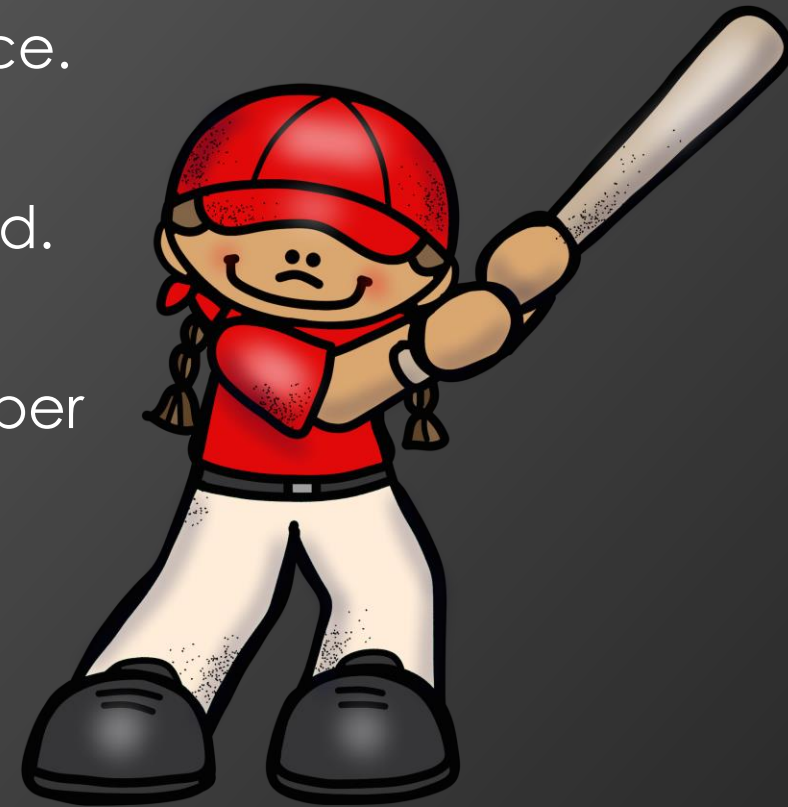# INVERSIONS OF PERMUTATIONS

Sonia Kovalevsky Day 2019

Dylan Heuer

PERMUTATIONS

- A permutation of size n is a way of arranging the numbers between 1 and n.

- Each number must be used, and can only be used once.

- Permutations appear all over the place in the real world.

- There is a really nice formula that counts the total number of permutations of a given size.

- *Goal: Come up with this formula!*

1 2 3 4 5 6

462513

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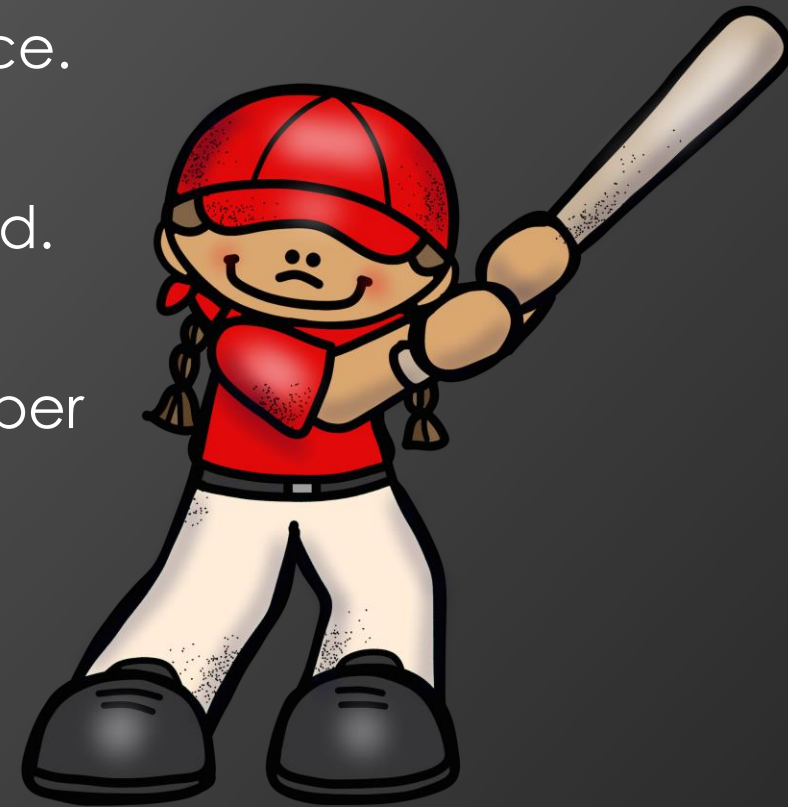# INVERSION NUMBER

- An *inversion* in a permutation is a pair of numbers such that the larger number appears to the left of the smaller one in the permutation.

- The *inversion number* of a permutation is the total number of inversions.

- One way to help calculate the inversion number is to look at each position in the permutation and count how many smaller numbers are to the right, and then add those numbers up.

- Inversion number can be thought of as a measure of how "out of order" a permutation is.

- *Goal: Find the inversion numbers of some permutations!*

462513

4 6 2 5 1 3

Inversion number: 3

4 **6** 2 5 1 3

Inversion number: 3 + 4

4 6 2 5 1 3

Inversion number: 3 + 4 + 1

4 6 2 5 1 3

Inversion number: 3 + 4 + 1 + 2

4 6 2 5 1 3

Inversion number: 3 + 4 + 1 + 2 + 0

4 6 2 5 1 3

Inversion number: 3 + 4 + 1 + 2 + 0 + 0

4 6 2 5 1 3

Inversion number: 3 + 4 + 1 + 2 + 0 + 0 = 10

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- Inversion number can be thought of as a measure of how "out of order" a permutation is.

- *Goal: Find the inversion numbers of some permutations!*

INVERSION NUMBER POLYNOMIALS

- We can use a polynomial to keep track of how many permutations of a given size n have a certain inversion number. Let's call this the "inversion number polynomial."

- Each permutation's contribution to this polynomial will be x raised to the power of that permutation's inversion number.

- To form the whole polynomial, we add up all these powers of x.

- It turns out these polynomials factor in a really nice, special way.

- *Goal: Construct these polynomials and find the nice factorization!*

INVERSION NUMBER POLYNOMIALS

$n = 2$:
$$x + 1$$

$n = 3$:
$$x^3 + 2x^2 + 2x + 1 = (x^2 + x + 1)(x + 1)$$

$n = 4$:
$$x^6 + 3x^5 + 5x^4 + 6x^3 + 5x^2 + 3x + 1 = (x^3 + x^2 + x + 1)(x^2 + x + 1)(x + 1)$$

General n:
$$(x^{n-1} + x^{n-2} + \cdots + x + 1)(x^{n-2} + x^{n-2} + \cdots + x + 1) \cdots (x^2 + x + 1)(x + 1)$$

DESCENTS AND MAJOR INDEX

- A *descent* in a permutation is a place where there are consecutive numbers such that first number is larger than the one immediately to its right.

- The position of the descent is how far from the beginning it occurs.

- If we keep track of all the positions where descents occur, and add up those numbers, we get something called the *major index*.

- *Goal: Find the major index of some permutations!*

Position: 1   2   3   4   5

4 6 2 5 1 3

Position: 1   2   3   4   5

4 6 2 5 1 3

Major index:

Position: 1 2 3 4 5

4 6 • 2 5 1 3

Major index: 2

Position: 1 2 3 4 5

4 6 2 5 1 3

Major index: 2

Position: 1   2   3   4   5

4 6 2 5 1 3

Major index: 2 + 4

Position: 1   2   3   4   5

4 6 2 5 1 3

Major index: 2 + 4 = 6

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- The position of the descent is how far from the beginning it occurs.

- If we keep track of all the positions where descents occur, and add up those numbers, we get something called the *major index*.

- ***Goal: Find the major index of some permutations!***

MAJOR INDEX POLYNOMIALS

- We can use a polynomial to keep track of how many permutations of a given size n have a certain major index. Let's call this the "major index polynomial."

- Each permutation's contribution to this polynomial will be x raised to the power of that major index.

- To form the whole polynomial, we add up all these powers of x.

- ***Goal: Construct these polynomials and make a surprising observation!***