Ordinary Differential Equations

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I. Introduction

The analysis of high-speed flow in variable-area passages is fundamental to numerous engineering applications including that of jet engines, rocket nozzles, and wind tunnels. To optimize performance and ensure efficient design, it is critical to understand the behavior of Mach number variations along variable-area passages. In this report, Mach number distribution along a conical flow passage is analyzed using a mathematical model consisting of ordinary differential equations (ODEs) and solved using fourth order Runge-Kutta.

The ODEs describe the relationship between the Mach number and the geometric (area and friction), and thermodynamic (heat transfer) effects. Thus, numerically approximating reduced ordinary differential equations as a result of specific set conditions produces valuable insight into how thermodynamic and geometric factors influence flow characteristics.

II. Objective/Mathematical Model

The governing ODE that describe the high-speed flow in the variable-area passage is

$$\frac{dM}{dx} = \frac{M(1+\frac{\gamma-1}{2}M^2)}{1-M^2}\left(-\frac{1}{A}\frac{dA}{dx} + \frac{1}{2}\gamma M^2\frac{4f}{D} + \frac{1}{2}(1 + \gamma M^2)\frac{1}{T}\frac{dT}{dx}\right) \quad (1)$$

where x is the distance along the passage, M = M(x) is the Mach number, γ is the ratio of specific heats, A is the cross-sectional flow area, f is the friction coefficient, D is the diameter, and T is the stagnation temperature. The variable-area of a circular cross section is described by the ODE

$$\frac{dA}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}D^2\right) = \frac{\pi}{4}\frac{d}{dx}(D_i + \alpha x)^2 = \frac{\pi}{2}(D_i + \alpha x)\alpha = \alpha\frac{\pi}{2}D \quad (2)$$

where

$$D(x) = D_i + \alpha x \quad (3)$$

describes the diameter and

$$A = \frac{\pi}{4}D^2 \quad (4)$$

describe the area of a circular cross-section.

The stagnation temperature T is described as

$$T(x) = T_i + \frac{Q(x)}{C} \tag{5}$$

where Q(x) is the heat transfer along the flow passage described by

$$Q(x) = Q_i + \beta x \tag{6}$$

for a linear heat transfer rate. Thus, the differential form of (Eq. 5) is described as

$$\frac{dT}{dx} = \frac{1}{C}\frac{d}{dx}(Q_i + \beta x) = \frac{\beta}{C}. \tag{7}$$

Note friction coefficient f is assumed to be constant as it is an empirical function of the Reynolds number and passage surface roughness.

The set conditions investigated in the domain x = [0, 5] cm are described by test cases:

a. $f = \beta = 0, \alpha = 0.25\ cm/cm, \gamma = 1.4, D_i = 1.0\ cm$
   i. M(x) = 0.7
   ii. M(x) = 1.5
b. $\alpha = \beta = 0, f = 0.005, \gamma = 1.4,\ D_i = 1.0\ cm$
   i. M(x) = 0.7
   ii. M(x) = 1.5
c. $\alpha = f = 0, T_i = 1000\ K, \beta = 50\ J/cm, C = 1.0\ kJ/(Kg - K),\ D_i = 1.0\ cm$
   i. M(x) = 0.5
   ii. M(x) = 2.0

The reduced ordinary differential equation for each test case is outlined by:

Governing ODE

$$\frac{dM}{dx} = \frac{M\left(1+\frac{\gamma-1}{2}M^2\right)}{1-M^2}\left(-\frac{1}{A}\frac{dA}{dx}+\frac{1}{2}\gamma M^2\frac{4f}{D}+\frac{1}{2}(1+\gamma M^2)\frac{1}{T}\frac{dT}{dx}\right)$$

$$\frac{dM}{dx} = \frac{M\left(1+\frac{\gamma-1}{2}M^2\right)}{1-M^2}\left(-\frac{\pi\alpha D}{2\left(\frac{\pi D^2}{4}\right)}+\frac{1}{2}\gamma M^2\frac{4f}{D}+\frac{(1+\gamma M^2)\beta}{2TC}\right)$$

Case "a" $(x=[0,5])$

$f=\beta=0 \quad \alpha=0.25\ \text{cm/cm} \quad \gamma=1.4 \quad D_i=1.0\ \text{cm}$

$$\frac{dM}{dx} = \frac{M\left(1+\frac{\gamma-1}{2}M^2\right)}{1-M^2}\left(-\frac{\pi\alpha D}{2\left(\frac{\pi D^2}{4}\right)}+\cancel{\frac{1}{2}\gamma M^2\frac{4f}{D}}+\cancel{\frac{(1+\gamma M^2)\beta}{2TC}}\right)$$

$$\frac{dM}{dx} = \left[\frac{M(1+0.2M^2)}{1-M^2}\right]\left[-\frac{\pi(0.25)(1+(0.25x))}{2\left(\frac{\pi(1+(0.25x))^2}{4}\right)}\right]$$

Final reduced ODE for case "a"

i) $M_i = 0.7 \rightarrow M(0) = 0.7$

Initial Conditions for RK4
(21 Points)

ii) $M_i = 1.5 \rightarrow M(0) = 1.5$

$D(x) = D_i + \alpha x$

Case "b" $x=[0,5]$

$\alpha=\beta=0 \quad f=0.005 \quad \gamma=1.4 \quad D_i=1.0\,\text{cm}$

$$\frac{dM}{dx}=\frac{M\left(1+\frac{\gamma-1}{2}M^2\right)}{1-M^2}\left(-\cancel{\frac{\dot{m}\alpha D}{2\left(\frac{\pi D^2}{4}\right)}}+\frac{1}{2}\gamma M^2\frac{4f}{D}+\cancel{\frac{(1+\gamma M^2)\beta}{2TC}}\right)$$

$$\frac{dM}{dx}=\frac{M(1+0.2M^2)}{1-M^2}\left(0.014M^2\right)$$ ← Final reduced ODE for case "b"

i) $M_i=0.7 \rightarrow M(0)=0.7$

ii) $M_i=1.5 \rightarrow M(0)=1.5$

Initial conditions for RK4 (21 Points)

Case "c" $x=[0,5]$

$\alpha=f=0 \quad T_i=1000\text{ K} \quad \beta=50\text{ J/cm} \quad C=1.0\text{ kJ/(kg-K)} \quad D_i=1.0\text{ cm}$

$$\frac{dM}{dx}=\frac{M\left(1+\frac{\gamma-1}{2}M^2\right)}{1-M^2}\left(-\cancel{\frac{\dot{m}\alpha D}{2\left(\frac{\pi D^2}{4}\right)}}+\cancel{\frac{1}{2}\gamma M^2\frac{4f}{D}}+\frac{(1+\gamma M^2)\beta}{2TC}\right)$$

$$\frac{dM}{dx}=\frac{M(1+0.2M^2)}{1-M^2}\left(\frac{(1+1.4M^2)(50)}{2(1000+50x)}\right)$$ ← Final reduced ODE for case "c"

i) $M_i=0.5$

ii) $M_i=2.0$

Initial conditions for RK4 (21 Points)

☆ $T(x)=T_i+Q(x)\frac{1}{c}$ ☆

☆ $\gamma=1.4$ is assumed. Same as other cases as gas is the same ☆

The objective of this report is to produce M(x) vs. x plots for all three cases to investigate Mach Number N given area, frictional, and heat transfer effects.

III. Numerical Method

Runge-Kutta-Fourth-Order (RK4) is used to solve ODEs in the described cases a, b, and c. The error per step is proportional to h^5 and thus is fourth order accurate. Considerably more than lower order methods such as Euler and Heun methods. It also only relies on the current step to compute all the "k" terms. Given an ODE

$$\frac{dy}{dx} = f(x,\ y) \tag{8}$$

and initial value

$$y(x_0) = y_0, \tag{9}$$

RK4 uses the average of four slope estimates k_1, k_2, k_3, k_4 defined by Eq. 10 to Eq. 13.

$$k_1 = f(x_n,\ y_n) \tag{10}$$

$$k_2 = f(x_n + \frac{h}{2},\ y_n + \frac{h}{2}k_1) \tag{11}$$

$$k_3 = f(x_n + \frac{h}{2},\ y_n + \frac{h}{2}k_2) \tag{12}$$

$$k_4 = f(x_n + h,\ y_n + hk_3) \tag{13}$$

The final value at the next step is computed/updated as

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4). \tag{14}$$

IV. Implementation

Algorithm

1. Define each case with parameters
    a. Setup each ODE for each case.
    b. Call RK4 solver for each case.
    c. Produce Plot for each result
2. RK4 solver
    a. Setup grid of 21 points
    b. Setup the step size
    c. Setup initial Mach value
    d. Approximate each point through iteration
        i. Compute k1, k2, k3, k4 terms
        ii. Update Mach number based on RK4 equation

MATLAB Code

```matlab
function main()
    % Case a
    alpha_A = 0.25; f_A = 0; beta_A = 0;
    [x_A1, M_A1] = RK4(alpha_A, f_A, beta_A, 0.7, 'A');
    [x_A2, M_A2] = RK4(alpha_A, f_A, beta_A, 1.5, 'A');
    % Case b
    alpha_B = 0; f_B = 0.005; beta_B = 0;
    [x_B1, M_B1] = RK4(alpha_B, f_B, beta_B, 0.7, 'B');
    [x_B2, M_B2] = RK4(alpha_B, f_B, beta_B, 1.5, 'B');
    % Case c
    alpha_C = 0; f_C = 0; beta_C = 50;
    [x_C1, M_C1] = RK4(alpha_C, f_C, beta_C, 0.5, 'C');
    [x_C2, M_C2] = RK4(alpha_C, f_C, beta_C, 2.0, 'C');
    % Plot Cases
    plotResults(x_A1, M_A1, x_A2, M_A2, 'Case A: Area Change');
    plotResults(x_B1, M_B1, x_B2, M_B2, 'Case B: Friction');
    plotResults(x_C1, M_C1, x_C2, M_C2, 'Case C: Heat Transfer');
end

% RK4 implementation
function [x, M] = RK4(alpha, f, beta, M_0, caseType)
    x = linspace(0, 5, 21);
    h = x(2) - x(1);
    M = zeros(size(x));
```

```
    M(1) = M_0;
    for i = 1:length(x)-1
        if caseType == 'A'
            dMdx = caseA_ODE(x(i), M(i), alpha, f, beta);
        elseif caseType == 'B'
            dMdx = caseB_ODE(x(i), M(i), alpha, f, beta);
        elseif caseType == 'C'
            dMdx = caseC_ODE(x(i), M(i), alpha, f, beta);
        end
        k1 = h * dMdx;
        k2 = h * caseSpecificODE(x(i)+h/2, M(i)+k1/2, alpha, f, beta, caseType);
        k3 = h * caseSpecificODE(x(i)+h/2, M(i)+k2/2, alpha, f, beta, caseType);
        k4 = h * caseSpecificODE(x(i)+h, M(i)+k3, alpha, f, beta, caseType);
        M(i+1) = M(i) + (k1 + 2*k2 + 2*k3 + k4)/6;
    end
end

function dMdx= caseSpecificODE(x, M, alpha, f, beta, caseType)
    if caseType == 'A'
        dMdx = caseA_ODE(x, M, alpha, f, beta);
    elseif caseType == 'B'
        dMdx = caseB_ODE(x, M, alpha, f, beta);
    elseif caseType == 'C'
        dMdx = caseC_ODE(x, M, alpha, f, beta);
    end
end

% Case A ODE
function dMdx = caseA_ODE(x, M, alpha, f, beta)
    D = 1.0 + alpha*x;
    A = pi*D^2 / 4;
    term = - (pi * alpha * D) / (2 * A)
    dMdx = (M*(1+ 0.2*M^2)) / (1-M^2) * term;
end

% Case B ODE
function dMdx = caseB_ODE(x, M, alpha, f, beta)
```

```
    term = 0.5 * 1.4 * M^2 * (4*f/1.0);
    dMdx = (M * (1+0.2*M^2)) / (1-M^2) * term;
end
% Case C ODE
function dMdx = caseC_ODE(x, M, alpha, f, beta)
    T = 1000 + beta*x / 1.0;
    term = (1 + 1.4*M^2) * beta / (2* T * 1.0);
    dMdx = (M*(1 + 0.2*M^2)) / (1-M^2) * term;
end

% Plot function
function plotResults(x1, M1, x2, M2, caseName)
    figure;
    plot(x1, M1, 'b-o', x2, M2, 'r-s', 'LineWidth', 1.5);
    xlabel('Distance x (cm)');
    ylabel('Mach Number M');
    title(sprintf('%s\nM_0 = %.1f (blue) vs M_0 = %.1f (red)', caseName, M1(1), M2(1)));
    grid on;
    legend(sprintf('M_0=%.1f', M1(1)), sprintf('M_0=%.1f', M2(1)));
End
```

V. Results

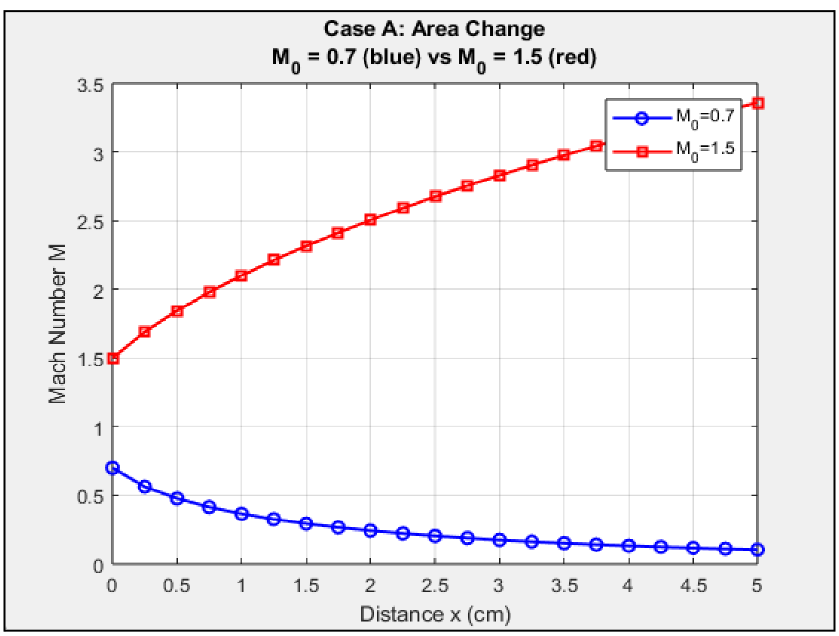

FIG. 1. Mach Number N results over Distance x of case a

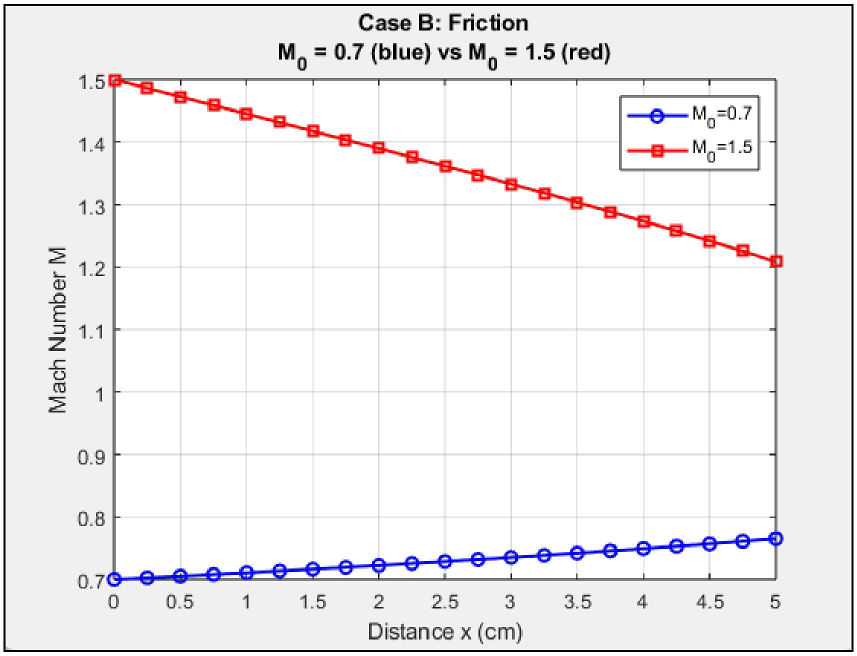

FIG. 2. Mach Number N results over Distance x of case b

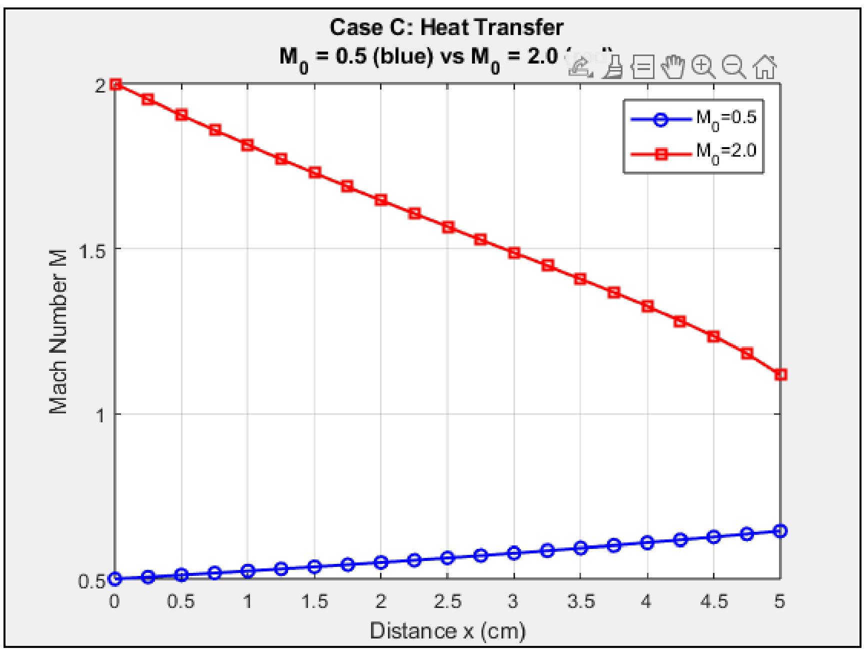

FIG. 3. Mach Number N results over Distance x of case c

Note: 21 RK4 iterations were performed for all three cases.

VI. Discussion

(Fig. 1), (Fig. 2), and (Fig. 3) illustrate interesting insight regarding subsonic flow (Mach Number < 1) compared to supersonic flow (Mach number 1.5 to 5) given parameters in cases a, b, and c.

Investigation of case a (Fig. 1) suggests that an overall Mach number increase given area change in subsonic flow and the Mach number decrease when flow transformed to supersonic. On the other hand, investigation of case b (Fig. 2) suggests a sharp decline of Mach number considering frictional effects in supersonic flow, and the opposite for subsonic flow. Similarly, investigation of case c (Fig. 3) suggests a sharper decline in Mach number in supersonic flow compared to frictional effects, and a similar slight increase in Mach number when flow is subsonic.

VII. Conclusion

The investigation of Mach number modelled by ODEs and solved using RK4 implemented using MATLAB code, aid in understanding of high-speed flow in variable-area passages. This is critical for the design of nozzles, diffusers, turbomachinery, etc. Moreover, It is discussed how effective RK4 is to numerically approximate such ODEs. The resulting Mach Number N vs. Distance x plots for cases a, b, and c investigated area, frictional, and heat transfer effects demonstrated to have an effect to consider on subsonic/supersonic flow. Future work could incorporate investigation of other parameters and interaction between parameters resulting in effects on high-speed flow.