CSE-423

Prove that, If $P_n = e^{-n\lambda t}$, then $P_{n-1} = n e^{-(n-1)\lambda t}(1-e^{-\lambda t})$

Solve:

$P_n = e^{-n\lambda t}$ ⟶ (1)

$P_n(0) = 1$

$P_{n-1}(0) = 0$ ⟶ (2)

Given,

$P_{n-1}' = n\lambda P_n - (n-1)\lambda P_{n-1}$

$\Rightarrow P_{n-1}' + (n-1)\lambda P_{n-1} = n\lambda P_n$

$\Rightarrow P_{n-1}' + (n-1)\lambda P_{n-1} = n\lambda e^{-n\lambda t}$ [from (1)]

Let, $y = P_{n-1}$

then, $y' = P_{n-1}'$

so, $y' + (n-1)\lambda y = n\lambda e^{-n\lambda t}$ ⟶ (3)

This equation (3) looks like $y' + cy = f(t)$

$\Rightarrow e^{ct}y' + ce^{ct}y = e^{ct}f(t)$

$\Rightarrow \frac{d}{dt}\left[e^{ct}y\right] = e^{ct}f(t)$

$\Rightarrow e^{ct}y = \int e^{ct}f(t)\,dt$ $\left[f(t) = n\lambda e^{-n\lambda t}\right]$

$\Rightarrow e^{ct}y = \int e^{ct}n\lambda e^{-n\lambda t}\,dt$

$= \int n\lambda\left(e^{ct-n\lambda t}\right)dt$

$= n\lambda\int e^{t(c-n\lambda)}\,dt$

$= n\lambda\int e^{((n-1)\lambda - n\lambda)t}\,dt$

$$= n\lambda \int e^{-\lambda t}\, dt$$

$$= n\lambda \left[\frac{e^{-\lambda t}}{-\lambda} + k \right]$$

$$= -ne^{-\lambda t} + n\lambda k$$

So, $e^{ct} y = -ne^{-\lambda t} + n\lambda k$

$y(0) = 0$

$P_{n-1}(0) = 0$

$0 = -ne^{-\lambda t} + n\lambda k$

$\Rightarrow 0 = -ne^{-\lambda \times 0} + n\lambda k$

$\Rightarrow 0 = -n + n\lambda k$

$\Rightarrow k = \frac{1}{\lambda}$

Then,

$e^{ct} y = -ne^{-\lambda t} + n\lambda k$

$= -ne^{-\lambda t} + n\lambda \cdot \frac{1}{\lambda}$

$= -ne^{-\lambda t} + n$

$= n(1 - e^{-\lambda t})$

$\Rightarrow e^{ct} y = n(1 - e^{-\lambda t})$

$\Rightarrow y = e^{-ct} n (1 - e^{-\lambda t})$

$\Rightarrow P_{n-1} = ne^{-(n-1)\lambda t}(1 - e^{-\lambda t})$ [Proved]