boldsymbol{S})} d\mu(\boldsymbol{S}),\ \mu \in \mathcal{X},\ \boldsymbol{S}\in\Omega. \tag{32}$$

We recall that:

$$\mu(\boldsymbol{S}) = \sum_i I_i \delta_{\boldsymbol{P_i}}(\boldsymbol{S}), \tag{33}$$

and by the sifting property given in (39), we derive the following solution:

$$\hat{\Phi}(\mu)(\boldsymbol{Q}) = \sum_{i=1}^{N} I_i \frac{F(\boldsymbol{Q},\boldsymbol{P_i})}{G(\boldsymbol{P_i})},\ \boldsymbol{Q}\in\partial\Omega. \tag{34}$$

Notice that:

$$F(\boldsymbol{Q},\boldsymbol{P_i}) = \phi_{\boldsymbol{P_i}}(\boldsymbol{Q}),\ \boldsymbol{Q}\in\partial\Omega, \tag{35}$$

where $\phi_{\boldsymbol{P_i}}(\boldsymbol{Q})$ is defined in (4). We now derive to the normalized forward solution, as follows:

$$\hat{\Phi}(\mu)(\boldsymbol{Q}) = \sum_{i=1}^{N} \frac{I_i}{G(\boldsymbol{P_i})} \phi_{\boldsymbol{P_i}}(\boldsymbol{Q}),\ \boldsymbol{Q}\in\partial\Omega. \tag{36}$$

0.4.2 Normalized adjoint problem

Using the normalized integral kernel, the adjoint problem becomes:

$$\hat{\Phi}^*(p) = \int_{\partial\Omega} \hat{F}(\boldsymbol{Q},\boldsymbol{S}) p(\boldsymbol{Q}) d\boldsymbol{Q},\ \boldsymbol{S}\in\Omega,\ \boldsymbol{Q}\in\partial\Omega,\ p\in L^2(\partial\Omega), \tag{37}$$

$$\text{or, } \hat{\Phi}^*(p) = \int_0^{2\pi}\int_0^{\pi} \frac{[\frac{2}{\sqrt{r^2+1-2r\cos\tilde{\gamma}}} - \log(1-r\cos\tilde{\gamma}+\sqrt{r^2+1-2r\cos\tilde{\gamma}})]}{\sqrt{\int_0^{2\pi}\int_0^{\pi}[\frac{2}{\sqrt{r^2+1-2r\cos\tilde{\gamma}}} - \log(1-r\cos\tilde{\gamma}+\sqrt{r^2+1-2r\cos\tilde{\gamma}})]^2 \sin\tilde{\theta} d\tilde{\theta} d\tilde{\psi}}} p(\tilde{\theta},\tilde{\psi}) \sin\tilde{\theta} d\tilde{\theta} d\tilde{\psi}, \tag{38}$$

where, $\cos\tilde{\gamma} := \cos\tilde{\theta}\cos\theta + \sin\tilde{\theta}\sin\theta\cos(\psi-\tilde{\psi})$.

0.5 Formulate the forward solution using the Dirac delta measure in Cartesian coordinates

Dirac delta measure has the following property:

$$\int f(y) d\delta_x(y) = f(x). \tag{39}$$

Let,

$$\boldsymbol{S} = (\tilde{x}, \tilde{y}, \tilde{z}),\ \boldsymbol{Q} = (x, y, z),\ \boldsymbol{P_i} = (x_i, y_i, z_i).$$

Define the source term in Equation (1) as a measure, as follows:

$$\mu(\boldsymbol{Q}) := \sum_{i=1}^{N} I_i \delta_{\boldsymbol{P}_i}(\boldsymbol{Q}), \tag{40}$$

where the dirac delta function is defined in Definition (2). Using Property (39) to Solution (3), we derive to:

$$\phi(\boldsymbol{Q}) = \int_{\Omega} F(\boldsymbol{Q}, \boldsymbol{S}) d\mu(\boldsymbol{S}) = \Phi(\mu),\ \boldsymbol{Q}, \boldsymbol{S} \in \Omega, \tag{41}$$

where,

$$\begin{aligned} F(\boldsymbol{Q}, \boldsymbol{S}) := & \frac{1}{\sqrt{(x-\tilde{x})^2 + (y-\tilde{y})^2 + (z-\tilde{z})^2}} + \frac{1}{\sqrt{x^2+y^2+z^2}\sqrt{(\tilde{x}-x')^2 + (\tilde{y}-y')^2 + (\tilde{z}-z')^2}} \\ & - \log(1 - x\tilde{x} + y\tilde{y} + z\tilde{z} + \sqrt{x^2+y^2+z^2}\sqrt{(x'-\tilde{x})^2 + (y'-\tilde{y})^2 + (z'-\tilde{z})^2}), \end{aligned} \tag{42}$$

where,

$$(x', y', z') = \frac{1}{\sqrt{x^2+y^2+z^2}}(x, y, z).$$

Apply this to the boundary solution, i.e., Solution (9), we have the following:

$$\phi(\boldsymbol{Q})|_{\boldsymbol{Q} \in \partial\Omega} = \int_{\Omega} F(\boldsymbol{Q}, \boldsymbol{S})|_{\boldsymbol{Q} \in \partial\Omega} d\mu(\boldsymbol{S}) = \Phi(\mu),\ \boldsymbol{S} \in \Omega, \tag{43}$$

where,

$$\Phi : \mathcal{X}(\Omega) \to L^2(\partial\Omega),\ \text{s.t.}, \tag{44}$$

$$\Phi(\mu) = \langle F(\boldsymbol{Q}, \boldsymbol{S})|_{\boldsymbol{Q} \in \partial\Omega}, \mu \rangle_{\Omega} = \int_{\Omega} F(\boldsymbol{Q}, \boldsymbol{S})|_{\boldsymbol{Q} \in \partial\Omega} d\mu(\boldsymbol{S}),\ \mu \in \mathcal{X},\ \boldsymbol{S} \in \Omega. \tag{45}$$

where $\mathcal{X}$ is a space of measure. For the operator Φ, its adjoint is given as follows:

$$\Phi^* : L^2(\partial\Omega) \to C^1(\Omega),\ \text{s.t.}, \tag{46}$$

$$\Phi^*(p) = \langle F(\boldsymbol{Q}, \boldsymbol{S}), p \rangle_{\partial\Omega} = \int_{\partial\Omega} F(\boldsymbol{Q}, \boldsymbol{S}) p(\boldsymbol{S}) d\boldsymbol{S},\ \boldsymbol{Q} \in \Omega,\ \boldsymbol{S} \in \partial\Omega,\ p \in L^2(\partial\Omega). \tag{47}$$

The restriction of $\boldsymbol{S} \in \partial\Omega$ yields the following identity:

$$\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 = 1, \tag{48}$$

it defines an implicit function, reads:

$$\tilde{z} = \tilde{z}(\tilde{x}, \tilde{y}).$$

Using the explicit form of $F(\boldsymbol{Q},\boldsymbol{S})$ and the definition of a surface integral, we expand Equation (47) as follows:

$$\Phi^*(p) = \int_{-1}^{1}\int_{-\sqrt{1-\tilde{x}^2}}^{\sqrt{1-\tilde{x}^2}}[\frac{1}{\sqrt{(x-\tilde{x})^2+(y-\tilde{y})^2+(z-\tilde{z})^2}} + \frac{1}{\sqrt{x^2+y^2+z^2}\sqrt{(\tilde{x}-x')^2+(\tilde{y}-y')^2+(\tilde{z}-z')^2}} \tag{49}$$
$$-\log(1-x\tilde{x}+y\tilde{y}+z\tilde{z}+\sqrt{x^2+y^2+z^2}\sqrt{(x'-\tilde{x})^2+(y'-\tilde{y})^2+(z'-\tilde{z})^2})]p(\tilde{x},\tilde{y},\tilde{z})\sqrt{\frac{\partial\tilde{z}}{\partial\tilde{x}}^2+\frac{\partial\tilde{z}}{\partial\tilde{y}}^2+1}d\tilde{x}d\tilde{y},$$

using the restriction given in Equation (48), Equation (49) simplifies to:

$$\begin{aligned}\Phi^*(p) = &\int_{-1}^{1}\int_{-\sqrt{1-\tilde{x}^2}}^{\sqrt{1-\tilde{x}^2}}[\frac{2}{\sqrt{(x-\tilde{x})^2+(y-\tilde{y})^2+(z-\tilde{z}(\tilde{x},\tilde{y}))^2}}- \\ &\log(1-x\tilde{x}+y\tilde{y}+z\tilde{z}(\tilde{x},\tilde{y})+\sqrt{(x-\tilde{x})^2+(y-\tilde{y})^2+(z-\tilde{z}(\tilde{x},\tilde{y}))^2})] \\ &p(\tilde{x},\tilde{y},\tilde{z}(\tilde{x},\tilde{y}))\frac{1}{|\tilde{z}(\tilde{x},\tilde{y})|}d\tilde{x}d\tilde{y},\end{aligned} \tag{50}$$

0.5.1 Normalization in Cartesian coordinates

Following the convention, we normalize the integral kernel $F(\boldsymbol{Q},\boldsymbol{S})$ as follows:

$$\hat{F}(\boldsymbol{Q},\boldsymbol{S})|_{\boldsymbol{Q}\in\partial\Omega} = \frac{F(\boldsymbol{Q},\boldsymbol{S})|_{\boldsymbol{Q}\in\partial\Omega}}{\sqrt{\int_{\partial\Omega}|F(\boldsymbol{Q},\boldsymbol{S})|^2 d\boldsymbol{Q}}}, \tag{51}$$

where,

$$F(\boldsymbol{Q},\boldsymbol{S})|_{\boldsymbol{Q}\in\partial\Omega} := \frac{2}{\sqrt{(x-\tilde{x})^2+(y-\tilde{y})^2+(z-\tilde{z})^2}} - \log(1-x\tilde{x}+y\tilde{y}+z\tilde{z}+\sqrt{(x-\tilde{x})^2+(y-\tilde{y})^2+(z-\tilde{z})^2}) \tag{52}$$

with a restriction: $x^2+y^2+z^2=1$.

Notice that the restriction in Definition (52) defines an implicit function that reads:

$$z = z(x,y). \tag{53}$$

The surface integral in Normalization (51) is defined as follows:

$$\int_{\partial\Omega}|F(\boldsymbol{Q},\boldsymbol{S})|^2 d\boldsymbol{Q} = \int_{-1}^{1}\int_{-\sqrt{1-\tilde{x}^2}}^{\sqrt{1-\tilde{x}^2}} F^2(\boldsymbol{Q},\boldsymbol{S})|_{\boldsymbol{Q}\in\partial\Omega}\sqrt{\frac{\partial z}{\partial x}^2+\frac{\partial z}{\partial y}^2+1}dxdy. \tag{54}$$

where $F(\boldsymbol{Q},\boldsymbol{S})|_{\boldsymbol{Q}\in\partial\Omega}$ is defined in Definition 52.

0.6 BLASSO scheme for the inverse problem

Definition 0.6.1. Source tracing problem in the space of measure.

Given the measurement data $\boldsymbol{\phi^d}\in L^2(\partial\Omega)$, we reconstruct a measure μ^* from:

$$\begin{aligned}&\mu^* = argmin_{\mu\in\mathcal{X}}\{\frac{1}{2}||\boldsymbol{\Phi}\mu-\boldsymbol{\phi^d}||_2^2+\lambda||\mu||\},\ ||\mu|| = ||\boldsymbol{I}||_1, \\ &\boldsymbol{I} := \{I_1, I_2, \cdots, I_N\},\end{aligned} \tag{55}$$

where I_i is the intensity of the ith source.

The optimality condition for Problem (55) is given as follows:

$$0\in\boldsymbol{\Phi}^*(\boldsymbol{\Phi}\mu-\boldsymbol{\phi^d})+\lambda\partial||\mu||,\ \text{where } \partial||\mu|| := \{g|\ ||g||_\infty\le 1\ \&\ \int g d\mu = ||\mu||\}. \tag{56}$$

0.7 Sliding Frank Wolfe algorithm

To solve Problem (55), we implement the Sliding Frank Wolfe algorithm. We present the algorithm in this section.

Result: Solve $\boldsymbol{I}_* = \{I_1, I_2 \cdots\}, \boldsymbol{P}_* = \{\boldsymbol{P}_1, \boldsymbol{P}_2, \cdots\}$

Initialize: $\boldsymbol{I}_*^0 = [], \boldsymbol{P}_*^0 = []$, $k = 0$;

while $||\eta_k||_\infty > 1$, **do**

$\mu_k = \sum_{i=0}^{k} I_i \delta_{\boldsymbol{P_i}}(\boldsymbol{S})$;

Solve $\boldsymbol{P}_{k+\frac{1}{2}}$ from:

$$\boldsymbol{P}_{k+\frac{1}{2}} = argmax_{\boldsymbol{P} \in R^3} ||\eta_k||_\infty, \text{ where } \eta_k = \frac{\Phi^*(p_k)}{\lambda}, \ p_k = \Phi\mu_k - \boldsymbol{\phi}^d. \tag{57}$$

$\boldsymbol{P}_*^{k+\frac{1}{2}} = [\boldsymbol{P}_*^k; \boldsymbol{P}_{k+\frac{1}{2}}]$, updates μ to $\mu_{k+\frac{1}{2}} = \sum_{i=0}^{k+\frac{1}{2}} I_i \delta_{\boldsymbol{P_i}}(\boldsymbol{S})$.

Solve $\boldsymbol{I}_*^{k+\frac{1}{2}} = \{I_1, I_2, \cdots, I_{k+\frac{1}{2}}\}$ from:

$$\boldsymbol{I}_*^{k+\frac{1}{2}} = argmin_{\boldsymbol{I} \in R^{k+1}} \{\frac{1}{2} ||\Phi\mu_k|_{\boldsymbol{P}=\boldsymbol{P}_*^{k+\frac{1}{2}}} - \boldsymbol{\phi}^d||_2^2 + \lambda ||\mu_k||_1\}. \tag{58}$$

Initialize with $\boldsymbol{I}_*^{k+\frac{1}{2}}, \boldsymbol{P}_*^{k+\frac{1}{2}}$, s.t., $\mu_{k+1} = \sum_{i=0}^{k+\frac{1}{2}} I_i \delta_{\boldsymbol{P}_i}(\boldsymbol{Q})$, and solve:

$$\boldsymbol{I}_*^{k+1}, \boldsymbol{P}_*^{k+1} = argmin_{\boldsymbol{I} \in R^{k+1}, \boldsymbol{P} \in R^{3(k+1)}} \{\frac{1}{2} ||\Phi\mu_{k+1} - \boldsymbol{\phi}^d||_2^2 + \lambda ||\mu_{k+1}||_1\}. \tag{59}$$

$k = k + 1$.

end

0.8 Some results