Deflecting an Asteroid

Team 452

Problem A

Abstract

There are innumerable asteroids in this galaxy and in our solar system there are countless asteroids that are usually detached from the asteroid belt and float in space. A planet's gravity leaves a chance for asteroids to hit that planet. In this paper we have tried to find a mathematical solution for deflecting an asteroid that could hit the Earth by a spacecraft at the right time. For this we have used different types of mathematical equations and theory including Kepler theorem, different mathematical equations of deflection, orbital equation etc. Finally, we used these to calculate the impact time of the asteroid from the lower earth orbit and the impact time from an estimated distance to the earth, and from this we determined the time that the asteroid would deflect before the spacecraft hit the earth.

Contents

1 Notations Used

Symbol	Meaning	Numerical Value
μ	Standard Gravitational Parameter	$3.986004418 \times 10^{14}$ m^3s^{-2}
$r_{perigee}$	Perigee Radius of Earth	
h	Angular Momentum	
R	Earth Radius	6378 km
M_n	Mean Anomaly	
v	Speed of Asteroid	

2 Problem Analysis

In this problem we are given a situation where an asteroid can hit the Earth's surface with a velocity of 25 km/s. But we have placed a spacecraft in our lower earth orbit that can deflect the asteroid so that the asteroid doesn't hit the earth but passes by. In this case, it is said that the diameter of the asteroid is 100 meters and the mass of the spacecraft we are using to deflect it is 20000 kg.

3 Introduction

Asteroids of various sizes have hit our Earth at various times over thousands of years. Because of this, the world has had a catastrophic impact every time. It cannot be ruled out that this is unlikely to happen in the future. About 65 million years ago, a large asteroid hit the Earth and the dinosaurs disappeared from the Earth. The impact of this injury was many more years.

Whether an asteroid will hit Earth depends on several factors. One such is the size of the asteroid. Small asteroids often enter our Earth's atmosphere, most of which burn up in the atmosphere before reaching the surface. Again, some small pieces reach the surface of the earth which are collected and used in various research. However, the probability of a large impact is less than the probability of a small asteroid impact. According to scientists, such a possibility occurs once every 300,000 years. Scientists have found out that the last time around 65 million years ago, the Earth was hit by a large asteroid, which caused the temperature of the whole Earth to rise, and the Earth faced an apocalyptic situation.

To prevent asteroids from hitting the earth, first, all the small and big asteroids near the earth must be identified. Those asteroids are usually called NEO (near earth object). After that, scientists check the probability of those asteroids hitting the earth. For this, the orbit of that asteroid is observed very well day after day after month after year after year. After thorough observation they are categorized based on their various characteristics. Then, if an asteroid is found that has the potential to hit Earth, its orbit is deflected so that it misses Earth. In this case, the more time we get, the easier it will be for us to deflect the asteroid because we can figure out different ways to do it. Some of these methods, such as a large rocket or spacecraft, can be used to alter the asteroid's orbit using gravity. An asteroid can also be hit by a satellite causing the impact to change its trajectory. Again, in this case nuclear detonation can also be used. But trying to destroy an asteroid close to the Earth's atmosphere can backfire as it can have a nuclear bomb impact on the Earth.

3.1 Asteroid

There is an asteroid belt in our solar system between the planets Mars and Jupiter. In this belt there are countless small and large asteroids that orbit the Sun. Objects smaller than 600 miles in diameter that orbit the Sun are generally considered asteroids. Most asteroids orbit the Sun in an elliptical path and rotate in the same direction as the planets. Some asteroids from these asteroids may come close to the earth to rotate in its orbit. When the asteroid's orbit meets the Earth's orbit at a point. Due to this, the gravitational force of the earth is effective on the asteroid, so the probability of it hitting the earth increases many times. Asteroids near the earth are called NEO (near earth object) or NEA (near earth asteroid). There are also some asteroids that follow abnormal orbits. Asteroids come in different sizes. Again, their mass is also different. Usually, the mass of the asteroid is small in most cases. Asteroid rotation can also be different.

3.2 Deflecting an Asteroid

An asteroid can be deflected in various ways, one of which is to send a rocket carrying a nuclear bomb into space and hit the asteroid. This may change the trajectory of the asteroid, but this impact of the nuclear bomb will also have the possibility of falling on the earth if it is close to the earth. Another method could be to send a rocket or spacecraft with a certain speed or mass in the direction of the asteroid so that the rocket causes the asteroid's trajectory to change by changing its momentum. This is called a kinetic impactor.

3.3 Impact of Asteroid on Earth

Although most asteroids burn up in the atmosphere before reaching the Earth's surface, there are many asteroids that could impact Earth if they did reach the Earth's surface. And the larger the diameter of the asteroid, the greater the amount of impact on the earth. If a large asteroid were to hit the Earth, dust and smoke would cover the Earth's atmosphere, preventing sunlight from reaching the Earth, causing the extinction of many life forms in an instant. Even small asteroids can cause serious damage. It can kill many people. The Chelyabinsk Event, The Tunguska Event, The Chicxulub Event are some such incidents when an asteroid hit the Earth causing damage and loss of life. Moreover, the cataclysmic event that hit the coast of Mexico 65 million years ago brought a cataclysm to Earth that caused the extinction of the dinosaurs.

3.4 Trajectories of an Object on Space

By trajectory we mean the path along which an object or asteroid completes its rotation or passes through. Gravitational-attraction properties of concentrated masses of material are usually used to determine the different trajectories that asteroids take. Also, laws of newton are also considered in this matter. Trajectories are generally spherical in shape. Different types of trajectories are circular, elliptical, parabolic and hyperbolic.

Conic Section	Equation	Eccentricity(e)	Semi-major axis	Energy
Circle	$x^2 + y^2 = a^2$	0	=radius	<0
Ellipse	$x^2/a^2 + y^2/b^2 = 1$	$0<e= \sqrt{(1-b^2/a^2)}<1$	>0	<0
Parabola	$y^2 = 4ax$, $x^2 = 4ay$	1	infinity	0
Hyperbola	$x^2/a^2 - y2/b^2 = 1$ $x^2/b^2 - y2/b^2 = -1$	$e= \sqrt{(1+b^2/a^2)}>1$	<0	>0

4 Assumptions

- Trajectory is elliptical that means $e < 1$
- $b = 1.5$ AU (astronomical unit) for both earth and asteroid
- $a = \sqrt{\frac{b^2}{1-e^2}}$
- $T^2 = a^3$

- $M = nt = \frac{2\pi}{T}t$
- $M = E - \Sigma \sin E$
- True anomaly, $\tan\frac{\theta}{2} = \sqrt{\frac{1+E}{1-E}} \tan\frac{E}{2}$
- Heliocentric distance, $r = a(1- \Sigma \cos E) = \frac{a(1-a^2)}{1+\Sigma \cos E}$

5 Theoretical Analysis

5.1 Two Body Problem

Two body problem deals with two particles of an isolated system, and there is a central potential by which they can interact. For this, equations of motions are considered as,

$$m_1\ddot{\boldsymbol{r}}_1 = \mathbf{F}_{21} \quad \ldots\ldots (1)$$

$$m_2\ddot{\boldsymbol{r}}_2 = \mathbf{F}_{12} \quad \ldots\ldots (2)$$

For example, the newtons law of gravitation is,

$$U_{12}(|r_1 - r_2|) = U_{12}(|r_2 - r_1|) = G\frac{m1m2}{|r1 - r2|2}$$

Here, $G = 6.673 \times 10^{-11}\ N\ m^2 / kg^2$

Now, the center of mass,

$$R = \frac{m1\ r1 + m2\ r2}{m1 + m2} = \frac{m1\ r1 + m2\ r2}{M}$$

The center of mass velocity,

$$v_{CM} = \frac{m1\ v1 + m2\ v2}{M}$$

Acceleration of the center of mass depends on the next center of mass,

$$F_{ext} = Ma_{CM}$$

Because of the system is isolated, the center of mass acceleration is zero, and center of mass velocity is a constant,

$$v_{CM}^{(0)} = \frac{m1\ v1 + m2\ v2}{M}$$

here, $v_1 = v_1^{(0)}$ & $v_2 = v_2^{(0)}$

Now the center of mass motion,

$$R(t) = v_{CM}^{(0)}t$$

The relative distance vector,

$$r = r_1 - r_2$$

So, now,

$$r_1 = R + \frac{m2}{m1+m2} r \quad \&$$

$$r_2 = R - \frac{m1}{m1+m2} r$$

Now, equation 1 multiplied by m_2 ,

$$m_1 m_2 \ddot{\boldsymbol{r}}_1 = m_2 \mathbf{F}_{21}$$

equation 1 multiplied by m_1 ,

$$m_1 m_2 \ddot{\boldsymbol{r}}_2 = m_1 \mathbf{F}_{12}$$

Now subtracting the second equation from the first,

$$m_1 m_2 (\ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2) = m_2 \mathbf{F}_{21} - m_1 \mathbf{F}_{12}$$

$$\Rightarrow \frac{m1\, m2}{m1+m1} \ddot{\boldsymbol{r}} = \mathbf{F}_{21}$$

Finally, we have,

$$m_* = \frac{m1\, m2}{m1+m1}$$

This m_* is the reduces mass of the system. So, the two-particle system is now have become a one particle system with position vector $\boldsymbol{r}$ and mass of m_*.

5.2 Kepler's Law

There are 3 planetary laws that Kepler gave. These are –

Kepler's first law –

"All the planets revolve around the sun in elliptical orbits having the sun at one of the foci". The closest point of the planet to the sun is Perihelion and the most distant point is Aphelion. For this elliptical orbit there are season change in the earth.

Kepler's second law –

"The radius vector drawn from the sun to the planet sweeps out equal areas in equal intervals of time". As the orbit is not circular, the planet's kinetic energy is not constant in its path. It has more kinetic energy near the perihelion, and less kinetic energy near the aphelion implies more speed at the perihelion and less speed (v_{min}) at the aphelion. If r is the distance of planet from sun, at perihelion (r_{min}) and at aphelion (r_{max}), then,

$r_{min} + r_{max} = 2a \times$ (length of major axis of an ellipse) . . . . (1)

Using the law of conservation of angular momentum, the law can be verified. At any point of time, the angular momentum can be given as, $L = mr^2\omega$.

Now consider a small area ΔA described in a small time interval Δt and the covered angle is $\Delta\theta$. Let the radius of curvature of the path be r, then the length of the arc covered = r $\Delta\theta$.

$\Delta A = 1/2[r.(r.\Delta\theta)] = 1/2r^2\Delta\theta$

Therefore, $\Delta A/\Delta t = [\ 1/2r^2]\Delta\theta/dt$

Taking limits on both sides as, $\Delta t \to 0$, we get;

$\lim_{\Delta t \to 0} \Delta A/\Delta t = \lim_{\Delta t \to 0} \frac{1}{2} r^2 \Delta\theta/\Delta t$
$dA/dt = \frac{1}{2} r^2\omega$
$dA/dt = L/2m$
Now, by conservation of angular momentum, L is a constant

Thus, dA/dt = constant

The area swept in equal intervals of time is a constant.

Kepler's second law can also be stated as "The areal velocity of a planet revolving around the sun in elliptical orbit remains constant, which implies the angular momentum of a planet remains constant". As the angular momentum is constant, all planetary motions are planar motions, which is a direct consequence of central force.

Kepler's third law –

"The square of the time period of revolution of a planet around the sun in an elliptical orbit is directly proportional to the cube of its semi-major axis".

$$T^2 \propto a^3$$

Shorter the orbit of the planet around the sun, the shorter the time taken to complete one revolution. Using the equations of Newton's law of gravitation and laws of motion, Kepler's third law takes a more general form:

$P^2 = 4\pi^2 /[G(M_1 + M_2)] \times a^3$

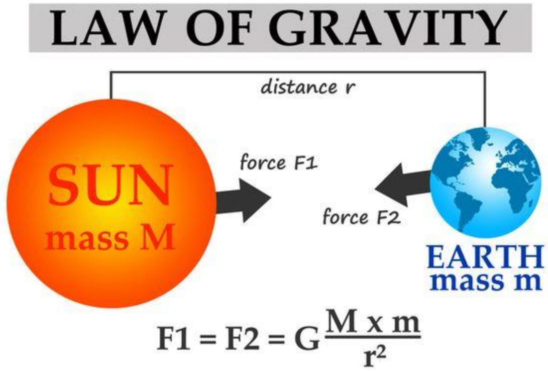
where M_1 and M_2 are the masses of the two orbiting objects in solar masses.

5.3 Newton's Gravitational Law

Newton's law of gravitation, statement that any particle of matter in the universe attracts any other with a force varying directly as the product of the masses and inversely as the square of the distance between them. In symbols, the magnitude of the attractive force F is equal to G (the gravitational constant, a number the size of which depends on the system of units used and which is a universal constant) multiplied by the product of the masses (m_1 and m_2) and divided by the square of the distance R:

$$F = G(m_1 m_2)/R^2$$

Here, $G = 6.673 \times 10^{-11}$ N m^2/ kg^2



5.4 Angular Momentum & The Orbit Formulas

The angular momentum of body m_2 relative to m_1 is the moment of m_1's relative linear momentum $m_2\dot{\boldsymbol{r}}$.

$\mathbf{H}_{2/1} = \dot{\boldsymbol{r}} \times m_2\, \dot{\boldsymbol{r}}$

where $\dot{\boldsymbol{r}}$ = v is the velocity of m_2 relative to m_1. Let us divide this equation through by m_2 and let h = $\mathbf{H}_{2/1}$ /m_2 , so that $\mathbf{h} = \mathbf{r} \times \dot{\boldsymbol{r}}$

h is the relative angular momentum of m_2 per unit mass, that is, the specific relative angular momentum. The units of **h**are $km^2\ s^{-1}$. Taking the time derivative of **h** yields

$dh/dt = \dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} + \mathbf{r} \times \ddot{\boldsymbol{r}}$

But $\dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} = 0$. Furthermore, $\ddot{\boldsymbol{r}} = -(\mu/r^3)r$,

$$h = r\, v$$

Now, we can get,

$$r = (h^2/\mu)\, \frac{1}{1+e\cos\theta}$$

Again,

$$v_r = \frac{\mu}{h}\, e\sin\theta$$

5.5 Hyperbolic Trajectories

When e > 1, the trajectory of a planet is hyperbola. There are two symmetric curves in this system.

If e > 1, the orbital formula is,

$$r = (h^2/\mu)\, \frac{1}{1+e\cos\theta}$$

The nearest point of the satellite from the earth is called perigee. Here,

$$r_{perigee} = (h^2/\mu)\, \frac{1}{1+e}$$

5.6 Mean Anomaly

The satellite's position, orbit's orientation and orbit's shape at a fixed time point that was last known is called an Epoch. The Mean Anomaly indicates where the satellite was at a particular Epoch.

The Mean Anomaly formulas are –

$$M_h = e\sinh F - F_\Theta \quad \&$$

$$M_h = (\mu^2/h^3)\,(e^2-1)\text{^}(3/2)\ t$$

5.7 Calculation of Impact Time from Lower Earth Orbital

The impact speed of the asteroid is given **v** = 25 km/s and the lower earth orbit is from earths surface to less or equal to 2000 km. The mass of the spacecraft is M = 20000 kg.

Let's assume he flight path angle, $\Theta = 45^o$

The horizontal component of **v,**

$v = \mathbf{v}\cos(-45^o)$

$= 17.678$ km/s

Now,

Angular momentum, $h = r\,v$

$= (R + \mathbf{h})\,v$

$= (6378 + 2000) \times 17.678$

$= 148106.284\ km^2/s$

Now, we know, separation distance between two bodies,

$$r = (h^2/\mu)\,\frac{1}{1+e\cos\Theta}$$

⇨ $1+e\cos\Theta = ((148106.284)^2 / (398600 \times 8378)) = 6.5685$

⇨ $e\cos\Theta = 5.5685$

Horizontal component of the velocity of asteroid is,

$v_r = \mathbf{v}\sin\Theta$

⇨ $v_r = 25 \times \sin(-45^o)$

⇨ $v_r = -17.6777$ km/s

Now,

$v_r = (\mu/h)e\sin\Theta$

⇨ $e\sin\Theta = (v_r \times h)/\mu$

⇨ $e\sin\Theta = -6.5684$

$\tan\Theta = \frac{e\sin\theta}{e\cos\theta} = (-6.5684/5.5685) = -1.1796$

Hyperbolic tangent angle, $\Theta_1 = -49.71^o$

Eccentricity of hyperbola, e = (-6.5684/sin(-49.71º)) = 8.61

Perigee Radius, $r_{perigee} = (h^2/\mu) \times \frac{1}{1+e}$

⇨ $r_{perigee} = ((148106.284)^2/398600) \times (1/1+8.61) = 5726.46$ km

Now,

$\tanh\frac{F}{2} = \sqrt{\frac{e-1}{e+1}}\tan(-49.71/2) = -0.4122$

⇨ $\frac{F}{2} = \tanh^{-1}(-0.4122)$

⇨ $\frac{F}{2} = ½ \ln(\frac{1+0.4122}{1-0.4122})$

⇨ $\frac{F}{2} = - 0.438$

⇨ F = - 0.8765

Mean Anomaly,

$M_h = e\sin F - F$

$= 8.61 \times (-0.9931) - (-0.8765)$

$= -7.764$

Again,

$M_h = (\mu^2/h^3)\ (e^2-1)$^(3/2) t

⇨ $-7.674 = ((398600)^2/(148106.284)^3) \times ((8.61)^2 - 1)$^(3/2) t

⇨ t = 250.9 sec

5.8 Calculation of Impact Time from an Estimated Orbit

Assuming the altitude of the orbit of the asteroid is 40000 km away from the earth.

Let’s assume he flight path angle, Θ = 45º

The horizontal component of **v,**

$v = \mathbf{v}\cos(-45^o)$

$= 17.678$ km/s

Angular momentum, $h = r\, v$

$= (R + \mathbf{h})\, v$

$= (6378 + 40000) \times 17.678$

$= 819823.91$ km^2/s

Now, we know, separation distance between two bodies,

$r = (h^2/\mu)\, \frac{1}{1+e\cos\theta}$

⇨ $1+e\cos\Theta = ((819823.91)^2 / (398600 \times 46378))$

⇨ $e\cos\Theta = 35.36$

Now,

$v_r = (\mu/h)e\sin\Theta$

⇨ $e\sin\Theta = (v_r \times h)/\mu$

⇨ $e\sin\Theta = -36.36$

$\tan\Theta = \frac{e\sin\theta}{e\cos\theta} = (-36.36/35.36)$

Hyperbolic tangent angle, $\Theta_1 = -45.8^o$

Eccentricity of hyperbola, $e = 50.718$

$r_{perigee} = (h^2/\mu) \times \frac{1}{1+e} = 32603.34$ km

$F = -0.88$

$M_h = -49.74$

Now,

$M_h = (\mu^2/h^3)\, (e^2-1)^{\wedge}(3/2)\, t$

$t = 1323.95$ sec

So, now the time needed for the spacecraft to deflect the asteroid is,

$t = 1323.95 – 250.9$

= 1073.05 sec

= 17.884 minutes

6 Simulation

6.1 Simulation of Kepler equation

Kepler equation relates various geometric properties of orbit of an object which is discussed in orbital mechanics. Kepler's equation is of a fundamental property in orbital mechanics, but cannot be directly used in order to determine where the planet will be at a given time. Let be M_h the mean anomaly (a parameterization of time) and E be the eccentric anomaly (a parameterization of polar angle) of a body orbiting on an ellipse with eccentricity, then

$$E - e \sin E = M_h$$

This transcendental equation cannot be solved directly for E. A rough value can be assumed. However, an accurate solution requires an iterative procedure like 'trial and error'. Newton Raphson method is one of the more common and efficient ways of finding the root of a well-behaved function. To apply Newton's method to the solution of Kepler's equation, we can form the function

$$f(E) = E - e \sin E - M_h$$

and from this function we can write eccentric anomaly that makes f (E) = 0. Since here newton's method is used, so firstly it is differentiated to

$$f'(E) = 1 - e \cos E$$

Lastly, we get the equation like below-

$$\mathrm{Ei} + 1 = \mathrm{Ei} - \frac{\mathrm{Ei} - e \sin \mathrm{Ei} - \mathrm{M}h}{1 - e \cos \mathrm{Ei}}$$

From here we get the equations with respect to time as we know,

$$Mh = \frac{2\mathrm{pi}}{\mathrm{T}} \times \mathrm{t}$$

We had simulated the second scenario where we considered that, both the asteroid and earth orbiting Against sun. We considered different value of eccentricity and special energy, £ for the asteroid which was about to impact the earth. The earth and asteroid have different trajectories and when both the trajectories (which we assumed elliptical) intersect with each other those are the possible places where asteroid could impact the earth only if they cross at the same time.

Thereby we had simulated the eccentricity of earth and asteroid as a function of time t where t is the time since perihelion of asteroid. From the simulated curve we could possibly find the point of intersect. Hence time t taken before this occurrence is known as well.

All the curves are plotted against special anomaly E vs time t-

Firstly, the graph is drawn for some values of e=0.1

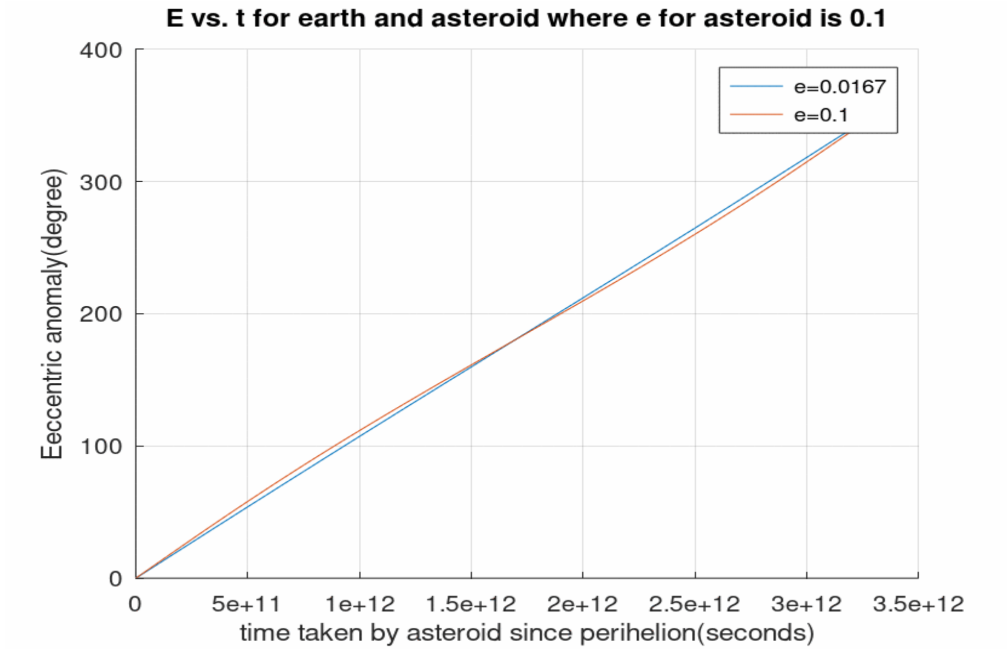


Figure 1 Eccentric anomaly with respect to time.

Then, the graph is for some values of e=0.2

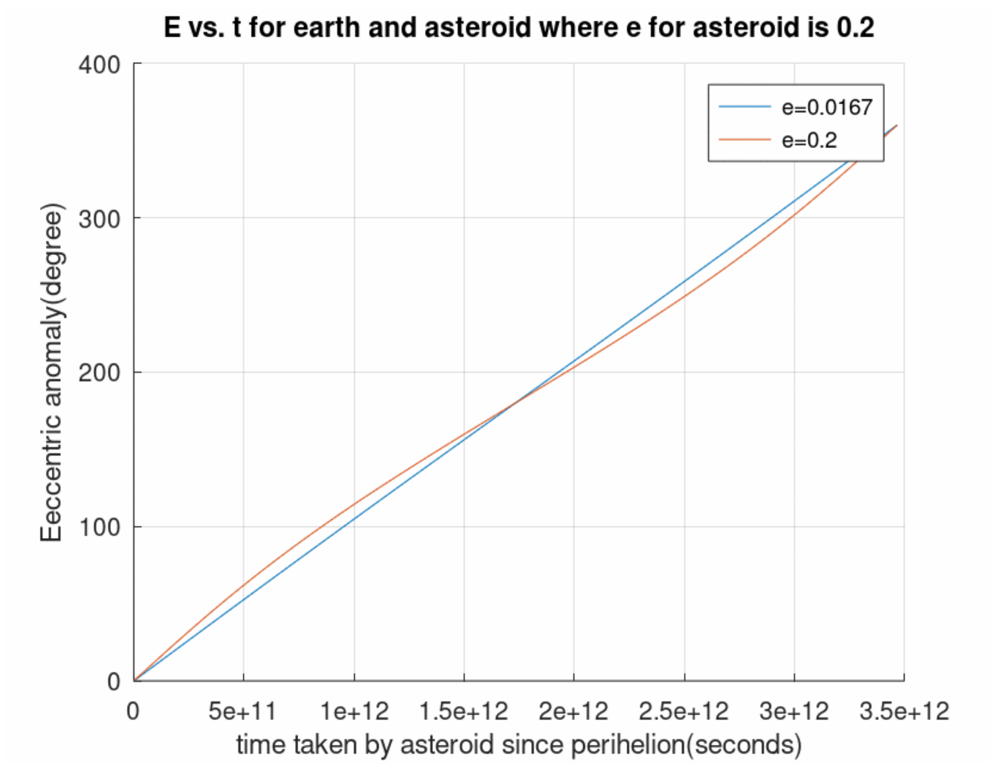


Figure 2 Eccentric anomaly with respect to time.

Then, the graph is for some values of e=0.4

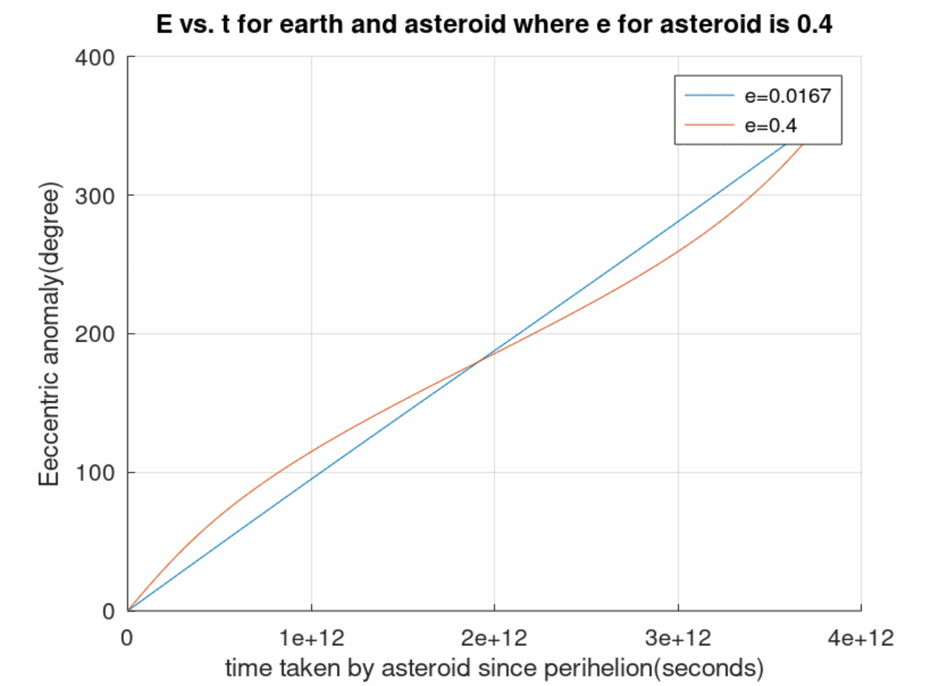


Figure 3 Eccentric anomaly with respect to time.

Then, the graph is for some values of e=0.6

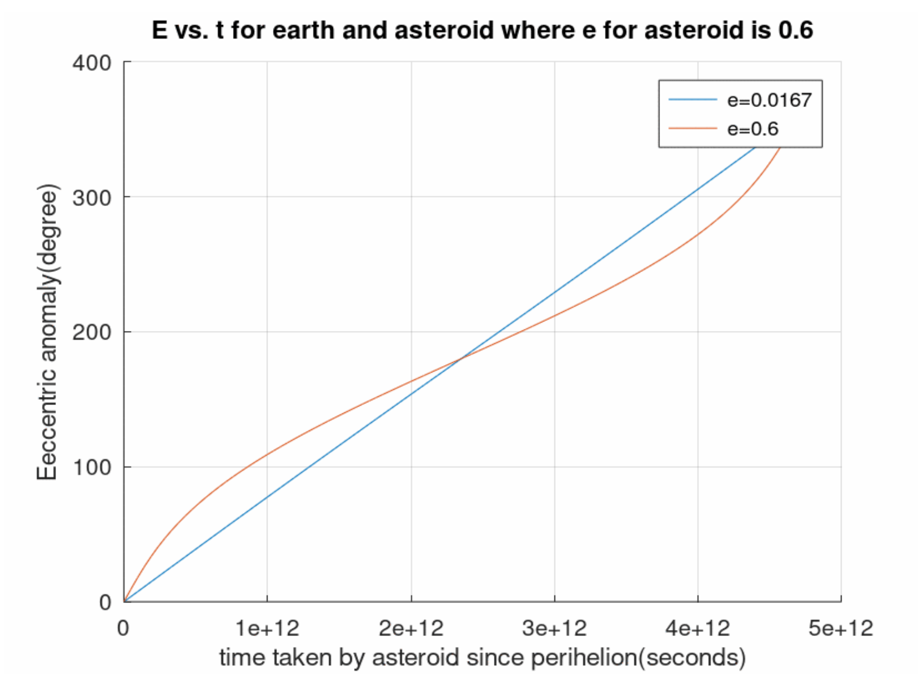


Figure 4 Eccentric anomaly with respect to time.

6.2 Code for simulation

The generated code for the simulation is attached here for better understanding-

```
clc
clear all

e = [0.0167,0.1]%,0.2,0.4,0.6,0.9];
M = linspace(0,2*pi,100);
delta = ones(6,100);
for i =1:6
    for n = 1:100
        E(i,n)= M(n);
    end
end

for m =1:2
    for n = 1:100
        while (abs(delta(m,n))>= 10^(-8))
            f(m,n) = E(m,n) - e(m)*sin(E(m,n)) - M(n);
            F(m,n) = 1- e(m)*cos(E(m,n));
            temp(m,n) = E(m,n) - ((f(m,n))/(F(m,n)));
            delta(m,n) = temp(m,n) - E(m,n);
            E(m,n) = temp(m,n);
        end
    end
end
e2= [1.8688e-12,1.855e-12]%1.8128e-12,1.64e-12,1.337e-12,5.399e-13];
 t=M/e2(2)   ;

hold on
for i = 1:2;
    plot(t,E(i,:)*180/pi)
    xlabel('time taken by asteroid since perihelion(seconds)')
    ylabel('Eeccentric anomaly(degree)')
    title('E vs. t for earth and asteroid where e for asteroid is 0.1')

    grid on
end
legend('e=0.0167','e=0.1');

    hold off;
```

6.3 Simulation for distance between two bodies

Firstly, the equation of relative motion can be stated here as:

$$\ddot{r} = -\left(\frac{\mu}{r3}\right)\boldsymbol{r}$$

We want to be able to integrate this equation to find a scalar equation. An analytical equation will be more accurate than the numerical techniques we used earlier, and a scalar equation is easier to work with than a vector one.

After all the calculation and derivation, we get the equation,

$$r = \frac{h * h}{\mu\,(1 \;+\; e \cos\theta)}$$

By plotting the equation with respect to flight angle, we get different values of distance between the two bodies which are orbiting in their orbits.

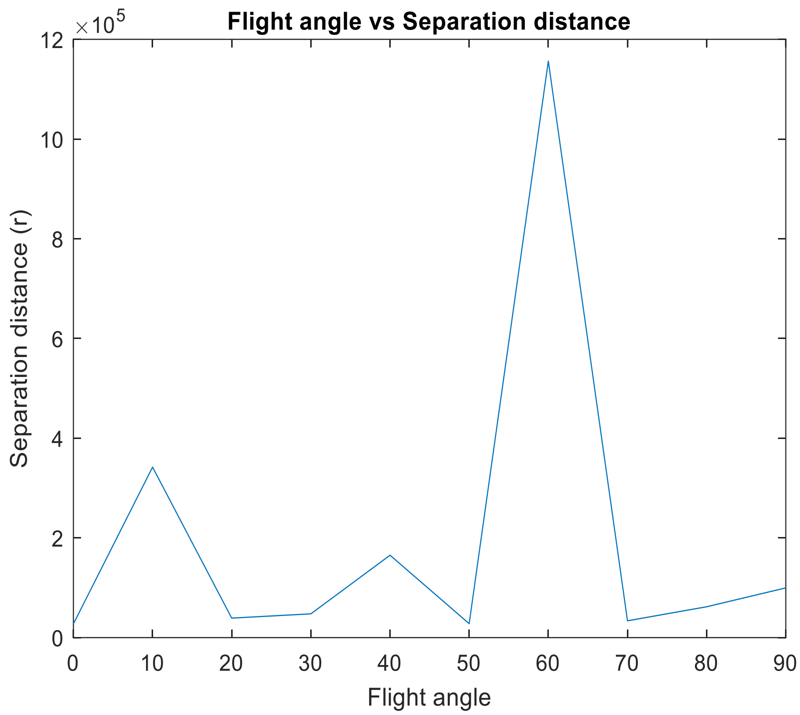


Figure 4 Eccentric anomaly with respect to time.

6.4 Code for simulation

The generated code for calculating r is given-

```
clc;
clear all;
a = 0:10:90;
u = 398600;
h = 148106.284;
r = (h*h)./(u*(1+cos(a)));
plot(a,r);
```

7 Results

We formulated a theory and calculated accordingly. According to this theory, the spacecraft will deflect the asteroid so that it avoids Earth. The spacecraft must correctly deflect the asteroid 17.884 minutes before the asteroids impact with Earth.

8 Discussion

Strength –

- We have tried to do all the calculations properly.
- We have tried to explain the theories we used.
- By using Kepler's theorem, we find a way to solve the situation logically.

Weakness –

- No separate calculation of air resistance is shown in our solution.

9 Conclusion

Finally, we figured out that the spacecraft could deflect the asteroid if it went according to our calculations.

10 References

- Orbital Mechanics for Engineering Students by Howard Curtis
- Kepler’s Law from https://byjus.com/jee/keplers-laws/
- Mean Anomaly from http://www.castor2.ca/03_Mechanics/02_Elements/06_Mean_Anom/index.html
- Newton’s Gravitational Law from https://www.britannica.com/science/Newtons-law-of-gravitation