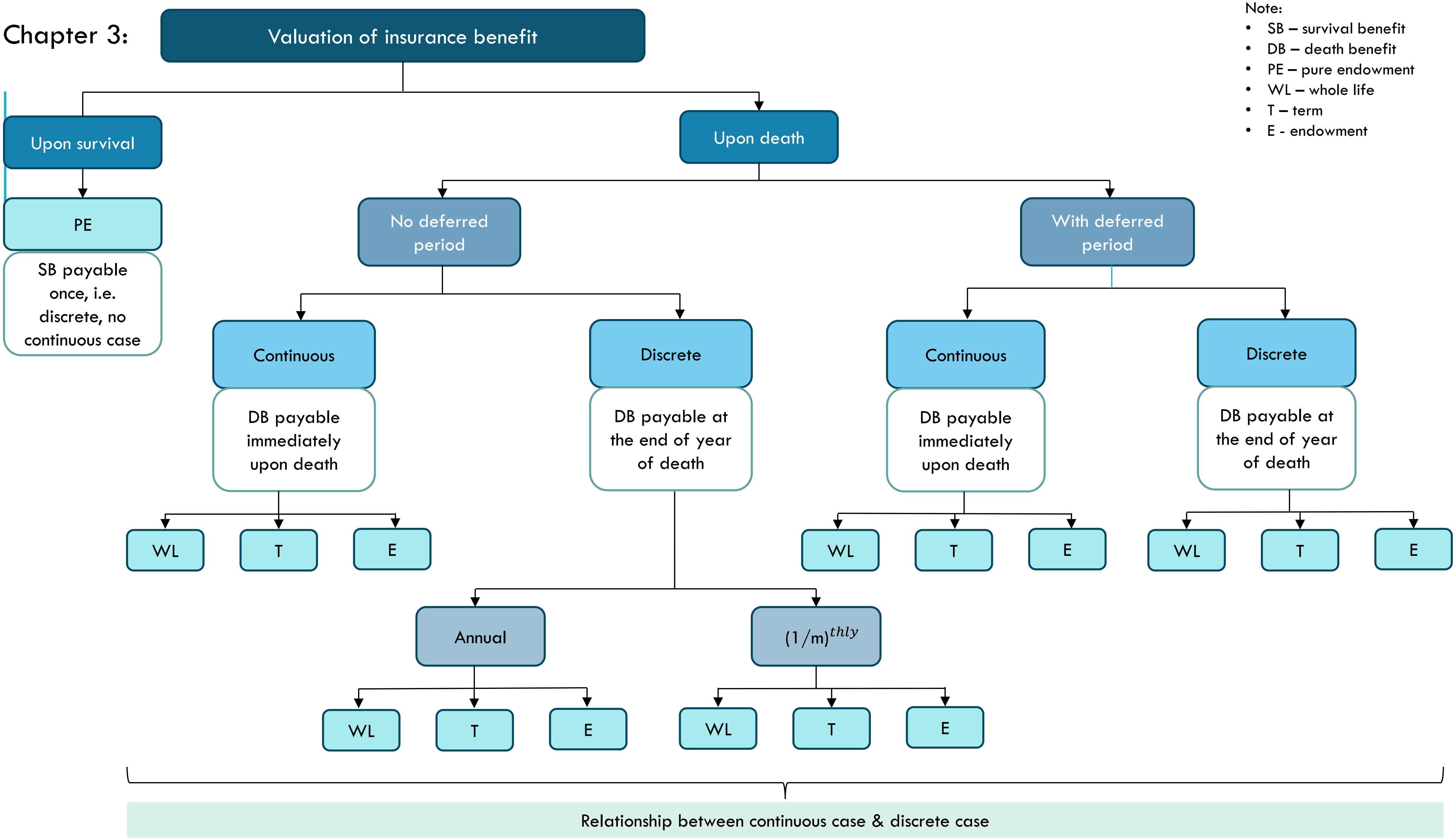
VALUATION OF LIFE INSURANCE

ACTUARIAL MATHEMATICS I
ASC425

OUTLINE

- Introduction
- Continuous case
- Discrete case
  - Annual
  - $(1/m)^{thly}$ → m times in 1 year
- Relationship between continuous case and discrete case
- Variable insurance benefit

Chapter 3:



Note:

- SB – survival benefit
- DB – death benefit
- PE – pure endowment
- WL – whole life
- T – term
- E - endowment

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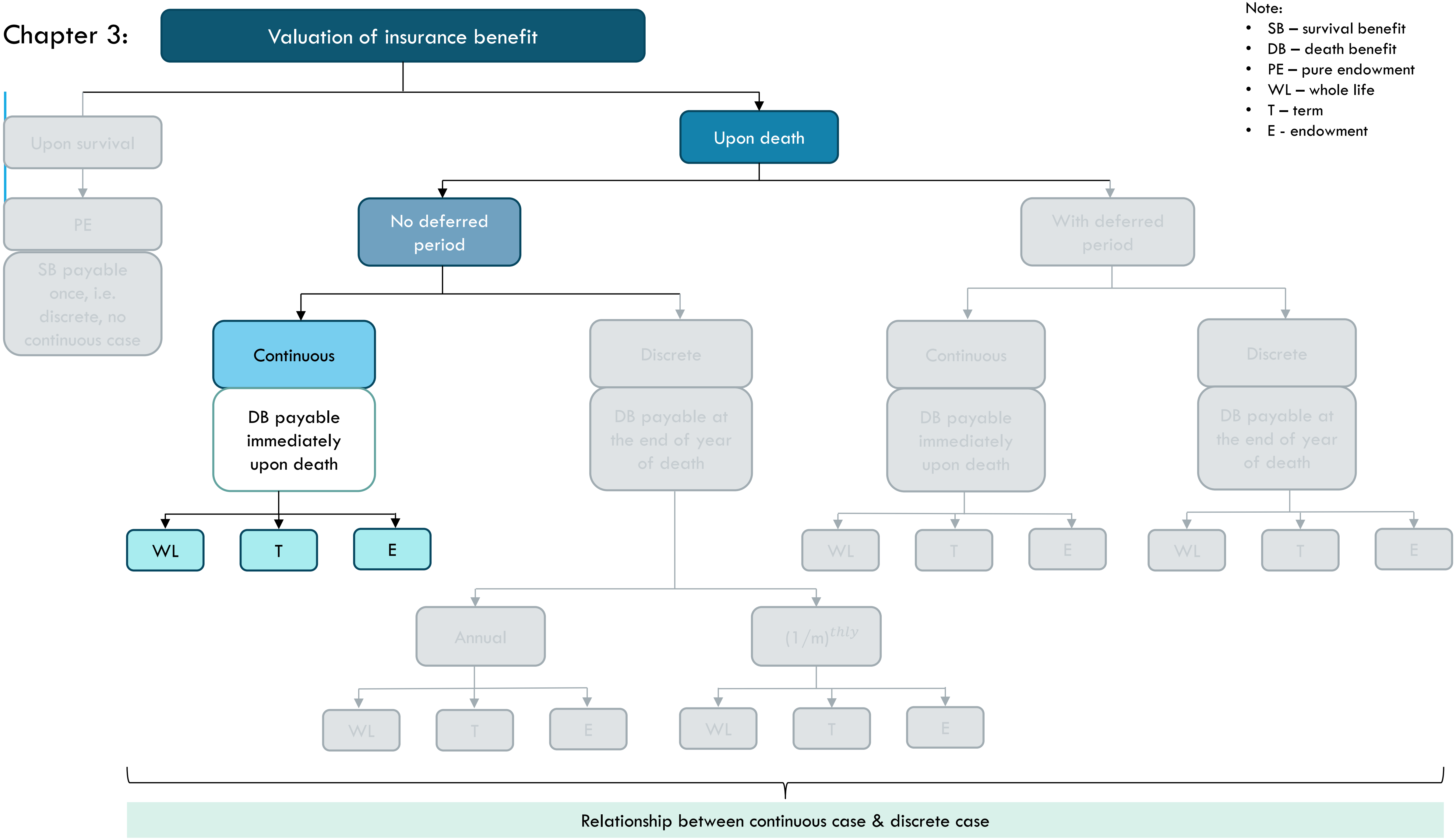
INTRODUCTION

- Life insurance provides **benefit**/payment that is contingent upon death/survival of the policyholder.
- Since the timing and possibility of benefit is uncertain, the **PV of the benefit** can be modelled as random variable.
  - PV of life contingent future benefit = PV of (“benefit function” × “time value of money” × “survival model”)
  - This is called, Actuarial Present Value (**APV**)/Expected Present Value (**EPV**)/Net Single Premium (**NSP**)
- Timing of the benefit can be in continuous time or discrete time.

Continuous time	Discrete time

• Assume that the death benefit is paid at the exact time of death, i.e. benefit is payable immediately on the death of (x)
• Based on T_x
    • payment is made exactly T_x years from now because T_x is time until exact death.
	• Assume that the death benefit is paid at the end of year of death
	• Based on K_x
	    • payment is made at $K_x + 1$, because K_x is the year of death
	• Applied in practice by insurers/actuaries

Chapter 3:



Note:
- SB – survival benefit
- DB – death benefit
- PE – pure endowment
- WL – whole life
- T – term
- E - endowment

CONTINUOUS – WHOLE LIFE INSURANCE

- Insurance where benefit is payable at the time of death of policyholder, no matter when it happens.
- Time at which the benefit will be paid is unknown, until the policyholder actually dies and the policy becomes a claim.
- Let:
  - b_t = benefit of \$1 (per unit of sum insured)
  - v^t = discount factor
  - Z = a random variable of the PV of benefit, where $Z = b_t v^t$ -- a function of time of death
  - $E[Z]$ = expected PV random variable of benefit, also called APV/EPV/NSP
- EPV of whole life insurance benefit payment of \$1 payable immediately on death is denoted by $\bar{A}_x$.
- Timeline:

CONTINUOUS - WHOLE LIFE INSURANCE

EPV & VARIANCE

EPV of Z, i.e. average cost of a whole life insurance:	Second moment of Z:	Variance of Z, i.e. variability in cost of a whole life insurance:
For benefit of \$1: $\bar{A}_x = E[Z]$ $= \int_0^\infty b_t v^t f_{T_x}(t)\,dt$ $= \int_0^\infty 1\left(e^{-\delta t}\right)\left({}_tp_x \mu_{x+t}\right)dt$ For benefit of \$S: EPV of \$S death benefit $= S(\bar{A}_x)$	For benefit of \$1: ${}^2\bar{A}_x = E[Z^2]$ $= \int_0^\infty (b_t v^t)^2 \, f_{T_x}(t)\,dt$ $= \int_0^\infty (b_t)^2 \,(v^t)^2 f_{T_x}(t)\,dt$ $= \int_0^\infty 1\left(e^{-2\delta t}\right)\left({}_tp_x \mu_{x+t}\right)dt$	For benefit of \$1: $V[Z] = E[Z^2] - E[Z]^2$ $= {}^2\bar{A}_x - \left(\bar{A}_x\right)^2$ For benefit of \$S: Variance of \$S death benefit $= S\left[{}^2\bar{A}_x - \left(\bar{A}_x\right)^2\right]$

CONTINUOUS – TERM INSURANCE

- Insurance where benefit is payable only if the policyholder dies within n years.
- Let:
  - b_t = benefit of $1 (per unit of sum insured)
  - v^t = discount factor
  - Z = a random variable of the PV of benefit, where $Z = b_t v^t$ -- a function of time of death
  - $E[Z]$ = expected PV random variable of benefit, also called APV/EPV/NSP
- PV of benefit $1:

$$Z = \begin{cases} b_t v^{T_x} = e^{-\delta\left(T_x\right)} & \text{if } T_x \leq n \\ 0 & \text{if } T_x > n \end{cases}$$

- EPV of whole life insurance benefit payment of $1 payable immediately on death is denoted by $\bar{A}^{1}_{x:\overline{n}|}$
- Timeline:

CONTINUOUS – TERM INSURANCE

EPV & VARIANCE

EPV of Z, i.e. average cost of a term insurance:	Second moment of Z:	Variance of Z, i.e. variability in cost of a term insurance:
For benefit of \$1: $\bar{A}^{1}_{x:\overline{n}\rvert} = E[Z]$ $=$ $=$ For benefit of \$S: EPV of \$S death benefit $= \mathrm{S}(\bar{A}^{1}_{x:\overline{n}\rvert})$	For benefit of \$1: ${}^{2}\bar{A}^{1}_{x:\overline{n}\rvert} = E[Z^2]$ $=$ $=$ $=$	For benefit of \$1: $V[Z] = E[Z^2] - E[Z]^2$ $= {}^{2}\bar{A}^{1}_{x:\overline{n}\rvert} - \left(\bar{A}^{1}_{x:\overline{n}\rvert}\right)^2$ For benefit of \$S: Variance of \$S death benefit $= \mathrm{S}\left[{}^{2}\bar{A}^{1}_{x:\overline{n}\rvert} - \left(\bar{A}^{1}_{x:\overline{n}\rvert}\right)^2\right]$

CONTINUOUS – ENDOWMENT INSURANCE

- Insurance where benefit is payable either upon death or survival of the policyholder, where death within n years (DB immediately upon death), survive for n years (SB payable at the end of n^{th} year).

- Let:
  - b_t = benefit of \$1 (per unit of sum insured)
  - v^t = discount factor
  - Z = a random variable of the PV of benefit, where $Z = b_t v^t$ -- a function of time of death
  - $E[Z]$ = expected PV random variable of benefit, also called APV/EPV/NSP

$$b_t = 1$$

$$v_t = \begin{cases} v^t & ; 0 < t < n \\ v^n & ; t \geq n \end{cases}$$

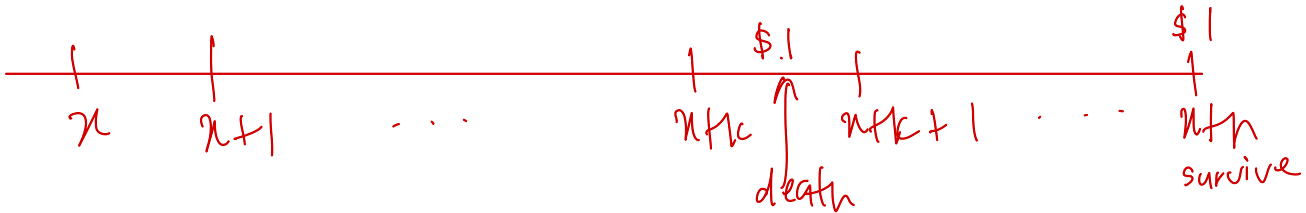
$$z_t = \begin{cases} 1 \cdot v^t & ; 0 < t < n \\ 1 \cdot v^n & ; t \geq n \end{cases}$$

- PV of benefit \$1:

$$Z = \begin{cases} b_t v^{T_x} = e^{-\delta(T_x)} & \text{if } T_x < n \\ b_t v^n = e^{-\delta(n)} & \text{if } T_x \geq n \end{cases}$$

- EPV of whole life insurance benefit payment of \$1 payable immediately on death is denoted by $\bar{A}_{x:\overline{n|}}$

- Timeline:



CONTINUOUS – ENDOWMENT INSURANCE

EPV & VARIANCE

EPV of Z, i.e. average cost of an endowment insurance:	Second moment of Z:	Variance of Z, i.e. variability in cost of an endowment insurance:
For benefit of \$1: $\bar{A}_{x:\overline{n}\vert} = E[Z]$ $=$ $=$ For benefit of \$S: EPV of \$S death benefit $= S(\bar{A}_{x:\overline{n}\vert})$	For benefit of \$1: ${}^{2}\bar{A}_{x:\overline{n}\vert} = E[Z^2]$ $=$ $=$ $=$	For benefit of \$1: $V[Z] = E[Z^2] - E[Z]^2$ $= {}^{2}\bar{A}_{x:\overline{n}\vert} - \left(\bar{A}_{x:\overline{n}\vert}\right)^2$ For benefit of \$S: Variance of \$S death benefit $= S\left[{}^{2}\bar{A}_{x:\overline{n}\vert} - \left(\bar{A}_{x:\overline{n}\vert}\right)^2\right]$

$$\bar{A}_{x:\overline{n}|} = \int_0^n v^t \, {}_tp_x \, \mu_{x+t} \, dt + \int_n^\infty v^n \, {}_tp_x \, \mu_{x+t} \, dt$$

$$= \int_0^n e^{-\delta t} \, {}_tp_x \, \mu_{x+t} \, dt + \int_n^\infty v^n \, {}_tp_x \, \mu_{x+t} \, dt$$

$$\boxed{= \underbrace{\int_0^n e^{-\delta t} \, {}_tp_x \, \mu_{x+t} \, dt}_{\text{term insurance}} + \underbrace{e^{-\delta n} \, {}_np_x}_{\text{pure endowment}}}$$

$$\boxed{\bar{A}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}}$$

$$\boxed{{}^2\bar{A}_{x:\overline{n}|} = \int_0^n e^{-2\delta t} \, {}_tp_x \, \mu_{x+t} \, dt + e^{-2\delta n} \, {}_np_x}$$

$$V(\bar{Z}_{x:\overline{n}|}) = {}^2\bar{A}_{x:\overline{n}|} - \left(\bar{A}_{x:\overline{n}|}\right)^2$$

Example 26.16

The lifetime of a group of people has the following survival function associated with it: $s(x) = 1 - \frac{x}{100}$, $0 \leq x \leq 100$. Paul, a member of the group, is currently 40 years old and has a 15-year endowment insurance policy, which will pay him \$50,000 upon death. Find the actuarial present value of this policy. Assume an annual force of interest $\delta = 0.05$.

Answer : 26 510

De Moivre's $\rightarrow$ uniform

$$s(x) = \frac{\omega - x}{\omega}$$

$$_tp_x \, \mu_{x+t} = \frac{1}{\omega - x}$$

$$_tp_x = \frac{\omega - x - t}{\omega - x}$$

$$s(x) = 1 - \frac{x}{100} \quad ; 0 < x < 100 \ (\omega)$$

same

$$s(x) = \frac{100 - x}{100}$$

$$\bar{A}_{40:\overline{15}|} = 50\,000 \left[\int_0^n e^{-\delta t} \, _tp_x \, \mu_{x+t} \, dt + e^{-\delta n} \, _np_x \right]$$

$$= 50\,000 \left[\int_0^{15} e^{-0.05t} \left(\frac{1}{100-40} \right) + e^{-0.05(15)} \, _{15}p_{40} \right]$$

$$= 50\,000 \left[\frac{1}{60} \left(\frac{e^{-0.05t}}{-0.05} \right)_0^{15} + e^{-0.05(15)} \left(\frac{100-40-15}{100-40} \right) \right]$$

$\vdots$

$= 26\,510$ #

THANK YOU

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Reference:

Dickson, Hardy & Waters (2009). Actuarial Mathematics for Life Contingent Risks. Cambridge University Press.