Gradient Computation

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This document contains the gradient formula for the linear and nonlinear lasso problem.

1 Gradient computation

Define two points in spherical coordinates as follows:

$$\boldsymbol{Q} = (1, \theta, \psi), \ \boldsymbol{P_i} = (r_i, \theta_i, \psi_i). \tag{1}$$

The normalized forward solution reads:

$$Est(\boldsymbol{Q}) = \hat{\Phi}(\mu)(\boldsymbol{Q}) = \sum_{i=1}^{N} \frac{I_i}{G_{\boldsymbol{P_i}}} \phi_{\boldsymbol{P_i}}(\boldsymbol{Q}), \ \boldsymbol{Q} \in \partial\Omega, \ where, \tag{2}$$

$$G_{\boldsymbol{P_i}} = \sqrt{\int_{\partial\Omega} |F(\boldsymbol{Q}, \boldsymbol{P_i})|^2 d\boldsymbol{Q}} = \sqrt{\int_{\partial\Omega} \phi^2_{\boldsymbol{P_i}}(\boldsymbol{Q}) d\boldsymbol{Q}} \tag{3}$$

where,

$$\begin{aligned}
\phi_{\boldsymbol{P_i}}(\boldsymbol{Q}) &= \frac{2}{||\boldsymbol{Q} - \boldsymbol{P_i}||_2} - \log(1 - \langle \boldsymbol{P}_i, \ \boldsymbol{Q} \rangle + ||\boldsymbol{Q} - \boldsymbol{P_i}||_2) \\
&= \frac{2}{\sqrt{1 + r_i^2 - 2r_i \cos\gamma_i}} - \log(1 - r_i \cos\gamma_i + \sqrt{1 + r_i^2 - 2r_i \cos\gamma_i}), \ where, \\
\cos\gamma_i &= \cos\theta \cos\theta_i + \sin\theta \sin\theta_i (\psi - \psi_i).
\end{aligned} \tag{4}$$

Define the source parameter vector as follows:

$$\boldsymbol{V} = \{\overbrace{I_1, I_2, \cdots}^{N}, \overbrace{r_1, r_2, \cdots}^{N}, \overbrace{\theta_1, \theta_2, \cdots}^{N}, \overbrace{\psi_1, \psi_2, \cdots}^{N}\}, \tag{5}$$

$$I_{tot} = \{I_1, I_2, \cdots, I_N\} \tag{6}$$

For the linear lasso problem, we compute the pseudo gradient of the term:

$$Ojb = \frac{1}{2} ||Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})||_2^2 + \lambda ||I_{tot}||_1, \tag{7}$$

$$= \int_{\partial\Omega} [Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})]^2 d\boldsymbol{Q} + \lambda \sum_i |I_i|. \tag{8}$$

For convenience, we introduce the following two terms:

$$f_{des}(\boldsymbol{Q}) := \frac{1}{2} ||Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})||_2^2 = \frac{1}{2} \int_{\partial\Omega} [Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})]^2 d\boldsymbol{Q}, \tag{9}$$

$$f_{reg} := \lambda ||I_{tot}||_1. \tag{10}$$

1.1 Subgradient of the regularization term

$$D^i f_{reg} = \lambda \ sign(I_i). \tag{11}$$

1.2 Linear lasso problem

FBS

For the FBS solver, only the gradient of the discrepancy term is required, and locations of the source are given. Therefore we only compute:

$$\begin{aligned} Grad_i = D^i f_{des}(\boldsymbol{Q}) &= \int_{\partial\Omega} (Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})) D^i Est(\boldsymbol{Q}) d\boldsymbol{Q}, \\ &= \int_{\partial\Omega} (Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})) \frac{1}{G_{\boldsymbol{P_i}}} \phi_{\boldsymbol{P_i}}(\boldsymbol{Q}) d\boldsymbol{Q}, \\ i &= 1, 2, \cdots, N. \end{aligned} \tag{12}$$

In the minimization iteration, source location $\boldsymbol{P_i}$ is provided by the solver in each search step.

Quasi Newton

For quasi newton solver, gradient of the whole objective function needs to be computed.

$$PseudoGrad_i = \int_{\partial\Omega} (Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})) \frac{1}{G_{\boldsymbol{P_i}}} \phi_{\boldsymbol{P_i}}(\boldsymbol{Q}) d\boldsymbol{Q} + \lambda\ sign(I_i). \tag{13}$$

1.3 Nonlinear lasso problem

In the nonlinear lasso problem, the locations are also unknown. The pseudo gradient of the nonlinear lasso problem consists of two component: the gradient of the discrepancy term and the subgradient of the l1 regularizatoin term.

The discrepancy gradient

Firstly we compute:

$$D^i \phi_{\boldsymbol{P_i}} = -\frac{2}{||\boldsymbol{Q} - \boldsymbol{P_i}||_2^2} D^i ||\boldsymbol{Q} - \boldsymbol{P_i}||_2 - \frac{1}{1 - \langle \boldsymbol{P}_i,\ \boldsymbol{Q} \rangle + ||\boldsymbol{Q} - \boldsymbol{P_i}||_2} D^i (-\langle \boldsymbol{P}_i,\ \boldsymbol{Q} \rangle + ||\boldsymbol{Q} - \boldsymbol{P_i}||_2). \tag{14}$$

Recall that,

$$G_{\boldsymbol{P_i}} = \sqrt{\int_{\partial\Omega} |F(\boldsymbol{Q}, \boldsymbol{P_i})|^2 d\boldsymbol{Q}} = \sqrt{\int_{\partial\Omega} \phi^2_{\boldsymbol{P_i}}(\boldsymbol{Q}) d\boldsymbol{Q}},$$

thus, we derive that:

$$D^i G_{\boldsymbol{P_i}} = \frac{\int_{\partial\Omega} \phi_{\boldsymbol{P_i}}(\boldsymbol{Q}) d\boldsymbol{Q}}{G_{\boldsymbol{P_i}}} D^i \phi_{\boldsymbol{P_i}}(\boldsymbol{Q}). \tag{15}$$

Next we compute the following terms:

$$D^i ||\boldsymbol{Q} - \boldsymbol{P_i}||_2, \quad D^i \langle \boldsymbol{Q}, \boldsymbol{P_i} \rangle.$$

for $i = N+1 \to 2N$:

$$D^i \langle \boldsymbol{Q}, \boldsymbol{P_i} \rangle = 0, \tag{16}$$

$$D^i ||\boldsymbol{Q} - \boldsymbol{P_i}||_2 = \frac{\partial ||\boldsymbol{Q} - \boldsymbol{P_i}||_2}{\partial r_i} = \frac{r_i - \cos\gamma_i}{||\boldsymbol{Q} - \boldsymbol{P_i}||_2}, \tag{17}$$

for $i = 2N+1 \to 3N$:

$$D^i \langle \boldsymbol{Q}, \boldsymbol{P_i} \rangle = r_i (\cos\theta \sin\theta_i - \sin\theta \sin\theta_i \cos(\psi - \psi_i), \tag{18}$$

$$D^i ||\boldsymbol{Q} - \boldsymbol{P_i}||_2 = \frac{\partial ||\boldsymbol{Q} - \boldsymbol{P_i}||_2}{\partial \theta_i} = \frac{r_i (\cos\theta \sin\theta_i - \sin\theta \sin\theta_i \cos(\psi - \psi_i))}{||\boldsymbol{Q} - \boldsymbol{P_i}||_2}, \tag{19}$$

for $i = 3N+1 \to 4N$:

$$D^i \langle \boldsymbol{Q}, \boldsymbol{P_i} \rangle = -r_i (\cos\theta \cos\theta_i - \sin\theta \sin\theta_i \sin(\psi - \psi_i), \tag{20}$$

$$D^i ||\boldsymbol{Q} - \boldsymbol{P_i}||_2 = \frac{\partial ||\boldsymbol{Q} - \boldsymbol{P_i}||_2}{\partial \psi_i} = \frac{-r_i (\cos\theta \cos\theta_i - \sin\theta \sin\theta_i \sin(\psi - \psi_i))}{||\boldsymbol{Q} - \boldsymbol{P_i}||_2}. \tag{21}$$

$$
\begin{aligned}
&for\ i = 1 \rightarrow N,\\
&Grad_i = D^i f_{des}(\boldsymbol{Q}) = D^i f_{des}(\boldsymbol{Q}) = \int_{\partial\Omega} (Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})) \frac{1}{G_{\boldsymbol{P_i}}} \phi_{\boldsymbol{P_i}}(\boldsymbol{Q}) d\boldsymbol{Q}.
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
&for\ i = N+1 \rightarrow 4N,\\
&Grad_i = D^i f_{des}(\boldsymbol{Q}) = D^i f_{des}(\boldsymbol{Q}) = \int_{\partial\Omega} (Est(\boldsymbol{Q}) - \phi^d(\boldsymbol{Q})) D^i Est(\boldsymbol{Q}) d\boldsymbol{Q},
\end{aligned} \tag{23}
$$

$$
D^i Est(\boldsymbol{Q}) = I_i \frac{G_{\boldsymbol{P_i}} D^i \phi_{\boldsymbol{P_i}} - \phi_{\boldsymbol{P_i}} D^i G_{\boldsymbol{P_i}}}{G^2_{\boldsymbol{P_i}}}, \tag{24}
$$

Pseudo Gradient

Whenever the gradient of the whole objective function is required, we compute the discrepancy gradient and add the subgradient of the regularization term to it, i.e.,

$$
PsudoGrad_i = \begin{cases} Grad_i + \lambda * sign(I_i),\ \ i = 1 \rightarrow N, \\ Grad_i,\ \ i > N. \end{cases} \tag{25}
$$