Foundations of Linear Algebra & PCA Application

Exploring the mathematical foundations that power modern data science and engineering applications

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Vector Spaces & Linear Transformations

Vector Space

Set with addition & scalar multiplication operations (e.g., $\mathbb{R}^2$)

Basis & Dimension

Minimal spanning set determines the space's dimension

Linear Transformation

Map $T: V \rightarrow W$ preserving addition & scalar multiplication

Example applications include 2D rotation matrices and scaling transformations that preserve linear structure.

Eigenvalues & Eigenvectors

$$Av = \lambda v$$

Where v is a nonzero eigenvector and λ is the corresponding eigenvalue

Fundamental Directions

Capture the essential directions of linear transformations

Real Applications

System stability analysis, facial recognition, quantum mechanics

Example: Matrix A = [[2,0],[0,3]] has eigenvalues $\lambda_1=2$, $\lambda_2=3$

Singular Value Decomposition

$$A = U\Sigma V^T$$

01

U Matrix

Left singular vectors representing column space directions

02

Σ Matrix

Diagonal matrix containing singular values in descending order

03

V Matrix

Right singular vectors representing row space directions

Critical for data compression, image processing, and recommendation systems.

PCA for Data Reduction

Principal Component Analysis transforms high-dimensional data into lower dimensions while preserving maximum variance.

1. **Standardize Data**

   Center and scale features to unit variance

2. **Covariance Matrix**

   Compute relationships between variables

3. **Eigendecomposition**

   Find eigenvectors and eigenvalues

4. **Project Data**

   Transform onto top k principal components

Worked Example

Find eigenvalues and eigenvectors of matrix A = [[2,0],[0,3]]

Step 1: Characteristic Equation

$\det(A-\lambda I) = 0 \rightarrow (2-\lambda)(3-\lambda) = 0$

Therefore: $\lambda = 2, 3$

Step 2: Eigenvector for λ=2

$(A-2I)v = 0 \rightarrow$ eigenvector [1,0]

Step 3: Eigenvector for λ=3

$(A-3I)v = 0 \rightarrow$ eigenvector [0,1]

Key Takeaways

Foundation

Vector spaces and linear transformations form the mathematical backbone

Structure

Eigenvalues and eigenvectors reveal hidden mathematical structure

Decomposition

SVD provides powerful matrix factorization capabilities

Application

PCA enables practical data reduction in engineering systems