VALUATION OF LIFE INSURANCE

ACTUARIAL MATHEMATICS I

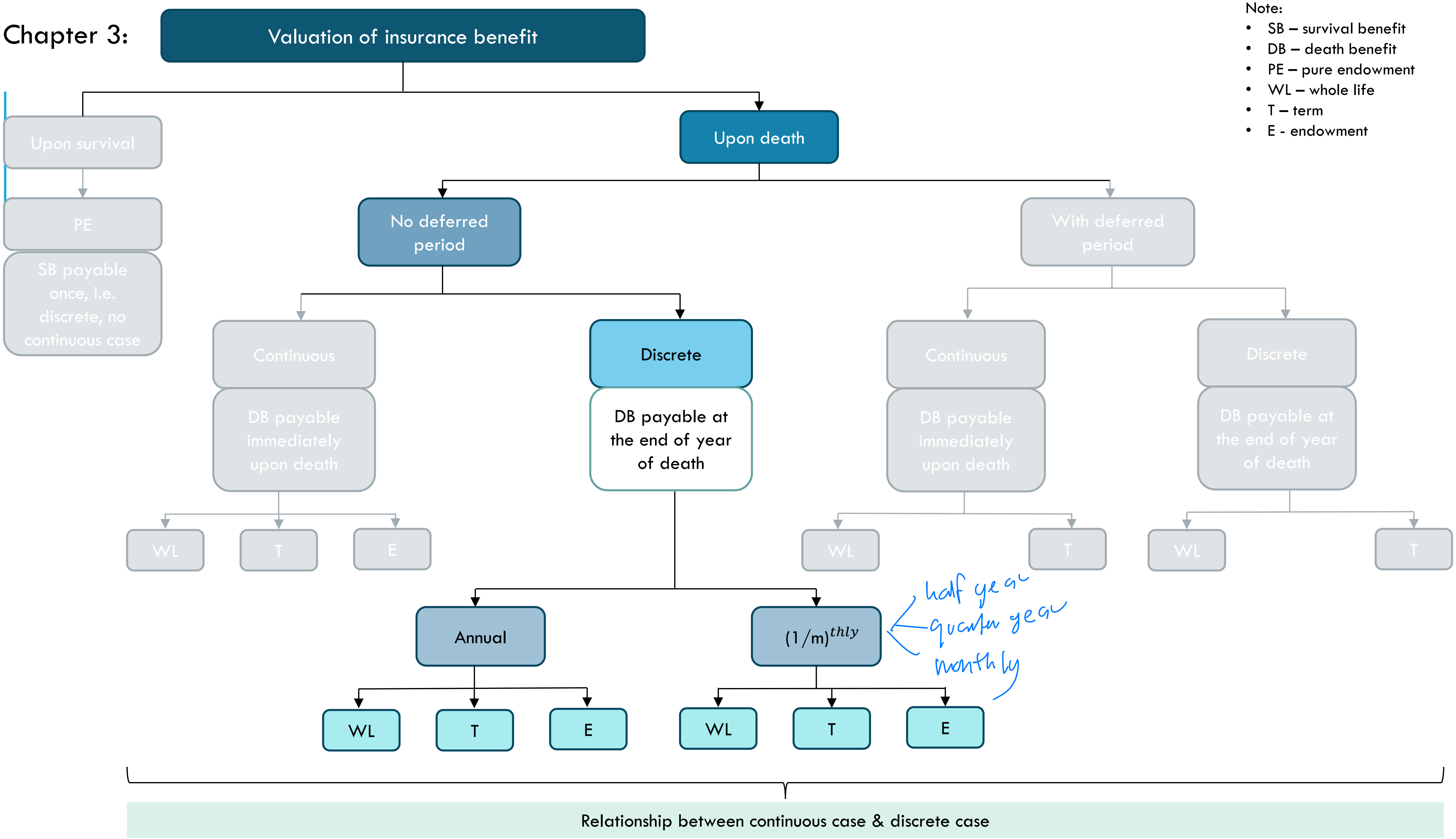
ASC425

OUTLINE

- Introduction
- Continuous case
- Discrete case
  - Annual
  - $(1/m)^{thly}$ → m times in 1 year
- Relationship between continuous case and discrete case
- Variable insurance benefit

Chapter 3:

Note:
- SB – survival benefit
- DB – death benefit
- PE – pure endowment
- WL – whole life
- T – term
- E - endowment

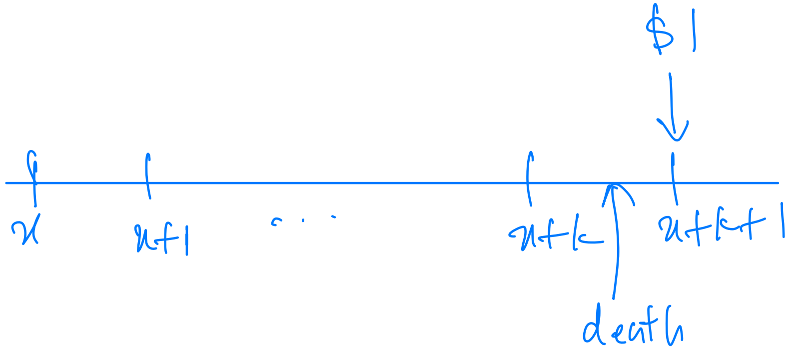


Norkhairunnisa Redzwan, 2024

ANNUAL CASE

DISCRETE – ANNUAL CASE – WHOLE LIFE INSURANCE

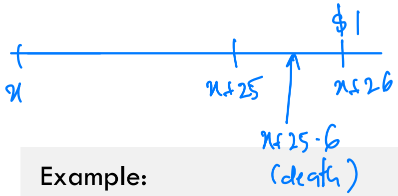
- Insurance where death benefit is payable at the end of year of death of policyholder; i.e. $K_x + 1$.
- Let:
  - Z = a random variable of the PV of benefit \$1 (per unit of sum insured), where $Z = b_t v^{K_x+1}$ -- a function of time of death
  - $E[Z]$ = expected PV random variable of benefit, also called APV/EPV/NSP
- EPV of whole life insurance benefit payment of \$1 payable at the end of year of death is denoted by A_x.
- Timeline:



$b_{k+1} = 1 \quad ; k=0,1,2,\dots$

$v_{k+1} = v^{k+1}$
$= e^{-\delta(k+1)} \quad ; k=0,1,2\dots$

$z_{k+1} = 1 \cdot v^{k+1} \quad ; k=0,1,2\dots$



Example:
(x) lived 25.6 years from the issue of the policy.
$T_x = 25.6$
$K_x = 25$
Death payable at $K_x + 1 = 26$

DISCRETE – ANNUAL - WHOLE LIFE INSURANCE

EPV & VARIANCE

EPV of Z, i.e. average cost of a whole life insurance:	Second moment of Z:	Variance of Z, i.e. variability in cost of a whole life insurance:
For benefit of \$1: $A_x = E[Z]$ $= E[b_t v^{K_x+1}]$ $= \sum_{0}^{\infty} b_t v^{K_x+1} \; {}_kp_x q_{x+k}$ $\boxed{= \sum_{0}^{\infty} 1 \cdot v^{k+1} \; {}_kp_x q_{x+k}}$ *	For benefit of \$1: ${}^2A_x = E[Z^2]$ $= \sum_{0}^{\infty} (b_t)^2 v^{2(K_x+1)} \; {}_kp_x q_{x+k}$ $\boxed{= \sum_{0}^{\infty} v^{2(k+1)} \; {}_kp_x q_{x+k}}$ *	For benefit of \$1: $V[Z] = E[Z^2] - E[Z]^2$ $= {}^2A_x - (A_x)^2$
For benefit of \$S: EPV of \$S death benefit $= S(A_x)$		For benefit of \$S: Variance of \$S death benefit $= S[\,{}^2A_x - (A_x)^2]$

Example 27.1

Let the remaining lifetime at birth random variable X be uniform on $[0,100]$. Let Z_{30} be the contingent payment random variable for a life aged $x = 30$. Find A_{30}, ${}^2A_{30}$, and $\mathrm{Var}(Z_{30})$ if $\delta = 0.05$.

De Moivre's → uniform

$$s(x) = \frac{\omega - x}{\omega}$$

$$_tp_x\,\mu_{x+t} = \frac{1}{\omega - x}$$

$$_tp_x = \frac{\omega - x - t}{\omega - x}$$

$$A_{30} = \sum_{k=0}^{\overset{\omega - x - 1}{\infty}} v^{k+1}\,{}_kp_x\,q_{x+k}$$

$$= \sum_{k=0}^{100-30-1} v^{k+1}\left(\frac{1}{100-30}\right)$$

$$= \frac{1}{70}\sum_{k=0}^{69} v^{k+1}$$

$$= \frac{1}{70}\sum_{k=0}^{\overset{(70-1)\;n=70}{69}} \underset{a}{v}\cdot \underset{r^k}{v^k}$$

$$= \frac{1}{70}\left[v\cdot\frac{1-v^{70}}{1-v}\right]$$

Geometric Series

$$\sum_{k=0}^{n-1} ar^k = a\cdot\frac{1-r^n}{1-r}$$

$a = v$, $r = v$, $n = 70$

$$v = e^{-\delta} = e^{-0.05}$$

$$v^{70} = e^{-\delta(70)} = e^{-0.05(70)}$$

$$= \frac{e^{-0.05}}{70}\left[\frac{1-e^{-0.05(70)}}{1-e^{-0.05}}\right]$$

$\vdots$

$$= 0.27022$$

${}_{k}p_{x}\, q_{x+k}$

Answer

$$\boxed{{}^{2}A_{30} = 0.1357}$$

$${}^{2}A_{30} = \sum_{0}^{69} v^{2(k+1)} \quad \frac{1}{70}$$

$$= \sum_{0}^{69} v^{2} \cdot v^{2k} \cdot \frac{1}{70}$$

$$= \frac{1}{70}\left[v^{2} \cdot \frac{1-v^{2(70)}}{1-v^{2}}\right]$$

$$= \frac{1}{70}\left[e^{-0.05(2)} \cdot \left(\frac{1-e^{-0.05(2)(70)}}{1-e^{-0.05(2)}}\right)\right]$$

$\vdots$

$$= 0.1357$$

DISCRETE – ANNUAL CASE – TERM INSURANCE

- Insurance where death benefit is payable at the end of year of death of policyholder, provided that death occurs within n years .
- Let:
  - Z = a random variable of the PV of benefit \$1 (per unit of sum insured), where $Z = b_t v^{K_x+1}$ -- a function of time of death
  - $E[Z]$ = expected PV random variable of benefit, also called APV/EPV/NSP

$$b_{k+1} = \begin{cases} 1 & ;\ k = 0, 1, \cdots, n-1 \\ 0 & ;\ k = n, n+1, \cdots \end{cases}$$

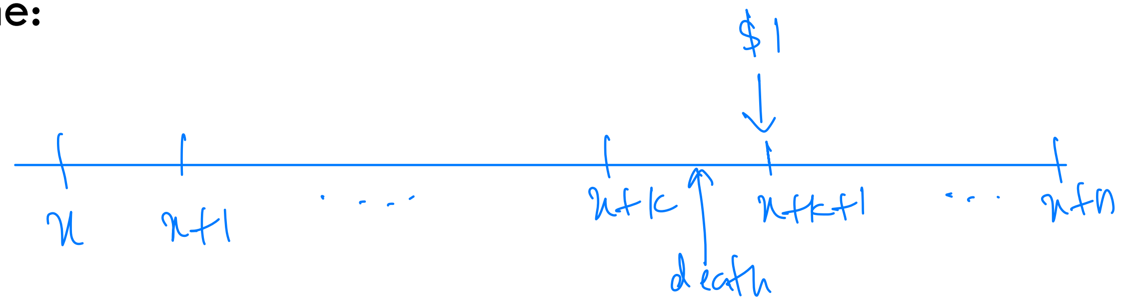
- PV random variable for \$1 benefit:

$$Z = \begin{cases} b_t v^{K_x+1} & \text{if } K_x \le n-1 \\ 0 & \text{if } K_x \ge n \end{cases}$$

$$v_{k+1} = \begin{cases} v^{k+1} & ;\ k = 0, 1, \cdots, n-1 \\ v^{k+1} & ;\ k = n, n+1, \cdots \end{cases}$$

- EPV of term insurance benefit payment of \$1 payable at the end of year of death is denoted by $A^{1}_{x:\overline{n|}}$.
- Timeline:

$$z_{k+1} = (b_{k+1} \cdot v_{k+1}) = \begin{cases} v^{k+1} & ;\ k = 0, 1, \cdots, n-1 \\ 0 & ;\ k = n, n+1, \cdots \end{cases}$$



DISCRETE – ANNUAL – TERM INSURANCE

EPV & VARIANCE

EPV of Z, i.e. average cost of a term insurance:	Second moment of Z:	Variance of Z, i.e. variability in cost of a term insurance:
For benefit of \$1: $A^{1}_{x:\overline{n}\vert} = E[Z]$ $= \sum_{k=0}^{n-1} 1 \cdot v^{k+1} {}_kp_x q_{x+k} + \cancelto{0}{\sum_{k=n}^{\infty} 0 \cdot v^{k+1} {}_kp_x q_{x+k}}$ $= \sum_{k=0}^{n-1} v^{k+1} {}_kp_x q_{x+k}$ For benefit of \$S: EPV of \$S death benefit $= S(A^{1}_{x:\overline{n}\vert})$	For benefit of \$1: ${}^2A^{1}_{x:\overline{n}\vert} = E[Z^2]$ $= \sum_{k=0}^{n-1} v^{2(k+1)} {}_kp_x q_{x+k}$	For benefit of \$1: $V[Z] = E[Z^2] - E[Z]^2$ $= {}^2A^{1}_{x:\bar{n}\vert} - \left(A^{1}_{x:\overline{n}\vert}\right)^2$ For benefit of \$S: Variance of \$S death benefit $= S\left[\underbrace{{}^2A^{1}_{x:\bar{n}\vert}}_{2^{nd}\text{ moment}} - \underbrace{\left(A^{1}_{x:\overline{n}\vert}\right)}_{1^{st}\text{ moment}}^2\right]$

$A^{1}_{30:\overline{10}|} = 0.10963$ $\quad {}^{2}A^{1}_{30:\overline{10}|} = 0.08587$

$V(Z^{1}_{30:\overline{10}|}) = 0.07385$

Example 27.2

Let the remaining lifetime at birth random variable X be uniform on [0,100]. Let $Z^{1}_{30:\overline{10}|}$ be the contingent payment random variable for a life aged $x = 30$. Find $A^{1}_{30:\overline{10}|}$, ${}^{2}A^{1}_{30:\overline{10}|}$, and $\mathrm{Var}(Z^{1}_{30:\overline{10}|})$ if $\delta = 0.05$.

$$A^{1}_{30:\overline{10}|} = \sum_{0}^{n-1} v^{k+1}\, {}_{k}p_{x}\, q_{x+k}$$

$$= \sum_{0}^{10-1} v \cdot v^{k} \left(\frac{1}{100-30}\right)$$

$$= \sum_{0}^{10-1} v \cdot v^{k} \cdot \left(\frac{1}{70}\right)$$

$$= \frac{1}{70}\left[v \cdot \left(\frac{1 - v^{10}}{1 - v}\right)\right]$$

$$= \frac{1}{70}\left[e^{-0.05} \left(\frac{1 - e^{-0.05(10)}}{1 - e^{-0.05}}\right)\right]$$

$$\vdots$$

$$= 0.10963$$

De Moivre's → uniform

$$s(x) = \frac{\omega - x}{\omega}$$

$${}_{t}p_{x}\,\mu_{x+t} = \frac{1}{\omega - x}$$

$${}_{t}p_{x} = \frac{\omega - x - t}{\omega - x}$$

Geometric Series

$$\sum_{k=0}^{n-1} a r^{k} = a \cdot \frac{1 - r^{n}}{1 - r}$$

DISCRETE – ANNUAL CASE – ENDOWMENT INSURANCE

- Insurance where benefit is payable either on death or survival of the policyholder, where death within n years (DB at the end of year of death death), survive for n years (SB payable at the end of n^{th} year).
- Known as n-year endowment insurance.
- Let:
  - Z = a random variable of the PV of benefit, where $Z = b_t v^t$ -- a function of time of death
  - $E[Z]$ = expected PV random variable of benefit, also called APV/EPV/NSP
- PV random variable of benefit \$1:

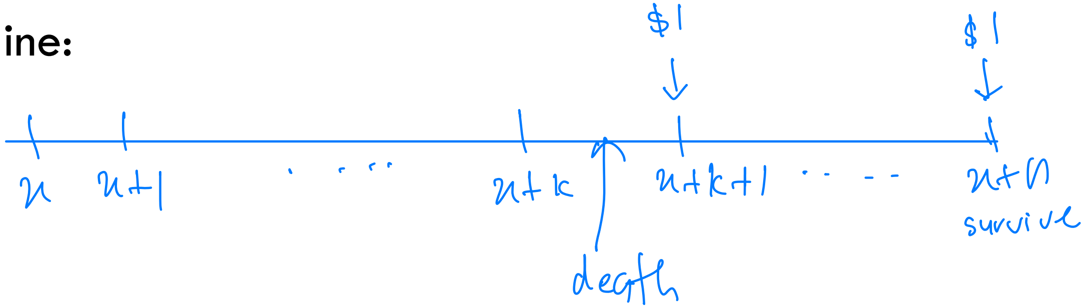
$$Z = \begin{cases} b_t v^{K_x+1} & \text{if } K_x \le n-1 \\ b_t v^n & \text{if } K_x \ge n \end{cases}$$

$$b_{k+1} = 1$$

$$v_{k+1} = \begin{cases} v^{k+1} & ; k = 0, 1, \dots, n-1 \\ v^n & ; k = n, n+1, \dots \end{cases}$$

$$z_{k+1} = \begin{cases} v^{k+1} & ; k = 0, 1, \dots, n-1 \\ v^n & ; k = n, n+1, \dots \end{cases}$$

- EPV of endowment insurance benefit payment of \$1 payable at the end of year of death is denoted by $A_{x:\overline{n}|}$
- Timeline:



DISCRETE – ANNUAL – ENDOWMENT INSURANCE

EPV & VARIANCE

EPV of Z, i.e. average cost of an endowment insurance:

For benefit of \$1:

$$A_{x:\overline{n}|} = E[Z]$$

$$= \underbrace{\sum_{k=0}^{n-1} v^{k+1} \, {}_kp_x \, q_{x+k}}_{\text{term}} + \underbrace{\sum_{k=n}^{\infty} v^{n} \, {}_kp_x \, q_{x+k}}_{\text{pure endowment}}$$

$$= \sum_{k=0}^{n-1} v^{k+1} \, {}_kp_x \, q_{x+k} + v^{n} \, {}_np_x$$

$$= A^{1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}$$

For benefit of \$S:

EPV of \$S death benefit $= S(A_{x:\overline{n}|})$

Second moment of Z:

For benefit of \$1:

$${}^2A_{x:\overline{n}|} = E[Z^2]$$

$$= \sum_{k=0}^{n-1} v^{2(k+1)} \, {}_kp_x \, q_{x+k} + v^{2n} \, {}_np_x$$

Variance of Z, i.e. variability in cost of an endowment insurance:

For benefit of \$1:

$$V[Z] = E[Z^2] - E[Z]^2$$

$$= {}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2$$

For benefit of \$S:

Variance of \$S death benefit

$$= S\left[{}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2\right]$$

$A_{30:\overline{10}|} = 0.6295$ $\quad {}^2A_{30:\overline{10}|} = 0.4012$ $\quad V[Z_{30:\overline{10}|}] = 0.0049$

Example 27.4

Let the remaining lifetime at birth random variable X be uniform on [0,100]. Let $Z_{30:\overline{10}|}$ be the contingent payment random variable for a life aged $x = 30$. Find $A_{30:\overline{10}|}$, ${}^2A_{30:\overline{10}|}$ and $\text{Var}(Z_{30:\overline{10}|})$ if $\delta = 0.05$.

$$A_{30:\overline{10}|} = \sum_{k=0}^{10-1} v^{k+1} \underbrace{{}_kp_x\, q_{x+k}} + v^{10}\, {}_{10}p_{30}$$

$$= \sum v \cdot v^k \cdot \left(\frac{1}{100-30}\right)$$

$$+ v^{10} \left(\frac{100-30-10}{100-30}\right)$$

$$=$$

De Moivre's $\rightarrow$ uniform

$$s(x) = \frac{\omega - x}{\omega}$$

$${}_tp_x\, \mu_{x+t} = \frac{1}{\omega - x}$$

$${}_tp_x = \frac{\omega - x - t}{\omega - x}$$

Geometric Series

$$\sum_{k=0}^{n-1} ar^k = a \cdot \frac{1-r^n}{1-r}$$

$\left({}^{1}\!/_{m}\right)^{thly}$ CASE

DISCRETE – $\left({}^{1}/_{m}\right)^{thly}$ CASE – WHOLE LIFE INSURANCE

- Insurance where death benefit is payable at the end of the ${}^{1}/_{m}{}^{th}$ year of death of policyholder, assuming that the year is divided into m periods and benefits are paid m^{thly}.
- $K_x^{(m)} = k$ indicates that (x) dies in the interval $[k, k + {}^{1}/_{m}]$, for $k = 0, \frac{1}{m}, \frac{2}{m}, \ldots$
- Define $K_x^{(m)}$ as a random variable where $m > 1$ is an integer and the future lifetime of (x) in years rounded to the lower $\left({}^{1}/_{m}\right)^{th}$ of a year. For example, $m = 2, 4, 12$.
- Let:
  - Z = a random variable of the PV of benefit \$1 (per unit of sum insured), where $Z = b_t v^{K_x^{(m)} + \left(\frac{1}{m}\right)}$ -- a function of time of death
  - $E[Z]$ = expected PV random variable of benefit, also called APV/EPV/NSP
- EPV of whole life insurance benefit payment of \$1 payable at the end of m^{thly} period is denoted by $A_x^{(m)}$.
- Timeline:

$m=2$

$m=4$

$m=12$

For example, suppose (x) lives at exactly 23.675 years. Then

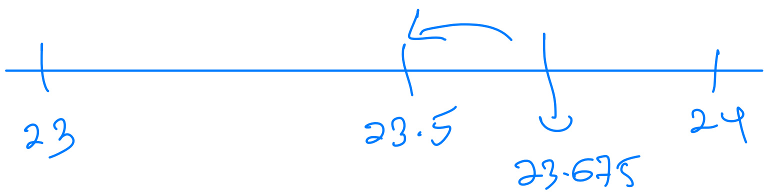
$$K_x = 23$$

$$K_x^{(2)} = 23.5$$

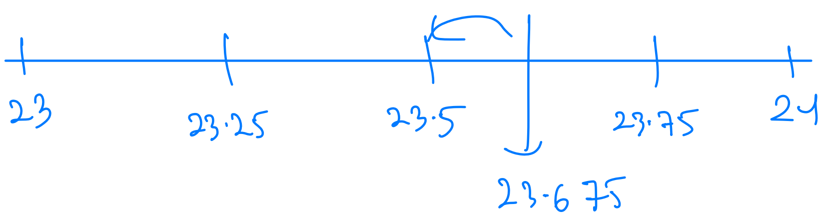
$$K_x^{(4)} = 23.5$$

$$K_x^{(12)} = 23\tfrac{8}{12}$$

$m = 2$



$m = 4$



$$J = \lfloor (T - K) m \rfloor$$

J = no. of complete m-ths of a year
K = no. of complete years lived prior to death
T = time of death
m = m^{th} time interval

$$K_x^{(m)} = K \frac{J}{m}$$

$m = 2$

$$J = \lfloor (23.675 - 23) 2 \rfloor = 1.35 \approx 1$$

$$K_x^{(2)} = 23 \frac{1}{2}$$

$m = 4$

$$J = 2.7 \approx 2$$

$$K_x^{(4)} = 23 \frac{2}{4}$$

$m = 12$

$$J = 8.1 \approx 8$$

$$K_x^{(12)} = 23 \frac{8}{12}$$

DISCRETE – $\left({}^{1}/_{m}\right)^{thly}$ - WHOLE LIFE INSURANCE

EPV & VARIANCE

EPV of Z, i.e. average cost of a $\left({}^{1}/_{m}\right)^{thly}$ whole life insurance:	Second moment of Z:	Variance of Z, i.e. variability in cost of a $\left({}^{1}/_{m}\right)^{thly}$ whole life insurance:
For benefit of \$1:	For benefit of \$1:	For benefit of \$1:
$A_x^{(m)} = E[Z]$ $\boxed{= \sum_0^{\infty} b_t v^{\frac{k+1}{m}} {}_{\frac{k}{m}\mid\frac{1}{m}}q_x}$	${}^2A_x^{(m)} = E[Z^2]$ $\boxed{= \sum_0^{\infty} (b_t)^2 v^{2\left(\frac{k+1}{m}\right)} {}_{\frac{k}{m}\mid\frac{1}{m}}q_x}$	$V[Z] = E[Z^2] - E[Z]^2$ $= {}^2A_x^{(m)} - \left(A_x^{(m)}\right)^2$
For benefit of \$S:		For benefit of \$S:
EPV of \$S death benefit $= S(A_x^{(m)})$		Variance of \$S death benefit $= S\left[{}^2A_x^{(m)} - \left(A_x^{(m)}\right)^2\right]$

DISCRETE $-({}^{1}/_{m})^{thly}$ CASE – TERM INSURANCE

- Insurance where death benefit is payable at the end of year of ${}^{1}/_{m}{}^{th}$ year of death of policyholder, provided that death occurs within n years.

- Let:
  - Z = a random variable of the PV of benefit \$1 (per unit of sum insured), where $Z = b_t v^{K_x^{(m)}+\left(\frac{1}{m}\right)}$ -- a function of time of death
  - $E[Z]$ = expected PV random variable of benefit, also called APV/EPV/NSP

- PV random variable for \$1 benefit:

$$Z = \begin{cases} b_t v^{K_x^{(m)}+{}^{1}/_{m}} & \text{if } K_x^{(m)} \le n - {}^{1}/_{m} \\ 0 & \text{if } K_x^{(m)} \ge n \end{cases}$$

- EPV of term insurance benefit payment of \$1 payable at the end of m^{thly} period is denoted by $A^{(m)\,1}_{x:\overline{n}|}$.

- Timeline:

$A^{(m)\,1}_{x:\overline{n}|} = \sum_{k=0}^{mn-1} v^{\frac{k+1}{m}} \; {}_{\frac{k}{m}|\frac{1}{m}}q_x$ | ${}^{2}A^{(m)\,1}_{x:\overline{n}|} = \sum_{k=0}^{mn-1} v^{2\left(\frac{k+1}{m}\right)} \; {}_{\frac{k}{m}|\frac{1}{m}}q_x$ | $V\left(Z^{(m)\,1}_{x:\overline{n}|}\right) = {}^{2}A^{(m)\,1}_{x:\overline{n}|} - \left(A^{(m)\,1}_{x:\overline{n}|}\right)^2$

DISCRETE – $(^1/_m)^{thly}$ CASE – ENDOWMENT INSURANCE

- Insurance where benefit is payable at the end of $^1/_m{}^{th}$ year death or survival of the policyholder, where death within n years (DB at the end of year of death death), survive for n years (SB payable at the end of n^{th} year).
- Known as n-year endowment insurance.
- Let:
  - Z = a random variable of the PV of benefit, where $Z = b_t v^t$ -- a function of time of death
  - $E[Z]$ = expected PV random variable of benefit, also called APV/EPV/NSP
- PV random variable of benefit \$1:

$$Z = \begin{cases} b_t v^{\left[K_x^{(m)} + ^1/_m\right]} & \text{if } \ K_x^{(m)} \le n - {}^1/_m \\ b_t v^n & \text{if } \ K_x^{(m)} \ge n \end{cases}$$

- EPV of endowment insurance benefit payment of \$1 payable at the end of m^{thly} period is denoted by $A^{(m)}_{x:\overline{n}|}$
- Timeline:

$A^{(m)}_{x:\overline{n}|} = \sum_{k=0}^{mn-1} v^{\frac{k+1}{m}} \; {}_{\frac{k}{m}|\frac{1}{m}}q_x + v^n \, {}_np_x$

${}^2A^{(m)}_{x:\overline{n}|} = \sum_{k=0}^{mn-1} v^{2\left(\frac{k+1}{m}\right)} \; {}_{\frac{k}{m}|\frac{1}{m}}q_x + v^{2n} \, {}_np_x$

$V\left(Z^{(m)}_{x:\overline{n}|}\right) = {}^2A^{(m)}_{x:\overline{n}|} - \left(A^{(m)}_{x:\overline{n}|}\right)^2$

THANK YOU

Norkhairunnisa Mohamed Redzwan

Reference:

Dickson, Hardy & Waters (2009). Actuarial Mathematics for Life Contingent Risks. Cambridge University Press.