METHODS OF PROOF

PROOFS

- an argument supporting the validity of the statement
- proof of the theorem:
  - shows that the **conclusion** follows from **premises**
  - may use:
    - Premises
    - Axioms (***Axiom*** is a rule or a statement that is accepted as true without proof. An ***axiom*** is also called a postulate)
    - Results of other theorems

Formal proofs:

- steps of the proofs follow logically from the set of premises and axioms
- we assume ***formal proofs*** in propositional logic

Direct Proof

- Direct Proofs lead from premises of a theorem to the conclusion.

Example:

$P \rightarrow Q$

- We only need to consider the case P is true because when its false, the argument is true (by default)
- Assume that P is true. Next, we use axioms, definitions, and previously proven theorems, together with the rules of inference, to show that Q is true.
- If we can deduce that Q is true, therefore $P \rightarrow Q$ is true.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Direct Proof

Example:

Give a direct proof "If n is an odd integer, then n^2 is odd"

- Assume hypothesis "n is an odd integer" is true
- Definition of odd integer is $n = 2k + 1$, *where k is some integer*
- Show that n^2 is odd :

$$
\begin{aligned}
n^2 &= (2k+1)^2 \\
&= 4k^2 + 4k + 1 \\
&= 2(2k^2 + 2k) + 1
\end{aligned}
$$

Therefore n^2 is odd.

Consequently, we have proven that "If n is an odd integer, then n^2 is odd" is true.

Direct Proof

Example:

Give a direct proof "If *m*, *n* are odd integers, then *m x n* is odd"

- Assume hypothesis "*m*, *n* are odd integers" is true
- Definition of odd integer is $n = 2k + 1$, $m = 2l + 1$ *where k, l is some integer*
- Show that *m x n* is odd:

$$
\begin{aligned}
m \times n &= (2k+1) \times (2l+1) \\
&= 2kl + 2k + 2l + 1 \\
&= 2(kl + k + l) + 1
\end{aligned}
$$

Therefore m x n is odd.

Consequently, we have proven that "If *m*, *n* are odd integers, then *m x n* is odd" is true.

Indirect Proof

- Proof by contraposition.

Example:

$P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$

- Assume that $\neg Q$ is true. Next, we use axioms, definitions, and previously proven theorems, together with the rules of inference, to show that $\neg P$ is true.
- If we can deduce that $\neg P$ is true, therefore $P \rightarrow Q$ is true.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Indirect Proof

Example:

Give an indirect proof *"If **3n + 2 is odd** then n is odd"*

- Assume hypothesis "*n* is even" is true
- Definition of even integer is $n = 2k$, *where k is some integer*
- Show that **3n + 2** is even:

$$3n + 2 = 3(2k) + 2$$

$$= 6k + 2$$

$$= 2(3k + 1)$$

Therefore 3n + 2 is even.

Consequently, we have proven that *"If **3n + 2 is odd** then n is odd"* is true.

Proof by Cases

$P \rightarrow Q$, where $P = P_1 \vee P_2 \vee P_3 \vee P_4 \vee P_5 \vee \ldots\ldots. \vee P_n$

- if the hypothesis naturally breaks down into parts ($P_1 \vee P_2 \vee P_3 \vee P_4 \vee P_5 \vee \ldots\ldots. \vee P_n$), we prove $P_1 \rightarrow Q$, $P_2 \rightarrow Q$, $P_3 \rightarrow Q, \ldots\ldots, P_n \rightarrow Q$
- Hence, P (the whole parts) is true, so the proposition is correct.

Proof by Cases

Example: Show that $|x||y|=|xy|$

Proof:

- 4 cases:
- $x>=0, y>=0$ $xy >0$ and $|xy|=xy=|x||y|$
- $x>= 0, y<0$ $xy <0$ and $|xy|=-xy =x (-y)=|x||y|$
- $x<0, y>=0$ $xy <0$ and $|xy|=-xy =(-x) y=|x||y|$
- $x<0, y<0$ $xy >0$ and $|xy|= (-x)(-y) =|x||y|$

All cases proved.

Proving Universally Quantified Statements

- To prove $\forall x\ P(x)$ is true, we have to **exhaustively** show that for every x in the universe of discourse, $P(x)$ is true.
- To prove $\forall x\ P(x)$ is false, we provide proof there exist a value for x in the universe of discourse, that makes $P(x)$ false.

Proving Existentially Quantified Statements

- To prove ∃x P(x) is true, we provide proof there exist a value for x in the universe of discourse, that makes P(x) true.
- To prove ∃x P(x) is false, we have to **exhaustively** show that for every x in the universe of discourse, P(x) is false.