SET and FUNCTIONS

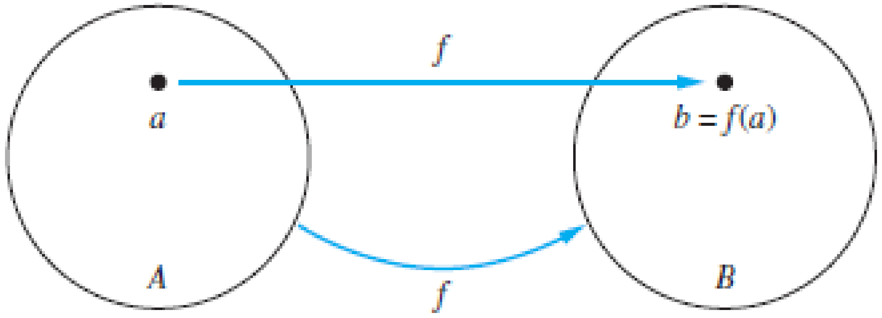
FUNCTIONS

- Let A and B be nonempty sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A.
- We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.
- If f is a function from A to B, we write

$$f : A \rightarrow B$$

FUNCTIONS

- Functions are sometimes also called **mappings** or **transformations**.
- Every element in A will be used in the mapping, but not all elements in B needs to be used.
- Each element in A must be used only once.



FUNCTIONS

Example

Suppose that each student in a discrete mathematics class is assigned a letter grade from the set $\{A, B, C, D, F\}$.

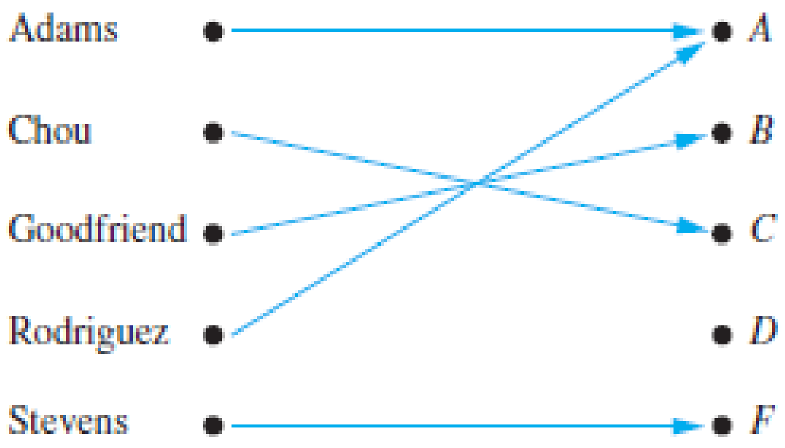
Adams – A

Chou – C

Goodfriend – B

Rodriguez – A

Stevens - F



FUNCTIONS

Example

Let A = {1, 2, 3 } and B = $\{A, B, C, D, F\}$

Assume f is defined as:

- 1 $\longrightarrow$ A
- 1 $\longrightarrow$ B
- 3 $\longrightarrow$ A

Is f a function?

NO – $f(1)$ is assigned both A and B

Representing FUNCTIONS

Representing Functions

- Explicitly state the assignments between elements in the two sets. Roster notation.
- Set builder notation
  - e.g $F(x) = x^2$
- Tabular
- Digraph
- Mathematical graph

Representing FUNCTIONS

- Mathematical graph

Display the graph of the function *f (x)* = *x*2 from the set of integers to the set of integers.

- *Solution*: The graph of *f* is the set of ordered pairs of the form *(x, f (x))* = *(x, x*2*)*, where *x* is an integer.

FUNCTIONS

- If f is a function from A to B, we say that A is the *domain* of f and B is the *codomain* of f.
- If $f(a) = b$, we say that b is the *image* of a and a is a *preimage* of b.
- The *range*, or *image*, of f is the set of all images of elements of A.
- Also, if f is a function from A to B, we say that *f maps A to B.*

FUNCTIONS

Example

Let G be the function that assigns a grade to a student in our discrete mathematics class.

For instance $G($Adams$) = A$

Domain of G = {Adams, Chou, Goodfriend, Rodriguez, Stevens}

Codomain = $\{A,B,C,D, F\}$

Range of G = $\{A,B,C, F\}$, because each grade except D is assigned to some student.

Image of Subset

- Let f be a function from A to B and let S be a subset of A. The *image* of S under the function f is the subset of B that consists of the images of the elements of S.
- We denote the image of S by $f\ (S)$

$$f\ (S) = \{t \mid \exists s \in S\ (t = f\ (s))\}.$$

- We also use the shorthand $\{f\ (s) \mid s \in S\}$ to denote this set.

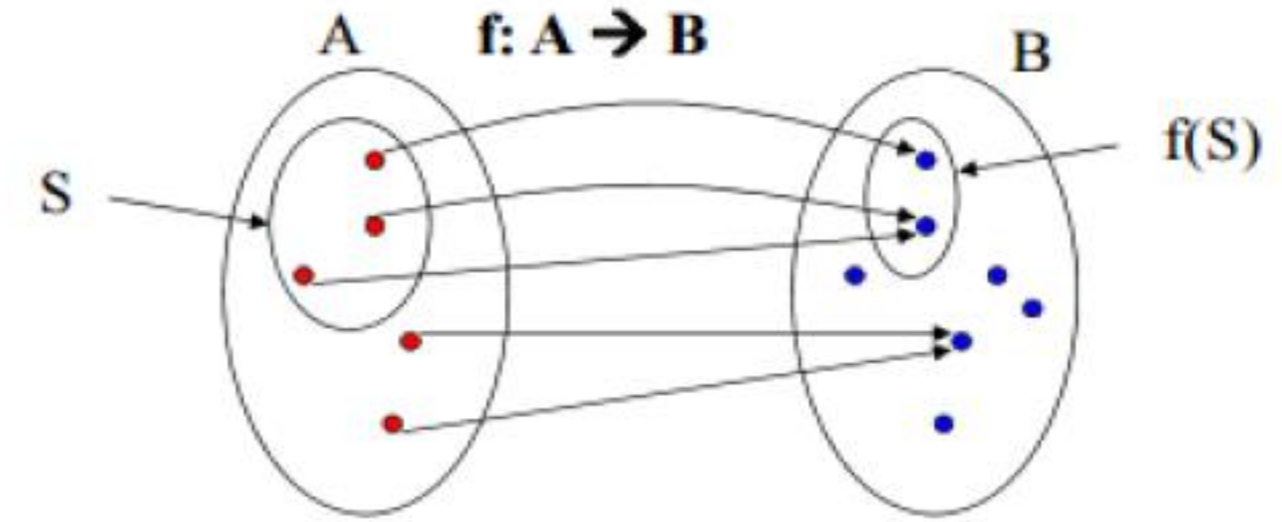


Image of Subset

Example

Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$

$f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$,

$f(e) = 1$.

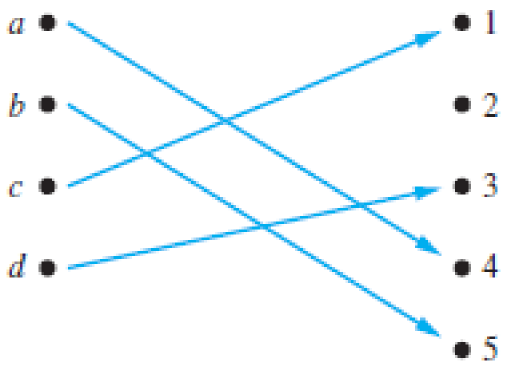
The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$.

Types of Functions

- Injective
- Surjective
- Bijective
- Identity
- Inverse

Injective / one-to-one

- A function f is said to be *one-to-one*, or injective, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.

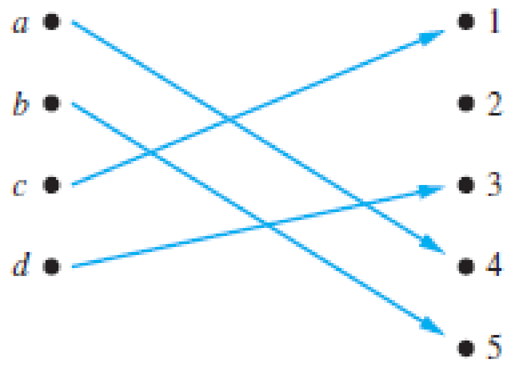


Injective / one-to-one

Example

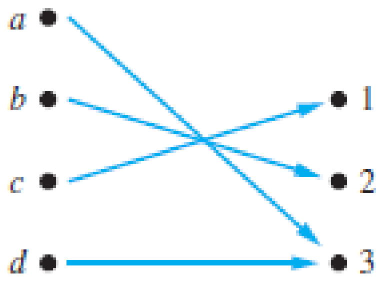
Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.

- *Solution:* The function f is one-to-one because f takes on different values at the four elements of its domain.



Surjective / onto

- A function f from A to B is called *onto*, or a *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$

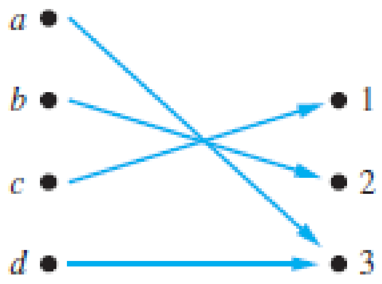


Surjective / onto

Example

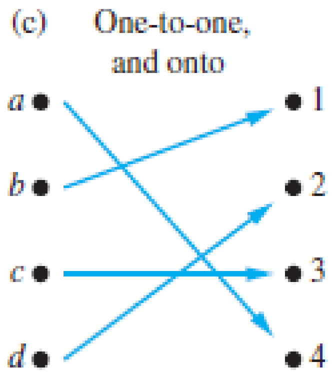
Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

- *Solution:* Because all three elements of the codomain are images of elements in the domain, we see that f is onto.



Bijective / one-to-one and onto

- The function f is a *one-to-one correspondence*, or a *bijective*, if it is both one-to-one and onto.
- Also known as isomorphism

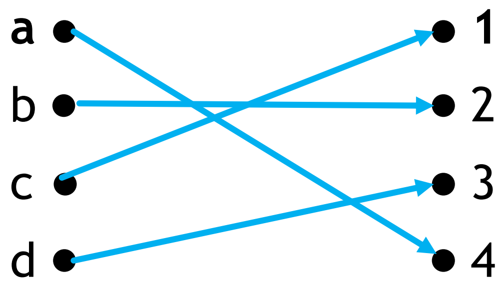


Bijective / one-to-one and onto

Example

Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f\ (a) = 4$, $f\ (b) = 2$, $f\ (c) = 1$, and $f\ (d) = 3$. Is f a bijection?

- *Solution:* The function f is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value.
- It is onto because all four elements of the codomain are images of elements in the domain. Hence, f is a bijection.



Tips

Suppose that $f : A \rightarrow B$.

- *To show that f is injective* Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x = y$,then $x = y$.
- *To show that f is not injective* Find particular elements $x, y \in A$ such that $x = y$ and $f(x) = f(y)$.
- *To show that f is surjective* Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.
- *To show that f is not surjective* Find a particular $y \in B$ such that $f(x) = y$ for all $x \in A$.

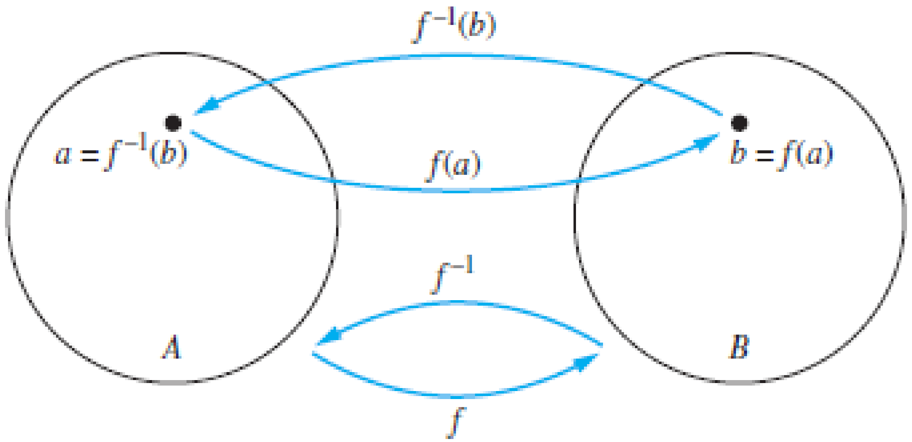
Identity

- Let A be a set. The *identity function* on A is the function $\iota A : A \to A$, where $\iota A(x) = x$ for all $x \in A$.
- In other words, the identity function ιA is the function that assigns each element to itself.
- The function ιA is one-to-one and onto, so it is a bijection.

(Note that ι is the Greek letter iota.)

Inverse

- Let f be a one-to-one correspondence from the set A to the set B. The ***inverse function of f*** is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$.
- The inverse function of f is denoted by f^{-1}.
- Hence, $f^{-1}(b) = a$ when $f(a) = b$.

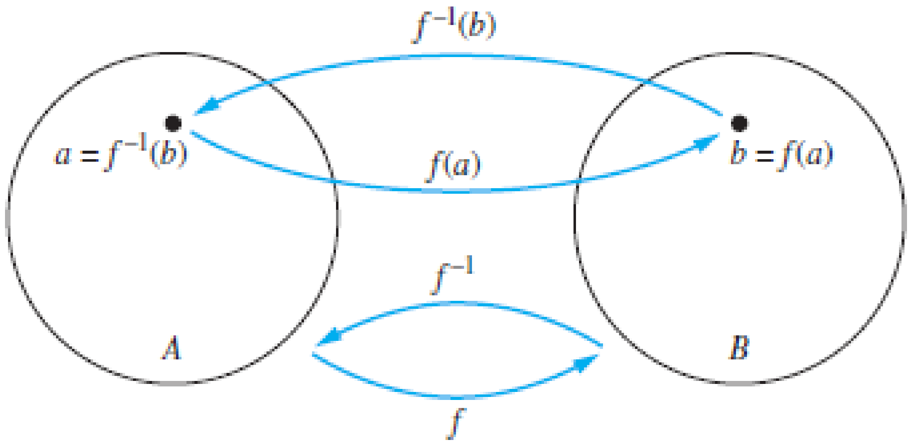


Inverse

Example

Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

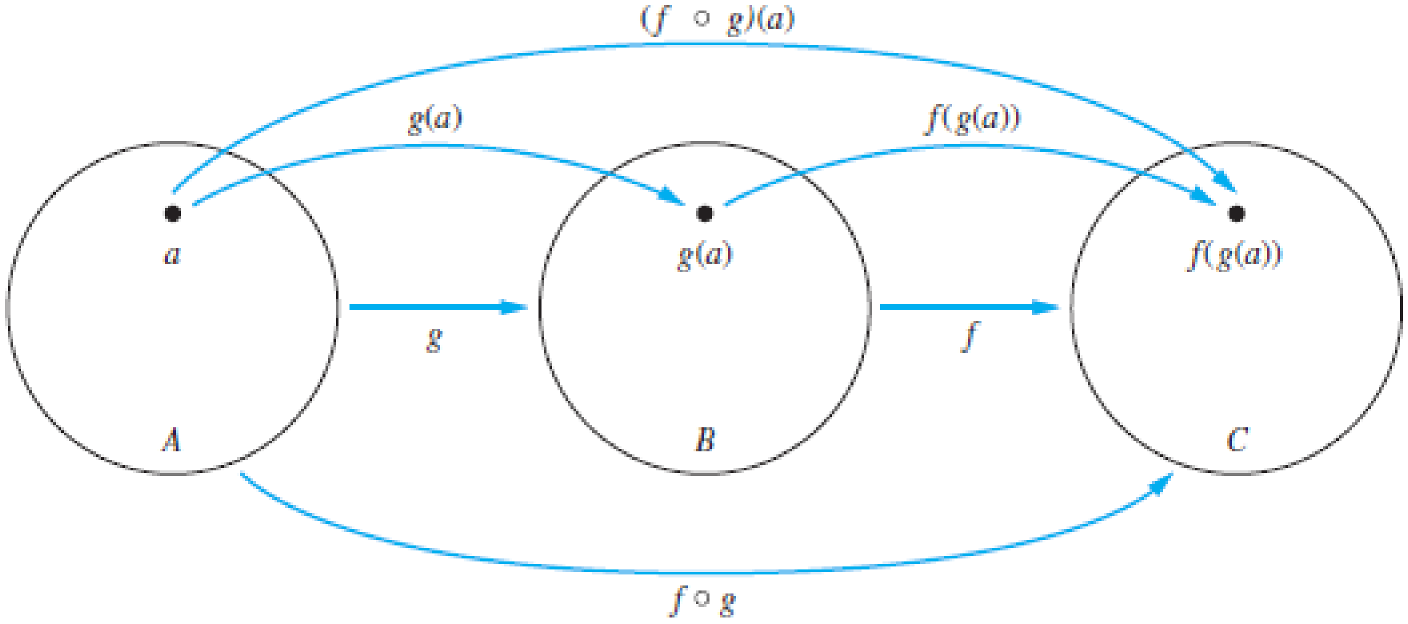
- *Solution:* The function f is invertible because it is a one-to-one correspondence.
- The inverse function f^{-1} reverses the correspondence given by f
- so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.



Composition

- Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The *composition* of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by

$$(f \circ g)(a) = f\,(g(a))$$



Composition

Example

Let A = $\{1,2,3\}$ and B = $\{a,b,c,d\}$

$\boldsymbol{f: A \rightarrow}$ **A**	**g: A → B**
$1 \rightarrow 3$	$1 \rightarrow b$
$2 \rightarrow 1$	$2 \rightarrow a$
$3 \rightarrow 2$	$3 \rightarrow d$

$\boldsymbol{(f \circ g)}$

$1 \rightarrow d$

$2 \rightarrow b$

$3 \rightarrow a$

Composition

Example

Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

What is the composition of f and g?

- $(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

What is the composition of g and f ?

- $(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$

Note: $f \circ g$ and $g \circ f$ are not equal, hence they are not commutative.